Annals of Mathematics, **160** (2004), 1057–1097

Quiver varieties and t-analogs of q-characters of quantum affine algebras

By Hiraku Nakajima*

Abstract

We consider a specialization of an untwisted quantum affine algebra of type ADE at a nonzero complex number, which may or may not be a root of unity. The Grothendieck ring of its finite dimensional representations has two bases, simple modules and standard modules. We identify entries of the transition matrix with special values of "computable" polynomials, similar to Kazhdan-Lusztig polynomials. At the same time we "compute" q-characters for all simple modules. The result is based on "computations" of Betti numbers of graded/cyclic quiver varieties. (The reason why we use " " will be explained at the end of the introduction.)

Contents

Introduction

Let $\mathfrak{g}$ be a simple Lie algebra of type ADE over $\mathbb{C}$, $\mathbf{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ be its loop algebra, and $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ be its quantum universal enveloping algebra, or the quantum loop algebra for short. It is a subquotient of the quantum

*Supported by the Grant-in-aid for Scientific Research (No.11740011), the Ministry of Education, Japan.

affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$, i.e., without central extension and degree operator. Let $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ be its specialization at $q = \varepsilon$, a nonzero complex number. (See §1 for definition.)

It is known that $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ is a Hopf algebra. Therefore the category $\mathcal{R}ep\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ of finite dimensional representations of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ is a monoidal (or tensor) abelian category. Let $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ be its Grothendieck ring. It is known that $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ is commutative (see e.g., [15, Cor. 2]).

The ring $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ has two natural bases, simple modules $L(P)$ and standard modules $M(P)$, where P is the Drinfeld polynomial. The latter were introduced by the author [33].

The purpose of this article is to "compute" the transition matrix between these two bases. More precisely, we define certain "computable" polynomials $Z_{PQ}(t)$, which are analogs of Kazhdan-Lusztig polynomials for Weyl groups. Then we show that the multiplicity $[M(P) : L(Q)]$ is equal to $Z_{PQ}(1)$. This generalizes a result of Arakawa [1] who expressed the multiplicities by Kazhdan-Lusztig polynomials when $\mathfrak{g}$ is of type A_n and ε is not a root of unity. Furthermore, coefficients of $Z_{PQ}(t)$ are equal to multiplicities of simple modules of subquotients of standard modules with respect to a Jantzen filtration if we combine our result with [16], where the transversal slice is as given in [33].

Since there is a slight complication when ε is a root of unity, we assume ε is *not* so in this introduction. Then the definition of $Z_{PQ}(t)$ is as follows. Let $\mathbf{R}_t \overset{\text{def.}}{=} \operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$, which is a t-analog of the representation ring. By [33], $\mathbf{R}_t$ is identified with the dual of the Grothendieck group of a category of perverse sheaves on affine graded quiver varieties (see Section 4 for the definition) so that (1) $\{M(P)\}$ is the specialization at $t = 1$ of the dual base of constant sheaves of strata, extended by 0 to the complement, and (2) $\{L(P)\}$ is that of the dual base of intersection cohomology sheaves of strata. A property of intersection cohomology complexes leads to the following combinatorial definition of $Z_{PQ}(t)$: Let $\overline{}$ be the involution on $\mathbf{R}_t$, dual to the Grothendieck-Verdier duality. We denote the two bases of $\mathbf{R}_t$ by the same symbols $M(P)$, $L(P)$ at the specialization at $t = 1$ for simplicity. Let us express the involution in the basis $\{M(P)\}_P$, classes of standard modules:

$$\overline{M(P)} = \sum_{Q: Q \le P} u_{PQ}(t) M(Q),$$

where $\le$ is a certain ordering $<$ among P's. We then define an element $L(P)$ by

$$\overline{L(P)} = L(P), \qquad L(P) \in M(P) + \sum_{Q: Q < P} t^{-1}\mathbb{Z}[t^{-1}] M(Q). \tag{0.1}$$

The above polynomials $Z_{PQ}(t) \in \mathbb{Z}[t^{-1}]$ are given by

$$M(P) = \sum_{Q: Q \le P} Z_{PQ}(t) L(Q).$$

The existence and uniqueness of $L(P)$ (and hence of $Z_{PQ}(t)$) is proved exactly as in the case of the Kazhdan-Lusztig polynomial. In particular, it gives us a combinatorial algorithm computing $Z_{PQ}(t)$, once $u_{PQ}(t)$ is given.

In summary, we have the following analogy:

$\mathbf{R}_t$	the Iwahori-Hecke algebra H_q
standard modules $\{M(P)\}_P$	$\{T_w\}_{w\in W}$
simple modules $\{L(P)\}_P$	Kazhdan-Lusztig basis $\{C'_w\}_{w\in W}$

See [22] for definitions of H_q, T_w, C'_w.

The remaining task is to "compute" $u_{PQ}(t)$. For this purpose we introduce a t-analog $\widehat{\chi}_{\varepsilon,t}$ of the q-character, or ε-character. The original ε-character χ_ε, which is a specialization of our t-analog at $t=1$, was introduced by Knight [23] (for Yangian and generic ε) and Frenkel-Reshetikhin [15] (for generic ε) and Frenkel-Mukhin [13] (when ε is a root of unity). It is an injective ring homomorphism from $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ to $\mathbb{Z}[Y_{i,a}^{\pm}]_{i\in I,a\in\mathbb{C}^*}$, a ring of Laurent polynomials of infinitely many variables. It is an analog of the ordinary character homomorphism of the finite dimensional Lie algebra $\mathfrak{g}$. Our t-analog is an injective $\mathbb{Z}[t,t^{-1}]$-linear map

$$\widehat{\chi}_{\varepsilon,t}\colon \mathbf{R}_t \to \widehat{\mathcal{Y}}_t \overset{\text{def.}}{=} \mathbb{Z}[t,t^{-1},V_{i,a},W_{i,a}]_{i\in I,a\in\mathbb{C}^*}.$$

We have a simple, explicit definition of an involution $\overline{}$ on $\widehat{\mathcal{Y}}_t$ (see (2.3)). The involution on $\mathbf{R}_t$ is the restriction. Therefore the matrix $(u_{PQ}(t))$ can be expressed in terms of values of $\widehat{\chi}_{\varepsilon,t}(M(P))$ for all P.

We define $\widehat{\chi}_{\varepsilon,t}$ as the generating function of Betti numbers of nonsingular graded/cyclic quiver varieties. We axiomatize its properties. The axioms are purely combinatorial statements in $\widehat{\mathcal{Y}}_t$, involving no geometry nor representation theory of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$. Moreover, the axioms uniquely characterize $\widehat{\chi}_{\varepsilon,t}$, and give us an algorithm for computation. Therefore the axioms can be considered as a definition of $\widehat{\chi}_{\varepsilon,t}$. When $\mathfrak{g}$ is not of type E_8, we can directly prove the existence of $\widehat{\chi}_{\varepsilon,t}$ satisfying the axioms without using geometry or representation theory of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$.

Two of the axioms are most important. One is the characterization of the image of $\widehat{\chi}_{\varepsilon,t}$. Another is the multiplicative property.

The former is a modification of Frenkel-Mukhin's result [12]. They give a characterization of the image of χ_ε, as an analog of the Weyl group invariance of the ordinary character homomorphism. And they observed that the characterization gives an algorithm computing χ_ε at l-fundamental representations. This property has no counterpart in the ordinary character homomorphism for $\mathfrak{g}$, and is one of the most remarkable features of χ_ε. We use a t-analog of their characterization to "compute" $\widehat{\chi}_{\varepsilon,t}$ for l-fundamental representations.

A standard module $M(P)$ is a tensor product of l-fundamental representations in $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ (see Corollary 3.7 or [39]). If $\widehat{\chi}_{\varepsilon,t}$ would be a ring homomorphism, then $\widehat{\chi}_{\varepsilon,t}(M(P))$ is just a product of $\widehat{\chi}_{\varepsilon,t}$ of l-fundamental representations. This is *not* true under the usual ring structures on $\mathbf{R}_t$ and $\widehat{\mathcal{Y}}_t$. We introduce 'twistings' of multiplications on $\mathbf{R}_t$, $\widehat{\mathcal{Y}}_t$ so that $\widehat{\chi}_{\varepsilon,t}$ is a ring homomorphism. The resulting algebras are *not* commutative.

We can add another column to the table above by [25].

$\mathbf{U}_q^-$: the $-$ part of the quantized enveloping algebra
PBW basis
canonical basis

In fact, when $\mathfrak{g}$ is of type A, affine graded quiver varieties are varieties used for the definition of the canonical base [25]. Therefore it is more natural to relate $\mathbf{R}_t$ to the dual of $\mathbf{U}_q^-$. In this analogy, $\widehat{\chi}_{\varepsilon,t}$ can be considered as an analog of Feigin's map from $\mathbf{U}_q^-$ to the skew polynomial ring ([18], [19], [2], [38]). We also have an analog of the monomial base, $(E((\mathbf{c}))$ in [25, 7.8]. See also [7], [38].)

This article is organized as follows. In Section 1 we recall results on quantum loop algebras and their finite dimensional representations. In Section 2 we introduce a twisting of the multiplication on $\widehat{\mathcal{Y}}_t$. In Section 3 we give axioms which $\widehat{\chi}_{\varepsilon,t}$ satisfies and derive their consequences. In particular, $\widehat{\chi}_{\varepsilon,t}$ is uniquely determined from the axioms. In Section 4 we introduce graded and cyclic quiver varieties, which will be used to prove the existence of $\widehat{\chi}_{\varepsilon,t}$ satisfying the axioms. In Sections 5, 6, 7 we check that a generating function of Betti numbers of nonsingular graded/cyclic quiver varieties satisfies the axioms. In Section 8 we prove the characterization of simple modules mentioned above. In Section 9 we study the case $\varepsilon = \pm 1$ in detail. In Section 10 we state a conjecture concerning finite dimensional representations studied in the literature [37], [17].

In this introduction and also in the main body of this article, we enclose the word compute in quotation marks. What we actually do in this article is to give a purely combinatorial algorithm to compute something. The author wrote a computer program realizing the algorithm for computing $\widehat{\chi}_{\varepsilon,t}$ for l-fundamental representations when $\mathfrak{g}$ is of type E. Up to this moment (2001, April), the program produces the answer except two l-fundamental representations of E_8. It took three days for the last successful one, and the remaining ones are inaccessible so far. In this sense, our character formula is *not* computable in a strict sense.

The result of this article for generic ε was announced in [34].

Acknowledgement. The author would like to thank D. Hernandez and E. Frenkel for pointing out mistakes in an earlier version of this paper.

1. Quantum loop algebras

1.1. *Definition.* Let $\mathfrak{g}$ be a simple Lie algebra of type ADE over $\mathbb{C}$. Let I be the index set of simple roots. Let $\{\alpha_i\}_{i\in I}$, $\{h_i\}_{i\in I}$, $\{\Lambda_i\}_{i\in I}$ be the sets of simple roots, simple co-roots and fundamental weights of $\mathfrak{g}$ respectively. Let P be the weight lattice, and P^* be its dual. Let P^+ be the semigroup of dominant weights.

Let q be an indeterminant. For nonnegative integers $n \geq r$, define

$$[n]_q \stackrel{\text{def.}}{=} \frac{q^n - q^{-n}}{q - q^{-1}},$$
$$[n]_q! \stackrel{\text{def.}}{=} \begin{cases} [n]_q[n-1]_q \cdots [2]_q[1]_q & (n > 0), \\ 1 & (n = 0), \end{cases} \qquad \begin{bmatrix} n \\ r \end{bmatrix}_q \stackrel{\text{def.}}{=} \frac{[n]_q!}{[r]_q![n-r]_q!}.$$

Later we consider another indeterminant t. We define a t-binomial coefficient $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_t$ by replacing q by t.

Let $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ be the quantum loop algebra associated with the loop algebra $\mathbf{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ of $\mathfrak{g}$. It is an associative $\mathbb{Q}(q)$-algebra generated by $e_{i,r}$, $f_{i,r}$ ($i \in I$, $r \in \mathbb{Z}$), q^h ($h \in P^*$), $h_{i,m}$ ($i \in I$, $m \in \mathbb{Z} \setminus \{0\}$) with the following defining relations:

$$q^0 = 1, \quad q^h q^{h'} = q^{h+h'}, \quad [q^h, h_{i,m}] = 0, \quad [h_{i,m}, h_{j,n}] = 0,$$
$$q^h e_{i,r} q^{-h} = q^{\langle h, \alpha_i \rangle} e_{i,r}, \quad q^h f_{i,r} q^{-h} = q^{-\langle h, \alpha_i \rangle} f_{i,r},$$
$$(z - q^{\pm\langle h_j, \alpha_i \rangle} w) \psi_i^s(z) x_j^{\pm}(w) = (q^{\pm\langle h_j, \alpha_i \rangle} z - w) x_j^{\pm}(w) \psi_i^s(z),$$
$$\left[x_i^+(z), x_j^-(w)\right] = \frac{\delta_{ij}}{q - q^{-1}} \left\{ \delta\left(\frac{w}{z}\right) \psi_i^+(w) - \delta\left(\frac{z}{w}\right) \psi_i^-(z) \right\},$$
$$(z - q^{\pm 2\langle h_j, \alpha_i \rangle} w) x_i^{\pm}(z) x_j^{\pm}(w) = (q^{\pm 2\langle h_j, \alpha_i \rangle} z - w) x_j^{\pm}(w) x_i^{\pm}(z),$$
$$\sum_{\sigma \in S_b} \sum_{p=0}^{b} (-1)^p \begin{bmatrix} b \\ p \end{bmatrix}_q x_i^{\pm}(z_{\sigma(1)}) \cdots x_i^{\pm}(z_{\sigma(p)}) x_j^{\pm}(w) \cdots x_i^{\pm}(z_{\sigma(p+1)}) x_j^{\pm}(z_{\sigma(b)}) = 0, \quad \text{if } i \neq j,$$

where $s = \pm$, $b = 1 - \langle h_i, \alpha_j \rangle$, and S_b is the symmetric group of b letters. Here $\delta(z)$, $x_i^+(z)$, $x_i^-(z)$, $\psi_i^{\pm}(z)$ are generating functions defined by

$$\delta(z) \stackrel{\text{def.}}{=} \sum_{r=-\infty}^{\infty} z^r, \qquad x_i^+(z) \stackrel{\text{def.}}{=} \sum_{r=-\infty}^{\infty} e_{i,r} z^{-r}, \qquad x_i^-(z) \stackrel{\text{def.}}{=} \sum_{r=-\infty}^{\infty} f_{i,r} z^{-r},$$
$$\psi_i^{\pm}(z) \stackrel{\text{def.}}{=} q^{\pm h_i} \exp\left(\pm(q - q^{-1}) \sum_{m=1}^{\infty} h_{i,\pm m} z^{\mp m} \right).$$

We also need the following generating function

$$p_i^{\pm}(z) \stackrel{\text{def.}}{=} \exp\left(-\sum_{m=1}^{\infty} \frac{h_{i,\pm m}}{[m]_q} z^{\mp m}\right).$$

Also, $\psi_i^{\pm}(z) = q^{\pm h_i} p_i^{\pm}(qz)/p_i^{\pm}(q^{-1}z)$.

Let $e_{i,r}^{(n)} \stackrel{\text{def.}}{=} e_{i,r}^n/[n]_q!$, $f_{i,r}^{(n)} \stackrel{\text{def.}}{=} f_{i,r}^n/[n]_q!$. Let $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$ be the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by $e_{i,r}^{(n)}$, $f_{i,r}^{(n)}$ and q^h for $i \in I$, $r \in \mathbb{Z}$, $h \in P^*$.

Let $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^+$ (resp. $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^-$) be the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by $e_{i,r}^{(n)}$ (resp. $f_{i,r}^{(n)}$) for $i \in I$, $r \in \mathbb{Z}$, $n \in Z_{>0}$. Now, $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^0$ is the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by q^h, the coefficients of $p_i^{\pm}(z)$ and

$$\begin{bmatrix} q^{h_i}; n \\ r \end{bmatrix} \stackrel{\text{def.}}{=} \prod_{s=1}^{r} \frac{q^{h_i} q^{n-s+1} - q^{-h_i} q^{-n+s-1}}{q^s - q^{-s}}$$

for all $h \in P$, $i \in I$, $n \in \mathbb{Z}$, $r \in \mathbb{Z}_{>0}$. Thus, $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g}) = \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^+ \cdot \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^0 \cdot \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^-$ ([5, 6.1]).

Let ε be a nonzero complex number. The specialization $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$ with respect to the homomorphism $\mathbb{Z}[q,q^{-1}] \ni q \mapsto \varepsilon \in \mathbb{C}^*$ is denoted by $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$. Set

$$\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^{\pm} \stackrel{\text{def.}}{=} \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^{\pm} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}, \qquad \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0 \stackrel{\text{def.}}{=} \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^0 \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}.$$

It is known that $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ is isomorphic to a subquotient of the quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$ defined in terms of Chevalley generators e_i, f_i, q^h ($i \in I \cup \{0\}$, $h \in P^* \oplus \mathbb{Z}c$). (See [11], [2].) Using this identification, we define a coproduct on $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ by

$$\Delta q^h = q^h \otimes q^h, \quad \Delta e_i = e_i \otimes q^{-h_i} + 1 \otimes e_i,$$
$$\Delta f_i = f_i \otimes 1 + q^{h_i} \otimes f_i.$$

Note that this is different from one in [27], although there is a simple relation between them [20, 1.4]. The results in [33] hold for either co-multiplication (tensor products appear in (1.2.19) and (14.1.2)). In [34, §2] another co-multiplication was used.

