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Isomonodromy transformations of linear systems of difference equations

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Abstract

We introduce and study "isomonodromy" transformations of the matrix linear difference equation $Y(z+1) = A(z)Y(z)$ with polynomial $A(z)$. Our main result is construction of an isomonodromy action of $\mathbb{Z}^{m(n+1)-1}$ on the space of coefficients $A(z)$ (here m is the size of matrices and n is the degree of $A(z)$). The (birational) action of certain rank n subgroups can be described by difference analogs of the classical Schlesinger equations, and we prove that for generic initial conditions these difference Schlesinger equations have a unique solution. We also show that both the classical Schlesinger equations and the Schlesinger transformations known in isomonodromy theory, can be obtained as limits of our action in two different limit regimes.

Similarly to the continuous case, for $m = n = 2$ the difference Schlesinger equations and their q-analogs yield discrete Painlevé equations; examples include dPII, dPIV, dPV, and q-PVI.

Introduction

In recent years there has been considerable interest in analyzing a certain class of discrete probabilistic models which in appropriate limits converge to well-known models of random matrix theory. The sources of these models are quite diverse, they include combinatorics, representation theory, percolation theory, random growth processes, tiling models and others.

One quantity of interest in both discrete models and their random matrix limits is the *gap probability* – the probability of having no particles in a given set. It is known, due to works of many people (see [JMMS], [Me], [TW], [P], [HI], [BD]), that in the continuous (random matrix type) setup these probabilities can be expressed through solution of an associated isomonodromy problem for a linear system of differential equations with rational coefficients.

The goal of this paper is to develop a general theory of "isomonodromy" transformations for linear systems of *difference* equations with rational coefficients. This subject is of interest in its own right. As an application of

the theory, we show in a subsequent publication that the gap probabilities in the discrete models mentioned above are expressible through solutions of isomonodromy problems for such systems of difference equations. In the case of one-interval gap probability this has been done (in a different language) in [Bor], [BB]. One example of the probabilistic models in question can be found at the end of this introduction.

Consider a matrix linear difference equation

$$Y(z+1) = A(z)Y(z). \tag{1}$$

Here

$$A(z) = A_0 z^n + A_1 z^{n-1} + \cdots + A_n, \qquad A_i \in \mathrm{Mat}(m, \mathbb{C}),$$

is a matrix polynomial and $Y : \mathbb{C} \to \mathrm{Mat}(m, \mathbb{C})$ is a matrix meromorphic function. [1] We assume that the eigenvalues of A_0 are nonzero and that their ratios are not real. Then, without loss of generality, we may assume that A_0 is diagonal.

It is a fundamental result proved by Birkhoff in 1911, that the equation (1) has two canonical meromorphic solutions $Y^l(z)$ and $Y^r(z)$, which are holomorphic and invertible for $\Re z \ll 0$ and $\Re z \gg 0$ respectively, and whose asymptotics at $z = \infty$ in any left (right) half-plane has a certain form. Birkhoff further showed that the ratio

$$P(z) = (Y^r(z))^{-1} Y^l(z),$$

which must be periodic for obvious reasons, is, in fact, a rational function in $\exp(2\pi i z)$. This rational function has just as many constants involved as there are matrix elements in $A_1, \ldots, A_n$. Let us call $P(z)$ the *monodromy matrix* of (1).

Other results of Birkhoff show that for any periodic matrix P of a specific form, there exists an equation of the form (1) with prescribed A_0, which has P as the monodromy matrix. Furthermore, if two equations with coefficients $A(z)$ and $\widetilde{A}(z)$, $\widetilde{A}_0 = A_0$, have the same monodromy matrix, then there exists a rational matrix $R(z)$ such that

$$\widetilde{A}(z) = R(z+1)A(z)R^{-1}(z). \tag{2}$$

The first result of this paper is a construction, for generic $A(z)$, of a homomorphism of $\mathbb{Z}^{m(n+1)-1}$ into the group of invertible rational matrix functions, such that the transformation (2) for any $R(z)$ in the image, does not change the monodromy matrix.

If we denote by $a_1, \ldots, a_{mn}$ the roots of the equation $\det A(z) = 0$ (called *eigenvalues of* $A(z)$) and by $d_1, \ldots, d_n$ certain uniquely defined exponents of the asymptotic behavior of a canonical solution $Y(z)$ of (1) at $z = \infty$, then

[1] Changing $Y(z)$ to $(\Gamma(z))^k Y(z)$ readily reduces a rational $A(z)$ to a polynomial one.

the action of $\mathbb{Z}^{m(n+1)-1}$ is uniquely defined by integral shifts of $\{a_i\}$ and $\{d_j\}$ with the total sum of all shifts equal to zero. (We assume that $a_i - a_j \notin \mathbb{Z}$ and $d_i - d_j \notin \mathbb{Z}$ for any $i \neq j$.)

The matrices $R(z)$ depend rationally on the matrix elements of $\{A_i\}_{i=1}^n$ and $\{a_i\}_{i=1}^{mn}$ (A_0 is always invariant), and define birational transformations of the varieties of $\{A_i\}$ with given $\{a_i\}$ and $\{d_j\}$.

There exist remarkable subgroups $\mathbb{Z}^n \subset \mathbb{Z}^{m(n+1)-1}$ which define birational transformations on the space of all $A(z)$ (with fixed A_0 and with no restrictions on the roots of $\det A(z)$), but to see this we need to parametrize $A(z)$ differently.

To define the new coordinates, we split the eigenvalues of $A(z)$ into n groups of m numbers each:

$$\{a_1, \dots, a_{mn}\} = \{a_1^{(1)}, \dots, a_m^{(1)}\} \cup \dots \cup \{a_1^{(n)}, \dots, a_m^{(n)}\}.$$

The splitting may be arbitrary. Then we define B_i to be the uniquely determined (remember, everything is generic) element of $\mathrm{Mat}(m, \mathbb{C})$ with eigenvalues $\left\{a_j^{(i)}\right\}_{j=1}^m$, such that $z - B_i$ is a right divisor of $A(z)$:

$$A(z) = (A_0 z^{n-1} + A_1' z^{n-1} + \dots + A_{n-1}')(z - B_i).$$

The matrix elements of $\{B_i\}_{i=1}^n$ are the new coordinates on the space of $A(z)$.

The action of the subgroup $\mathbb{Z}^n$ mentioned above consists of shifting the eigenvalues in any group by the same integer assigned to this group, and also shifting the exponents $\{d_i\}$ by the same integer (which is equal to minus the sum of the group shifts). If we denote by $\{B_i(k_1, \dots, k_n)\}$ the result of applying $k \in \mathbb{Z}^n$ to $\{B_i\}$, then the following equations are satisfied:

$$B_i(\dots) - B_i(\dots, k_j + 1, \dots) = B_j(\dots) - B_j(\dots, k_i + 1, \dots), \tag{3}$$

$$B_j(\dots, k_i + 1, \dots) B_i(\dots) = B_i(\dots, k_j + 1, \dots) B_j(\dots), \tag{4}$$

$$B_i(k_1 + 1, \dots, k_n + 1) = A_0^{-1} B_i(k_1, \dots, k_n) A_0 - I, \tag{5}$$

where $i, j = 1, \dots, n$, and dots in the arguments mean that other k_l's remain unchanged. We call them the *difference Schlesinger equations* for the reasons that will be clarified below. Note that (3) and (4) can be rewritten as

$$\big(z - B_i(\dots, k_j + 1, \dots)\big)\big(z - B_j(\dots)\big) = \big(z - B_j(\dots, k_i + 1, \dots)\big)\big(z - B_i(\dots)\big).$$

Independently of Birkhoff's general theory, we prove that the difference Schlesinger equations have a unique solution satisfying

$$\mathrm{Sp}(B_i(k_1, \dots, k_n)) = \mathrm{Sp}(B_i) - k_i, \qquad i = 1, \dots, n, \tag{6}$$

for an arbitrary nondegenerate A_0 and generic initial conditions $\{B_i = B_i(0)\}$. (The notation means that the eigenvalues of $B_i(k)$ are equal to those of B_i shifted by $-k_i$.) Moreover, the matrix elements of this solution are rational

functions in the matrix elements of the initial conditions. This is our second result.

In order to prove this claim, we introduce yet another set of coordinates on $A(z)$ with fixed A_0, which is related to $\{B_i\}$ by a birational transformation. It consists of matrices $C_i \in \mathrm{Mat}(m,\mathbb{C})$ with $\mathrm{Sp}(C_i) = \mathrm{Sp}(B_i)$ such that

$$A(z) = A_0(z - C_1)\cdots(z - C_n).$$

In these coordinates, the action of $\mathbb{Z}^n$ is described by the relations

$$\begin{aligned}(7)\quad &\big(z+1-C_i\big)\cdots\big(z+1-C_n\big)A_0\big(z-C_1\big)\cdots\big(z-C_{i-1}\big)\\ &\qquad=\big(z+1-\widetilde{C}_{i+1}\big)\cdots\big(z+1-\widetilde{C}_n\big)A_0\big(z-\widetilde{C}_1\big)\cdots\big(z-\widetilde{C}_i\big),\\ &C_j = C_j(k_1,\dots,k_n),\quad \widetilde{C}_j = C_j(k_1,\dots,k_{i-1},k_i+1,k_{i+1},\dots,k_n) \text{ for all } j.\end{aligned}$$

Again, we prove that there exists a unique solution to these equations satisfying $\mathrm{Sp}(C_i(k)) = \mathrm{Sp}(C_i) - k_i$, for an arbitrary invertible A_0 and generic $\{C_i = C_i(0)\}$. The solution is rational in the matrix elements of the initial conditions.

The difference Schlesinger equations have an autonomous limit which consists of (3), (4), and

$$B_i(k_1+1,\dots,k_n+1) = A_0^{-1}B_i(k_1,\dots,k_n)A_0, \tag{5-aut}$$

$$\mathrm{Sp}(B_i(k_1,\dots,k_n)) = \mathrm{Sp}(B_i), \qquad i = 1,\dots,n. \tag{6-aut}$$

The equation (7) then becomes

$$\begin{aligned}(7\text{-aut})\qquad &\big(z-C_i\big)\cdots\big(z-C_n\big)A_0\big(z-C_1\big)\cdots\big(z-C_{i-1}\big)\\ &\qquad=\big(z-\widetilde{C}_{i+1}\big)\cdots\big(z-\widetilde{C}_n\big)A_0\big(z-\widetilde{C}_1\big)\cdots\big(z-\widetilde{C}_i\big).\end{aligned}$$

The solutions of these equations were essentially obtained in [V] via a general construction of commuting flows associated with set-theoretical solutions of the quantum Yang-Baxter equation; see [V] for details and references.

The autonomous equations can also be explicitly solved in terms of abelian functions associated with the spectral curve $\{(z,w) : \det(A(z) - wI) = 0\}$,[2] very much in the spirit of [MV, §1.5]. We hope to explain the details in a separate publication.

The whole subject bears a strong similarity (and not just by name!) to the theory of isomonodromy deformations of linear systems of differential equations with rational coefficients:

$$\frac{d\mathcal{Y}(\zeta)}{d\zeta} = \left(\mathcal{B}_\infty + \sum_{k=1}^{n}\frac{\mathcal{B}_i}{\zeta - x_i}\right)\mathcal{Y}(\zeta), \tag{8}$$

[2]It is easy to see that the curve is invariant under the flows.

which was developed by Schlesinger around 1912 and generalized by Jimbo, Miwa, and Ueno in [JMU], [JM] to the case of higher order singularities. If we analytically continue any fixed (say, normalized at a given point) solution $\mathcal{Y}(\zeta)$ of (8) along a closed path γ in $\mathbb{C}$ avoiding the singular points $\{x_k\}$ then the columns of $\mathcal{Y}$ will change into their linear combinations: $\mathcal{Y} \mapsto \mathcal{Y}M_\gamma$. Here M_γ is a constant invertible matrix which depends only on the homotopy class of γ. It is called the *monodromy matrix* corresponding to γ. The monodromy matrices define a linear representation of the fundamental group of $\mathbb{C}$ with n punctures. The basic isomonodromy problem is to change the differential equation (8) so that the monodromy representation remains invariant.

There exist isomonodromy deformations of two types: continuous ones, when x_i move in the complex plane and $\mathcal{B}_i = \mathcal{B}_i(x)$ form a solution of a system of partial differential equations called *Schlesinger equations*, and discrete ones (called *Schlesinger transformations*), which shift the eigenvalues of $\mathcal{B}_i$ and exponents of $\mathcal{Y}(\zeta)$ at $\zeta = \infty$ by integers with the total sum of shifts equal to 0.

We prove that in the limit when

$$B_i = x_i\varepsilon^{-1} + \mathcal{B}_i, \qquad \varepsilon \to 0,$$

our action of $\mathbb{Z}^{m(n+1)-1}$ in the discrete case converges to the action of Schlesinger transformations on $\mathcal{B}_i$. This is our third result.

Furthermore, we argue that the "long-time" asymptotics of the $\mathbb{Z}^n$-action in the discrete case (that is, the asymptotics of $B_i([x_1\varepsilon^{-1}], \dots, [x_n\varepsilon^{-1}])$), ε small, is described by the corresponding solution of the Schlesinger equations. More exactly, we conjecture that the following is true.

Take $B_i = B_i(\varepsilon) \in \mathrm{Mat}(m, \mathbb{C})$, $i = 1, \dots, n$, such that

$$B_i(\varepsilon) - y_i\varepsilon^{-1} + \mathcal{B}_i \to 0, \qquad \varepsilon \to 0.$$

Let $B_i(k_1, \dots, k_n)$ be the solution of the difference Schlesinger equations (3.1)–(3.3) with the initial conditions $\{B_i(0) = B_i\}$, and let $\mathcal{B}_i(x_1, \dots, x_n)$ be the solution of the classical Schlesinger equations (5.4) with the initial conditions $\{\mathcal{B}_i(y_1, \dots, y_n) = \mathcal{B}_i\}$. Then for any $x_1, \dots, x_n \in \mathbb{R}$ and $i = 1, \dots, n$, we have

$$B_i\big([x_1\varepsilon^{-1}], \dots, [x_n\varepsilon^{-1}]\big) + [x_i\varepsilon^{-1}] - y_i\varepsilon^{-1} + \mathcal{B}_i(y_1 - x_1, \dots, y_n - x_n) \to 0, \ \ \varepsilon \to 0.$$

In support of this conjecture, we explicitly show that the difference Schlesinger equations converge to the conventional Schlesinger equations in the limit $\varepsilon \to 0$.

Note that the monodromy representation of $\pi_1(\mathbb{C} \setminus \{x_1, \dots, x_n\})$ which provides the integrals of motion for the Schlesinger flows, has no obvious analog in the discrete situation. On the other hand, the obvious differential analog of the periodic matrix P, which contains all integrals of motion in the case of difference equations, gives only the monodromy information at infinity and does not carry any information about local monodromies around the poles $x_1, \dots, x_n$.

Most of the results of the present paper can be carried over to the case of q-difference equations of the form $Y(qz) = A(z)Y(z)$. The q-difference Schlesinger equations are, cf. (3)–(6),

$$B_i(\dots) - B_i(\dots, q^{k_j+1}, \dots) = B_j(\dots) - B_j(\dots, q^{k_i+1}, \dots), \tag{3q}$$

$$B_j(\dots, q^{k_i+1}, \dots)B_i(\dots) = B_i(\dots, q^{k_j+1}, \dots)B_j(\dots), \tag{4q}$$

$$B_i(q^{k_1+1}, \dots, q^{k_n+1}) = q^{-1}A_0^{-1}B_i(q^{k_1}, \dots, q^{k_n})A_0, \tag{5q}$$

$$\operatorname{Sp}(B_i(q^{k_1}, \dots, q^{k_n})) = q^{-k_i}\operatorname{Sp}(B_i), \qquad i = 1, \dots, n. \tag{6q}$$

The q-analog of (7) takes the form

$$\begin{aligned}\big(z - q^{-1}C_i\big)\cdots\big(z - q^{-1}C_n\big)A_0\big(z - C_1\big)\cdots\big(z - C_{i-1}\big)\\ = \big(z - q^{-1}\widetilde{C}_{i+1}\big)\cdots\big(z - q^{-1}\widetilde{C}_n\big)A_0\big(z - \widetilde{C}_1\big)\cdots\big(z - \widetilde{C}_i\big),\end{aligned} \tag{7q}$$

$$C_j = C_j(q^{k_1}, \dots, q^{k_n}), \quad \widetilde{C}_j = C_j(q^{k_1}, \dots, q^{k_{i-1}}, q^{k_i+1}, q^{k_{i+1}}, \dots, q^{k_n}) \text{ for all } j.$$

A more detailed exposition of the q-difference case will appear elsewhere.

Similarly to the classical case, see [JM], discrete Painlevé equations of [JS], [Sak] can be obtained as reductions of the difference and q-difference Schlesinger equations when both m (the size of matrices) and n (the degree of the polynomial $A(z)$) are equal to two. For examples of such reductions see [Bor, §3] for difference Painlevé II equation (dPII), [Bor, §6] and [BB, §9] for dPIV and dPV, and [BB, §10] for q-PVI. This subject still remains to be thoroughly studied.

As was mentioned before, the difference and q-difference Schlesinger equations can be used to compute the gap probabilities for certain probabilistic models. We conclude this introduction by giving an example of such a model. We define the *Hahn orthogonal polynomial ensemble* as a probability measure on all l-point subsets of $\{0, 1, \dots, N\}$, $N > l > 0$, such that

$$\operatorname{Prob}\{(x_1, \dots, x_l)\} = \operatorname{const} \cdot \prod_{1 \le i < j \le l} (x_i - x_j)^2 \cdot \prod_{i=1}^{l} w(x_i),$$

where $w(x)$ is the weight function for the classical Hahn orthogonal polynomials:

$$w(x) = \binom{\alpha + x}{x}\binom{\beta + N - x}{N - x}, \qquad \alpha, \beta > -1 \text{ or } \alpha, \beta < -N.$$

This ensemble came up recently in harmonic analysis on the infinite-dimensional unitary group [BO, §11] and in a statistical description of tilings of a hexagon by rhombi [Joh, §4].

