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Reduction of the singularities of codimension one singular foliations in dimension three

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0. Introduction

The reduction of the singularities of a codimension one holomorphic foliation over an ambient space of dimension two has been achieved by Seidenberg in [26]. Here we give a complete answer to this problem over an ambient space of dimension three, as stated in the following theorem.

THEOREM (reduction to simple singularities). *Let X be a three-dimensional germ, around a compact analytic subset, of nonsingular complex analytic space. Let $\mathcal{F}$ be a holomorphic singular foliation of codimension one and D a normal crossings divisor on X. Then there is a morphism $\pi : X' \to X$ composition of a finite sequence of blowing-ups with nonsingular centers such that*:

(1) *Each center is invariant for the strict transform of $\mathcal{F}$ and has normal crossings with the total transform of D.*

(2) *The strict transform $\mathcal{F}'$ of $\mathcal{F}$ in X' has normal crossings with the total transform D' of D and it has at most "simple singularities" adapted to D'.*

This result has been announced in [8]. In [7], [9] we have proved a similar result, under the additional assumption of *nondicriticalness*. In this paper we treat in a unified way both dicritical and nondicritical foliations.

The statement of the Theorem of Reduction to Simple Singularities makes sense for ambient spaces X of any dimension n. Let us explain it.

Singular foliations of codimension one: A holomorphic singular foliation $\mathcal{F}$ of codimension one over X is locally given by a differential equation $\omega = 0$, where

$$\omega = \sum_{i=1}^{n} a_i(x)dx_i$$

is an integrable 1-form, that is $\omega \wedge d\omega = 0$, and the coefficients a_i have no common factor. The *singular locus* $\mathrm{Sing}\mathcal{F}$ is locally given by the common zeroes of the coefficients a_i. Hence $\mathrm{Sing}\mathcal{F}$ is a closed analytic subset of X of codimension at least two. At any nonsingular point $P \in X - \mathrm{Sing}\mathcal{F}$, the usual Frobenius theorem states that $\mathcal{F}$ is given by the differential equation $dx_1 = 0$, for certain coordinates defined in a suitable open set $U \ni P$. The *local pieces of leaf* $x_1 = \lambda$, for $\lambda \in \mathbb{C}$, can be pasted by connectedness with other local pieces of leaf: the *leaves* obtained in this way give to us the foliated structure of $X - \mathrm{Sing}\mathcal{F}$ given by $\mathcal{F}$.

Let us note that if $\phi = f/g$ is a meromorphic function, the differential equation $\phi\omega = 0$ is the same one as $\omega = 0$ outside of the zeroes and poles of ϕ. Then any integrable meromorphic 1-form determines a singular foliation up to multiplication by a suitable meromorphic function. This remark will be important in the description and control of singular foliations in this paper and in general we shall use local meromorphic 1-forms instead of holomorphic ones.

A closed analytic hypersurface H locally defined by a reduced equation $f = 0$ is an *integral hypersurface* of $\mathcal{F}$ if $\omega \wedge df$ is a holomorphic 2-form divisible by f. That is, the 1-forms ω and df define tangent co-vectors that are linearly dependent at the points of H, in particular $H \cap (X - \mathrm{Sing}\mathcal{F})$ is a leaf of $\mathcal{F}$. In the real two-dimensional case the integral curves are called *separatrices*: the dynamical behavior of the foliation around a singular point is organized in regions separated by them.

In the complex case, for $n = 2$ there is always at least one integral curve at any singular point as it was proved by Camacho and Sad in [5]. For $n \geq 3$, the same statement is true (see [9], [11]) under the additional assumption of *nondicriticalness*, to be explained below. The proofs of these results depend strongly on statements of reduction of singularities of foliations: the two-dimensional case [26], the nondicritical three-dimensional case [7], [9] and a generic equireduction in any ambient dimension [11].

We will always consider a normal crossings divisor D in the ambient space as an additional datum for the reduction of the singularities. It is a finite union of nonsingular hypersurfaces $D = \cup_{i=1}^{k} H_i$ that are locally "like coordinate hyperplanes" (we give the precise definition in Section 1.1). The divisor D comes in fact from the exceptional divisors of the blowing-ups in the process of reduction of singularities. We say that an irreducible component H_i of D is *nondicritical* if it is an integral hypersurface. The other ones, called *dicritical components,* are generically transversal to $\mathcal{F}$.

Blowing-ups: The main tool in the reduction of singularities are the blowing-up morphisms $\pi : X' \to X$ with nonsingular centers $Y \subset X$. Let us recall that π is a proper morphism between nonsingular ambient spaces that induces an isomorphism outside Y and the exceptional divisor $\pi^{-1}(Y)$. The *strict transform* $\mathcal{F}'$ of $\mathcal{F}$ by π is the foliation locally defined by the pull-back $\pi^*\omega$. We say that π is *nondicritical* if the exceptional divisor $\pi^{-1}(Y)$ is an integral hypersurface of $\mathcal{F}'$. Otherwise π is called *dicritical*. The blowing-up morphism modifies the singularities of $\mathcal{F}$ contained in the center Y. The aim of the reduction of singularities is to perform these transformations until we get the simplest possible kind of singularities.

We require some basic properties to the centers of the blowing-ups that we are going to use. The first one is that the center Y is tangent to $\mathcal{F}$. This means that either $Y \subset \mathrm{Sing}\mathcal{F}$ or $Y - \mathrm{Sing}\mathcal{F}$ is contained in a single leaf. In particular, the case of a center that is a point satisfies this property. Note that if we do not require tangency between Y and $\mathcal{F}$ we can perform *superfluous* dicritical blowing-ups. By example, take on $\mathbb{C}^3$ the nonsingular foliation $dz = 0$ and consider the transversal center $x = y = 0$ that cuts all the leaves: we get a "nonjustified" dicritical blowing-up. The second basic property is that Y has *normal crossings* with the divisor D. This allows us to consider a new normal crossings divisor $D' = \pi^{-1}(D \cup Y)$ on X'. In this way we can continue the blowing-up process.

Dicriticalness: A singular foliation $\mathcal{F}$ is *dicritical* if there is a finite sequence of blowing-ups with nonsingular invariant centers such that the last blowing-up is dicritical: that is, the last exceptional divisor is generically transversal to the strict transform of $\mathcal{F}$.

If $n = 2$, being dicritical is equivalent to the property of having infinitely many integral curves. Once we get a generically transversal exceptional divisor, we see transversal invariant curves through any point in the divisor, except for finitely many. They project by the sequence of blowing-ups over distinct-to-each-other invariant curves at the initial singular point. The converse of this statement is a consequence of Seidenberg's reduction of singularities in dimension two [26].

In ambient dimension $n \geq 3$, to verify the nondicriticalness property we need an infinite process if we do not dispose of a reduction of the singularities [6]. For instance, being dicritical does not mean to have infinitely many integral hypersurfaces: the foliation produces by transversality a codimension one foliation on a dicritical component, but the leaves are not necessarily closed and hence we do not have an integral hypersurface in the ambient space. In fact, there are dicritical singular foliations without integral hypersurfaces [9]. For the real case, in the paper [10] we have shown that the appearance of such dicritical components is important in order to understand transcendence properties of the leaves of the foliation.

Simple singularities in dimension two: Let us recall the definition of a simple singularity for the case of a two-dimensional ambient space. It was given in [26], but without the *adapted to the divisor* view point. A singularity is said to be *simple* or *elementary* if the foliation is locally given by a 1-form of the type

$$ydx - \lambda xdy + \text{ higher degree terms}$$

where λ is not a positive rational number. An important example is the foliation $pydx + qxdy = 0$, that corresponds to the level sets of the monomial $x^p y^q$. In particular, simple singularities are a generalization of functions locally given by a monomial. Note that getting locally a monomial is the main objective in the problem of reduction of singularities of varieties, both in zero and positive characteristic [15], [1], [28], [29], [4]. However for the saddle-nodes, that correspond to $\lambda = 0$, the leaves are far from being comparable to the level sets of a function. For instance, if we take Euler's equation

$$(y - x^2)dx - x^2 dy = 0$$

we have a formal integral curve $y = \sum_{k=1}^{\infty}(k!)x^{k+1}$ that is not convergent. Anyway, the behavior of simple singularities under blowing-up has many characteristics of a monomial $pydx + qxdy = 0$:

- The blowing-up is nondicritical.
- One gets exactly two singularities after blowing-up (that are simple ones).

As a formal consequence of these properties, we deduce that a simple singularity has exactly two (formal) integral curves Γ_1 and Γ_2 that correspond to the two points above and, in particular, they are nonsingular and transversal to each other.

If $\mathcal{F}$ is given by $bdx - ady = 0$, the leaves of the foliation are also the trajectories of the vector field $\xi = a\partial/\partial x + b\partial/\partial y$. Moreover, as explained in [23], in the case of a simple singularity, by formal jordanization of ξ we get formal coordinates x, y such that $\mathcal{F}$ is given by one of the *formal normal forms*:

a) $xy\left\{\frac{dx}{x} + \lambda\frac{dy}{y}\right\}$, with $p + q\lambda \neq 0$, for $p, q \in \mathbb{Z}_{>0}$.

b-1) $xy\left\{\frac{dx}{x} + \psi(x)\frac{dy}{y}\right\}$, where $\psi(0) = 0$.

b-2) $xy\left\{p\frac{dx}{x} + q\frac{dx}{x} + \psi(x^p y^q)\frac{dy}{y}\right\}$, where $\psi(0) = 0$.

Let us remark that the normal form a) is the pull-back of du/u under the multivalued function $u = xy^\lambda$ and the normal form b-2) is the pull back of the saddle node

$$\frac{du}{u} + \psi(u)\frac{dv}{v}$$

under the map $u = x^p y^q$, $v = y$.

The classical Seidenberg's theorem [26] states that after finitely many blowing-ups with center in the nonsimple singularities we get that the strict transform of the foliation has at most simple singularities. Note that in dimension two the singular points are isolated points and then we know exactly what center to choose: any singular point that is not a simple point.

Even in the relatively easy case where $n = 2$, it is interesting to consider the role of a normal crossings divisor D in the ambient space, to which we add each time the exceptional divisor of the blowing-up. Let E be the divisor of the nondicritical components of D and denote $e(E, P)$ the number of irreducible components of E passing through a point P. We say that $P \in \mathrm{Sing}\mathcal{F}$ is a *simple singularity adapted* to D if in addition to the property of being simple we have that $1 \leq e(D, P) \leq 2$. In the case $e(E, P) = 1$ we say that we have a *trace singularity*: there is exactly one integral curve of the foliation outside the divisor. If $e(E, P) = 2$ we have a *simple corner* and the integral curves at P are exactly the two irreducible components of the divisor.

On the other hand, let D^* be the union of the dicritical components, we say that $\mathcal{F}$ and D have normal crossings at P if either $P \notin D^*$ or there are local coordinates (x, y) such that $\mathcal{F}$ is given by $dx = 0$ and $D^* = \{y = 0\}$. Note that if $P \in D^*$, then necessarily P is a nonsingular point of $\mathcal{F}$.

Now we can state Seidenberg's result [26] as follows:

> Performing finitely many blowing-ups centered at (possibly nonsingular) points, the strict transform of the foliation has normal crossings with the divisor D and has at most simple singularities adapted to D.

The Theorem of Reduction to Simple Singularities in this paper corresponds to this statement in ambient dimension three.

Pre-simple singularities "versus" simple singularities in dimension two: The properties defining simple singularities split in two conditions:

(1) The condition that the linear part of the vector field $\xi = a\partial/\partial x + b\partial/\partial y$ is nonnilpotent. That is, there is at least one nonzero eigenvalue. To verify this it is enough to consider invariants as multiplicity of ideals, that we call geometrical invariants. The singularities that fulfill this condition are called *presimple singularities.*

(2) The nonresonance condition $\lambda \notin \mathbb{Q}_{>0}$. To get this property starting from a presimple singularity, we perform blowing-ups that will act on λ as in Euclid's algorithm.

Hence the reduction to simple singularities in an ambient space of dimension two splits two steps: first to get presimple singularities, second to *destroy* the resonances. This will also be the process in dimension three.

Simple and presimple singularities: In the last section of Chapter 1 we give the definition of *presimple singularity* in any ambient dimension. It corresponds to the two-dimensional property of having a nonnilpotent linear part. In Chapter 4 we give the precise definition of *simple singularity,* adding the necessary nonresonance conditions. In order to facilitate the reader's task we recall it in the Appendix. These definitions already appear in dimension three in [9]. For a simple singularity, there is a $\tau \leq n$ such that we can write a formal generator $\hat{\omega}$ of $\mathcal{F}$ in formal coordinates in one of the following ways:

a) There are $\lambda_i \in \mathbb{C}^*$ such that

$$\hat{\omega} = \left(\prod_{i=1}^{\tau} x_i\right) \sum_{i=1}^{\tau} \lambda_i \frac{dx_i}{x_i},$$

where $\sum_{i=1}^{\tau} m_i \lambda_i \neq 0$, for any nonzero vector $(m_i) \in (\mathbb{Z}_{\geq 0})^{\tau}$.

b-k) There are positive integers $p_1, \ldots, p_k$ and $\lambda_i \in \mathbb{C}$, such that

$$\hat{\omega} = \left(\prod_{i=1}^{\tau} x_i\right) \left\{ \sum_{i=1}^{k} p_i \frac{dx_i}{x_i} + \psi(x_1^{p_1} \cdots x_k^{p_k}) \sum_{i=2}^{\tau} \lambda_i \frac{dx_i}{x_i} \right\},$$

where $\sum_{i=k+1}^{\tau} m_i \lambda_i \neq 0$ for any nonzero vector $(m_i) \in (\mathbb{Z}_{\geq 0})^{\tau-k}$.

The *formal linear* case given in a) is the pullback of du/u under the multiform map $u = x_1^{\lambda_1} \cdots x_\tau^{\lambda_\tau}$. The *formally ramified saddle-node* cases of b-k) are the pullback of the saddle node

$$\frac{du}{u} + \psi(u)\frac{dv}{v}$$

under the map $u = x_1^{p_1} \cdots x_k^{p_k}$, $v = x_2^{\lambda_2} \cdots x_\tau^{\lambda_\tau}$.

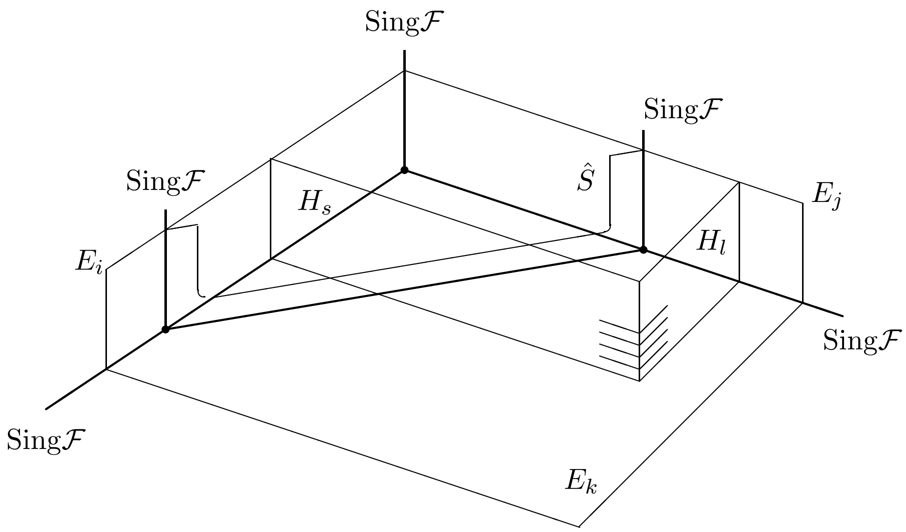
Normal crossings with a foliation: The number τ in the above definition is the *dimensional type* $\tau(\mathcal{F}, P)$: it is the minimum integer τ such that $\mathcal{F}$ is locally given by a 1-form in the first τ variables. That is, $\mathcal{F}$ is locally an analytic cylinder over a codimension-one foliation in an ambient space of dimension τ. The dimensional type is one for a nonsingular point, the singular points have at least dimensional type two. Let D^* be the union of the dicritical components of D. We say that the foliation $\mathcal{F}$ and the divisor D have *normal crossings* at P if there are coordinates $x_1, \ldots, x_n$ such that D^* is contained in $x_{\tau+1} \cdots x_n = 0$ and $\mathcal{F}$ is given by a 1-form in the variables $x_1, \ldots, x_\tau$. This definition is compatible with the above one in ambient dimension two, in particular, for a nonsingular point P this means that D^* has normal crossings with the unique integral hypersurface of $\mathcal{F}$ through P.

Adapted simple singularities: Denote $E = \mathrm{Nd}(D, \mathcal{F})$ the union of the nondicritical components of D and $\mathrm{e}(E, P)$ the number of irreducible components of E at P. Note that $\mathrm{e}(E, P) \leq \tau(\mathcal{F}, P)$. Assume that P is either a nonsingular point or a simple singularity. We say that P is *simple adapted to* D if we have that

$$\tau(\mathcal{F}, P) - 1 \leq \mathrm{e}(E, P) \leq \tau(\mathcal{F}, P).$$

Note that this condition holds at a nonsingular point.

The final picture: The only formal integral hypersurfaces of a simple singularity are the components of $x_1 \cdots x_\tau = 0$. In particular, the nondicritical divisor E is a union of some of these components. The *simple corners*, defined by the property $\mathrm{e}(E, P) = \tau(\mathcal{F}, P)$, have no integral hypersurfaces outside E. The *trace singularities*, with $\mathrm{e}(E, P) = \tau(\mathcal{F}, P) - 1$, have exactly one integral hypersurface not contained in the divisor. It is of a *transversely formal nature*, see [9], that allows to continue it along the divisor E to get a global object $\hat{S}$. Jointly with the normal crossings property with dicritical components, we can provide a picture of the final situation that we get after reduction of singularities:



In the picture we have represented three points of dimensional type 3, one of them is a simple corner, three nondicritical components E_i, E_j and E_k of the divisor, two dicritical components H_s and H_l and a part of a connected component of trace singularities, that supports an integral hypersurface $\hat{S}$.

Structure of the proof: We divide the proof of the theorem in two parts. The first part is the reduction to presimple singularities (Theorem 1) and most of this paper (Chapters 1, 2 and 3) is devoted to it. The centers that we use in the reduction to presimple singularities are *permissible centers* that are contained in the singular locus Sing$\mathcal{F}$. The second part (Theorem 3) is the passage from presimple singularities to simple ones and finally to the property of normal crossings between the foliation and the exceptional divisor. It is only in this last step that we use invariant centers which may not be contained in the singular locus.

Let us describe the main ideas used in the reduction to presimple singularities. We denote by $\mathrm{Sing}^*(\mathcal{F}, D)$ the set of points that are not presimple singularities. It is an analytic set. The objective is to make it disappear. As in most of the results on reduction of singularities, we organize our proof in the following steps:

(1) Description of the *invariants* used to measure the complexity of the singularity and to determine the permissible centers to be used.

(2) To give a *global strategy of blowing-up* that allows us to choose the next center, in a global way, provided we still have points in $\mathrm{Sing}^*(\mathcal{F}, D)$.

(3) To provide *local control* that implies that the global sequence of blowing-ups must end after finitely many steps.

Invariants: The local invariants for $\mathcal{F}$ at a point P are defined in an adapted way with respect to the nondicritical divisor E. If E is locally given by the equation $\prod_{i\in A} x_i = 0$, then a local generator ω of $\mathcal{F}$ is written as

$$\omega = \left(\prod_{i\in A} x_i\right)\left\{\sum_{i\in A} b_i(x_1,\ldots,x_n)\frac{dx_i}{x_i} + \sum_{i\notin A} b_i(x_1,\ldots,x_n)dx_i\right\}.$$

The first invariants are the *adapted order* $r = \nu(\mathcal{F}, E; P)$, that is the minimum of the orders of the coefficients b_i, and the *adapted multiplicity* $m = \mu(\mathcal{F}, E; P)$ defined as the minimum of the orders of b_i, for $i \in A$ and $x_i b_i$, for $i \notin A$. They have been already used in [7]. We put $m^* = m + 1$ in the case that $m = r$ and there is a *resonance* with integer coefficients between the parts of degree r of b_i, for $i \in A$, otherwise we put $m^* = m$. The pair (r, m^*) is the main invariant of control as the Hilbert-Samuel function is in the case of varieties in characteristic zero. Note that in the nondicritical case [7] it was enough to use the invariant (r, m).

We distinguish two kinds of invariants: the invariants of transversality or contact and the resonance invariants. The resonances give in fact a pathological behaviour that has many parallels with the situations arising in the reduction of singularities in positive characteristic [12], [28]. Among the contact invariants let us mention the *directrix* that plays a similar role to Hironaka's strict tangent space. The contact invariants are defined in fact for any dimension and in terms of coherent ideals, but they are not necessarily upper-semicontinuous, since the definition of the ideal depends on the point. Anyway, the equimultiplicity of certain ideals gives us the definition of *permissible center* (semicontinuous) and the more restrictive notion of *appropriate center* (nonsemicontinuous).

Our invariants, lexicographically ordered as usual, will exhibit *vertical stability*; that is, they do not increase under the kind of blowing-up that we are going to use, but may not exhibit *horizontal stability*. In Hironaka's strategies [15], [2], [3] the invariants should simultaneously have both types of good behaviour. In our situation this is not possible, neither for the control invariants nor for the definition of permissible centers. We give a simultaneous horizontal-vertical control for the generic part of $\mathrm{Sing}^*(\mathcal{F}, D)$, that we call *good points* (this terminology is inspired by [1] but it has not the same meaning). The control of the *bad points* is done just in a vertical way.

Global strategy of blowing-up: The *good points* are reduced in an essentially two-dimensional way and they are stable under the global blowing-ups we do. The *bad points* are finitely many and our strategy is concentrated in the destruction of them. We do it step by step, by looking at the maximum invariant (r, m^*) over the bad points. We select the kind of blowing-ups we want to perform, with centers contained in the so-called *influence locus*, by the main property that they are globally permissible and also they are appropriate at the bad points. We need some global preparations to insure that the influence locus has *weak normal crossings* and to avoid the existence of cycles formed with our future centers and with nodes in the bad points having the fixed invariant (r, m^*).

After this, our main *global criterion of blowing-up* will always work unless we have destroyed the bad points with the fixed (r, m^*). In this way we create a possibly infinite sequence of global blowing-ups. It is enough now to prove that is is always a finite sequence. If it is not, we get an infinite chain (or *bamboo*) of bad points, each obtained from the preceding one by a local blowing-up respecting certain *local rules*. The hardest result in this paper is the local control theorem (Theorem 2) that says that such an infinite bamboo cannot exist.

In our global strategy we choose mainly the centers having maximal dimension, but with some priorities (related with the resonant cases) very specific for dicritical foliations: destroy first the *radially dicritical points*, that give a dicritical component after one quadratic blowing-up.

Local control: It corresponds to the proof of Theorem 2 stated in Section 2.5. We have to prove that a vertical sequence $\mathcal{S}$ of bad points respecting the local rules cannot exist. We use all our vertical invariants and we divide the study of the sequence $\mathcal{S}$ in two cases:

(1) *The m-stable case.* The adapted multiplicity m is stable in the sequence. To control this situation we use the notions of differential idealistic exponent and vertical maximal contact, inspired in the analogous ideas introduced by Hironaka. In that way we are able to project our problem in an essentially two-dimensional one. Let us remark the vertical character of our maximal contact surfaces, in contrast with the horizontal-vertical properties of the classical maximal contact theory [2]. We also need accurate control of the characteristic polygons associated to a foliation, both to get maximal contact in the cases we cannot get it directly and to develop new invariants. The resonances in the vertices of the characteristic polygon play an important role in this case.

(2) *The jumping situation.* The adapted multiplicity is not stable. This case is very specific for dicritical foliations. We deal with it by describing a partially ordered set of twelve levels and give an algorithm for going from higher levels to lower ones. These levels are defined by combining our vertical invariants with secondary resonances.

Destroying resonances: The passage from presimple singularities to simple ones (and getting the normal crossings property, Theorem 3) is easier that the previous results. We need to destroy the resonances. Essentially, starting with a form of the type

$$\sum_{i=1}^{n} \lambda_i \frac{dx_i}{x_i}$$

such that $\sum p_i \lambda_i = 0$ for some nonnegative integers p_i, we have to get a form of the same type, but without this resonant property. To do this we use *resonant centers*. The problem has an easy solution in dimension three, but not in general (it is closely related to the study of toric varieties).

General comments: The main difference between the proof presented in this paper and the one for the nondicritical case in [7] is that we have to deal with resonances of several types, most of them not present in the nondicritical situations. Moreover, the nonresonant dicriticalness (appearing in the study of good points) is responsible for the failure of the semi-continuity of the property of being an appropriate center. Hence we need to separate clearly the generic reduction of the singularities, simultaneously controlled in a horizontal and vertical way and the reduction at the bad or specific points, controlled just vertically.

The local control is the main obstruction to generalizing our proof to ambient dimension n. It should be possible to imagine reasonable global strategies, inspired for instance in the positive characteristic results for varieties [28], but it is not clear how to generalize the differential idealistic exponents, the different levels in the jumping situation and mainly the use of the characteristic polygon that becomes very difficult in higher dimensions [19].

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1. Blowing-up singular foliations

1.1. *Adapted singular foliations.* Let X be a nonsingular complex analytic space of dimension n; it is the *ambient space*. Consider a normal crossings divisor D on X. Note that D is locally given at a point $P \in X$ by

$$D = \left(\prod_{i \in A} x_i = 0 \right)$$

for some coordinates $(x_1,\dots,x_n)$ and a set $A\subset\{1,\dots,n\}$ of $e=\mathrm{e(D,P)}$ elements, where e is the number of global irreducible components of D through P. Call these coordinates *adapted to* D. Denote by $\Omega_X[D]$ the sheaf of germs of logarithmic 1-forms along D. A local basis at P is given by:

$$\left\{\frac{dx_i}{x_i}\right\}_{i\in A}\cup\{dx_i\}_{i\notin A}.$$

A *singular integrable hyperplane field adapted to* D is an invertible submodule $\mathcal{H}$ of $\Omega_X[D]$, locally generated at each point P by a 1-form

$$\omega=\sum_{i\in A}a_i\frac{dx_i}{x_i}+\sum_{i\notin A}a_idx_i;\ a_i\in\mathcal{O}_{X,P}$$

satisfying the integrability condition $\omega\wedge d\omega=0$. We say that $\mathcal{H}$ is a *singular foliation adapted to* D if and only if it is saturated in the sense that the quotient $\Omega_X[D]/\mathcal{H}$ has no torsion (this locally means that $\gcd\{a_i\}_{i=1}^n=1$). Denote by $\mathbb{H}(\mathbb{X},\mathbb{D})$, respectively by $\mathbb{F}(\mathbb{X},\mathbb{D})$, the set of singular integrable hyperplane fields, respectively singular foliations, adapted to D. In the case $D=\emptyset$ we put $\mathbb{F}(\mathbb{X},\emptyset)=\mathbb{F}(\mathbb{X})$ and get the set of (usual) singular foliations of codimension one over X. There is a map

$$\mathrm{Sat}(\_,D):\mathbb{H}(X,D)\to\mathbb{F}(X,D)$$

(where "Sat" stands for "saturation") defined by the property $\mathrm{Sat}(\mathcal{H},D)\supset\mathcal{H}$. Actually

$$\mathcal{H}=\mathcal{J}_{(\mathcal{H},D)}\mathrm{Sat}(\mathcal{H},D)$$

where $\mathcal{J}_{(\mathcal{H},\mathcal{D})}$ is the ideal sheaf locally generated by $\gcd\{a_i\}_{i=1}^n$.

Given another normal crossings divisor $D'\subset D$, we have $\mathbb{H}(\mathbb{X},\mathbb{D}')\subset\mathbb{H}(\mathbb{X},\mathbb{D})$, but in general $\mathbb{F}(\mathbb{X},\mathbb{D}')\not\subset\mathbb{F}(\mathbb{X},\mathbb{D})$ (for example, take $D'=\emptyset$, $D=\{x=0\}$ and $\mathcal{H}$ locally given by dx). Anyway, the map $\mathrm{Sat}(\_,D)$ provides a bijection between $\mathbb{F}(\mathbb{X},\mathbb{D}')$ and $\mathbb{F}(\mathbb{X},\mathbb{D})$. In particular to any $\mathcal{H}\in\mathbb{H}(\mathbb{X},\mathbb{D})$ we can associate a singular foliation $\mathrm{Fol}(\mathcal{H})\in\mathbb{F}(\mathbb{X})$ such that $\mathrm{Sat}(\mathrm{Fol}(\mathcal{H}),D)=\mathrm{Sat}(\mathcal{H},D)$.

Recall that an irreducible germ $f\in\mathcal{O}_{X,P}$ defines a germ of *integral hypersurface* of a singular foliation $\mathcal{F}\in\mathbb{F}(X)$ if and only if f divides $\omega\wedge df$, where ω is a local generator of $\mathcal{F}$. We define in the same way *formal integral hypersurfaces*. A global irreducible hypersurface $H\subset X$ is an *integral hypersurface* of $\mathcal{F}$ if and only if each local irreducible component of the germ of H at any point is an integral hypersurface (by connectedness, it is enough to verify this at a single point of H). In the case of a normal crossings divisor we use special terminology. We call *dicritical components* of D the irreducible components that are not integral hypersurfaces; the *nondicritical components* are the ones which are integral hypersurfaces. Denote by $\mathrm{Nd}(D,\mathcal{F})$ the normal crossings divisor on X which is the union of the nondicritical components of D. Then

$$\mathrm{Sat}(\mathcal{F},D)=\mathrm{Sat}(\mathcal{F},\mathrm{Nd}(D,\mathcal{F})).$$

In local terms, assume that $\mathcal{H} = \mathrm{Sat}(\mathcal{F}, D)$ is given by ω as above and that

$$\mathrm{Nd}(D,\mathcal{F}) = \left(\prod_{i\in A^*} x_i = 0\right) \subset \left(\prod_{i\in A} x_i = 0\right) = D.$$

Then $\mathcal{F}$ is generated by the holomorphic 1-form $\eta = (\prod_{i\in A^*} x_i)\omega$.

Denote by $\Theta_X[D]$ the sheaf of germs of holomorphic vector fields that are tangent to D. A local basis of $\Theta_{X,P}[D]$ is given by

$$\left\{x_i\frac{\partial}{\partial x_i}\right\}_{i\in A} \cup \left\{\frac{\partial}{\partial x_i}\right\}_{i\notin A}.$$

Moreover, there is a perfect pairing $\Omega_X[D]\times\Theta_X[D] \to \mathcal{O}_X$. For any $\mathcal{H} \in \mathbb{H}(\mathbb{X},\mathbb{D})$ we get a coherent ideal sheaf $\mathcal{U}(\mathcal{H},D)\subset\mathcal{O}_X$ defined by

$$\mathcal{U}(\mathcal{H},D) = \{\alpha(\mathcal{X});\ \alpha\in\mathcal{H},\ \mathcal{X}\in\Theta_X[D]\}.$$

It is locally generated by the coefficients a_i of a generator ω of $\mathcal{H}$. Given an irreducible analytic subspace Y of X, the *adapted order* $\nu(\mathcal{H},D;Y)$ is the Y-adic order of $\mathcal{U}(\mathcal{H},D)$ along Y. The map that assigns to each point $P\in X$ the adapted order $\nu(\mathcal{H},D;P)\in\mathbb{N}$ is analytically upper-semicontinuous, since the ideal $\mathcal{U}(\mathcal{H},D)$ is coherent (actually, it is locally given by the minimum of the orders of the coefficients a_i, for $i=1,\dots,n$). Then, for any $r\in\mathbb{Z}$ the set

$$\mathrm{S}_r(\mathcal{H},D) = \{P\in X;\ \nu(\mathcal{H},D;P)\geq r\}$$

is a closed analytic subset of X. Note that $\mathcal{H}$ is saturated if and only if $\mathrm{codim}_X \mathrm{S}_1(\mathcal{H},D)\geq 2$. For a singular foliation $\mathcal{F}$ we denote $\mathrm{Sing}\mathcal{F} = \mathrm{S}_1(\mathcal{F},\emptyset)$ and call this set the *singular locus* of $\mathcal{F}$. We also adopt the notation

$$\nu(\mathcal{F},D;Y) = \nu(\mathrm{Sat}(\mathcal{F},D),D;Y)$$

and call it *the adapted order of* $\mathcal{F}$ *along* Y. Given a germ of logarithmic 1-form $\omega\in\Omega_{X,P}[D]$ at a point $P\in Y$ as above, we define the adapted order $\nu(\omega,D;Y)$ by

$$\nu(\omega,D;Y) = \min\{\nu_{\mathcal{J}}(a_i);\ i=1,\dots,n\}$$

where $\mathcal{J} = \mathcal{J}_{Y,P}\subset\mathcal{O}_{X,P}$ is the stalk at P of the ideal sheaf defining Y. Obviously $\nu(\mathcal{H},D;Y) = \nu(\omega,D;Y)$ if ω locally generates $\mathcal{H}$.

Let us define the *dimensional type* $\tau(\mathcal{F},P)$ of a singular foliation $\mathcal{F}$ at a point $P\in X$. Denote by $\mathcal{D}(\mathcal{F})$ the $\mathcal{O}_X$-submodule of Θ_X given by the germs of vector fields that are tangent to $\mathcal{F}$ and by $\mathcal{D}(\mathcal{F})(P)$ the $\mathbb{C}$-vector subspace of T_PX formed by the $\mathcal{X}(P)$, where $\mathcal{X}\in\mathcal{D}(\mathcal{F})$. We define

$$\tau(\mathcal{F},P) = \dim_{\mathbb{C}} T_PX/\mathcal{D}(\mathcal{F})(P).$$

Note that $\mathcal{D}(\mathcal{F})\subset\Theta_X[\mathrm{Nd}(D,\mathcal{F})]$ for any normal crossings divisor D. If $\tau = \tau(\mathcal{F},P)$ there are local coordinates $(x_1,\dots,x_n)$ adapted to $\mathrm{Nd}(D,\mathcal{F})$ and a local generator of $\mathcal{F}$ at P given by

$$\sum_{i=1}^{\tau} a_i(x_1, \ldots, x_\tau) dx_i.$$

The foliation $\mathcal{F}$ is then locally trivial (an analytic cylinder) over a foliation on $(\mathbb{C}^\tau, 0)$.