It is known that the subalgebra $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$ is preserved under Δ. Therefore $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ also has an induced coproduct.

For $a \in \mathbb{C}^*$, there is a Hopf algebra automorphism τ_a of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$, given by

$$\tau_a(e_{i,r}) = a^r e_{i,r}, \quad \tau_a(f_{i,r}) = a^r f_{i,r}, \quad \tau_a(h_{i,m}) = a^m h_{i,m}, \quad \tau_a(q^h) = q^h,$$

which preserves $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}[q,q^{-1}]$ and induces an automorphism of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$, which is denoted also by τ_a.

We define an algebra homomorphism from $\mathbf{U}_\varepsilon(\mathfrak{g})$ to $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ by

$$e_i \mapsto e_{i,0}, \quad f_i \mapsto f_{i,0}, \quad q^h \mapsto q^h \qquad (i \in I, h \in P^*). \tag{1.2}$$

(See [33, §1.1] for the definition of $\mathbf{U}_\varepsilon(\mathfrak{g})$.)

1.2. *Finite dimensional representation of* $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$. Let V be a $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module. For $\lambda \in P$, we define

$$V_\lambda \stackrel{\text{def.}}{=} \left\{ v \in V \,\middle|\, q^h v = \varepsilon^{\langle h,\lambda\rangle} v, \begin{bmatrix} q^{h_i}; 0 \\ r \end{bmatrix} v = \begin{bmatrix} \langle h_i, \lambda\rangle \\ r \end{bmatrix}_\varepsilon v \right\}.$$

The module V is said to be of *type* 1 if $V = \bigoplus_\lambda V_\lambda$. In what follows we consider only modules of type 1.

By (1.2) any $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module V can be considered as a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module. This is denoted by $\operatorname{Res} V$. The above definition is based on the definition of type 1 representation of $\mathbf{U}_\varepsilon(\mathfrak{g})$, i.e., V is of type 1 if and only if $\operatorname{Res} V$ is of type 1.

A $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module V is said to be an *l-highest weight module* if there exists a vector v such that $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^+ \cdot v = 0$, $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0 \cdot v \subset \mathbb{C}v$ and $V = \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) \cdot v$. Such v is called an *l-highest weight vector*.

THEOREM 1.3 ([5]). *A simple l-highest weight module V with an l-highest weight vector v is finite dimensional if and only if there exists an I-tuple of polynomials $P = (P_i(u))_{i\in I}$ with $P_i(0) = 1$ such that*

$$q^h v = \varepsilon^{\langle h, \sum_i \deg P_i \Lambda_i\rangle} v, \qquad \begin{bmatrix} q^{h_i}; 0 \\ r \end{bmatrix} v = \begin{bmatrix} \deg P_i \\ r \end{bmatrix}_\varepsilon v,$$
$$p_i^+(z) v = P_i(1/z) v, \quad p_i^-(z) v = c_{P_i}^{-1} z^{\deg P_i} P_i(1/z) v,$$

where c_{P_i} is the top term of P_i, i.e., the coefficient of $u^{\deg P_i}$ in P_i.

The I-tuple of polynomials P is called the *l-highest weight*, or the Drinfeld polynomial of V. We denote the above module V by $L(P)$ since it is determined by P.

For $i \in I$ and $a \in \mathbb{C}^*$, the simple module $L(P)$ with

$$P_i(u) = 1 - au, \qquad P_j(u) = 1 \quad \text{if } j \neq i,$$

is called an *l-fundamental representation* and denoted by $L(\Lambda_i)_a$.

Let V be a finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module with the weight space decomposition $V = \bigoplus V_\lambda$. Since the commutative subalgebra $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0$ preserves each V_λ, we can further decompose V into a sum of generalized simultaneous eigenspaces of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0$.

THEOREM 1.4 ([15, Prop. 1], [13, Lemma 3.1], [33, 13.4.5]). *Simultaneous eigenvalues of* $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0$ *have the following forms*:

$$\varepsilon^{\langle h, \deg Q_i^1 - \deg Q_i^2\rangle} \quad \text{for } q^h, \qquad \begin{bmatrix} \deg Q_i^1 - \deg Q_i^2 \\ r \end{bmatrix}_\varepsilon \quad \text{for } \begin{bmatrix} q^{h_i}; 0 \\ r \end{bmatrix},$$
$$\frac{Q_i^1(1/z)}{Q_i^2(1/z)} \quad \text{for } p_i^+(z), \qquad \frac{c_{Q_i^1}^{-1} z^{\deg Q_i^1} Q_i^1(1/z)}{c_{Q_i^2}^{-1} z^{\deg Q_i^2} Q_i^2(1/z)} \quad \text{for } p_i^-(z),$$

where Q_i^1, Q_i^2 *are polynomials with* $Q_i^1(0) = Q_i^2(0) = 1$ *and* $c_{Q_i^1}$, $c_{Q_i^2}$ *are as above.*

We simply write the I-tuple of rational functions $(Q_i^1(u)/Q_i^2(u))$ by Q. A generalized simultaneous eigenspace is called an *l-weight space*. The corresponding I-tuple of rational functions is called an *l-weight*. We denote the l-weight space by V_Q.

The q-character, or ε-character [15], [13] of a finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module V is defined by

$$\chi_\varepsilon(V) = \sum_Q \dim V_Q \, e^Q.$$

The precise definition of e^Q will be explained in the next section.

1.3. *Standard modules.* We will use another family of finite dimensional l-highest weight modules, called standard modules.

Let $\mathbf{w} \in P^+$ be a dominant weight. Let $w_i = \langle h_i, \mathbf{w}\rangle \in \mathbb{Z}_{\ge 0}$. Let $G_\mathbf{w} = \prod_{i\in I} \mathrm{GL}(w_i, \mathbb{C})$. Its representation ring $R(G_\mathbf{w})$ is the invariant part of the Laurent polynomial ring:

$$R(G_\mathbf{w}) = \mathbb{Z}[x_{1,1}^\pm, \dots, x_{1,w_1}^\pm]^{\mathfrak{S}_{w_1}} \otimes \mathbb{Z}[x_{2,1}^\pm, \dots, x_{2,w_2}^\pm]^{\mathfrak{S}_{w_2}} \otimes \cdots \otimes \mathbb{Z}[x_{n,1}^\pm, \dots, x_{n,w_n}^\pm]^{\mathfrak{S}_{w_n}},$$

where we put a numbering $1, \dots, n$ to I. In [33], we constructed a $\mathbf{U}_q^\mathbb{Z}(\mathbf{L}\mathfrak{g}) \otimes_\mathbb{Z} R(G_\mathbf{w})$-module $M(\mathbf{w})$ such that it is free of finite rank over $R(G_\mathbf{w}) \otimes \mathbb{Z}[q, q^{-1}]$ and has a vector $[0]_\mathbf{w}$ satisfying

$$\begin{aligned} e_{i,r}[0]_\mathbf{w} &= 0 \quad \text{for any } i \in I,\ r \in \mathbb{Z}, \\ M(\mathbf{w}) &= \left(\mathbf{U}_q^\mathbb{Z}(\mathbf{L}\mathfrak{g})^- \otimes_\mathbb{Z} R(G_\mathbf{w})\right)[0]_\mathbf{w}, \\ q^h[0]_\mathbf{w} &= q^{\langle h, \mathbf{w}\rangle}[0]_\mathbf{w}, \\ p_i^+(z)[0]_\mathbf{w} &= \prod_{p=1}^{w_i}\left(1 - \frac{x_{i,p}}{z}\right)[0]_\mathbf{w}, \\ p_i^-(z)[0]_\mathbf{w} &= \prod_{p=1}^{w_i}\left(1 - \frac{z}{x_{i,p}}\right)[0]_\mathbf{w}. \end{aligned}$$

If an I-tuple of monic polynomials $P(u) = (P_i(u))_{i\in I}$ with $\deg P_i = w_i$ is given, then we define a *standard module* by the specialization

$$M(P) = M(\mathbf{w}) \otimes_{R(G_{\mathbf{w}})[q,q^{-1}]} \mathbb{C},$$

where the algebra homomorphism $R(G_{\mathbf{w}})[q,q^{-1}] \to \mathbb{C}$ sends q to ε and $x_{i,1}, \dots, x_{i,w_k}$ to roots of P_i. The simple module $L(P)$ is the simple quotient of $M(P)$.

The original definition of the universal standard module [33] is geometric. However, it is not difficult to give an algebraic characterization. Let $M(\Lambda_i)$ be the universal standard module for the dominant weight Λ_i. It is a $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})[x,x^{-1}]$-module. Let $W(\Lambda_i) = M(\Lambda_i)/(x-1)M(\Lambda_i)$. Then we have:

THEOREM 1.5 ([35, 1.22]). *Put a numbering $1,\dots,n$ on I. Let $w_i = \langle h_i, \mathbf{w}\rangle$. The universal standard module $M(\mathbf{w})$ is the $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}} R(G_\lambda)$-submodule of*

$$W(\Lambda_1)^{\otimes w_1} \otimes \cdots \otimes W(\Lambda_n)^{\otimes w_n} \otimes \mathbb{Z}[q,q^{-1}, x_{1,1}^{\pm}, \dots, x_{1,w_1}^{\pm}, \cdots, x_{n,1}^{\pm}, \dots, x_{n,w_n}^{\pm}]$$

(the tensor product is over $\mathbb{Z}[q,q^{-1}]$) generated by $\bigotimes_{i\in I}[0]_{\Lambda_i}^{\otimes\lambda_i}$. (The result holds for the tensor product of any order.)

It is not difficult to show that $W(\Lambda_i)$ is isomorphic to a module studied by Kashiwara [21] ($V(\lambda)$ in his notation). Since his construction is algebraic, the standard module $M(\mathbf{w})$ has an algebraic construction.

We also prove that $M(P^1P^2)$ is equal to $M(P^1) \otimes M(P^2)$ in the representation ring $\operatorname{Rep} \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ later. (See Corollary 3.7.) Here the I-tuple of polynomials $(P_iQ_i)_i$ for $P = (P_i)_i$, $Q = (Q_i)_i$ is denoted by PQ for brevity.

2. A modified multiplication on $\widehat{\mathcal{Y}}_t$

We use the following polynomial rings in this article:

$$\begin{aligned}
\widehat{\mathcal{Y}}_t &\overset{\text{def.}}{=} \mathbb{Z}[t,t^{-1}, V_{i,a}, W_{i,a}]_{i\in I, a\in\mathbb{C}^*},\\
\mathcal{Y}_t &\overset{\text{def.}}{=} \mathbb{Z}[t,t^{-1}, Y_{i,a}, Y_{i,a}^{-1}]_{i\in I, a\in\mathbb{C}^*},\\
\mathcal{Y} &\overset{\text{def.}}{=} \mathbb{Z}[Y_{i,a}, Y_{i,a}^{-1}]_{i\in I, a\in\mathbb{C}^*},\\
\overline{\mathcal{Y}} &\overset{\text{def.}}{=} Z[y_i, y_i^{-1}]_{i\in I}.
\end{aligned}$$

We consider $\widehat{\mathcal{Y}}_t$ as a polynomial ring in infinitely many variables $V_{i,a}$, $W_{i,a}$ with coefficients in $\mathbb{Z}[t,t^{-1}]$. So a *monomial* means a monomial only in $V_{i,a}$, $W_{i,a}$, containing no t, t^{-1}. The same convention applies also to $\mathcal{Y}_t$.

For a monomial $m \in \widehat{\mathcal{Y}}_t$, let $w_{i,a}(m)$, $v_{i,a}(m) \in \mathbb{Z}_{\geq 0}$ be the degrees in $V_{i,a}$, $W_{i,a}$; i.e.,

$$m = \prod_{i,a} V_{i,a}^{v_{i,a}(m)} W_{i,a}^{w_{i,a}(m)}.$$

We also define

$$u_{i,a}(m) \overset{\text{def.}}{=} w_{i,a}(m) - v_{i,a\varepsilon^{-1}}(m) - v_{i,a\varepsilon}(m) + \sum_{j:C_{ji}=-1} v_{j,a}(m).$$

When ε is not a root of unity, we define $(\tilde{u}_{i,a}(m))_{i\in I, a\in\mathbb{C}^*}$ for a monomial m in $\widehat{\mathcal{Y}}_t$, as the solution of

$$u_{i,a}(m) = \tilde{u}_{i,a\varepsilon^{-1}}(m) + \tilde{u}_{i,a\varepsilon}(m) - \sum_{j:a_{ij}=-1} \tilde{u}_{j,a}(m).$$

To solve the system, we may assume that $u_{i,a}(m) = 0$ unless a is a power of q. Then the above is a recursive system, since q is not a root of unity. Thus, it has a unique solution such that $\tilde{u}_{i,q^s}(m) = 0$ for sufficiently small s. Note that $\tilde{u}_{i,a}(m)$ is nonzero for possibly infinitely many a's, although $u_{i,a}(m)$ is not.

If m^1, m^2 are monomials, we set

$$\begin{aligned}(2.1)\qquad d(m^1, m^2) &\overset{\text{def.}}{=} \sum_{i,a} \left(v_{i,a\varepsilon}(m^1) u_{i,a}(m^2) + w_{i,a\varepsilon}(m^1) v_{i,a}(m^2) \right) \\ &= \sum_{i,a} \left(u_{i,a}(m^1) v_{i,a\varepsilon^{-1}}(m^2) + v_{i,a}(m^1) w_{i,a\varepsilon^{-1}}(m^2) \right).\end{aligned}$$

From the definition, $d(\ ,\ )$ satisfies

$$\begin{aligned}(2.2)\qquad d(m^1 m^2, m^3) &= d(m^1, m^3) + d(m^2, m^3), \\ d(m^1, m^2 m^3) &= d(m^1, m^2) + d(m^1, m^3).\end{aligned}$$

When ε is not a root of unity, we also define

$$\tilde{d}(m^1, m^2) \overset{\text{def.}}{=} -\sum_{i,a} u_{i,a}(m^1) \tilde{u}_{i,a\varepsilon^{-1}}(m^2).$$

Since $u_{i,a}(m^2) = 0$ except for finitely many a's, this is well-defined. Moreover, we have

$$\tilde{d}(m^1, m^2) = d(m^1, m^2) + \tilde{d}_W(m^1, m^2),$$

where $\tilde{d}_W$ is defined as $\tilde{d}$ by replacing $u_{i,a}$ by $w_{i,a}$. Here we have used $\tilde{u}_{i,a}(m) = \tilde{w}_{i,a}(m) - v_{i,a}(m)$.

We define a ring involution $\overline{}$ on $\widehat{\mathcal{Y}}_t$ by

$$(2.3)\qquad \overline{t} = t^{-1}, \quad \overline{m} = t^{2d(m,m)} m,$$

where m is a monomial in $V_{i,a}$, $W_{i,a}$. We define a ring involution $\overline{}$ on $\mathcal{Y}_t$ by $\overline{t} = t^{-1}$, $\overline{Y_{i,a}^{\pm}} = Y_{i,a}^{\pm}$.

We define a new multiplication $*$ on $\widehat{\mathcal{Y}}_t$ by

$$m^1 * m^2 \overset{\text{def.}}{=} t^{2d(m^1,m^2)} m^1 m^2,$$

where m^1, m^2 are monomials and m^1m^2 is the usual multiplication of m^1 and m^2. By (2.2) it is associative. (**NB**: The multiplication in [34] was $m^1 * m^2 \stackrel{\text{def.}}{=} t^{2d(m^2,m^1)}m^1m^2$. This is because the coproduct is changed.)

From the definition we have

$$\overline{m^1 * m^2} = \overline{m^2} * \overline{m^1}. \tag{2.4}$$

Let us give an example which will be important later. Suppose that m is a monomial with $u_{i,a}(m) = 1$, $u_{i,b}(m) = 0$ for $b \neq a$ for some i. Then

$$\begin{aligned}[m(1 + V_{i,a\varepsilon})]^{*n} &\stackrel{\text{def.}}{=} \underbrace{m(1 + V_{i,a\varepsilon}) * \cdots * m(1 + V_{i,a\varepsilon})}_{n \text{ times}} \\ &= m^n \sum_{r=0}^{n} t^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_t V_{i,a\varepsilon}^r.\end{aligned} \tag{2.5}$$

When ε is not a root of unity, there is another multiplication $\tilde{*}$ defined by

$$m^1 \tilde{*}\, m^2 \stackrel{\text{def.}}{=} t^{\tilde{d}(m^1,m^2) - \tilde{d}(m^2,m^1)} m^1m^2.$$

We define a $\mathbb{Z}[t, t^{-1}]$-linear homomorphism $\widehat{\Pi}\colon \widehat{\mathcal{Y}}_t \to \mathcal{Y}_t$ by

$$m = \prod_{i,a} V_{i,a}^{v_{i,a}(m)} W_{i,a}^{w_{i,a}(m)} \longmapsto t^{-d(m,m)} \prod_{i,a} Y_{i,a}^{u_{i,a}(m)}. \tag{2.6}$$

This is not a ring homomorphism with respect to either the ordinary multiplication or $*$. However, when ε is not a root of unity, we can define a new multiplication on $\mathcal{Y}_t$ so that the above is a ring homomorphism with respect to this multiplication and $\tilde{*}$. It is because $\varepsilon(m^1, m^2)$ involves only $u_{i,a}(m^1)$, $u_{i,a}(m^2)$. We denote also by $\tilde{*}$ the new multiplication on $\mathcal{Y}_t$. We have

$$\begin{aligned}&\widehat{\Pi}(m^1 * m^2) = t^{\tilde{d}_W(m^1,m^2) - \tilde{d}_W(m^2,m^1)} \widehat{\Pi}(m^1) \tilde{*} \widehat{\Pi}(m^2), \\ &\widehat{\Pi} \circ {}^{-} = {}^{-} \circ \widehat{\Pi}.\end{aligned} \tag{2.7}$$

Further we define homomorphisms $\Pi_t\colon \mathcal{Y}_t \to \mathcal{Y}$, $\overline{\Pi}\colon \mathcal{Y} \to \mathbb{Z}[y_i^{\pm}]$ by

$$\Pi_t\colon \mathcal{Y}_t \ni \begin{array}{c} t \longmapsto 1 \\ Y_{i,a} \longmapsto Y_{i,a} \end{array} \in \mathcal{Y}, \qquad \overline{\Pi}\colon \mathcal{Y} \ni Y_{i,a} \longmapsto y_i \in \mathbb{Z}[y_i, y_i^{-1}]_{i\in I}.$$

The composition $\widehat{\mathcal{Y}}_t \to \mathcal{Y}$ or $\widehat{\mathcal{Y}}_t \to \mathbb{Z}[y_i^{\pm}]$ is a ring homomorphism with respect to both the usual multiplication and $*$.