The quantity of interest is the probability that the point configuration $(x_1, \dots, x_l)$ does not intersect a disjoint union of intervals $[k_1, k_2] \sqcup \dots \sqcup [k_{2s-1}, k_{2s}]$. As a function in the endpoints $k_1, \dots, k_{2s} \in \{0, 1, \dots, N\}$; this

probability can be expressed through a solution of the difference Schlesinger equations (3)–(6) for 2×2 matrices with $n = \deg A(z) = s + 2$, $A_0 = I$,

$$\operatorname{Sp}(B_i) = \{-k_i, -k_i\}, \quad i = 1, \dots, 2s,$$
$$\operatorname{Sp}(B_{2s+1}) \sqcup \operatorname{Sp}(B_{2s+2}) = \{0, -\alpha, N+1, N+1+\beta\},$$

and with certain explicit initial conditions. The equations are also suitable for numerical computations, and we refer to [BB, §12] for examples of those in the case of a one interval gap.

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1. Birkhoff's theory

Consider a matrix linear difference equation of the first order

$$Y(z+1) = A(z)Y(z). \tag{1.1}$$

Here $A : \mathbb{C} \to \operatorname{Mat}(m, \mathbb{C})$ is a rational function (i.e., all matrix elements of $A(z)$ are rational functions of z) and $m \geq 1$. We are interested in matrix meromorphic solutions $Y : \mathbb{C} \to \operatorname{Mat}(m, \mathbb{C})$ of this equation.

Let n be the order of the pole of $A(z)$ at infinity, that is,

$$A(z) = A_0 z^n + A_1 z^{n-1} + \text{ lower order terms }.$$

We assume that (1.1) has a formal solution of the form

$$Y(z) = z^{nz} e^{-nz} \left(\hat{Y}_0 + \frac{\hat{Y}_1}{z} + \frac{\hat{Y}_2}{z^2} + \dots \right) \operatorname{diag}\left(\rho_1^z \, z^{d_1}, \dots, \rho_m^z \, z^{d_m} \right) \tag{1.2}$$

with $\rho_1, \dots, \rho_m \neq 0$ and $\det \hat{Y}_0 \neq 0$.[3]

It is easy to see that if such a formal solution exists then $\rho_1, \dots, \rho_m$ must be the eigenvalues of A_0, and the columns of $\hat{Y}_0$ must be the corresponding eigenvectors of A_0.

Note that for any invertible $T \in \operatorname{Mat}(m, \mathbb{C})$, $(TY)(z)$ solves the equation

$$(TY)(z+1) = \left(TA(z)T^{-1}\right)(TY)(z).$$

Thus, if A_0 is diagonalizable, we may assume that it is diagonal without loss of generality. Similarly, if $A_0 = I$ and A_1 is diagonalizable, we may assume that A_1 is diagonal.

[3]Substituting (1.2) in (1.1) we use the expansion $\left(\frac{z+1}{z}\right)^{nz} = e^{nz \ln(1+z^{-1})} = e^n - \frac{ne^n}{2z} + \dots$ to compare the two sides.

PROPOSITION 1.1. *If $A_0 = \operatorname{diag}(\rho_1, \dots, \rho_m)$, where $\{\rho_i\}_{i=1}^m$ are nonzero and pairwise distinct, then there exists a unique formal solution of* (1.1) *of the form* (1.2) *with $\hat{Y}_0 = I$.*

Proof. It suffices to consider the case $n = 0$; the general case is reduced to it by considering $(\Gamma(z))^n Y(z)$ instead of $Y(z)$, because

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z} \left(1 + \frac{1}{12}\, z^{-1} + \dots\right).$$

(More precisely, this expression formally solves $\Gamma(z+1) = z\Gamma(z)$.)

Thus, we assume $n = 0$. Then we substitute (1.2) into (1.1) and compute $\hat{Y}_k$ one by one by equating the coefficients of z^{-l}, $l = 0, 1, \dots$. If $\hat{Y}_0 = I$ then the constant coefficients of both sides are trivially equal. The coefficients of z^{-1} give

$$\hat{Y}_1 A_0 + \operatorname{diag}(\rho_1 d_1, \dots, \rho_m d_m) = A_0 \hat{Y}_1 + A_1. \tag{1.3}$$

This equality uniquely determines $\{d_i\}$ and the off-diagonal entries of $\hat{Y}_1$, because

$$[\hat{Y}_1, A_0]_{ij} = (\rho_j - \rho_i)(\hat{Y}_1)_{ij}\,.$$

Comparing the coefficients of z^{-2} we obtain

$$(\hat{Y}_2 - \hat{Y}_1)A_0 + \hat{Y}_1 \operatorname{diag}(\rho_1 d_1, \dots, \rho_m d_m) + \dots = A_0 \hat{Y}_2 + A_1 \hat{Y}_1 + \dots\,,$$

where the dots stand for the terms which we already know (that is, those which depend only on ρ_i's, d_i's, A_i's, and $\hat{Y}_0 = I$). Since the diagonal values of A_1 are exactly $\rho_1 d_1, \dots \rho_n d_n$ by (1.3), we see that we can uniquely determine the diagonal elements of $\hat{Y}_1$ and the off-diagonal elements of $\hat{Y}_2$ from the last equality.

Now let us assume that we already determined $\hat{Y}_1, \dots, \hat{Y}_{l-2}$ and the off-diagonal entries of $\hat{Y}_{l-1}$ by satisfying (1.1) up to order $l-1$. Then comparing the coefficients of z^{-l} we obtain

$$(\hat{Y}_l - (l-1)\hat{Y}_{l-1})A_0 + \hat{Y}_{l-1} \operatorname{diag}(\rho_1 d_1, \dots, \rho_m d_m) + \dots = A_0 \hat{Y}_l + A_1 \hat{Y}_{l-1} + \dots\,,$$

where the dots denote the terms depending only on ρ_i's, d_i's, A_i's, and $\hat{Y}_0, \dots, \hat{Y}_{l-2}$. This equality allows us to compute the diagonal entries of Y_{l-1} and the off-diagonal entries of Y_l. Induction on l completes the proof. □

The condition that the eigenvalues of A_0 are distinct is not necessary for the existence of the asymptotic solution, as our next proposition shows.

PROPOSITION 1.2. *Assume that $A_0 = I$ and $A_1 = \operatorname{diag}(r_1, \dots, r_n)$ where $r_i - r_j \notin \{\pm 1, \pm 2, \dots\}$ for all $i, j = 1, \dots, n$. Then there exists a unique formal solution of* (1.1) *of the form* (1.2) *with $\hat{Y}_0 = I$.*

Proof. As in the proof of Proposition 1.1, we may assume that $n = 0$. Comparing constant coefficients we see that $\rho_1 = \cdots = \rho_m = 1$. Then equating the coefficients of z^{-1} we find that $d_i = r_i$, $i = 1, \dots, m$. Furthermore, equating the coefficients of z^{-l}, $l \geq 2$ we find that

$$[\hat{Y}_{l-1}, A_1] - (l-1)\hat{Y}_{l-1}$$

is expressible in terms of A_i's and $\hat{Y}_1, \dots, \hat{Y}_{l-2}$. This allows us to compute all $\hat{Y}_i$'s recursively. □

We call two complex numbers z_1 and z_2 *congruent* if $z_1 - z_2 \in \mathbb{Z}$.

THEOREM 1.3 (G. D. Birkhoff [Bi1, Th. III]). *Assume that*

$$A_0 = \operatorname{diag}(\rho_1, \dots, \rho_m),$$

$$\rho_i \neq 0, \quad i = 1, \dots, m, \qquad \rho_i/\rho_j \notin \mathbb{R} \ \text{ for all } \ i \neq j.$$

Then there exist unique solutions $Y^l(z)$ $(Y^r(z))$ *of* (1.1) *such that*:

(a) *The function* $Y^l(z)$ $(Y^r(z))$ *is analytic throughout the complex plane except possibly for poles to the right* (*left*) *of and congruent to the poles of* $A(z)$ (*respectively,* $A^{-1}(z-1)$);

(b) *In any left* (*right*) *half-plane* $Y^l(z)$ $(Y^r(z))$ *is asymptotically represented by the right-hand side of* (1.2).

Remark 1.4. Part (b) of the theorem means that for any $k = 0, 1, \dots,$

$$\left| Y^{l,r}(z)\, z^{-nz} e^{nz} \operatorname{diag}(\rho_1^{-z} z^{-d_1}, \dots, \rho_m^{-z} z^{-d_m}) - \hat{Y}_0 - \frac{\hat{Y}_1}{z} - \cdots - \frac{\hat{Y}_{k-1}}{z^{k-1}} \right| \leq \frac{\text{const}}{z^k}$$

for large $|z|$ in the corresponding domain.

Theorem 1.3 holds for any (fixed) choices of branches of $\ln(z)$ in the left and right half-planes for evaluating $z^{-nz} = e^{-nz\ln(z)}$ and $z^{-d_k} = e^{-d_k \ln(z)}$, and of a branch of $\ln(\rho)$ with a cut not passing through $\rho_1, \dots, \rho_m$ for evaluating $\rho_k^{-z} = e^{-z\ln\rho_k}$. Changing these branches yields the multiplication of $Y^{l,r}(z)$ by a diagonal periodic matrix on the right.

Remark 1.5. Birkhoff states Theorem 1.3 under a more general assumption: he only assumes that the equation (1.1) has a formal solution of the form (1.2). However, as pointed out by P. Deligne, Birkhoff's proof has a flaw in case one of the ratios ρ_i/ρ_j is real. The following counterexample was kindly communicated to me by Professor Deligne.

Consider the equation (1.1) with $m = 2$ and

$$A(z) = \begin{bmatrix} 1 & 1/z \\ 0 & 1/e \end{bmatrix}.$$

The formal solution (1.2) has the form

$$Y(z) = \left(I + \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} z^{-1} + \dots \right) \begin{bmatrix} 1 & 0 \\ 0 & e^{-z} \end{bmatrix}$$

with $a = e/(1-e)$.

Actual solutions that we care about have the form

$$Y(z) = \begin{bmatrix} 1 & u(z) \\ 0 & e^{-z} \end{bmatrix}$$

where $u(z)$ is a solution of $u(z+1) = u(z) + e^{-z}/z$. In a right half-plane we can take

$$u^r(z) = -\sum_{n=0}^{\infty} \frac{e^{-(z+n)}}{z+n}\,.$$

The first order approximation of $u^r(z)$ anywhere except near nonpositive integers is

$$u^r(z) \sim -\sum_{n=0}^{\infty} \frac{e^{-(z+n)}}{z} = \frac{ae^{-z}}{z}\,.$$

Next, terms can be obtained by expanding $1/(z+n)$.

In order to obtain a solution which behaves well on the left, it suffices to cancel the poles:

$$u^l(z) = u^r(z) + \frac{2\pi i}{e^{2\pi i z} - 1}\,.$$

The corresponding solution $Y^l(z)$ has the needed asymptotics in sectors of the form $\pi/2 + \varepsilon < \arg z < 3\pi/2 + \varepsilon$, but it has the wrong asymptotic behavior as $z \to +i\infty$. Indeed, $\lim_{z \to +i\infty} u^l(z) = -2\pi i$.

On the other hand, we can take

$$\widetilde{u}^l(z) = u^l(z) + 2\pi i = u^r(z) + \frac{2\pi i\, e^{2\pi i z}}{e^{2\pi i z} - 1}\,,$$

which has the correct asymptotic behavior in $\pi/2 - \varepsilon < \arg z < 3\pi/2 - \varepsilon$, but fails to have the needed asymptotics at $-i\infty$.

Remark 1.6. In the case when $|\rho_1| > |\rho_2| > \dots > |\rho_m| > 0$, a result similar to Theorem 1.3 was independently proved by R. D. Carmichael [C]. He considered the asymptotics of solutions along lines parallel to the real axis only. Birkhoff also referred to [N] and [G] where similar results had been proved somewhat earlier.

Now let us restrict ourselves to the case when $A(z)$ is a polynomial in z. The general case of rational $A(z)$ is reduced to the polynomial case by the following transformation. If $(z - x_1) \cdots (z - x_s)$ is the common denominator of $\{A_{kl}(z)\}$ (the matrix elements of $A(z)$), then

$$\bar{Y}(z) = \Gamma(z - x_1) \cdots \Gamma(z - x_s) \cdot Y(z)$$

solves $\bar{Y}(z+1) = \bar{A}(z)\bar{Y}(z)$ with polynomial

$$\bar{A}(z) = (z - x_1)\cdots(z - x_s)A(z).$$

Note that the ratio $P(z) = (Y^r(z))^{-1}\, Y^l(z)$ is a periodic function. (The relation $P(z+1) = P(z)$ immediately follows from the fact that $Y^{l,r}$ solves (1.1).) From now on let us fix the branches of $\ln(z)$ in the left and right half-planes mentioned in Remark 1.4 so that they coincide in the upper half-plane. Then the structure of $P(z)$ can be described more precisely.

THEOREM 1.7 ([Bi1, Th. IV]). *With the assumptions of Theorem* 1.3, *the matrix elements $p_{kl}(z)$ of the periodic matrix $P(z) = (Y^r(z))^{-1}Y^l(z)$ have the form*

$$p_{kk}(z) = 1 + c_{kk}^{(1)}e^{2\pi iz} + \cdots + c_{kk}^{(n-1)}e^{2\pi(n-1)iz} + e^{2\pi id_k}e^{2\pi niz},$$
$$p_{kl}(z) = e^{2\pi\lambda_{kl}z}\left(c_{kl}^{(0)} + c_{kl}^{(1)}e^{2\pi iz} + \cdots + c_{kl}^{(n-1)}e^{2\pi(n-1)iz}\right) \quad (k \neq l),$$

where $c_{kl}^{(s)}$ are some constants, and λ_{kl} denotes the least integer as great as the real part of $(\ln(\rho_l) - \ln(\rho_k))/2\pi i$.

Thus, starting with a matrix polynomial $A(z) = A_0z^n + A_1z^{n-1} + \cdots + A_n$ with nondegenerate $A_0 = \operatorname{diag}(\rho_1, \ldots, \rho_m)$, $\rho_k \neq \rho_l$ for $k \neq l$, we construct the *characteristic constants* $\{d_k\}$, $\{c_{kl}^{(s)}\}$ using Proposition 1.1 and Theorems 1.3, 1.7.

Note that the total number of characteristic constants is exactly the same as the number of matrix elements in matrices $A_1, \ldots, A_n$. Thus, it is natural to ask whether the map

$$(A_1, \ldots, A_n) \mapsto \left(\{d_k\}, \{c_{kl}^{(s)}\}\right)$$

is injective or surjective (the constants $\rho_1, \ldots, \rho_n$ being fixed). The following partial results are available.

THEOREM 1.8 ([Bi2, §17]). *For any nonzero $\rho_1, \ldots, \rho_m$, $\rho_i/\rho_j \notin \mathbb{R}$ for $i \neq j$, there exist matrices $A_1, \ldots, A_n$ such that the equation* (1.1) *with $A_0 = \operatorname{diag}(\rho_1, \ldots \rho_m)$ either possesses the prescribed characteristic constants $\{d_k\}$, $\{c_{kl}^{(s)}\}$, or else constants $\{d_k + l_k\}$, $\{c_{kl}^{(s)}\}$, where $l_1, \ldots, l_m$ are integers.*

THEOREM 1.9 ([Bi1, Th. VII]). *Assume there are two matrix polynomials $A'(z) = A'_0z^n + \cdots + A'_n$ and $A''(z) = A''_0z^n + \cdots + A''_n$ with*

$$A'_0 = A''_0 = \operatorname{diag}(\rho_1, \ldots \rho_m), \qquad \rho_k \neq 0, \quad \rho_k/\rho_l \notin \mathbb{R} \text{ for } k \neq l,$$

such that the sets of the characteristic constants for the equations $Y'(z+1) = A'(z)Y'(z)$ and $Y''(z+1) = A''(z)Y''(z)$ are equal. Then there exists a rational matrix $R(z)$ such that

$$A''(z) = R(z+1)A'(z)R^{-1}(z),$$

and the left and right canonical solutions $Y^{l,r}$ of the second equation can be obtained from those of the first equation by multiplication by R on the left:

$$(Y'')^{l,r} = R\,(Y')^{l,r}.$$

2. Isomonodromy transformations

The goal of this section is to construct explicitly, for given $A(z)$, rational matrices $R(z)$ such that the transformation $A(z) \mapsto R(z+1)A(z)R^{-1}(z)$, cf. Theorem 1.9 above, preserves the characteristic constants (more generally, preserves $\{c_{kl}^{(s)}\}$ and shifts d_k's by integers).

Let $A(z)$ be a matrix polynomial of degree $n \geq 1$, $A_0 = \operatorname{diag}(\rho_1, \dots, \rho_m)$, and ρ_i's are nonzero and their ratios are not real. Fix mn complex numbers $a_1, \dots, a_{mn}$ such that $a_i - a_j \notin \mathbb{Z}$ for any $i \neq j$. Denote by $\mathcal{M}(a_1, \dots, a_{mn}; d_1, \dots, d_m)$ the algebraic variety of all n-tuples of m by m matrices $A_1, \dots, A_n$ such that the scalar polynomial

$$\det A(z) = \det(A_0 z^n + A_1 z^{n-1} + \cdots + A_n)$$

of degree mn has roots $a_1, \dots, a_{mn}$, and $\rho_i \left(d_i - \frac{n}{2}\right) = (A_1)_{ii}$ (this comes from the analog of (1.3) for arbitrary n).