1.2. *Permissible centers.* Let Y be a connected nonsingular analytic subspace of X that has normal crossings with the divisor D. That is, locally at each $P \in X$ there are coordinates $(x_1, \ldots, x_n)$ and two sets $A, B \subset \{1, \ldots, n\}$ such that

$$D = \left(\prod_{i\in A} x_i = 0\right);\ Y = (x_i = 0;\ i \in B)\,.$$

Call these coordinates *adapted to* D, Y. Let $\pi : X' \to X$ be the blowing-up of X with center Y. Put $D' = \pi^{-1}(D \cup Y)$. It is a normal crossings divisor on X'.

Given $\mathcal{H} \in \mathbb{H}(X, D)$, define the *total transform* $\pi^{-1}\mathcal{H}$ to be the element of $\mathbb{H}(X', D')$ locally given by the pull back $\pi^*\omega$ of a local generator ω of $\mathcal{H}$. The *multiplicity* $\mu(\mathcal{H}; Y)$ of $\mathcal{H}$ along Y is defined by

$$\mu(\mathcal{H}; Y) = \nu_{\pi^{-1}(Y)}\left(\mathcal{J}_{(\pi^{-1}\mathcal{H}, D')}\right) = \nu_{\pi^{-1}(Y)}\left(\mathcal{U}(\pi^{-1}\mathcal{H}, D')\right).$$

It depends only on $\mathcal{H}$ and not on the particular divisor such that $\mathcal{H} \in \mathbb{H}(X, D)$. To see this, compute it at a point $P' \in \pi^{-1}(Y)$ not in the strict transform of the components of D. We sometimes use the notation $\mu(\omega, D; Y) = \mu(\mathcal{H}; Y)$ for a local generator ω of $\mathcal{H}$.

If $\mathcal{F} \in \mathbb{F}(X)$, define the *adapted multiplicity* $\mu(\mathcal{F}, D; Y)$ by

$$\mu(\mathcal{F}, D; Y) = \mu\left(\mathrm{Sat}(\mathcal{F}, D); Y\right).$$

The foliation $\mathcal{F}' = \mathrm{Fol}(\pi^{-1}\mathcal{F}) = \mathrm{Fol}(\pi^{-1}\mathrm{Sat}(\mathcal{F}, D))$ is called the *strict transform of* $\mathcal{F}$ by the blowing-up π. It is the only singular foliation over X' such that

$$\mathcal{F}' \mid_{X' - \pi^{-1}(Y)} = \mathcal{F} \mid_{X-Y}$$

under the isomorphism $\pi : X' - \pi^{-1}(Y) \to X - Y$. Note also that

$$\mathrm{Sat}(\mathcal{F}', D') = \left(\mathcal{J}_{\pi^{-1}(Y)}\right)^{-\mu(\mathcal{F}, D; Y)} \pi^{-1}\mathrm{Sat}(\mathcal{F}, D)$$

where $\mathcal{J}_{\pi^{-1}(Y)}$ is the ideal sheaf defining the exceptional divisor $\pi^{-1}(Y)$.

We say that the blowing-up is *dicritical for the singular foliation* $\mathcal{F}$ if and only if the exceptional divisor $\pi^{-1}(Y)$ is a dicritical component for $\mathcal{F}'$; otherwise, we say that π is a nondicritical blowing-up. Denote the strict transform of D by D^*: thus $D' = D^* \cup \pi^{-1}(Y)$. Then the blowing-up π is dicritical if and only if

$$\mathrm{Sat}(\mathcal{F}', D') \in \mathbb{H}(X', D^*).$$

In fact, in this case we have that $\mathrm{Sat}(\mathcal{F}', D') = \mathrm{Sat}(\mathcal{F}', D^*)$.

Let us give another interpretation of the multiplicity $\mu(\mathcal{H}; Y)$. Denote by $\Theta_X[Y]$ the sheaf of the germs of vector fields that are tangent to Y and let $\mathcal{U}(\mathcal{H}, D, Y)$ be the ideal sheaf given by the image of

$$\mathcal{H} \times (\Theta_X[Y] \cap \Theta_X[D]) \to \mathcal{O}_X.$$

The ideal sheaf $\mathcal{U}(\pi^{-1}\mathcal{H}, D')$ is the total transform of $\mathcal{U}(\mathcal{H}, D, Y)$ by π (compute on the generators). This implies that $\nu_Y \mathcal{U}(\mathcal{H}, D, Y) = \nu_{\pi^{-1}(Y)} \mathcal{U}(\pi^{-1}\mathcal{H}, D')$ and thus

$$\mu(\mathcal{H}; Y) = \nu_Y \mathcal{U}(\mathcal{H}, D, Y).$$

Consider a point $P \in Y$. Locally at P, take a normal crossings divisor $\widetilde{D} \supset D$ such that $\mathrm{e}(\widetilde{D}, P) = n$ and Y is the intersection of some irreducible components of $\widetilde{D}$. Then $\Theta_X[Y] \supset \Theta_X[\widetilde{D}]$ and hence $\mathcal{U}(\mathcal{H}, \widetilde{D}, Y) = \mathcal{U}(\mathcal{H}, \widetilde{D})$. Thus $\mu(\mathcal{H}; Y)$ may be computed as the adapted order $\nu(\mathcal{H}, \widetilde{D}; Y)$, that is

$$\mu(\mathcal{H}; Y) = \nu_Y \left(\mathcal{U}(\mathcal{H}, \widetilde{D})\right) = \nu(\mathcal{H}, \widetilde{D}; Y).$$

Note also that $\mathcal{U}(\mathcal{H}, D) \supset \mathcal{U}(\mathcal{H}, D, Y) \supset \mathcal{U}(\mathcal{H}, \widetilde{D})$. Putting $r = \nu(\mathcal{H}, D; P)$, we get

$$\begin{aligned} r &\leq \nu_P \left(\mathcal{U}(\mathcal{H}, D, Y)\right) \\ &\leq \mu(\mathcal{H}; P) = \nu_P \left(\mathcal{U}(\mathcal{H}, \widetilde{D})\right) = \nu(\mathcal{H}, \widetilde{D}; P) \leq r + 1. \end{aligned}$$

Remark 1. The map $P \mapsto \mu(\mathcal{H}; P)$ is not necessarily upper-semicontinuous. Take $D = (x = 0)$ and $\mathcal{H}$ given by

$$y^2 \frac{dx}{x} + z^2 dy - 2yzdz.$$

Then $\mu(\mathcal{H}, (0,0,0)) = 2$ and $\mu(\mathcal{H}, (\lambda, 0, 0)) = 3$ if $\lambda \neq 0$.

Remark 2. We know that $\nu_Y \mathcal{U}(\mathcal{H}, D; Y) = \nu_Y \mathcal{U}(\mathcal{H}, D^\sharp; Y)$ for $D^\sharp \supset D$, but the ideals $\mathcal{U}(\mathcal{H}, D; Y)$ and $\mathcal{U}(\mathcal{H}, D^\sharp; Y)$ are not necessarily equal to each other. For example, take $D = \emptyset$, $D^\sharp = (x = 0)$, $Y = (y = z = 0)$ and $\mathcal{H}$ given by dx. Then $\mathcal{U}(\mathcal{H}, D; Y) = \mathcal{O}_X$ and $\mathcal{U}(\mathcal{H}, D^\sharp; Y) = x\mathcal{O}_X$.

We take the equimultiplicity of $\mathcal{U}(\mathcal{H}, D, Y)$ along Y as the main property in order to define permissible and appropriate centers.

Definition 1. Consider a singular foliation $\mathcal{F}$ over X, a normal crossings divisor $D \subset X$ and a closed irreducible analytic subspace Y of X. Assume that Y is tangent to $\mathcal{F}$. Put $\mathcal{H} = \mathrm{Sat}(\mathcal{F}, D)$. We say that Y is a *permissible* *center for $\mathcal{F}$ adapted to D* at a point $P \in Y$ if and only if Y is nonsingular, has normal crossings with D at P and

$$\nu_P \mathcal{U}(\mathcal{H}, D, Y) = \nu_Y \mathcal{U}(\mathcal{H}, D, Y).$$

(Note that $\nu_Y \mathcal{U}(\mathcal{H}, D, Y) = \mu(\mathcal{H}, Y) = \mu(\mathcal{F}, D; Y)$.) We say that Y is an *appropriate center* at P if, in addition, we have that

$$\mu(\mathcal{F}, D; Y) = \mu(\mathcal{F}, D; P).$$

A permissible center is a center permissible at every point.

Remark 3. Put $E = \mathrm{Nd}(D, \mathcal{F})$. Assume that Y has normal crossings with D at P. Then Y is an appropriate center for $\mathcal{F}$ at P adapted to D if and only if it is an appropriate center for $\mathcal{F}$ at P adapted to E. To see this note that $\mu(\mathcal{F}, D; P) = \mu(\mathcal{F}, E; P)$ and $\mu(\mathcal{F}, D; Y) = \mu(\mathcal{F}, E; Y)$ in view of the above remark.

Remark 4. Assume that Y is tangent to $\mathcal{F}$ (this is a global condition that one can verify at a single point of Y). Then Y is a permissible center outside a closed analytic subset $Y' \neq Y$. Nevertheless, the condition for appropriate centers is nonopen inside Y. In the example of the above remark $Y = (y = z = 0)$ is a permissible center that is appropriate only at the origin.

PROPOSITION 1. *Let $\pi : X' \to X$ be the blowing-up with a permissible center Y for $\mathcal{F}$ adapted to D. Denote by $\mathcal{F}'$ the strict transform of $\mathcal{F}$. Put $D' = \pi^{-1}(D \cup Y)$ and consider a point $P \in Y$. Then there is a point $Q' \in \pi^{-1}(P)$ with $\nu(\mathcal{F}', D'; Q') = 0$. For any point $P' \in \pi^{-1}(P)$ we have that $\nu(\mathcal{F}', D'; P') \leq \nu(\mathcal{F}, D; P)$. Moreover, if $e(D', P') \geq e(D, P)$ then $\mu(\mathcal{F}', D'; P') \leq \mu(\mathcal{F}, D; P)$.*

Proof. Denote

$$\mathcal{H} = \mathrm{Sat}(\mathcal{F}, D), \qquad \mathcal{H}' = \mathrm{Sat}(\pi^{-1}\mathcal{H}, D') = \mathrm{Sat}(\mathcal{F}', D'),$$
$$r = \nu(\mathcal{F}, D; P), \qquad \alpha = \nu_P \mathcal{U}(\mathcal{H}, D, Y) = \nu_Y \mathcal{U}(\mathcal{H}, D, Y).$$

We have either $\alpha = r$ or $\alpha = r + 1$. Recall that

$$\mathcal{U}(\mathcal{H}', D') = \mathcal{J}^{-\alpha}_{\pi^{-1}(Y)} \mathcal{U}(\pi^{-1}\mathcal{H}, D') = \mathcal{J}^{-\alpha}_{\pi^{-1}(Y)} \pi^{-1} \mathcal{U}(\mathcal{H}, D, Y).$$

We know that $\mathcal{U}(\mathcal{H}, D, Y)$ is equimultiple along Y at P with multiplicity α. In particular, there is an element $h \in \mathcal{U}(\mathcal{H}, D, Y)$ that is equimultiple along Y with multiplicity α, locally at P. Then, recalling that $\mathcal{U}(\mathcal{H}', D') = \mathcal{J}^{-\alpha}_{\pi^{-1}(Y)} \pi^{-1} \mathcal{U}(\mathcal{H}, D, Y)$, we get a point $Q' \in \pi^{-1}(P)$, such that $\nu_{Q'} \mathcal{U}(\mathcal{H}', D') = 0$ and for any point $P' \in \pi^{-1}(P)$ we have that $\nu_{P'} \mathcal{U}(\mathcal{H}', D') \leq \alpha$. This ends the case $\alpha = r$.

Assume that $\alpha = r + 1$. Take a generator ω of $\mathcal{H}$. There is a germ of a vector field $\mathcal{X} \in \Theta_X[D]$ such that if $f = \omega(\mathcal{X})$ then $\nu_P(f) = r$. For any $g \in \mathcal{J}_Y$ we have $gf = \omega(g\mathcal{X}) \in \mathcal{U}(\mathcal{H}, D, Y)$. Pick $g \in \mathcal{J}_Y$ such that $\nu_{P'}(\mathcal{J}^{-1}_{\pi^{-1}(Y)}(g \circ \pi)) = 0$. We conclude that

$$\nu_{P'}\left(\mathcal{J}^{-\alpha}_{\pi^{-1}(Y)}(gf \circ \pi)\right) = \nu_{P'}\left(\mathcal{J}^{-r}_{\pi^{-1}(Y)}(f \circ \pi)\right) \leq r$$

and thus $\nu_{P'}\mathcal{U}(\mathcal{H}', D') \leq r$. This proves the first part. If $m = \mu(\mathcal{F}, D; P) = r + 1$, the second part is obvious. Assume that $m = r$, and hence $\alpha = r$. Change locally D by $\widetilde{D}$ with $\mathrm{e}(\widetilde{D}, P) = n$ and $\mathrm{e}(\widetilde{D}', P') = n$. Since $m = r$, the center Y is also permissible for $\mathcal{F}$ adapted to $\widetilde{D}$. Thus,

$$r \geq \nu(\mathcal{F}', \widetilde{D}'; P') = \mu(\mathcal{F}', \widetilde{D}'; P') = \mu(\mathcal{F}', D'; P')$$

and the proof is finished. □

We will need results on the "vertical stability" of being appropriate and permissible for a curve that we summarize below.

Fix an irreducible curve Γ on X and a point $P \in \Gamma$. Assume that Γ is tangent to the singular foliation $\mathcal{F}$. Recall that the sequence $\{P_i \in X_i\}$ of *infinitely near points* of Γ at P is obtained by blowing-up $\pi_{i+1} : X_{i+1} \to X_i$ centered at the only point P_i over P_{i-1} in the strict transform Γ_i of Γ_{i-1} by π_{i-1}; we start with $X_0 = X$, $\Gamma_0 = \Gamma$ and $P_0 = P$. Given a normal crossings divisor D, let us denote $D_{i+1} = \pi^{-1}(D_i \cup \{P_i\})$ and $D_0 = D$. Also denote by $\mathcal{F}_{i+1}$ the strict transform of $\mathcal{F}_i$ by π_{i+1} and $\mathcal{F}_0 = \mathcal{F}$. Put $\mathcal{H}_i = \mathrm{Sat}(\mathcal{F}_i, D_i)$ and $m_i = \mu(\mathcal{F}_i, D_i; P_i)$, $r_i = \nu(\mathcal{F}_i, D_i; P_i)$.

Recall that if Γ_i is nonsingular and has normal crossings with D_i at P_i, then the same is true for Γ_{i+1}. Moreover there is always an index N such that this property is true for any $i \geq N$.

Proposition 2. *Assume that* Γ *is nonsingular and has normal crossings with* D *at* P*. Then* $m_1 \leq m_0$*. Moreover, if* $m_1 = m_0$ *then* Γ_1 *is appropriate for* $\mathcal{F}_1$ *at* P_1 *adapted to* D_1 *if and only if* Γ *is appropriate for* $\mathcal{F}$ *at* P *adapted to* D*.*

Proof. Since $\mathrm{e}(D_1, P_1) \geq \mathrm{e}(D, P)$, then $m_1 \leq m_0$ by the above proposition. Note that Γ is appropriate at P if and only if $\nu_\Gamma\mathcal{U}(\mathcal{H}, D, \Gamma) = m_0$. Looking at a generic point of Γ, we get $\nu_\Gamma\mathcal{U}(\mathcal{H}, D, \Gamma) = \nu_{\Gamma_1}\mathcal{U}(\mathcal{H}_1, D_1, \Gamma_1)$ and the result follows. □

Proposition 3. *There exists* M *such that* Γ_i *is appropriate at* P_i *for all* $i \geq M$*.*

Proof. Take M' such that for any index $i \geq M'$ we have that $m_i = m$, the curve Γ_i is nonsingular, has normal crossings with D_i at P_i and moreover there is an irreducible component of D_i transversal to Γ_i. Computing over the generators, we get that $\nu_{P_i}\mathcal{U}(\mathcal{H}_i, D_i, \Gamma_i) = m$ and

$$\mathcal{U}(\mathcal{H}_{i+1}, D_{i+1}, \Gamma_{i+1}) = \mathcal{J}^{-m}_{\pi^{-1}(P_i)}\pi^{-1}\mathcal{U}(\mathcal{H}_i, D_i, \Gamma_i).$$

Then there is $M \geq M'$ that gives equimultiplicity for $i \geq M$. □

1.3. *Vertical invariants.* We present here the list of the main invariants that will serve us to control the vertical evolution of the singularities under our process of reduction of singularities.

Consider a point $P \in X$, a normal crossings divisor $D \subset X$ and a nonsingular subspace Y having normal crossings with D such that $P \in Y$. Denote by $\pi : X' \to X$ the blowing-up with center Y and put $D' = \pi^{-1}(D \cup Y)$. Consider a singular foliation $\mathcal{F}$ over X. Let $\mathcal{H} = \mathrm{Sat}(\mathcal{F}, D)$ and put

$$r = \nu(\mathcal{F}, D; P);\ m = \mu(\mathcal{F}, D; P).$$

Denote by $\mathcal{F}'$ the strict transform of $\mathcal{F}$ by π and $\mathcal{H}' = \mathrm{Sat}(\mathcal{F}', D')$. In all this section we will assume that $r > 0$. If $r = 0$, we have a "presimple" singularity, which will be considered in the next section.

Let $\mathcal{O} = \mathcal{O}_{X,P}$ be the local ring of X at P and $\mathcal{M}$ its maximal ideal. Let F be the intersection of the irreducible components of D passing through P. We can choose local coordinates $(x_1, \ldots, x_n)$ at P such that

$$\mathcal{J}_{D,P} = \left(\prod_{i\in A} x_i\right)\mathcal{O};\ \mathcal{J}_{F,P} = \sum_{i\in A} x_i\mathcal{O};\ \mathcal{J}_{Y,P} = \sum_{i\in B} x_i\mathcal{O}.$$

Put $\widetilde{\mathcal{O}} = \mathcal{O}/\mathcal{J}_{F,P}$ and $\widetilde{\mathcal{M}} = \mathcal{M}/\mathcal{J}_{F,P}$. We have the graded rings

$$\mathrm{Gr}_{\mathcal{M}}(\mathcal{O}) \simeq \mathbb{C}[T_1, \ldots, T_n],\ \mathrm{Gr}_{\widetilde{\mathcal{M}}}(\widetilde{\mathcal{O}}) \simeq \mathbb{C}[\widetilde{T}_i;\ i \notin A]$$

where $T_i = x_i + \mathcal{M}^2$ and $\widetilde{T}_i = x_i + \widetilde{\mathcal{M}}^2$. The initial ideal $I_F = \mathrm{In}(\mathcal{J}_{F,P})$ is the kernel of the natural map

$$\xi : \mathrm{Gr}_{\mathcal{M}}(\mathcal{O}) \to \mathrm{Gr}_{\widetilde{\mathcal{M}}}(\widetilde{\mathcal{O}}).$$

Recall that $\mathrm{Gr}_{\mathcal{M}}(\mathcal{O})$ is the coordinate ring of the tangent space T_PX.

The directrix and related invariants. Let us first recall the construction of the Hironaka strict tangent space, or directrix, for a homogeneous polynomial. Given a homogeneous element $h \in \mathrm{Gr}_{\mathcal{M}}(\mathcal{O})$ we denote by $\mathrm{Dir}(h)$ the biggest linear subspace of T_PX that leaves $h = 0$ invariant by translation. Denote by $\mathcal{J}\mathrm{Dir}(h)$ the ideal of $\mathrm{Dir}(h)$ in $\mathrm{Gr}_{\mathcal{M}}(\mathcal{O})$. It is generated by $n - l$ independent linear forms $\phi_1, \ldots, \phi_{n-l}$, where $l = \dim_{\mathbb{C}} \mathrm{Dir}(h)$, such that $h = \Phi(\phi_1, \ldots, \phi_{n-l})$ for a certain homogeneous polynomial Φ in $n - l$ variables. Moreover, the codimension $n - l$ is minimal for that property. For a homogeneous ideal $I \subset \mathrm{Gr}_{\mathcal{M}}(\mathcal{O})$ let us denote by $\mathcal{J}\mathrm{Dir}(I)$ the ideal of $\mathrm{Gr}_{\mathcal{M}}(\mathcal{O})$ generated by $\mathcal{J}\mathrm{Dir}(h)$, where $h \in I$, and denote by $\mathrm{Dir}(I)$ the corresponding linear subspace of T_PX.

Fix $s \in \mathbb{Z}$. Consider an ideal $\mathcal{U} \subset \mathcal{O}$ such that $\nu_P \mathcal{U} \geq s$. Let $\mathrm{In}^s\mathcal{U}$ be the homogeneous ideal of $\mathrm{Gr}_{\mathcal{M}}(\mathcal{O})$ generated by $\mathrm{In}^s(f) = f + \mathcal{M}^{s+1}$, where $f \in \mathcal{U}$. Put $\mathcal{J}\mathrm{Dir}^s\mathcal{U} = \mathcal{J}\mathrm{Dir}\,(\mathrm{In}^s\mathcal{U})$ and $\mathrm{Dir}^s\mathcal{U} = \mathrm{Dir}\,(\mathrm{In}^s\mathcal{U})$. Define also the *adapted codimensions* $\mathrm{d}^s(\mathcal{U}, D)$ and $\mathrm{t}^s(\mathcal{U}, D)$ by

$$\mathrm{d}^s(\mathcal{U}, D) = \dim_{\mathbb{C}} \left\{ \xi(\mathcal{J}\mathrm{Dir}^s\mathcal{U}) \cap \mathrm{Gr}^1_{\mathcal{M}}(\widetilde{\mathcal{O}}) \right\},$$
$$\mathrm{t}^s(\mathcal{U}, D) = \dim_{\mathbb{C}} \left\{ (\xi(\mathrm{In}^s\mathcal{U})) \cap \mathrm{Gr}^1_{\mathcal{M}}(\widetilde{\mathcal{O}}) \right\}.$$

The dimensions are taken as dimensions of $\mathbb{C}$-vector subspaces of $\mathrm{Gr}^1_{\mathcal{M}}(\widetilde{\mathcal{O}})$, the homogeneous part of degree 1 of the graded ring $\mathrm{Gr}_{\mathcal{M}}(\widetilde{\mathcal{O}})$. Note that $n - l \geq \mathrm{d}^s(\mathcal{U}, D) \geq \mathrm{t}^s(\mathcal{U}, D)$ where $l = \dim_{\mathbb{C}} \mathrm{Dir}^s\mathcal{U}$.

Lemma 1. *Let $\pi^{-1}\mathcal{U}$ be the total transform of $\mathcal{U}$ by π. Assume that $\nu_Y\mathcal{U} \geq s$ and put $\mathcal{U}' = \mathcal{J}^{-s}_{\pi^{-1}(Y)}\pi^{-1}\mathcal{U}$. Then $T_PY \subset \mathrm{Dir}^s\mathcal{U}$ and for any point $P' \in \pi^{-1}(P)$ such that $\nu_{P'}\mathcal{U}' \geq s$ we have that*

$$P' \in \mathbb{P}\mathrm{roj}\,(\mathrm{Dir}^s\mathcal{U}/T_PY) \subset \pi^{-1}(P) = \mathbb{P}\mathrm{roj}\,(T_PX/T_PY)\,.$$

Moreover $\dim_{\mathbb{C}} \mathrm{Dir}^s\mathcal{U}'_{P'} \leq \dim_{\mathbb{C}} \mathrm{Dir}^s\mathcal{U}$ *and in case of equality we have that*

$$T_{P'}\pi^{-1}(P) \cap T_{P'}\,(\mathbb{P}\mathrm{roj}\,(\mathrm{Dir}^s\mathcal{U}/T_PY)) = \mathrm{Dir}^s\mathcal{U}'_{P'} \cap T_{P'}\pi^{-1}(P).$$

Finally $\mathrm{d}^s(\mathcal{U}'_{P'}, D') \geq \mathrm{d}^s(\mathcal{U}, D)$ *and* $\mathrm{t}^s(\mathcal{U}'_{P'}, D') \geq \mathrm{t}^s(\mathcal{U}, D)$.

Proof. We verify these statements first for a hypersurface, using classical results on Hironaka strict tangent space. The case of an ideal $\mathcal{U}$ follows by work on the elements of $\mathcal{U}$. □

Definition 2. Let $\mathcal{F} \in \mathbb{F}(X)$ be a singular foliation over X. Put $\mathcal{H} = \mathrm{Sat}(\mathcal{F}, D)$, $r = \nu(\mathcal{F}, D; P)$ and $m = \mu(\mathcal{F}, D; P)$. Let $\mathcal{U}$ be the ideal of $\mathcal{O}$ given either by $\mathcal{U} = \mathcal{U}(\mathcal{H}, D)$ if $m = r+1$ or by $\mathcal{U} = \mathcal{U}(\mathcal{H}, D, P)$ if $m = r$. We define the *directrix* $\mathrm{Dir}(\mathcal{F}, D; P)$, the ideal $\mathcal{J}\mathrm{Dir}(\mathcal{F}, D; P)$ and the *codimensions* $d(\mathcal{F}, D; P)$ and $t(\mathcal{F}, D; P)$ by

$$\mathrm{Dir}(\mathcal{F}, D; P) = \mathrm{Dir}^r\mathcal{U} \quad ; \quad \mathcal{J}\mathrm{Dir}(\mathcal{F}, D; P) = \mathcal{J}\mathrm{Dir}^r\mathcal{U}.$$
$$d(\mathcal{F}, D; P) = d^r(\mathcal{U}, D) \quad ; \quad t(\mathcal{F}, D; P) = t^r(\mathcal{U}, D).$$

Finally, we put $l(\mathcal{F}, D; P) = \dim_{\mathbb{C}} \mathrm{Dir}(\mathcal{F}, D; P)$.

Note that $\nu_P\mathcal{U} = r > 0$ and thus $\mathrm{Dir}(\mathcal{F}, D; P)$ is a strict linear subspace of T_PX.

Behaviour under blowing-up. Assume that Y is a permissible center for $\mathcal{F}$ adapted to D, appropriate at P. This implies that $T_PY \subset \mathrm{Dir}(\mathcal{F}, D; P)$. Denote by $\mathcal{F}'$ the strict transform of $\mathcal{F}$ by π and put $\mathcal{H}' = \mathrm{Sat}(\mathcal{F}', D')$.

Let us fix a point $P' \in \pi^{-1}(P)$. In order to simplify the notation, denote

$$\begin{aligned} r &= \nu(\mathcal{F}, D; P), & r' &= \nu(\mathcal{F}', D'; P'), \\ m &= \mu(\mathcal{F}, D; P), & m' &= \mu(\mathcal{F}', D'; P'), \\ l &= l(\mathcal{F}, D; P), & l' &= l(\mathcal{F}', D'; P'), \\ d &= d(\mathcal{F}, D; P), & d' &= d(\mathcal{F}', D'; P'), \\ t &= t(\mathcal{F}, D; P), & t' &= t(\mathcal{F}', D'; P'). \end{aligned}$$

Proposition 4 (Directrix theorem). *If $r = r'$, then*

$$P' \in \mathbb{P}\mathrm{roj}\left(\mathrm{Dir}(\mathcal{F}, D; P)/T_P Y\right).$$

Proof. We treat separately the cases $m = r+1$ and $m = r$. If $m = r+1$, note that $\mathcal{J}_Y \mathcal{U}(\mathcal{H}, D) \subset \mathcal{U}(\mathcal{H}, D, Y)$ and $\nu_Y \mathcal{U}(\mathcal{H}, D) \geq r$. We deduce that

$$\mathcal{U}' = \mathcal{J}^{-r}_{\pi^{-1}(Y)} \pi^{-1} \mathcal{U}(\mathcal{H}, D) \subset \mathcal{J}^{-m}_{\pi^{-1}(Y)} \pi^{-1} \mathcal{U}(\mathcal{H}, D, Y) = \mathcal{U}(\mathcal{H}', D').$$

Since $\nu_{P'} \mathcal{U}(\mathcal{H}', D') = r' = r$, then $\nu_{P'} \mathcal{U}' \geq r$ and we conclude by the lemma. Consider the case $m = r$, then $\mathrm{In}^r \mathcal{U}(\mathcal{H}, D, P) \subset \mathrm{In}^r \mathcal{U}(\mathcal{H}, D, Y)$ and thus $\mathrm{Dir}(\mathcal{F}, D; P) \supset \mathrm{Dir}^r \mathcal{U}(\mathcal{H}, D, Y)$. We conclude Lemma 1. □

Proposition 5. *If $m = m' = r+1$ (and hence $r = r'$), then $l' \leq l$, $d' \geq d$ and $t' \geq t$. Moreover, in the case $l' = l$ we have that*

$$T_{P'} \pi^{-1}(P) \cap T_{P'}\left(\mathbb{P}\mathrm{roj}\left(\mathrm{Dir}(\mathcal{F}, D; P)/T_P Y\right)\right) = \mathrm{Dir}(\mathcal{F}', D'; P') \cap T_{P'} \pi^{-1}(P).$$

Proof. Keep the notation of the above proof. The result follows from Lemma 1 and the fact that $\mathcal{U}' \subset \mathcal{U}(\mathcal{H}', D')$. □

Let us consider now the case $m = r$. A germ of vector field $R \in \Theta_{X,P}$ is called a *radial vector field* if and only if for any $s \in \mathbb{N}$ and any $f \in \mathcal{M}^s$ we have the Euler identity $\mathrm{In}^s(Rf) = s\mathrm{In}^s f$. For example, given a coordinate system $(x_1, \ldots, x_n)$ at P, any vector field of the type

$$\sum x_i \partial/\partial x_i + \sum b_i \partial/\partial x_i$$

where $\nu(b_i) \geq 2$ is a radial vector field (and conversely). In particular, there are radial vector fields $R \in \Theta_{X,P}[D]$. We say that the foliation $\mathcal{F}$ is *radially dicritical* at P if for any element $\alpha \in \mathcal{F}$ and any (or one) radial vector field R we have that

$$\nu_P\left(\alpha(R)\right) \geq \nu(\mathcal{F}, \emptyset; P) + 2 \quad \left(= \mu(\mathcal{F}, \emptyset; P) + 1\right).$$

This is equivalent to saying that after the blowing-up with center the point P, the exceptional divisor is a dicritical component of the strict transform of $\mathcal{F}$.

Lemma 2. *Consider an integrable homogeneous 1-form*

$$W = \sum_{i \in A} A_i(X_1, \ldots, X_n) \frac{dX_i}{X_i}$$

where $A \subset \{1, \ldots, n\}$ and the $A_i \in \mathbb{C}[X_1, \ldots, X_n]$ are homogeneous polynomials of degree r. Then there is a common factor $T \in \mathbb{C}[X_1, \ldots, X_n]$ such that $A_i = T\widetilde{A_i}$, where $\widetilde{A_i} \in \mathbb{C}[X_j; j \in A]$.

Proof. Fix indices $i, j \in A$ and $k \notin A$. The integrability condition gives

$$A_i \frac{\partial A_j}{\partial X_k} = A_j \frac{\partial A_i}{\partial X_k}$$

and thus the rational function A_i/A_j does not depend on X_k. □

PROPOSITION 6. *Assume that $\mathcal{F}$ is nonradially dicritical at P and that $m = r$. Then $m' \leq m$. If $r' = m' = m = r$, then $d' \geq d$ and $t' \geq t$. If in addition $e(D', P') \geq e(D, P)$, then $l' \leq l$.*

Proof. Take a radial vector field $R \in \Theta_{X,P}[Y] \cap \Theta_{X,P}[D]$ and a local generator ω of $\mathcal{H} = \mathrm{Sat}(\mathcal{F}, D)$ at P. Consider the ideal $\mathcal{A} = \omega(R)\mathcal{O}$. Then $\nu_P \mathcal{A} = \nu_Y \mathcal{A} = r$. In view of the above lemma, looking at the initial part of ω, we get $d = d^r(\mathcal{A}, D)$ and $t = t^r(\mathcal{A}, D)$. Let R' be the total transform of R by π (consider R as a derivation over meromorphic functions). Then $R' \in \Theta_{X',P'}[D']$ and $R'(P') = 0$. In particular $\omega'(R') \in \mathcal{U}(\mathcal{H}', D', P')$, where ω' is a local generator of $\mathcal{H}'$ at P'. But $\omega'(R')$ generates $\mathcal{A}' = \mathcal{J}^{-r}_{\pi^{-1}(Y)} \pi^{-1}\mathcal{A}$. Thus, $m' = \nu_{P'} \mathcal{U}(\mathcal{H}', D', P') \leq r = m$. If $r' = m' = m$, then $\nu_{P'} \mathcal{A}' = r$ and by the preceding lemmas we get

$$d' \geq \mathrm{d}^r(\mathcal{A}', D') \geq d; \; t' \geq \mathrm{t}^r(\mathcal{A}', D') \geq t.$$

Assume now that in addition $e(D', P') \geq e(D, P)$. Then for any (not necessarily radial) germ of vector field $\mathcal{X} \in \Theta_{X,P}[D]$ such that $\mathcal{X}(P) = 0$, the total transform $\mathcal{X}'$ satisfies $\mathcal{X}' \in \Theta_{X',P'}[D']$ and $\mathcal{X}'(P') = 0$. This implies that $\mathcal{J}^{-r}_{\pi^{-1}(Y)} \pi^{-1} \mathcal{U}(\mathcal{H}, D, P) \subset \mathcal{U}(\mathcal{H}', D', P')$ and hence $l' \leq l$. □

Contact type invariant. Given a linear subspace $L \subset T_P X$, denote by $\Theta_{X,P}[D * L]$ the $\mathcal{O}$-module of the germs of vector fields $\mathcal{X} \in \Theta_{X,P}[D]$ such that $\mathcal{X}(P) \in L$. Denote by $\mathcal{U}(\mathcal{H}, D * L)$ the ideal generated by $\omega(\mathcal{X})$, where ω is a generator of $\mathcal{H}$ and $\mathcal{X} \in \Theta_{X,P}[D * L]$. We define the *contact type* $\delta(\mathcal{F}, D; P)$ adapted to D between the foliation $\mathcal{F}$ and its directrix by

$$\begin{aligned} \delta(\mathcal{F}, D; P) &= r, && \text{if } \mathrm{In}^r\left(\mathcal{U}(\mathcal{H}, D * \mathrm{Dir}(\mathcal{F}, D; P))\right) \neq 0. \\ \delta(\mathcal{F}, D; P) &= r + 1, && \text{if } \mathrm{In}^r\left(\mathcal{U}(\mathcal{H}, D * \mathrm{Dir}(\mathcal{F}, D; P))\right) = 0. \end{aligned}$$

This is of interest only in the case $m = r + 1$.