Definition 2.8. A monomial $m \in \widehat{\mathcal{Y}}_t$ is said to be *i-dominant* if $u_{i,a}(m) \geq 0$ for any $i \in I$. A monomial $m \in \widehat{\mathcal{Y}}_t$ is said to be *l-dominant* if it is i-dominant for all $i \in I$, i.e., $\widehat{\Pi}(m)$ contains only nonnegative powers of $Y_{i,a}$. Similarly a monomial $m \in \mathcal{Y}$ is called *l-dominant* if it contains only nonnegative powers of $Y_{i,a}$. Note that a monomial $m \in \mathbb{Z}[y_i, y_i^{-1}]_{i\in I}$ contains only nonnegative powers of y_i if and only if it is dominant as a weight of $\mathfrak{g}$.

Let

$$m = \prod_{i,a} Y_{i,a}^{u_{i,a}}$$

be a monomial in $\mathcal{Y}$ with $u_{i,a} \in \mathbb{Z}$. We associate to m an I-tuple of rational functions $Q = (Q_i)$ by

$$Q_i(u) = \prod_a (1 - au)^{u_{i,a}}.$$

Conversely an I-tuple of rational functions $Q = (Q_i)$ with $Q_i(0) = 1$ determines a monomial in $\mathcal{Y}$. We denote it by e^Q. This is the e^Q mentioned in the previous section. Note that e^Q is l-dominant if and only if Q is an I-tuple of polynomials.

We also use a similar identification between an I-tuple of polynomials $P = (P_i)$ with $P_i(0) = 1$ and a monomial m in $W_{i,a}$ ($i \in I$, $a \in \mathbb{C}^*$):

$$m = \prod_{i,a} W_{i,a}^{w_{i,a}} \longleftrightarrow P = (P_i); \quad P_i(u) = \prod_a (1 - au)^{w_{i,a}}.$$

We denote m also by e^P, hoping that it makes no confusion.

Definition 2.9. Let m, m' be monomials in $\widehat{\mathcal{Y}}_t$. We say that $m \le m'$ if m/m' is a monomial in $V_{i,a}$ ($i \in I$, $a \in \mathbb{C}^*$). We say $m < m'$ if $m \le m'$ and $m \neq m'$. It defines a partial order among monomials in $\widehat{\mathcal{Y}}_t$. Similarly for monomials m, m' in $\mathcal{Y}$, we say $m \le m'$ if m/m' is a monomial in $\widehat{\Pi}(V_{i,a})$ ($i \in I$, $a \in \mathbb{C}^*$). For two I-tuples of rational functions Q, Q', we say $Q \le Q'$ if $e^Q \le e^{Q'}$. Finally for monomials m, m' in $\mathbb{Z}[y_i, y_i^{-1}]_{i \in I}$, we say $m \le m'$ if m/m' is a monomial in $\overline{\Pi} \circ \Pi_t \circ \widehat{\Pi}(V_{i,a})$ ($i \in I$, $a \in \mathbb{C}^*$). But this is nothing but the usual order on weights.

3. A t-analog of the q-character: Axioms

A main tool in this article is a t-analog of the q-character:

$$\widehat{\chi}_{\varepsilon,t} \colon \mathbf{R}_t = \operatorname{Rep} \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}] \to \widehat{\mathcal{Y}}_t.$$

For the definition we need geometric constructions of standard modules, so we will postpone it to Section 4. In this section, we explain properties of $\widehat{\chi}_{\varepsilon,t}$ as axioms. Then we show that these axioms uniquely characterize $\widehat{\chi}_{\varepsilon,t}$, and in fact, give us an algorithm for "computation". Thus we may consider the axioms as the definition of $\widehat{\chi}_{\varepsilon,t}$.

Our first axiom is the highest weight property:

Axiom 1. *The value of $\widehat{\chi}_{\varepsilon,t}$ at a standard module $M(P)$ has a form*

$$\widehat{\chi}_{\varepsilon,t}(M(P)) = e^P + \sum a_m(t) m,$$

where each monomial m satisfies $m < e^P$.

Composing maps $\widehat{\mathcal{Y}}_t \to \mathcal{Y}_t$, $\mathcal{Y}_t \to \mathcal{Y}$, $\mathcal{Y} \to \mathbb{Z}[y_i^\pm]$ in Section 2, we define maps

$$\chi_{\varepsilon,t} = \widehat{\Pi} \circ \widehat{\chi}_{\varepsilon,t} \colon \mathbf{R}_t \to \mathcal{Y}_t,$$
$$\chi_\varepsilon = \Pi_t \circ \chi_{\varepsilon,t} \colon \operatorname{Rep} \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) \to \mathcal{Y}, \quad \chi = \overline{\Pi} \circ \chi_\varepsilon \colon \operatorname{Rep} \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) \to \mathbb{Z}[y_i, y_i^{-1}]_{i \in I}.$$

$\widehat{\chi}_{\varepsilon,t}$ is a homomorphism of $\mathbb{Z}[t,t^{-1}]$-modules, not of rings.

Frenkel-Mukhin [12, 5.1, 5.2] proved that the image of χ_ε is equal to

$$\bigcap_{i \in I} \left(\mathbb{Z}[Y_{j,a}^\pm]_{j : j \neq i, a \in \mathbb{C}^*} \otimes \mathbb{Z}[Y_{i,b}(1 + V_{i,b\varepsilon})]_{b \in \mathbb{C}^*} \right).$$

We define its t-analog, replacing $(1 + V_{i,b\varepsilon})^n$ by

$$\sum_{r=0}^{n} t^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_t V_{i,b\varepsilon}^r.$$

More precisely, for each $i \in I$, let $\widehat{\mathcal{K}}_{t,i}$ be the $\mathbb{Z}[t,t^{-1}]$-linear subspace of $\widehat{\mathcal{Y}}_t$ generated by elements

$$E_i(m) \overset{\text{def.}}{=} m \prod_a \sum_{r_a=0}^{u_{i,a}(m)} t^{r_a(u_{i,a}(m) - r_a)} \begin{bmatrix} u_{i,a}(m) \\ r_a \end{bmatrix}_t V_{i,a\varepsilon}^{r_a}, \tag{3.1}$$

where m is an i-dominant monomial, i.e., $u_{i,a}(m) \geq 0$ for all $a \in \mathbb{C}^*$. Let

$$\widehat{\mathcal{K}}_t \overset{\text{def.}}{=} \bigcap_i \widehat{\mathcal{K}}_{t,i}, \qquad \mathcal{K}_t \overset{\text{def.}}{=} \widehat{\Pi}(\widehat{\mathcal{K}}_t) \subset \mathcal{Y}_t.$$

Axiom 2. *The image of $\widehat{\chi}_{\varepsilon,t}$ is contained in $\widehat{\mathcal{K}}_t$.*

The next axiom is about the multiplicative property of $\widehat{\chi}_{\varepsilon,t}$. As explained in the introduction, it is not multiplicative under the usual product structure on $\mathbf{R}_t$.

Axiom 3. *Suppose that two I-tuples of polynomials $P^1 = (P_i^1)$, $P^2 = (P_i^2)$ with $P_i^1(0) = P_i^2(0) = 1$ satisfy the following conditions*:

(3.2) $a/b \notin \{\varepsilon^n \mid n \in \mathbb{Z}, n \geq 2\}$ *for any pair a, b with* $P_i^1(1/a) = 0$, $P_j^2(1/b) = 0$ $(i, j \in I)$.

Then

$$\widehat{\chi}_{\varepsilon,t}(M(P^1 P^2)) = \widehat{\chi}_{\varepsilon,t}(M(P^1)) * \widehat{\chi}_{\varepsilon,t}(M(P^2)).$$

We have the following special case

$$\widehat{\chi}_{\varepsilon,t}(M(P^1 P^2)) = \widehat{\chi}_{\varepsilon,t}(M(P^1)) \widehat{\chi}_{\varepsilon,t}(M(P^2))$$

under the stronger condition $a/b \notin \varepsilon^{\mathbb{Z}}$ by the definition of $*$.

The last axiom is about specialization at a root of unity. Suppose that ε is a primitive s-th root of unity. We choose and fix q, which is not a root of unity. The axiom will say that $\widehat{\chi}_{\varepsilon,t}(M(P))$ can be written in terms of $\widehat{\chi}_{q,t}(M(P_q))$ for some P_q.

By Axiom 3, more precisely, the sentence following Axiom 3, we may assume that inverses of roots of $P_i(u) = 0$ ($i \in I$) are contained in $a\varepsilon^{\mathbb{Z}}$ for some $a \in \mathbb{C}^*$. Therefore

$$P_i(u) = \prod_{n=0}^{s-1} (1 - a\varepsilon^n u)^{N_{i,n}},$$

with $N_{i,n} \in \mathbb{Z}_{\geq 0}$. We define $P_q = ((P_q)_i)$ by

$$(P_q)_i(u) = \prod_{n=0}^{s-1} (1 - aq^n u)^{N_{i,n}},$$

and set $N_{i,n} = 0$ if $n \notin \{0, \dots, s-1\}$.

Let

$$\widehat{\chi}_{q,t}(M(P_q)) = \sum a_m(t) m.$$

By previous axioms, each m is written as

$$m = e^{P_q} \prod_{i \in I, n \in \mathbb{Z}} V_{i,aq^n}^{M_{i,n}} = \prod_{i \in I, n \in \mathbb{Z}} W_{i,aq^n}^{N_{i,n}} V_{i,aq^n}^{M_{i,n}} \tag{3.3}$$

with $M_{i,n} \in \mathbb{Z}_{\geq 0}$. By previous axioms $M_{i,n}$ is independent of q (cf. Theorem 3.5(4)). We define monomials $m|_{q=\varepsilon}$, $m[k]$ by

$$\begin{aligned} m|_{q=\varepsilon} &\overset{\text{def.}}{=} \prod_{i \in I, n \in \mathbb{Z}} W_{i,a\varepsilon^n}^{N_{i,n}} V_{i,a\varepsilon^n}^{M_{i,n}}, \\ m[k] &\overset{\text{def.}}{=} \prod_{i \in I, n \in \mathbb{Z}} W_{i,aq^n}^{N_{i,n+k}} V_{i,aq^n}^{M_{i,n+k}}. \end{aligned} \tag{3.4}$$

Note that $m|_{q=\varepsilon} = m[k]|_{q=\varepsilon}$ if $k \equiv 0 \mod s$. Now,

$$D^-(m) \overset{\text{def.}}{=} \sum_{k<0} d_q(m, m[ks]),$$

where we define d_q as d in (2.1) replacing ε by q.

Axiom 4.

$$\widehat{\chi}_{\varepsilon,t}(M(P)) = \sum t^{2D^-(m)} a_m(t)\, m|_{q=\varepsilon}.$$

We can consider similar axioms for $\chi_\varepsilon = \Pi_t \circ \widehat{\Pi} \circ \widehat{\chi}_{\varepsilon,t}$. Axioms 3 and 4 are simplified when $t = 1$. Axiom 3 is $\chi_\varepsilon(M(P^1P^2)) = \chi_\varepsilon(M(P^1))\chi_\varepsilon(M(P^2))$. Axiom 4 says $\chi_\varepsilon(M(P)) = \chi_q(M(P))|_{q=\varepsilon}$. The original χ_ε defined in [15], [13] satisfies those axioms: Axioms 1 and 2 were proved in [12, Th. 4.1, Th. 5.1]. Axiom 3 was proved in [15, Lemma 3]. Axiom 4 was proved in [13, Th. 3.2].

Let us give few consequences of the axioms.

THEOREM 3.5. (1) *The map* $\chi_{\varepsilon,t}$ (*and hence also* $\widehat{\chi}_{\varepsilon,t}$) *is injective. The image of* $\chi_{\varepsilon,t}$ *is equal to* $\mathcal{K}_t$.

(2) *Suppose that a* $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-*module* M *has the following property*: $\widehat{\chi}_{\varepsilon,t}(M)$ *contains only one l-dominant monomial* m_0. *Then* $\widehat{\chi}_{\varepsilon,t}(M)$ *is uniquely determined from* m_0 *and the condition* $\widehat{\chi}_{\varepsilon,t}(M) \in \widehat{\mathcal{K}}_t$.

(3) *Let* m *be an l-dominant monomial in* $\mathcal{Y}_t$, *considered as an element of the dual of* $\mathbf{R}_t$ *by taking the coefficient of* $\chi_{\varepsilon,t}$ *at* m. *Then* $\{m \mid m \text{ is } l\text{-dominant}\}$ *is a base of the dual of* $\mathbf{R}_t$.

(4) *The* $\widehat{\chi}_{\varepsilon,t}$ *is unique*, *if it exists.*

(5) $\widehat{\chi}_{\varepsilon,t}(\tau_a^*(V))$ *is obtained from* $\widehat{\chi}_{\varepsilon,t}(V)$ *by replacing* $W_{i,b}$, $V_{i,b}$ *by* $W_{i,ab}$, $V_{i,ab}$.

(6) *The coefficient of a monomial* m *in* $\widehat{\chi}_{\varepsilon,t}(M(P))$ *is a polynomial in* t^2. (*In fact*, *it will become clear that it is a polynomial in* t^2 *with nonnegative coefficients.*)

Proof. These are essentially proved in [15], [12]. So our proof is sketchy.

(1) Since $\chi_{\varepsilon,t}(M(P))$ equals $\widehat{\Pi}(e^P)$ plus the sum of lower monomials, the first assertion follows by induction on $<$. The second assertion follows from the argument in [12, 5.6], where we use the standard module $M(P)$ instead of simple modules.

(2) Let m be a monomial appearing in $\widehat{\chi}_{\varepsilon,t}(M)$, which is not m_0. It is not l-dominant by the assumption. By Axiom 2, m appears in $E_i(m')$ for some monomial m' appearing in $\widehat{\chi}_{\varepsilon,t}(M)$. In particular, we have $m < m'$. Repeating the argument for m', we have $m < m_0$.

The coefficient of m in $\widehat{\chi}_{\varepsilon,t}(M)$ is equal to the sum of coefficients of m in $E_i(m')$ for all possible m''s. (i is fixed.) Again by induction on $<$, we can determine the coefficient inductively.

(3) By Axiom 1, the transition matrix between $\{M(P)\}$ and the dual base of $\{m\}$ above is upper-triangular with diagonal entries 1.

(4) By Axiom 4, we may assume that ε is not a root of unity. Consider the case $P_i(u) = 1 - au$, $P_j(u) = 1$ for $j \neq i$ for some i. By [12, Cor. 4.5], Axiom 1 implies that the $\widehat{\chi}_{\varepsilon,t}(M(P))$ for P does not contains l-dominant terms other than e^P. (See Proposition 4.13 below for a geometric proof.) In particular, $\widehat{\chi}_{\varepsilon,t}(M(P))$ is uniquely determined by (2) above in this case. We use Axiom 3 to "calculate" $\widehat{\chi}_{\varepsilon,t}(M(P))$ for arbitrary P as follows. We order inverses of roots (counted with multiplicities) of $P_i(u) = 0$ ($i \in I$) as a_1, a_2, … , so that $a_p/a_q \neq \varepsilon^n$ for $n \geq 2$ if $p < q$. This is possible since ε is not a root of unity.

For each a_p, we define a Drinfeld polynomial Q^p by

$$Q^p_{i_p}(u) = (1 - a_p u), \qquad Q^p_j(u) = 1 \quad (j \neq i_p),$$

if $1/a_p$ is a root of $P_{i_p}(u) = 0$. Therefore we have $P_i = \prod_p Q^p_i$. By our choice,

$$\widehat{\chi}_{\varepsilon,t}(M(P)) = \widehat{\chi}_{\varepsilon,t}(M(Q^1)) * \widehat{\chi}_{\varepsilon,t}(M(Q^2)) * \cdots$$

by Axiom 3. Each $\widehat{\chi}_{\varepsilon,t}(M(Q^p))$ is uniquely determined by the above discussion. Therefore $\widehat{\chi}_{\varepsilon,t}(M(P))$ is also uniquely determined.