THEOREM 2.1. *For any $\kappa_1, \dots, \kappa_{mn} \in \mathbb{Z}$, $\delta_1, \dots, \delta_m \in \mathbb{Z}$,*

$$\sum_{i=1}^{mn} \kappa_i + \sum_{j=1}^{m} \delta_j = 0,$$

there exists a nonempty Zariski open subset $\mathcal{A}$ of $\mathcal{M}(a_1, \dots, a_{mn}; d_1, \dots, d_m)$ such that for any $(A_1, \dots, A_n) \in \mathcal{A}$ there exists a unique rational matrix $R(z)$ with the following properties:

$$\widetilde{A}(z) = R(z+1)A(z)R^{-1}(z) = \widetilde{A}_0 z^n + \widetilde{A}_1 z^{n-1} + \cdots + \widetilde{A}_n, \quad \widetilde{A}_0 = A_0,$$
$$(\widetilde{A}_1, \dots, \widetilde{A}_n) \in \mathcal{M}(a_1 + \kappa_1, \dots, a_{mn} + \kappa_{mn}; d_1 + \delta_1, \dots, d_m + \delta_m),$$

and the left and right canonical solutions of $\widetilde{Y}(z+1) = \widetilde{A}(z)\widetilde{Y}(z)$ have the form

$$\widetilde{Y}^{l,r} = R\,Y^{l,r},$$

where $Y^{l,r}$ are left and right canonical solutions of $Y(z+1) = A(z)Y(z)$.

The map $(A_1, \dots, A_n) \mapsto (\widetilde{A}_1, \dots, \widetilde{A}_n)$ is a birational map of algebraic varieties.

Remark 2.2. The theorem implies that the characteristic constants $\{c_{kl}^{(s)}\}$ for the difference equations with coefficients A and $\widetilde{A}$ are the same, while the constants d_k are being shifted by $\delta_k \in \mathbb{Z}$.

Note also that if we require that all d_k's do not change, then, by virtue of Theorem 1.9, Theorem 2.1 provides all possible transformations which preserve the characteristic constants. Indeed, if $A''(z) = R(z+1)A'(z)R^{-1}(z)$ then zeros of $\det A''(z)$ must be equal to those of $\det A'(z)$ shifted by integers.

Proof. Let us prove the uniqueness of R first. Assume that there exist two rational matrices R_1 and R_2 with needed properties. This means, in particular, that the determinants of the matrices

$$\widetilde{A}^{(1)} = R_1(z+1)A(z)R_1^{-1}(z) \quad \text{and} \quad \widetilde{A}^{(2)} = R_2(z+1)A(z)R_2^{-1}(z)$$

vanish at the same set of mn points $\widetilde{a}_i = a_i + \kappa_i$, none of which are different by an integer. Denote by $\widetilde{Y}_1^r = R_1 Y^r$ and $\widetilde{Y}_2^r = R_2 Y^r$ the right canonical solutions of the corresponding equations. Then $\widetilde{Y}_1^r(\widetilde{Y}_2^r)^{-1} = R_1 R_2^{-1}$ is a rational matrix which tends to I at infinity. Moreover,

$$\left(\widetilde{Y}_1^r(\widetilde{Y}_2^r)^{-1}\right)(z+1) = \widetilde{A}^{(1)}(z)\,\left(\widetilde{Y}_1^r(\widetilde{Y}_2^r)^{-1}\right)(z)\,\left(\widetilde{A}^{(2)}(z)\right)^{-1}.$$

Since $\widetilde{Y}_1^r(\widetilde{Y}_2^r)^{-1}$ is holomorphic for $\Re z \ll 0$, the equation above implies that this function may only have poles at the points which are congruent to $\widetilde{a}_i$ (zeros of $\det \widetilde{A}^{(2)}(z)$) and to the right of them. (Recall that two complex numbers are congruent if their difference is an integer.) But since $\widetilde{Y}_1^r(\widetilde{Y}_2^r)^{-1}$ is also holomorphic for $\Re z \gg 0$, the same equation rewritten as

$$\left(\widetilde{Y}_1^r(\widetilde{Y}_2^r)^{-1}\right)(z) = \left(\widetilde{A}^{(1)}(z)\right)^{-1}\,\left(\widetilde{Y}_1^r(\widetilde{Y}_2^r)^{-1}\right)(z+1)\,\widetilde{A}^{(2)}(z)$$

implies that this function may only have poles at the points $\widetilde{a}_i$ (zeros of $\det \widetilde{A}^{(1)}(z)$) or at the points congruent to them and to the left of them. Thus, $\widetilde{Y}_1^r(\widetilde{Y}_2^r)^{-1} = R_1 R_2^{-1}$ is entire, and by Liouville's theorem it is identically equal to I. The proof of uniqueness is complete.

To prove the existence we note, first of all, that it suffices to provide a proof if one of the κ_i's is equal to ± 1 and one of the δ_j's is equal to ∓ 1 with all other κ's and δ's equal to zero. The proof will consist of several steps.

LEMMA 2.3. *Let $A(z)$ be an m by m matrix-valued function holomorphic near $z = a$, and $\det A(z) = c(z-a) + O\left((z-a)^2\right)$ as $z \to a$, where $c \neq 0$. Then there exists a unique (up to a constant) nonzero vector $v \in \mathbb{C}^m$ such that $A(a)v = 0$. Furthermore, if $B(z)$ is another matrix-valued function which is holomorphic near $z = a$, then $(BA^{-1})(z)$ is holomorphic near $z = a$ if and only if $B(a)v = 0$.*

Proof. Let us denote by E_1 the matrix unit which has 1 as its $(1,1)$-entry and 0 as all other entries. Since $\det A(a) = 0$, there exists a nondegenerate constant matrix C such that $A(a)CE_1 = 0$ (the first column of C must be a 0-eigenvector of $A(a)$). This implies that

$$H(z) = A(z)C(E_1(z-a)^{-1} + I - E_1)$$

is holomorphic near $z = a$. On the other hand, $\det H(a) = c \det C \neq 0$. Thus, $A(a) = H(a)(I - E_1)C^{-1}$ annihilates a vector v if and only if $C^{-1}v$ is proportional to $(1, 0, \dots, 0)^t$. Hence, v must be proportional to the first column of C. The proof of the first part of the lemma is complete.

To prove the second part, we notice that

$$(BA^{-1})(z) = B(z)C(E_1(z-a)^{-1} + I - E_1)H^{-1}(z)$$

which is bounded at $z = a$ if and only if $B(a)CE_1 = 0$. □

More generally, we will denote by E_i the matrix unit defined by

$$(E_i)_{kl} = \begin{cases} 1, & k = l = i, \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 2.4 ([JM, §2 and Appendix A]). *For any nonzero vector* $v = (v_1, \dots, v_m)^t$, $Q \in \mathrm{Mat}(m, \mathbb{C})$, $a \in \mathbb{C}$, *and* $i \in \{1, \dots, m\}$, *there exists a linear matrix-valued function* $R(z) = R_{-1}(z - a) + R_0$ *with the properties*

$$R(z)\left(I + Qz^{-1} + O\left(z^{-2}\right)\right) z^{-E_i} = I + O(z^{-1}), \quad z \to \infty,$$
$$R_0\, v = 0,$$

if and only if $v_i \neq 0$. *In this case,* $R^{\pm 1}(z)$ *is given by*

$$R(z) = E_i(z-a) + R_0, \quad R^{-1}(z) = I - E_i + R_1(z-a)^{-1}, \quad \det R(z) = z - a,$$

$$(R_0)_{kl} = \begin{cases} v_i^{-1} \sum_{s \neq i} Q_{is} v_s, & k = l = i, \\ -Q_{il}, & k = i,\ l \neq i, \\ -v_i^{-1} v_k, & k \neq i,\ l = i, \\ \delta_{kl}, & k \neq i,\ l \neq i, \end{cases}$$

$$R_1 = v_i^{-1} v \left[Q_{i1}, \dots, Q_{i,i-1}, 1, Q_{i,i+1}, \dots, Q_{im}\right].$$

The proof is straightforward.

Now we return to the proof of Theorem 2.1. Assume that $\kappa_1 = -1$, $\delta_i = 1$ for some $i = 1, \dots, m$, and all other κ's and δ's are zero. Since a_1 is a simple root of $\det A(z)$, by Lemma 2.3 there exists a unique (up to a constant) vector v such that $A(a)v = 0$. Clearly, the condition $v_i \neq 0$ defines a nonempty Zariski open subset of $\mathcal{M}(a_1, \dots, a_{mn}; \delta_1, \dots; \delta_m)$. On this subset, let us take $R(z)$ to be the matrix afforded by Lemma 2.4 with $a = a_1$ and $Q = \hat{Y}_1$ (we assume that $\hat{Y}_0 = I$, see Proposition 1.1). Then by the second part of Lemma 2.3, $(A(z)R^{-1}(z))^{-1} = R(z)A^{-1}(z)$ is holomorphic and invertible near $z = a_1$ (the invertibility follows from the fact that $\det R(z)A^{-1}(z)$ tends to a nonzero value as $z \to a_1$). Thus, $\widetilde{A}(z) = R(z+1)A(z)R^{-1}(z)$ is entire, hence, it is a polynomial. Since

$$\det \widetilde{A}(z) = \frac{z + 1 - a_1}{z - a_1} \det A(z) = c\,(z + 1 - a_1)(z - a_2) \cdots (z - a_{mn}), \quad c \neq 0,$$

the degree of $A(z)$ is $\geq n$. Looking at the asymptotics at infinity, we see that $\deg A(z) \leq n$, which means that $\widetilde{A}$ is a polynomial of degree n:

$$\widetilde{A}(z) = \widetilde{A}_0 z^n + \cdots + \widetilde{A}_n, \quad \widetilde{A}_0 \neq 0.$$

Denote by $Y^{l,r}$ the left and right canonical solutions of $Y(z+1) = A(z)Y(z)$ (see Theorem 1.3 above). Then $\widetilde{Y}^{l,r} := RY^{l,r}$ are solutions of $\widetilde{Y}(z+1) = \widetilde{A}(z)\widetilde{Y}(z)$. Moreover, their asymptotics at infinity at any left (right) half-plane, by Lemma 2.4, is given by an expansion of the form (1.2) with $\hat{\widetilde{Y}}_0 = I$, $\widetilde{\rho}_k = \rho_k$ for all $k = 1, \dots, m$, and

$$\widetilde{d}_k = \begin{cases} d_k + 1, & k = i, \\ d_k, & k \neq i. \end{cases}$$

This implies that $\widetilde{A}_0 = \operatorname{diag}(\rho_1, \dots, \rho_m)$, and that $\widetilde{Y}^{l,r}$ are the left and right canonical solutions of the equation $\widetilde{Y}(z+1) = \widetilde{A}(z)\widetilde{Y}(z)$. Indeed, their asymptotic expansion at infinity must also be a formal solution of the equation, the fact that $\widetilde{Y}^{l,r}$ are holomorphic for $\Re z \ll 0$ ($\gg 0$) follows from the analogous property for $Y^{l,r}$, and the location of possible poles of $\widetilde{Y}^r$ is easily determined from the equation.

For future reference let us also find a (unique up to a constant) vector $\widetilde{v}$ such that $\widetilde{A}^t(a_1 - 1)\,\widetilde{v} = 0$. This means that $R^{-t}(a_1 - 1)A^t(a_1 - 1)R^t(a_1)\,\widetilde{v} = 0$. Lemma 2.4 then implies that

$$\widetilde{v} = \left[(\hat{Y}_1)_{i1}, \dots, (\hat{Y}_1)_{i,i-1}, 1, (\hat{Y}_1)_{i,i+1}, \dots, (\hat{Y}_1)_{im}\right]^t$$

is a solution. Note that $\widetilde{v}_i \neq 0$.

Now let us assume that $\kappa_1 = 1$ and $\delta_i = -1$ for some $i = 1, \dots, m$. By Lemma 2.3, there exists a unique (up a to a constant) vector w such that $A^t(a_1)w = 0$. The condition $w_i \neq 0$ defines a nonempty Zariski open subset of $\mathcal{M}(a_1, \dots, a_{mn}; \delta_1, \dots \delta_m)$. On this subset, denote by $R'(z)$ the rational matrix-valued function afforded by Lemma 2.4 with $a = a_1$, $v = w$, and $Q = -\hat{Y}_1^t$ (again, we assume that $\hat{Y}_0 = I$). Set

$$R(z) := (R')^{-t}(z - 1).$$

Then by Lemma 2.4

$$R(z)\left(I + \hat{Y}_1 z^{-1} + O\left(z^{-2}\right)\right) z^{E_i} = I + O(z^{-1}), \quad z \to \infty.$$

Furthermore, by Lemma 2.3, $R^{-t}(z+1)A^{-t}(z)$ is holomorphic and invertible near $z = a_1$. Hence, $\widetilde{A}(z) = R(z+1)A(z)R^{-1}(z)$ is entire (note that $R^{-1}(z) = (R')^t(z-1)$ is linear in z). The rest of the argument is similar to the case $\kappa_1 = -1$, $\delta_i = 1$ considered above.

Finding a solution $\widetilde{w}$ to $\widetilde{A}(a_1+1)\widetilde{w}=0$ is equivalent to finding a solution to $R'(a_1)\widetilde{w}=0$. One such solution has the form

$$\widetilde{w}=\left[-(\hat{Y}_1)_{1i},\dots,-(\hat{Y}_1)_{i-1,i},1,-(\hat{Y}_1)_{i+1,i},\dots,-(\hat{Y}_1)_{mi}\right]^t$$

and all others are proportional to it. Note that its ith coordinate is nonzero.

From what was said above, it is obvious that the image of the map

$$\mathcal{M}(a_1,\dots,a_{mn};\delta_1,\dots,\delta_m)\to\mathcal{M}(a_1-1,\dots,a_{mn};\delta_1,\dots,\delta_i+1,\dots,\delta_m)$$

is in the domain of definition of the map

$$\mathcal{M}(a_1-1,\dots,a_{mn};\delta_1,\dots,\delta_i+1,\dots,\delta_m)\to\mathcal{M}(a_1,\dots,a_{mn};\delta_1,\dots,\delta_m)$$

and the other way around. On the other hand, the composition of these maps in either order must be equal to the identity map due to the uniqueness argument in the beginning of the proof. Hence, these maps are inverse to each other, and they establish a bijection between their domains of definition. The rationality of the maps follows from the explicit formula for $R(z)$ in Lemma 2.4. The proof of Theorem 2.1 is complete. □

Remark 2.5. Quite similarly to Lemma 2.4, the multiplier $R(z)$ can be computed in the cases when two κ's are equal to ±1 or two δ's are equal to ±1 with all other κ's and δ's being zero; cf. [JM].

Assume $\kappa_i=-1$ and $\kappa_j=1$. Denote by v and w the solutions of $A(a_i)\,v=0$ and $A^t(a_j)\,w=0$. Then R exists if and only if $(v,w):=v^tw=w^tv\neq0$, in which case

$$R(z)=I+\frac{R_0}{z-a_j-1},\quad R^{-1}(z)=I-\frac{R_0}{z-a_i},\quad \det R(z)=\frac{z-a_i}{z-a_j-1},$$
$$R_0=\frac{a_j-a_i+1}{(v,w)}\,vw^t.$$

Now assume $\delta_i=1$, $\delta_j=-1$. Then we must have $\det R(z)=1$ and

$$R(z)\left(I+\hat{Y}_1z^{-1}+\hat{Y}_2z^{-2}+O(z^{-3})\right)z^{E_j-E_i}=I+O(z^{-1}),\quad z\to\infty.$$

The solution exists if and only if $(\hat{Y}_1)_{ij}\neq0$, in which case it has the form

$$R(z)=E_iz+R_0,\quad R^{-1}(z)=E_jz+R_0^{-1},$$

with $(R_0)_{kl}$ given by

	$l=i$	$l=j$	$l\neq i,j$
$k=i$	$\frac{-(\hat{Y}_2)_{ij}+\sum_{s\neq i}(\hat{Y}_1)_{is}(\hat{Y}_1)_{sj}}{(\hat{Y}_1)_{ij}}$	$-(\hat{Y}_1)_{ij}$	$-(\hat{Y}_1)_{il}$
$k=j$	$\frac{1}{(\hat{Y}_1)_{ij}}$	0	0
$k\neq i,j$	$-\frac{(\hat{Y}_1)_{kj}}{(\hat{Y}_1)_{ij}}$	0	δ_{kl},

and $(R_0^{-1})_{kl}$ given by

$$\begin{array}{cccc} & l=i & l=j & l\neq i,j \\ k=i & 0 & (\hat{Y}_1)_{ij} & 0 \\ k=j & -\frac{1}{(\hat{Y}_1)_{ij}} & -\frac{(\hat{Y}_2)_{ij}}{(\hat{Y}_1)_{ij}}+(\hat{Y}_1)_{jj} & -\frac{(\hat{Y}_1)_{il}}{(\hat{Y}_1)_{ij}} \\ k\neq i,j & 0 & (\hat{Y}_1)_{kj} & \delta_{kl}\ . \end{array}$$

3. Difference Schlesinger equations

In this section we give a different description for the transformations $A \mapsto \widetilde{A}$ of Theorem 2.1 with

$$\kappa_{i_1} = \cdots = \kappa_{i_m} = \pm 1, \quad \delta_1 = \cdots = \delta_m = \mp 1,$$

and all other κ_i's equal to zero, and for compositions of such transformations.

In what follows we always assume that our matrix polynomials $A(z) = A_0 z^n + \dots$ have nondegenerate highest coefficients: $\det A_0 \neq 0$. We also assume that mn roots of the equation $\det A(z) = 0$ are pairwise distinct; we will call them the *eigenvalues* of $A(z)$. For an eigenvalue a, there exists a (unique) nonzero vector v such that $A(a)\,v = 0$, see Lemma 2.3. We will call v the *eigenvector* of $A(z)$ corresponding to the eigenvalue a. The word *generic* everywhere below stands for "belonging to a Zariski open subset" of the corresponding algebraic variety.

We start with few simple preliminary lemmas.