LEMMA 3. *Let $L \subset T_P X$ be a linear subspace such that $T_P Y \subset L$. Assume that $P' \in \mathbb{P}\mathrm{roj}\,(L/T_P Y)$ and consider a linear subspace $L' \subset T_{P'} X'$ such that $\dim_{\mathbb{C}} L = \dim_{\mathbb{C}} L'$ and $T_{P'} \mathbb{P}\mathrm{roj}\,(L/T_P Y) = L' \cap T_{P'} \pi^{-1}(P)$. Then, for any $\mathcal{X} \in \Theta_{X,P}[D * L]$ the total transform $\mathcal{X}'$ satisfies $\mathcal{J}_{\pi^{-1}(Y)} \mathcal{X}' \subset \Theta_{X',P'}[D' * L']$.*

Proof. This is a straightforward computation. □

PROPOSITION 7. *Assume that* $m = r + 1$, $m' = m$ *and that* $l' = l$. *Then* $\delta' \leq \delta$.

Proof. Put $L = \mathrm{Dir}(\mathcal{F}, D; P)$ and $L' = \mathrm{Dir}(\mathcal{F}', D'; P')$. Assume that $\delta = r$. Then there is a local generator ω of $\mathcal{H}$ and a germ of vector field $\mathcal{X} \in \Theta_{X,P}[D * L]$ such that if $f = \omega(\mathcal{X})$ then $\mathrm{In}^r(f) \neq 0$. Note that $\nu_Y f = r$. Taking the situation of the above lemma, we have that $f \circ \pi = \pi^* \omega(\mathcal{X}')$ and

$$\mathcal{J}^{-r}_{\pi^{-1}(Y)}(f \circ \pi) = \left(\mathcal{J}^{-m}_{\pi^{-1}(Y)} \pi^* \omega \right) \left(\mathcal{J}_{\pi^{-1}(Y)} \mathcal{X}' \right);$$

hence $\mathcal{J}^{-r}_{\pi^{-1}(Y)}(f \circ \pi) \subset \mathcal{U}(\mathcal{H}, D' * L')$. Then $\delta' = r$, since $\nu_{P'}(\mathcal{J}^{-r}_{\pi^{-1}(Y)} f \circ \pi) = r$. □

Contact and essential components. Assume that $m = r$ and $d = 0$. Then the directrix is *hidden* in the equations of the divisor D, in the sense that

$$\mathrm{Dir}(\mathcal{F}, D; P) = \{\phi_1 = \cdots = \phi_{n-l} = 0\}\,; \ \phi_j \in \mathbb{C}[T_i; i \in A].$$

Fix an irreducible component H of D through P. Denote by $\widetilde{D}$ the normal crossings divisor obtaining from D by eliminating H. Let $\widetilde{F}$ be the intersection of the irreducible components of $\widetilde{D}$ through P and put $I_{\widetilde{F}} = \mathrm{In}(\mathcal{J}_{\widetilde{F},P})$. Denote by $\mathcal{U}(\mathcal{H}, D, H; P)$ the ideal of $\mathcal{O}$ generated by the elements $\omega(\mathcal{X})$, where ω is a local generator of $\mathcal{H}$ and $\mathcal{X} \in (\mathcal{J}_H \Theta_{X,P}) \cap \Theta_{X,P}[D]$. We say that H is a *contact component* of D for $\mathcal{F}$ at P if and only if

$$\mathrm{In}^r \left(\mathcal{U}(\mathcal{H}, D, H; P) \right) \not\subset I_{\widetilde{F}}.$$

If there are no contact components, we say that H is an *essential* component if and only if

$$\mathrm{In}^r \left(\mathcal{U}(\mathcal{H}, D; P) \right) \not\subset I_{\widetilde{F}}.$$

Note that there is *always* at least one essential component.

LEMMA 4. *Consider an integrable homogeneous* 1*-form*

$$W = \sum_{i=1}^{n} A_i(X_1, \ldots, X_n) \frac{dX_i}{X_i}$$

and put $P = \sum_{i=1}^{n} A_i$. *If* $P \neq 0$, *the following statements hold*:

a) *Assume that* $\partial A_n / \partial X_n \neq 0$ *and that* $v = \sum_{i=1}^{n} \lambda_i \partial / \partial X_i$ *is an element of the directrix* $\mathrm{Dir}(A_n)$ *with* $\lambda_n \neq 0$. *Then there is an index* j, *with* $\lambda_j \neq 0$, *such that* $\partial P / \partial X_j \neq 0$.

b) *Assume that* $\partial A_i / \partial X_i = 0$ *for all* $i = 1, \ldots, n$ *and that there is an index* $j \neq n$ *such that* $\partial A_j / \partial X_n \neq 0$. *Then* $\partial P / \partial X_n \neq 0$.

Proof. The integrability condition $W \wedge dW = 0$, gives that

$$0 = A_i \left(X_j \frac{\partial A_k}{\partial X_j} - X_k \frac{\partial A_j}{\partial X_k} \right) + A_j \left(X_k \frac{\partial A_i}{\partial X_k} - X_i \frac{\partial A_k}{\partial X_i} \right) + A_k \left(X_i \frac{\partial A_j}{\partial X_i} - X_j \frac{\partial A_i}{\partial X_j} \right)$$

for any set of indices i, j, k. Making $i = n$ and taking the sum over all k, we get that

$$X_j \left(A_n \frac{\partial P}{\partial X_j} - P \frac{\partial A_n}{\partial X_j} \right) = X_n \left(A_j \frac{\partial P}{\partial X_n} - P \frac{\partial A_j}{\partial X_n} \right).$$

Let us prove a). Let us reason by contradiction assuming that $\partial P/\partial X_j = 0$ if $\lambda_j \neq 0$. Since $P \neq 0$, we have

$$X_j \frac{\partial A_n}{\partial X_j} = X_n \frac{\partial A_j}{\partial X_n}; \quad \text{if } \lambda_j \neq 0.$$

Thus X_n divides $\partial A_n/\partial X_j$ if $\lambda_j \neq 0$. We can decompose A_n as follows: $A_n = A'_n X_n^s + A''_n$, with $s \geq 1$, where X_n does not divide A'_n and $v(A''_n) = 0$. Now

$$0 = v(A_n) = X_n^{s-1} \left(X_n v(A'_n) + s\lambda_n A'_n \right).$$

This implies that X_n divides A'_n. Contradiction.

Let us prove b). Assume that $\partial P/\partial X_n = 0$. Then X_n divides $A_n \partial P/\partial X_j - P\partial A_n/\partial X_j$. The only possibility is that

$$0 = A_n \frac{\partial P}{\partial X_j} - P \frac{\partial A_n}{\partial X_j}$$

since $(\partial/\partial X_n)(A_n \partial P/\partial X_j - P\partial A_n/\partial X_j) = 0$. Then

$$0 = A_j \frac{\partial P}{\partial X_n} - P \frac{\partial A_j}{\partial X_n}$$

and thus $\partial P/\partial X_n \neq 0$, since $0 \neq P\partial A_j/\partial X_n$. □

PROPOSITION 8. *Assume that the singular foliation $\mathcal{F}$ is nonradially dicritical at P, that $m = r$, $d = 0$, $m' = m$, $r' = r$ and $d' = d = 0$. Let H be an irreducible component of D through P and denote by H' the strict transform of H by π.*

a) *If H is a contact component, then $Y \subset H$, $P' \in H'$ and H' is a contact component of D' for $\mathcal{F}'$ at P'.*

b) *If there are no contact components of D at P and H is an essential component, then $Y \subset H$ and $P' \in H'$. Moreover, if D' has no contact components for $\mathcal{F}'$ at P', then H' is an essential component for $\mathcal{F}'$ at P'.*

Proof. Note that $\mathcal{U}(\mathcal{H}, D, P) \supset \mathcal{U}(\mathcal{H}, D, H; P)$. Both in cases a) and b) we find an element $\phi \in \mathrm{In}^r(\mathcal{U}(\mathcal{H}, D, P)) - I_{\widetilde{F}}$. If $Y \not\subset H$, we get that $T_P Y \not\subset \mathrm{Dir}(\phi) \supset \mathrm{Dir}(\mathcal{F}, D; P)$. Thus $Y \subset H$. Assume that $P' \in H'$. Given $\mathcal{X} \in \Theta_{X,P}$, denote by $\mathcal{X}'$ the total transform of $\mathcal{X}$ at P' (consider $\mathcal{X}$ as a derivation over meromorphic functions). We see that

(i) If $\mathcal{X} \in (\mathcal{J}_H \Theta_{X,P}) \cap \Theta_{X,P}[D]$, then $\mathcal{X}' \in (\mathcal{J}_{H'} \Theta_{X',P'}) \cap \Theta_{X',P'}[D']$.

(ii) If $\mathcal{X} \in \Theta_{X,P}[D]$ and $\mathcal{X}(P) = 0$, then $\mathcal{X}' \in \Theta_{X',P'}[D']$ and $\mathcal{X}'(P') = 0$.

Take $\mathcal{X}$ as in (i) in case a) and as in (ii) in case b), in such a way that $\mathrm{In}^r(f) \notin I_{\widetilde{F}}$, where $f = \omega(\mathcal{X})$ and ω is a local generator of $\mathcal{H}$. The ideal

$$\mathcal{J}^{-r}_{\pi^{-1}(Y)} f \circ \pi = \mathcal{J}^{-r}_{\pi^{-1}(Y)} \pi^* \omega(\mathcal{X}')$$

is contained in $\mathcal{U}(\mathcal{H}', D', H'; P')$ in case a) and in $\mathcal{U}(\mathcal{H}', D', P')$ in case b). We see immediately that $\mathrm{In}^r(\mathcal{J}^{-r}_{\pi^{-1}(Y)} f \circ \pi) \not\subset I_{\widetilde{F}'}$ and we get our conclusion. It remains to show that $P' \in H'$. Assume that $P' \notin H'$. Let $R \in \Theta_{X,P}[D]$ be a radial vector field. Put $\mathcal{A} = \omega(R)\mathcal{O}$. There is an element

$$v \in \mathrm{Dir}(\mathcal{F}, D, P) \subset \mathrm{Dir}^r \mathcal{A}$$

defining the point P' (modulo $T_P Y$) and such that $v \notin T_P H$. Lemma 4-a) applies then in case a) and we get that $\mathrm{In}^r\left(\mathcal{A}' = \mathcal{J}^{-r}_{\pi^{-1}(Y)}(\mathcal{A} \circ \pi)\right) \not\subset I_{F'}$. This implies that $d' \geq 1$. In case b), by Lemma 4-b) we know that $\mathrm{In}^r \mathcal{A} \not\subset I_{\widetilde{F}}$. Hence $\mathrm{In}^r \mathcal{A}' \not\subset I_{\widetilde{F}'} = I_{F'}$ and thus $d' \geq 1$. This completes the proof. □

Resonance. Let us introduce the *resonance invariant* $\mathrm{Rs}(\mathcal{F}, D; P)$. It will take the values 0, 1 or 2. We say that $\mathrm{Rs}(\mathcal{F}, D; P) = 1$ if $m = r$ and $\mathcal{F}$ is radially dicritical at P. We say that $\mathrm{Rs}(\mathcal{F}, D; P) = 2$ if $m = r$, $\mathrm{Rs}(\mathcal{F}, D; P) \neq 1$ and there are local coordinates $(x_1, \ldots, x_n)$ at P and a map $\phi : A \to \mathbb{Z}_{>0}$ such that

$$D = \left\{ \prod_{i \in A} x_i = 0 \right\};\ \nu_P \left(\omega \left(\sum_{i \in A} \phi(i) x_i \frac{\partial}{\partial x_i} \right) \right) \geq r + 1$$

for a local generator ω of $\mathcal{H}$. Otherwise, we say that $\mathrm{Rs}(\mathcal{F}, D; P) = 0$.

Remark 5. If $\mathrm{Rs}(\mathcal{F}, D; P) \neq 0$, there is always a map $\phi : A \to \mathbb{Z}_{>0}$ such that

$$\nu_P \left(\omega \left(\sum_{i \in A} \phi(i) x_i \frac{\partial}{\partial x_i} \right) \right) \geq r + 1.$$

(Take $\phi(i) = 1$ for the case of a radially dicritical foliation.) If $m = r$, the converse is also true.

Proposition 9. *If* $\mathrm{Rs}(\mathcal{F}, D; P) \neq 1$, $r' = r$ *and* $m' = m$, *then*

$$\mathrm{Rs}(\mathcal{F}', D'; P') \leq \mathrm{Rs}(\mathcal{F}, D; P).$$

Proof. We have only to prove that if $\mathrm{Rs}(\mathcal{F}', D'; P') \neq 0$, then $\mathrm{Rs}(\mathcal{F}, D; P) \neq 0$. Note that $m = r$, since $m' = r'$. Select coordinates $(x_1, \ldots, x_n)$ such that

$$D = \left\{ \prod_{i \in A} x_i = 0 \right\};\ Y = \bigcap_{i \in B} \{x_i = 0\}$$

locally at P. Let B' be the set of indices $i \in B$ such that P' is in the strict transform of $x_i = 0$. Take $i_0 \in B - B'$ and coordinates $(x'_1, \ldots, x'_n)$ at P' given by:

$$\begin{array}{ll} x_i = x'_i, & \text{for } i \notin B - \{i_0\} \\ x_i = x'_{i_0} x'_i, & \text{for } i \in B' \\ x_i = x'_{i_0}(x'_i + \zeta_i),\ \zeta_i \neq 0, & \text{for } i \in (B - \{i_0\}) - B'. \end{array}$$

Put $A' = (A - B) \cup (B' \cap A) \cup \{i_0\}$. Since $\mathrm{Rs}(\mathcal{F}', D'; P') \neq 0$, there is a function $\phi' : A' \to \mathbb{Z}_{>0}$ such that

$$\nu_{P'} \left(\omega' \left(\mathcal{X}' = \sum_{i \in A'} \phi'(i) x'_i \frac{\partial}{\partial x_i} \right) \right) \geq r + 1$$

for any local generator ω' of $\mathcal{H}'$ at P'. Let $\mathcal{X} \in \Theta_{X,P}$ be given by

$$\mathcal{X} = \sum_{i \in A' - \{i_0\}} \phi'(i) x_i \frac{\partial}{\partial x_i} + \phi'(i_0) \sum_{i \in B} x_i \frac{\partial}{\partial x_i}.$$

Note that the total transform of $\mathcal{X}$ is $\mathcal{X}'$. Put $f = \omega(\mathcal{X})$, where ω is a local generator of $\mathcal{H}$. Then $f \circ \pi = {x'_{i_0}}^{r} \omega'(\mathcal{X}')$. Thus $\nu_P f \geq r + 1$. Define $\phi : A \to \mathbb{Z}_{>0}$ by

$$\mathcal{Z} = \sum_{i \in A} \phi(i) x_i \frac{\partial}{\partial x_i} = \mathcal{X} - \phi'(i_0) \sum_{i \in B - A} x_i \frac{\partial}{\partial x_i} = \mathcal{X} - \mathcal{Y}.$$

Since $\nu_P(\omega(\mathcal{Y})) \geq r + 1$ and $\nu_P f \geq r + 1$, we get $\nu_P(\omega(\mathcal{Z})) \geq r + 1$. □

Putting together the resonance invariant and the adapted multiplicity, we define the *resonant adapted multiplicity* $\mu^*(\mathcal{F}, D; P)$ by

$$\begin{array}{ll} \mu^*(\mathcal{F}, D; P) = \mu(\mathcal{F}, D; P), & \text{if } \mathrm{Rs}(\mathcal{F}, D; P) = 0, \\ \mu^*(\mathcal{F}, D; P) = \mu(\mathcal{F}, D; P) + 1, & \text{if } \mathrm{Rs}(\mathcal{F}, D; P) \neq 0. \end{array}$$

PROPOSITION 10. $\mu^*(\mathcal{F}', D'; P') \leq \mu^*(\mathcal{F}, D; P)$.

Proof. Straightforward consequence of the preceding propositions. □

Put $m^* = \mu^*(\mathcal{F}, D; P)$. Note that $r \leq m \leq m^* \leq r + 1$. The main invariant in our control of the singularities is the pair $(\nu(\mathcal{F}, E; P), \mu^*(\mathcal{F}, E; P))$ where $E = \mathrm{Nd}(D, \mathcal{F})$ is the divisor given by the nondicritical components of D.

Forgetting the dicritical components. The next result shows that the above propositions are still true when D' is replaced by $\mathrm{Nd}(D', \mathcal{F}')$.

Proposition 11. *Put* $E = \mathrm{Nd}(D, \mathcal{F})$. *Denote by* $r^\vee$, $m^\vee$, $m^{*\vee}$, $l^\vee$, $d^\vee$, $t^\vee$ *and* $\delta^\vee$ *the invariants of* $\mathcal{F}$ *at* P *adapted to* E *instead of* D. *Then*:

$$\left(r^\vee, m^\vee, l^\vee, -d^\vee, -t^\vee, \delta^\vee\right) \leq (r, m, l, -d, -t, \delta)$$

for the lexicographical ordering. Moreover $m^\vee = m$ *and*

(1) *If* $(r^\vee, m^\vee, l^\vee) = (r, m, l)$ *then* $\mathrm{Dir}(\mathcal{F}, E; P) = \mathrm{Dir}(\mathcal{F}, D; P)$.

(2) *If* $(r^\vee, m^\vee) = (r, m)$ *and* $m = r$, *then* $l^\vee = l$.

(3) *Assume* $(r^\vee, m^\vee, d^\vee) = (r, m, d)$, $m = r$ *and* $d^\vee = d = 0$. *Let* H *be a contact component of* D *for* $\mathcal{F}$ *at* P. *Then* H *is a nondicritical component of* D *for* $\mathcal{F}$ *and* H *is also a contact component of* E *for* $\mathcal{F}$ *at* P. *Conversely, any contact component* H *of* E *for* $\mathcal{F}$ *at* P *is also a contact component of* D *for* $\mathcal{F}$. *Same statement* "mutatis mutandis" *for the essential components in the case that there are no contact components.*

(4) *If* $(r^\vee, m^\vee) = (r, m)$, *then* $\mathrm{Rs}(\mathcal{F}, E; P) = \mathrm{Rs}(\mathcal{F}, D; P)$.

In particular, we have that $(r^\vee, m^{*\vee}) \leq (r, m^*)$ *for the lexicographical ordering.*

Proof. Recall that $\mathcal{H} = \mathrm{Sat}(\mathcal{F}, D) = \mathrm{Sat}(\mathcal{F}, D^*)$. Write locally

$$D = \left\{\prod_{i\in A} x_i = 0\right\};\ E = \left\{\prod_{i\in A^*} x_i = 0\right\}$$

where $A^* \subset A$. Take a local generator ω of $\mathcal{H}$ given by

$$\omega = \sum_{i\in A} a_i \frac{dx_i}{x_i} + \sum_{i\notin A} a_i dx_i = \sum_{i\in A^*} b_i \frac{dx_i}{x_i} + \sum_{i\notin A^*} b_i dx_i.$$

Then $a_i = x_i b_i$ for $i \in A - A^*$ and the proposition follows. □

1.4. *First properties of presimple singularities.* Consider a singular foliation $\mathcal{F} \in \mathbb{F}(X)$ and a normal crossings divisor E on X such that all the components of E are nondicritical components.

Definition 3. We say that a point $P \in \mathrm{Sing}(\mathcal{F})$ is a *presimple singularity* for $\mathcal{F}$ adapted to E if and only if one of the two following possibilities holds:

a) $\nu(\mathcal{F}, E; P) = 0$.

b) $\nu(\mathcal{F}, E; P) = \mu^*(\mathcal{F}, E; P) = 1$ and $\mathrm{d}(\mathcal{F}, E; P) \geq 1$.

Denote $\mathrm{Sing}^*(\mathcal{F}, E)$ the set of points of $\mathrm{Sing}\mathcal{F}$ that are not presimple ones.

The condition b) above is equivalent to saying that $\nu(\mathcal{F}, E; P) = \mu(\mathcal{F}, E; P) = 1$, $\mathrm{d}(\mathcal{F}, E; P) \geq 1$ and $\mathrm{Rs}(\mathcal{F}, E; P) = 0$.

PROPOSITION 12 (vertical stability). *Let us consider a blowing-up $\pi : X' \to X$ permissible for $\mathcal{F}$, E, with center Y. Denote by $\mathcal{F}'$ the strict transform of $\mathcal{F}$ by π and put $E' = \mathrm{Nd}(\pi^{-1}(E \cup Y), \mathcal{F}')$. Then*

$$\pi\left(\mathrm{Sing}^*(\mathcal{F}', E')\right) \subset \mathrm{Sing}^*(\mathcal{F}, E).$$

Proof. Pick a point $P \in X - \mathrm{Sing}^*(\mathcal{F}, E)$. Put $r = \nu(\mathcal{F}, E; P)$, $m = \mu(\mathcal{F}, E; P)$ and $m^* = \mu^*(\mathcal{F}, E; P)$. We have to prove that $P' \notin \mathrm{Sing}^*(\mathcal{F}', E')$ for any $P' \in \pi^{-1}(P)$. If $r = 0$, we are done in view of the vertical stability of the adapted order under permissible blowing-ups. Thus, we have only to consider the case $r = m^* = 1$ and $\mathrm{d}(\mathcal{F}, E; P) \geq 1$. In particular the foliation $\mathcal{F}$ is nonradially dicritical at P and $r = m = 1$. The center Y is appropriate at P and, in view of the propositions of the above paragraph, we get (with the obvious notation)

$$(r', m^{*\prime}, -d') \leq (r, m^*, -d)$$

for the lexicographical ordering. Hence $P' \notin \mathrm{Sing}^*(\mathcal{F}', E')$. $\square$

The next results are devoted to proving the *horizontal stability* for presimple singularities. Later on we shall give a more detailed description of the presimple singularities, as well as the reduction to simple ones. Fix a point $P \in X$ and put $r = \nu(\mathcal{F}, E; P)$, $m = \mu(\mathcal{F}, E; P)$, $m^* = \mu^*(\mathcal{F}, E; P)$ and $d = \mathrm{d}(\mathcal{F}, E; P)$.

LEMMA 5. *Assume that either $r = 0$ or $r = m = 1$, $d \geq 1$. Then the dimensional type is $\tau(\mathcal{F}, P) = \nu(\mathcal{F}, \emptyset; P) + 1$.*

Proof. By induction on the ambient dimension, assume $n = \tau(\mathcal{F}, P)$ and hence $\mathcal{D}(\mathcal{F})(P) = 0$. We have to prove that $\nu(\mathcal{F}, \emptyset; P) = n - 1$. Put $e = \mathrm{e}(E, P)$. If $e = n$, then $d = 0$, $r = 0$ and the usual algebraic order $\nu(\mathcal{F}, \emptyset; P)$ is $n - 1$. Assume that $e < n$. Take local coordinates $(x_1, \ldots, x_n)$ such that $E = \{\prod_{i=1}^e x_i = 0\}$ and a local generator of $\mathcal{H} = \mathrm{Sat}(\mathcal{F}, E)$ is given by

$$\omega = \sum_{i=1}^{e} a_i \frac{dx_i}{x_i} + \sum_{i=e+1}^{n} a_i dx_i.$$

Consider first the case $r = 0$. Up to reordering the variables, either $a_1 = 1$ or $a_{e+1} = 1$. If $a_1 = 1$, we get $a_{e+1}(0) x_1 \partial/\partial x_1 - \partial/\partial x_{e+1}$ in $\mathcal{D}(\mathcal{F})(P)$. Thus $a_{e+1} = 1$. If $e \leq n - 2$, the tangent vector $a_{e+2}(0) x_{e+1} \partial/\partial x_{e+1} - \partial/\partial x_{e+2}$ is in $\mathcal{D}(\mathcal{F})(P)$. Then $e = n - 1$, $a_n = 1$ and it is obvious that $\nu(\mathcal{F}, \emptyset; P) = n - 1$. Consider the case $r = m = 1$, $d \geq 1$. Since $r = m = 1$, we get that $\nu(\mathcal{F}, \emptyset; P) = e$. It remains to prove that $e = n - 1$. Up to reordering the variables, the condition $d \geq 1$ implies that

$$\frac{\partial a_1}{\partial x_{e+1}}(P) \neq 0.$$

Assume that $e \leq n-2$. The coefficient of $dx_1 \wedge dx_{e+1} \wedge dx_n$ in the integrability condition $\omega \wedge d\omega = 0$ gives that

$$\left(\frac{\partial a_n}{\partial x_{e+1}} - \frac{\partial a_{e+1}}{\partial x_n}\right) x_1 \frac{\partial}{\partial x_1} + \left(\frac{\partial a_1}{\partial x_{e+1}} - x_1 \frac{\partial a_n}{\partial x_1}\right) \frac{\partial}{\partial x_{e+1}} + \left(-\frac{\partial a_1}{\partial x_{e+1}} + x_1 \frac{\partial a_{e+1}}{\partial x_1}\right) \frac{\partial}{\partial x_n}$$

trivializes the foliation. Contradiction. □

Proposition 13. *Assume that $r = m = 1$ and $d \geq 1$. Then*:

a) *There is a unique formal hypersurface $\hat{H}$ at P, with $\hat{H} \not\subset E$, such that $\hat{H}$ is a formal integral hypersurface of $\mathcal{F}$ and $E \cup \hat{H}$ defines a formal normal crossings divisor at P.*

b) $\mathrm{Rs}(\mathcal{F}, E; P) = 0$.

Proof. a) (See also [9]). We can assume that $\tau(\mathcal{F}, P) = n$ and hence $e = \mathrm{e}(E, P) = n-1$. Take local coordinates $(x_1, \ldots, x_n)$ such that $E = \{\prod_{i=1}^{n-1} x_i = 0\}$ and a local generator of $\mathcal{H} = \mathrm{Sat}(\mathcal{F}, E)$ is given by

$$\omega = \sum_{i=1}^{n-1} a_i \frac{dx_i}{x_i} + a_n dx_n$$

where $\nu(a_n) \geq 1$ and the initial form $\mathrm{In}^1(a_1) = T_n = \mathrm{In}^1(x_n)$. Let $\lambda_I x^I = \lambda_I x_1^{i_1} \ldots x_n^{i_n}$ be the first monomial appearing in $a_1 - x_n$, for the ordering given by

$$I \leq I' \Leftrightarrow (|I|, I) \leq (|I'|, I') \text{ (lexicographically)},$$

where $|I| = i_1 + \ldots i_n$. Do the coordinate change $x_n \mapsto x_n + \lambda_I x^I$. The new *first monomial* has strictly higher multi-index I. Repeating this operation, we get formal coordinates $(x_1, \ldots, x_{n-1}, \hat{x}_n)$ such that

$$\omega = \hat{x}_n \frac{dx_1}{x_1} + \sum_{i=2}^{n-1} \hat{a}_i \frac{dx_i}{x_i} + \hat{a}_n d\hat{x}_n.$$

The integrability condition $\omega \wedge d\omega = 0$ implies that $\hat{x}_n$ divides $\hat{a}_i$, for $i = 2, \ldots, n-1$ and we can write

$$\omega = \hat{x}_n \left(\frac{dx_1}{x_1} + \sum_{i=2}^{n-1} \hat{b}_i \frac{dx_i}{x_i} + \hat{a}_n \frac{d\hat{x}_n}{\hat{x}_n} \right).$$

Now, take $\hat{H} = \{\hat{x}_n = 0\}$. The uniqueness is a direct computation.

b) Put $\lambda_1 = 1$, $\lambda_i = \hat{b}_i(0)$, for $i = 2, \ldots, n-1$. If $\mathrm{Rs}(\mathcal{F}, E; P) \neq 0$, by Remark 5 there is a function $\Phi : \{1, \ldots, n-1\} \to \mathbb{Z}_{>0}$ such that

$\sum_{j=1}^{n-1}\Phi(j)\lambda_j = 0$. Since $\hat{x}_n$ does not divide $\hat{a}_n$ (otherwise $r = 0$) there is a first monomial $\alpha x_1^{i_1}\ldots x_{n-1}^{i_{n-1}}$ appearing in $\hat{a}_n$. Looking at the coefficient of $dx_1 \wedge dx_j \wedge d\hat{x}_n$ in the integrability condition and making $\hat{x_n} = 0$ we get that

$$\left\{x_j\frac{\partial \hat{a}_n}{\partial x_j} + x_1\left(\hat{a}_n\frac{\partial b_j}{\partial x_1} - b_j\frac{\partial \hat{a}_n}{\partial x_1}\right)\right\}\Bigg|_{\hat{x}_n=0} = 0.$$

Considering the coefficient of $x_1^{i_1}\ldots x_{n-1}^{i_{n-1}}$, we conclude that $i_j = i_1\lambda_j$, for $j = 2,\ldots,n-1$. Then $i_1 \neq 0$ and $\sum_{j=1}^{n-1}\Phi(j)i_j = 0$. Contradiction. $\square$

COROLLARY. *A point $P \in \mathrm{Sing}\mathcal{F}$ is a presimple singularity for $\mathcal{F}$ adapted to E if and only if either $\nu(\mathcal{F},E;P) = 0$ or $\nu(\mathcal{F},E;P) = \mu(\mathcal{F},E;P) = 1$ and* $\mathrm{d}(\mathcal{F},E;P) \geq 1$.

LEMMA 6. *Let Γ be a (germ) of irreducible analytic curve at $P \in \Gamma$. Assume that $\Gamma - \{P\} \subset \mathrm{Sing}^*(\mathcal{F},E)$. Then $P \in \mathrm{Sing}^*(\mathcal{F},E)$.*

Proof. The vertical stability for presimple singularities allows us to assume that Γ is nonsingular and has normal crossings with E. To see this, blow-up the point P repeatedly. If Γ is contained in all the irreducible components of E through P, the results follows from the (easy) semicontinuity of the invariants r, m and d along Γ in this case. Choose coordinates $(x_1,\ldots,x_n)$ centered at P such that

$$E = \{x_1\cdots x_e = 0\} \text{ and } \Gamma = \{x_i = 0;\, i = 2,\ldots,n\}$$

and a generator

$$\omega = \sum_{i=1}^{e} a_i\frac{dx_i}{x_i} + \sum_{i=e+1}^{n} a_i dx_i$$

of $\mathcal{H} = \mathrm{Sat}(\mathcal{F},E)$. We reason by contradiction assuming that P is a presimple singularity but $\Gamma - \{P\}$ is contained in $\mathrm{Sing}^*(\mathcal{F},E)$. Because of the semicontinuity of the adapted order, necessarily $\nu(\mathcal{F},E;P) = 1$ and thus $\mu(\mathcal{F},E;P) = 1$ and $\mathrm{d}(\mathcal{F},E;P) \geq 1$. Moreover, since $\Gamma - \{P\}$ is contained in $\mathrm{Sing}^*(\mathcal{F},E)$ we have that $a_i(x_1,0,\ldots,0) \equiv 0$ for $i = 1,\ldots,n$ and that

$$\frac{\partial a_j}{\partial x_s}(x_1,0,\ldots,0) \equiv 0, \quad \text{for } j \in \{2,\ldots,e\}, s \in \{1\}\cup\{e+1,\ldots,n\}.$$

Up to reordering the variables, we can also asume that $\partial a_1/\partial x_n(0) \neq 0$. In this situation, and in view of the above propositions, we can also assume that the dimensional type is n, that $e = n-1$ and, passing to formal coordinates, that

$$\omega = x_n\left(\frac{dx_1}{x_1} + \sum_{i=2}^{n-1} b_i\frac{dx_i}{x_i} + b_n\frac{dx_n}{x_n}\right).$$

with $\nu(b_2,\ldots,b_{n-1},b_n)\geq 1$. Knowing that x_2 does not divide b_2, we can write

$$b_2 = x_1^t\left(\phi(x_3,\ldots,x_n)+x_1(\ldots)\right)+x_2(\ldots)$$

where $\phi\not\equiv 0$ and $\phi(0)=0$. In particular, there is an index $j\geq 3$ such that $\partial\phi/\partial x_j\not\equiv 0$. Looking at the coefficient of $dx_1\wedge dx_2\wedge dx_j$ in the integrability condition $\omega\wedge d\omega=0$, we get that

$$-x_j\frac{\partial b_2}{\partial x_j}+x_2\frac{\partial b_j}{\partial x_2}-x_1\left(b_2\frac{\partial b_j}{\partial x_1}-b_j\frac{\partial b_2}{\partial x_1}\right)=0$$

Put $\psi(x_3,\ldots,x_n)=b_j(0,0,x_3,\ldots,x_n)$, then $x_j\partial\phi/\partial x_j=t\psi\phi$ which is not possible, since $\psi(0)=0$ and $\phi(0)=0$. □

Proposition 14 (horizontal stability). *The set* $\mathrm{Sing}^*(\mathcal{F},E)$ *is a closed analytic subset of* X*.*

Proof. We may work on a relatively compact open set. The divisor E is a finite union of components $E=\bigcup_{j\in J}E_j$. For any $J'\subset J$, put

$$F_{J'}=\bigcap_{j\in J'}E_j \quad\text{and}\quad R_{J'}=\bigcup_{k\in J-J'}F_{J'}\cap E_k.$$

Let $S_{J'}\subset X$ be the topological closure of $\mathrm{Sing}^*(\mathcal{F},E)\cap F_{J'}-R_{J'}$. Let us prove that it is a closed analytic set. Take coordinates $(x_1,\ldots,x_n)$ such that $E=\{\prod_{i\in A}x_i=0\}$, $F_{J'}=\{x_i=0;\, i\in A'\}$ and $\mathcal{H}=\mathrm{Sat}(\mathcal{F},E)$ is locally generated by

$$\omega=\sum_{i\in A}a_i\frac{dx_i}{x_i}+\sum_{i\notin A}a_i dx_i.$$

Put $T_{J'}=\left\{Q\in F_{J'};\ \nu(\mathcal{F},E;Q)\geq 1,\ \frac{\partial a_j}{\partial x_s}(Q)=0,\ \text{for any } j\in A',\ s\notin A'\right\}$. It is a closed analytic subset of $F_{J'}$ intrinsically defined. Moreover, for any point $Q\in F_{J'}-R_{J'}$ the fact that $Q\in T_{J'}$ is equivalent to saying that:

1) $\nu(\mathcal{F},E;Q)\geq 1$.