(5) It is enough to check the case $V = M(P)$. In this case, $\tau_a^*(M(P))$ is the standard module with Drinfeld polynomial $P(au)$. The assertion follows from the axioms.

(6) This also follows from the axioms. By Axiom 4, we may assume ε is a root of unity. By Axiom 3, we may assume $M(P)$ is an l-fundamental representation. In this case, the assertion follows from Axiom 2, since $t^{r(n-r)} \left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]_t$ is a polynomial in t^2. □

In [12, §5.5], Frenkel-Mukhin gave an explict combinatorial algorithm to "compute" $\widehat{\chi}_{\varepsilon,t}(M)$ for M as in (2). We will give a geometric interpretation of their algorithm in Section 5.

By the uniqueness, we get:

Corollary 3.6. *The χ_ε coincides with the ε-character defined in* [15], [13].

By [15, Th. 3], χ is the ordinary character of the restriction of a $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module to a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module.

As promised, we prove:

Corollary 3.7. *In the representation ring* $\operatorname{Rep} \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$,

$$M(P^1 P^2) = M(P^1) \otimes M(P^2)$$

for any I-tuples of polynomials P^1, P^2.

Proof. Since χ_ε is injective, it is enough to show that $\chi_\varepsilon(M(P^1P^2)) = \chi_\varepsilon(M(P^1))\chi_\varepsilon(M(P^2))$.

In fact, it is easy to prove this equality directly from the geometric definition in (4.12). However, we prove it only from the axioms.

By Axiom 4, we may assume ε is not a root of unity. We order inverses of roots (counted with multiplicities) of $P^1_i P^2_i(u) = 0$ $(i \in I)$ as in the proof of Theorem 3.5(4). Then we have

$$\chi_\varepsilon(M(P^1P^2)) = \prod_p \chi_\varepsilon(M(Q^p))$$

by Axiom 3. The product can be taken in any order, since $\operatorname{Rep} \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ is commutative. Each a_p is either the inverse of a root of $P^1_i(u) = 0$ or $P^2_i(u) = 0$.

We divide a_p's into two sets accordingly. Then the products of $\chi_\varepsilon(M(Q^a))$ over groups are equal to $\chi_\varepsilon(M(P^1))$ and $\chi_\varepsilon(M(P^2))$ again by Axiom 3. Therefore we get the assertion. □

We also give another consequence of the axioms.

THEOREM 3.8. *The* $\widehat{\mathcal{K}}_t$ *is invariant under the multiplication* $*$ *and the involution* $\overline{}$ *on* $\widehat{\mathcal{Y}}_t$. *Moreover*, $\mathbf{R}_t$ *has an involution induced from one on* $\widehat{\mathcal{Y}}_t$. *When* ε *is not a root of unity, it also has a multiplication induced from that on* $\mathcal{Y}_t$.

The following proof is elementary, but less conceputal. We will give another geometric proof in Section 6.

Remark 3.9. The multiplication on $\mathbf{R}_t$ in an earlier version was not associative, although it works for the computation of tensor product decompositions of two simple modules. A modification of the multiplication here was inspired by a paper of Varagnolo-Vasserot [40].

Proof. For simplicity, we assume that ε is not a root of unity. The proof for the case when ε is a root of unity can be given by a straightforward modification.

Let us show $f * g \in \widehat{\mathcal{K}}_t$ for f, $g \in \widehat{\mathcal{K}}_t$. By induction and (2.5) we may assume that f is of the form

$$m' \left(1 + V_{i,b\varepsilon}\right),$$

where m' is a monomial with $u_{i,b}(m') = 1$, $u_{i,c}(m') = 0$ for $c \neq b$, and that $g = E_i(m)$ is as in (3.1). By a direct calculation, we get

$$\begin{aligned} & t^{-2d(m',m)} f * g - E_i(mm') \\ &= \left(t^{2n} - 1\right) mm' \prod_{a \neq b\varepsilon^{-2}} \sum_{r_a=0}^{u_{i,a}(m)} t^{r_a(u_{i,a}(m)-r_a)} \begin{bmatrix} u_{i,a}(m) \\ r_a \end{bmatrix}_t V_{i,a\varepsilon}^{r_a} \sum_{s=0}^{n-1} t^{s(n-s)} \begin{bmatrix} n-1 \\ s \end{bmatrix}_t V_{i,b\varepsilon^{-1}}^{s+1} \end{aligned}$$

where $n = u_{i,b\varepsilon^{-2}}(m)$. If $n = 0$, then the right-hand side is zero, so the assertion is obvious. If $n \neq 0$, then

$$u_{i,a}\left(mm' V_{i,b\varepsilon^{-1}}\right) = \begin{cases} u_{i,b\varepsilon^{-2}}(m) - 1 & \text{if } a = b\varepsilon^{-2}, \\ u_{i,a}(m) & \text{otherwise.} \end{cases}$$

Therefore the above expression is equal to $\left(t^{2n} - 1\right) E_i\left(mm' V_{i,b\varepsilon^{-1}}\right)$.

Next we show the closedness of the image under the involution. By (2.4) and the above assertion, we may assume $f = m'\left(1 + V_{i,b\varepsilon}\right)$ as above. We further assume m' does not contain t, t^{-1}. Then we get

$$\overline{f} = t^{2d(m',m')} f.$$

This is contained in $\widehat{\mathcal{K}}_t$.

Now we can define $\tilde{*}$ and $\overline{}$ on $\mathbf{R}_t$ so that

$$\begin{aligned} \chi_{\varepsilon,t}(\overline{V}) &= \widehat{\Pi}\left(\overline{\widehat{\chi}_{\varepsilon,t}(V)}\right) = \overline{\chi_{\varepsilon,t}(V)}, \\ \chi_{\varepsilon,t}(V_1 \tilde{*} V_2) &= \chi_{\varepsilon,t}(V_1) \tilde{*} \chi_{\varepsilon,t}(V_2), \end{aligned}$$

where we have assumed that ε is not a root of unity for the second equality. By the above discussion together with (2.7), the right-hand sides are contained in $\mathcal{K}_t$, and therefore in the image of $\chi_{\varepsilon,t}$ by Theorem 3.5(1). Since $\chi_{\varepsilon,t}$ is injective by Theorem 3.5(1), $\overline{V}$, $V_1 * V_2$ are well-defined. □

Remark 3.10. In this article, the existence of $\widehat{\chi}_{\varepsilon,t}$ satisfying the axioms is provided by a geometric theory of quiver varieties. But the author conjectures that there exists a purely combinatorial proof of the existence, independent of quiver varieties or the representation theory of quantum loop algebras. When $\mathfrak{g}$ is of type A or D, such a combinatorial construction is possible [36]. When $\mathfrak{g}$ is E_6, E_7, an explict construction of $\widehat{\chi}_{\varepsilon,t}$ is possible with the use of a computer.

4. Graded and cyclic quiver varieties

Suppose that a finite graph (I, E) of type ADE is given. The set I is the set of vertices, while E is the set of edges.

Let H be the set of pairs consisting of an edge together with its orientation. For $h \in H$, we denote by $\mathrm{in}(h)$ (resp. $\mathrm{out}(h)$) the incoming (resp. outgoing) vertex of h. For $h \in H$ we denote by $\overline{h}$ the same edge as h with the reverse orientation. We choose and fix a function $\varepsilon\colon H \to \mathbb{C}^*$ such that $\varepsilon(h) + \varepsilon(\overline{h}) = 0$ for all $h \in H$.

Let V, W be $I \times \mathbb{C}^*$-graded vector spaces such that the $(i \times a)$-component, denoted by $V_i(a)$, is finite dimensional and 0 for all but finitely many times $i \times a$. In what follows we consider only $I \times \mathbb{C}^*$-graded vector spaces with this condition. For an integer n, we define vector spaces by

$$\begin{aligned} \mathrm{L}^\bullet(V,W)^{[n]} &\overset{\text{def.}}{=} \bigoplus_{i \in I, a \in \mathbb{C}^*} \mathrm{Hom}\left(V_i(a), W_i(a\varepsilon^n)\right), \\ \mathrm{E}^\bullet(V,W)^{[n]} &\overset{\text{def.}}{=} \bigoplus_{h \in H, a \in \mathbb{C}^*} \mathrm{Hom}\left(V_{\mathrm{out}(h)}(a), W_{\mathrm{in}(h)}(a\varepsilon^n)\right). \end{aligned} \tag{4.1}$$

If V and W are $I \times \mathbb{C}^*$-graded vector spaces as above, we consider the vector spaces

$$\mathbf{M}^\bullet \equiv \mathbf{M}^\bullet(V,W) \overset{\text{def.}}{=} \mathrm{E}^\bullet(V,V)^{[-1]} \oplus \mathrm{L}^\bullet(W,V)^{[-1]} \oplus \mathrm{L}^\bullet(V,W)^{[-1]}, \tag{4.2}$$

where we use the notation $\mathbf{M}^\bullet$ unless we want to specify V, W. The above three components for an element of $\mathbf{M}^\bullet$ is denoted by B, α, β respectively. (**NB**: In [33] α and β were denoted by i, j respectively.) The $\operatorname{Hom}(V_{\mathrm{out}(h)}(a), V_{\mathrm{in}(h)}(a\varepsilon^{-1}))$-component of B is denoted by $B_{h,a}$. Similarly, we denote by $\alpha_{i,a}$, $\beta_{i,a}$ the components of α, β.

We define a map $\mu\colon \mathbf{M}^\bullet \to \mathrm{L}^\bullet(V,V)^{[-2]}$ by

$$\mu_{i,a}(B,\alpha,\beta) = \sum_{\mathrm{in}(h)=i} \varepsilon(h) B_{h,a\varepsilon^{-1}} B_{\overline{h},a} + \alpha_{i,a\varepsilon^{-1}}\beta_{i,a},$$

where $\mu_{i,a}$ is the (i,a)-component of μ.

Let $G_V \overset{\text{def.}}{=} \prod_{i,a} \mathrm{GL}(V_i(a))$. It acts on $\mathbf{M}^\bullet$ by

$$(B,\alpha,\beta) \mapsto g\cdot(B,\alpha,\beta) \overset{\text{def.}}{=} \left(g_{\mathrm{in}(h),a\varepsilon^{-1}} B_{h,a} g^{-1}_{\mathrm{out}(h),a},\ g_{i,a\varepsilon^{-1}}\alpha_{i,a},\ \beta_{i,a} g^{-1}_{i,a}\right).$$

The action preserves the subvariety $\mu^{-1}(0)$ in $\mathbf{M}^\bullet$.

Definition 4.3. A point $(B,\alpha,\beta) \in \mu^{-1}(0)$ is said to be *stable* if the following condition holds:

> If an $I \times \mathbb{C}^*$-graded subspace S of V is B-invariant and contained in $\operatorname{Ker}\beta$, then $S = 0$.

Let us denote by $\mu^{-1}(0)^{\mathrm{s}}$ the set of stable points.

Clearly, the stability condition is invariant under the action of G_V. Hence we may say an orbit is stable or not.

We consider two kinds of quotient spaces of $\mu^{-1}(0)$:

$$\mathfrak{M}_0^\bullet(V,W) \overset{\text{def.}}{=} \mu^{-1}(0)/\!\!/G_V, \qquad \mathfrak{M}^\bullet(V,W) \overset{\text{def.}}{=} \mu^{-1}(0)^{\mathrm{s}}/G_V.$$

Here $/\!\!/$ is the affine algebro-geometric quotient, i.e., the coordinate ring of $\mathfrak{M}_0^\bullet(V,W)$ is the ring of G_V-invariant functions on $\mu^{-1}(0)$. In particular, it is an affine variety. It is the set of closed G_V-orbits. The second one is the set-theoretical quotient, but coincides with a quotient in the geometric invariant theory (see [32, §3]). The action of G_V on $\mu^{-1}(0)^{\mathrm{s}}$ is free thanks to the stability condition ([32, 3.10]). By a general theory, there exists a natural projective morphism

$$\pi\colon \mathfrak{M}^\bullet(V,W) \to \mathfrak{M}_0^\bullet(V,W).$$

(See [32, 3.18].) The inverse image of 0 under π is denoted by $\mathfrak{L}^\bullet(V,W)$. We call these varieties *cyclic quiver varieties* or *graded quiver varieties*, according as ε is a root of unity or not.

Let $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W) \subset \mathfrak{M}_0^\bullet(V,W)$ be a possibly empty open subset of $\mathfrak{M}_0^\bullet(V,W)$ consisting of free G_V-orbits. It is known that π is an isomorphism on $\pi^{-1}(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W))$ [32, 3.24]. In particular, $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$ is nonsingular and is pure dimensional.

A G_V-orbit through (B,α,β), considered as a point of $\mathfrak{M}^\bullet(V,W)$, is denoted by $[B,\alpha,\beta]$.

We associate polynomials e^W, $e^V \in \widehat{\mathcal{Y}}_t$ to graded vector spaces V, W by

$$e^W = \prod_{i\in I, a\in\mathbb{C}^*} W_{i,a}^{\dim W_i(a)}, \quad e^V = \prod_{i\in I, a\in\mathbb{C}^*} V_{i,a}^{\dim V_i(a)}. \tag{4.4}$$

Suppose that we have two $I\times\mathbb{C}^*$-graded vector spaces V, V' such that $V_i(a)\subset V_i'(a)$ for all i, a. Then $\mathfrak{M}_0^\bullet(V,W)$ can be identified with a closed subvariety of $\mathfrak{M}_0^\bullet(V',W)$ by the extension by 0 to the complementary subspace (see [33, 2.5.3]). We consider the limit

$$\mathfrak{M}_0^\bullet(\infty,W) \overset{\text{def.}}{=} \bigcup_V \mathfrak{M}_0^\bullet(V,W).$$

It is known that the above stabilizes at some V (see [33, 2.6.3, 2.9.4]). The complement $\mathfrak{M}_0^\bullet(V,W)\setminus\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$ consists of a finite union of $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V',W)$ for smaller V''s [32, 3.27, 3.28]. Therefore we have a decomposition

$$\mathfrak{M}_0^\bullet(\infty,W) = \bigsqcup_{[V]} \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W), \tag{4.5}$$

where $[V]$ denotes the isomorphism class of V. The transversal slice to each stratum was constructed in [33, §3.3]. Using it, we can check

(4.6) If $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)\neq\emptyset$, then $e^V e^W$ is l-dominant.

(4.7) If $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)\subset\overline{\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V',W)}$, then $e^{V'}\le e^V$.

On the other hand, we consider the disjoint union for $\mathfrak{M}^\bullet(V,W)$:

$$\mathfrak{M}^\bullet(W) \overset{\text{def.}}{=} \bigsqcup_{[V]} \mathfrak{M}^\bullet(V,W).$$

Note that there are no obvious morphisms between $\mathfrak{M}^\bullet(V,W)$ and $\mathfrak{M}^\bullet(V',W)$ since the stability condition is not preserved under the extension. We have a morphism $\mathfrak{M}^\bullet(W)\to\mathfrak{M}_0^\bullet(\infty,W)$, still denoted by π.

The original quiver varieties [30], [32] are the special case when $\varepsilon=1$ and $V_i(a)=W_i(a)=0$ except $a=1$. On the other hand, the above varieties $\mathfrak{M}^\bullet(W)$, $\mathfrak{M}_0^\bullet(\infty,W)$ are fixed point set of the original quiver varieties with respect to a semisimple element in a product of general linear groups. (See [33, §4].) In particular, it follows that $\mathfrak{M}^\bullet(V,W)$ is nonsingular, since the corresponding original quiver variety is so. This can also be checked directly.

Since the action is free, V and W can be considered as $I\times\mathbb{C}^*$-graded vector bundles over $\mathfrak{M}^\bullet(V,W)$. We denote them by the same notation. We consider $\mathrm{E}^\bullet(V,V)$, $\mathrm{L}^\bullet(W,V)$, $\mathrm{L}^\bullet(V,W)$ as vector bundles defined by the same formula as in (4.1). By the definition, B, α, β can be considered as sections of those bundles.

We define a three-term sequence of vector bundles over $\mathfrak{M}^\bullet(V,W)$ by

$$C^\bullet_{i,a}(V,W) : V_i(a\varepsilon) \xrightarrow{\sigma_{i,a}} \bigoplus_{h:\mathrm{in}(h)=i} V_{\mathrm{out}(h)}(a) \oplus W_i(a) \xrightarrow{\tau_{i,a}} V_i(a\varepsilon^{-1}), \tag{4.8}$$

where

$$\sigma_{i,a} = \bigoplus_{\mathrm{in}(h)=i} B_{\overline{h},a\varepsilon} \oplus \beta_{i,a\varepsilon}, \qquad \tau_{i,a} = \sum_{\mathrm{in}(h)=i} \varepsilon(h) B_{h,a} + \alpha_{i,a}.$$

This is a complex thanks to the equation $\mu(B,\alpha,\beta)=0$. We assign the degree 0 to the middle term. By the stability condition, $\sigma_{i,a}$ is injective.

We define the rank of complex $C^\bullet$ by $\sum_p (-1)^p \operatorname{rank} C^p$. Then

$$\operatorname{rank} C^\bullet_{i,a}(V,W) = u_{i,a}(e^V e^W).$$

We denote the right-hand side by $u_{i,a}(V,W)$ for brevity.