LEMMA 3.1. *The sets of eigenvalues and corresponding eigenvectors define $A(z)$ up to multiplication by a constant nondegenerate matrix on the left.*

Proof. If there are two matrix polynomials A' and A'' with the same eigenvalues and eigenvectors, then $(A'(z))^{-1}A''(z)$ has no singularities in the finite plane. Moreover, since the degrees of $A'(z)$ and $A''(z)$ are equal, $(A'(z))^{-1}A''(z) \sim (A'_0)^{-1}A''_0$ as $z \to \infty$. Liouville's theorem concludes the proof. □

We will say that $z - B$, $B \in \mathrm{Mat}(m, \mathbb{C})$, is a *right divisor* of $A(z)$ if $A(z) = \hat{A}(z)(z - B)$, where $\hat{A}(z)$ is a polynomial of degree $n - 1$.

LEMMA 3.2. *A linear function $z - B$ is a right divisor of $A(z)$ if and only if*

$$A_0 B^n + A_1 B^{n-1} + \cdots + A_n = 0.$$

Proof. See, e.g., [GLR]. □

LEMMA 3.3. *Let $\alpha_1, \dots, \alpha_m$ be eigenvalues of $A(z)$ and $v_1, \dots, v_m$ be the corresponding eigenvectors. Assume that $v_1, \dots, v_m$ are linearly independent. Take $B \in \mathrm{Mat}(m, \mathbb{C})$ such that $Bv_i = \alpha_i v_i$, $i = 1, \dots, m$. Then $z - B$ is a right divisor of $A(z)$. Moreover, B is uniquely defined by the conditions that $z - B$ is a right divisor of $A(z)$ and $\mathrm{Sp}(B) = \{\alpha_1, \dots, \alpha_m\}$.*

Proof. For all $i = 1, \dots, m$,

$$(A_0 B^n + A_1 B^{n-1} + \dots + A_n)v_i = (A_0 \alpha_i^n + A_1 \alpha_i^{n-1} + \dots + A_n)v_i = A(\alpha_i)v_i = 0.$$

Lemma 3.2 shows that $z - B$ is a right divisor of $A(z)$.

To show uniqueness, assume that

$$A(z) = \hat{A}'(z)(z - B') = \hat{A}''(z)(z - B'').$$

This implies $(A''(z))^{-1}A'(z) = (z - B'')(z - B')^{-1}$. Possible singularities of the right-hand side of this equality are $z = \alpha_i$, $i = 1, \dots, m$, while possible singularities of the left-hand side are all other eigenvalues of $A(z)$. Since the eigenvalues of $A(z)$ are pairwise distinct, both sides are entire. But $(z - B'')(z - B')^{-1}$ tends to I as $z \to \infty$. Hence, by Liouville's theorem, $B' = B''$. □

Now let us assume that the eigenvalues $a_1, \dots, a_{mn}$ of $A(z)$ are divided into n groups of m numbers:

$$\{a_1, \dots, a_{mn}\} = \{a_1^{(1)}, \dots, a_m^{(1)}\} \cup \dots \cup \{a_1^{(n)}, \dots, a_m^{(n)}\}.$$

Lemma 3.3 shows that for a generic $A(z)$ we can construct uniquely defined $B_1, \dots, B_n \in \mathrm{Mat}(m, \mathbb{C})$ such that for any $i = 1, \dots, n$, $\mathrm{Sp}(B_i) = \{a_1^{(i)}, \dots, a_m^{(i)}\}$ and $z - B_i$ is a right divisor of $A(z)$.[4] By Lemma 3.1, $B_1, \dots, B_n$ define $A(z)$ uniquely up to a left constant factor, because the eigenvectors of B_i must be eigenvectors of $A(z)$.

LEMMA 3.4. *For generic $B_1, \dots, B_n \in \mathrm{Mat}(m, \mathbb{C})$ with $\mathrm{Sp}(B_i) = \{a_j^{(i)}\}$, there exists a unique monic degree n polynomial $A(z) = z^n + A_1 z^{n-1} + \dots$ such that $z - B_i$ are its right divisors. The matrix elements of $A_1, \dots, A_n$ are rational functions of the matrix elements of $B_1, \dots, B_n$ and eigenvalues $\{a_j^{(i)}\}$.*

Remark 3.5. 1. Later on we will show that, in fact, these rational functions do not depend on $\{a_j^{(i)}\}$.

2. Clearly, the condition of $A(z)$ being monic can be replaced by the condition of $A(z)$ having a prescribed nondegenerate highest coefficient A_0.

[4] It is obvious that the condition on $A(z)$, used in Lemma 3.3, is an open condition. The corresponding set is nonempty because it contains diagonal $A(z)$ where the $\{a_i^{(k)}\}$ are the roots of $A_{kk}(z)$. Similar remarks apply to all appearances of the word "generic" below.

Proof. The uniqueness follows from Lemma 3.1. To prove the existence part, we use induction on n. For $n = 1$ the claim is obvious. Assume that we have already constructed $\hat{A}(z) = z^{n-1} + \hat{A}_1 z^{n-1} + \dots$ such that $B_1, \dots, B_{n-1}$ are its right divisors. Let $\{v_i\}$ be the eigenvectors of B_n with eigenvalues $\{a_i^{(n)}\}$. Set $w_i = \hat{A}(a_i^{(n)})v_i$ and take $X \in \mathrm{Mat}(m, \mathbb{C})$ such that $Xw_i = a_i^{(n)} w_i$ for all $i = 1, \dots, m$. (The vectors $\{w_i\}$ are linearly independent generically.) Then $A(z) = (z - X)\hat{A}(z)$ has all needed properties. Indeed, we just need to check that $z - B_n$ is its right divisor (the rationality follows from the fact that computing the eigenvectors with known eigenvalues is a rational operation). For any $i = 1, \dots, m$,

$$\begin{aligned}(B_n^n + A_1 B_n^{n-1} + \dots + A_n)v_i &= \left(\left(a_i^{(n)}\right)^n + A_1 \left(a_i^{(n)}\right)^{n-1} + \dots + A_n \right) v_i \\ &= (a_i^{(n)} - X)\hat{A}(a_i^{(n)})v_i = (a_i^{(n)} - X)w_i = 0.\end{aligned}$$

Lemma 3.2 concludes the proof. □

Thus, we have a birational map between matrix polynomials $A(z) = A_0 z^n + \dots$ with a fixed nondegenerate highest coefficient and fixed mutually distinct eigenvalues divided into n groups of m numbers each, and sets of right divisors $\{z - B_1, \dots, z - B_n\}$ with B_i having the eigenvalues from the ith group. We will treat $\{B_i\}$ as a different set of coordinates for $A(z)$.

It turns out that in these coordinates some multipliers $R(z)$ of Theorem 2.1 take a very simple form. We will redenote by $\kappa_j^{(i)}$ the numbers $\kappa_1, \dots, \kappa_{mn}$ used in Theorem 2.1 in accordance with our new notation for the eigenvalues of $A(z)$. Denote the transformation of Theorem 2.1 with

$$\kappa_j^{(i)} = -k_i \in \mathbb{Z}, \quad i = 1, \dots, n,\ j = 1, \dots, m; \qquad \delta_1 = \dots = \delta_m = \sum_{i=1}^{n} k_i$$

by $\mathcal{S}(k_1, \dots, k_n)$.

PROPOSITION 3.6. *The multiplier $R(z)$ for $\mathcal{S}(0, \dots, 0, \overset{(i)}{1}, 0, \dots, 0)$ is equal to the right divisor $z - B_i$ of $A(z)$ corresponding to the eigenvalues $a_1^{(i)}, \dots, a_m^{(i)}$.*

Proof. It is easy to see that if B_i has eigenvalues $a_1^{(i)}, \dots, a_m^{(i)}$, and $z - B_i$ is a right divisor of $A(z)$ then $R(z) = z - B_i$ satisfies all the conditions of Theorem 2.1.

Conversely, if $R(z)$ is the corresponding multiplier then $R(z)$ is a product of n elementary multipliers with one κ equal to -1 and one δ equal to $+1$. The explicit construction of the proof of Theorem 2.1 shows that all these multipliers are polynomials; hence, $R(z)$ is a polynomial. The fact that $\delta_1 = \dots = \delta_m$ implies that $R(z)$ is a linear polynomial of the form $z - B$ for some $B \in$

$\mathrm{Mat}(m,\mathbb{C})$ (to see this, it suffices to look at the asymptotics of the canonical solutions). We have

$$A(z)=R^{-1}(z+1)\widetilde{A}(z)R(z)=(z+I-B)^{-1}\widetilde{A}(z)(z-B).$$

Comparing the determinants of both sides we conclude that $\mathrm{Sp}(B)=\{a_1^{(i)},\dots,a_m^{(i)}\}$. Since no two eigenvalues are different by an integer, B and $B-I$ have no common eigenvalues. This implies that $(z+I-B)^{-1}\widetilde{A}(z)$ must be a polynomial, and hence $z-B$ is a right divisor of $A(z)$. □

For any $k=(k_1,\dots,k_n)\in\mathbb{Z}^n$ we introduce matrices $B_1(k),\dots,B_n(k)$ such that the right divisors of $\mathcal{S}(k_1,\dots,k_n)A(z)$ have the form $z-B_i(k)$ with

$$\mathrm{Sp}(B_i(k))=\{a_1^{(i)}-k_i,\dots,a_n^{(i)}-k_i\},\quad i=1,\dots,n.$$

They are defined for generic $A(z)$ from the varieties $\mathcal{M}(\cdots)$ introduced in the previous section.

Proposition 3.7 (difference Schlesinger equations). *The matrices* $\{B_i(k)\}$ (*whenever they exist*) *satisfy the following equations*:

$$B_i(\dots)-B_i(\dots,k_j+1,\dots)=B_j(\dots)-B_j(\dots,k_i+1,\dots),\tag{3.1}$$

$$B_j(\dots,k_i+1,\dots)B_i(\dots)=B_i(\dots,k_j+1,\dots)B_j(\dots),\tag{3.2}$$

$$B_i(k_1+1,\dots,k_n+1)=A_0^{-1}B_i(k_1,\dots,k_n)A_0-I,\tag{3.3}$$

where $i,j=1,\dots,n$, *and dots in the arguments mean that other* k_l*'s remain unchanged.*

Remark 3.8. The first two equations above are equivalent to

$$\begin{aligned}\big(z-B_i(\dots,k_j+1,\dots)\big)\big(z-B_j(\dots)\big)\\ =\big(z-B_j(\dots,k_i+1,\dots)\big)\big(z-B_i(\dots)\big).\end{aligned}\tag{3.4}$$

Proof of Proposition 3.7. The uniqueness part of Theorem 2.1 implies that

$$\begin{aligned}&\mathcal{S}(0,\dots,0,\overset{(i)}{1},0,\dots,0)\circ\mathcal{S}(0,\dots,0,\overset{(j)}{1},0,\dots,0)\circ\mathcal{S}(k_1,\dots,k_n)\\ &=\mathcal{S}(0,\dots,0,\overset{(j)}{1},0,\dots,0)\circ\mathcal{S}(0,\dots,0,\overset{(i)}{1},0,\dots,0)\circ\mathcal{S}(k_1,\dots,k_n).\end{aligned}$$

Thus, the corresponding products of the multipliers are equal, which gives (3.4). This proves (3.1), (3.2). The relation (3.3) follows from the fact that the multiplier for $\mathcal{S}(1,\dots,1)$ is equal to $A_0^{-1}A(z)$, and $\widetilde{A}(z)=\mathcal{S}(1,\dots,1)=A_0^{-1}A(z+1)A_0$. This means that the right divisors for $\widetilde{A}(z)$ can be obtained from those for $A(z)$ by shifting z by 1 and conjugating by A_0. □

THEOREM 3.9. *Fix mn complex numbers $\{a_j^{(i)}\}_{i=1,j=1}^{n,m}$ such that no two of them are different by an integer, and an integer $M > 0$. Then for generic $B_1, \dots, B_n \in \mathrm{Mat}(m, \mathbb{C})$, $\mathrm{Sp}(B_i) = \{a_j^{(i)}\}_{j=1}^{m}$, there exists a unique solution*

$$\{B_i(k_1, \dots, k_n) : \max_{i=1,\dots,n} |k_i| \le M\}$$

of the difference Schlesinger equations (3.1)–(3.3) *with*

$$\mathrm{Sp}(A_0) = \{\rho_1, \dots, \rho_n\}, \quad \rho_i/\rho_j \notin \mathbb{R} \ \text{ for } \ i \ne j, \quad \rho_i \ne 0 \ \text{ for } \ i = 1, \dots, n,$$

such that

$$\mathrm{Sp}(B_i(k_1, \dots, k_n)) = \mathrm{Sp}(B_i) - k_i \text{ and } B_i(0, \dots, 0) = B_i \quad \text{for all } \ i = 1, \dots, n.$$

The matrix elements of $B_i(k)$ are rational functions of the matrix elements of the initial conditions $\{B_i\}_{i=1}^n$. Moreover, these rational functions do not depend on the eigenvalues $\{a_i^{(j)}\}$.

Remark 3.10. As we will see later, this theorem also extends to the case of arbitrary invertible A_0.

Proof. The existence and rationality of the flows have already been proved. Indeed, without loss of generality we can assume that A_0 is diagonal (the equations (3.1)–(3.3) remain intact if we conjugate all $B_i(k)$ and A_0 by the same constant matrix). By Lemma 3.4 we can construct a (unique) degree n polynomial $A(z)$ with the highest coefficient A_0, such that $\{B_i\}$ is the set of its right divisors. Then, using Theorem 2.1, we can define $\mathcal{S}(k)$ and hence $\{B_i(k)\}$. By Proposition 3.7 they will satisfy (3.1)–(3.3). Moreover, all operations involved in this construction are rational.

Thus, it remains to prove uniqueness and the fact that the rational functions involved do not depend on the eigenvalues. A simple computation shows that for any $X, Y, S, T \in \mathrm{Mat}(m, \mathbb{C})$, the relation $(z-X)(z-Y) = (z-S)(z-T)$ implies

$$Y = (X-S)^{-1}S(X-S), \qquad T = (X-S)^{-1}X(X-S), \tag{3.5}$$

$$X = (Y-T)T(Y-T)^{-1}, \qquad S = (Y-T)Y(Y-T)^{-1}, \tag{3.6}$$

whenever the corresponding matrices are invertible; cf. [GRW]. Applying this observation to (3.4), we see that, generically, $\{B_i = B_i(0, \dots, 0)\}$ uniquely define all

$$B_i(\varepsilon_1^{(i)}, \dots, \varepsilon_{i-1}^{(i)}, \overset{(i)}{0}, \varepsilon_{i+1}^{(i)}, \dots, \varepsilon_n^{(i)}), \qquad \varepsilon_j^{(i)} = 0, 1. \tag{3.7}$$

Moreover, they are all given by rational expressions involving the initial conditions $\{B_i\}$ only. To move further, we need the following lemma.

LEMMA 3.11. *For generic $X, Y \in \mathrm{Mat}(m, \mathbb{C})$ with fixed disjoint spectra, there exist unique $S, T \in \mathrm{Mat}(m, \mathbb{C})$ such that*

$$(z - X)(z - Y) = (z - S)(z - T), \qquad \mathrm{Sp}(S) = \mathrm{Sp}(Y), \quad \mathrm{Sp}(T) = \mathrm{Sp}(X).$$

The matrix elements of S and T are rational functions of the matrix elements of X and Y which do not depend on the spectra of X and Y.

Proof 1. Lemma 3.3 proves the uniqueness and shows how to construct T if we know the eigenvalues $x_1, \dots, x_m$ of X and vectors v_i such that $(x_i - X)(x_i - Y)v_i = 0$. If we normalize v_i's in the same way, for example, by requiring the first coordinate to be equal to 1 (this can be done generically), then using the construction of Lemma 3.3 we obtain the matrix elements of T as rational functions in the matrix elements of X, Y and $x_1, \dots, x_n$. However, it is easy to see that these rational functions are symmetric with respect to the permutations of $x_1, \dots, x_n$, which means that they depend only on the elementary symmetric functions $\sum_{i_1 < \dots < i_k} x_{i_1} \cdots x_{i_k}$ of x_i's. But these are the coefficients of the characteristic polynomial of X, and hence they are expressible as polynomials in the matrix elements of X. □

Proof 2 (see [O]). The uniqueness follows from Lemma 3.3. To prove the existence, denote by Λ the solution of the equation $Y\Lambda - \Lambda X = I$. Generically, it exists, it is unique and invertible. Set

$$S = X + \Lambda^{-1}, \qquad T = Y - \Lambda^{-1}.$$

Then it is easy to see that $(z - X)(z - Y) = (z - S)(z - T)$. Furthermore, if Y and T have a common eigenvalue then they must have a common eigenvector, which contradicts the invertibility of $Y - T = \Lambda^{-1}$. Hence, $\mathrm{Sp}(T) = \mathrm{Sp}(X)$ and $\mathrm{Sp}(S) = \mathrm{Sp}(Y)$. □

Remark 3.12. In the case of 2 by 2 matrices, it is not hard to produce an explicit formula for S and T in terms of X and Y:

$$\begin{aligned} S &= (X + Y - \operatorname{Tr} Y)Y(X + Y - \operatorname{Tr} Y)^{-1}, \\ T &= (X + Y - \operatorname{Tr} X)^{-1}X(X + Y - \operatorname{Tr} X). \end{aligned} \tag{3.8}$$

Now let us return to the proof of Theorem 3.9. Recall that we already proved that the initial conditions define (3.7) uniquely. Now let us use (3.4) with

$$(k_1, \dots, k_n) = (1, \dots, 1, \overset{(j)}{0}, 1, \dots, 1), \quad j \neq i.$$

By (3.3), we know what $B_i(1, \dots, 1)$ is. Thus, we know both matrices on the left-hand side of (3.4), and hence, by Lemma 3.11, we can compute both matrices on the right-hand side of (3.4), in particular, $B_i(1, \dots, 1, \overset{(j)}{0}, 1, \dots, 1)$.