2) If $\nu(\mathcal{F},E;Q)=1$, then $\mathrm{d}(\mathcal{F},E;Q)=0$.

Thus, we have

$$T_{J'}-R_{J'}=\mathrm{Sing}^*(\mathcal{F},E)\cap F_{J'}-R_{J'}$$

in view of the corollary to Proposition 13. Thus $S_{J'}$ is the topological closure of $T_{J'}-R_{J'}$ and hence it is a closed analytic subset of X. Moreover

$$S_{J'}\subset\mathrm{Sing}^*(\mathcal{F},E).$$

(Take, for any point $P\in S_{J'}-R_{J'}$ a curve $\Gamma\subset T_{J'}-R_{J'}$ and apply Lemma 6.) Now, it is evident that

$$\mathrm{Sing}^*(\mathcal{F},E)=\bigcup_{J'\subset J}S_{J'}$$

and the proof is finished. □

Considering a general normal crossings divisor D on X, we adopt the notation $\mathrm{Sing}^*(\mathcal{F}, D) = \mathrm{Sing}^*(\mathcal{F}, E)$, where $E = \mathrm{Nd}(D, \mathcal{F})$. We say that a point $P \in \mathrm{Sing}\mathcal{F}$ is a *presimple singularity* of $\mathcal{F}$ adapted to D if and only if it is a presimple singularity of $\mathcal{F}$ adapted to E.

2. Global strategy

2.1. *Reduction to presimple singularities. Statement.* Let us make below a precise statement of our result of reduction to presimple singularities:

THEOREM 1. *Let $\mathcal{F}$ be a singular foliation over an ambient space X and let D be a normal crossings divisor on X. Assume that* $\dim X = 3$ *and X is a germ along a compact analytic subset Z. Then there is a morphism $\pi : X' \to X$ such that*

(1) *The morphism π is the composition of a finite sequence of blowing-ups with nonsingular closed analytic centers. Moreover, each center is permissible for the strict transform of $\mathcal{F}$ adapted to the total transform of D.*

(2) *If $\mathcal{F}'$ is the strict transform of $\mathcal{F}$ and D' is the total transform of D in X', then $\mathcal{F}'$ has at most presimple singularities adapted to D'.*

Two important examples for the ambient space X are $(\mathbb{C}^3, 0)$ and any compact ambient space of dimension three.

In this chapter we provide a proof of this result under an additional assumption that we call the local control theorem. Chapter 3 is devoted to the proof of the local control theorem.

In all this chapter we consider a three-dimensional ambient space X that is a germ along a compact core Z, a singular foliation $\mathcal{F}$ and a normal crossings divisor D on X. We also denote $E = \mathrm{Nd}(D, \mathcal{F})$.

2.2. *Good points. Bad points. Equi-reduction.* In this section we develop the *horizontal generic strategy* for the reduction of the singularities of $\mathcal{F}$ to presimple ones.

Definition 4. Consider a point $P \in \mathrm{Sing}^*(\mathcal{F}, D)$. We say that P is a *pre-good* point for $\mathcal{F}$ adapted to D if and only if the following properties hold:

a) The germ of $\mathrm{Sing}^*(\mathcal{F}, D)$ at P is a nonsingular curve having normal crossings with D that is permissible for $\mathcal{F}$ adapted to D.

b) Each component of D at P contains the germ of $\mathrm{Sing}^*(\mathcal{F}, D)$ at P.

c) There is an open set $U \ni P$ such that for any point $Q \in U \cap \mathrm{Sing}^*(\mathcal{F}, D)$ we have that $\nu(\mathcal{F}, E; Q) = \nu(\mathcal{F}, E; P)$ and $\mu(\mathcal{F}, E; Q) = \mu(\mathcal{F}, E; P)$.

Proposition 15. *The non-pre-good points in* $\mathrm{Sing}^*(\mathcal{F}, D)$ *are a finite set.*

Proof. The properties a) and b) above are true outside a zero-dimensional analytic subset of $\mathrm{Sing}^*(\mathcal{F}, D)$. The adapted order is analytically upper semi-continuous. Moreover the adapted multiplicity is also analytically upper semi-continuous under the special conditions of b). This ends the proof. $\square$

Remark 6. In the above proof we have used the fact that the ambient space is a germ along a *compact* core. Without this assumption we would get a locally finite set. In fact, in a way similar to the Hironaka results of reduction of singularities for analytic spaces [16], [2], [3], we could try to do a reduction of the singularities over any ambient space of dimension three. The sequence of blowing-ups we could eventually get in this way should be locally finite on the first ambient space, but it should be necessary to solve also some additional problems of coherence. Anyway, we do not treat this case in this paper.

Let us introduce some terminology about *local blowing-ups*. A local blowing-up at $P \in X$ is the selection of an open set $U \subset X$, a closed non-singular analytic subset $Y \subset U$, with $P \in Y$ and a point $P' \in \pi^{-1}(P)$, where $\pi : X' \to U$ is the blowing-up with center Y. The point P' is called an *infinitely near point* of P. A finite sequence of local blowing-ups is represented by a diagram as follows

$$
\begin{array}{ccccccccccc}
X = X_0 \supset & U_0 & \overset{\pi_1}{\longleftarrow} & X_1 \supset & U_1 & \overset{\pi_2}{\leftarrow} & \dots & \overset{\pi_{k-1}}{\leftarrow} & X_{k-1} \supset & U_{k-1} & \overset{\pi_k}{\longleftarrow} & X_k \\
 & \cup & & & \cup & & & & & \cup & & \\
 & Y_0 & & & Y_1 & & & & & Y_{k-1} & & \Cup \\
 & \Cup & & & \Cup & & & & & \Cup & & \\
P = & P_0 & \longleftarrow & & P_1 & \leftarrow & \dots & \leftarrow & & P_{k-1} & \longleftarrow & P_k
\end{array}
$$

Call this sequence $\mathcal{S}$. For short, we write,

$$\mathcal{S} = \{(X_i, U_i, Y_i, P_i; \pi_{i+1})\}_{i=0}^{k-1} \cup \{(X_k, P_k)\}\,.$$

If we consider infinite sequences of local blowing-ups, we write

$$\mathcal{S} = \{(X_i, U_i, Y_i, P_i; \pi_{i+1})\}_{i=0}^{\infty}\,.$$

Given such sequences we adopt the following notation. Put $\mathcal{F}_0 = \mathcal{F}$, $D_0 = D$ and $E_0 = E$. Denote by $\mathcal{F}_i$ the strict transform of $\mathcal{F}$ in X_i, put $D_i = \pi_i^{-1}(D_{i-1} \cup Y_{i-1})$ and $E_i = \mathrm{Nd}(D_i, \mathcal{F}_i)$. Note that we also have $E_i = \mathrm{Nd}(\pi_i^{-1}(E_{i-1} \cup Y_{i-1}), \mathcal{F}_i)$. We will always deal with situations where the D_i are normal crossings divisors.

Definition 5. Consider a pre-good point $P \in \mathrm{Sing}^*(\mathcal{F}, D)$. We say that P is a *good point* for $\mathcal{F}$ adapted to D if and only if the following property holds: for any sequence $\mathcal{S}$ of local blowing-ups of length k such that $Y_i = U_i \cap \mathrm{Sing}^*(\mathcal{F}_i, D_i)$, for $i = 0, 1, \ldots, k-1$, the last infinitely near point P_k of $\mathcal{S}$ satisfies that either $P_k \notin \mathrm{Sing}^*(\mathcal{F}_k, D_k)$ or P_k is a pre-good point for $\mathcal{F}_k$ adapted to D_k and the morphism

$$\pi_1 \circ \cdots \circ \pi_k : \mathrm{Sing}^*(\mathcal{F}_k, D_k) \to \mathrm{Sing}^*(\mathcal{F}, D)$$

is a local isomorphism at P_k. A *bad point* is a point in $\mathrm{Sing}^*(\mathcal{F}, D)$ which is not a good point. We denote by $\mathrm{Bd}(\mathcal{F}, D)$ the set of bad points.

Remark 7. Note that, by definition, the good points remain good points after the blowing-up with center the singular locus (this property is not true for the pre-good points). The process of blowing-up the singular locus at the good points can be iterated. We will prove below that the process stops after finitely many steps. That is, we get no more points in $\mathrm{Sing}^*(\mathcal{F}, E)$ over the original good points: we get the so-called *equi-reduction property*. As a consequence of this, we shall deduce in Section 2.3 the finiteness of the set of bad points.

Generic equi-reduction. In [10] we have proved the existence of a *generic equi-reduction* along each irreducible component of codimension two of $\mathrm{Sing}^*(\mathcal{F}, D)$ in any ambient dimension. Here we are only interested in the case $\dim X = 3$ and we present a proof for the sake of completeness.

Consider a pre-good point $P \in \mathrm{Sing}^*(\mathcal{F}, D)$. Let Y be the germ of $\mathrm{Sing}^*(\mathcal{F}, D)$ at P. Define the invariant α^* by

$$\alpha^* = 1 + \mu(\mathcal{F}, E; Y) - \mu(\mathcal{F}, E; P).$$

Then, either $\alpha^* = 0$ or $\alpha^* = 1$. The condition $\alpha^* = 1$ is equivalent to saying that Y is an appropriate center at P. If $\alpha^* = 0$, the blowing-up with center Y is dicritical but $\mathrm{Rs}(\mathcal{F}, E; P) = 0$.

LEMMA 7. *The resonance invariant is constant along Y. If $\mathrm{Rs}(\mathcal{F}, E; P) = 1$, the blowing-up with center Y is dicritical.*

Proof. Note that $\mathrm{e}(E, P) \leq 2$ in view of the property b) of Definition 4. If either $\mathrm{e}(E, P) = 1$ or $m = r+1$, where $r = \nu(\mathcal{F}, E; P)$ and $m = \mu(\mathcal{F}, E; P)$, the resonance invariant is zero. Hence, the only case to be considered is $\mathrm{e}(E, P) = 2$ and $m = r$, where $r = \nu(\mathcal{F}, E; P)$ and $m = \mu(\mathcal{F}, E; P)$. Take local coordinates (x, y, z) such that $E = \{yz = 0\}$ and $Y = \{y = z = 0\}$. Let

$$\omega = a dx + b\frac{dy}{y} + c\frac{dz}{z}$$

be a local generator of $\mathrm{Sat}(\mathcal{F}, E)$. Note that $\nu_{(y,z)}(b, c) = r$. Write $b = \sum_{ij} b_{ij}(x) y^i z^j$ and put $B = \sum_{i+j=r} b_{ij}(x) y^i z^j$; same thing with the coefficient c. Looking at the lowest degree terms in the integrability condition

$\omega \wedge d\omega = 0$ we get that

$$B\frac{\partial C}{\partial x} = C\frac{\partial B}{\partial x}.$$

The quotient B/C does not depend on x and thus the resonance is constant along $y = z = 0$. The fact that $\mathrm{Rs}(\mathcal{F}, E; P) = 1$ means that $B(0, y, z) + C(0, y, z) \equiv 0$, then $B + C \equiv 0$ and the blowing-up with center $y = z = 0$ is dicritical. □

Let $\Delta \simeq (\mathbb{C}^2, 0)$ be a germ at P of the subspace of X. We say that Δ is transversal to $\mathcal{F}$ if and only if it is transversal to Y and $\mathcal{F}|_\Delta \in \mathbb{F}(\Delta)$, where $\mathcal{F}|_\Delta$ is generated by the restriction to Δ of a local generator of $\mathcal{F}$. The *transversality theorem* in [25] assures the existence of Δ transversal to $\mathcal{F}$. Let $\pi : X' \to X$ be the (local) blowing-up with center Y and $\overline{\pi} : \Delta' \to \Delta$ the blowing-up with center P. Note that $\Delta' \subset X'$ is the strict transform of Δ by π and $\overline{\pi}$ is the corresponding restriction of π. Denote by $\mathcal{F}'$ the strict transform of $\mathcal{F}$ by π and by $(\mathcal{F}|_\Delta)'$ the strict transform of $\mathcal{F}|_\Delta$ by $\overline{\pi}$. Clearly

$$(\mathcal{F}|_\Delta)' = \mathrm{Sat}(\mathcal{F}'|_{\Delta'}, \emptyset)$$

but $\mathcal{F}'|_{\Delta'}$ is not necessarily a (saturated) singular foliation on Δ'.

Lemma 8. *Assume that $\alpha^* = 1$ and Δ is transversal to $\mathcal{F}$. Then*

$$\begin{aligned} \Delta \cap D &= \mathrm{Nd}(\Delta \cap E, \mathcal{F}|_\Delta) \\ \overline{r} = \nu(\mathcal{F}|_\Delta, \Delta \cap E; P) &= \nu(\mathcal{F}, E; P) = r \\ \overline{m} = \mu(\mathcal{F}|_\Delta, \Delta \cap E; P) &= \mu(\mathcal{F}, E; P) = m \\ \mathrm{Rs}(\mathcal{F}|_\Delta, \Delta \cap E; P) &= \mathrm{Rs}(\mathcal{F}, E; P) \end{aligned}$$

and $\overline{\pi}$ is dicritical for $\mathcal{F}|_\Delta$ if and only if π is dicritical for $\mathcal{F}$. Moreover, the restriction $\mathcal{F}'|_{\Delta'}$ is a (saturated) singular foliation on Δ'; in fact $\mathcal{F}'|_{\Delta'} = (\mathcal{F}|_\Delta)'$.

Proof. Take local coordinates (x, y, z) at P such that $Y = \{y = z = 0\}$ and $E = \{y^{\varepsilon_1} z^{\varepsilon_2} = 0\}$, with $\varepsilon_i \in \{0, 1\}$ and $e = \varepsilon_1 + \varepsilon_2$. Let

$$\omega = a dx + b\frac{dy}{y^{\varepsilon_1}} + c\frac{dz}{z^{\varepsilon_2}}$$

be a local generator of $\mathrm{Sat}(\mathcal{F}, E)$. Then $\eta = y^{\varepsilon_1} z^{\varepsilon_2}\omega$ is a local generator of $\mathcal{F}$. By transversality, the coefficients of

$$\eta|_\Delta = y^{\varepsilon_1} z^{\varepsilon_2}\omega|_\Delta = z^{\varepsilon_2} b(0, y, z)dy + y^{\varepsilon_1} c(0, y, z)dz$$

have no common factor. Then, the components of $\Delta \cap E$ are nondicritical components for $\mathcal{F}|_\Delta$. Let us prove that $\overline{r} = r$. Note first that $\nu_{(y,z)}(a, b, c) = r$. If $\overline{r} > r$, then $\nu_{(y,z)}(b(0, y, z), c(0, y, z)) > r$ and this implies that $\nu_{(y,z)}(b, c) > r$. We get both $m = r + 1$ and $\nu_{(x,y,z)}(a) = r$; this contradicts the fact that $\alpha^* = 1$. Thus $\overline{r} = r$. A similar argument shows that $\overline{m} = m$. The result on the resonance invariant is obvious, when we note that $\mathrm{In}^r(b(0,y,z)) = \mathrm{In}^r(b)$ and the same thing with the coefficient c. Let us prove the statement about dicriticalness. Assume that $m = r$. We know that the blowing-up $\overline{\pi}$ (centered in a point) is dicritical if and only if $\mathrm{Rs}(\mathcal{F}|_\Delta, \Delta \cap E; P) = 1$; moreover, in view of the preceding lemma, we also have that π is dicritical if and only if $\mathrm{Rs}(\mathcal{F}, E; P) = 1$. Then we are done. Assume now that $m = r+1$. Note that $\nu_{(y,z)}(a) \geq r+1$ since $\alpha^* = 1$. Put $\widetilde{E} = \{yz = 0\} \supset E$. In fact we artificially add some possibly dicritical components to E. Write

$$\omega = adx + \tilde{b}\frac{dy}{y} + \tilde{c}\frac{dz}{z} = adx + y^{1-\varepsilon_1} b\frac{dy}{y} + z^{1-\varepsilon_2} c\frac{dz}{z}.$$

Note that $r+1 = \nu(\omega, \widetilde{E}; P) = \mu(\omega, \widetilde{E}; P)$. We can repeat exactly the above arguments (and the arguments in the preceding lemma) with this new set of coefficients a, $\tilde{b}$ and $\tilde{c}$. Thus $\overline{\pi}$ is dicritical if and only if π is. Let us show finally that $\mathcal{F}'|_{\Delta'} = (\mathcal{F}|_\Delta)'$. Note from the above that $\mathcal{H}|_\Delta = \mathrm{Sat}(\mathcal{F}|_\Delta, \Delta \cap E)$. Since $\alpha^* = 1$, a local generator ω' of $\mathcal{H}' = \mathrm{Sat}(\mathcal{F}', \pi^{-1}(E \cup Y)$ is given by $\omega' = y'^{-m}\pi^*\omega$, where y' is a reduced local equation of the exceptional divisor. Thus, a local generator η' of $\mathcal{F}'$ is given either by $y'^{1-e-m}\pi^*\eta$ if π is nondicritical or by $y'^{-e-m}\pi^*\eta$ if π is dicritical. We use the same procedure and the same invariants to compute a local generator $(\eta|_\Delta)'$ of $(\mathcal{F}|_\Delta)'$. Then $(\eta|_\Delta)' = \eta'|_{\Delta'}$. This ends the proof. □

Proposition 16 (*generic equi-reduction*). *Let P be a good point for $\mathcal{F}$ adapted to D. Then, there is no infinite sequence of local blowing-ups*

$$\mathcal{S} = \{(X_i, U_i, Y_i, P_i; \pi_{i+1})\}_{i=0}^{\infty}$$

with $P_0 = P$, such that for any $i = 0, 1, \ldots,$ we have $Y_i = U_i \cap \mathrm{Sing}^*(\mathcal{F}_i, D_i)$.

Proof. Let us reason by contradiction. For an index $i \geq 0$, denote

$$\begin{aligned} r_i &= \nu(\mathcal{F}_i, E_i; P_i); & m_i &= \mu(\mathcal{F}_i, E_i; P_i); & \alpha_i &= \mu(\mathcal{F}_i, E_i; Y_i) \\ \mathrm{Rs}_i &= \mathrm{Rs}(\mathcal{F}_i, E_i; P_i); & m_i^* &= \mu^*(\mathcal{F}_i, E_i; P_i); & \alpha_i^* &= 1 + \alpha_i - m_i \\ e_i &= \mathrm{e}(E_i, P_i); & \mathcal{H}_i &= \mathrm{Sat}(\mathcal{F}_i, E_i) \end{aligned}$$

and let ω_i be a local generator of $\mathcal{H}_i$ at P_i that we will choose below in a convenient way.

Let us first show that $(r_{i+1}, m_{i+1}^*, \alpha_{i+1}^*) \leq (r_i, m_i^*, \alpha_i^*)$ for the lexicographical ordering. The fact that $r_{i+1} \leq r_i$ is given by the vertical stability of the adapted order under permissible blowing-ups. Assume that $r_{i+1} = r_i = r$. If $\alpha_i^* = 1$, we have an appropriate center and $m_{i+1}^* \leq m_i^*$. If $\alpha_i^* = 0$, then $m_i = r+1$ and *a fortiori* we have $m_{i+1}^* \leq m_i^*$. Assume also that $m_{i+1}^* = m_i^*$ and let us show that $\alpha_{i+1}^* \leq \alpha_i^*$. We have only to consider the case $\alpha_i^* = 0$. Choose local coordinates (x, y, z) at P_i such that $Y_i = \{y = z = 0\}$ and $E_i \subset \{yz = 0\}$. The fact that $\alpha_i = 0$ is equivalent to saying that $m_i = r+1$ and $\nu_{P_i}(a_i) = r$, where $a_i = \omega_i(\partial/\partial x)$. After a suitable coordinate change, we may assume in addition that (x', y', z') are local coordinates in P_{i+1}, where $x = x'$, $y = y'$, $z = y'(z' + \zeta)$ and $Y_{i+1} = \{y' = z' = 0\}$. Since $\alpha_i = 0$, the blowing-up π_{i+1} is dicritical and $e_{i+1} \leq 1$; thus $\mathrm{Rs}_{i+1} = 0$ and hence $m_{i+1} = m^*_{i+1} = r + 1$. Now $a_{i+1} = y'^{-r}(a_i \circ \pi_{i+1})$, hence $\nu_{P_{i+1}}(a_{i+1}) \leq r$ and thus $\alpha^*_{i+1} = 0$.

We can assume that $(r_i, m^*_i, \alpha^*_i) = (r, m^*, \alpha^*)$ for all indices $i \geq 0$. We shall distinguish two cases: $\alpha^* = 1$ and $\alpha^* = 0$.

Case $\alpha^* = 1$. (*Control by a transversal section*). Let $\Delta = \Delta_0$ be a two dimensional transversal section for $\mathcal{F}_0$ at P_0. Denote by Δ_{i+1} the strict transform of Δ_i by π_{i+1} at P_{i+1}. The above lemma implies that each Δ_i is transversal to $\mathcal{F}_i$ at P_i and $\mathcal{F}_{i+1}|_{\Delta_{i+1}}$ is the strict transform of $\mathcal{F}_i|_{\Delta_i}$ by the blowing-up $\Delta_{i+1} \to \Delta_i$. Invoking the reduction of the singularities in dimension two, there is an index $i \geq 0$ such that P_i is a presimple singularity for $\mathcal{F}_i|_{\Delta_i}$ adapted to $\Delta_i \cap E_i$. This easily implies (write the equations) that P_i is presimple for $\mathcal{F}_i$ adapted to E_i, in contradiction to the fact that $P_i \in \mathrm{Sing}^*(\mathcal{F}_i, D_i)$.

Case $\alpha^* = 0$. (*Nonresonant dicritical case*). Since each π_i is a dicritical blowing-up, we have that $1 \geq e_i \geq e_{i+1} \geq 0$, for $i \geq 1$. Thus assume that $e = e_i$, for all $i \geq 0$, with $e \in \{0, 1\}$. Let us do the proof first under the assumption that there are systems of coordinates (x_i, y_i, z_i) at each P_i such that $Y_i = \{y_i = z_i = 0\}$, $E_i = \{z_i^\varepsilon = 0\}$, where $\varepsilon \in \{0, 1\}$, $x_i = x_{i+1}$, $y_i = y_{i+1}$ and $z_i = y_{i+1} z_{i+1}$. Put $(x, y, z) = (x_0, y_0, z_0)$ and let

$$\omega = a dx + b dy + c \frac{dz}{z^\varepsilon}$$

be a local generator of $\mathcal{H}_0$. Making the blowing-ups given by the above equations, we find that if $\varepsilon = 1$ then z^r must divide a, b, c and that if $\varepsilon = 0$ then z^r must divide a, b and yz^{r-1} must divide c. Since $r \geq 1$, the only case to be considered is $r = 1$ and $\varepsilon = 0$. In that case we necessarily have that $\omega = z dx + z b^* dy + y c^* dz$ which is not an integrable form, which is a contradiction. To end the proof, let us assure the existence of the coordinate systems (x_i, y_i, z_i). We do the case $e = 0$, the case $e = 1$ is similar. Start with (x, y, z) such that $Y_0 = \{y = z = 0\}$ and π_1 is given by $x = x'$, $y = y'$ and $z = y'(z' + \zeta)$. Since $r_1 = r$, we necessarily have that $\mathrm{In}^r(a) = (z - \zeta y)^r$. Performing a linear change of coordinates $z \mapsto z - \zeta y$, we can assume that $\zeta = 0$. Moreover, the new center Y_1 is given by $y' = z' - \phi_1(x) = 0$. Up to a change $z \mapsto z - y\phi_1(x)$ we get that $Y_1 = \{y' = z' = 0\}$. Now, since $a_1 = y'^{-r}(a \circ \pi_1)$, we see that $\mathrm{In}^r(a_1) = (z' - \zeta_1 y')^r$. Repeat. We get in this way our coordinate systems up to a formal initial coordinate change of the type $z \mapsto z - \sum \zeta_i y^{i+1} - \sum y^i \phi_i(x)$. □

2.3. *Finiteness of bad points.* Here we shall prove that the set of bad points $\mathrm{Bd}(\mathcal{F}, D)$ is a finite subset of the ambient space X. We shall deduce from this a proof of Theorem 1 of reduction to presimple singularities in the case when there are no bad points. In order to give a stratified description of the bad points, we shall construct a canonical sequence of global blowing-ups that we call *the equi-reduction sequence*.

Let us note that all the finiteness properties (statements of Lemma 9, Proposition 19 and its first corollary) to be proved in this section are due to the fact that we work on an ambient space that is a germ along a *compact* core. Otherwise we would get properties of local finiteness, as we pointed out in Remark 6.

Define the *badness set of order zero* B_0 to be the set of points in $\mathrm{Sing}^*(\mathcal{F}, D)$ that are not pre-good points. We know that B_0 is a finite set. Let S^* be the set of irreducible components Γ of $\mathrm{Sing}^*(\mathcal{F}, D)$ such that $\dim \Gamma = 1$. It is a finite set. Put $S^* = \{\Gamma_j; j \in I_0\}$ where $I_0 = \{1, 2, \ldots, s\}$. Our first task is to arrive at the situation when each Γ_j is permissible and that $\Gamma_i \cap \Gamma_j = \emptyset$ if $i \neq j$. We get this by a finite sequence of blowing-ups centered at points.

PROPOSITION 17. *There is a unique morphism $\pi'_0 : X'_0 \to X_0$ that is a finite composition of point blowing-ups*

$$X_0 = X_{00} \overset{\pi_{01}}{\leftarrow} X_{01} \overset{\pi_{02}}{\leftarrow} \cdots \overset{\pi_{0k}}{\leftarrow} X_{0k} = X'_0$$

of centers $P_{0j} \in X_{0j}$, for $j = 1, \ldots, k-1$, satisfying the following properties.

a) *Denote by Γ_i^j the* strict *transform of Γ_i in X_{0j}. Let $\mathcal{F}_{0j}$ and D_{0j} be the transforms of $\mathcal{F}$ and D, as usual. Then, either there are two indices i, l such that $P_{0j} \in \Gamma_i^j \cap \Gamma_l^j$ or $P_{0j} \in \Gamma_i^j$ for a single index i and Γ_i^j is nonpermissible for $\mathcal{F}_{0j}$ at P_{0j} adapted to D_{0j}.*

b) *Put $\Gamma'_i = \Gamma_i^k$, for $i = 1, \ldots, s$. Then each Γ'_i is permissible for $\mathcal{F}'_0 = \mathcal{F}_{0k}$ adapted to $D'_0 = D_{0k}$ and moreover $\Gamma'_i \cap \Gamma'_l = \emptyset$ if $i \neq l$.*

Proof. By applying repeatedly the criteria indicated in property a), we get first a reduction of the singularities of the curve $\Gamma_1 \cup \cdots \cup \Gamma_s$ and second each irreducible component becomes permissible as was indicated in Proposition 3. The uniqueness is obvious from the fact that we only deal with centers that are isolated points. □

Remark 8. The morphism π'_0 induces an isomorphism between $X'_0 - {\pi'_0}^{-1}(B_0)$ and $X_0 - B_0$. In particular, each center P_{0j} is projected over a point in B_0.

Let $\sigma_1 : X_1 \to X'_0$ be the blowing-up with center the union of the curves Γ'_i, for $i = 1, \ldots, s$. Denote by $\mathcal{F}_1$ and D_1 the transforms of $\mathcal{F}'_0$ and D'_0. We get a morphism $\rho_1 : X_1 \to X_0$ given by the composition $\rho_1 = \pi_0' \circ \sigma_1$. Let B_1^1 be the set of points $Q \in \mathrm{Sing}^*(\mathcal{F}_1, D_1)$ such that either Q is not a pre-good point for $\mathcal{F}_1$ adapted to D_1 or the morphism ρ_1 is not a local isomorphism at Q between $\mathrm{Sing}^*(\mathcal{F}_1, D_1)$ and $\mathrm{Sing}^*(\mathcal{F}_0, D_0)$. Define the *badness set of order one* B_1 by $B_1 = \rho_1(B_1^1) - B_0$.

LEMMA 9. *The set B_1 is a finite subset of* $\mathrm{Sing}^*(\mathcal{F}, D)$.

Proof. The set B_1^1 is a closed analytic subset of $\mathrm{Sing}^*(\mathcal{F}_1, D_1)$. Note that in view of the vertical stability of presimple singularities under permissible blowing-ups, we have that $\rho_1\left(\mathrm{Sing}^*(\mathcal{F}_1, D_1)\right) \subset \mathrm{Sing}^*(\mathcal{F}, D)$. Now, it is enough to prove that $\rho_1(\Gamma)$ is a single point for any one-dimensional irreducible component Γ of B_1^1. Assume not; then $\rho_1(\Gamma) = \Gamma_j \in S^*$. Then, there is only a finite set of points Q in Γ such that $\Gamma \to \Gamma_j$ is not a local isomorphism at Q. Add the non-pre-good points to these; then Γ should be a finite set, contradiction. $\square$

Put $S_0^* = S^*$. Let S_1^* be the set of irreducible components Γ of $\mathrm{Sing}^*(\mathcal{F}_1, D_1)$ such that $\dim \rho_1(\Gamma) = 1$. Given any $\Gamma \in S_1^*$, then $\rho_1(\Gamma) \in S_0^*$. For any index $j \in I_0$, let

$$S_1^*(j) = \left\{\Gamma_{j1}, \ldots, \Gamma_{js_j}\right\} = \{\Gamma_\alpha\}_{\alpha \in \{j\} \times I(j)}$$

be the set of elements $\Gamma_{jl} \in S_1^*$ such that $\rho_1(\Gamma_{jl}) = \Gamma_j$. Then

$$S_1^* = \bigcup_{j \in I_0} S_1^*(j) = \bigcup_{\alpha \in I_1} \{\Gamma_\alpha\}$$

where $I_1 = \bigcup_{j \in I_0} \{j\} \times I(j) \subset \mathbb{N}^2$. Note that each $I(j)$ is finite, possibly empty. If $S_1^* = \emptyset$, we stop and we say that the two morphisms

$$X_0 \overset{\pi_0'}{\leftarrow} X_0' \overset{\sigma_1}{\leftarrow} X_1$$

define the *equi-reduction sequence*. Assume that $S_1^* \neq \emptyset$ or, equivalently, that the set of indices I_1 is nonempty. Then, we repeat the above procedure starting with S_1^*. To be precise, we construct first a morphism $\pi_1' : X_1' \to X_1$, which is a finite composition of point blowing-ups

$$X_1 = X_{10} \overset{\pi_{11}}{\leftarrow} X_{11} \overset{\pi_{12}}{\leftarrow} \cdots \overset{\pi_{1k_1}}{\leftarrow} X_{1k_1} = X_1'$$

centered at points $P_{1j} \in X_{1j}$ with the following properties. Denote by Γ_α^j the strict transform of Γ_α in X_{1j}, then

a) Given an index j with $0 \leq j \leq k_1 - 1$, we have that either there are two indices $\alpha \neq \beta$ in I_1 such that $P_{1j} \in \Gamma_\alpha^j \cap \Gamma_\beta^j$ or $P_{1j} \in \Gamma_\alpha^j$ for a single α and Γ_α^j is nonpermissible at P_{1j} for $\mathcal{F}_{1j}$ adapted to D_{1j}.

b) Put $\Gamma_\alpha' = \Gamma_\alpha^{k_1}$. Each Γ_α' is permissible for $\mathcal{F}_1'$ adapted to D_1' and $\Gamma_\alpha' \cap \Gamma_\beta' = \emptyset$ if $\alpha \neq \beta$.

Let $\sigma_2 : X_2 \to X_1'$ be the blowing-up with center the (disjoint) union of the Γ_α', where $\alpha \in I_1$. Put $\rho_2 = \pi_1' \circ \sigma_2$. Let B_2^2 be the set of points $Q \in \mathrm{Sing}^*(\mathcal{F}_2, D_2)$ such that either Q is not a pre-good point for $\mathcal{F}_2$ adapted to D_2 or the morphism ρ_2 is not a local isomorphism at Q between $\mathrm{Sing}^*(\mathcal{F}_2, D_2)$ and $\mathrm{Sing}^*(\mathcal{F}_1, D_1)$. Put $B_1^2 = \rho_2(B_2^2) - B_1^1$. It is a finite set. The *badness set of order two* B_2 is defined by

$$B_2 = \rho_1(B_1^2) - B_0 \cup B_1.$$

It is obviously a finite set. Let S_2^* be the set of irreducible components Γ of $\mathrm{Sing}^*(\mathcal{F}_2, D_2)$ such that $\dim \rho_1\rho_2(\Gamma) = 1$. This is equivalent to saying that $\rho_2(\Gamma) \in S_1^*$. For any index $\alpha \in I_1$, let

$$S_2^*(\alpha) = \{\Gamma_{\alpha 1}, \ldots, \Gamma_{\alpha s_\alpha}\} = \{\Gamma_\beta\}_{\beta \in \{\alpha\} \times I(\alpha)}$$

be the set of elements $\Gamma_{\alpha l} \in S_2^*$ such that $\rho_1(\Gamma_{\alpha l}) = \Gamma_\alpha$. Then

$$S_2^* = \bigcup_{\alpha \in I_1} S_1^*(\alpha) = \bigcup_{\beta \in I_2} \{\Gamma_\beta\}$$

where $I_2 = \bigcup_{\alpha \in I_1} \{\alpha\} \times I(\alpha) \subset \mathbb{N}^3$. If $S_2^* = \emptyset$, we stop and the morphisms

$$X_0 \overset{\pi_0'}{\leftarrow} X_0' \overset{\sigma_1}{\leftarrow} X_1 \overset{\pi_1'}{\leftarrow} X_1' \overset{\sigma_2}{\leftarrow} X_2$$

define the equi-reduction sequence. Otherwise, continuing, we get in this way a finite or infinite sequence of morphisms

$$X_j \overset{\pi_j'}{\leftarrow} X_j' \overset{\sigma_{j+1}}{\leftarrow} X_{j+1},\ \rho_j = \pi_j' \circ \sigma_j$$

for $j = 0, 1, \ldots$, called *the equi-reduction sequence*.