There exists a three term complex of vector bundles over $\mathfrak{M}^\bullet(V^1,W^1)\times \mathfrak{M}^\bullet(V^2,W^2)$:

$$\begin{array}{ccc} & \mathrm{E}^\bullet(V^1,V^2)^{[-1]} & \\ & \oplus & \\ \mathrm{L}^\bullet(V^1,V^2)^{[0]} \xrightarrow{\sigma^{21}} & \mathrm{L}^\bullet(W^1,V^2)^{[-1]} & \xrightarrow{\tau^{21}} \mathrm{L}^\bullet(V^1,V^2)^{[-2]} \\ & \oplus & \\ & \mathrm{L}^\bullet(V^1,W^2)^{[-1]}, & \end{array} \tag{4.9}$$

where

$$\begin{aligned} \sigma^{21}(\xi) &= (B^2\xi - \xi B^1) \oplus (-\xi\alpha^1) \oplus \beta^2\xi, \\ \tau^{21}(C\oplus I\oplus J) &= \varepsilon B^2 C + \varepsilon C B^1 + \alpha^2 J + I\beta^1. \end{aligned}$$

We assign the degree 0 to the middle term. By the same argument as in [32, 3.10], σ^{21} is injective and τ^{21} is surjective. Thus the quotient $\operatorname{Ker}\tau^{21}/\operatorname{Im}\sigma^{21}$ is a vector bundle over $\mathfrak{M}^\bullet(V^1,W^1)\times\mathfrak{M}^\bullet(V^2,W^2)$. Its rank is given by

$$d(e^{V^1}e^{W^1}, e^{V^2}e^{W^2}). \tag{4.10}$$

If $V^1=V^2$, $W^1=W^2$, then the restriction of $\operatorname{Ker}\tau^{21}/\operatorname{Im}\sigma^{21}$ to the diagonal is isomorphic to the tangent bundle of $\mathfrak{M}^\bullet(V,W)$ (see [33, Proof of 4.1.4]). In particular, we have

$$\dim \mathfrak{M}^\bullet(V,W) = d(e^V e^W, e^V e^W). \tag{4.11}$$

Let us give the definition of $\widehat{\chi}_{\varepsilon,t}$. We define $\widehat{\chi}_{\varepsilon,t}$ for all standard modules $M(P)$. Since $\{M(P)\}_P$ is a basis of $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$, we can extend it linearly to any finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules.

The relation between standard modules and graded/cyclic quiver varieties is as follows (see [33, §13]): Choose W so that $e^W = e^P$, i.e.,

$$P_i(u) = \prod_a (1-au)^{\dim W_i(a)}.$$

Then a standard module $M(P)$ is defined as $H_*(\mathfrak{L}^\bullet(W), \mathbb{C})$, which is equipped with a structure of a $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module by the convolution product. Moreover, its l-weight space $M(P)_Q$ is

$$\bigoplus_{V : e^V e^W = e^Q} H_*(\mathfrak{L}^\bullet(V, W), \mathbb{C}).$$

Here $H_k(\ , \mathbb{C})$ denotes the Borel-Moore homology with complex coefficients. If ε is not a root of unity, then V is determined from Q. So the above has only one summand.

Let

$$\widehat{\chi}_{\varepsilon,t}(M(P)) \stackrel{\text{def.}}{=} \sum_{[V]} (-t)^k \dim H_k(\mathfrak{L}^\bullet(V, W), \mathbb{C})\, e^V e^W. \tag{4.12}$$

Since $H_k(\mathfrak{L}^\bullet(V, W), \mathbb{C})$ vanishes for odd k [33, §7], we may replace $(-t)^k$ by t^k. In particular, it is clear that coefficients of $\widehat{\chi}_{\varepsilon,t}(M(P))$ are polynomials in t^2 with positive coefficients.

In subsequent sections we prove that the above $\widehat{\chi}_{\varepsilon,t}$ satisfies the axioms. By definition, it is clear that $\widehat{\chi}_{\varepsilon,t}$ satisfies Axiom 1.

Note that Corollary 3.6 follows directly from this geometric definition ([33, 13.4.5]).

We give a simple consequence of the definition:

Proposition 4.13. *Assume ε is not a root of unity. Suppose that all roots of $P_i(u) = 0$ have the same value (e.g., $P_i(u) = 1 - au$, $P_j(u) = 1$ for $j \neq i$ for some i). Then $M(P)$ has no l-dominant term other than e^P.*

This was proved in [12, Cor. 4.5]. But we give a geometric proof.

Proof. Take W so that $e^W = e^P$. It is enough to show that $u_{i,a}(V, W) < 0$ for some i, a if $\mathfrak{M}^\bullet(V, W) \neq \emptyset$ and $V \neq 0$.

By the assumption, there is a nonzero constant a such that $W_i(b) = 0$ for all i, $b \neq a$. By the stability condition, we have $V_i(b) = 0$ if $b \neq a\varepsilon^n$ for some $n \in \mathbb{Z}_{>0}$. Let n_0 be the maximum of such n, and suppose $V_i(a\varepsilon^{n_0}) \neq 0$. Since $W_i(a\varepsilon^{n_0+1}) = V_i(a\varepsilon^{n_0+1}) = V_i(a\varepsilon^{n_0+2}) = 0$, we have

$$u_{i,a\varepsilon^{n_0+1}}(V, W) = \operatorname{rank} C^\bullet_{i,a\varepsilon^{n_0+1}}(V, W) < 0. \qquad \square$$

5. Proof of Axiom 2: Analog of the Weyl group invariance

For a complex algebraic variety X, let $e(X; x, y)$ denote the virtual Hodge polynomial defined by Danilov-Khovanskii [9] using a mixed Hodge strucuture of Deligne [10]. It has the following properties.

(1) $e(X; x, y)$ is a polynomial in x, y with integral coefficients.

(2) If X is a nonsingular projective variety, then

$$e(X;x,y) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X) x^p y^q,$$

where the $h^{p,q}(X)$ are the Hodge numbers of X.

(3) If Y is a closed subvariety in X, then

$$e(X;x,y) = e(Y;x,y) + e(X \setminus Y;x,y).$$

(4) If $f\colon Y \to X$ is a fiber bundle with fiber F which is locally trivial in the Zarisky topology, then $e(Y;x,y) = e(X;x,y)e(F;x,y)$.

We define the virtual Poincaré polynomial of X by $p_t(X) \overset{\text{def.}}{=} e(X;t,t)$. (In fact, this reduction does not loose any information. The argument in 5.2 shows that $e(X;x,y)$ appearing here is a polynomial in xy.) The actual Poincaré polynomial is defined as

$$P_t(X) = \sum_{k=0}^{2\dim X} (-t)^k \dim H_k(X,\mathbb{C}),$$

where $H_k(X,\mathbb{C})$ is the Borel-Moore homology of X with complex coefficients.

Remark 5.1. Instead of virtual Poincaré polynomials, we can use numbers of rational points in the following argument, if we define graded/cyclic varieties over an algebraic closure of a finite field $\mathbf{k}$. As a consequence, those numbers are special values of "computable" polynomials $P(t)$ at $t = \sqrt{\#\mathbf{k}}$.

LEMMA 5.2. *The virtual Poincaré polynomial of $\mathfrak{L}^\bullet(V,W)$ is equal to the actual Poincaré polynomial. Moreover, it is a polynomial in t^2. The same holds for $\mathfrak{M}^\bullet(V,W)$.*

Proof. In [33, §7] we showed that $\mathfrak{L}^\bullet(V,W)$ has a partition into locally closed subvarieties $X_1,\dots,X_n$ with the following properties:

(1) $X_1 \cup X_2 \cup \dots \cup X_i$ is closed in $\mathfrak{L}^\bullet(V,W)$ for each i.

(2) Each X_i is a vector bundle over a nonsingular projective variety whose homology groups vanish in odd degrees.

A partition satisfying property (1) is called an *α-partition.* (More precisely, it was shown in [33, §7] that X_i is a fiber bundle with an affine space fiber over the base with the above property. The statement above was shown in [35].)

By the long exact sequence in homology groups, we have $P_t(\mathfrak{L}^\bullet(V,W)) = \sum_i P_t(X_i)$. On the other hand, by the property of the virtual Poincaré polynomial, $p_t(\mathfrak{L}^\bullet(V,W)) = \sum_i p_t(X_i)$. Since X_i satisfies the required properties in the statement, it follows that $\mathfrak{L}^\bullet(V,W)$ satisfies the same property.

There is an α-partition with property (2) also for $\mathfrak{M}^\bullet(V,W)$, so that we have the same assertion. $\square$

Recall the complex (4.8). For a $\mathbb{C}^*$-tuple of nonnegative integers $(n_a) \in \mathbb{Z}_{\geq 0}^{\mathbb{C}^*}$, let

$$\mathfrak{M}^\bullet_{i;(n_a)}(V,W) \overset{\text{def.}}{=} \left\{ [B,\alpha,\beta] \in \mathfrak{M}^\bullet(V,W) \;\middle|\; \operatorname{codim}_{V_i(\varepsilon^{-1}a)} \operatorname{Im}\tau_{i,a} = n_a \text{ for each } a \in \mathbb{C}^* \right\}.$$

This is a locally closed subset of $\mathfrak{M}^\bullet(V,W)$. We also set

$$\mathfrak{L}^\bullet_{i;(n_a)}(V,W) \overset{\text{def.}}{=} \mathfrak{M}^\bullet_{i;(n_a)}(V,W) \cap \mathfrak{L}^\bullet(V,W).$$

There are partitions

$$\mathfrak{M}^\bullet(V,W) = \bigsqcup_{(n_a)} \mathfrak{M}^\bullet_{i;(n_a)}(V,W), \qquad \mathfrak{L}^\bullet(V,W) = \bigsqcup_{(n_a)} \mathfrak{L}^\bullet_{i;(n_a)}(V,W).$$

Let $Q_{i,a}(V,W)$ be the middle cohomology of the complex $C^\bullet_{i,a}(V,W)$ (4.8); i.e.,

$$Q_{i,a}(V,W) \overset{\text{def.}}{=} \operatorname{Ker}\tau_{i,a}/\operatorname{Im}\sigma_{i,a}.$$

Over each stratum $\mathfrak{M}^\bullet_{i;(n_a)}(V,W)$ it defines a vector bundle. In particular, over the open stratum $\mathfrak{M}^\bullet_{i;(0)}(V,W)$, i.e., points where $\tau_{i,a}$ is surjective for all i, its rank is equal to

$$\operatorname{rank} C^\bullet_{i,a}(V,W) = u_{i,a}(V,W). \tag{5.3}$$

Suppose that a point $[B,\alpha,\beta] \in \mathfrak{M}^\bullet_{i;(n_a)}(V,W)$ is given. We define a new graded vector space V' by $V'_i(\varepsilon^{-1}a) \overset{\text{def.}}{=} \operatorname{Im}\tau_{i,a}$. The restriction of (B,i,j) to V' also satisfies the equation $\mu = 0$ and the stability condition and thus defines a point in $\mathfrak{M}^\bullet(V',W)$. It is clear that this construction defines a map

$$p \colon \mathfrak{M}^\bullet_{i;(n_a)}(V,W) \to \mathfrak{M}^\bullet_{i;(0)}(V',W). \tag{5.4}$$

Let $G(n_a, Q_{i,\varepsilon^{-2}a}(V',W)|_{\mathfrak{M}^\bullet_{i;(0)}(V',W)})$ denote the Grassmann bundle of n_a-planes in the vector bundle obtained by restricting $Q_{i,\varepsilon^{-2}a}(V',W)$ to $\mathfrak{M}^\bullet_{i;(0)}(V',W)$. Let

$$\prod_a G(n_a, Q_{i,\varepsilon^{-2}a}(V',W)|_{\mathfrak{M}^\bullet_{i;(0)}(V',W)})$$

be their fiber product over $\mathfrak{M}^\bullet_{i;(0)}(V',W)$. By [33, 5.5.2] there exists a commutative diagram

$$\begin{array}{ccc}
\prod_a G(n_a, Q_{i,\varepsilon^{-2}a}(V',W)|_{\mathfrak{M}^\bullet_{i;(0)}(V',W)}) & \xrightarrow{\pi} & \mathfrak{M}^\bullet_{k;(0)}(V',W) \\
\downarrow{\cong} & & \Vert \\
\mathfrak{M}^\bullet_{i;(n_a)}(V,W) & \xrightarrow{p} & \mathfrak{M}^\bullet_{i;(0)}(V',W),
\end{array}$$

where π is the natural projection. (The assumption $\varepsilon \neq \pm 1$ there was unnecessary. See Section 9.)

Since the projection (5.4) factors through π, it induces

$$p\colon \mathfrak{L}^\bullet_{i;(n_a)}(V,W) \to \mathfrak{L}^\bullet_{i;(0)}(V',W).$$

Therefore,

$$\begin{aligned} &p_t\left(\mathfrak{L}^\bullet_{i;(n_a)}(V,W)\right) \\ &\quad = \prod_a t^{n_a(\operatorname{rank} C^\bullet_{i,\varepsilon^{-2}a}(V',W)-n_a)} \begin{bmatrix} \operatorname{rank} C^\bullet_{i,\varepsilon^{-2}a}(V',W) \\ n_a \end{bmatrix}_t p_t\left(\mathfrak{L}^\bullet_{i;(0)}(V',W)\right). \end{aligned}$$

Using (5.3), we get

$$\begin{aligned} \widehat{\chi}_{\varepsilon,t}(M(P)) = &\sum_{[V']} p_t\left(\mathfrak{L}^\bullet_{i;(0)}(V',W)\right) e^{V'} e^W \\ &\times \prod_a t^{r_a(u_{i,a}(V',W)-r_a)} \sum_{r_a=0}^{u_{i,a}(V',W)} \begin{bmatrix} u_{i,a}(V',W) \\ r_a \end{bmatrix}_t V_{i,a\varepsilon}^{r_a}. \end{aligned}$$

This shows that $\widehat{\chi}_{\varepsilon,t}(M(P))$ is contained $\widehat{\mathcal{K}}_t$ and completes the proof of Axiom 2.

As promised, we give an algorithm computing $\widehat{\chi}_{\varepsilon,t}(M)$ for M as in Theorem 3.5(2). Although we will explain it only when M is a standard module $M(P)$, a modification to the general case is straightforward, if we interpret $p_t(\mathfrak{L}^\bullet_{i;(n_a)}(V,W))$ suitably. Moreover, we use the Grassmann bundle (5.4) instead of the condition $\widehat{\chi}_{\varepsilon,t}(M) \in \widehat{\mathcal{K}}_t$.

We "compute" the virtual Poincaré polynomials $p_t(\mathfrak{L}^\bullet_{i;(n_a)}(V,W))$ by induction. The first step of the induction is the case $V = 0$. In this case, $\mathfrak{L}^\bullet(0,W)$ is a single point, so that $p_t(\mathfrak{L}^\bullet(0,W)) = 1$. Moreover, $\mathfrak{L}^\bullet(0,W) = \mathfrak{L}^\bullet_{i;(0)}(0,W)$ for all i.

Suppose that we have already "computed" all $p_t(\mathfrak{L}^\bullet_{i;(n'_\lambda)}(V',W))$ for $\dim V' < \dim V$. If $(n_a) \neq (0)$, then by the Grassmann bundle (5.4) and the induction hypothesis, we can "compute" $p_t(\mathfrak{L}^\bullet_{i;(n_a)}(V,W))$. By the assumption in Theorem 3.5(2), $e^V e^W$ is *not* l-dominant; hence there exist i, a such that $u_{i,a}(V,W) < 0$. Then $\mathfrak{L}^\bullet_{i;(0)}(V,W)$ is the empty set by [33, 5.5.5]. Therefore,

$$p_t\left(\mathfrak{L}^\bullet(V,W)\right) = \sum_{(n_a)\neq(0)} p_t\left(\mathfrak{L}^\bullet_{i;(n_a)}(V,W)\right).$$

We have already "computed" the right-hand side. Now, of course,

$$p_t\left(\mathfrak{L}^\bullet_{i;(0)}(V,W)\right) = 0.$$

For $j \neq i$,

$$p_t\left(\mathfrak{L}^\bullet_{j;(0)}(V,W)\right) = p_t\left(\mathfrak{L}^\bullet(V,W)\right) - \sum_{(n_a)\neq(0)} p_t\left(\mathfrak{L}^\bullet_{j;(n_a)}(V,W)\right).$$

The right-hand side is already "computed".

Remark 5.5. (1) Note that the above argument shows that $p_t(\mathfrak{L}^\bullet(V,W))$ is a polynomial in t^2 without appealing to [33, §7] as in Lemma 5.2. In fact, nonsingular quasi-projective varieties appearing in Lemma 5.2 are examples of graded quiver varieties such that the above argument can be applied; i.e., the corresponding standard modules satisfy the condition in Theorem 3.5(2). Therefore, the above gives a new proof of the vanishing of odd homology groups.

(2) If the reader carefully compares our algorithm with Frenkel-Mukhin's one [12], he/she finds a difference. The *coloring* s_i of a monomial $m = e^V e^W$ is

$$\sum_{(n_a)\neq(0)} p_{t=1}\left(\mathfrak{L}^\bullet_{i;(n_a)}(V,W)\right)$$

in our algorithm. This might possibly be a *negative* integer, while it is assumed to be nonnegative in [12]. Therefore, we must modify their definition of the *admissibility* of a monomial m. Let us consider all values s_i such that m is not i-dominant. We say m is *admissible* if all values are the same. In our case, s_i is $p_{t=1}\left(\mathfrak{L}^\bullet(V,W)\right)$ if m is not i-dominant; hence $\mathfrak{L}^\bullet_{i;(0)}(V,W) = \emptyset$. Therefore it is independent of i.