Now take (3.4) with

$$(k_1,\dots,k_n)=(1,\dots,1,\overset{(j)}{0},1,\dots,1,\overset{(l)}{0},1,\dots,1),$$

where i,j,l are pairwise distinct. Applying Lemma 3.11 again, we find all

$$B_i(1,\dots,1,\overset{(j)}{0},1,\dots,1,\overset{(l)}{0},1,\dots,1).$$

Continuing the computations in this fashion (changing one more 1 to 0 in $(k_1,\dots,k_n)$ on each step), we obtain all

$$B_i(\varepsilon_1^{(i)},\dots,\varepsilon_{i-1}^{(i)},\overset{(i)}{1},\varepsilon_{i+1}^{(i)},\dots,\varepsilon_n^{(i)}),\qquad \varepsilon_j^{(i)}=0,1.$$

Together with (3.3) (and (3.7)) this computes all $B_i(k)$ with $\max|k_i|\le 1$. Iterating this procedure, we complete the proof. □

4. An alternative description of the Schlesinger flows

The goal of this section is to provide yet another set of coordinates for the polynomials $A(z)$, in which the flows described in the previous section can be easily defined. In particular, this will lead to a different proof of Theorem 3.9, which will be valid for an arbitrary invertible A_0.

PROPOSITION 4.1. *With the assumptions of Theorem* 3.9, *the monic degree n polynomial*

$$(z-B_1(0,1,\dots,1))(z-B_2(0,0,1,\dots,1))\cdots(z-B_n(0,\dots,0))$$

has $z-B_i$, $i=1,\dots,n$, *as its right divisors.*

This statement and Theorem 3.9 provide a proof for Remark 3.5(1).

Proof. Using (3.4) we obtain, for $(j>i)$,

$$\begin{aligned}&\left(z-B_i(0,\dots,0,\overset{(j)}{1},\dots,1)\right)\left(z-B_j(0,\dots,0,\overset{(j+1)}{1},\dots,1)\right)\\&\quad=\left(z-B_j(0,\dots,0,\overset{(i)}{1},0,\dots,0,\overset{(j+1)}{1},\dots,1)\right)\left(z-B_i(0,\dots,0,\overset{(j+1)}{1},\dots,1)\right).\end{aligned}$$

Using this commutation relation, we can move the factor $(z-B_i(\cdots))$, in the product above, to the right most position, where it will turn into

$$(z-B_i(0,\dots,0))=(z-B_i).$$ □

Let us introduce the notation $(l_1,\dots,l_n\in\mathbb{Z})$

$$C_i(l_1,\dots,l_n):=B_i(l_1,\dots,l_i,l_{i+1}+1,\dots,l_n+1),\qquad i=1,\dots,n.$$

If we denote by $A(z)$ the polynomial of degree n with highest coefficient A_0 such that the $\{z-B_i\}$ are its left divisors, then the definition of $B_i(k)$ and Proposition 4.1 imply that for any $l=(l_1,\dots,l_n)\in\mathbb{Z}^n$,

$$\mathcal{S}(l_1,\dots,l_n)A(z)=A_0\big(z-C_1(l)\big)\cdots\big(z-C_n(l)\big). \tag{4.1}$$

(To apply Proposition 4.1, we also used an easy fact that for any solution $\{B_i(k)\}$ of (3.1)–(3.3) and any $l_1,\dots,l_n\in\mathbb{Z}$,

$$B_i'(k_1,\dots,k_n):=B_i(k_1+l_1,\dots,k_n+l_n),\qquad i=1,\dots,n,$$

also form a solution of (3.1)–(3.3).)

LEMMA 4.2. *The map $\{B_i\}\mapsto\{C_i\}$ is birational.*

Proof. The rationality of the forward map follows from Theorem 3.9. The rationality of the inverse map follows from Lemma 3.3 (indeed, we just need to find the right divisors of the known matrix $\mathcal{S}(l_1,\dots,l_n)A(z)$). Even though it looks like to construct B_i we need to know the eigenvalues of C_i, it is clear that by normalizing the eigenvectors of $A(z)$ corresponding to these eigenvalues, in the same way, we will obtain a formula for B_i which will be symmetric with respect to the permutations of these eigenvalues. Thus we can rewrite it through the matrix elements of C_i's only (this argument was used in the first proof of Lemma 3.11 above). □

Our goal is to describe the transformations $\mathcal{S}(k)$ in terms of $\{C_i\}$. We need a preliminary lemma which generalizes Lemma 3.11.

LEMMA 4.3. *For generic $X_1,\dots,X_N\in\mathrm{Mat}(m,\mathbb{C})$ with fixed disjoint spectra and any permutation $\sigma\in S_N$, there exist unique $Y_1,\dots,Y_N\in\mathrm{Mat}(m,\mathbb{C})$ such that $\mathrm{Sp}(Y_i)=\mathrm{Sp}(X_i)$ for all $i=1,\dots,N$, and*

$$(z-X_1)\cdots(z-X_N)=(z-Y_{\sigma(1)})\cdots(z-Y_{\sigma(N)}).$$

The matrix elements of $\{Y_i\}$ are rational functions of the matrix elements of $\{X_i\}$ which do not depend on the spectra of $\{X_i\}$.

Proof. The existence and rationality claims follow from Lemma 3.11, because elementary transpositions $(i,i+1)$ generate the symmetric group S_N. To show uniqueness, we rewrite the equality

$$(z-Y_1')\cdots(z-Y_N')=(z-Y_1'')\cdots(z-Y_N''),\qquad \mathrm{Sp}(Y_i')=\mathrm{Sp}(Y_i''),$$

in the form

$$(z-Y_1'')^{-1}(z-Y_1')=\Big((z-Y_2'')\cdots(z-Y_N'')\Big)\Big((z-Y_2')\cdots(z-Y_N')\Big)^{-1}.$$

If the spectrum of Y_1'' is disjoint with the spectra of $Y_2',\dots,Y_m'$, then both sides of the last equality are entire because they cannot possibly have common poles.

Since both sides tend to I as $z \to \infty$, by Liouville's theorem we conclude that both sides are identically equal to I, and $Y_1' = Y_1''$. Induction on N concludes the proof. □

PROPOSITION 4.4. *With the assumptions of Theorem* 3.9, *the* $\{C_i(l)\}$ *satisfy the equations*

$$\begin{aligned}(4.2)\qquad &\bigl(z+1-C_i\bigr)\cdots\bigl(z+1-C_n\bigr)A_0\bigl(z-C_1\bigr)\cdots\bigl(z-C_{i-1}\bigr)\\ &\qquad=\bigl(z+1-\widetilde{C}_{i+1}\bigr)\cdots\bigl(z+1-\widetilde{C}_n\bigr)A_0\bigl(z-\widetilde{C}_1\bigr)\cdots\bigl(z-\widetilde{C}_i\bigr),\\ &C_j=C_j(l_1,\dots,l_n),\quad \widetilde{C}_j=C_j(l_1,\dots,l_{i-1},l_i+1,l_{i+1},\dots,l_n)\ \textit{for all } j,\end{aligned}$$

and

$$(4.3)\qquad C_i(l_1+1,\dots,l_n+1)=A_0^{-1}C_i(l_1,\dots,l_n)A_0-I.$$

In both equations $i=1,\dots,n$ *is arbitrary.*

Proof. The relation (4.3) is a direct corollary of (3.3). Let us prove (4.2). Proposition 3.6 implies that the multiplier for the shift

$$(l_1,\dots,l_n)\mapsto(l_1,\dots,l_{i-1},l_i+1,l_{i+1},\dots,l_n)$$

has the form $R(z)=z-B_i(l)$. Thus, (4.1) gives

$$A_0(z-\widetilde{C}_1)\cdots(z-\widetilde{C}_n)(z-B_i(l))=(z+1-B_i(l))A_0(z-C_1)\cdots(z-C_n).$$

Comparing the spectra of factors on both sides and applying Lemma 4.3, we get

$$\begin{gathered}A_0(z-\widetilde{C}_1)\cdots(z-\widetilde{C}_i)=(z+1-B_i(l))A_0(z-C_1)\cdots(z-C_{i-1}),\\ (z-\widetilde{C}_{i+1})\cdots(z-\widetilde{C}_n)(z-B_i(l))=(z-C_i)\cdots(z-C_n).\end{gathered}$$

Combining these two relations and shifting $z\mapsto z+1$ in the second one, we arrive at (4.2). □

THEOREM 4.5. *Fix* mn *complex numbers* $\bigl\{a_j^{(i)}\bigr\}_{i=1,j=1}^{n,m}$ *such that no two of them are different by an integer, an integer* $M>0$, *and any nondegenerate* $A_0\in\mathrm{Mat}(m,\mathbb{C})$. *Then for generic* $C_1,\dots,C_n\in\mathrm{Mat}(m,\mathbb{C})$, $\mathrm{Sp}(C_i)=\bigl\{a_j^{(i)}\bigr\}_{j=1}^{m}$, *there exists a unique solution*

$$\{C_i(l_1,\dots,l_n):\max_{i=1,\dots,n}|l_i|\le M\}$$

of the equations (4.2) *and, consequently,* (4.3), *such that*

$$\mathrm{Sp}(C_i(l_1,\dots,l_n))=\mathrm{Sp}(C_i)-l_i\quad\textit{and}\quad C_i(0,\dots,0)=C_i\quad\textit{for all}\ \ i=1,\dots,n.$$

The matrix elements of $C_i(l)$ are rational functions of the matrix elements of the initial conditions $\{C_i\}_{i=1}^n$, and these rational functions do not depend on the eigenvalues $\{a_i^{(j)}\}$. Moreover,

$$B_i(k_1,\dots,k_n) := C_i(k_1,\dots,k_i,k_{i+1}-1,\dots,k_n-1), \qquad i=1,\dots,n,$$

solves the difference Schlesinger equations (3.1)–(3.3).

Remark 4.6. If the ratios of eigenvalues of A_0 are not real then Theorem 3.9, Lemma 4.2, and Proposition 4.4 provide a proof of Theorem 4.5. However, our goal is to provide an independent proof of this theorem, thus giving a different proof of Theorem 3.9 with arbitrary invertible A_0; cf. Remark 3.10.

To prove Theorem 4.5 we will develop a rather general formalism.

(a) *Semigroup.* Let P be a semigroup and P_0 be its subset. We assume that every element of P_0 has a *type*. The types of two different elements $p_1, p_2 \in P_0$ may be the same, which will be denoted by $t(p_1) = t(p_2)$, and may be *disjoint*, which will be denoted by $t(p_1) \perp t(p_2)$. The types may also be neither equal nor disjoint.

Assumption 4.7. *For any elements $p_1,\dots,p_N \in P_0$ such that their types are pairwise disjoint*:

$$t(p_i) \perp t(p_j), \qquad i\neq j, \quad 1\le i,j\le N,$$

and for any permutation $\sigma \in S_N$ there exist unique elements $\hat p_1,\dots \hat p_N \in P_0$ such that $t(\hat p_i) = t(p_i)$, $i=1,\dots,N$, and

$$p_1\cdots p_N = \hat p_{\sigma(1)}\cdots \hat p_{\sigma(N)}.$$

We will be interested in the situation when

$$P = P^{\mathrm{Mat}(m,\mathbb{C})} = \left\{ z^k + Q_1 z^{k-1} + \cdots + Q_k \,|\, Q_i \in \mathrm{Mat}(m,\mathbb{C}),\, 1\le i\le k \right\},$$
$$P_0 = P_0^{\mathrm{Mat}(m,\mathbb{C})} = \{ z - Q \,|\, Q\in \mathrm{Mat}(m,\mathbb{C})\},$$
$$t(z-Q) = \{ z\in\mathbb{C} \,|\, \det(z-Q)=0\} = \mathrm{Sp}(Q).$$

The notions of equality and disjointness for types are the natural ones for the m-point subsets of $\mathbb{C}$. Lemma 4.3 shows that $P^{\mathrm{Mat}(m,\mathbb{C})}$ and $P_0^{\mathrm{Mat}(m,\mathbb{C})}$ satisfy Assumption 4.7 generically.

Proposition 4.8. *Let P be a semigroup satisfying Assumption* 4.7. *Assume that there is an equality in P of the form*

$$\left(p_1^{(1)}\cdots p_{m_1}^{(1)}\right)\cdots\left(p_1^{(k)}\cdots p_{m_k}^{(k)}\right) = \left(q_1^{(1)}\cdots q_{m_1}^{(1)}\right)\cdots\left(q_1^{(k)}\cdots q_{m_k}^{(k)}\right), \tag{4.4}$$

where all $p_i^{(j)}, q_i^{(j)}$ *are from* P_0*, the types of all elements on the left-hand side are pairwise disjoint, the types of all elements on the right-hand side are pairwise disjoint, and*

$$\left\{t(p_1^{(j)}),\dots,t(p_{m_j}^{(j)})\right\} = \left\{t(q_1^{(j)}),\dots,t(q_{m_j}^{(j)})\right\}$$

for all $j = 1,\dots,k$*. Then*

$$p_1^{(j)}\cdots p_{m_j}^{(j)} = q_1^{(j)}\cdots q_{m_j}^{(j)}, \qquad j = 1,\dots,k.$$

Proof. By Assumption 4.7, for any $j = 1,\dots,k$, there exist $\hat q_1^{(j)},\dots,\hat q_{m_j}^{(j)}$ such that

$$q_1^{(j)}\cdots q_{m_j}^{(j)} = \hat q_1^{(j)}\cdots \hat q_{m_j}^{(j)}$$

and $t(p_i^{(j)}) = t(\hat q_i^{(j)})$. Then by the uniqueness part of Assumption 4.7 applied to (4.4) we obtain $p_i^{(j)} = \hat q_i^{(j)}$ for all i, j. This immediately implies the claim. □

We, essentially, used Proposition 4.8 in the proof of Proposition 4.4 above.

(b) *Commuting flows on sequences.* Denote by $\mathcal{P}$ the set of all sequences $\{p_k\}_{k\in\mathbb{Z}} \subset P_0$ such that the types of all elements of a sequence are pairwise disjoint.

Fix an integer $n > 0$. For any $l \in \mathbb{Z}$ we define a map $F_l : \mathcal{P} \to \mathcal{P}$ as follows:

$$F_l : \{p_k\}_{k\in\mathbb{Z}} \mapsto \{q_k\}_{k\in\mathbb{Z}},$$

$$p_{l+\mu n}p_{l+\mu n+1}\cdots p_{l+(\mu+1)n-1} = q_{l+\mu n+1}q_{l+\mu n+2}\cdots q_{l+(\mu+1)n}, \quad \mu\in\mathbb{Z},$$

$$t(q_j) = \begin{cases} t(p_{j-n}), & \text{if } (j-l) \text{ divides } n,\\ t(p_j), & \text{otherwise.}\end{cases}$$

In this definition μ ranges over all integers, and for each μ we use Assumption 4.7 for $\sigma = (12\cdots n) \in S_n$. Clearly, F_l is invertible.

It is convenient to denote

$$F_{l_1}\Big(F_{l_2}\big(\cdots F_{l_m}(\{p_k\})\big)\Big) = \{p_k^{l_1,\dots,l_m}\}.$$

Then the second line in the definition above takes the form

$$p_{l+\mu n}p_{l+\mu n+1}\cdots p_{l+(\mu+1)n-1} = p^l_{l+\mu n+1}p^l_{l+\mu n+2}\cdots p^l_{l+(\mu+1)n}.$$

For example, for $n = 2$ we have

$$p_{2s-1}p_{2s} = p^1_{2s}p^1_{2s+1}, \qquad t(p^1_{2s}) = t(p_{2s}), \qquad t(p^1_{2s+1}) = t(p_{2s-1}), \ \ s\in\mathbb{Z},$$
$$p_{2s}p_{2s+1} = p^2_{2s+1}p^2_{2s+2}, \quad t(p^2_{2s+1}) = t(p_{2s+1}), \quad t(p^2_{2s+2}) = t(p_{2s}), \ \ s\in\mathbb{Z}.$$

It is immediately seen from the definition that $F_{l+\mu n} = F_l$ and

$$p_k^{l,l+1,\dots,l+n-1} = p_{k-n} \tag{4.5}$$

for any $k, l \in \mathbb{Z}$.

THEOREM 4.9. (i) *For any $i, j \in \mathbb{Z}$, F_i and F_j commute. That is,*

$$p_k^{i,j} \equiv p_k^{j,i} \text{ for any } \{p_k\} \in \mathcal{P}.$$

(ii) *For any $\{q_k\} \in \mathcal{P}$ and any $i, j \in \mathbb{Z}$ such that $0 < j - i < n$, set $p_k = q_k^{i+1,\dots,j}$. Then*

$$q_i^j \, p_j = p_j^i \, q_i. \tag{4.6}$$

Remark 4.10. Part (i) of this theorem means that we have defined an action of $\mathbb{Z}^n$ on $\mathcal{P}$. There is a much larger group which acts on $\mathcal{P}$. Let $\pi : \mathbb{Z} \to \mathbb{Z}$ be a bijection such that for any $k \in \mathbb{Z}$ the sets

$$I_k = \{i \in \mathbb{Z} : i < k,\, \pi(i) > \pi(k)\}, \qquad J_k = \{j \in \mathbb{Z} : j > k,\, \pi(j) < \pi(k)\}$$

are finite: $I = \{i_1, \dots, i_s\}$, $J = \{j_1, \dots, j_t\}$. Then, given a sequence $\{p_k\} \in \mathcal{P}$, we define $\{p_k^\pi\} \in \mathcal{P}$ by

$$p_{i_1} \cdots p_{i_s} \; p_k \; p_{j_1} \cdots p_{j_t} = p'_{j_1} \cdots p'_{j_t} \; p^\pi_{\pi(k)} \; p'_{i_1} \cdots p'_{i_s}$$

where $t(p'_l) = t(p_l)$ and $t\big(p^\pi_{\pi(k)}\big) = t(p_k)$. One can show that this defines an action of the group of all π satisfying the condition above on the space $\mathcal{P}$. The maps $\{F_l\}$ correspond to shifts by n along n nonintersecting arithmetic progressions $\{l + \mu n : \mu \in \mathbb{Z}\}$, hence they must commute.