Let us recall the objects we have constructed up till now. Consider indices $0 \leq i \leq j$, where $j \geq 1$ is not greater than the length of the equi-reduction sequence. We have a finite set $B_0 = B_0^0$ and sets B_i^j, that are finite for $i < j$, defined from B_j^j by

$$B_i^j = \rho_{i+1}(B_{i+1}^j) - B_i^i \cup B_i^{i+1} \cup \cdots \cup B_i^{j-1}.$$

The *badness set of order* j is the set $B_j = B_0^j$. We have families S_j^* indexed by finite sets $I_j \subset \mathbb{N}^{j+1}$, whose elements are the irreducible components Γ_β of $\mathrm{Sing}^*(\mathcal{F}_j, D_j)$ such that $\rho_j(\Gamma_\beta) \in S_{j-1}^*$. The indexing has the property that $\rho_j(\Gamma_\beta) = \Gamma_\alpha$, where α is the projection of β over the first j entries.

Proposition 18. *The set of bad points* $\mathrm{Bd}(\mathcal{F}, D)$ *is the union of all the badness sets* B_j *of order* j, *smaller than or equal to the length of the equi-reduction sequence.*

Proof. Take a point $P \in B_k$. Let us show that $P \in \mathrm{Bd}(\mathcal{F}, D)$. If $k = 0$, then P is not a pre-good point and we are done. Assume $k \geq 1$. Then we have a finite sequence of points $P_i \in \mathrm{Sing}^*(\mathcal{F}_i, D_i)$, for $i = 0, 1, \ldots, k$ such that $P = P_0$, $P_k \in B_k$ and

$$P_i = \rho_i(P_{i-1}) \notin B_i^i \cup \ldots \cup B_i^{k-1}$$

for $i = 1, \ldots, k-1$. Let $\alpha = (\alpha_0, \ldots, \alpha_{k-1}) \in I_{k-1}$ be such that $P_{k-1} \in \Gamma_\alpha$. Then $P_i \in \Gamma_{\alpha(i)}$, where $\alpha(i) = (\alpha_0, \ldots, \alpha_i)$, for $0 \leq i \leq k-1$. For an index i with $0 \leq i \leq k-1$, define the open sets $U_{i,\alpha}^k \subset X_i$ by

$$U_{i,\alpha}^k = \left(X_i - \bigcup_{\beta \in I_i; \beta \neq \alpha(i)} \Gamma_\beta \right) - B_i^i \cup \ldots \cup B_i^{k-1}$$

with the following properties:

(1) $U_{i,\alpha}^k \subset \rho_i^{-1}(U_{i-1,\alpha}^k)$, for $1 \leq i \leq k-1$.

(2) $P_i \in U_{i,\alpha}^k$, for $0 \leq i \leq k-1$.

(3) $\mathrm{Sing}^*(\mathcal{F}_i, D_i) \cap U_{i,\alpha}^k = \Gamma_{\alpha(i)} \cap U_{i,\alpha}^k$ and $\Gamma_{\alpha(i)} \cap U_{i,\alpha}^k$ is permissible for the restriction of $\mathcal{F}_i$ and D_i to $U_{i,\alpha}^k$.

(4) For any $i = 0, 1, \ldots, k-1$, the morphism ρ_{i+1} induces by restriction the blowing-up $\pi_{i+1} : \rho_{i+1}^{-1}(U_{i,\alpha}^k) \to U_{i,\alpha}^k$ with center $\Gamma_{\alpha(i)} \cap U_{i,\alpha}^k$.

Putting $\widetilde{X}_0 = X_0$ and $\widetilde{X}_i = \rho_i^{-1}(U_{i-1,\alpha}^k)$ for $1 \leq i \leq k$, we get a finite sequence of local blowing-ups

$$\left\{ \left(\widetilde{X}_i, U_{i,\alpha}^k, \Gamma_{\alpha(i)} \cap U_{i,\alpha}^k, P_i; \pi_{i+1} \right) \right\}_{i=0}^{k-1} \cup \left\{ \widetilde{X}_k, P_k \right\}$$

that allows us to show that P is not a good point, since P_k fails to be pre-good or to give a local isomorphism between $\mathrm{Sing}^*(\mathcal{F}_k, D_k)$ and $\mathrm{Sing}^*(\mathcal{F}, D)$.

Conversely, let $P \in \mathrm{Sing}^*(\mathcal{F}, D)$ be a point not in B_j for every $j \geq 0$. Since $P \notin B_0$, it is a pre-good point. Consider a finite sequence of local blowing-ups

$$\mathcal{S} = \left\{ \left(\widetilde{X}_i, \widetilde{U}_i, \widetilde{Y}_i, \widetilde{P}_i; \widetilde{\pi}_{i+1} \right) \right\}_{i=0}^{k-1} \cup \left\{ \widetilde{X}_k, \widetilde{P}_k \right\}$$

where $\widetilde{X}_0 = X$, $\widetilde{P}_0 = P$ and $\widetilde{Y}_i = \widetilde{U}_i \cap \mathrm{Sing}^*(\mathcal{F}_i, D_i)$ for $i = 0, 1, \ldots, k-1$. Let $\alpha_0 = \alpha(0) \in I_0$ be such that $P \in \Gamma_{\alpha(0)}$ and consider the open set $U_{0,\alpha(0)}^k$ defined as above. Then $\widetilde{P}_0 \in U_{0,\alpha(0)}^k$. We can restrict the sequence $\mathcal{S}$ to the open set $\widetilde{W}_0 = \widetilde{U}_0 \cap U_{0,\alpha(0)}^k$ by considering the inverse image of $\widetilde{W}_0$. The new sequence has the same local properties as the old one at the points $\widetilde{P}_i$, $0 \leq i \leq k$. Thus, we may assume without loss of generality that $\widetilde{U}_0 \subset U_{0,\alpha(0)}^k$. Now, the morphism $\widetilde{\pi}_1$ is equal (up to isomorphisms) to the restriction of ρ_1 to $\rho_1^{-1}(\widetilde{U}_0) \to \widetilde{U}_0$. Then

$$\widetilde{X}_1 = \rho_1^{-1}(\widetilde{U}_0) \subset X_1$$

and hence $\widetilde{P_1} = P_1 \in X_1$. Since $P \notin B_1$, we get that $P_1 \notin B_1^1$. Thus, P_1 is a pre-good point and there is a local isomorphism at P_1 between $\mathrm{Sing}^*(\mathcal{F}_1, D_1)$ and $\mathrm{Sing}^*(\mathcal{F}, D)$. Let $\alpha(1) = (\alpha_0, \alpha_1) \in I_1$ be such that $P_1 \in \Gamma_{\alpha(1)}$. Making a restriction as above, we can assume that $\widetilde{U}_1 \subset U_{1,\alpha(1)}$. Repeating the procedure k times, if P_k is in $\mathrm{Sing}^*(\mathcal{F}_k, D_k)$ and is a pre-good point, we have a local isomorphism at P_k between $\mathrm{Sing}^*(\mathcal{F}_k, D_k)$ and $\mathrm{Sing}^*(\mathcal{F}, D)$. This proves that P is a good point. □

The next result (a version of Koenig's lemma [21]) is a key logical tool in our arguments.

Lemma 10. *Let $\{I_j\}_{j=0}^{\infty}$ be a sequence of nonempty finite sets $I_j \subset \mathbb{N}^{j+1}$, such that $\mathrm{pr}_j(I_j) \subset I_{j-1}$, for $j \geq 1$, where pr_j is the projection over the first j-entries. Then there is a sequence $\alpha = (\alpha_0, \alpha_1, \dots)$ such that $\alpha(j) = (\alpha_0, \dots, \alpha_j) \in I_j$ for all $j \geq 0$.*

Proof. The set $I = \bigcup_{j=0}^{\infty} I_j$ is infinite, but I_0 is finite. Thus there is $\alpha_0 \in I_0$ such that the elements in I projecting over α_0 form an infinite set I^1. Repeat the arguments with the new sequence $I_1 \cap I^1, I_2 \cap I^1, \dots$ to get $(\alpha_0, \alpha_1) \in I_1 \cap I^1$ such that the elements in I^1 projecting over (α_0, α_1) form an infinite set. We obtain inductively in this way the desired sequence α. □

Proposition 19. *The equi-reduction sequence is finite.*

Proof. Assume the contrary to get a contradiction. The sets $S_j^* = \{\Gamma_\beta\}_{\beta \in I_j}$ are finite and nonempty. By the lemma, we get a sequence $\alpha = (\alpha_0, \alpha_1, \dots)$ such that $\alpha(j) = (\alpha_0, \dots, \alpha_j) \in I_j$ for all $j \geq 0$. The curve $\Gamma_{\alpha(0)}$ is a uncountable set of points. Thus there is at least one good point $P \in \Gamma_{\alpha(0)}$ since the set of bad points $\mathrm{Bd}(\mathcal{F}, D) = \cup_{j=0}^{\infty} B_j$ is a countable set. The sets

$$\mathcal{L}_j = \Gamma_{\alpha(j)} \cap (\rho_1 \circ \cdots \circ \rho_j)^{-1}(P)$$

are nonempty, finite and $\rho_j(\mathcal{L}_j) \subset \mathcal{L}_{j-1}$ for all $j \geq 1$. By the lemma, there is a sequence of points $P_j \in \Gamma_{\alpha(j)}$ such that $P_0 = P$ and $\rho_j(P_j) = P_{j-1}$ for $j \geq 1$.

Consider as in the above proposition, the infinite sequence of local blowing-ups

$$\mathcal{S} = \left\{ \widetilde{X}_j, U^{j+1}_{j,\alpha(j)}, Y_j, P_j; \pi_{j+1} \right\}_{j=1}^{\infty}$$

where $\widetilde{X}_0 = X$, $\widetilde{X}_j = \rho_j^{-1}\left(U^j_{j-1,\alpha(j-1)}\right)$, $Y_j = \Gamma_{\alpha(j)} \cap U^{j+1}_{j,\alpha(j)}$ and π_{j+1} is the corresponding restriction of ρ_{j+1}. This sequence $\mathcal{S}$ contradicts the generic equi-reduction statement of Proposition 16. □

Corollary. *The set of bad points* $\mathrm{Bd}(\mathcal{F}, D)$ *is finite.*

Proof. It is a finite union of the finite sets B_j. □

Corollary. *If* $\mathrm{Bd}(\mathcal{F}, D) = \emptyset$ *then the equi-reduction sequence gives the reduction to presimple singularities.*

Proof. Let k be the length of the equi-reduction sequence. Then $S_k^* = \emptyset$. Moreover $B_k^k = \emptyset$ since otherwise B_k and hence $\mathrm{Bd}(\mathcal{F}, D)$ would be nonempty. We get that $\emptyset = \mathrm{Sing}^*(\mathcal{F}_k, D_k) = B_k^k \cup \bigcup \{\gamma; \gamma \in S_k^*\}$ as desired. □

In view of this corollary, we will get the reduction to presimple singularities by first eliminating the bad points and then applying the equi-reduction sequence.

2.4. *The influency locus.* Given a point $P \in X$, define the Samuel invariant $\mathrm{ISam}(\mathcal{F}, D; P)$ to be the pair

$$\mathrm{ISam}(\mathcal{F}, D; P) = (\nu(\mathcal{F}, E; P), \mu^*(\mathcal{F}, E; P)) \,.$$

The global Samuel invariant $\mathrm{ISam}(\mathcal{F}, D)$ is the maximum of these invariants over the bad points, for the lexicographical ordering. Given a pair (p, q), denote $\mathrm{Bd}^{p,q}(\mathcal{F}, D)$ the set of bad points having exactly the pair (p, q) as Samuel invariant. Our objective is to decrease the global Samuel invariant by a finite sequence of blowing-ups with *very good centers* that are either bad points or curves in the *Influency Locus*, which we will define below. Let us use the notation $\mathrm{ISam}(\mathcal{F}, D) = (r, m^*)$ in this section.

Definition 6. Let Y be an irreducible analytic subspace of $\mathrm{Sing}\mathcal{F}$. We say that Y is a *good center* for $\mathcal{F}$ adapted to D if and only if it is a permissible center that contains at least one bad point and is appropriate at each bad point in Y.

Remark 9. A bad point is a good center; a good point is not a good center.

Proposition 20. *Let* $\pi : X' \to X$ *be the blowing-up with good center* Y*. Denote by* $\mathcal{F}'$ *and* D' *the transforms of* $\mathcal{F}$ *and* D*. Then for any bad point* P' *in* $\mathrm{Bd}(\mathcal{F}', D')$ *the image* $P = \pi(P')$ *is a bad point for* $\mathcal{F}$, D *and*

$$\mathrm{ISam}(\mathcal{F}', D'; P') \leq \mathrm{ISam}(\mathcal{F}, D; P).$$

In particular $\mathrm{ISam}(\mathcal{F}', D') \leq \mathrm{ISam}(\mathcal{F}, D)$*.*

Proof. For the first part, it is enough to note that the germ of Y at any good point is equal to the germ of $\mathrm{Sing}\mathcal{F}$. The second part is the vertical stability of the Samuel invariant under appropriate blowing-ups. □

Definition 7. Consider a point $P \in \mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$ and let Y be a one-dimensional irreducible subspace of X containing P. We say that Y is *influent* for P relatively to $\mathcal{F}$, D if $\mu(\mathcal{F}, E; Y) \geq r$ and the following properties hold:

(1) If $\mu(\mathcal{F}, E; Y) = r < m^*$, then $Y = E_1 \cap E_2$ for two irreducible components of the divisor E.

(2) If $r = m^* = 1$, then Y is contained in any contact or essential component of E at P.

Denote by $\mathrm{Fl}(\mathcal{F}, D; P)$ the set whose elements are the point P and the influent subspaces at P. Let $\mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ be the union of $\mathrm{Fl}(\mathcal{F}, D; P)$ for $P \in \mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$. We say that a good center Y is a *very good center* if and only if it is either a bad point or it belongs to $\mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$.

Let U be an open set in X with $P \in U$. Given an influent component Y at P, any irreducible component of $Y \cap U$ containing P is also an influent component for P relatively to $\mathcal{F}|_U$, $D \cap U$. We shall denote $\mathrm{Fl}_P(\mathcal{F}, D; P)$ the set whose elements are the point P and the irreducible components of the germs at P of the influent components for P.

Proposition 21. *Let $\pi : X' \to X$ be the blowing-up with a very good center Y. Denote by $\mathcal{F}'$ and D' the transforms of $\mathcal{F}$ and D. Then there is at most one element Y' in $\mathrm{Fl}^{r,m^*}(\mathcal{F}', D')$ such that $\dim Y' = 1$ and $Y' \subset \pi^{-1}(Y)$. Moreover, if $Y = \{P\}$, a point, then Y' is a linear subspace $Y' \subset \pi^{-1}(Y) = \mathbb{P}^2$ and if $\dim Y = 1$ the morphism $Y' \to Y$ induced by π is an isomorphism.*

Proof. Denote by C' the union of the Y' of the statement. We shall assume that $C' \neq \emptyset$. We shall prove that either C' is a linear subspace of the exceptional divisor of π or the morphism $C' \to Y$ is one-to-one and hence an isomorphism.

First case: $Y = \{P\}$. Assume first $m^* = r + 1$. Given an irreducible component Γ' of C' we have that either $\mu(\mathcal{F}', E'; \Gamma') = r + 1$ or $\mu(\mathcal{F}', E'; \Gamma') = r$ and $\Gamma' = E'_1 \cap E'_2$ for two irreducible components of E'. In both cases $\nu(\mathcal{F}', E'; \Gamma') = r$. Applying Proposition 4 (the Directrix Theorem) we get that

$$C' \subset \mathbb{P}\mathrm{roj}(\mathrm{Dir}(\mathcal{F}, E; P)).$$

(In this proof we will make systematic use of the Directrix Theorem stated in Proposition 4.) Assume now that $m^* = r$. Let ω be a local generator of $\mathcal{H} = \mathrm{Sat}(\mathcal{F}, E)$ and $R \in \Theta_{X,P}[D]$ a germ of radial vector field. Put $f = \omega(R)$. Since $\mathrm{Rs}(\mathcal{F}, E; P) = 0$, we get that π is a nondicritical blowing-up and $\nu_P(f) = r$. By a local computation, the points of C' are r-multiple points of the strict transform of f. Thus C' is contained in the projective line on $\pi^{-1}(Y)$ determined by $\mathrm{Dir}(\mathrm{In}^r(f))$.

Second case: $\dim Y = 1$. Assume first that $m^* = r + 1$. Let us show that Y is appropriate at each point $P \in Y$. If P is a bad point we are done in view of Definition 6 of a good center. We also get that Y is appropriate if $\mu(\mathcal{F}, E; Y) = r+1$. The remaining case is $Y = E_1 \cap E_2$ and $\mu(\mathcal{F}, E; Y) = r$. Assume by contradiction that Y is not an appropriate center at $P \in Y$. Then $\mu(\mathcal{F}, E; P) = r+1 \geq 2$ and P is not a presimple singularity. In particular $P \in \mathrm{Sing}^*(\mathcal{F}, D)$ is a good point and $Y = \mathrm{Sing}^*(\mathcal{F}, D) = E_1 \cap E_2$ near P. We deduce that each point $R \in Y - \mathrm{Bd}(\mathcal{F}, D)$ is a good point and $\mu(\mathcal{F}, E; R) = r+1$. Consider a bad point $Q \in Y \cap \mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$. If $\mathrm{e}(D, Q) = 2$, we see that $\mu(\mathcal{F}, E; Q) = r+1$ and hence Y is nonappropriate at Q. If $\mathrm{e}(E, Q) = 3$, we get that $\mathrm{Rs}(\mathcal{F}, E; Q) = 0$, since $\nu_Q(\omega(\mathcal{X})) \geq r+1$ for any germ of vector field $\mathcal{X}$ tangent to Y. This gives a contradiction with the fact $m^* = r+1$. Hence Y is appropriate at each point $P \in Y$. Since $\nu(\mathcal{F}', E'; P') \geq r$, for any $P' \in C'$, applying Proposition 4, we get that the morphism $C' \to Y$ is one-to-one.

Assume that $m^* = r \geq 2$. The morphism $C' \to Y$ is finite-to-one. If not, there is a point $Q \in Y$ such that $\pi^{-1}(Q) \subset C'$ and we contradict a general property of permissible centers, since the points in C' have adapted order greater than or equal to $r-1 \geq 1$. Then $C' \to Y$ is finite-to-one and onto, by properness. Now it is enough to show that $C' \cap \pi^{-1}(P)$ is a single point for a generic $P \in Y$. Note that $r \geq 2$ and hence any nonbad point in Y, or in C', is a good point. Let us distinguish two cases: $\nu(\mathcal{F}, E; Y) = r-1$ and $\nu(\mathcal{F}, E; Y) = r$. In the first case, the center Y is appropriate at any point, and we end as above by applying Proposition 4, since any point in C' has adapted order greater than or equal to $r-1$ and then there is at most one point of C' over each point of Y. Assume that $\nu(\mathcal{F}, E; Y) = r$. Take a good point $P \in Y$, a germ of nonsingular vector field $\mathcal{X} \in \Theta[E]$ tangent to Y and a local generator ω of $\mathcal{H}$. Consider the function $a = \omega(\mathcal{X})$. Choosing the point P outside a suitable finite set, we have exactly one of the following two possibilities: either $\nu_P(a) = \nu_Y(a) = r$ or $\nu_Y(a) \geq r+1$. In the first case, the only possible points in C' are r-multiple points of the strict transform of the function a. By considering the directrix of $\mathrm{In}^r(a)$, we get at most one point of C' over each point of Y. Assume finally that $\nu_Y(a) \geq r+1$. Note that $\mathrm{e}(E, P) \geq 1$, otherwise we should get $\mu(\mathcal{F}, E; Y) = r+1 > m^*$. Choose coordinates at P such that $Y = \{y = z = 0\}$ and $E = \{yz^\varepsilon = 0\}$, where $\varepsilon \in \{0, 1\}$. Put $h = \omega(y\partial/\partial y + z\partial/\partial z)$. Taking the point P generic enough, we have exactly one of the following possibilities: either $\nu_P(h) = \nu_Y(h) = r$ or $\nu_Y(h) \geq r+1$. In the first case, we have a nondicritical blowing-up and the only point $Q' \in C' \cap \pi^{-1}(P)$ must be an r-multiple point of the strict transform of h. In the second case the blowing-up is dicritical. Moreover, consider a bad point $Q \in \mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$. We can choose local coordinates (x, y, z) at Q with $Y = \{y = z = 0\}$ and $E = \{x^{\varepsilon_1} y z^{\varepsilon_2} = 0\}$. Write a local generator of $\mathcal{H}$ at Q as follows:

$$\omega = a\frac{dx}{x^{\varepsilon_1}} + b\frac{dy}{y} + c\frac{dz}{z^{\varepsilon_2}}.$$

Then $\nu_Y(a) \geq r+1$ and $\nu_Y(b+z^{(1-\varepsilon_2)}c) \geq r+1$. Since $\nu_Q(a,b,c)=r$, we get that $\varepsilon_2 = 1$ and thus $\nu_Y(b+c) \geq r+1$. In particular we have a resonance in contradiction to the fact that $m^* = r$.

Assume finally that $r = m^* = 1$. Let $\mathrm{e}(E,Y)$ be the number of irreducible components of E containing Y. By definition of influent subspace, we know that $\mathrm{e}(E,Y) \geq 1$. Assume first that $1 = \mathrm{e}(E,Y)$, with $Y \subset E_1$ and E_1 is an irreducible component of E. The vertical stability of the contact and essential components imply that $C' = E_1' \cap \pi^{-1}(Y)$, where E_1' is the strict transform of E_1. Assume now that $Y = E_1 \cap E_2$, for two irreducible components E_1 and E_2 of E. Reasoning as above $C' \subset \Gamma_1' \cup \Gamma_2'$, where $\Gamma_i' = E_i' \cap \pi^{-1}(Y)$. Note that $\Gamma_1' \cap \Gamma_2' = \emptyset$. Let us show that it is not possible to have $C' = \Gamma_1' \cup \Gamma_2'$. Assume the contrary. Consider first the case that π is nondicritical and hence $E' \supset \pi^{-1}(Y) = E_0'$. Then $\Gamma_i' = E_0' \cap E_i'$ and $\mathrm{e}(E', \Gamma_i') = 2$. A local computation shows that $\nu(\mathcal{F}', E'; \Gamma_i') = 1$. Take any bad point $Q \in Y$; by consideration of the directrix, there is at most one point $Q' \in \pi^{-1}(Q)$ with $\nu(\mathcal{F}', E'; Q') \geq 1$. This contradicts the fact that $(\Gamma_1' \cup \Gamma_2') \cap \pi^{-1}(Q)$ contains two points. Assume finally that π is a dicritical blowing-up. Choose a bad point $Q \in Y$. Since $\mathrm{Rs}(\mathcal{F}, E; Q) = 0$, the only possibility is to have $\mathrm{e}(E,P) = 3$ and local coordinates (x,y,z) at Q such that $Y = \{y = z = 0\}$ and $E = \{xyz = 0\}$ with a local generator

$$\omega = a\frac{dx}{x} + b\frac{dy}{y} + c\frac{dz}{z}$$

of $\mathcal{H}$ at Q such that $\nu_Y(b+c) \geq 2$ and $\nu_Q(a) = 1$. Note that $E_1 \cup E_2 = \{yz = 0\}$. Now we see that the two points in $(\Gamma_1' \cup \Gamma_2') \cap \pi^{-1}(Q)$ should be in the strict transform of $a = 0$, which is not possible. □

Weak normal crossings. Let $\mathcal{A}$ be a finite set whose elements are points and irreducible curves in X. We say that $\mathcal{A}$ has *weak normal crossings* if given a point P in $\mathcal{A}$ any curve $\Gamma \in \mathcal{A}$ with $P \in \Gamma$ is nonsingular at P; there are at most two such curves and in this case they are transversal at P.

PROPOSITION 22. *Let $\pi : X' \to X$ be a blowing-up with a very good center Y. If $\mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ has weak normal crossings, then $\mathrm{Fl}^{r,m^*}(\mathcal{F}', D')$ also has weak normal crossings.*

Proof. If $Y \notin \mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ we are done by Proposition 21. Assume that $Y \in \mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$. Note that given $\Gamma' \in \mathrm{Fl}^{r,m^*}(\mathcal{F}', D')$ not contained in $\pi^{-1}(Y)$, we have $\pi(\Gamma') \in \mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$. Take a point $Q' \in \mathrm{Bd}^{r,m^*}(\mathcal{F}', D')$. Put $Q = \pi(Q') \in \mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$. By the weak normal crossings property of $\mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ there is at most one $\Gamma' \in \mathrm{Fl}^{r,m^*}(\mathcal{F}', D')$ not contained in $\pi^{-1}(Y)$, such that $Q' \in \Gamma'$. Moreover Γ' is nonsingular and transversal to $\pi^{-1}(Y)$ at Q'. Now, in view of Proposition 21, the only possible new influent curve $Y' \subset \pi^{-1}(Y)$ is nonsingular. □

Proposition 23. *There is a morphism* $\pi : X' \to X$ *which is the composition of a finite sequence of blowing-ups centered at bad points such that* $\mathrm{Fl}^{r,m^*}(\mathcal{F}', D')$ *has weak normal crossings.*

Proof. Blow-up repeatedly the points in $\mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$ until the strict transforms of the curves in $\mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ become nonsingular and transversal to the exceptional divisor at the bad points with Samuel invariant (r, m^*). The new possible elements added to the influence locus are linear curves each one in one exceptional divisor. This assures the weak normal crossings property. □

Remark 10. In the process defined in the above proof, some curves in the influence locus may "disappear", in the sense that their strict transform is no more in the new influence locus. Anyway, this does not affect the argument in the proof.

Thus, in order to eliminate the Samuel Stratum $\mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$, where $(r, m^*) = \mathrm{ISam}(\mathcal{F}, D)$, we can assume that the influence locus $\mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ has weak normal crossings and this is stable under any very good blowing-up.

2.5. *The local control theorem.* Here we state the local control theorem to be proved in the next chapter. Put $(r, m^*) = \mathrm{ISam}(\mathcal{F}, D)$ and assume that $\mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ has weak normal crossings.

Theorem 2 (local control). *There is no infinite sequence of local blowing-ups*

$$\mathcal{S} = \{(X_i, U_i, Y_i, P_i; \pi_{i+1})\}_{i=1}^{\infty}$$

such that $P_i \in \mathrm{Bd}^{r,m^*}(\mathcal{F}_i, D_i)$ *and the sequence satisfies the following rules on the center selection*:

LR1. *If* $\mathrm{Rs}(\mathcal{F}_i, E_i; P_i) = 1$ *and* $\mathrm{e}(E_i, P_i) = 3$, *the center* Y_i *is the point* P_i.

LR2. *If the assumption of* LR1 *does not hold and there is no curve* $\Gamma_i \in \mathrm{Fl}_{P_i}(\mathcal{F}_i, D_i)$ *that is appropriate at* P_i, *the center* Y_i *is the point* P_i.

LR3. *If neither the assumption of* LR1 *nor the assumption of* LR2 *holds*, *the center defines a germ of curve* $Y_i \in \mathrm{Fl}_{P_i}(\mathcal{F}_i, D_i)$ *appropriate at the point* P_i, *with maximal* $\mathrm{e}(E_i, Y_i)$.

We shall refer to the above rules as the *Local Criteria of Blowing-up* and we say that a finite or infinite sequence of local blowing-ups *respects the Local Criteria of Blowing-up* if the above rules are satisfied. Moreover, to say that *the rule* LRi *applies*, for $i = 1, 2, 3$, means that the assumption of the rule LRi holds. Hence for any step in a sequence of local blowing-ups that respects the Local Criteria of Blowing-up, there is a unique index $i \in \{1, 2, 3\}$ such that the rule LRi applies.

2.6. *Destroying cycles.* The existence of cycles is the most important obstruction in applying the local control theorem to get a global reduction of the singularities. Here we prove we can destroy the cycles and they do not reappear under very good blowing-ups. We do that under the assumption that the local control theorem is proved and that $\mathrm{Fl}^{r,m^*}(\mathcal{F},D)$ has weak normal crossings, where $(r,m^*)=\mathrm{ISam}(\mathcal{F},D)$.

Definition 8. A *cycle* for $\mathcal{F}$, D is a finite sequence $c=(P_1,\Gamma_1,\dots,P_k,\Gamma_k)$ of length $k\geq 2$ formed with distinct points $P_i\in \mathrm{Bd}^{r,m^*}(\mathcal{F},D)$ and curves $\Gamma_i\in \mathrm{Fl}^{r,m^*}(\mathcal{F},D)$, such that $\Gamma_k\ni P_1\in\Gamma_1$ and $\Gamma_{s-1}\ni P_s\in\Gamma_s$ for $s=2,\dots,k$. We denote by $\mathrm{Cycl}^{r,m^*}(\mathcal{F},D)$ the set of cycles.

We note that $\mathrm{Cycl}^{r,m^*}(\mathcal{F},D)$ is a finite set.

Proposition 24. *Let $\pi: X'\to X$ be a blowing-up with very good center Y. Then there is an injective map $\Phi:\mathrm{Cycl}^{r,m^*}(\mathcal{F}',D')\to \mathrm{Cycl}^{r,m^*}(\mathcal{F},D)$.*

Proof. Consider $c'=(P'_1,\Gamma'_1,\dots,P'_k,\Gamma'_k)\in \mathrm{Cycl}^{r,m^*}(\mathcal{F}',D')$. Assume first that $\Gamma'_j\not\subset\pi^{-1}(Y)$ for all $j=1,\dots,k$. Then we put

$$\Phi(c')=(P_1,\Gamma_1,\dots,P_k,\Gamma_k)$$

where $P_j=\pi(P'_j)$ and $\Gamma_j=\pi(\Gamma'_j)$, for $j=1,\dots,k$. It is a cycle for $\mathcal{F}$, D: use the weak normal crossings property of the influence locus to see that $P_i\neq P_j$ for $i\neq j$. Moreover, the map Φ is injective over these cycles. Assume second that there is an index j such that $\Gamma'_j\subset\pi^{-1}(Y)$. We separate two cases: $Y=\{P\}$ and $\dim Y=1$. If $Y=\{P\}$, because of the weak normal crossings property we get that $k\geq 3$, in fact the strict transform of an element in the influence locus intersects the exceptional divisor in at most one point. We put

$$\Phi(c')=(P_1,\Gamma_1,\dots,P_{j-1},\Gamma_{j-1},P_{j+1},\Gamma_{j+1},\dots,P_k,\Gamma_k)$$

which is a cycle of length $k-1$. The uniqueness of the Γ'_j, created after blowing-up, guarantees that Φ is injective over these cycles. If $\dim Y=1$, then $\pi(\Gamma'_j)=Y$ and we define $\Phi(c')$ as in the first situation; also the uniqueness of Γ'_j implies that Φ is injective over these cycles. The injectivity of Φ is deduced now from the weak normal crossings property of the influence locus. $\square$

Proposition 25. *There is a finite sequence of blowing-ups with very good centers*

$$X=X_0\xleftarrow{\pi_1}X_1\xleftarrow{\pi_2}\cdots\xleftarrow{\pi_N}X_N=X'$$

such that either $\mathrm{ISam}(\mathcal{F}',D')<(r,m^*)$ *or* $\mathit{Cycl}^{r,m^*}(\mathcal{F}',D')=\emptyset$.

Proof. Induction on the number of cycles. It is enough to make one cycle disappear, $c=(P_1,\Gamma_1,\dots,P_k\Gamma_k)$. Define the center Y of the first blowing-up π_1 by means of the following priorities:

I. Assume that either $\mathrm{Rs}(\mathcal{F}, D; P_1) = 1$ and $\mathrm{e}(D, P_1) = 3$ or neither Γ_k nor Γ_1 is appropriate at P_1. Then put $Y = \{P_1\}$.

II. Condition I does not hold and either Γ_k or Γ_1 is a good center. Then we put $Y = \Gamma_k$ or $Y = \Gamma_1$ respectively.

III. None of the above and either Γ_k or Γ_1 is not an appropriate center at a bad point $Q \neq P_1$. Take $Y = \{Q\}$.

If the number of cycles does not decrease, let c' be such that $\Phi(c') = c$, as in the statement of the above proposition. Repeat. Note that case III occurs only finitely many times repeatedly: after finitely many steps Γ_k or Γ_1 becomes appropriate. Then the point P_1 is modified infinitely many times and at each step we respect the Local Criteria of Blowing-up. We conclude by the local control theorem. $\square$

Thus, before starting a global procedure of elimination of the bad points, we can assume that there are no cycles and this condition is stable under very good blowing-ups.

2.7. *Global criteria of blowing-up.* Here we give a criteria for choosing global centers of very good blowing-ups in order to eliminate the bad points and we give the end of the proof of the main Theorem 1 of reduction to presimple singularities. We do that under the assumption that the local control theorem is proved, that $\mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ has weak normal crossings, where $(r, m^*) = \mathrm{ISam}(\mathcal{F}, D)$ and that there are no cycles.

Consider the following statements:

S1. There are a point $P \in \mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$ such that $\mathrm{Rs}(\mathcal{F}, E; P) = 1$ and $\mathrm{e}(E, P) = 3$.

S2. There are a point $P \in \mathrm{Bd}(\mathcal{F}, D)$ but $P \notin \mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$ and a curve $\Gamma \in \mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ such that Γ is nonappropriate at P.

S3. There is a curve $\Gamma \in \mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ which is a good center.

S4. There is a point $P \in \mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$ such that first there is exactly one curve $\Gamma \in \mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ such that $P \in \Gamma$ and second Γ is nonappropriate at P.

S5. There is a point $P \in \mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$ such that first there are exactly two curves Γ_1 and Γ_2 in $\mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ such that $P \in \Gamma_1 \cap \Gamma_2$ and second neither Γ_1 nor Γ_2 is appropriate at P.

S6. There is a point $P \in \mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$ such that no curve of $\mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ passes through it.

Proposition 26. *The above statements*, S1–S6, *cover all the possibilities.*

Proof. Get a contradiction by constructing a cycle as follows. There are a point $P_1 \in \mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$ and a curve $\Gamma_1 \in \mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ such that $P_1 \in \Gamma_1$ and Γ_1 is nonappropriate at P_1. Necessarily (otherwise S4) there is another curve $\Gamma_2 \in \mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$ passing through P_1. Now Γ_2 is appropriate at P_1, otherwise S5 (by the weak normal crossings property of the influence locus). Since Γ_2 is not a good center (otherwise S3), there is a bad point P_2 in Γ_2 such that Γ_2 is nonappropriate at P_2. Necessarily P_2 is in $\mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$, otherwise S2. Repeat with P_2. In this way we create a sequence P_1, Γ_1, P_2, Γ_2 and so on. At the first step, repeating an element, we get a cycle. □

Definition 9. Let $\pi : X' \to X$ be a blowing-up with center Y. We say that π *respects the global criteria of blowing-up* if and only if the center Y has been selected according to the following rules:

R1. If S1, put $Y = \{P\}$.