6. Proof of Axiom 3: Multiplicative property

By [31] it has been known that Betti numbers of arbitrary quiver varieties are determined by special cases corresponding to fundamental weights. We will use the same idea in this section.

Let W^1, W^2, W be $I\times\mathbb{C}^*$-graded vector spaces such that $W_i(a) = W_i^1(a)\oplus W_i^2(a)$ for $i\in I$, $a\in\mathbb{C}^*$. Let P^1, P^2 be I-tuples of polynomials corresponding to W^1, W^2. Then W corresponds to P^1P^2.

We define a map $\mathfrak{M}^\bullet(W^1)\times\mathfrak{M}^\bullet(W^2)\to\mathfrak{M}^\bullet(W)$ by

$$\left([B^1,\alpha^1,\beta^1],[B^2,\alpha^2,\beta^2]\right) \longmapsto [B^1\oplus B^2, \alpha^1\oplus\alpha^2, \beta^1\oplus\beta^2]. \tag{6.1}$$

We define a $\prod_{i,a}\operatorname{GL}(W_i(a))$-action on $\mathfrak{M}^\bullet(W)$ by

$$s * [B,\alpha,\beta] \overset{\text{def.}}{=} [B, \alpha s^{-1}, s\beta].$$

We define a one-parameter subgroup $\lambda\colon \mathbb{C}^*\to\prod_{i,a}\operatorname{GL}(W_i(a))$ by

$$\lambda(t) = \bigoplus_{i,a} \operatorname{id}_{W_i^1(a)} \oplus t\operatorname{id}_{W_i^2(a)}.$$

Then (6.1) is a closed embedding and the fixed point set $\mathfrak{M}^\bullet(W)^{\lambda(\mathbb{C}^*)}$ is its image by [35, 3.2]. We identify $\mathfrak{M}^\bullet(W^1) \times \mathfrak{M}^\bullet(W^2)$ with its image hereafter. The fixed point set $\mathfrak{M}^\bullet(V,W)^{\lambda(\mathbb{C}^*)}$ is the union of $\mathfrak{M}^\bullet(V^1,W^1) \times \mathfrak{M}^\bullet(V^2,W^2)$ with $V \cong V^1 \oplus V^2$.

Let

$$\begin{aligned}&\mathfrak{Z}^\bullet(V^1,W^1;V^2,W^2)\\ &\stackrel{\text{def.}}{=} \left\{[B,\alpha,\beta] \in \mathfrak{M}^\bullet(W) \,\middle|\, \lim_{t\to 0} \lambda(t) * [B,\alpha,\beta] \in \mathfrak{M}^\bullet(V^1,W^1) \times \mathfrak{M}^\bullet(V^2,W^2)\right\}.\end{aligned}$$

We also define $\widetilde{\mathfrak{Z}}^\bullet(V^1,W^1;V^2,W^2)$ by replacing $\mathfrak{M}^\bullet(V^1,W^1) \times \mathfrak{M}^\bullet(V^2,W^2)$ by $\mathfrak{L}^\bullet(V^1,W^1) \times \mathfrak{L}^\bullet(V^2,W^2)$. These are studied in [35]. They are nonsingular, locally closed subvarieties of $\mathfrak{M}^\bullet(W)$ [loc. cit., 3.7].

Let $\mathfrak{Z}^\bullet(W^1;W^2)$, $\widetilde{\mathfrak{Z}}^\bullet(W^1,W^2)$ be their union over $[V^1]$, $[V^2]$ respectively. These are closed subvarieties of $\mathfrak{M}^\bullet(W)$ [loc. cit., 3.6]. By [loc. cit., 3.7, 3.13], the partition

$$\mathfrak{Z}^\bullet(W^1;W^2) = \bigsqcup_{[V^1],[V^2]} \mathfrak{Z}^\bullet(V^1,W^1;V^2,W^2)$$

is an α-partition such that each stratum $\mathfrak{Z}^\bullet(V^1,W^1;V^2,W^2)$ is isomorphic to the total space of the vector bundle $\operatorname{Ker}\tau^{21}/\operatorname{Im}\sigma^{21}$ over $\mathfrak{M}^\bullet(V^1,W^1) \times \mathfrak{M}^\bullet(V^2,W^2)$ in (4.9). (More precisely, we restrict the result of [loc. cit.] to the fixed point set.)

Similarly

$$\widetilde{\mathfrak{Z}}^\bullet(W^1;W^2) = \bigsqcup_{V^1,V^2} \widetilde{\mathfrak{Z}}^\bullet(V^1,W^1;V^2,W^2)$$

is an α-partition such that each stratum $\mathfrak{Z}^\bullet(V^1,W^1;V^2,W^2)$ is isomorphic to the restriction of $\operatorname{Ker}\tau^{21}/\operatorname{Im}\sigma^{21}$ to $\mathfrak{L}^\bullet(V^1,W^1) \times \mathfrak{L}^\bullet(V^2,W^2)$.

PROPOSITION 6.2. (1) *The virtual Poincaré polynomial of* $\widetilde{\mathfrak{Z}}^\bullet(W^1;W^2)$ (*more precisely that of each connected component of* $\widetilde{\mathfrak{Z}}^\bullet(W^1;W^2)$) *is equal to its actual Poincaré polynomial. Moreover, it is a polynomial in* t^2. *The same holds for* $\mathfrak{Z}^\bullet(W^1;W^2)$.

(2) *Now,*

$$\widehat{\chi}_{\varepsilon,t}(M(P^1)) * \widehat{\chi}_{\varepsilon,t}(M(P^2)) = \sum_{[V^1],[V^2]} P_t(\widetilde{\mathfrak{Z}}^\bullet(V^1,W^1;V^2,W^2))\, e^{V^1} e^{V^2} e^{W^1} e^{W^2}. \tag{6.3}$$

(3) *The above expression is contained in* $\widehat{\mathcal{K}}_t$.

Proof. (1) This can be shown exactly as in Lemma 5.2.

(2) The rank of the vector bundle $\operatorname{Ker}\tau^{21}/\operatorname{Im}\sigma^{21}$ is equal to $d(e^{W^1}e^{V^1}, e^{W^2}e^{V^2})$ (see (4.10)). By the property of virtual Poincaré polynomials, we get the assertion.

(3) Exactly the same as Section 5. □

Axiom 3 follows from the above and the following assertion proved in [35, 6.12]:

$$\widetilde{\mathfrak{Z}}^\bullet(W^1;W^2) = \mathfrak{L}^\bullet(W)$$

under the condition (3.2).

As promised, we give

A different proof of Theorem 3.8. We only prove the second statement. In fact, it is not difficult to show that the following argument also implies the first statement.

We will prove that our $\widehat{\chi}_{\varepsilon,t}$ satisfies Axiom 4 in the next section. Therefore, it is enough to check the assertion for $\widehat{\chi}_{\varepsilon,t}$ given by the geometric definition (4.12).

By Theorem 3.5(1) and Proposition 6.2, we get the statement regarding the multiplication.

Similarly, for the proof of the statement regarding the involution, it is enough to show

$$\overline{\widehat{\chi}_{\varepsilon,t}(M(P))} \in \widehat{\mathcal{K}}_t.$$

This follows from

$$\overline{\widehat{\chi}_{\varepsilon,t}(M(P))} = \sum_{[V]} P_t(\mathfrak{M}^\bullet(V,W))\, e^V e^W. \tag{6.4}$$

In fact, the same argument as in the proof of Section 5 shows that the right-hand side is contained in $\widehat{\mathcal{K}}_t$.

Let us prove (6.4). Since $\mathfrak{L}^\bullet(V,W)$ is homotopic to $\mathfrak{M}^\bullet(V,W)$ [33, 4.1.2], its usual homology group is isomorphic to that of $\mathfrak{M}^\bullet(V,W)$. Since $\mathfrak{L}^\bullet(V,W)$ is compact, the usual homology group is isomorphic to the Borel-Moore homology. Therefore, the Poincaré duality for $\mathfrak{M}^\bullet(V,W)$, which is applicable since $\mathfrak{M}^\bullet(V,W)$ is nonsingular, implies

$$t^{2\dim \mathfrak{M}^\bullet(V,W)} P_{1/t}\left(\mathfrak{L}^\bullet(V,W)\right) = P_t\left(\mathfrak{M}^\bullet(V,W)\right).$$

Since $\dim \mathfrak{M}^\bullet(V,W) = d(e^V e^W, e^V e^W)$ (see (4.11)), we get (6.4). □

7. Proof of Axiom 4: Roots of unity

In this section, we use a $\mathbb{C}^*$-action on $\mathfrak{M}^\bullet(V,W)$ to calculate Betti numbers. This idea originally appeared in [31] and [30, §5].

We assume that ε is a primitive s-th root of unity ($s \in \mathbb{Z}_{>0}$).

We may assume $\alpha = 1$ in the setting of Axiom 4. We consider V, W as $I \times (\mathbb{Z}/s\mathbb{Z})$-graded vector spaces.

We define a $\mathbb{C}^*$-action on $\mathbf{M}^\bullet(V, W)$ by

$$t \star (B, \alpha, \beta) = (tB, t\alpha s(t)^{-1}, ts(t)\beta), \qquad (t \in \mathbb{C}^*),$$

where $s(t) \in \prod \operatorname{GL}(W_i(a))$ is defined by

$$s(t) = \bigoplus_{i \in I,\, 0 \le n < s} t^n \operatorname{id}_{W_i(\varepsilon^n)}.$$

Now, $s(t)$ preserves the equation $\mu(B, \alpha, \beta) = 0$ and commutes with the action of G_V. Therefore it induces an action on the affine cyclic quiver variety $\mathfrak{M}_0^\bullet(V, W)$. The action preserves the stability condition. Therefore it induces action on $\mathfrak{M}^\bullet(V, W)$. These induced actions are also denoted by $\star$. The map $\pi \colon \mathfrak{M}^\bullet(V, W) \to \mathfrak{M}_0^\bullet(V, W)$ is equivariant.

Lemma 7.1. *Let $[B, \alpha, \beta] \in \mathfrak{M}^\bullet(V, W)$. The flow $t \star [B, \alpha, \beta]$ for $t \in \mathbb{C}^*$ has a limit when $t \to 0$.*

Proof. By a generality theory, it is enough to show that $t \star [B, \alpha, \beta]$ stays in a compact set. Since π is proper, we only need to show that $\pi(t \star [B, \alpha, \beta]) = t \star \pi([B, \alpha, \beta])$ stays in a compact set.

By [28] the coordinate ring of $\mathfrak{M}_0^\bullet(V, W)$ is generated by functions of forms

$$\langle \chi, \beta_{i, \varepsilon^{n+1}} B_{h_1, \varepsilon^{n+2}} \dots B_{h_N, \varepsilon^{n+N+1}} \alpha_{j, \varepsilon^{n+N+2}} \rangle$$

where χ is a linear form on $\operatorname{Hom}\left(W_{i,\varepsilon^n}, W_{j,\varepsilon^{n+N+2}}\right)$, and $i = \operatorname{in}(h_1)$, $\operatorname{out}(h_1) = \operatorname{in}(h_2)$, … , $\operatorname{out}(h_N) = j$. By the $\mathbb{C}^*$-action, this function is multiplied by

$$t^{n+1} t^N t^{-r+1} = t^{n+N+2-r}$$

where we assume $0 \le n < s$ and r is the integer such that $0 \le r < s$ and $r \equiv n + N + 2 \mod s$. Then $n + N + 2 - r$ is nonnegative. Therefore $t \star \pi([B, \alpha, \beta])$ stays in a compact set for any $[B, \alpha, \beta]$. □

We want to identify a fixed point set in $\mathfrak{M}^\bullet(V, W)$ with some quiver variety $\mathfrak{M}^\bullet(V_q, W_q)$ defined for q which is *not* a root of unity. We first explain a morphism from $\mathfrak{M}^\bullet(V_q, W_q)$ to $\mathfrak{M}^\bullet(V, W)$. Corresponding vector spaces $\mathfrak{M}^\bullet(V_q, W_q)$ are $I \times \mathbb{Z}$-graded vector spaces. Suppose that V_q, W_q are $I \times \mathbb{Z}$-graded vector spaces such that $(W_q)_i(q^k) = 0$ unless $0 \le k < s$ (no condition for V_q). We consider W_q as an $I \times (\mathbb{Z}/s\mathbb{Z})$-graded vector space simply identifying $\mathbb{Z}/s\mathbb{Z}$ with $\{0, 1, \dots, s-1\}$. Let us denote by W the resulting $I \times (\mathbb{Z}/s\mathbb{Z})$-graded vector space. We define an $I \times (\mathbb{Z}/s\mathbb{Z})$-graded vector space V by

$$V_i(\varepsilon^n) \overset{\text{def.}}{=} \bigoplus_{k \not\equiv n \mod s} (V_q)_i(q^k).$$

If a point in $\mathbf{M}^\bullet(V_q, W_q)$ is given, it defines a point in $\mathbf{M}^\bullet(V, W)$ in an obvious way. The map $\mathbf{M}^\bullet(V_q, W_q) \to \mathbf{M}^\bullet(V, W)$ preserves the equation $\mu = 0$ and the stability condition. It is equivariant under the G_{V_q} action, where $G_{V_q} \to G_V$ is an obvious homomorphism. Therefore, we have a morphism

$$\mathfrak{M}^\bullet(V_q, W_q) \to \mathfrak{M}^\bullet(V, W). \tag{7.2}$$

Note that W_q is uniquely determined by W, while V_q is *not* determined from V.

LEMMA 7.3. *A point $[B, \alpha, \beta] \in \mathfrak{M}^\bullet(V, W)$ is fixed by the $\mathbb{C}^*$-action if and only if it is contained in the image of* (7.2) *for some V_q. Moreover, the map* (7.2) *is a closed embedding.*

Proof. Fix a representatitve (B, α, β) of $[B, \alpha, \beta]$. Then $[B, \alpha, \beta]$ is a fixed point if and only if there exists $\lambda(t) \in G_V$ such that

$$t \star (B, \alpha, \beta) = \lambda(t)^{-1} \cdot (B, \alpha, \beta).$$

Such a $\lambda(t)$ is unique since the action of G_V is free. In particular, $\lambda\colon \mathbb{C}^* \to G_V$ is a group homomorphism.

Let $V_i(\varepsilon^n)[k]$ be the weight space of $V_i(\varepsilon^n)$ with eigenvalue t^k. The above equation means that

$$B_{h,\varepsilon^{n+1}}\left(V_{\mathrm{out}(h)}(\varepsilon^{n+1})[k+1]\right) \subset V_{\mathrm{in}(h)}(\varepsilon^n)[k], \qquad \alpha_{i,\varepsilon^{n+1}}(W_i(\varepsilon^{n+1})) \subset V_i(\varepsilon^n)[n],$$
$$\beta_{i,\varepsilon^n}\left(V_i(\varepsilon^n)[k]\right) = 0 \quad \text{if } k \neq n.$$

Let us define an $I \times \mathbb{C}^*$-graded subspace S of V by

$$S_i(\varepsilon^n) \overset{\text{def.}}{=} \bigoplus_{k \not\equiv n \mod s} V_i(\varepsilon^n)[k].$$

The above equations imply that S is contained in $\operatorname{Ker}\beta$ and B-invariant. Therefore $S = 0$ by the stability condition. This means that $[B, \alpha, \beta]$ is in the image of (7.2) if we set

$$(V_q)_i(q^k) \overset{\text{def.}}{=} V_i(\varepsilon^k)[k].$$

Conversely, a point in the image is a fixed point. Since λ is unique, the map (7.2) is injective.

Let us consider the differential of (7.2). The tangent space of $\mathfrak{M}^\bullet(V, W)$ at $[B, i, j]$ is the middle cohomology group of the complex

$$\mathrm{L}^\bullet(V,V)^{[0]} \xrightarrow{\sigma^{21}} \begin{matrix} \mathrm{E}^\bullet(V,V)^{[-1]} \\ \oplus \\ \mathrm{L}^\bullet(W,V)^{[-1]} \\ \oplus \\ \mathrm{L}^\bullet(V,W)^{[-1]} \end{matrix} \xrightarrow{\tau^{21}} \mathrm{L}^\bullet(V,V)^{[-2]}. \tag{7.4}$$

Similarly the tangent space of $\mathfrak{M}^\bullet(V_q, W_q)$ is the middle cohomology of a complex with V, W replaced by V_q, W_q. We have a natural morphism between the complexes so that the induced map between cohomology groups is the differential of (7.2). It is not difficult to show the injectivity by using the stability condition. $\square$

Let us consider the tangent space T of $\mathfrak{M}^\bullet(V,W)$ at $[B,\alpha,\beta] \in \mathfrak{M}^\bullet(V_q, W_q) \subset \mathfrak{M}^\bullet(V,W)$, which is the middle cohomology of (7.4). Let $V = \bigoplus_k V[k]$ be the weight space decomposition as in the proof of the above lemma. The tangent space T has a weight decomposition $T = \bigoplus_k T[k]$, where $T[k]$ is the middle cohomology of

$$\bigoplus_n \mathrm{L}^\bullet(V[n], V[n+k])^{[0]} \xrightarrow{\sigma^{21}} \begin{matrix} \bigoplus_n \mathrm{E}^\bullet(V[n], V[n+k-1])^{[-1]} \\ \oplus \\ \bigoplus_n \mathrm{L}^\bullet(W[n], V[n+k-1])^{[-1]} \\ \oplus \\ \bigoplus_n \mathrm{L}^\bullet(V[n], W[n+k-1])^{[-1]}, \end{matrix} \xrightarrow{\tau^{21}} \bigoplus_n \mathrm{L}^\bullet(V[n], V[n+k-2])^{[-2]}$$

where $W[n] = W(\varepsilon^n)$ if $0 \le n < s$ and 0 otherwise. The rank of the complex is equal to

$$\begin{cases} d_q(e^{V_q}e^{W_q}, e^{V_q}e^{W_q}[k]) & \text{if } k \equiv 0 \mod s, \\ 0 & \text{otherwise.} \end{cases}$$

Here $e^{V_q}e^{W_q}[k]$ is defined as in (3.4).