Proof of Theorem 4.9. Because $F_l = F_{l+n}$, it suffices to assume that $0 < j - i < n$. Consider the product

$$\Pi = p_i p_{i+1} \dots p_{j+2n-1}.$$

On one hand, we have

$$\begin{aligned}
\Pi &= \big(p_i \cdots p_{i+n-1}\big)\big(p_{i+n} \cdots p_{i+2n-1}\big) p_{i+2n} \cdots p_{j+2n-1} \\
&= \big(p^i_{i+1} \cdots p^i_{i+n}\big)\big(p^i_{i+n+1} \cdots p^i_{i+2n}\big) p_{i+2n} \cdots p_{j+2n-1} \\
&= p^i_{i+1} \cdots p^i_{j-1} \big(p^i_j \cdots p^i_{j+n-1}\big) p^i_{j+n} \cdots p^i_{i+2n} \, p_{i+2n} \cdots p_{j+2n-1} \\
&= p^i_{i+1} \cdots p^i_{j-1} \, p^{j,i}_{j+1} \cdots p^{j,i}_{j+n} \, p^i_{j+n} \cdots p^i_{i+2n} \, p_{i+2n} \cdots p_{j+2n-1}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Pi &= p_i \cdots p_{j-1} \big(p_j \cdots p_{j+n-1}\big)\big(p_{j+n} \cdots p_{j+2n-1}\big) \\
&= p_i \cdots p_{j-1} \big(p^j_{j+1} \cdots p^j_{j+n}\big)\big(p^j_{j+n+1} \cdots p^j_{j+2n}\big) \\
&= p_i \cdots p_{j-1} \, p^j_{j+1} \cdots p^j_{i+n-1} \big(p^j_{i+n} \cdots p^j_{i+2n-1}\big) p^j_{i+2n} \cdots p^j_{j+2n} \\
&= p_i \cdots p_{j-1} \, p^j_{j+1} \cdots p^j_{i+n-1} \, p^{i,j}_{i+n+1} \cdots p^{i,j}_{i+2n} \, p^j_{i+2n} \cdots p^j_{j+2n}.
\end{aligned}$$

Thus, we obtain

(4.7)
$$\begin{aligned}&\left(p^i_{i+1}\cdots p^i_{j-1}\,p^{j,i}_{j+1}\cdots p^{j,i}_{i+n}\right)\left(p^{j,i}_{i+n+1}\cdots p^{j,i}_{j+n}\right)\left(p^i_{j+n}\cdots p^i_{i+2n}\,p_{i+2n}\cdots p_{j+2n-1}\right)\\ &=\left(p_i\cdots p_{j-1}\,p^j_{j+1}\cdots p^j_{i+n-1}\right)\left(p^{i,j}_{i+n+1}\cdots p^{i,j}_{j+n}\right)\left(p^{i,j}_{j+n+1}\cdots p^{i,j}_{i+2n}\,p^j_{i+2n}\cdots p^j_{j+2n}\right).\end{aligned}$$

Comparing the types in the three factors on the left and on the right, we see that we are in a position to apply Proposition 4.8. It implies, in particular, that the middle factors are equal. Since the order of the types in the middle factors is the same, these middle factors must be equal termwise:

$$p^{j,i}_k = p^{i,j}_k, \qquad i+n+1 \le k \le j+n.$$

Because $F_l = F_{l+n}$ for all n, and i and j are arbitrary, we see that $p^{j,i}_k = p^{i,j}_k$, $\quad i+1 \le k \le j$. Switching from (i,j) to $(j, i+n)$, we get $p^{i,j}_k = p^{j,i}_k$ for $j+1 \le k \le i+n$. Thus, the commutativity relation is proved for $i+1 \le k \le i+n$, and thus for all $k \in \mathbb{Z}$. The proof of the first part of Theorem 4.9 is complete.

In order to prove Theorem 4.9(ii), we need to compare the first and the third factors of the two sides of (4.7). The first factors give

$$p^i_{i+1}\cdots p^i_{j-1}\,p^{j,i}_{j+1}\cdots p^{j,i}_{i+n} = p_i\cdots p_{j-1}\,p^j_{j+1}\cdots p^j_{i+n-1}.$$

Commuting p_i on the right-hand side to the right and using Proposition 4.8, we see that

$$p_i\cdots p_s = p^i_{i+1}\cdots p^i_s\,x_{s,i} \tag{4.8}$$

where $i+1 \le s \le j-1$, $x_{s,i} \in P_0$, and $t(x_{s,i}) = t(p_i)$. Since j is arbitrary (but $j-i<n$), we can assume that $i+1 \le s \le i+n-2$. Note that (4.8) also holds for $s = i+n-1$ with $x_{i+n-1,i} = p^i_{i+n}$, as follows from the definition of F_i.

Similarly, looking at the third factors and substituting $F_j^{-1}\{p_k\}$ for $\{p_k\}$, we get

$$p_t\cdots p_j = y_{t,j}\,\left(F_j^{-1}\{p_k\}\right)_t\cdots\left(F_j^{-1}\{p_k\}\right)_{j-1} \tag{4.9}$$

where $j-n+2 \le t \le j-1$, $y_{t,j} \in P_0$, and $t(y_{t,j}) = t(p_j)$. Again, this also holds for $t = j-n+1$ with $y_{j+n-1,j} = \left(F_j^{-1}\{p_k\}\right)_{j-n}$.

Lemma 4.11. *For $i+1 \le s \le i+n-1$,*

$$x_{s,i} = \left(F_{i+1}^{-1}\circ\cdots\circ F_s^{-1}\{p_k\}\right)_i.$$

Proof. Induction on $s-i$. To prove both the base $s = i+1$ of the induction and the induction step we first use (4.9) to write

$$p_i\cdots p_s = y_{i,s}\,\left(F_s^{-1}\{p_k\}\right)_i\cdots\left(F_s^{-1}\{p_k\}\right)_{s-1}$$

and now use the induction hypothesis on the factors after $y_{i,s}$ to obtain

$$p_i \cdots p_s = y_{i,s} \left(F_i F_s^{-1}\{p_k\}\right)_{i+1} \cdots \left(F_i F_s^{-1}\{p_k\}\right)_{s-1} \left(F_{i+1}^{-1} \circ \cdots \circ F_s^{-1}\{p_k\}\right)_i .$$

(If $s = i+1$ then the second step is empty.) Since

$$t\left(\left(F_{i+1}^{-1} \circ \cdots \circ F_s^{-1}\{p_k\}\right)_i\right) = t(x_{s,i}) = t(p_i),$$

comparing with (4.8) we conclude that $x_{s,i} = \left(F_{i+1}^{-1} \circ \cdots \circ F_s^{-1}\{p_k\}\right)_i$.

This argument works for $s \le i+n-2$. For $s = i-n+1$ the lemma follows from (4.5). □

Now we return to the second part of Theorem 4.9. Applying Lemma 4.11 to all but one factor in $p_i \cdots p_j$, and then to all factors in $p_i \cdots p_j$, we obtain

$$\begin{aligned} p_i \cdots p_j &= p^i_{i+1} \cdots p^i_{j-1} \left(F_{i+1}^{-1} \circ \cdots \circ F_{j-1}^{-1}\{p_k\}\right)_i p_j \\ &= p^i_{i+1} \ldots p^i_{j-1} p^i_j \left(F_{i+1}^{-1} \circ \cdots \circ F_j^{-1}\{p_k\}\right)_i . \end{aligned}$$

In the last two products all but the last two factors coincide. By Proposition 4.8, this means that the products of the last two also coincide:

$$\left(F_{i+1}^{-1} \circ \cdots \circ F_{j-1}^{-1}\{p_k\}\right)_i p_j = p^i_j \left(F_{i+1}^{-1} \circ \cdots \circ F_j^{-1}\{p_k\}\right)_i .$$

Renaming $F_{i+1}^{-1} \circ \cdots \circ F_j^{-1}\{p_k\}$ by $\{q_k\}$ we arrive at (4.6). The proof of Theorem 4.9 is complete. □

(c) *Proof of Theorem* 4.5. Let us concentrate on the case $P = P^{\mathrm{Mat}(m,\mathbb{C})}$, $P_0 = P_0^{\mathrm{Mat}(m,\mathbb{C})}$; see (a) above. Since Assumption 4.7 generically holds in this case (see Lemma 4.3), we will be acting as if it always holds, keeping in mind that all the claims we prove hold only generically. Set $p_i = z - C_i$, $i = 1, \ldots, n$, where $C_i = C_i(0, \ldots, 0)$ are as in Theorem 4.5. More generally, define

$$p_{i+\mu n} = z - \mu - A_0^\mu C_i(0, \ldots, 0) A_0^{-\mu}, \qquad i = 1, \ldots, n, \quad \mu \in \mathbb{Z}, \tag{4.10}$$

where A_0 is an arbitrary invertible element of $\mathrm{Mat}(m, \mathbb{C})$. The assumption that no two numbers of the set $\{a_j^{(i)}\}$ are different by an integer guarantees that $\{p_k\} \in \mathcal{P}$. Now define $\{C_i(l_1, \ldots, l_n)\}$ by

$$\left(F_1^{l_1} \cdots F_n^{l_n}\{p_k\}\right)_{i+\mu n} = z - \mu - A_0^\mu C_i(l_1, \ldots, l_n) A_0^{-\mu}, \qquad i = 1, \ldots, n, \quad \mu \in \mathbb{Z}.$$

(It is immediately seen that the subset of $\mathcal{P}$ consisting of sequences $\{p_k = z - Q_k\}$ such that $Q_{k+n} = I + A_0 Q_k A_0^{-1}$ is stable under the flows $F_1, \ldots, F_n$ which shows that the $C_i(l)$ are well-defined.) The very definition of F_i implies (4.2). Furthermore, (4.3) is a direct corollary of (4.5). It is easy to see that $\mathrm{Sp}(C_i(l)) = \mathrm{Sp}(C_i) - l_i$, and this gives the existence part of

Theorem 4.5. The uniqueness and rationality claims follow from Lemma 4.3. Finally, let us show that

$$B_i(k_1,\dots,k_n) = C_i(k_1,\dots,k_i,k_{i+1}-1,\dots,k_n-1)$$

solves (3.1)–(3.3). The relation (3.3) is equivalent to (4.3). We will derive (3.4) (and hence (3.1), (3.2)) from Theorem 4.9(ii).

Fix $1 \le i < j \le n$ and define

$$\{\tilde p_k\} = F_1^{k_1}\cdots F_n^{k_n}\{p_k\}, \quad \{\tilde q_k\} = F_{i+1}^{-1}\circ\cdots\circ F_j^{-1}\{\tilde p_k\}.$$

Then

$$\begin{aligned}
\tilde p_j &= z - C_j(k_1,\dots,k_n) = z - B_j(k_1,\dots,k_j,k_{j+1}+1,\dots,k_n+1),\\
\tilde q_i &= z - B_i(k_1,\dots,k_j,k_{j+1}+1,\dots,k_n+1),\\
\tilde p_j^i &= z - B_j(k_1,\dots,k_{i-1},k_i+1,k_{i+1},\dots,k_j,k_{j+1}+1,\dots,k_n+1),\\
\tilde q_i^j &= z - B_i(k_1,\dots,k_{j-1},k_j+1,\dots,k_n+1).
\end{aligned}$$

If we apply the shift

$$k_{j+1}\mapsto k_{j+1}-1,\dots,k_n\mapsto k_n-1,$$

then the equality $\tilde q_i^j\tilde p_j = \tilde p_j^i\tilde q_i$ turns into (3.4). This completes the proof of Theorem 4.5. □

Remark 4.12. The set of sequences $\{p_k = z - Q_k\}$ with $Q_{k+n} = I + A_0Q_kA_0^{-1}$ is also stable under the action of permutations $\pi : \mathbb{Z}\to\mathbb{Z}$ (see Remark 4.10) of the form

$$\pi(i+\mu n) = \sigma(i) + \mu n, \qquad \sigma\in S_n, \quad i = 1,\dots,n, \quad \mu\in\mathbb{Z}.$$

Defining $\hat C_i(l) = C_i(l) + l_iI$, we obtain a birational action of the semidirect product $\mathbb{Z}^n \ltimes S_n$ on $\{\hat C_1,\dots,\hat C_n\}\in(\mathrm{Mat}(m,\mathbb{C}))^n$ which preserves the spectra of $\hat C_i's$.

Remark 4.13. If instead of (4.10) we use periodic initial conditions

$$p_{i+\mu n} = z - C_i, \qquad i = 1,\dots,n, \quad \mu\in\mathbb{Z},$$

which corresponds to the autonomous limit of the difference Schlesinger equations mentioned in the introduction, then the maps $F_1,\dots,F_n$ are exactly the *monodromy maps* constructed by Veselov [V] in the framework of set-theoretical solutions of the quantum Yang-Baxter equation. We refer to [V] for details and further references on the subject.

Remark 4.14. Solutions of the q-difference Schlesinger equations mentioned in the introduction are obtained from consideration of $\{p_k = z - Q_k\}$ with $Q_{k+n} = qA_0Q_kA_0^{-1}$.

5. Continuous limit

We start with a brief survey of the classical deformation theory for linear matrix differential equations, which is due to Riemann, Schlesinger, Fuchs, and Garnier; see [JMU], [JM] for details.

Consider a first order matrix system of ordinary linear differential equations

$$\frac{d\mathcal{Y}}{d\zeta} = \mathcal{B}(\zeta)\mathcal{Y}(\zeta), \qquad \mathcal{B}(\zeta) = \mathcal{B}_\infty + \sum_{k=1}^{n} \frac{\mathcal{B}_k}{\zeta - x_k}\,. \tag{5.1}$$

Here all matrices are in $\mathrm{Mat}(m, \mathbb{C})$. We will assume that all $\mathcal{B}_k$'s can be diagonalized:

$$\mathcal{B}_k = G_k T_k G_k^{-1}, \qquad T_k = \operatorname{diag}(t_1^{(k)}, \dots, t_m^{(k)}), \quad k = 1, \dots, n,$$
$$\mathcal{B}_\infty = G_\infty \operatorname{diag}(s_1, \dots, s_n) G_\infty^{-1}$$

where $t_i^{(k)} - t_j^{(k)} \notin \mathbb{Z}$, $i \neq j$, for all $k \neq \infty$, and $s_i \neq s_j$, $i \neq j$.

Alternatively, we may also consider

$$\frac{d\mathcal{Y}}{d\zeta} = \mathcal{B}(\zeta)\mathcal{Y}(\zeta), \qquad \mathcal{B}(\zeta) = \sum_{k=1}^{n} \frac{\mathcal{B}_k}{\zeta - x_k}\,, \tag{5.2}$$

in which case we assume (in addition to the above assumption on $\mathcal{B}_k$, $k = 1, \dots, m$) that

$$-\sum_{k=1}^{n} \mathcal{B}_k = G_\infty T_\infty G_\infty^{-1}, \qquad T_\infty = \operatorname{diag}(t_1^{(\infty)}, \dots, t_m^{(\infty)}),$$

with $t_i^{(\infty)} - t_j^{(\infty)} \notin \mathbb{Z}$ for $i \neq j$.

Since we can conjugate $\mathcal{Y}$ and $\{\mathcal{B}_k\}$ by G_∞, we may set $G_\infty = I$ without loss of generality.

One can show, see e.g. [JMU, Prop. 2.1], that there exists a unique formal solution $\mathcal{Y}(\zeta)$ of (5.1) or (5.2) of the form

$$\mathcal{Y}(\zeta) = \hat{\mathcal{Y}}(\zeta) \exp(T(\zeta)), \qquad \hat{\mathcal{Y}}(\zeta) = I + \hat{\mathcal{Y}}_1 \zeta^{-1} + \hat{\mathcal{Y}}_2 \zeta^{-2} + \dots, \tag{5.3}$$

where

$$T(\zeta) = \begin{cases} \operatorname{diag}(s_1, \dots, s_n) z + T_\infty \ln(z^{-1}) & \text{for (5.1)}, \\ T_\infty \ln(z^{-1}) & \text{for (5.2)}, \end{cases}$$

with $T_\infty = \operatorname{diag}(t_1^{(\infty)}, \dots, t_m^{(\infty)})$. (This formula is also the definition of T_∞ for (5.1).) This is the analog of Propositions 1.1 and 1.2.

It turns out that for (5.2) the series in (5.3) is convergent, and after multiplication by $\exp(T(\zeta))$ it defines a holomorphic (near $\zeta = \infty$) multivalued function $\mathcal{Y}^\infty(\zeta)$. However, in the case of (5.1) this series is, generally speaking, divergent. Then the analog of Theorem 1.3 holds. Namely,

there exist unique holomorphic solutions $\mathcal{Y}^{l,r}$ of (5.1), defined for $\Re\zeta \ll 0$ and $\Re\zeta \gg 0$, respectively, such that they have the asymptotic expansion (5.3) as $\zeta \to \infty$.[5] Since both these functions solve the same differential equation, there exist constant matrices $S^{\pm}$ such that the analytic continuations of $\mathcal{Y}^{l,r}$ in $\Im\zeta \gg 0$ ($\ll 0$) are related by $\mathcal{Y}^l = \mathcal{Y}^r S^{\pm}$. The matrices $S^{\pm}$ are called the *Stokes multipliers*.