R2. If S2 and not S1, put $Y = \{P\}$.

R3. If S3 and not S1, S2, put $Y = \{\Gamma\}$, with maximal $e(E, \Gamma)$.

R4. If S4 and not S1, S2, S3, put $Y = \{P\}$.

R5. If S5 and not S1, S2, S3, S4, put $Y = \{P\}$.

R6. If S6 and not S1, S2, S3, S4, S5, put $Y = \{P\}$.

Remark 11. We can always find a center respecting the above rules. The center Y contains at least one bad point in $\mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$ except when we apply rule R2: note that in the case of rule R3, any curve in the influence locus contains at least one bad point.

Proof of main Theorem 1 (*under the assumption that the local control theorem is proved*). Assume that the theorem is not true. Then, there is an infinite sequence of global blowing-ups

$$\mathcal{S} : \; X = X_0 \overset{\pi_1}{\leftarrow} X_1 \overset{\pi_1}{\leftarrow} \dots$$

such that we have the same Samuel Invariant (r, m^*) at each step, the influence locus has weak normal crossings, there are no cycles and each blowing-up respects the global criteria of blowing-up. It is constructed as follows: first do blowing-ups to get weak normal crossings and to eliminate cycles; second do repeated blowing-ups respecting the global criteria. If the Samuel invariant decreases, restart with a new Samuel Invariant (this can be done only finitely many times). We do not eliminate the bad points, since otherwise we can use the equi-reduction sequence to get presimple singularities.

Now let us prove that the sequence $\mathcal{S}$ does not exist. We construct a *tree $\mathcal{T}$ of bad points* associated to the sequence $\mathcal{S}$ as follows. Consider the disjoint union of all the sets of bad points $\mathrm{Bd}^{r,m^*}(\mathcal{F}_i, D_i)$, for $i = 0, 1, \ldots$. Let $\mathcal{T}$ be the quotient set under the equivalence relation that identifies two points having locally isomorphic neighborhoods under the blowing-up morphism (one point not in the center of the blowing-up and its inverse image). The sequence of the blowing-up morphisms provides an obvious partial order on $\mathcal{T}$. This makes a tree $\mathcal{T}$ such that each level is finite. For instance, the first level has exactly as many points as $\mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$. Take a branch $\mathcal{B}$ of the tree $\mathcal{T}$. It corresponds to a sequence of local blowing-ups that respects the local criteria of blowing-up. Hence the branch $\mathcal{B}$ is a finite branch in view of the local control theorem. The tree $\mathcal{T}$ should be finite, since it has finite levels and finite branches. Note that:

> If the rule R2 fails to be applied infinitely many times, then the tree $\mathcal{T}$ is infinite. In fact, in view of Remark 11 given an index k, we get $k' \geq k$ such that the center of $\pi_{k'+1}$ contains at least one point in $\mathrm{Bd}^{r,m^*}(\mathcal{F}, D)$; that is, we modify bad points infinitely many times.

Then, in order to get a contradiction, it is enough to show that the rule R2 fails to be applied infinitely many times. But we know that by blowing-up bad points not in $\mathrm{Bd}^{r,m^*}(\mathcal{F}_i, D_i)$ we do not add new elements to the strict transforms of the elements of the influence locus. Moreover these curves will be appropriate at these points after finitely many steps and we must stop, at least at one step, the use of rule R2.

3. Local control

This chapter is devoted to the proof of the local control theorem. Recall that we have a three-dimensional ambient space X, a singular foliation $\mathcal{F}$ on X and a normal crossings divisor D over X. Let us put $(r, m^*) = \mathrm{ISam}(\mathcal{F}, D)$ and assume that the influence locus $\mathrm{Fl}^{r,m^*}(\mathcal{F}, D)$ has weak normal crossings. We reason by contradiction, assuming that there is an infinite sequence of local blowing-ups

$$\mathcal{S} = \{(X_i, U_i, Y_i, P_i; \pi_{i+1})\}_{i=0}^{\infty}$$

starting at $X_0 = X$, $\mathcal{F}_0 = \mathcal{F}$, $D_0 = D$, such that $P_i \in \mathrm{Bd}^{r,m^*}(\mathcal{F}_i, D_i)$ for each index $i \geq 0$ which respects the local criteria of blowing-up.

Put $m_i = \mu(\mathcal{F}_i, E_i; P_i)$. We know that $r \leq m_i \leq m^* \leq r + 1$. We shall divide our study in two cases:

The m-stable case. That is $m_i = m_{i+1}$ for all $i \geq 0$. For this case we develop a vertical maximal contact theory and we shall control it by means of our vertical invariants and new invariants mainly obtained from the characteristic polygon.

The m-unstable case. We also call it the *jumping situation*. That is, for any $k \geq 0$ there is an index $i \geq k$ such that $m_i \neq m_{i+1}$. Here we shall describe a finite set of situations ordered by a natural hierarchy, that do not persist under the blowing-ups and that drop over the lower levels after finitely many steps.

3.1. *Two-dimensional differential idealistic exponents.* Let H be a two-dimensional ambient space and $L \subset H$ a normal crossings divisor on H. A *differential idealistic exponent* on H adapted to L is a triple

$$\mathcal{E} = \left(\{\mathcal{H}_i, \mathcal{C}_i\}_{i=0}^{m-1}, L, m \right)$$

where $\mathcal{H}_i \subset \Omega_H[L]$ and $\mathcal{C}_i \subset \mathcal{O}_H$ are either invertible or zero and m is a positive integer. A nonsingular subspace $Y \subset H$, with $\dim Y \leq 1$ is said to be *permissible* for $\mathcal{E}$ if and only if Y has normal crossings with L and

$$\mu(\mathcal{H}_i; Y) \geq m - i;\ \nu_Y(\mathcal{C}_i) \geq m - i; \quad \text{for } i = 1, \ldots, m-1.$$

Let $\pi : H' \to H$ be the blowing-up with center Y, permissible for $\mathcal{E}$. Let $\mathcal{J}_\pi$ be the reduced ideal sheaf defining $\pi^{-1}(Y)$ and put

$$\mathcal{H}'_i = \mathcal{J}_\pi^{i-m} \pi^{-1} \mathcal{H}_i;\ \mathcal{C}'_i = \mathcal{J}_\pi^{i-m} \pi^{-1} \mathcal{C}_i;\ L' = \pi^{-1}(L \cup Y).$$

Then we say that $\mathcal{E}' = \left(\{\mathcal{H}'_i, \mathcal{C}'_i\}_{i=0}^{m-1}, L', m \right)$ is the transform of $\mathcal{E}$ by π.

A point $P \in H$ is a *singular point* of $\mathcal{E}$ if $\{P\}$ is a permissible center for $\mathcal{E}$. We denote by $\mathrm{Sing}\mathcal{E}$ the set of singular points of $\mathcal{E}$. Note that $\mathrm{Sing}\mathcal{E}$ is not necessarily an analytic subset of H: for example, take $m = 1$, $\mathcal{C}_0 = 0$ and $\mathcal{H}_0$ given by $\omega = dx/x + dy/y$ with $L = \{xy = 0\}$; then $\mathrm{Sing}\mathcal{E} = H - L$.

Definition 10. Let $P \in \mathrm{Sing}\mathcal{E}$. For each $\mathcal{H}_i \neq 0$, put $\mathcal{G}_i = \mathrm{Fol}(\mathcal{H}_i)$.We say that $\mathcal{E}$ is *differentially trivial* at the point P if either $P \notin \mathrm{Sing}\mathcal{G}_i$ or P is a simple singularity in the sense of Seidenberg for each $\mathcal{G}_i$ and moreover the (formal) divisor L^* at P defined by

$$L^* = L \cup \{ \text{ formal integral curves of all the } \mathcal{G}_i\}$$

is a normal crossings divisor. (*To be a simple singularity means that a local generator of* $\mathcal{G}_i$ *has the form* $y dx - \mu x dy + \ldots$ *with* $\mu \notin \mathbb{Q}_+$; *in particular there are exactly two formal integral curves at* P.)

Assume that $\mathcal{E}$ is differentially trivial at P. Let us work in formal coordinates at P: this may be necessary since the integral curves of a foliation are possibly nonconvergent. Consider the principal ideals $\mathcal{J}_i \subset \widehat{\mathcal{O}}_{H,P}$ given by

$$\mathcal{H}_i = \mathcal{J}_i \mathrm{Sat}\,(\mathcal{H}_i, L^*)$$

if $\mathcal{H}_i \neq 0$ and $\mathcal{J}_i = 0$ if $\mathcal{H}_i = 0$. Put $\mathcal{H}_i^* = \mathrm{Sat}\,(\mathcal{H}_i, L^*) = \mathrm{Sat}\,(\mathcal{G}_i, L^*)$. Since $\mathcal{E}$ is differentially trivial at P, we get that $\mu\,(\mathcal{H}_i^*; Y) = 0$ for any $Y \ni P$. Hence

$$\mu\,(\mathcal{H}_i; Y) = \nu_Y \mathcal{J}_i + \mu\,(\mathcal{H}_i^*; Y) = \nu_Y \mathcal{J}_i.$$

Now, we consider the formal idealistic exponent $\mathcal{I}_P(\mathcal{E}) = (\mathcal{J}_P(\mathcal{E}), m!)$, where

$$\mathcal{J}_P(\mathcal{E}) = \sum_{i=0}^{m-1} \mathcal{J}_i^{\frac{m!}{m-1}} + \sum_{i=0}^{m-1} \mathcal{C}_i^{\frac{m!}{m-1}}.$$

LEMMA 11. *Let $\mathcal{E}$ be differentially trivial at $P \in \mathrm{Sing}\mathcal{E}$ and let $\pi : H' \to H$ be the blowing-up with a permissible center Y, with $P \in Y \subset L$. Then Y is also permissible for the formal idealistic exponent $\mathcal{I}_P(\mathcal{E})$. Moreover, the transform $\mathcal{E}'$ is differentially trivial at any point $P' \in \pi^{-1}(Y) \cap \mathrm{Sing}\mathcal{E}'$ and the formal idealistic exponent $\mathcal{I}_{P'}(\mathcal{E}')$ is the transform of $\mathcal{I}_P(\mathcal{E})$ by π at the point P'.*

Proof. The first part follows directly from the fact that $\nu_Y \mathcal{J}_i \geq m - i$ and $\nu_Y \mathcal{C}_i \geq m - i$. Take a point $P' \in \pi^{-1}(Y) \cap \mathrm{Sing}\mathcal{E}'$. We know that it is either a nonsingular point or a simple singularity for $\mathcal{G}_i'$. Since $Y \subset L$, then $L'^* = \pi^{-1}(L^* \cup L) = \pi^{-1}(L^*)$ has normal crossings and thus $\mathcal{E}'$ is differentially trivial at P'. Finally, since $\mu(\mathcal{H}_i^*; Y) = 0$ we have that $\mathcal{H}_i'^* = \pi^{-1}(\mathcal{H}_i^*)$ and thus $\mathcal{J}_i' = \mathcal{J}_\pi^{i-m} \pi^{-1}(\mathcal{J}_i)$. This shows that $\mathcal{I}_{P'}(\mathcal{E}')$ is the transform of $\mathcal{I}_P(\mathcal{E})$. □

Definition 11. A point $P \in \mathrm{Sing}\mathcal{E}$ is said to be a *simple singularity* if $\mathcal{E}$ is differentially trivial at P and the formal ideal $\mathcal{J}^{**}(\mathcal{E}) = \mathcal{J}_{L^*} \mathcal{J}_P(\mathcal{E})$ is a principal ideal generated by a monomial (hence it defines a formal normal crossings divisor L^{**} that contains L^*).

By Lemma 11, to be a simple singularity is also a stable property under permissible blowing-ups with centers $Y \subset L$.

PROPOSITION 27. *Let $\mathcal{E}$ be a differential idealistic exponent over H adapted to L and consider an infinite sequence of local blowing-ups*

$$\mathcal{B} = \{(H_i, U_i, Y_i, P_i; \pi_{i+1})\}$$

starting at $H_0 = H$ and such that the center Y_i is permissible for the transform $\mathcal{E}_i$ of $\mathcal{E}_{i-1}$, where $\mathcal{E}_0 = \mathcal{E}$. Assume that $P_i \in \mathrm{Sing}\mathcal{E}_i$ for all $i \geq 0$. Then:

(1) *The sequence is residually quadratic. That is, for any $j \geq 0$ there is $k \geq j$ such that the center Y_k is the point P_k.*

(2) *There is an index $N \geq 0$ such that $Y_i \subset L_i$ and P_i is a simple singularity for $\mathcal{E}_i$, with $i \geq N$.*

Proof. In order to prove that $\mathcal{B}$ is residually quadratic, assume the contrary. Restricting attention to the part of the blowing-up sequence for i sufficiently large, we may assume that $\dim Y_i = 1$ for all $i \geq 0$. Then each π_{i+1} is an isomorphism. If L has fewer than two irreducible components, a component may be added to it at a given step. Since it can happen at most two times, by looking at i sufficiently large we may assume that $L_{i+1} = L_i$ for all $i \geq 0$. One of the components of L_i, say Y, is repeated infinitely many times as a center. Putting

$$\delta = \min\{\mu(\mathcal{H}_j; Y), \nu_Y(\mathcal{C}_j)\}_{j=0}^{m-1},$$

we have that $1 \leq \delta < \infty$ and δ decreases at least one unit each time we do a blowing-up. This is a contradiction.

Let us prove the second part of the proposition. By Seidenberg's reduction of the singularities and since $\mathcal{B}$ is residually quadratic, we may assume that $P = P_0$ and each of the P_i is either a nonsingular point or a simple singularity for the $\mathcal{G}_j$. Also, after the first blowing-up, $L_i \neq \emptyset$; thus, we assume without loss of generality that $L_i \neq \emptyset$, for $i \geq 0$. Denote by $\mathrm{S}^*_P(\mathcal{E})$ the set of germs of irreducible formal curves Γ at P such that

$$\mu(\mathcal{H}_j; \Gamma) \geq m - j,\ \nu_\Gamma(\mathcal{C}_j) \geq m - j.$$

It is a finite set (of at most two elements). Denote by s_i the number of elements Γ in $\mathrm{S}^*_P(\mathcal{E}_i)$ such that $\Gamma \not\subset L_i$. Then $s_i \geq s_{i+1}$ and $s_i > s_{i+1}$ if $Y_i \not\subset L_i$, since in this case $Y_i \in \mathrm{S}^*_P(\mathcal{E}_i)$. This proves that $Y_i \subset L_i$ for large i. Assume that $Y_i \subset L_i$ for all i. Then

$$L^*_{i+1} = \pi^{-1}(L^*_i)$$

and thus L^*_i has normal crossings for $i \gg 0$, since $\mathcal{B}$ is residually quadratic. This shows that up to a finite number of steps $\mathcal{E}$ is differentially trivial at P. Getting a simple singularity is a consequence of Lemma 11 and the principalization of $\mathcal{J}_{L^*}\mathcal{J}_P(\mathcal{E})$ according to Hironaka's theory. □

3.2. *Maximal contact.* Let X be a three-dimensional ambient space and D a normal crossings divisor on X. Fix a point $P \in X$ and consider an infinite sequence of local blowing-ups

$$\mathcal{S} = \{(X_i, U_i, Y_i, P_i; \pi_{i+1})\}_{i=0}^{\infty}$$

starting at $x_0 = X$, $P_0 = P$ and such that the centers Y_i have normal crossings with the divisors $D_i = \pi_i^{-1}(D_{i-1} \cup Y_{i-1})$ where $D_0 = D$ and moreover $\dim Y_i \leq 1$.

Definition 12. We say that a germ H at P of a nonsingular (possibly formal) surface has *maximal contact* with $\mathcal{S}$ adapted to D if and only if first H has normal crossings with D and second for each $i \geq 0$ the strict transform H_i of H in X_i contains the center Y_i. (*Note that in particular* $P_i \in H_i$ *and* H_i *has normal crossings with* D_i.)

Coordinate data. We will choose coordinates at each step in a normalized way. This will allow us to control the evolution of a singular foliation in a maximal contact situation, in terms of invariants obtained from the equations.

To be precise, assume that H has maximal contact with $\mathcal{S}$ adapted to D. Then we can choose two systems of coordinates

$$(x_i, y_i^*, z_i) \text{ and } (x_i, y_i, z_i) \text{ with } y_i = y_i^* + \sum_{s\geq 1} \lambda_{is} x_i^s$$

at each point P_i satisfying the following conditions for all $i \geq 0$:

CD1. $H_i = \{z_i = 0\}$; $D_i \subset \{x_i y_i^* z_i = 0\}$; $D_{i+1} \supset \{x_{i+1} = 0\}$.

CD2. If $\dim Y_i = 1$, then either $Y_i = \{x_i = z_i = 0\}$ or $Y_i = \{y_i = z_i = 0\}$. Moreover:

a) If $Y_i = \{x_i = z_i = 0\}$ then $y_i = y_i^*$ and $x_i = x_{i+1}$; $y_i = y_{i+1}^*$; $z_i = x_{i+1} z_{i+1}$.

b) If $Y_i = \{y_i = z_i = 0\}$ then $x_i = x_{i+1}$; $y_i = y_{i+1}^*$; $z_i = y_{i+1}^* z_{i+1}$.

CD3. If $Y_i = \{P_i\}$ then either i) or ii) where

i) $\lambda_{is} = 0$ for $s \geq 2$ and $x_i = x_{i+1}$; $y_i = x_{i+1} y_{i+1}^*$; $z_i = x_{i+1} z_{i+1}$.

ii) $y_i = y_i^*$ and $x_i = x_{i+1} y_{i+1}^*$; $y_i = y_{i+1}^*$; $z_i = y_{i+1}^* z_{i+1}$.

We say that the family

$$\mathcal{R} = \left\{ (x_i, y_i^*, z_i),\, (x_i, y_i, z_i);\; y_i = y_i^* + \sum_{s\geq 1} \lambda_{is} x_i^s \right\}_{i=0}^{\infty}$$

presents *coordinate data* associated to H, $\mathcal{S}$ and D. Conversely, given such an $\mathcal{R}$ the surface $\{z = 0\}$ has maximal contact with $\mathcal{S}$ adapted to D.

Note that the coordinate data are chosen according to the following principles: the divisor is always given by a monomial, the coordinate x is "as fixed as possible", the maximal contact surface is $z = 0$, the local blowing-ups have the simplest possible equation and (in particular) the center is "rectified". We make the coordinate change $y_i^* \mapsto y_i$ in order either to rectify the center or to get a centered equation for the blowing-up.

Remark 12. For further reference, following the classical notations of Hironaka in the Bowdoin College notes [20], we say that a coordinate system (x', y', z') at a point P' is obtained from (x, y, z) at P by the transformations T1, T2, T3 or T4 if and only if

$$\begin{array}{llll} \mathrm{T1}: & x = x', & y = x'y', & z = x'z'. \\ \mathrm{T2}: & x = x'y', & y = y', & z = y'z'. \\ \mathrm{T3}: & x = x', & y = y', & z = x'z'. \\ \mathrm{T4}: & x = x', & y = y', & z = y'z'. \end{array}$$

The transformations T1 and T2 correspond to local blowing-ups with center P (each one in the origin P' of one of the standard charts containing the strict transform of $z = 0$). The transformations T3 and T4 correspond to local blowing-ups centered respectively at $x = z = 0$ and $y = z = 0$ and following the strict transform of $z = 0$. Note that in CD2 and CD3 above $(x_{i+1}, y^*_{i+1}, z_{i+1})$ is obtained from (x_i, y_i, z_i) by one of the transformations T1, T2, T3 or T4.

Maximal contact in the m-stable case. In this paragraph we assume that the infinite sequence $\mathcal{S}$ of local blowing-ups corresponds to the m-stable case and moreover there is a nonsingular surface H having maximal contact with $\mathcal{S}$ adapted to D. We shall prove that this situation gives rise to a contradiction. To do that, we construct a differential idealistic exponent $\mathcal{E}$ on H and show that the points P_i cannot all of them be singular points for the successive transforms of $\mathcal{E}$.

Let us fix a coordinate data $\mathcal{R}$ associated to H, $\mathcal{S}$ and D and a local generator ω of $\mathcal{H} = \mathrm{Sat}(\mathcal{F}, D)$ at $P = P_0$. Put $(x, y, z) = (x_0, y^*_0, z_0)$ in order to simplify the notation. Since $D \subset \{xyz = 0\}$, we can write ω as follows:

$$\omega = a(x,y,z)\frac{dx}{x} + b(x,y,z)\frac{dy}{y} + c(x,y,z)\frac{dz}{z}$$

where $m = \nu_{(x,y,z)}(a, b, c)$. Let us decompose ω as follows:

$$\omega = \sum_{j=0}^{m-1} z^j \left\{ a_j(x,y)\frac{dx}{x} + b_j(x,y)\frac{dy}{y} + c_j(x,y)\frac{dz}{z} \right\} + z^m \eta$$

where $\eta \in \Omega[xyz = 0]$. Put $w_j = a_j(x,y)\frac{dx}{x} + b_j(x,y)\frac{dy}{y}$ and let $\mathcal{H}_j \in \mathbb{H}(H, L)$ be the hyperplane fields defined by the ω_j, where L is the normal crossings divisor on H defined by $H \cap \overline{D - H}$. Now let $\mathcal{C}_j \subset \mathcal{O}_H$ be the ideals generated by the coefficients $c_j(x, y)$. Then we put

$$\mathcal{E} = \mathcal{E}\left(\omega, (x,y,z), D, m\right) = \left(\{\mathcal{H}_j, \mathcal{C}_j\}_{j=0}^{m-1}, L, m \right)$$

and we call it the *differential idealistic exponent* associated to ω, (x, y, z), m and D.

Note that $\mathcal{S}$ induces a sequence $\mathcal{S}|_H$ of local blowing-ups on H by

$$\mathcal{S}|_H = \{(H_i, U_i \cap H_i, Y_i, P_i; \overline{\pi}_{i+1})\}_{i=0}^{\infty}.$$

LEMMA 12. *Denote by $\mathcal{E}_i$ the transform of $\mathcal{E}$ at step* i) *of $\mathcal{S}|_H$. Then $P_i \in \mathrm{Sing}\mathcal{E}_i$ and Y_i is a permissible center for $\mathcal{E}_i$. Moreover a germ Γ of the curve at P_i is permissible for $\mathcal{E}_i$ if and only if it is appropriate at P_i for $\mathcal{F}_i, D_i$.*

Proof. Write either $u_{i+1} = x_{i+1}$ or $u_{i+1} = y^*_{i+1}$ in such a way that $\pi^{-1}_{i+1}(Y_i) = \{u_{i+1} = 0\}$. Put $\omega^{(0)} = \omega$. Then $\omega^{(i+1)} = u^{-m}_{i+1}\pi^*_{i+1}\omega^{(i)}$ is a local generator of $\mathrm{Sat}(\mathcal{F}_{i+1}, D_{i+1})$. If $\pi(i) = \pi_1 \circ \cdots \circ \pi_i$, then

$$\omega^{(i)} = \sum_{j=0}^{m-1} z_i^j \left\{ \omega_j^{(i)} + c_j^{(i)} \pi(i)^* \left(\frac{dz}{z} \right) \right\} + z_i^m \pi(i)^* \eta$$

where $\omega_j^{(i+1)} = u_{i+i}^{j-m}\overline{\pi}_{i+1}^*\omega_j^{(i)}$ and $c_j^{(i+1)} = u_{i+i}^{j-m}\overline{\pi}_{i+1}^* c_j^{(i)}$. This means that $\omega_j^{(i)}$ and $c_j^{(i)}$ generate respectively $\mathcal{H}_j^{(i)}$ and $\mathcal{C}_j^{(i)}$, where

$$\mathcal{E}_i = \left(\left\{\mathcal{H}_j^{(i)}, \mathcal{C}_j^{(i)}\right\}_{j=0}^{m-1}, L_i, m\right).$$

Noting that $D_i \subset \{x_i y_i^* z_i = 0\}$ and $L_i \subset \{x_i y_i^* = 0\}$, let us write

$$\omega_j^{(i)} = a_j^{(i)}(x_i, y_i^*)\frac{dx_i}{x_i} + b_j^{(i)}(x_i, y_i^*)\frac{dy_i^*}{y_i^*}$$

and

$$\pi(i)^*\left(\frac{dz}{z}\right) = \frac{dz_i}{z_i} + f^{(i)}(x_i, y_i^*)\frac{dx_i}{x_i} + g^{(i)}(x_i, y_i^*)\frac{dy_i^*}{y_i^*}.$$

Then $\mu(\mathcal{F}_i, D_i; \Gamma) \geq m$ if and only if $\nu_\Gamma(a_j^{(i)}, b_j^{(i)}, c_j^{(i)}) \geq m - j$. This ends the proof. □

PROPOSITION 28. *Let us consider X, $\mathcal{F}$, D and an infinite sequence of blowing-ups $\mathcal{S}$ that respects the local criteria of blowing-up as above. Assume that $\mathcal{S}$ corresponds to the m-stable case. Then it is not possible to get a non-singular surface H having maximal contact with $\mathcal{S}$ adapted to D.*

Proof. Let us find a contradiction by assuming that H exists. Let $\mathcal{R}$ be coordinate data associated to H, $\mathcal{S}$ and D. Choose a local generator ω of $\mathrm{Sat}(\mathcal{F}, D)$ and consider the corresponding differential idealistic exponent

$$\mathcal{E} = \mathcal{E}\left(\omega, (x, y^*, z), D, m\right)$$

where $(x, y^*, z) = (x_0, y_0^*, z_0)$. Denote by $\mathcal{E}_i$ the successive transforms of $\mathcal{E}$ at the points P_i and put as above

$$\mathcal{E}_i = \left(\left\{\mathcal{H}_j^{(i)}, \mathcal{C}_j^{(i)}\right\}_{j=0}^{m-1}, L_i, m\right).$$

We know that $Y_i \subset L_i$ and P_i is a simple singularity for $\mathcal{E}_i$, for $i \gg 0$. For the sake of simplicity we assume that this is true for $i \geq 0$.

Note that, except for $i = 0$, the differential idealistic exponent $\mathcal{E}_i$ does not coincide with the idealistic exponent associated to $\mathcal{F}_i$ and the coordinate system (x_i, y_i^*, z_i). We have seen this situation in the statement and proof of Lemma 12. Anyway, the properties stated in Lemma 12 for $\mathcal{F}_i$ are the ones we will use in order to get a contradiction to the existence of H. Also, up to an initial coordinate change, we assume that $L^{**} \subset \{xy^* = 0\}$ and thus $L_i^{**} \subset \{x_i y_i^* = 0\}$ for all $i \geq 0$ (recall that L^* denotes the union of the components of L with the integral curves of the partial foliations $\mathcal{G}_j$ and that L^{**} corresponds to the union of L^* with the zeroes of the ideal $\mathcal{J}_P(\mathcal{E})$).

Then, the ideal $\mathcal{J}(\mathcal{E}_i)$ corresponding to the formal idealistic exponent $\mathcal{I}_{P_i}(\mathcal{E}_i)$ is generated by a monomial

$$\mathcal{J}(\mathcal{E}_i) = x_i^{m!\alpha_i}(y_i^*)^{m!\beta_i}\widehat{\mathcal{O}}_{X_i,P_i}.$$

Let us recall that $\alpha_i + \beta_i \geq 1$ and that

$$\begin{array}{lcl} \alpha_i \geq 1 & \iff & \{x_i = z_i = 0\} \text{ is appropriate at } P_i. \\ \beta_i \geq 1 & \iff & \{y_i^* = z_i = 0\} \text{ is appropriate at } P_i. \end{array}$$

Now, let us show that the evolution of the pairs (α_i, β_i) leads to a contradiction, in view of the rules LR1, LR2 and LR3 of the local criteria of blowing-up. Note that

$$(\alpha_{i+1}, \beta_{i+1}) = \tau_{i+1}(\alpha_i, \beta_i)$$

where:

$$\begin{array}{llr} \tau_{i+1} = \sigma_1 : (u,v) \mapsto (u+v-1, v) & \text{if} & \lambda_{i1} = 0 \text{ and T1.} \\ \tau_{i+1} = \sigma_1^* : (u,v) \mapsto (u+v-1, 0) & \text{if} & \lambda_{i1} \neq 0 \text{ and T1.} \\ \tau_{i+1} = \sigma_2 : (u,v) \mapsto (u, u+v-1) & \text{if} & \text{T2.} \\ \tau_{i+1} = \sigma_3 : (u,v) \mapsto (u-1, v) & \text{if} & \text{T3.} \\ \tau_{i+1} = \sigma_4 : (u,v) \mapsto (u, v-1) & \text{if} & \text{T4.} \end{array}$$

Recall that the local criteria of blowing-up depends on the resonance invariant (because of the rule LR1 and also of the definition of the influence locus). Thus we shall consider two cases: the nonresonant case and the resonant one. Put $m^* = \mu^*(\mathcal{F}_i, E_i; P_i)$ and recall that $m = \mu(\mathcal{F}_i, E_i; P_i)$ for all $i \geq 0$. We know that $m \leq m^* \leq m+1$. Moreover $m = m^*$ if and only if $\mathrm{Rs}(\mathcal{F}_i, E_i; P_i) = 0$ for all $i \geq 0$ and $m+1 = m^*$ if and only if $\mathrm{Rs}(\mathcal{F}_i, E_i; P_i) \neq 0$ for all $i \geq 0$. (Recall $E_i = \mathrm{Nd}(D_i, \mathcal{F}_i)$.)

A. *The nonresonant case*: $m = m^*$. The rule LR1 never applies. The rules LR2 and LR3 imply the following properties:

$$\begin{array}{lcr} \tau_{i+1} \in \{\sigma_1, \sigma_1^*, \sigma_2\} & \Leftrightarrow & \max\{\alpha_i, \beta_i\} < 1. \\ \tau_{i+1} = \sigma_3 & \Rightarrow & \alpha_i \geq 1. \\ \tau_{i+1} = \sigma_4 & \Rightarrow & \beta_i \geq 1. \end{array} \tag{1}$$

(In fact, the properties in the equation 1 are true each time that LR1 does not apply.) Note that $(\alpha_i, \beta_i) \in (1/m!)\mathbb{Z}^2$. The properties in equation 1 imply that

$$\alpha_{i+1} + \beta_{i+1} \leq \alpha_i + \beta_i - 1/m!.$$

Hence we find that $\alpha_i + \beta_i < 1$ after finitely many steps, a contradiction.

B. *The resonant case*: $m + 1 = m^*$. We can assume that the following properties hold for any index $i \geq 0$:

a) $H_i \subset E_i$. Otherwise we should have that $e_i = \mathrm{e}(E_i, P_i) \leq 2$ (actually $e_i = 2$, because of the resonance property), rule LR1 never applies and we make the same argument as in the nonresonant case.

b) $\max\{\alpha_i, \beta_i\} \geq 1$. In fact, if $\max\{\alpha_i, \beta_i\} < 1$ then $\tau_{i+1} \in \{\sigma_1, \sigma_1^*, \sigma_2\}$, even if rule LR1 applies. We conclude as in the nonresonant case.

c) $\tau_{i+1} \neq \sigma_1^*$. If $\tau_{i+1} = \sigma_1^*$, then $\beta_{i+1} = 0$ and $E_{i+1} \not\supset \{y_{i+1}^* = 0\}$. Thus $e_{i+1} = 2$ and LR1 does not apply at the step $i+1$, in particular the properties of equation 1 hold. Now, since $\alpha_{i+1} \geq 1$ and $\beta_{i+1} = 0$, we necessarily have $\tau_{i+2} = \sigma_3$. We get that $\alpha_{i+2} = \alpha_{i+1} - 1$, $\beta_{i+2} = \beta_{i+1} = 0$ and also $E_{i+2} \not\supset \{y_{i+2}^* = 0\}$. The situation is the same one as above, but with $\alpha_{i+2} = \alpha_{i+1} - 1$. We cannot repeat this infinitely many times.