We consider the Bialynicki-Birula decomposition of $\mathfrak{M}^\bullet(V,W)$:

$$\mathfrak{M}^\bullet(V,W) = \bigsqcup_{[V_q]} S(V_q, W_q),$$
$$S(V_q, W_q) \overset{\text{def.}}{=} \{x \in \mathfrak{M}^\bullet(V,W) \mid \lim_{t\to 0} t \star x \in \mathfrak{M}^\bullet(V_q, W_q)\}\ .$$

By a general theory, each $S(V_q, W_q)$ is a locally closed subvariety of $\mathfrak{M}^\bullet(V,W)$, and the natural map $S(V_q,W_q) \to \mathfrak{M}^\bullet(V_q,W_q)$ is a fiber bundle whose fiber is an affine space of dimension equal to $\sum_{k>0} \dim T[k]$. By the above formula, it is equal to

$$\sum_{k>0} d_q(e^{V_q}e^{W_q}, e^{V_q}e^{W_q}[ks]).$$

We write this number as $D^+(e^{V_q}e^{W_q})$.

By a property of virtual Poincaré polynomials, we have

$$P_t(\mathfrak{M}^\bullet(V,W)) = \sum_{[V_q]} t^{2D^+(e^{V_q}e^{W_q})} P_t(\mathfrak{M}^\bullet(V_q,W_q)).$$

(Recall that the virtual Poincaré polynomials coincide with the actual Poincaré polynomials for these varieties.) Combining this with an argument in the proof of (6.4), we have

$$\begin{aligned}
P_t(\mathfrak{L}^\bullet(V,W)) &= t^{2d(e^Ve^W,e^Ve^W)}P_{1/t}(\mathfrak{M}^\bullet(V,W)) \\
&= \sum_{[V_q]} t^{2d(e^Ve^W,e^Ve^W)-2D^+(e^{V_q}e^{W_q})}P_{1/t}(\mathfrak{M}^\bullet(V_q,W_q)) \\
&= \sum_{[V_q]} t^{2d(e^Ve^W,e^Ve^W)-2D^+(e^{V_q}e^{W_q})-2d_q(e^{V_q}e^{W_q},e^{V_q}e^{W_q})}P_t(\mathfrak{L}^\bullet(V_q,W_q)).
\end{aligned}$$

Since

$$d(e^Ve^W,e^Ve^W) = \dim T = \sum_k \dim T[k] = \sum_k d_q(e^{V_q}e^{W_q},e^{V_q}e^{W_q}[ks]),$$

we have

$$\begin{aligned}
&d(e^Ve^W,e^Ve^W) - D^+(e^{V_q}e^{W_q}) - d_q(e^{V_q}e^{W_q},e^{V_q}e^{W_q}) \\
&\qquad = \sum_{k<0} d_q(e^{V_q}e^{W_q},e^{V_q}e^{W_q}[ks]) = D^-(e^{V_q}e^{W_q}).
\end{aligned}$$

Thus we have checked Axiom 4.

Remark 7.5. When $\varepsilon = 1$, there is a different $\mathbb{C}^*$-action so that the index $D^-(m)$ can be read off from $a_m(t)$. See [34, §7].

8. Perverse sheaves on graded/cyclic quiver varieties

The following is the main result of this article:

THEOREM 8.1. (1) *There exists a unique base $\{L(P)\}$ of $\mathbf{R}_t$ such that*

$$\overline{L(P)} = L(P), \qquad L(P) \in M(P) + \sum_{Q:Q<P} t^{-1}\mathbb{Z}[t^{-1}]M(Q).$$

(2) *The specialization of $L(P)$ at $t=1$ coincides with the simple module with Drinfeld polynomial P.*

As mentioned in the introduction, the relation between $M(P)$ and $L(P)$ in $\mathbf{R}_t$ (not in its specialization) can be understood by a Jantzen filtration [16].

For a later purpose we define matrices in the Laurent polynomial ring of t:

$$\begin{aligned}
c_{PQ}(t) &\overset{\text{def.}}{=} \text{the coefficient of } e^Q \text{ in } \chi_{\varepsilon,t}(M(P)), \\
(c^{PQ}(t)) &\overset{\text{def.}}{=} (c_{PQ}(t))^{-1}, \\
M(P) &= \sum_Q Z_{PQ}(t)L(Q).
\end{aligned}$$

When ε is not a root of unity, there is an isomorphism between $\mathbf{R}_t$ and the dual of the Grothendieck group of a category of perverse sheaves on affine graded quiver varieties [33, §14]. The full detailed proof of the above theorem was explained in [34]. However, the latter group becomes larger when ε is a root of unity. So we modify $\mathbf{R}_t$ to $\widetilde{\mathbf{R}}_t$, and give a proof of the above theorem in this $\widetilde{\mathbf{R}}_t$.

Let us fix an I-tuple of polynomials P throughout this section. Let $\mathcal{I}$ be the set of l-dominant monomials $m \in \widehat{\mathcal{Y}}_t$ such that $m \le e^P$. We consider a $\mathbb{Z}[t,t^{-1}]$-module with basis $\mathcal{I}$, and denote it by $\widetilde{\mathbf{R}}_t$.

For each monomial $m \in \mathcal{I}$, let P_m be an I-tuple of polynomials given by $(P_m)_i(u) \overset{\text{def.}}{=} \prod_a (1-ua)^{u_{i,a}(m)}$. In other words, P_m is determined so that $\widehat{\Pi}(e^{P_m}) = \widehat{\Pi}(m)$. If ε is not a root of unity, then $P_m = P_{m'}$ implies $m = m'$ by the invertibility of the ε-analog of the Cartan matrix. But it is not true in general. This is the reason why we need a modification.

We modify $\widehat{\chi}_{\varepsilon,t}$ of the standard module $M(P_m)$ so that it has the image in $\widetilde{\mathbf{R}}_t$ as follows: If

$$\widehat{\chi}_{\varepsilon,t}(M(P_m)) = \sum_n a_{n,m}(t) e^{P_m} n,$$

then, we define

$$M_m \overset{\text{def.}}{=} \sum_{n^*} a_{n^*,m}(t)\, t^{-d(e^{P_m} n^*, e^{P_m} n^*)} m n^*,$$

where the summation runs *only* over n^* such that mn^* is l-dominant. The M_m is contained in $\widetilde{\mathbf{R}}_t$ by Axiom 1. And $\widehat{\chi}_{\varepsilon,t}(M(P_m))$ is recovered from M_m. Let us denote the coefficient of mn^* by $c_{mn}(t)$, where $n = mn^* \in \mathcal{I}$; that is,

$$M_m = \sum_n c_{mn}(t) n. \tag{8.2}$$

We have $c_{mm} = 1$ and $c_{mn}(t) = 0$ for $n \not\le m$. In particular, $\{M_m\}_m$ is a base of $\widetilde{\mathbf{R}}_t$.

We define an involution $\overline{}$ on $\widetilde{\mathbf{R}}_t$ by

$$\overline{t} = t^{-1}, \quad \overline{m} = m.$$

We define a map $\widetilde{\mathbf{R}}_t \to \mathbf{R}_t$ by $M_m \mapsto M(P_m)$. When ε is not a root of unity, this map is injective and the image is the submodule spanned by $M(P')$'s such that $\widehat{\Pi}(e^{P'}) \le \widehat{\Pi}(e^P)$. The map intertwines the involutions.

Also,

$$\overline{M_m} = \sum_n c_{mn}(t^{-1}) n = \sum_{n,s} c_{mn}(t^{-1}) c^{ns}(t) M_s,$$

where $(c^{ns}(t))$ is the inverse matrix of $(c_{mn}(t))$. Let

(8.3)

$$u_{mn}(t) \overset{\text{def.}}{=} \sum_s c_{ms}(t^{-1}) c^{sn}(t), \qquad \text{or equivalently } \overline{M_m} = \sum_n u_{mn}(t) M_n.$$

By the axioms, $u_{mm}(t) = 1$ and $u_{mn}(t) = 0$ if $n \not\le m$.

Lemma 8.4. *There exists a unique element $L_m \in \widetilde{\mathbf{R}}_t$ such that*

$$\overline{L_m} = L_m, \qquad L_m \in M_m + \sum_{n:n<m} t^{-1}\mathbb{Z}[t^{-1}]M_n.$$

Although the proof is exactly the same as the one in [25, 7.10], we give it for the convenience of the reader.

Proof. Let

$$M_m = \sum_{n\le m} Z_{mn}(t) L_n.$$

Then the condition for $\{L_m\}$ is equivalent to the following system:

$$Z_{mm}(t) = 1, \quad Z_{mn}(t) \in t^{-1}\mathbb{Z}[t^{-1}] \text{ for } n < m, \tag{8.5a}$$

$$Z_{mn}(t^{-1}) = \sum_{s:n\le s\le m} u_{ms}(t) Z_{sn}(t). \tag{8.5b}$$

The equation can be rewritten as

$$Z_{mn}(t^{-1}) - Z_{mn}(t) = \sum_{s:n\le s<m} u_{ms}(t) Z_{sn}(t).$$

Let $F_{mn}(t)$ be the right-hand side. We can solve this system uniquely by induction: If $Z_{sn}(t)$'s are given, $Z_{mn}(t)$ is uniquely determined by the above equation and $Z_{mn}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$, provided $F_{mn}(t^{-1}) = -F_{mn}(t)$. We can check this condition by the induction hypothesis:

$$\begin{aligned} F_{mn}(t^{-1}) &= \sum_{s:n\le s<m} u_{ms}(t^{-1}) Z_{sn}(t^{-1}) \\ &= \sum_{s:n\le s<m} \sum_{t:n\le t\le s} u_{ms}(t^{-1}) u_{st}(t) Z_{tn}(t) \\ &= -\sum_{t:n\le t<m} u_{mt}(t) Z_{tn}(t) = -F_{mn}(t), \end{aligned}$$

where $\sum_{s:t\le s\le m} u_{ms}(t^{-1}) u_{st}(t) = 0$ for $t < m$. □

The proof of Theorem 8.1(1) is exactly the same. Since the map $\widetilde{\mathbf{R}}_t \to \mathbf{R}_t$ intertwines the involution, the image of L_m is equal to $L(P_m)$. Therefore Theorem 8.1(2) is equivalent to the following statement:

Theorem 8.6. *The multiplicity $[M(P) : L(Q)]$ is equal to*

$$\sum_n Z_{e^P,n}(1),$$

where the summation is over the set $\{n \mid e^Q = \widehat{\Pi}(n)\}$.

The following proof is just a modification of that given in [34].

We choose W so that $e^P = e^W$ as before. Let $D^b(\mathfrak{M}_0^\bullet(\infty, W))$ be the bounded derived category of complexes of sheaves whose cohomology sheaves are constant along each connected component of a stratum $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$ of (4.5). (The connectedness of $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$ is *not* known.) If $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)^\alpha$ is a connected component of $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$, then $IC(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)^\alpha)$ is the intersection homology complex associated with the constant local system $\mathbb{C}_{\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)^\alpha}$ on $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)^\alpha$. Then $D^b(\mathfrak{M}_0^\bullet(\infty,W))$ is the category of a complex of sheaves which are finite direct sums of complexes of the forms $IC(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)^\alpha)[d]$ for various V, α and $d \in \mathbb{Z}$, thanks to the existence of transversal slices [33, §3].

We associate a monomial $m = e^V e^W$ to each $[V]$. It gives us a bijective correspondence between the set of monomials m with $m \le e^P$ and the set of isomorphism classes of $I \times \mathbb{C}^*$-graded vector spaces. If $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W) \neq \emptyset$, the corresponding monomial m is l-dominant, i.e., $m \in \mathcal{I}$. We choose a point in $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$ and denote it by x_m.

Let $\mathbb{C}_{\mathfrak{M}^\bullet(V',W)}$ be the constant local system on $\mathfrak{M}^\bullet(V',W)$. Then $\pi_*\mathbb{C}_{\mathfrak{M}^\bullet(V',W)}$ is an object of $D^b(\mathfrak{M}_0^\bullet(\infty,W))$ again by the transversal slice argument. From the decomposition theorem of Beilinson-Bernstein-Deligne, we have

(8.7)
$$\pi_*(\mathbb{C}_{\mathfrak{M}^\bullet(V',W)}[\dim_{\mathbb{C}} \mathfrak{M}^\bullet(V',W)]) \cong \bigoplus_{V,\alpha,k} L_{V,\alpha,k}(V',W) \otimes IC(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)^\alpha)[k]$$

for some vector space $L_{V',\alpha,k}(V,W)$ [33, 14.3.2]. We set

$$L_{mn}(t) \overset{\text{def.}}{=} \sum_k \dim L_{V,\alpha,k}(V',W)\, t^{-k}, \tag{8.8}$$

where V, V' are determined so that $m = e^V e^W$, $n = e^{V'} e^W$. By the description of the transversal slice [33, §3], $\dim L_{V,\alpha,k}$ is independent of α. So α can disappear on the left-hand side. Applying the Verdier duality to both sides of (8.7) and using the self-duality of $\pi_*(\mathbb{C}_{\mathfrak{M}^\bullet(V',W)}[\dim_{\mathbb{C}} \mathfrak{M}^\bullet(V',W)])$ and $IC(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)^\alpha)$, we find $L_{m'm}(t) = L_{m'm}(t^{-1})$.

By our definition of M_m, we have

$$M_m = \sum_{[V_n]} t^{-\dim \mathfrak{M}^\bullet(V_n,W_m)} P_t(\mathfrak{L}^\bullet(V_n,W_m))\, m e^{V_n},$$

where W_m is given by $\dim(W_m)_{i,a} = u_{i,a}(m)$. By [33, §3], this is equal to

(8.9)
$$\begin{aligned} M_m =& \sum_{[V_n]} t^{-\dim \mathfrak{M}^\bullet(V_n,W_m)} P_t\left(\pi^{-1}(x_m) \cap \mathfrak{M}^\bullet(V_n \oplus V_m, W)\right)\, m e^{V_n} \\ =& \sum_{[V']} \sum_k t^{\dim \mathfrak{M}^\bullet(V_m,W)-k} \dim H^k(i_{x_m}^! \pi_* \mathbb{C}_{\mathfrak{M}^\bullet(V',W)}[\dim \mathfrak{M}^\bullet(V',W)])\, e^{V'} e^W, \end{aligned}$$

where V_m is given so that $e^{V_m} e^W = m$.

Let

$$Z_{mn}(t) \stackrel{\text{def.}}{=} \sum_{k,\alpha} \dim H^k(i_{x_m}^! IC(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)^\alpha))\, t^{\dim \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V_m,W)-k},$$

where $n = e^V e^W$. By the defining property of the intersection homology, we have (8.5a) and $Z_{mn}(t) = 0$ if $n \not\leq m$.

Substituting (8.7) into (8.9), we get

$$c_{mn}(t) = \sum_s Z_{ms}(t) L_{sn}(t). \tag{8.10}$$

Now $L_{sn}(t) = L_{sn}(t^{-1})$ and (8.3) imply (8.5b).

Let $Z^\bullet(W)$ be the fiber product $\mathfrak{M}^\bullet(W) \times_{\mathfrak{M}_0^\bullet(\infty,W)} \mathfrak{M}^\bullet(W)$. Let $\mathcal{A} = H_*(Z^\bullet(W),\mathbb{C})$ be its Borel-Moore homology group, equipped with an algebra structure by the convolution (see [33, 14.2]). Taking direct sum with respect to V' in (8.7), we have a linear isomorphism (forgetting gradings)

$$\pi_*(\mathbb{C}_{\mathfrak{M}^\bullet(W)}) = \bigoplus_{V,\alpha} L_{V,\alpha} \otimes IC(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)^\alpha),$$

where $L_{V,\alpha} = \bigoplus_{[V'],k} L_{V,\alpha,k}(V',W)$. By a general theory (see [8] or [33, 14.2]), $\{L_{V,\alpha}\}$ is a complete set of mutually nonisomorphic simple $\mathcal{A}$-modules. Moreover, taking $H^*(i_{x_m}^!\ )$ of both sides, we have

$$H(\pi^{-1}(x_m),\mathbb{C})) = \bigoplus_{V,\alpha} L_{V,\alpha} \otimes H^*(i_{x_m}^! IC(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)^\alpha)),$$

which is an equality in the Grothendieck group of $\mathcal{A}$-modules. Here the $\mathcal{A}$-module structure on the right-hand side is given by $a : \xi \otimes \xi' \mapsto a\xi \otimes \xi'$.