It is also possible to determine the nature of solutions of (5.1), (5.2) near the poles $x_1, \dots, x_k$. Namely, one can show that there exist locally holomorphic functions

$$\hat{\mathcal{Y}}^{(k)}(\zeta) = I + \mathcal{Y}_1^{(k)}(z - x_k) + \mathcal{Y}_2^{(k)}(z - x_k)^2 + \dots$$

such that for any solution $\mathcal{Y}(\zeta)$, there exist constant matrices C_k such that locally near $\zeta = x_k$

$$\mathcal{Y}(\zeta) = G_k \hat{\mathcal{Y}}^{(k)}(\zeta) \exp\bigl(T_k \ln(z - x_k)\bigr) C_k.$$

(Recall that the $\{G_k\}$ were defined above by $\mathcal{B}_k = G_k T_k G_k^{-1}$.) In particular, if we fix paths from $\zeta = \infty$ (or $\pm\infty$ for (5.1)), then we can define $\{C_k\}$ for the (analytic continuations of the) canonical solutions $\mathcal{Y}^{\infty}$ or $\mathcal{Y}^{l,r}$.

Thus, to any equation of the form (5.1) or (5.2), we associate the following *monodromy data*: $\{T_k\}_{k=1}^n$ and T_∞, $\{C_k\}_{k=1}^n$ computed for the canonical solution $\mathcal{Y}^{\infty}$ or $\mathcal{Y}^{l,r}$, and in the case of (5.1) we also add the Stokes multipliers $S^{\pm}$ and the exponents $s_1, \dots, s_m$.

If we analytically continue any solution $\mathcal{Y}(\zeta)$ of (5.1) or (5.2) along a closed path γ in $\mathbb{C}$ avoiding the singular points $\{x_k\}$ then the columns of $\mathcal{Y}$ will change into their linear combinations: $\mathcal{Y} \mapsto \mathcal{Y} M_\gamma$. Here M_γ is a constant invertible matrix which depends only on the homotopy class of γ. It is called the *monodromy matrix* corresponding to γ. If γ is a positive loop around x_k then the corresponding monodromy matrix M_k for the canonical solution $\mathcal{Y}^{\infty}$ or $\mathcal{Y}^{l,r}$ can be computed using the monodromy data introduced above:

$$M_k = C_k^{-1} \exp(2\pi i T_k) C_k.$$

The basic problem of the *isomonodromy deformation* of the linear system (5.1) or (5.2) is to change $\mathcal{B}(\zeta)$ in such a way that the monodromy data, or, more generally, the monodromy matrices $\{M_k\}$ remain invariant.

There are two types of isomonodromy deformations, both discovered by Schlesinger [Sch] and later generalized to the case of singularities of higher order in [JMU], [JM].

The first type is a continuous deformation which allows the singularities $x_1, \dots, x_n$ to move and describes the $\{\mathcal{B}_k\}$ as functions of x_j's. This deformation leaves the whole set of monodromy data intact. The evolution of the $\{\mathcal{B}_k\}$

[5] As in the case of difference equations, one has to be careful in choosing the sector where ζ may tend to ∞. One may always take $\arg\zeta \in (\pi/2 + \varepsilon, 3\pi/2 - \varepsilon)$ for $\Re\zeta \ll 0$ and $\arg\zeta \in (-\pi/2 + \varepsilon, \pi/2 - \varepsilon)$ for $\Re\zeta \gg 0$. If $s_i - s_j \notin \mathbb{R}$, these sectors may be extended.

is described by a system of partial differential equations called the *Schlesinger equations*:

(5.4)
$$\frac{\partial \mathcal{B}_l}{\partial x_j} = \frac{[\mathcal{B}_j, \mathcal{B}_l]}{x_j - x_l}, \qquad \frac{\partial \mathcal{B}_j}{\partial b_j} = \sum_{\substack{1 \le l \le n \\ l \ne j}} \frac{[\mathcal{B}_j, \mathcal{B}_l]}{x_l - x_j} - [\mathcal{B}_j, \mathcal{B}_\infty], \qquad j, l = 1, \dots, n,$$

where for (5.2) the term with $\mathcal{B}_\infty$ is dropped.

It is not hard to show that this system has a local solution for arbitrary initial conditions $\{\mathcal{B}_k(x_1^o, \dots, x_n^o)\}$. It is a much deeper fact (proved independently in [Mal], [Miw]) that the Schlesinger equations with arbitrary initial conditions have a *global* meromorphic solution on the universal covering space of

$$\{(x_1, \dots, x_n) \in \mathbb{C}^n : x_i \ne x_j \text{ for } i \ne j\}.$$

To describe this fact, one often says that the system of Schlesinger equations enjoys the *Painlevé property*.

The second deformation (or, better to say, transformation) is an action of $\mathbb{Z}^{m(n+1)-1}$ on the space of $\mathcal{B}(z)$, which consists of multiplying $\mathcal{Y}(z)$ by an appropriate rational function on the left: $\mathcal{Y}(z) \mapsto \mathcal{R}(z)\mathcal{Y}(z)$. Such a transformation (called *Schlesinger transformation*) is uniquely determined by the shifts

$$t_j^{(k)} \mapsto t_j^{(k)} + \lambda_j^{(k)}, \qquad k = 1, \dots, n, \infty,$$

of the eigenvalues of $\mathcal{B}_k$. Here all $\lambda_j^{(k)}$ are integers, and their total sum is equal to zero. Schlesinger transformations exist for generic $\{\mathcal{B}_k\}$; see [JM]. Clearly, these transformations change the monodromy data, but they do not change the monodromy matrices $\{M_k\}$ and the Stokes multipliers $S^{\pm}$.

Now let us take a difference equation of the type considered earlier:

$$Y(z+1) = A(z)Y(z), \qquad A(z) = A_0 z^n + A_1 z^{n-1} \cdots + A_n. \tag{5.5}$$

We distinguish two cases (cf. Propositions 1.1 and 1.2):

- A_0 is diagonal and has pairwise distinct nonzero eigenvalues;
- $A_0 = I$, A_1 is diagonal and no two eigenvalues of A_1 are different by an integer.

As explained in Section 4, see Proposition 4.1, we can generically represent $A(z)$ in the form

$$A(z) = A_0(z - C_1) \cdots (z - C_n),$$

where the eigenvalues $\{a_j^{(i)}\}$ of $\{C_i\}$ are zeros of $\det A(z)$ divided into n groups of m numbers. We assume, as usual, that no two eigenvalues are different by an integer.

Suppose that $A(z)$ depends on a small parameter ε, and as $\varepsilon \to 0$,

(5.6)
$$C_i - y_i\varepsilon^{-1} + \mathcal{B}_i \to 0, \quad i = 1, \dots, n, \qquad (A_0 - I)\varepsilon^{-1} - \mathcal{B}_\infty \to 0, \qquad \varepsilon \to 0,$$

for some pairwise distinct complex numbers $y_1, \dots, y_n$ and some $\mathcal{B}_1, \dots, \mathcal{B}_n$, $\mathcal{B}_\infty \in \mathrm{Mat}(m, \mathbb{C})$. (The limit relation for A_0 is omitted in the case $A_0 = I$.)

Note that if we multiply the unknown function $Y(z)$ in (5.5) by $\prod_i \Gamma(z - y_i\varepsilon^{-1})$, then (5.5) takes the form

$$\left(\prod_i \Gamma(z + 1 - y_i\varepsilon^{-1})Y(z+1)\right) = (I + B_\infty\varepsilon + o(\varepsilon)) \times \left(I + \frac{\mathcal{B}_1 + o(1)}{z - y_1\varepsilon^{-1}}\right) \cdots \left(I + \frac{\mathcal{B}_n + o(1)}{z - y_n\varepsilon^{-1}}\right) \left(\prod_i \Gamma(z - y_i\varepsilon^{-1})Y(z)\right).$$

If we now assume that $\prod_i \Gamma((\zeta + y_i)\varepsilon^{-1})Y(\zeta\varepsilon^{-1})$ tends to a holomorphic function $\mathcal{Y}(\zeta)$, then the difference equation above in the limit $\varepsilon \to 0$ turns into the differential equation (5.1) (or (5.2)) with $x_i = y_i$.

Substituting the asymptotic relations (5.6) into (4.2), we see that for any fixed $l_1, \dots, l_n \in \mathbb{Z}$,

$$C_i(l_1, \dots, l_n) + l_i - y_i\varepsilon^{-1} + \mathcal{B}_i \to 0, \quad i = 1, \dots, n, \qquad \varepsilon \to 0.$$

(This conclusion is based on the fact that if $X = x\varepsilon^{-1} + X_0 + o(1)$, $Y = y\varepsilon^{-1} + Y_0 + o(1)$, where $x, y \in \mathbb{C}$, $x \neq y$, and $(z - X)(z - Y) = (z - S)(z - T)$ with $\mathrm{Sp}(S) = \mathrm{Sp}(Y)$, $\mathrm{Sp}(T) = \mathrm{Sp}(X)$, then $S = Y + o(1)$, $T = X + o(1)$; see the explicit construction of T in Lemma 3.3.)

In particular, for any $k_1, \dots, k_n \in \mathbb{Z}$,

$$B_i(k_1, \dots, k_n) + k_i - y_i\varepsilon^{-1} + \mathcal{B}_i \to 0, \quad i = 1, \dots, n, \qquad \varepsilon \to 0.$$

(See §4 for the relation of $\{B_i\}$ and $\{C_i\}$.) Thus, on finite intervals $B_i(k) + k_i - y_i\varepsilon^{-1}$ is approximately constant. However, if we assume that the $\{B_i(k) + k_i - y_i\varepsilon^{-1}\}$ for k of size ε^{-1} approach some smooth functions of εk_j:

$$B_i\big([x_1\varepsilon^{-1}], \dots, [x_n\varepsilon^{-1}]\big) + [x_i\varepsilon^{-1}] - y_i\varepsilon^{-1} + \mathcal{B}_i(y_1 - x_1, \dots, y_n - x_n) \to 0, \quad \varepsilon \to 0.$$

$$\mathcal{B}_i(0, \dots, 0) = \mathcal{B}_i, \qquad i = 1, \dots, n,$$

then the corresponding equation (5.5) converges to (5.1) with $\{\mathcal{B}_i = \mathcal{B}_i(x)\}$ and x_i replaced by $y_i - x_i$. Furthermore, the difference Schlesinger equations (3.1)–(3.3) tend to

$$\frac{\partial \mathcal{B}_l}{\partial x_j} = \frac{[\mathcal{B}_l, \mathcal{B}_j]}{(y_j - x_j) - (y_l - x_l)}, \qquad \sum_{l=1}^{n} \frac{\partial \mathcal{B}_l}{\partial x_j} = [\mathcal{B}_l, \mathcal{B}_\infty], \qquad j, l = 1, \dots, n.$$

Comparing these equations to (5.4), we are led to the following:

CONJECTURE 5.1. *For generic $\mathcal{B}_1,\dots,\mathcal{B}_n,\mathcal{B}_\infty$ and pairwise distinct $y_1,\dots,y_n\in\mathbb{C}$, take $B_i=B_i(\varepsilon)\in\mathrm{Mat}(m,\mathbb{C})$, $i=1,\dots,n$ such that*

$$B_i(\varepsilon)-y_i\varepsilon^{-1}+\mathcal{B}_i\to 0,\qquad \varepsilon\to 0.$$

Let $B_i(k_1,\dots,k_n)$ be the solution of the difference Schlesinger equations (3.1)–(3.3) *with the initial conditions $\{B_i(0)=B_i\}$, and let $\mathcal{B}_i(x_1,\dots,x_n)$ be the solution of the classical Schlesinger equations* (5.4) *with the initial conditions*

$$\mathcal{B}_i(y_1,\dots,y_n)=\mathcal{B}_i,\qquad i=1,\dots,n.$$

Then for any $x_1,\dots,x_n\in\mathbb{R}$ and $i=1,\dots,n$,

$$B_i\bigl([x_1\varepsilon^{-1}],\dots,[x_n\varepsilon^{-1}]\bigr)+[x_i\varepsilon^{-1}]-y_i\varepsilon^{-1}+\mathcal{B}_i(y_1-x_1,\dots,y_n-x_n)\to 0,\quad \varepsilon\to 0.$$

As for isomonodromy deformations of the second kind (Schlesinger transformations), we are able to prove an asymptotic result rigorously. We will consider the case of equation (5.1); for (5.2) the situation is similar.

Fix

$$\bigl\{t_j^{(k)}\bigr\}_{1\le k\le n,\ 1\le j\le m}\subset\mathbb{C},\qquad t_{j_1}^{(k_1)}-t_{j_2}^{(k_2)}\notin\mathbb{Z}\quad\text{unless}\quad j_1=j_2,\ k_1=k_2.$$

For any

$$\mathcal{B}_k\subset\mathrm{Mat}(m,\mathbb{C}),\quad \mathrm{Sp}(\mathcal{B}_k)=\bigl\{t_j^{(k)}\bigr\}_{j=1}^m,\qquad k=1,\dots,n,$$

and pairwise distinct $x_1,\dots,x_n\in\mathbb{C}$, we define

$$B_k(\varepsilon)=x_k\varepsilon^{-1}-\mathcal{B}_k,\qquad k=1,\dots,n,\quad \varepsilon\ne 0.$$

Then

$$\mathrm{Sp}(B_k(\varepsilon))=\left\{a_j^{(k)}\right\}_{j=1}^m,\quad a_j^{(k)}=x_k\varepsilon^{-1}-t_j^{(k)},\qquad j=1,\dots,m.$$

We also fix $\mathcal{B}_\infty=\mathrm{diag}(s_1,\dots,s_m)$, $s_i\ne s_j$ for $i\ne j$. Set

$$A_0(\varepsilon)=I+\varepsilon B_\infty.$$

LEMMA 5.2. *For generic $\mathcal{B}_1,\dots,\mathcal{B}_n$ and $|\varepsilon|$ small enough, there exists a unique degree n polynomial $A(z,\varepsilon)=A_0(\varepsilon)z^n+A_1(\varepsilon)z^{n-1}+\dots$ having $\{z-B_k(\varepsilon)\}$ as its right divisors.*

Proof. According to Lemma 3.4, the statement is true for large $|\varepsilon|$. On the other hand, for fixed $\{\mathcal{B}_k\}$ the existence of $A(z)$ is an open condition on ε, and if it holds for large $|\varepsilon|$, it also holds for $|\varepsilon|$ small enough. □

THEOREM 5.3. *Fix any integers $\bigl\{\lambda_j^{(i)}\bigr\}_{j=1}^m$, $i=1,\dots,n,\infty$, of total sum 0:*

$$\sum_{j=1}^m\left(\sum_{i=1}^n\lambda_j^{(i)}+\lambda_j^{(\infty)}\right)=0.$$

Then for generic $\mathcal{B}_1, \dots \mathcal{B}_n$ *and small enough* $|\varepsilon|$, *there exists the transformation of Theorem* 2.1 *for the equation* $Y(z+1) = A(z,\varepsilon)Y(z)$ *with*

$$\kappa_j^{(i)} = -\lambda_j^{(i)}, \quad 1 \le k \le n, \qquad \delta_j = -\lambda_j^{(\infty)}, \qquad 1 \le j \le m.$$

Furthermore, denote by $\{\widetilde{\mathcal{B}}_k\}$ *the coefficients of* (5.1) *after the corresponding Schlesinger transformation, and denote by* $\{\widetilde{B}_k(\varepsilon)\}$ *the matrices such that* $\{z - \widetilde{B}_k(\varepsilon)\}$ *are the right divisors of the transformed* $\widetilde{A}(z,\varepsilon)$. *Then*

$$\Delta\, \widetilde{B}_i(\varepsilon)\Delta^{-1} - x_i\varepsilon^{-1} + \widetilde{\mathcal{B}}_i \to 0, \quad \varepsilon \to 0, \quad i = 1, \dots, n,$$

where $\Delta = \operatorname{diag}(\varepsilon^{\delta_1}, \dots, \varepsilon^{\delta_n})$.

Remark 5.4. It is easy to verify the statement of Theorem 5.3 if $\lambda_j^{(i)} = -\lambda_j^{(\infty)} = \pm 1$ for some fixed i and all $j = 1, \dots, m$, with all other λ's being zero. Then $\widetilde{B}_i(\varepsilon) = B_i(\varepsilon; 0, \dots, 0, \overset{(i)}{\pm 1}, 0, \dots, 0)$. As mentioned above, for any fixed $k_1, \dots, k_n \in \mathbb{Z}$, we have the asymptotics

$$B_i(\varepsilon; k_1, \dots, k_n) + k_i - x_i\varepsilon^{-1} + \mathcal{B}_i \to 0, \qquad \varepsilon \to 0.$$

Hence, Theorem 5.3 implies that

$$\widetilde{\mathcal{B}}_k = \begin{cases} \mathcal{B}_k, & k \ne i, \\ \mathcal{B}_i \pm I, & k = i. \end{cases}$$

This is immediately verified by the fact that the multiplier $\mathcal{R}(\zeta)$ for the (continuous) Schlesinger transformation in this case equals $\mathcal{R}(\zeta) = (\zeta - x_i)^{\pm 1}$.

Proof of Theorem 5.3. Arguing as in the proof of Lemma 5.2, we can show that all the statements used in this proof which hold generically (like the existence of the polynomial with given right divisors), also hold for generic $\mathcal{B}_1, \dots, \mathcal{B}_n$ and small enough ε. Thus, we will ignore the questions of genericity from now on.