Recall now that the fact $\mathrm{Rs}(\mathcal{F}_i, E_i; P_i) \neq 0$ implies that $e_i \geq 2$. Moreover, each time we apply the rule LR1, the blowing-up is dicritical and thus

$$2 = e_{i+1} = e_i - 1.$$

We do this infinitely many times; otherwise we reason as in the nonresonant case. Thus, we have two sequences of indices $\{s_j\}_{j=1}^{\infty}$ and $\{t_j\}_{j=1}^{\infty}$ with $s_j < t_j < s_{j+1}$ for all $j \geq 1$, such that

$$\begin{array}{lll} e_i = 2 & \text{for} & s_j \leq i < t_j. \\ e_i = 3 & \text{for} & t_j \leq i < s_{j+1}. \end{array}$$

Let us define the invariant γ_j by

$$\gamma_j = \begin{cases} \alpha_{s_j} & \text{if} \quad \left\{y_{s_j}^* z_{s_j} = 0\right\} \subset E_{s_j} \\ \beta_{s_j} & \text{if} \quad \left\{x_{s_j} z_{s_j} = 0\right\} \subset E_{s_j}. \end{cases}$$

We shall show that $\gamma_{j+1} < \gamma_j - 1$ to get the desired contradiction. In order to simplify notation, put $u_j = t_j - 1$, $v_j = s_{j+1} - 1$. We shall assume that E_{s_j} is given by $y_{s_j}^* z_{u_j} = 0$. The case $x_{s_j} z_{u_j} = 0$ is handled in a symmetric way. We have the following properties:

i) $\tau_{i+1} \in \{\sigma_3, \sigma_4\}$, for $s_j \leq i \leq v_j - 1$. Note that rule LR1 does not apply (and hence the properties of the equation 1 are true) for the indices i with $s_j \leq i \leq v_j - 1$, since it is not possible to have that $e_i = 3$ and $e_{i+1} = 2$. Then, the property (b) implies that $\tau_{i+1} \in \{\sigma_3, \sigma_4\}$.

ii) $\alpha_i \geq 1$ and $E_i = \{y_i^* z_i = 0\}$, for $s_j \leq i \leq u_j$. Assume that $\alpha_{s_j} < 1$. Then $\beta_{s_j} \geq 1$ and $\tau_{1+s_j} = \sigma_4$. This implies that $\alpha_{1+s_j} = \alpha_{s_j}$, $\beta_{1+s_j} = \beta_{s_j} - 1$, $E_{1+s_j} = \{y_{1+s_j}^* z_{1+s_j} = 0\}$ and in particular $e_{1+s_j} = 2$. The situation repeats at the step $1 + s_j$ and thus we get that $e_i = 2$ for $i \geq s_j$, a contradiction (we also see a contradiction looking at the evolution of β_{s_j}). Then $\alpha_{s_j} \geq 1$. If $s_j = u_j$ we are done. Assume that $s_j < u_j$ and let us show that $\alpha_{1+s_j} \geq 1$ and $E_{1+s_j} = \{y_{1+s_j}^* z_{1+s_j} = 0\}$ to conclude by an evident induction. If $\tau_{1+s_j} = \sigma_4$, we are done since $\alpha_{1+s_j} = \alpha_{s_j}$, $\beta_{1+s_j} = \beta_{s_j} - 1$ and $E_{1+s_j} = \{y_{1+s_j}^* z_{1+s_j} = 0\}$ as above. If$\tau_{1+s_j} = \sigma_3$, the fact that $e_{1+s_j} = 2$ means that the exceptional divisor $\{x_{1+s_j} = 0\}$

is dicritical and thus $E_{1+s_j} = \{y^*_{1+s_j} z_{1+s_j} = 0\}$, moreover necessarily $\alpha_{1+s_j} = \alpha_{s_j} - 1 \geq 1$ since otherwise we get a contradiction as above.

iii) $\alpha_{u_j} \leq \alpha_{s_j} = \gamma_j$ and $\beta_{u_j} < 1$. The first part is evident by (i). For the second part, if $\beta_{u_j} \geq 1$, in view of the priorities of rule LR3, we have that $\tau_{1+u_j} = \sigma_4$. Since $E_{u_j} = \{y^*_{u_j} z_{u_j} = 0\}$ this implies that $2 = e_{1+u_j} = e_{t_j}$, contradiction.

iv) $\tau_{i+1} = \sigma_3$, for $u_j \leq i \leq v_j - 1$. If $\tau_{u_j} = \sigma_4$ then $e_{t_j} = 2$, contradiction. Thus $\tau_{u_j} = \sigma_3$. We get that $\alpha_{t_j} = \alpha_{u_j} - 1$ and $\beta_{t_j} = \beta_{u_j} < 1$. If $u_j = v_j - 1$ we are done. Assume that $u_j < v_j - 1$. Since rule LR1 does not apply for $u_j \leq i \leq v_j - 1$ and $\beta_{t_j} < 1$ we get $\tau_{1+u_j} = \sigma_3$ and end by finite induction.

v) $\alpha_{v_j} = \alpha_{u_j} - \{v_j - u_j\} \leq \alpha_{s_j} - 1 = \gamma_j - 1$ and $\beta_{v_j} = \beta_{u_j} < 1$. In fact, by (iv) we have the more precise result that $\alpha_i = \alpha_{u_j} - \{i - u_j\}$ and $\beta_i = \beta_{u_j}$, for $u_j \leq i \leq v_j$.

Now we have three possibilities for $\tau_{s_{j+1}}$, since $\beta_{u_j} < 1$ and thus $\tau_{s_{j+1}} \neq \sigma_4$:

(1) $\tau_{s_{j+1}} = \sigma_1$. Then $\{x_{s_{j+1}} = 0\}$ is a dicritical component and thus $\gamma_{j+1} = \alpha_{s_{j+1}} = \alpha_{v_j} + \beta_{v_j} - 1 < \alpha_{v_j} \leq \gamma_j - 1$.

(2) $\tau_{s_{j+1}} = \sigma_3$. Then $\{x_{s_{j+1}} = 0\}$ is a dicritical component and thus $\gamma_{j+1} = \alpha_{s_{j+1}} = \alpha_{v_j} - 1 < \alpha_{v_j} \leq \gamma_j - 1$.

(3) $\tau_{s_{j+1}} = \sigma_2$. Then $\{y^*_{s_{j+1}} = 0\}$ is a dicritical component and thus $\gamma_{j+1} = \beta_{s_{j+1}} = \alpha_{v_j} + \beta_{v_j} - 1 < \alpha_{v_j} \leq \gamma_j - 1$.

This completes the proof. □

3.3. *Local control in the first m-stable cases.* Assume that $\mathcal{S}$ corresponds to the m-stable case. We divide the m-stable situation in three cases:

(1) Adapted multiplicity bigger than the adapted order.

(2) The resonant m-stable case.

(3) Adapted multiplicity equal to the adapted order without resonances.

In this section we shall treat all the possibilities except for the following two:

I*: $m^* = r + 1;\ m = r;\ d = \mathrm{d}(\mathcal{F}_i, E_i; P_i) = 1;\ l = \mathrm{l}(\mathcal{F}_i, E_i; P_i) = 2.$

II*: $m^* = m = r;\ d = \mathrm{d}(\mathcal{F}_i, E_i; P_i) = 1;\ t = \mathrm{t}(\mathcal{F}_i, E_i; P_i) = 1.$

These two cases are the more complicated ones and will be considered in the next section.

In order to describe all the situations we shall use the vertical invariants. The situations in this section will be treated either by a direct consideration, or by exhibiting a surface H having maximal contact with $\mathcal{S}$ adapted to D.

3.3.1. *Adapted multiplicity bigger than the adapter order.* Let us suppose here that $m^* = m = r + 1$. We assume without loss of generality that the vertical invariants l, d and δ are stable, that is

$$l = \mathrm{l}(\mathcal{F}_i, E_i; P_i);\ d = \mathrm{d}(\mathcal{F}_i, E_i; P_i);\ \delta = \delta(\mathcal{F}_i, E_i; P_i).$$

for all $i \geq 0$. We shall consider the following cases that cover all the possibilities:

$$\begin{array}{l} l \leq 1. \\ l = 2,\ d = 0. \\ l = 2,\ d = 1,\ \delta = r. \\ l = 2, d = 1, \delta = r + 1. \end{array}$$

Proposition 29. *The case $l \leq 1$ does not occur.*

Proof. The centers of the blowing-ups must be points since the dimension of the directrix is one, by Proposition 4 (the Directrix Theorem). We shall see that the points P_i are in fact the infinitely near points of a curve Γ. This implies that the strict transform Γ_k of Γ at P_k is an appropriate center for large k. We get a contradiction to rule LR3 (note that LR1 does not apply because we have no resonance). Now, it is enough to show that $P_{i+2} = \mathbb{P}\mathrm{roj}(\mathrm{Dir}(\mathcal{F}_{i+1}, E_{i+1}; P_{i+1}))$ is not in the strict transform of $\pi_{i+1}^{-1}(P_i)$ under π_{i+2}. But this follows from the fact that

$$0 = \dim \mathrm{T}_{P_{i+1}} \pi^{-1}(P_i) \cap \mathrm{Dir}(\mathcal{F}_{i+1}, E_{i+1}; P_{i+1}),$$

which was shown in Proposition 5. This completes the proof. $\square$

Proposition 30. *The case $l = 2$, $d = 0$ does not occur.*

Proof. Let us simplify the notation by putting P, P' instead of P_i, P_{i+1} and so on. We shall prove that there is an irreducible component H of E having maximal contact with $\mathcal{S}$ adapted to D. Put $e = \mathrm{e}(E, P)$. Since $m = r + 1$ and $d = 0$ we have that $e \leq 2$ and $1 \leq e$. Thus we have the two possibilities $e = 1$ and $e = 2$. Moreover, we can take coordinates (x, y, z) at P such that

$$\begin{array}{l} \{z = 0\} \subset E \subset \{xz = 0\}, \\ \mathrm{Dir}(\mathcal{F}, E; P) = \{\bar{z} + \lambda \bar{x} = 0\}, \\ \text{If } e = 1, \text{ then } \lambda = 0, \end{array}$$

where $\bar{x}$ is the initial form of the coordinate x and so on. Note that we can write a local generator ω of $\mathrm{Sat}(\mathcal{F}, E)$ as follows:

$$\omega = adx + bdy + c\frac{dz}{z} \quad \text{if } e = 1,$$
$$\omega = a\frac{dx}{x} + bdy + c\frac{dz}{z} \quad \text{if } e = 2,$$

where $\nu(c) \geq r+1$ if $e = 1$ and $\nu(a, c) \geq r+1$ if $e = 2$. Moreover

$$(\mathrm{In}^r(a), \mathrm{In}^r(b)) = (\bar{z} + \lambda\bar{x})^r(\alpha, \beta)$$

with $(\alpha, \beta) \neq (0, 0)$. First, let us make the following observation:

The case $\lambda \neq 0$ *does not occur*: Assume $\lambda \neq 0$. We shall show that $d' \geq 1$, if the other invariants remain stable, to obtain in this way a contradiction. Necessarily $e = 2$, since $\lambda \neq 0$ and $d = 0$. We have $\mathrm{In}^r(b) = (\bar{z} + \lambda\bar{x})^r$. If π is quadratic (that is, the center of π is given by the point P) then up to a coordinate change $y \mapsto y + \eta x$ we get coordinates (x', y', z') at P' given by one of the following equations

$$\text{T-2}: \quad x = x'y',\ y = y',\ z = y'z',$$
$$\text{T-1,}(0, \lambda): \quad x = x',\ y = x'y',\ z = x'(z' - \lambda).$$

If T-1,$(0, \lambda)$ we get that $d' \geq 1$, since $E' \subset \{x' = 0\}$ and $\mathrm{In}^r(b') = \bar{z}'^r + \bar{x}'(\dots)$, where $b' = \omega'(\partial/\partial y')$ and ω' is a local generator of $\mathrm{Sat}(\mathcal{F}', E')$ at P'. If T-2 and π is nondicritical, then $e' = 3$ and the adapted multiplicity drops. If π is dicritical, we get a monomial of the type yz^r either in the coefficient a or c and the adapted multiplicity drops. If π is monoidal (the center is a curve), necessarily the center is given by $Y = \{x = z = 0\}$ and we have coordinates (x', y', z') at P' given by $x = x'$, $y = y'$ and $x = x'(z' - \lambda)$. We get $d' \geq 1$.

Hence $\lambda = 0$, that is, the directrix is exactly the tangent space of $H = \{z = 0\}$. We shall show that H has maximal contact with $\mathcal{S}$ adapted to D. Let Y be the center of π. Since Y has normal crossings with D and its tangent space is contained in the directrix, we deduce that $Y \subset H$. Also, we deduce from the Directrix Theorem that $P' \in H'$, where H' is the strict transform of H by π. Thus, we are done if we find local coordinates (x', y', z') at P' satisfying the same properties as (x, y, z) above.

Up to a coordinate change in x, y not affecting the situation, we assume that either $Y = \{P\}$ or $Y = \{x = z = 0\}$ or $Y = \{y = z = 0\}$ and moreover we have local coordinates (x', y', z') at P' given by one of the transformations T1, T2, T3 or T4. Also, up to a possible reordering of x, y we can assume that $\mathrm{In}^r(b) = \bar{z}^r$. Let us consider the effect of the above four transformations:

T1 and T3. First, we have $H' = \{z' = 0\} \subset E' \subset \{x'z' = 0\}$. Moreover $\mathrm{In}^r(b') = \bar{z}'^r + \bar{x}'(\dots) + \bar{y}'(\dots)$, where $b' = \omega'(\partial/\partial y')$ and ω' is a local generator of $\mathrm{Sat}(\mathcal{F}', E')$ at P'. Since $\mathrm{In}^r(b')$ must be a power of a linear form, because of $l=2$, we get that $\mathrm{In}^r(b')=(\bar{z}'+\lambda'\bar{x}'+\mu'\bar{y}')^r$ and

$$\mathrm{Dir}(\mathcal{F}',E';P')=\{\bar{z}'+\lambda'\bar{x}'+\mu'\bar{y}'=0\}.$$

Since $d=0$, we have that $\mu'=0$ and $\lambda'=0$ if $e'=1$. Thus we are done.

T2 and T4. Assume first that $e=1$. Without loss of generality we can assume that $\nu(a)\geq r+1$; otherwise we reorder x,y and repeat the argument of T1, T3. If π is dicritical then $\nu(yb+c)\geq r+2$ and we find a monomial of the type $-yz^r$ in the coefficient c; the adapted multiplicity drops. Thus π is nondicritical and

$$b'=\omega'(y'\partial/\partial y')=y'^{-m}\{x'y'(a\circ\pi)+y'(b\circ\pi)+(c\circ\pi)\}\quad \text{if T2.}$$

$$b'=\omega'(y'\partial/\partial y')=y'^{-m}\{y'(b\circ\pi)+(c\circ\pi)\}\quad \text{if T4.}$$

Since y' divides $y'^{-r}(a\circ\pi)$ (in the case T2) and $\nu(y'^{-m}(c\circ\pi))\geq r+1$, we find a monomial of the type z'^r in b' and thus the adapted multiplicity drops. This ends with the case $e=1$. If $e=2$ and π is dicritical, we find a monomial of the type yz^r either in a or in c and the adapted multiplicity drops; if π is nondicritical, then $e'=3$ and the adapted multiplicity drops. □

Proposition 31. *The case $l=2$, $d=1$, $\delta=r$ does not occur.*

Proof. We shall prove that the centers of the blowing-ups are the points P_i and moreover they are the infinitely near points of a curve Γ. We get in this way a contradiction as in the study of the case $l\leq 1$.

Let us consider an index $i\geq 0$. Put $e_i=\mathrm{e}(E_i,P_i)$. We note that $0\leq e_i\leq 1$. In fact, if $e_i=3$ then the adapted multiplicity should be r and if $e_i=2$, $l=2$, $d=1$ then necessarily $\delta=r+1$.

We can take local coordinates (x,y,z) at P_i and a local generator ω of $\mathrm{Sat}(\mathcal{F}_i,E_i)$ at P_i such that

$$\{x=0\}\subset E_i$$
$$\mathrm{Dir}(\mathcal{F}_i,E_i;P_i)=\{\bar{z}=0\}$$
$$\mathrm{In}^r(b)=\bar{z}^r \text{ if } b=\omega(\partial/\partial y).$$

Show now that the blowing-up π_{i+1} has the center $Y_i=\{P_i\}$. Assume $\dim Y_i=1$. Take the coordinates (x,y,z) above such that in addition we have either $Y_i=\{x=z=0\}$ or $Y_i=\{y=z=0\}$. In fact, the center $\{x=z=0\}$ is nonappropriate at P_i and thus $Y_i=\{y=z=0\}$. Put $c=\omega(\partial/\partial z)$. Since $l=2$, $\mathrm{In}^r(c)=\alpha\bar{z}^r$ and thus $\mathrm{In}^r(yb+zc)=\bar{y}\bar{z}^r+\alpha\bar{z}^{r+1}$. Then π_{i+1} is nondicritical and the adapted multiplicity drops after the blowing-up.

Thus the centers $Y_i=\{P_i\}$ for each index $i\geq 0$. Let us consider now the two possible situations:

A) For all indices $i\geq 0$ we have that $1=e_i=e_{i+1}$.

B) There is an index $i\geq 0$ such that $e_i=0$.

Assume first that we are in case A). Write ω as follows

$$\omega = a\frac{dx}{x} + bdy + cdz.$$

Up to a linear coordinate change of the type $y \mapsto y + \zeta x$, the equations of π_{i+1} are given either by T1 or T2. Let us show that T2 is not possible. If T2, then π_{i+1} is dicritical, otherwise we should have $e_{i+1} = 2$. Thus $\nu(a+yb+zc) \geq r+2$. Then there is a monomial of the type $-yz^r$ in the coefficient a and the adapted multiplicity drops. Then we have T1. Moreover π_{i+1} is nondicritical since otherwise $e_{i+1} = 0$. This implies both that P_{i+1} is not in the strict transform of E_i and also that E_{i+1} is precisely the exceptional divisor of the blowing-up, given by $\pi^{-1}(P_i)$. Repeating the argument for all indices $i \geq 0$, we have that P_{i+2} is not in the strict transform by π_{i+2} of the exceptional divisor of π_{i+1}. This condition (even only for large i) assures that P_i are the infinitely near points of a curve Γ.

Before starting with the case B), let us describe a situation that is stable under blowing-up and that assures that $e_i = 1$ for large i. Once we get this situation, we end as in the case A). We say that the singularity of $\mathcal{F}_i$ at P_i is of the type *strong*-A if and only if there are local coordinates (x, y, z) such that $E_i = \{x = 0\}$, the directrix is given by $\bar{z} = 0$ and a local generator

$$\omega = a\frac{dx}{x} + bdy + cdz$$

of $\mathrm{Sat}(\mathcal{F}_i, E_i)$ satisfies $\mathrm{In}^r(b) = \bar{z}^r$ and the monomial yz^r appears in a with a coefficient $q \in \mathbb{Z}_{>0}$.

Assume that we are in a situation of the type strong-A. The existence of the monomial qyz^r in a shows that π_{i+1} is nondicritical. Up to a linear change of coordinates $y \mapsto y+\zeta x$, not affecting our situation, the equations of π_{i+1} are either T1 or T2. Moreover T2 is not possible, since otherwise $e_{i+1} = 2$. Thus we have T1. Now, up to a coordinate change of the type $z' \mapsto z' + \alpha x'$ we get at P_{i+1} a situation of the type strong-A where the coefficient $a' = \omega'(x'\partial/\partial x')$ contains the monomial $(q+1)y'z'^r$. In particular $e_{i+1} = 1$. Repeating this, we get that $e_j = 1$ for any $j \geq i$. Then we can apply the arguments of A).

Let us now consider the case B), where $\omega = adx + bdy + cdz$. Then π_{i+1} is nondicritical, otherwise $\nu(xa + by + zc) \geq r + 2$ and this contradicts $l = 2$. Moreover, up to a linear change of coordinates $y \mapsto y + \zeta x$, not affecting our situation, the equations of π_{i+1} are either T-1 or T-2. If T-2, the adapted multiplicity drops, since the initial part $\mathrm{In}^r(xa + by + zc)$ must be of the form

$$\mathrm{In}^r(xa + by + zc) = \beta\bar{x}\bar{z}^r + \bar{y}\bar{z}^r + \alpha\bar{z}^{r+1}.$$

Thus we have T-1. Then $\mathrm{Sat}(\mathcal{F}_{i+1}, E_{i+1})$ is locally generated at P_{i+1} by

$$\omega' = a'\frac{dx'}{x'} + b'dy' + c'dz'$$

with a monomial $+y'z'^r$ in the coefficient a'. Up to a coordinate change $z' \mapsto z' + \alpha' x'$, we get a situation of the type strong-A, with $q = 1$. This ends the proof. $\square$

PROPOSITION 32. *The case* $l = 2$, $d = 1, \delta = r + 1$ *does not occur.*

Proof. We shall find a nonsingular surface H having maximal contact with $\mathcal{S}$ adapted to the total transform of E_0, up to eliminate finitely many steps. This gives the desired contradiction.

Let (x, y, z) be a local system of coordinates at P_0 such that $E_0 \subset \{xy = 0\}$ and $\mathrm{Dir}(\mathcal{F}_0, E_0; P_0) = \{\bar{z} = 0\}$. Take a local generator ω of $\mathrm{Sat}(\mathcal{F}_0, E_0)$ such that $\mathrm{In}^r(c) = \bar{z}^r$, where $c = \omega(\partial/\partial z)$. By the Weierstrass preparation theorem, up to multiplication ω by a unit,

$$c(x, y, z) = z^r + c_1(x, y)z^{r-1} + \ldots + c_r(x, y).$$

Now, let us do a Tchirnhausen transformation $z \mapsto z^* = z + (1/r)c_1(x, y)$ and let $\{\partial^*/\partial x, \partial^*/\partial y, \partial^*/\partial z^*,\}$ be the basis of vector fields corresponding to the coordinate system (x, y, z^*). Then

$$c^* = \omega(\partial^*/\partial z^*) = {z^*}^r + c_2^*(x, y){z^*}^{r-2} + \ldots + c_r^*(x, y).$$

Put $H = \{z^* = 0\}$. The classical theory says that H has maximal contact with the surface given by c^*. But in this situation, it is enough to assure the maximal contact with $\mathcal{S}$ adapted to the total transform of E_0. $\square$

3.3.2. *The resonant m-stable case.* This big case is defined by the property $m^* = r + 1$ and $m = r$. Then the resonance invariant $Rs_i = \mathrm{Rs}(\mathcal{F}_i, E_i; P_i) \neq 0$ for all $i \geq 0$. We shall treat here all the possibilities except for the one defined by

$$m^* = r + 1;\ m = r;\ d = \mathrm{d}(\mathcal{F}_i, E_i; P_i) = 1;\ l = \mathrm{l}(\mathcal{F}_i, E_i; P_i) = 2,$$

to be studied in the next section by means of the characteristic polygons. We shall consider here the behaviour of the vertical invariants $d_i = \mathrm{d}(\mathcal{F}_i, E_i; P_i)$ and $l_i = \mathrm{l}(\mathcal{F}_i, E_i; P_i)$ as well as the number of contact and essential components in order to organize our control of this situation. We know the vertical stability of these invariants when $Rs_i \neq 1$. It will be true even when $Rs_i = 1$ by using the additional assumption that we are in the resonant m-stable case. Let us denote $e_i = \mathrm{e}(E_i, P_i)$ for all $i \geq 0$.

PROPOSITION 33. *For any* $i \geq 0$, $d_i \leq d_{i+1}$.

Proof. If $Rs_i \neq 1$, we are done. Assume $Rs_i = 1$. Note that $2 \leq e_i \leq 3$, because we are in a resonant case. Thus $0 \leq d_i \leq 1$. The only possible bad case is that $d_i = 1$ (and hence $e_i = 2$) and $d_{i+1} = 0$. Assume we are in this case. If π_i is quadratic (that is, the center $Y_i = \{P_i\}$), then the blowing-up is dicritical. Let F be the intersection of the two components of E_i. The fact that $d_i = 1$ implies that

$$T_{P_i}F \not\subset \mathrm{Dir}(\mathcal{F}_i, E_i; P_i).$$

This, jointly with the fact that π_i is dicritical, implies that $e_{i+1} \leq 1$, a contradiction. Consider now the case $\dim Y_i = 1$. We already know that $Y_i \neq F$ since the tangent space of F is not contained in the directrix. Moreover, the directrix has dimension two and if H is the component of E_i containing Y_i we also know that $T_{P_i}H \neq \mathrm{Dir}(\mathcal{F}_i, E_i; P_i)$. In particular the directrix and E_i have normal crossings. More precisely, we can take local coordinates (x, y, z) at P_i and a local generator ω of $\mathrm{Sat}(\mathcal{F}_i, E_i)$ such that $E_i = \{xy = 0\}$, $Y_i = \{x = z = 0\}$, the directrix is given by $\{\bar{z} = 0\}$ and $\mathrm{In}^r(a) = -\mathrm{In}^r(b) = \bar{z}^r$, where

$$\omega = a\frac{dx}{x} + b\frac{dy}{y} + cdz.$$

Then π_i is given by the equations T3 and $\mathrm{Sat}(\mathcal{F}_{i+1}, E_{i+1})$ is generated by

$$\omega' = a'\frac{dx'}{x'} + b'\frac{dy'}{y'} + c'dz',$$

where $B'(0, 0, \bar{z}') = -\bar{z}'^r$, if $B' = \mathrm{In}^r(b')$. In particular $d_{i+1} = 1$, a contradiction. □

The above result allows us to assume that $d = d_i$ for all $i \geq 0$. Thus we have exactly two possibilities: $d = 1$ and $d = 0$.

Lemma 13. *If $d = 1$ then $l_{i+1} \leq l_i$.*

Proof. Note that $d = 1$ jointly with the resonance property imply that $e_i = 2$ for all $i \geq 0$. In particular $e_{i+1} \geq e_i$ and we are done if $\mathrm{Rs}_i \neq 1$ by our general results. Assume that $\mathrm{Rs}_i = 1$. The only possible bad case is $l_i = 1$ and $l_{i+1} = 2$. Then π_{i+1} is quadratic (since $l_i = 1$) and dicritical. Moreover, as in the preceding proposition, the point P_{i+1} is not in the strict transform of F, where F is the intersection of the two irreducible components of E_i. This, jointly with the dicriticalness of π_{i+1}, implies that $e_{i+1} \leq 1$, a contradiction. □

We can assume that if $d = 1$ then $l = l_i$, for all $i \geq 0$, up to eliminating finitely many steps of the sequence $\mathcal{S}$. Then either $l = 1$ or $l = 2$. We shall treat the case $l = 2$ in the next section.

Proposition 34. *The case $d = 1$ and $l = 1$ does not occur.*

Proof. Note that each π_{i+1} is a quadratic blowing-up. Choose coordinates (x, y, z) at P_i such that $E_i = \{xy = 0\}$ and $\mathrm{Dir}(\mathcal{F}_i, E_i; P_i) \subset \{\bar{z} = 0\}$. Let

$$\omega = a\frac{dx}{x} + b\frac{dy}{y} + cdz$$

be a local generator of $\mathrm{Sat}(\mathcal{F}_i, E_i)$ at P_i. We know that there are relatively prime positive integers p and q such that $p\mathrm{In}^r(a)+q\mathrm{In}^r(b) = 0$. Since $\gamma_i = p+q$, it is an invariant not depending on ω, (x, y, z). Because the blowing-up is quadratic, $e_i = 2$ and P_{i+1} is not in the strict transform of $x = y = 0$, we can choose equations T1 for the blowing-up and a direct computation shows that $\gamma_{i+1} < \gamma_i$. □

Next, we study the case $d = 0$ by means of the contact and essential components.

PROPOSITION 35. *The case $d = 0$ does not occur.*

Proof. Fix an index $i \geq 0$. We shall consider two cases:

First case: *there is a contact component H of E_i.* We shall show that H has maximal contact with the sequence $\mathcal{S}$ of local blowing-ups. To do this it is enough to show that the following properties hold:

(1) The center $Y_i \subset H$.

(2) If H' is the strict transform of H by π_{i+1}, then $P_{i+1} \in H'$.

(3) The component H' is also a contact component of E_{i+1} at P_{i+1}.

We already know that these properties are true if $\mathrm{Rs}_i \neq 1$. Assume $\mathrm{Rs}_i = 1$. Let us first show that π_{i+1} must be a quadratic blowing-up, that is $Y_i = \{P_i\}$. Assume that $\dim Y_i = 1$ to get a contradiction. We know that $e_i = 2$, since $e_i = 3$ is not compatible with rule LR1. Take local coordinates (x, y, z) at P_i such that $E_i = \{yz = 0\}$. The dimension of the directrix should be 2 since the blowing-up is monoidal. Up to a reordering of y, z the directrix is $\bar{z} + \lambda\bar{y} = 0$ and $Y_i = \{y = z = 0\}$. Since $e_{i+1} \geq 2$, we deduce that $\lambda = 0$ and π_{i+1} is nondicritical, given by the equations T4. Let ω be a local generator of $\mathrm{Sat}(\mathcal{F}_i, E_i)$ at P_i such that $C = -B = \bar{z}^r$, where $C = \mathrm{In}^r(\omega(z\partial/\partial z))$ and $B = \mathrm{In}^r(\omega(y\partial/\partial y))$. We get a generator ω' of $\mathrm{Sat}(\mathcal{F}_{i+1}, E_{i+1})$ at P_{i+1} such that

$$B'(0, 0, \bar{z}') = 0;\ C'(0, 0, \bar{z}') = \bar{z}'^r,$$

if $C' = \mathrm{In}^r(\omega'(z'\partial/\partial z'))$ and $B' = \mathrm{In}^r(\omega'(y'\partial/\partial y'))$. Hence $\mathrm{Rs}_{i+1} = 0$, a contradiction.

Thus, the blowing-up π_{i+1} is quadratic and hence dicritical, since $\mathrm{Rs}_i = 1$. In particular $e_{i+1} = 2$ and P_{i+1} is in the strict transform of the intersection of two irreducible components of E_i. One of them is necessarily H (the directrix cannot contain the tangent space of the eventual third *coordinate line*). Thus we can choose coordinates (x, y, z) at P_i in such a way such that $\{yz = 0\} \subset E_i \subset \{xyz = 0\}$, $H = \{z = 0\}$, the blowing-up π_{i+1} is given by T1 and

$$C(\bar{x}, \bar{y}, \bar{z}) \notin \mathbb{C}[\bar{x}, \bar{y}]$$

where $C = \mathrm{In}^r(c)$ and $c = \omega(z\partial/\partial z)$, for a local generator ω of $\mathrm{Sat}(\mathcal{F}_i, E_i)$. We get a generator ω' of $\mathcal{F}_{i+1}$ at P_{i+1} such that if $c' = \omega'(z'\partial/\partial z')$ then $c' = {x'}^{-r}(c \circ \pi_{i+1})$ and hence $C'(\bar{x}', \bar{y}', \bar{z}') \notin \mathbb{C}[\bar{x}', \bar{y}']$. This implies that $z' = 0$ is a contact component of $E_{i+1} = \{y'z' = 0\}$.

Second case: *there are no contact components.* Let us first show that $e_i = 3$ for all $i \geq 0$. If $e_i = 2$, take local coordinates (x, y, z) at P_i such that $E_i = \{yz = 0\}$, let ω be a local generator of $\mathrm{Sat}(\mathcal{F}_i, E_i)$ at P_i and put

$$q\Phi(\bar{x}, \bar{y}, \bar{z}) = \mathrm{In}^r(\omega(y\partial/\partial y));\ -p\Phi(\bar{x}, \bar{y}, \bar{z}) = \mathrm{In}^r(\omega(z\partial/\partial z)).$$

Since $d = 0$, we have that $\Phi \in \mathbb{C}[\bar{y}, \bar{z}]$. Moreover, up to a reordering of y, z we can assume that $\Phi \notin \mathbb{C}[\bar{y}]$ and thus $\{z = 0\}$ is a contact component, a contradiction. Then $e_i = 3$ for all $i \geq 0$. By LR1, it is not possible to have $\mathrm{Rs}_i = 1$, since then π_{i+1} would be quadratic; so it is dicritical and $e_{i+1} = 2$. Thus $\mathrm{Rs}_i \neq 1$ for all $i \geq 0$. Then, by Proposition 8 there is at least one essential component H of E_i and it has maximal contact with the sequence $\mathcal{S}$. □

3.3.3. *Adapted multiplicity equal to the adapted order.* This big case is defined by the property $m^* = m = r$. We shall treat here all the possibilities except for the one defined by

$$m^* = m = r;\ d = \mathrm{d}(\mathcal{F}_i, E_i; P_i) = 1;\ t = \mathrm{t}(\mathcal{F}_i, E_i; P_i) = 1,$$

that will be studied in the next section by means of the characteristic polygons.

Note that we are in a nonresonant situation; that is, $\mathrm{Rs}_i = 0$, for all $i \geq 0$, in particular, we are never radially dicritical. In this way we have good behaviour of our vertical invariants and we can assume without loss of generality that there are integers d and t, with $d \geq t$, such that

$$d = \mathrm{d}(\mathcal{F}_i, E_i; P_i);\ t = \mathrm{t}(\mathcal{F}_i, E_i; P_i)$$

for all $i \geq 0$. Note also that if $e_i = \mathrm{e}(E_i, P_i)$, then $e_i \geq 1$ since $m = r$. This implies that $2 \geq d \geq t \geq 0$. Moreover, in the case $d = 0$ we have good behaviour of the contact and essential components (which are never radially dicritical) and thus if we get a contact component, it has maximal contact with the sequence $\mathcal{S}$. Otherwise we have an essential component having also maximal contact with $\mathcal{S}$. The remaining cases are then: i) $d = 2$. ii) $d = 1$ and $t = 0$.

PROPOSITION 36. *The case $d = 2$ does not occur.*

Proof. Necessarily $e_i = 1$ and the directrix has dimension one. In particular each blowing-up π_{i+1} is quadratic. Moreover it is nondicritical since nonradially dicritical. This, jointly with the fact $e_i = 1$, means that P_{i+1} is not in the strict transform of the preceding exceptional divisor. This implies that the sequence of points P_i contains the strict transforms of a nonsingular curve Γ that should be appropriate at one step and then used as a center, a contradiction. $\square$

PROPOSITION 37. *The case $d = 1$, $t = 0$ does not occur.*

Proof. Assume first that $e_i = 2$. Consider local coordinates (x, y, z) at P_i such that $E_i = \{xy = 0\}$ and $\{\bar{z} = 0\}$ contains $\mathrm{Dir}(\mathcal{F}_i, E_i; P_i)$. Since $t = 0$, the dimension of the directrix is one (if it is two, we get $t = d = 1$) and, up to a reordering of x, y, we have that

$$\mathrm{Dir}(\mathcal{F}_i, E_i; P_i) = \{\bar{y} + \eta\bar{x} = \bar{z} = 0\}.$$

Let ω be a local generator of $\mathrm{Sat}(\mathcal{F}_i, E_i)$ at P_i. Put $A = \mathrm{In}^r(a)$ and $B = \mathrm{In}^r(b)$, where $a = \omega(x\partial/\partial x)$ and $b = \omega(y\partial/\partial y)$. Write $A = \Phi(\bar{y} + \eta\bar{x}, \bar{z})$ and $B = \Psi(\bar{y} + \eta\bar{x}, \bar{z})$. The fact $t = 0$ implies that $\bar{y} + \eta\bar{x}$ divides both A and B. Moreover, in view of the integrability of ω, we know that there is a decomposition

$$A = T(\bar{x}, \bar{y}, \bar{z})\widetilde{A}(\bar{x}, \bar{y})$$
$$B = T(\bar{x}, \bar{y}, \bar{z})\widetilde{B}(\bar{x}, \bar{y}).$$

Combining both facts, we get

$$A = \lambda(\bar{y} + \eta\bar{x})^s\widetilde{T}(\bar{y} + \eta\bar{x}, \bar{z})$$
$$B = \mu(\bar{y} + \eta\bar{x})^s\widetilde{T}(\bar{y} + \eta\bar{x}, \bar{z})$$

with $s \geq 1$ and $(\lambda, \mu) \neq (0, 0)$. The nonresonance implies that $\lambda + \mu \neq 0$.