By [33, §13], there exists an algebra homomorphism $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) \to \mathcal{A}$. Moreover [33, §14.3], each $L_{V,\alpha}$ is a simple l-highest weight $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module. Its Drinfeld polynomial is Q such that $\widehat{\Pi}(e^V e^W) = e^Q$. (It is possible to have two different V, V' give isomorphic $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules.) Combining this with the discussions above, we get Theorem 8.6.

Remark 8.11. If one enlarges the commutative subalgebra $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^0$ of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$, then one can recover a bijective correspondence between simple $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules and strata of affine quiver varieties. When $\mathfrak{g}$ is of type A_n, such an enlargement is $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{gl}_{n+1})$ (cf. [14]).

9. Specialization at $\varepsilon = \pm 1$

When $\varepsilon = \pm 1$, simple modules can be described explicitly [13, §4.8]. We study their $\widehat{\chi}_{\varepsilon,t}$ in this section.

Let P be an I-tuple of polynomials. We choose $I \times \mathbb{C}^*$-graded vector space W so that $e^W = e^P$ as before.

First consider the case $\varepsilon = 1$. The W can be considered as a collection of I-graded vector spaces $\{W^a\}_{a\in\mathbb{C}^*}$, where $W_i^a \overset{\text{def.}}{=} W_i(a)$. Then from the definition of cyclic quiver varieties, it is clear that

$$\mathfrak{M}^\bullet(W) \cong \prod_a \mathfrak{M}(W^a), \qquad \text{and} \qquad \mathfrak{M}_0^\bullet(\infty, W) \cong \prod_a \mathfrak{M}_0(\infty, W^a),$$

Here $\mathfrak{M}(W^a)$ and $\mathfrak{M}_0(\infty, W^a)$ are the original quiver varieties corresponding to W^a. Let P^a be an I-tuple polynomial defined by $P_i^a(u) = (1-au)^{\dim W_i(a)}$. The P^a is, of course, determined directly from P. From the above description, we have

$$M(P) = \bigotimes_a M(P^a), \qquad \widehat{\chi}_{\varepsilon,t}(M(P)) = \prod_a \widehat{\chi}_{\varepsilon,t}(M(P^a)). \tag{9.1}$$

The latter also follows directly from Axiom 3.

Next, consider the case $\varepsilon = -1$. We choose and fix a function $o\colon I \to \{\pm 1\}$ such that $o(i) = -o(j)$ if $a_{ij} \neq 0$, $i \neq j$. We define an I-graded vector space W^a by $W_i^a \overset{\text{def.}}{=} W_i(o(i)a)$. Then we have

$$\mathfrak{M}^\bullet(W) \cong \prod_a \mathfrak{M}(W^a), \qquad \mathfrak{M}_0^\bullet(\infty, W) \cong \prod_a \mathfrak{M}_0(\infty, W^a).$$

More precisely, $\mathfrak{M}^\bullet(V, W) = \prod_a \mathfrak{M}(V^a, W^a)$ with $V^a \overset{\text{def.}}{=} \bigoplus_i V_i(-o(i)a)$. Let P^a be an I-tuple of the polynomial defined by $P_i^a(u) = (1-o(i)au)^{\dim W_i(o(i)a)}$. The P^a is again determined directly from P. We have (9.1) also in this case.

Recall that we have an algebra homomorphism $\mathbf{U}_\varepsilon(\mathfrak{g}) \to \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ (1.2). By [27, §33], $\mathbf{U}_{-1}(\mathfrak{g})$ is isomorphic to $\mathbf{U}_1(\mathfrak{g})$. Moreover, the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ of $\mathfrak{g}$ is isomorphic to the quotient of $\mathbf{U}_1(\mathfrak{g})$ by the ideal generated by $q^h - 1$ ($h \in P^*$) [3, 9.3.10]. In particular, the category of type 1 finite dimensional $\mathbf{U}_\varepsilon(\mathfrak{g})$-modules is equivalent to the category of finite dimensional $\mathfrak{g}$-modules. Therefore we consider $\operatorname{Res} M(P)$ as a $\mathfrak{g}$-module.

Thanks to the fact that $\pi\colon \mathfrak{M}(W) \to \mathfrak{M}_0(\infty, W)$ is semismall, we have the following [33, §15]:

THEOREM 9.2. (1) $L(P) = \bigotimes_a L(P^a)$.

(2) *For each a, $\operatorname{Res}(L(P^a))$ is simple as a $\mathfrak{g}$-module. Its highest wight is $\Lambda^a = \sum_i \deg P_i^a \Lambda_i$.*

We want to interpret this result from $\widehat{\chi}_{\varepsilon,t}$. We may assume that there exists only one nontrivial P^a. All other P^b's are 1.

We identify an I-tuple of polynomials P whose roots are $1/a$ with a dominant weight by

$$P \mapsto \deg P \overset{\text{def.}}{=} \sum_i \deg P_i\, \Lambda_i, \qquad \lambda = \sum_i \lambda_i \Lambda_i \mapsto P_\lambda; (P_\lambda)_i(u) = (1-au)^{\lambda_i}.$$

We give an explicit formula of $Z_{PQ}(t)$, not based on inductive procedure:

Theorem 9.3. (1) *$Z_{PQ}(1)$ is equal to the multiplicity of the simple $\mathfrak{g}$-module $L(\deg Q)$ of highest weight $\deg Q$ in $\operatorname{Res} M(P)$.*

(2) *$c_{PQ}(t)$ is a polynomial in t^{-1}, so that $c_{PQ}(\infty)$ makes sense.*

(3) *We have* $\chi_{\varepsilon,t}(L(P)) = \sum_Q c_{PQ}(\infty)\, e^Q +$ *non l-dominant terms,*

$$\text{or equivalently } Z_{PQ}(t) = \sum_R c_{PR}(t) c^{RQ}(\infty).$$

(4) *The coefficient $c_{PQ}(\infty)$ is equal to the weight multiplicity of the dominant weight $\deg Q$ in $L(\deg P)$.*

Proof. (1) is clear.

(2), (3) By the fact that $\pi\colon \mathfrak{M}(W^a) \to \mathfrak{M}_0(\infty, W^a)$ is semismall, we have $L_{V,\alpha,k} = 0$ (in (8.7)) for $k \neq 0$, hence $L_{PQ}(t)$ (in (8.8)) is a constant. Then $c_{PQ}(t)$ is a polynomial in t^{-1} by (8.10) and (8.5a). Therefore $c_{PQ}(\infty)$ makes sense. We have $c_{PQ}(\infty) = L_{PQ}(t) = L_{PQ}(0)$ again by (8.10) and (8.5a). Thus we get the assertion.

(4) By (3), we have

$$\chi(L(P)) = \sum_Q c_{PQ}(\infty) e^{\deg Q} + \text{nondominant terms.}$$

Since χ is the ordinary character, the assertion is clear. $\square$

Note that the multiplicity of a simple (resp. Weyl) $\mathbf{U}_\varepsilon(\mathfrak{g})$-module in $\operatorname{Res} M(P)$ for generic (resp. a root of unity) ε is independent of ε. Therefore $Z_{PQ}(1)$ gives it. (See [34, §7].)

10. Conjecture

There is a large amount of literature on finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules. Some special classes of simple finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules are studied intensively: tame modules [37] and Kirillov-Reshetikhin modules [17] (see also the references therein). For tame modules, there are explicit formulae of χ_ε in terms of Young tableaux. For Kirillov-Reshetikhin modules, there are conjectural explicit formulae of χ (i.e., decomposition numbers of restrictions to $\mathbf{U}_\varepsilon(\mathfrak{g})$-modules).

Although our computation applies to *arbitrary* simple modules, our polynomials $Z_{PQ}(t)$ are determined recursively, and it is difficult to obtain explicit formulae in general. Thus those modules should have a very special feature among arbitrary modules. For Kazhdan-Lusztig polynomials, a special class is known to have explicit formulae. Those are Kazhdan-Lusztig polynomials for

Grassmannians studied in [24]. A geometric interpretation was given in [41]. Based on an analogy between Kazhdan-Lusztig polynomials and our polynomials, we propose a class of finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules. It is a class of *small* standard modules.

Definition 10.1. (1) A finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module M is called *special* if it satisfies the condition in Theorem 3.5(2), i.e., $\chi_\varepsilon(M)$ contains only one l-dominant monomial.

(2) Let $M(P)$ be a standard module with l-highest weight P. We say $M(P)$ is *small* if $c_{QR}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$ for any Q, $R \leq P$ with $Q \neq R$. Similarly $M(P)$ is called *semismall* if $c_{QR}(t) \in \mathbb{Z}[t^{-1}]$ for any $Q, R \leq P$.

Remark 10.2. (1) By the geometric definition of $\widehat{\chi}_{\varepsilon,t}$ (4.12), $M(P)$ is (semi)small if and only if $\pi\colon \mathfrak{M}^\bullet(V,W) \to \mathfrak{M}_0^\bullet(V,W)$ is (semi)small for any V such that $e^V e^W$ is l-dominant.

(2) By definition, $M(Q)$ is (semi)small if $M(P)$ is (semi)small and $Q \leq P$.

(3) The (semi)smallness of $M(P)$ is related to (semi)tightness of monomials in $\mathbf{U}_q^-$ [26].

Since a finite dimensional simple $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module contains at least one l-dominant monomial, namely the one corresponding to the l-highest weight vector, a special module is automatically simple. The converse is *not* true in general. For example, if $\mathfrak{g} = \mathfrak{sl}_2$, $P = (1-u)^2(1-\varepsilon^2 u)$, then one can compute (say, by our algorithm)

$$\chi_\varepsilon(L(P)) = Y_{1,1}^2 Y_{1,\varepsilon^2} + Y_{1,1} + Y_{1,1}^2 Y_{1,\varepsilon^4}^{-1} + 2Y_{1,1} Y_{1,\varepsilon^2}^{-1} Y_{1,\varepsilon^4}^{-1} + Y_{1,\varepsilon^2}^{-2} Y_{1,\varepsilon^4}^{-1}.$$

This has two l-dominant monomial terms.

THEOREM 10.3. *Suppose $M(P)$ is small. Then for any I-tuple of polynomials $Q \leq P$, the corresponding simple module $L(Q)$ is special.*

Proof. By the characterization of $Z_{QR}(t)$ in (8.5), we have $Z_{QR}(t) = c_{QR}(t)$ for all $Q, R \leq P$. Therefore

$$\sum_R Z_{QR}(t)\chi_{\varepsilon,t}(L(R)) = \chi_{\varepsilon,t}(M(Q)) = \sum_R Z_{QR}(t) e^R + \text{non } l\text{-dominant terms}.$$

Hence we have $\chi_{\varepsilon,t}(L(R))$ is e^R plus non l-dominant terms. $\square$

CONJECTURE 10.4. *Standard modules corresponding to tame modules and Kirillov-Reshetikhin modules are small.*

KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN
E-mail address: nakajima@math.kyoto-u.ac.jp

References

[1] T. Arakawa, Drinfeld functor and finite-dimensional representations of Yangian, *Comm. Math. Phys.* **205** (1999), 1–18.

[2] J. Beck, Braid group action and quantum affine algebras, *Comm. Math. Phys.* **165** (1994), 555–568.

[3] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge Univ. Press, Cambridge, 1994.

[4] ———, Quantum affine algebras and their representations, in *Representations of Groups* (Banff, AB, 1994), A.M.S., Providence, RI, 1995, 59–78.

[5] ———, Quantum affine algebras at roots of unity, *Represent. Theory* **1** (1997), 280–328.

[6] ———, Integrable and Weyl modules for quantum affine sl_2, in *QuantumGroups and Lie Theory* (Durham, 1999), 48–62, *London Math. Soc. Lecture Notes Series* **290**, Cambridge Univ., Press, Cambridge, 2001.

[7] V. Chari and N. Xi, Monomial bases of quantized enveloping algebras, in *Recent Developments in Quantum Affine Algebras and Related Topics*, *Contemp. Math.* **248**, A.M.S., Providence, RI, 1999, 69–81.

[8] N. Chriss and V. Ginzburg, *Representation Theory and Complex Geometry*, Birkhäuser, Boston, 1997.

[9] V. I. Danilov and A. G. Khovanskii, Newton polyhedra and an algorithm for calculating Hodge-Deligne numbers, *Math. USSR Izv.* **29** (1987), 279–298.

[10] P. Deligne, Théorie de Hodge. II, *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), 5–57; Théorie de Hodge. III, *op. cit.* **44**, (1974), 5–77.

[11] V. G. Drinfel'd, A new realization of Yangians and quantized affine algebras, *Soviet Math. Dokl.* **32** (1988), 212–216.

[12] E. Frenkel and E. Mukhin, Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras, *Comm. Math. Phys.* **216** (2001), 23–57; arXiv:math.QA/9911112.

[13] ———, The q-characters at roots of unity, *Adv. Math.* **171** (2002), 139–167; arXiv:math.QA/0101018.

[14] ———, The Hopf algebra $\operatorname{Rep} U_q\widehat{\mathfrak{gl}}_\infty$, *Selecta Math.* **8** (2002), 537–635; arXiv:math.QA/0103126.

[15] E. Frenkel and N. Reshetikhin, The q-characters of representations of quantum affine algebras and deformations of $\mathcal{W}$-algebras, in *Recent Developments in Quantum Affine Algebras and Related Topics* (Raleigh, NC, 1998), *Contemp. Math.* **248**, A.M.S., Providence, RI, 1999, 163–205.

[16] I. Grojnowski, Jantzen filtrations, preprint, 1996.

[17] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada, Remarks on fermionic formula, in *Recent Developments in Quantum Affine Algebras and Related Topics* (Raleigh, NC, 1998), *Contemp. Math.* **248**, A.M.S., Providence, RI, 1999, 243–291.

[18] K. Iohara and F. Malikov, Rings of skew polynomials and Gel'fand-Kirillov conjecture for quantum groups, *Comm. Math. Phys.* **164** (1994), 217–237.

[19] A. Joseph, Sur une conjecture de Feigin, *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995), 1441–1444.

[20] M. Kashiwara, Crystal bases of modified quantized enveloping algebra, *Duke Math. J.* **73** (1994), 383–413.

[21] M. Kashiwara, On level-zero representations of quantized enveloping algebras, *Duke Math. J.* **112** (2002), 117–175; arXiv:math.QA/0010293.

[22] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.

[23] H. Knight, Spectra of tensor products of finite-dimensional representations of Yangians, *J. Algebra* **174** (1995), 187–196.

[24] A. Lascoux and M.-P. Schützenberger, Polynômes de Kazhdan and Lusztig pour les grassmanniennes, *Astérisque* **87–88** (1981), 249–266.

[25] G. Lusztig, Canonical bases arising from quantized enveloping algebras, *J. Amer. Math. Soc.* **3** (1990), 447–498.

[26] ———, Tight monomials in quantized enveloping algebras, in *Quantum Deformations of Algebras and their Representations* (Ramat-Gan 1991/1992; Rehovot 1991/1992), *Israel Math. Conf. Proc.* **7**, 117–132, Bar-Ilan Univ., Ramat Gan, 1993.

[27] ———, *Introduction to Quantum Groups*, *Progr. Math.* **110**, Birkhäuser, Boston, 1993.

[28] ———, On quiver varieties, *Adv. Math.* **136** (1998), 141–182.

[29] ———, Fermionic form and Betti numbers, preprint, 2000; arXiv:math.QA/0005010.

[30] H. Nakajima, Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras, *Duke Math. J.* **76** (1994), 365–416.

[31] ———, Homology of moduli spaces instantons on ALE spaces I, *J. Differential Geom.* **40** (1994), 105–127.

[32] ———, Quiver varieties and Kac-Moody algebras, *Duke Math. J.* **91** (1998), 515–560.

[33] ———, Quiver varieties and finite-dimensional representations of quantum affine algebras, *J. Amer. Math. Soc.*, **14** (2001), 145–238.

[34] ———, T-analogue of the q-characters of finite dimensional representations of quantum affine algebras, in *Physics and Combinatorics*, 196–219 (Nagoya, 2000), World Sci. Publ., River Edge, NJ, 2001.

[35] ———, Quiver varieties and tensor products, *Invent. Math.* **146** (2001), 399–449; arXiv:math.QA/0103008.

[36] ———, t-analogs of q-characters of quantum affine algebras of type A_n, D_n, in *Combinatorial and Geometric Representation Theory* (Seoul, 2001), 141–160, *Contemp. Math.* **325**, A.M.S., Providence, RI, 2003.

[37] M. Nazarov and V. Tarasov, Representations of Yangians with Gelfand-Zetlin bases, *J. Reine Angew. Math.* **496** (1998), 181–212.

[38] M. Reineke, Feigin's map and monomial bases for quantized enveloping algebras, *Math. Z.* **237** (2001), 639–667.

[39] M. Varagnolo and E. Vasserot, Standard modules of quantum affine algebras, *Duke Math. J.* **111** (2002), 509–533; arXiv:math.QA/0006084.

[40] ———, Perverse sheaves and quantum Grothendieck rings, in *Studies in Memory of Issai Schur* (Chevaleret/Rehovot, 2000), 345–365, *Progr. Math.* **210**, Birkhäuser, Boston, 2003; arXiv:math.QA/0103182.

[41] A. V. Zelevinsky, Small resolutions of singularities of Schubert varieties, *Funct. Anal. Appl.* **17** (1983), 142–144.

(Received May 25, 2001)