Note that we can decompose the transformations of Theorem 5.3 in both discrete and continuous cases into compositions of elementary ones of the same type (those, for which exactly one of $\{\lambda_j^{(i)}\}$ is equal to ± 1, and exactly one of $\{\lambda_j^{(\infty)}\}$ is equal to ∓ 1, with all others being zero). It is clear that the claim of the theorem follows from a slightly more general claim for the elementary transformations: we assume that

$$\Delta_0\, B_l(\varepsilon)\Delta_0^{-1} - x_l\varepsilon^{-1} + \mathcal{B}_l \to 0, \qquad \varepsilon \to 0, \quad l = 1, \dots, n, \tag{5.7}$$

with some diagonal Δ_0 containing integral powers of ε on the diagonal. We also need to conclude that

$$\Delta\Delta_0\, \widetilde{B}_l(\varepsilon)\Delta_0^{-1}\Delta^{-1} - x_l\varepsilon^{-1} + \widetilde{\mathcal{B}}_l \to 0, \qquad \varepsilon \to 0, \quad l = 1, \dots, n. \tag{5.8}$$

Let us consider the elementary transformations with $\lambda_j^{(1)}=1$, $\lambda_i^{(\infty)}=-1$. Denote by $R(z,\varepsilon)$ and $\mathcal{R}(\zeta)$ the corresponding multipliers for the discrete and continuous equations. According to the proof of Theorem 2.1,

$$R(z,\varepsilon)=(z-x_1\varepsilon^{-1}+t_j^{(1)})E_i+R_0(\varepsilon),\qquad R^{-1}(z,\varepsilon)=I-E_i+\frac{R_1(\varepsilon)}{z-x_1\varepsilon^{-1}+t_j^{(1)}}$$

are given by the formulas of Lemma 2.4 with $Q=\hat{Y}_1(\varepsilon)$, and $v=v(\varepsilon)$ being an eigenvector of $B_1(\varepsilon)$ with the eigenvalue $x_1\varepsilon^{-1}-t_j^{(1)}$. Similarly, [JM, Appendix A] shows that

$$\mathcal{R}(\zeta)=(\zeta-x_1)E_i+\mathcal{R}_0,\qquad \mathcal{R}^{-1}(z)=I-E_i+\frac{\mathcal{R}_1}{\zeta-x_1}$$

are given by the same formulas with $Q=\hat{\mathcal{Y}}_1$ and v being an eigenvector of $\mathcal{B}_1$ with the eigenvalue $t_j^{(1)}$. (Note that only the off-diagonal elements of Q participate in the formulas.)

LEMMA 5.5. *Under the assumption* (5.7),

$$\Delta\Delta_0R_0(\varepsilon)\Delta_0^{-1}\to\mathcal{R}_0,\qquad \varepsilon\,\Delta_0R_1(\varepsilon)\Delta_0^{-1}\Delta^{-1}\to\mathcal{R}_1,\qquad \varepsilon\to 0$$

where $\Delta=\varepsilon^{E_i}$.

Proof. First we note that (5.7) implies that in the projective space the vector $\Delta_0v(\varepsilon)$ tends to v as $\varepsilon\to 0$. Next, it is easy to see that the difference Schlesinger equations preserve the asymptotics (5.7):

$$\Delta_0\,B_l(\varepsilon;k_1,\dots,k_n)\Delta_0^{-1}+k_l-x_l\varepsilon^{-1}+\mathcal{B}_l\to 0,\qquad \varepsilon\to 0, \tag{5.9}$$

for any $k_1,\dots,k_n\in\mathbb{Z}$. In particular,

$$\Delta_0\,C_l(\varepsilon)\Delta_0^{-1}-x_l\varepsilon^{-1}+\mathcal{B}_l\to 0,\qquad \varepsilon\to 0.$$

Thus, from

$$A(z,\varepsilon)=A_0(\varepsilon)(z-C_1(\varepsilon))\cdots(z-C_n(\varepsilon))=A_0(\varepsilon)z^n+A_1(\varepsilon)z^{n-1}+\dots$$

we conclude that

$$A_1(\varepsilon)=-\sum_{l=1}^n C_l(\varepsilon)=\text{diagonal part}+\Delta_0^{-1}\left(\sum_{l=1}^n\mathcal{B}_l+o(1)\right)\Delta_0.$$

By (1.3) we also know that $\varepsilon(s_j-s_i)(\hat{Y}_1(\varepsilon))_{kl}=(A_1(\varepsilon))_{kl}$ for all $k\neq l$. Since $\hat{\mathcal{Y}}_1=\sum_{l=1}^n\mathcal{B}_i$, the statement follows from the explicit formulas of Lemma 2.4. □

A direct computation shows that (here we use the fact that $\mathcal{R}_0\mathcal{B}_1\mathcal{R}_1 = t_j^{(1)}\mathcal{R}_0\mathcal{R}_1 = 0$, which follows from the explicit formulas of [JM, Appendix A])

$$\widetilde{\mathcal{B}}_1 = \mathcal{R}_0\left(\mathcal{B}_\infty + \sum_{k=2}^{n}\frac{\mathcal{B}_k}{x_1 - x_k}\right)\mathcal{R}_1 + E_i\mathcal{R}_1, \tag{5.10}$$

$$\widetilde{\mathcal{B}}_l = ((x_l - x_1)E_i + \mathcal{R}_0)\mathcal{B}_l\left(I - E_i + \frac{\mathcal{R}_1}{x_l - x_1}\right), \quad l = 2, \dots, n. \tag{5.11}$$

Let us prove (5.8) for $l \geq 2$ first. Consider the composition of the elementary transformation (for the difference equation) in question with $\mathcal{S}(0, \dots, 0, \overset{(l)}{1}, 0, \dots, 0)$; see Section 3 for the notation. By the uniqueness part of Theorem 2.1, this is equivalent to making $\mathcal{S}(0, \dots, 0, \overset{(l)}{1}, 0, \dots, 0)$ first, and applying the elementary transformation after that. Denote the multiplier of this second elementary transformation by $\hat{R}(z, \varepsilon)$. Now,

$$\hat{R}(z,\varepsilon) = (z - x_1\varepsilon^{-1} + t_j^{(1)})E_i + \hat{R}_0(\varepsilon), \qquad \hat{R}^{-1}(z,\varepsilon) = I - E_i + \frac{\hat{R}_1(\varepsilon)}{z - x_1\varepsilon^{-1} + t_j^{(1)}}.$$

Using Proposition 3.6, we obtain

$$(z - \widetilde{B}_l(\varepsilon))R(z,\varepsilon) = \hat{R}(z,\varepsilon)(z - B_l(\varepsilon)).$$

Substituting $z = x_l\varepsilon^{-1}$ and conjugating by Δ_0, we get

$$\begin{aligned}&\Delta_0\widetilde{B}_l(\varepsilon)\Delta_0^{-1} - x_l\varepsilon^{-1}\\ &\quad = \left(\Delta_0\hat{R}(x_l\varepsilon^{-1},\varepsilon)\Delta_0^{-1}\right)\left(\Delta_0 B_l(\varepsilon)\Delta_0^{-1} - x_l\varepsilon^{-1}\right)\left(\Delta_0 R^{-1}(x_l\varepsilon^{-1},\varepsilon)\Delta_0^{-1}\right).\end{aligned}$$

Because of (5.9), the limit relations of Lemma 5.5 also hold for $\hat{R}_0$, $\hat{R}_1$. Using them, Lemma 5.5 itself, (5.7) and (5.11), we arrive at (5.8) for $l \geq 2$.

Thus, it remains to prove (5.8) for $l = 1$. We have

$$\begin{aligned}\widetilde{A}(z,\varepsilon) &= R(z+1,\varepsilon)A_0(z - C_1(\varepsilon))\cdots(z - C_n(\varepsilon))R^{-1}(z,\varepsilon)\\ &= A_0(z - \widetilde{C}_1(\varepsilon))\cdots(z - \widetilde{C}_n(\varepsilon)).\end{aligned} \tag{5.12}$$

The relation (5.8) for $l \geq 2$ implies that

$$\Delta\Delta_0\,\widetilde{C}_l(\varepsilon)\Delta_0^{-1}\Delta^{-1} - x_l\varepsilon^{-1} + \widetilde{\mathcal{B}}_l \to 0, \qquad \varepsilon \to 0, \quad l = 2, \dots, n.$$

Substituting these estimates and similar ones for C_l and setting $z = (w + x_1)\varepsilon^{-1} - t_j^{(1)}$, we can rewrite (5.12) as follows (note that A_0 is diagonal and hence it commutes with Δ_0):

$$\begin{aligned}&((w - x_1 + \varepsilon)E_i + \Delta\Delta_0 R_0(\varepsilon)\Delta_0^{-1})(I + \varepsilon\mathcal{B}_\infty)\left(I + \varepsilon\,\frac{\mathcal{B}_1 + o(1)}{w - x_1}\right)\cdots\left(I + \varepsilon\,\frac{\mathcal{B}_n + o(1)}{w - x_n}\right)\\ &\times\left(I - E_i + \frac{\varepsilon\Delta_0 R_1(\varepsilon)\Delta_0^{-1}\Delta^{-1}}{w - x_1}\right) = (I + \varepsilon\mathcal{B}_\infty)\left(I + \varepsilon\,\frac{x_1\varepsilon^{-1} - \Delta\Delta_0\widetilde{C}_1(\varepsilon)\Delta_0^{-1}\Delta^{-1}}{w - x_1}\right)\\ &\qquad\times\left(I + \varepsilon\,\frac{\widetilde{\mathcal{B}}_2 + o(1)}{w - x_2}\right)\cdots\left(I + \varepsilon\,\frac{\widetilde{\mathcal{B}}_n + o(1)}{w - x_n}\right).\end{aligned}$$

Comparing the residues of both sides at $w = x_1$ and looking at terms of order ε, we see that

$$x_1\varepsilon^{-1} - \Delta\Delta_0\widetilde{C}_1(\varepsilon)\Delta_0^{-1}\Delta^{-1} \to \widetilde{B}_1,$$

where $\widetilde{B}_1$ is as given by (5.10). (We need to use Lemma 5.5 and the relation $R_0(\varepsilon)R_1(\varepsilon) = 0$ here.) Since the difference Schlesinger equations preserve the asymptotics (5.8) (cf. (5.9)), we get (5.8) for $l = 1$, and thus for all l.

The proof that (5.7) implies (5.8) in the case $\lambda_j^{(1)} = -\lambda_j^{(\infty)} = -1$ is very similar. Let us outline the necessary changes. The multipliers have the form

$$R(z,\varepsilon) = I - E_i + \frac{R_1^t(\varepsilon)}{z - 1 - x_1\varepsilon^{-1} + t_j^{(1)}},$$

$$R^{-1}(z,\varepsilon) = (z - 1 - x_1\varepsilon^{-1} + t_j^{(1)})E_i + R_0^t(\varepsilon),$$

where $R_0(\varepsilon)$, $R_1(\varepsilon)$ are as in Lemma 2.4 with $v = v(\varepsilon)$ a solution of $A^t(x_1\varepsilon^{-1} - t_j^{(1)}, \varepsilon)\, v(\varepsilon) = 0$ and $Q = -\hat{Y}_1^t(\varepsilon)$; see the proof of Theorem 2.1. Similarly,

$$\mathcal{R}(\zeta) = I - E_i + \frac{\mathcal{R}_1^t}{\zeta - x_1}, \qquad \mathcal{R}^{-1}(\zeta) = (\zeta - x_1)E_i + \mathcal{R}_0^t,$$

where $\mathcal{R}_0$ and $\mathcal{R}_1$ are as in Lemma 2.4 with v an eigenvector of $\mathcal{B}_1^t$ with the eigenvalue $t_j^{(1)}$, and $Q = -\hat{\mathcal{Y}}_1^t$. Similarly to Lemma 5.5, (5.7) implies

$$\Delta_0 R_0^t(\varepsilon)\Delta_0^{-1}\Delta^{-1} \to \mathcal{R}_0^t, \qquad \varepsilon\,\Delta\Delta_0 R_1^t(\varepsilon)\Delta_0^{-1} \to \mathcal{R}_1^t, \qquad \varepsilon \to 0$$

with $\Delta = \varepsilon^{-E_i}$. Similarly to (5.10), (5.11), we have

$$\widetilde{\mathcal{B}}_1 = \mathcal{R}_1^t\left(\mathcal{B}_\infty + \sum_{k=2}^{n}\frac{\mathcal{B}_k}{x_1 - x_k}\right)\mathcal{R}_0^t - \mathcal{R}_1^t E_i,$$

$$\widetilde{\mathcal{B}}_l = \left(I - E_i + \frac{\mathcal{R}_1^t}{x_l - x_1}\right)\mathcal{B}_l\,((x_l - x_1)E_i + \mathcal{R}_0^t), \quad l = 2,\dots,n.$$

Using the same argument, composing our elementary transformation with $\mathcal{S}(0,\dots,0,\overset{(l)}{1},0,\dots,0)$, we prove (5.8) for $l \ge 2$. Then substituting estimates for C_l's and $\widetilde{C}_l$'s into $\widetilde{A}(z,\varepsilon) = R(z+1,\varepsilon)A(z,\varepsilon)R^{-1}(z,\varepsilon)$, we get (with $z = (w + x_1)\varepsilon^{-1} - t_j^{(1)}$)

$$\left(I - E_i + \frac{\varepsilon\Delta\Delta_0 R_1^t(\varepsilon)\Delta_0^{-1}}{w - x_1}\right)(I + \varepsilon\mathcal{B}_\infty)\left(I + \varepsilon\,\frac{\mathcal{B}_1 + o(1)}{w - x_1}\right)\cdots\left(I + \varepsilon\,\frac{\mathcal{B}_n + o(1)}{w - x_n}\right)$$
$$\times((w - x_1 - \varepsilon)E_i + \Delta_0 R_0^t(\varepsilon)\Delta_0^{-1}\Delta^{-1}) = (I + \varepsilon\mathcal{B}_\infty)$$
$$\times\left(I + \varepsilon\,\frac{x_1\varepsilon^{-1} - \Delta\Delta_0\widetilde{C}_1(\varepsilon)\Delta_0^{-1}\Delta^{-1}}{w - x_1}\right)\left(I + \varepsilon\,\frac{\widetilde{\mathcal{B}}_2 + o(1)}{w - x_2}\right)\cdots\left(I + \varepsilon\,\frac{\widetilde{\mathcal{B}}_n + o(1)}{w - x_n}\right).$$

Comparing the residues of both sides at $w = x_1$ and taking terms of order ε, we recover the estimate of type (5.8) for $\widetilde{C}_1(\varepsilon)$, and hence for $\widetilde{B}_1(\varepsilon)$. The proof of Theorem 5.3 is complete. □

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References

[Bi1] G. D. Birkhoff, General theory of linear difference equations, *Trans. Amer. Math. Soc.* **12** (1911), 243–284.

[Bi2] ———, The generalized Riemann problem for linear differential equations and the allied problems for linear difference equations, *Amer. Acad. Proc.* **49**, 521–568.

[Bor] A. Borodin, Discrete gap probabilities and discrete Painlevé equations, *Duke Math. J.* **117** (2003), 489–542; `math-ph/0111008`.

[BB] A. Borodin and D. Boyarchenko, Distribution of the first particle in discrete orthogonal polynomial ensembles, *Comm. Math. Phys.* **234** (2003), 287–338; `math-ph/0204001`.

[BD] A. Borodin and P. Deift, Fredholm determinants, Jimbo-Miwa-Ueno tau-functions, and representation theory, *Comm. Pure Appl. Math.* **55** (2002), 1160–1230; `math-ph/0111007`.

[BO] A. Borodin and G. Olshanski, Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes, *Ann. of Math.* **161**, no. 2 (2005), to appear; `math.RT/0109194`.

[C] R. D. Carmichael, Linear difference equations and their analytic solutions, *Trans. Amer. Math. Soc.* **12** (1911), 99–134.

[G] Galbrun, Dissertation, Paris, 1910.

[GRW] I. Gelfand, V. Retakh, and R. L. Wilson, Quadratic linear algebras associated with factorizations of noncommutative polynomials and noncommutative differential polynomials, *Selecta Math.* **7** (2001), 493–523.

[GLR] I. Gohberg, P. Lancaster, and L. Rodman, *Matrix Polynomials*, Academic Press Inc., New York, 1982.

[HI] J. Harnad and A. R. Its, Integrable Fredholm operators and dual isomonodromic deformations, *Comm. Math. Phys.* **226** (2002), 497–530.

[JMMS] M. Jimbo, T. Miwa, T. Môri, and M. Sato, Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent, *Phys.* D **1** (1980), 80–158.

[JMU] M. Jimbo, T. Miwa, and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I, *Phys.* D **2** (1981), 306–352.

[JM] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II, *Phys.* D **2** (1981), 407–448.

[JS] M. Jimbo and H. Sakai, A q-analog of the sixth Painlevé equation, *Lett. Math. Phys.* **38** (1996), 145–154.

[Joh] K. Johansson, Non-intersecting paths, random tilings and random matrices, *Probab. Theory Related Fields* **123** (2002), 225–280; `math.PR/0011250`.

[Mal] B. Malgrange, Sur les déformations isomonodromiques. I. Singularités régulières, in *Mathematics and Physics*, 401–426, *Progr. Math.* **37**, Birkhäuser, Boston, 1983.

[Me] M. L. Mehta, A nonlinear differential equation and a Fredholm determinant, *J. Physique* I **2** (1992), 1721–1729.

[Miw] T. Miwa, Painlevé property of monodromy preserving deformation equations and the analyticity of τ-functions, *Publ. Res. Inst. Math. Sci.* **17** (1981), 703–721.

[MV] J. Moser and A. P. Veselov, Discrete versions of some classical integrable systems and factorization of matrix polynomials, *Comm. Math. Phys.* **139** (1991), 217–243.

[N] Nörlund, Dissertation, Copenhagen, 1911.

[O] A. Odesskii, Set-theoretical solutions to the Yang-Baxter relation from factorization of matrix polynomials and theta-functions, *Mosc. Math. J.* **3** (2003), 97–103; `math.QA/0205051`.

[P] J. Palmer, Deformation analysis of matrix models, *Phys.* D **78** (1994), 166–185.

[Sak] H. Sakai, Rational surfaces associated with affine root systems and geometry of the Painlevé equations, *Comm. Math. Phys.* **220** (2001), 165–229.

[Sch] L. Schlesinger, Über eine Klasse von Differentsial System Beliebliger Ordnung mit Festen Kritischer Punkten, *J. Reine Angew. Math.* **141** (1912), 96–145.

[TW] C. A. Tracy and H. Widom, Fredholm determinants, differential equations and matrix models, *Comm. Math. Phys.* **163** (1994), 33–72; `hep-th/9306042`.

[V] A. Veselov, Yang-Baxter maps and integrable dynamics, *Phys. Lett.* A **314** (2003), 214–221.

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