Assume that $\eta = 0$. Then π_{i+1} is quadratic, nondicritical and given at P_{i+1} by the equations T1. We get a generator ω' of $\mathrm{Sat}(\mathcal{F}_{i+1}, E_{i+1})$ at P_{i+1} such that if $a' = \omega'(x'\partial/\partial x')$, $b' = \omega'(y'\partial/\partial y')$, $A' = \mathrm{In}^r(a')$ and $B' = \mathrm{In}^r(b')$, then

$$A' = (\lambda + \mu)\bar{y}'^s\widetilde{T}(\bar{y}', \bar{z}') + \bar{x}'(\dots)$$
$$B' = \mu\bar{y}'^s\widetilde{T}(\bar{y}', \bar{z}') + \bar{x}'(\dots).$$

Up to making a coordinate change of the type $z' \mapsto z' + \xi x'$, the situation repeats with a certain coefficient η'. If we always get $\eta' = 0$, then the points P_i are the infinitely near singular points of a curve Γ whose strict transform is appropriate at a certain step and thus we have to choose it as a center, which contradicts the fact that π_{i+1} is quadratic. Then for some index $i \geq 0$ we have that $\eta \neq 0$. In that case π_{i+1} is given by the equations $x = x'$, $y = x'(y' - \eta)$, $z = x'z'$ and $E_{i+1} = \{x' = 0\}$. If $a' = \omega'(x'\partial/\partial x')$ and $A' = \mathrm{In}^r(a')$ as above,

$$A' = (\lambda + \mu)\bar{y}'^r\widetilde{T}(\bar{y}', \bar{z}') + \bar{x}'(\dots)$$

and thus $\mathrm{d}(\mathcal{F}_{i+1}, E_{i+1}; P_{i+1}) = 2$ contradicting the fact that $d = 1$.

It remains to consider the case $e_i = 1$. Consider local coordinates (x, y, z) at P_i such that $E_i = \{x = 0\}$ and $\{\bar{z} = 0\}$ contains $\mathrm{Dir}(\mathcal{F}_i, E_i; P_i)$. Since $t = 0$, the dimension of the directrix is one (if it is two, we get $t = d = 1$) and necessarily

$$\mathrm{Dir}(\mathcal{F}_i, E_i; P_i) = \{\bar{x} = \bar{z} = 0\}\,.$$

Since π_{i+1} is quadratic and nondicritical, we get $e_{i+1} = 2$ and we are done. □

3.4. *Local control and characteristic polygons.* Let us consider $\mathcal{S}$, $\mathcal{F}$ and D as in the beginning of Section 3. Assume that $\mathcal{S}$ is m-stable. We shall treat here the following situations:

I*: The resonant m-stable case $m^* = r + 1$, $m = r$ with $d = 1$ and $l = 2$.

II*: The nonresonant case $m^* = m = r$, with $d = t = 1$.

These two situations are the m-stable cases remaining from the previous section, and thus our study of m-stable cases will be complete. We shall use the characteristic polygons as Hironaka did in [20], both to find maximal contact surfaces and for introducing new vertical invariants.

3.4.1. *Reduction to nondicritical-like behaviour.* The next propositions are devoted to proving that up to eliminating finitely many steps of $\mathcal{S}$ we get one of the following two situations:

I. (Resonant case). The sequence $\mathcal{S}$ has the invariants $m^* = r + 1$, $m = r$, $d = 1$ and $l = 2$. Moreover, each blowing-up π_{j+1} is nondicritical and $E_j = D_j$.

II. (Nonresonant case). The sequence $\mathcal{S}$ has the invariants $m^* = m = r$ and $d = t = 1$. For any index $j \geq 0$ we have that

$$r = m = \nu(\mathcal{F}_j, E_j; P_j) = \nu(\mathcal{F}_j, D_j; P_j) = \mu(\mathcal{F}_j, E_j; P_j) = \mu(\mathcal{F}_j, D_j; P_j).$$
$$0 = \mathrm{Rs}(\mathcal{F}_j, E_j; P_j) = \mathrm{Rs}(\mathcal{F}_j, D_j; P_j).$$
$$1 = \mathrm{d}(\mathcal{F}_j, E_j; P_j) = \mathrm{d}(\mathcal{F}_j, D_j; P_j) = \mathrm{t}(\mathcal{F}_j, E_j; P_j) = \mathrm{t}(\mathcal{F}_j, D_j; P_j).$$

Moreover, the directrix has dimension two, each referred to as E_j and D_j resp.

Let us remark that in any of the above situations I and II, we always have that $1 \leq e_j^* \leq e_j \leq 2$, where $e_j^* = \mathrm{e}(E_j, P_j)$ and $e_j = \mathrm{e}(D_j, P_j)$.

Proposition 38. *Up to elimination of finitely many steps of* $\mathcal{S}$ *in case* I* *above, situation* I *in* 3.4.1 *holds.*

Proof. The fact that we are in a resonant case and $d = 1$ implies that $e_j^* = 2$. Let F be the intersection of the two irreducible components of E_j

at P_j. Since $d = 1$, the tangent space $T_{P_j}F$ is not contained in the directrix and so the next point P_{j+1} is not in the strict transform of F. This implies that $\pi_{j+1}^{-1}(E_j \cup Y_j)$ has at most two components, one of them is $\pi_{j+1}^{-1}(Y_j)$. If π_{j+1} is dicritical, $e_{j+1}^* \leq 1$, a contradiction. Note that we have also shown that if $D_j = E_j$ then $D_{j+1} = E_{j+1}$. Hence, up to elimination of finitely many steps of $\mathcal{S}$, we can assume that $D_j \neq E_j$ for all $j \geq 0$, in order to get a contradiction. Put $D_j = E_j \cup H_j$ and let us show that H_j gives a maximal contact surface. The center Y_j has normal crossings with D_j and $T_{P_j}F$ is not contained in the directrix, so $Y_j \subset H_j$. Moreover H_{j+1} must be the strict transform of H_j by π_{j+1}, since the blowing-up is nondicritical. □

Proposition 39. *Up to elimination of finitely many steps of $\mathcal{S}$ in the case* II * *above, the situation* II *in* 3.4.1 *holds.*

Proof. Since $m = r$, we have that $\nu(\mathcal{F}_j, D_j; P_j) = r$. The statements on the adapted multiplicity and the resonance invariant are evident. Also we have that

$$1 = d \geq \mathrm{d}(\mathcal{F}_j, D_j; P_j) \geq \mathrm{t}(\mathcal{F}_j, D_j; P_j) \geq 0.$$

In view of the vertical stability of our vertical invariants (which are non-radially dicritical) and up to eliminating finitely many steps of $\mathcal{S}$ we have either $\mathrm{t}(\mathcal{F}_j, D_j; P_j) = 1$ or $\mathrm{t}(\mathcal{F}_j, D_j; P_j) = 0$ for all $j \geq 0$. Assume that $\mathrm{t}(\mathcal{F}_j, D_j; P_j) = 0$ to get a contradiction. Note in particular that $D_j \neq E_j$, since otherwise $\mathrm{t}(\mathcal{F}_j, D_j; P_j) = 1$. Up to eliminating finitely many steps of $\mathcal{S}$, we have two cases to consider:

First case: $\mathrm{d}(\mathcal{F}_j, D_j; P_j) = 1$, *for all* $j \geq 0$. Since $m = r$, $\mathrm{d}(\mathcal{F}_j, D_j; P_j) = 1$ and $D_j \neq E_j$ then $1 = e_j^* < e_j = 2$. Put $D_j = E_j \cup H$. We shall prove that H has maximal contact with $\mathcal{S}$. Take local coordinates (x, y, z) at P_j such that $E_j = \{x = 0\}$ and $H = \{z = 0\}$. Let ω be a local generator of $\mathrm{Sat}(\mathcal{F}_j, D_j)$ and put $a = \omega(x\partial/\partial x)$ and $c = \omega(z\partial/\partial z)$. We know that $\nu(a) = r$ and $\nu(c) \geq r+1$. Then the directrix is given by $A = \mathrm{In}^r(a)$. The facts that $\mathrm{d}(\mathcal{F}_j, D_j; P_j) = 1$ and $\mathrm{t}(\mathcal{F}_j, D_j; P_j) = 0$ but $\mathrm{t}(\mathcal{F}_j, E_j; P_j) = 1$ imply that

$$\frac{\partial}{\partial \bar{y}} A(\bar{x}, \bar{y}, \bar{z}) \neq 0;\ A(0, \bar{y}, 0) = 0;\ A(0, 0, \bar{z}) \neq 0.$$

Then necessarily the dimension of the directrix is less than one. In particular π_{j+1} is quadratic and nondicritical. The only way to have $D_{j+1} \neq E_{j+1}$ is to have P_{j+1} in the strict transform of H. The situation repeats and we get the maximal contact.

Second case: $\mathrm{d}(\mathcal{F}_j, D_j; P_j) = 0$, *for all* $j \geq 0$. By a direct application of our study of the vertical invariants, we get either a contact or an essential component of D_j having maximal contact with $\mathcal{S}$.

Now, let us prove that the directrix has dimension two. Assume that there is an index $j \geq 0$ such that $l_j = \dim \mathrm{Dir}(\mathcal{F}_j, D_j; P_j) \leq 1$. We first show that we can suppose $e_j = 2$. If $e_j = 1$ then $e_{j+1} \geq e_j = 1$ and $l_{j+1} \leq l_j$ in view of Proposition 6. The fact $t = d = 1$ and $l = 1$ implies that the directrix is contained in the tangent space of D_j. Since π_{j+1} is quadratic and nondicritical, we get $e_{j+1} = 2$. Thus assume $e_j = 2$. Take coordinates (x, y, z) at P_j such that $D_j = \{xy = 0\}$ and $\mathrm{Dir}(\mathcal{F}_j, D_j; P_j) = \{\bar{y} + \lambda\bar{x} = \bar{z} = 0\}$. In view of Lemma 2 and the fact that $t = d = 1$ we can write a local generator ω of $\mathrm{Sat}(\mathcal{F}_j, D_j)$ as follows

$$\omega = (z^r + (y + \lambda x)\,\Phi\,(y + \lambda x, z))\left(\alpha\frac{dx}{x} + \beta\frac{dy}{y}\right) + \tilde{a}\frac{dx}{x} + \tilde{b}\frac{dy}{y} + c\frac{dz}{z}.$$

where $\nu(\tilde{a}, \tilde{b}) \geq r + 1$, $\nu(c) \geq r$ and $(\alpha, \beta) \in \mathbb{C}^2$ without resonances. If $\lambda \neq 0$, we get $e_{j+1} = 1$ and $d_{j+1} = 2$. If $\lambda = 0$ we repeat the situation at P_{j+1} and if this repeats indefinitely, the surface $\{x = 0\}$ would have maximal contact. $\square$

In this section we have either of the situations I or II above.

3.4.2. *Normalized coordinate data.* Consider a local coordinate system (x, y, z) at the point P_j. We say that (x, y, z) is *admissible* with respect to D_j and $\mathcal{F}_j$ if and only if the following two properties hold:

(1) $\{x = 0\} \subset D_j \subset \{xy = 0\}$.

(2) $\{\bar{x} = \bar{y} = 0\} \not\subset \mathrm{Dir}(\mathcal{F}_j, D_j; P_j)$.

Given an index $j \geq 0$ we can always find an admissible coordinate system at P_j, when we have either of the situations I or II. In fact, if $e_j = 2$ it is enough to take (x, y, z) such that $D_j = \{xy = 0\}$; the second property is automatically satisfied because $d = 1$ and $l = 2$. If $e_j = 1$, take (x, y, z) such that $D_j = \{x = 0\}$ and reverse if necessary the ordering in y, z to get the second property.

Let (x, y, z) be an admissible coordinate system at P_j. We say that (x, y, z) is *maximally combinatorial* with respect to the sequence $\mathcal{S}$ if and only if either $e_j = 2$ or the following property holds:

> For any index $s \geq j$ such that $1 = e_j = e_{j+1} = \cdots = e_s$ the point P_s is in the strict transform H_s of $\{y = 0\}$ and the center Y_s has normal crossings with $D_s \cup H_s$.

In the resonant situation I, any admissible coordinate system is maximally combinatorial, since we always have that $e_j = 2$. Note also that the property of being maximally combinatorial depends on the steps $s \geq j$ in the sequence $\mathcal{S}$ and not only on the local situation at P_j.

Remark 13. The following properties are useful for the sequel:

a) Let (x, y, z) be an admissible coordinate system at P_j. Then the directrix is given by $\mathrm{Dir}(\mathcal{F}_j, D_j; P_j) = \{\bar{z} + \alpha\bar{x} + \beta\bar{y} = 0\}$.

b) Consider any coordinate change of the type $z^* = z + \sum \xi_{ik} x^i y^k$. Then (x, y, z^*) is also an admissible coordinate system. Moreover, if (x, y, z) is maximally combinatorial then (x, y, z^*) is also maximally combinatorial.

c) Assume that (x, y, z) is maximally combinatorial; then there is a coordinate change $z \mapsto z^*$ as above in such a way that $\mathrm{Dir}(\mathcal{F}_j, D_j; P_j) = \{\bar{z}^* = 0\}$ and the center Y_j of π_{j+1} is given either by $Y_j = \{P_j\}$ or $Y_j = \{x = z^* = 0\}$ or $Y_j = \{y = z^* = 0\}$. To see this, note that if $\dim Y_j = 1$ then $Y_j \subset \{xy = 0\}$ and Y_j is transversal to $\{x = y = 0\}$.

LEMMA 14. *Let (x, y, z) be an admissible coordinate system at P_j. Assume the following property*:

For any index $s \geq j$ such that $1 = e_j = e_{j+1} = \cdots = e_s$ the point P_s is in the strict transform H_s of $\{y = 0\}$ and if $Y_s \not\subset D_s$ then $Y_s \subset H_s$.

Then (x, y, z) is maximally combinatorial with respect to $\mathcal{S}$.

Proof. Let us prove that if $1 = e_j = e_{j+1} = \cdots = e_s$, then Y_s has normal crossings with $D_s \cup H_s$, by induction on $s - j$. Consider first the case $s = j$. If $Y_j = \{P_j\}$ we are done. Assume $\dim Y_j = 1$. If $Y_j \subset D_j = \{x = 0\}$, the fact that Y_j is tangent to the directrix implies transversality with $\{x = y = 0\}$, hence with $H_j = \{y = 0\}$ and we are done. If $Y_j \not\subset D_j = \{x = 0\}$ we know that $Y_j \subset \{y = 0\}$ and it is transversal to D_j; thus we also get the normal crossings property. Now assume $s > j$. First, note that $Y_j \subset D_j$; otherwise $e_{j+1} = 2$. In view of the above remarks, up to a coordinate change $z \mapsto z^*$, not affecting $\{y = 0\}$, we can assume that $\mathrm{Dir}(\mathcal{F}_j, D_j; P_j) = \{\bar{z} = 0\}$ and Y_j is given either by $Y_j = \{P_j\}$ or $Y_j = \{x = z = 0\}$. When P_{j+1} is in the strict transform of $\{y = 0\}$, a coordinate system (x', y', z') at P_{j+1} is given either by T1 or T3. Note that $\{y' = 0\}$ is the strict transform of $\{y = 0\}$. If we prove that (x', y', z') is an admissible coordinate system at P_{j+1} then we are done by induction. Consider a local generator ω of $\mathrm{Sat}(\mathcal{F}_j, D_j)$ at P_j. We know that $\mathrm{In}^r(a) = \bar{z}^r$, where $a = \omega(x\partial/\partial x)$. We get a local generator ω' of $\mathrm{Sat}(\mathcal{F}_{j+1}, D_{j+1})$ at P_{j+1} such that

$$A'(0, 0, \bar{z}') = \bar{z}'^r$$

where $A' = \mathrm{In}^r(\omega'(x'\partial/\partial x'))$. Hence the second property of an admissible coordinate system is satisfied and we are done. $\square$

Proposition 40. *Let (x,y,z) be an admissible coordinate system at P_j. Then there is a coordinate change of the type*

$$y^* = y + \sum_{i\geq 1} \eta_i x^i$$

such that (x,y^*,z) is maximally combinatorial.

Proof. If $e_j = 2$, put $y^* = y$. Assume $e_j = 1$ and let us do the coordinate change $y \mapsto y^*$ in order to get the property of the above lemma. Note first that (x,y^*,z) is always admissible. Consider two cases:

First case: *there is an index $s \geq j$ such that $1 = e_j = e_{j+1} = \cdots = e_s$ and $Y_s \not\subset D_s$.* Then necessarily $e_{s+1} = 2$. Let $\Gamma \subset X_j$ be the projection of Y_s over the ambient space X_j. Then Γ is a nonsingular curve at P_j transversal to D_j. Choose y^* such that $\Gamma \subset \{y^* = 0\}$.

Second case: *for any index $s \geq j$ such that $1 = e_j = e_{j+1} = \cdots = e_s$ then $Y_s \subset D_s$.* By standard results on blowing-up sequences, we find a (maybe formal) nonsingular curve $\Gamma \subset X_j$, not necessarily unique, transversal to D_j such that each P_s with $1 = e_j = e_{j+1} = \cdots = e_s$ is in the strict transform of Γ. Choose y^* such that $\Gamma \subset \{y^* = 0\}$.

Now, the property of the above lemma is satisfied by construction. $\square$

Remark 14. Let (x,y,z) be a maximally combinatorial coordinate system at P_j. Assume that $e_{j+1} \geq e_j$. Assume also that the directrix at P_j is given by $\{\bar{z} = 0\}$ and that either $Y_j = \{P_j\}$ or $Y_j = \{x = z = 0\}$ or $Y_j = \{y = z = 0\}$. Then we have a coordinate system (x',y',z') at P_{j+1} given by one of the equations T1, T2, T3 or T4. Moreover, the coordinate system (x',y',z') is also maximally combinatorial. To see the first part, note that P_{j+1} is necessarily in the strict transform of $\{xy = 0\}$. For the second part, we see, as in the proof of the above lemma, that the directrix at P_{j+1} is necessarily given by $\{\bar{z}' + \alpha'\bar{x}' + \beta'\bar{y}' = 0\}$ and so it does not contain $\{\bar{x}' = \bar{y}' = 0\}$.

3.4.3. *Characteristic polygons.* Let us first introduce some notation concerning the characteristic polygons of Hironaka [20]. Let us denote by $\mathbb{R}_0$ the set of nonnegative real numbers. Given any subset $\Lambda \subset \mathbb{R}_0^2$, denote by $[[\Lambda]]$ the positive convex hull of Λ, that is, the convex hull of $\Lambda + \mathbb{R}_0^2$. Consider a formal power series $f = \sum f_{isk} x^i y^s z^k$ with $f_{isk} \in \mathbb{C}$ and a positive integer $m \in \mathbb{Z}_{>0}$ such that $\nu(f) \geq m$. The characteristic polygon $\Delta^m(f;x,y,z)$ is defined by

$$\Delta^m(f;x,y,z) = \left[\left[\left\{\left(\frac{i}{m-k}, \frac{s}{m-k}\right); f_{isk} \neq 0 \text{ and } k < m\right\}\right]\right].$$

It is a polygonal region with finitely many vertices, all of them having rational coordinates with the common denominator $m!$. In order to simplify notation, put $\Delta = \Delta^m(f;x,y,z)$ if there is no confusion. Denote by $\mathrm{Vert}(\Delta)$ the set of vertices of Δ. Given a point $(\alpha, \beta) \in \mathbb{R}_0^2$ which is either a vertex of Δ or a point in $\mathbb{R}_0^2 - \Delta$, the initial part $\mathrm{In}^{(\alpha,\beta)}(f; x, y, z)$ is the following polynomial in the variables $\bar{x}, \bar{y}, \bar{z}$:

$$\mathrm{In}^{(\alpha,\beta)}(f; x, y, z) = f_{0,0,m}\bar{z}^m + \sum_{k=0}^{m-1} f_{\alpha(m-k),\beta(m-k),k}\bar{x}^{\alpha(m-k)}\bar{y}^{\beta(m-k)}\bar{z}^k.$$

For a set S of formal series such that $\nu(f) \geq m$ if $f \in S$, define $\Delta^m(S; x, y, z)$ to be the positive convex hull of the union of $\Delta^m(f; x, y, z)$ for $f \in S$.

Fix an admissible coordinate system (x, y, z) at the point P_j in the sequence $\mathcal{S}$. Write a local generator ω of $\mathrm{Sat}(\mathcal{F}_j, D_j)$ as follows:

$$\omega = f_1\frac{dx}{x} + f_2\frac{dy}{y} + f_3\frac{dz}{z}.$$

Note that z divides f_3 and $\nu(f_3) \geq r + 1$, since $z = 0$ is not a component of D_j. Moreover $\nu(f_1, f_2) = r = m$. Define the characteristic polygon $\Delta = \Delta(\mathcal{F}_j, D_j; x, y, z)$ by

$$\Delta(\mathcal{F}_j, D_j; x, y, z) = \Delta^m\left(\{f_1, f_2, f_3\}; x, y, z\right).$$

Given a vertex $(\alpha, \beta) \in \mathrm{Vert}(\Delta)$, the initial part $\mathrm{In}^{(\alpha,\beta)}(\omega; x, y, z)$ of ω is

$$\mathrm{In}^{(\alpha,\beta)}(\omega; x, y, z) = F_1\frac{d\bar{x}}{\bar{x}} + F_2\frac{d\bar{y}}{\bar{y}} + f_3\frac{d\bar{z}}{\bar{z}}$$

where $F_i = \mathrm{In}^{(\alpha,\beta)}(f_i; x, y, z)$, for $i = 1, 2, 3$. Put $W = \mathrm{In}^{(\alpha,\beta)}(\omega; x, y, z)$. Taking the lowest degree terms in $\omega \wedge d\omega$ for a convenient grading, we get that $W \wedge dW = 0$. The integrability of the initial part will play an important role in the theory.

Remark 15. The following properties for $\Delta = \Delta(\mathcal{F}_j, D_j; x, y, z)$ are directly deduced from the corresponding ones for surfaces (see [20]):

(1) $\Delta \subset \{(u, v) \in \mathbb{R}_0^2; u + v \geq 1\}$. Moreover Δ does not intersect the line $u + v = 1$ if and only if $\mathrm{Dir}(\mathcal{F}_j, D_j; P_j) = \{\bar{z} = 0\}$.

(2) The curve $x = z = 0$ is appropriate at P_j if and only if Δ is contained in the region $\{(u, v) \in \mathbb{R}_0^2; u \geq 1\}$.

(3) The curve $y = z = 0$ is appropriate at P_j if and only if Δ is contained in the region $\{(u, v) \in \mathbb{R}_0^2; v \geq 1\}$.

Let us summarize in the following lemma the effect of a coordinate change of the type $z \mapsto z_1 = z + \lambda x^p y^q$, for $\lambda \in \mathbb{C}^*$. Put $\Delta_1 = \Delta(\mathcal{F}_j, D_j; x, y, z_1)$.

LEMMA 15. *If the pair (p, q) is not a vertex of Δ. Then*

$$\Delta_1 = [[\Delta \cup \{(p, q)\}]].$$

Moreover, for any $(\alpha, \beta) \in \mathrm{Vert}(\Delta) \cap \mathrm{Vert}(\Delta_1)$*, up to identifying* $\bar{z}$ *to* $\bar{z}_1$*,*

$$\mathrm{In}^{(\alpha,\beta)}(\omega; x, y, z) = \mathrm{In}^{(\alpha,\beta)}(\omega; x, y, z_1).$$

If the pair (p, q) *is a vertex of* Δ*, then* $\Delta_1 \subset \Delta$*. Any vertex* $(\alpha, \beta) \in \mathrm{Vert}(\Delta)$ *different from* (p, q) *is also a vertex of* Δ_1 *and*

$$\mathrm{In}^{(\alpha,\beta)}(\omega; x, y, z) = \mathrm{In}^{(\alpha,\beta)}(\omega; x, y, z_1).$$

Proof. Recall that $f_3 = zc$ and write

$$\omega = \tilde{f}_1 \frac{dx}{x} + \tilde{f}_2 \frac{dy}{y} + \tilde{f}_3 \frac{dz_1}{z_1}.$$

Then we have that $\tilde{f}_1 = f_1 - \lambda p x^p y^q c$, $\tilde{f}_2 = f_2 - \lambda q x^p y^q c$, $\tilde{f}_3 = z_1 c$. Now the result follows by doing the same kinds of elementary computations as in the preparation process in [19], [20]. Let us remark that for the first case it is important that $l = 2$ and

$$(z + \lambda x + \mu y)^r \left(\xi \frac{dx}{x} + \eta \frac{dy}{y} \right), \ \xi, \eta \in \mathbb{C},$$

gives the part of degree $m = r$ in the expression of ω. □

Following Hironaka's terminology, we say that a vertex $(\alpha, \beta) \in \mathrm{Vert}(\Delta)$ is *well prepared* if and only if either $(\alpha, \beta) \notin \mathbb{Z}^2$ or $(\alpha, \beta) = (p, q) \in \mathbb{Z}^2$, but for any $\lambda \in \mathbb{C}$ we always have that $\Delta_1 = \Delta$ after the coordinate change $z \mapsto z_1 = z + \lambda x^p y^q$ as above (that is, the vertex (α, β) does not disappear under the change $z \mapsto z_1$). The coordinate system (x, y, z) is said to be a *well prepared coordinate system* if all the vertices in Δ are well prepared. Note that the fact that a vertex $(\alpha, \beta) \in \mathrm{Vert}(\Delta)$ is or not well prepared depends only on the initial form $\mathrm{In}^{(\alpha,\beta)}(\omega; x, y, z)$.

Remark 16 (Preparation algorithm). Starting from an admissible coordinate system (x, y, z) we get (x, y, z^*) well prepared with $z^* = z + \sum \lambda_{pq} x^p y^q$, by means of the following algorithm.

> If all the vertices of Δ are well prepared, stop. Otherwise take a not well prepared vertex $(p, q) \in \mathrm{Vert}(\Delta)$ with minimum $p + q$. Make a coordinate change $z \mapsto z_1 = z + \lambda_{pq} x^p y^q$ such that $\Delta_1 \neq \Delta$ (the vertex (p, q) disappears). Repeat.

The resulting coordinate system may be a formal one, but this does not affect our arguments and we will not point this out again. If the starting (x, y, z) is maximally combinatorial, then (x, y, z^*) is also maximally combinatorial. Put $\Delta^* = \Delta(\mathcal{F}_j, D_j; x, y, z^*)$. Then $\Delta^* \subset \Delta$ and the common vertices $\mathrm{Vert}(\Delta) \cap \mathrm{Vert}(\Delta^*)$ are just the well prepared vertices of Δ.

Proposition 41. *Let (x,y,z) be a maximally combinatorial well prepared coordinate system at P_j. Then the directrix is given by $\bar{z}=0$ and the center Y_j is either the point P_j or $x=z=0$ or $y=z=0$. Assume in addition that $e_{j+1}\geq e_j$. Then*

$$\Delta' = \Delta(\mathcal{F}_{j+1}, D_{j+1}; x', y', z') = [[\sigma(\Delta)]]$$

where (x',y',z') is obtained from (x,y,z) by the transformation $T-i$, for $i\in\{1,2,3,4\}$ and $\sigma=\sigma_i$, with

$$\sigma_1(u,v) = (u+v-1, v) \qquad \sigma_2(u,v) = (u, u+v-1)$$
$$\sigma_3(u,v) = (u-1, v) \qquad \sigma_4(u,v) = (u, v-1).$$

Moreover (x',y',z') is maximally combinatorial and well prepared.

Proof. Since (x,y,z) is admissible, the directrix is given by $\bar{z}+\lambda\bar{x}+\mu y=0$. If $(\lambda,\mu)=(0,0)$ we are done. If $\lambda\neq 0$ then $(1,0)\in \mathrm{Vert}(\Delta)$ and it disappears after the coordinate change $z\mapsto z+\lambda x$, analogously if $\mu\neq 0$, in contradiction to the fact that all the vertices are well prepared.

Assume that $\dim Y_j=1$. The fact that (x,y,z) is maximally combinatorial implies that $Y_j\subset\{xy=0\}$; also Y_j is transversal to $x=y=0$. In particular, either $Y_j\subset\{x=0\}$ or $Y_j\subset\{y=0\}$. If $Y_j\subset\{x=0\}$ but $Y_j\not\subset\{z=0\}$ then

$$Y_j = \{x = z + \sum_{i\geq s}\lambda_i y^i = 0\},\ \lambda_s\neq 0.$$

Put $z_1 = z+\sum_{i\geq s}\lambda_i y^i$. Then $\Delta_1\subset\{(u,v); u\geq 1\}$. This implies that the vertex $(s,0)$ of Δ is not well prepared. We use the same argument if $Y_j\subset\{y=0\}$.

Let us prove the second part of the proposition. We already know that (x',y',z') is maximally combinatorial. The fact that $\Delta'=[[\sigma(\Delta)]]$ is an easy verification. Assume that (p',q') is not a well prepared vertex of Δ' and that the coordinate change $z'\mapsto z'_1 = z'+\lambda {x'}^{p'}{y'}^{q'}$ makes this vertex to disappear. Put $(p,q)=\sigma^{-1}(p',q')$ and $z_1=z+\lambda x^p y^q$. Note that (p,q) is a vertex of Δ. We keep the same properties as (x,y,z) for (x,y,z_1). Then $\Delta'_1=[[\sigma(\Delta_1)]]$ (with evident notation) and thus the vertex (p,q) is not in Δ_1; hence it is not a well prepared vertex of Δ. □

An index $k\geq 0$ is a *break index* for $\mathcal{S}$ if and only if $e_k=2$ and $e_{k+1}=1$. In the resonant situation I, there is no break index, since $e_j=2$ for all $j\geq 0$.

Corollary. *Assume that the sequence $\mathcal{S}$ has only finitely many break indices, then there exists a maximal contact surface after finitely many steps. In particular the resonant situation* I *does not occur.*

Proof. Assume that there are no break indices $s \geq j$. Take a coordinate system (x, y, z) at P_j maximally combinatorial and well prepared. Then $H = \{z = 0\}$ is a maximal contact surface. □

3.4.4. *Break indices.* Assume that the sequence $\mathcal{S}$ has infinitely many break indices. In particular we have the nonresonant situation II.

Let (x, y, z) be an admissible coordinate system at P_j and consider a vertex (α, β) of the characteristic polygon $\Delta = \Delta(\mathcal{F}_j, D_j; x, y, z)$. Let ω be a local generator of $\mathrm{Sat}(\mathcal{F}_j, D_j)$ at P_j, put $W = \mathrm{In}^{(\alpha,\beta)}(\omega; x, y, z)$ and write

$$W = F_1(\bar{x}, \bar{y}, \bar{z})\frac{d\bar{x}}{\bar{x}} + F_2(\bar{x}, \bar{y}, \bar{z})\frac{d\bar{y}}{\bar{y}} + F_3(\bar{x}, \bar{y}, \bar{z})\frac{d\bar{z}}{\bar{z}}.$$

We say that (α, β) is a *resonant vertex* for Δ relatively to $\mathcal{F}_j$, D_j and (x, y, z) if and only if $F_3 = 0$ and there are positive integers p and q such that $pF_1 + qF_2 = \lambda \bar{z}^r$ for some $\lambda \in \mathbb{C}$.

PROPOSITION 42. *Assume that the sequence $\mathcal{S}$ is in the nonresonant situation* II. *Let (x, y, z) be an admissible coordinate system at P_j. Then any vertex (α, β) of the characteristic polygon $\Delta = \Delta(\mathcal{F}_j, D_j; x, y, z)$ is not resonant.*

Proof. Assume that (α, β) is resonant to get a contradiction. Since $\mathrm{Rs}_j = 0$ we get that if $pF_1 + qF_2 = \lambda \bar{z}^r$ then $\lambda \neq 0$. Write

$$W = F_1(\bar{x}, \bar{y}, \bar{z})\left(\frac{d\bar{x}}{\bar{x}} - \frac{p}{q}\frac{d\bar{y}}{\bar{y}}\right) + \frac{\lambda}{q}\frac{d\bar{y}}{\bar{y}}.$$

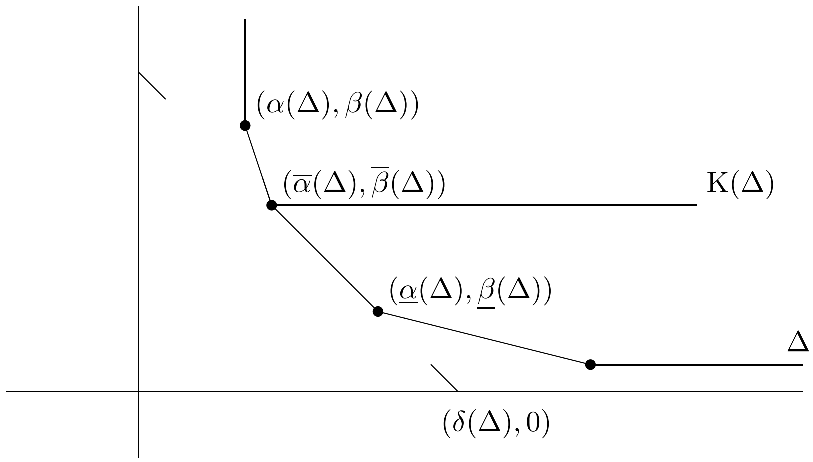
The integrability condition $W \wedge dW = 0$ implies that

$$\frac{\lambda}{q}\bar{z}^r\left(rF_1 - \bar{z}\frac{\partial F_1}{\partial \bar{z}}\right) = 0.$$

Hence $F_1 = \mu \bar{z}^r$, $F_2 = ((\lambda - \mu p)/q)\bar{z}^r$, $F_3 = 0$ in contradiction with the fact that (α, β) is a vertex of Δ. □

Now, let us introduce some notation concerning polygons. Fix a positively convex polygon $\Delta \subset \mathbb{R}^2_0$ with finitely many vertices. The *main vertex* of Δ is denoted by $(\alpha(\Delta), \beta(\Delta))$ to the vertex of lowest abscissa of Δ. The ordinate $\beta(\Delta)$ will be quite important for us. Let $\delta(\Delta)$ be the minimum number δ such that Δ cuts the line $u + v = \delta$. Denote by $(\overline{\alpha}(\Delta), \overline{\beta}(\Delta))$ and $(\underline{\alpha}(\Delta), \underline{\beta}(\Delta))$ respectively the highest and lowest vertices of Δ in the line $u + v = \delta(\Delta)$. Finally, denote by $\mathrm{K}(\Delta)$ the polygon whose vertices are exactly the vertices (α, β) of Δ such that $\beta \geq \overline{\beta}(\Delta)$ (equivalently, those such that $\alpha \leq \overline{\alpha}(\Delta)$).

Let us consider in the following lemma the effect over the polygon of a coordinate change of the type $y \mapsto y^* = y + \sum \zeta_i x^i$.



LEMMA 16. *Assume that* $e_j = 1$ *and let* (x, y, z) *be an admissible coordinate system at* P_j. *Consider any coordinate change* $y^* = y + \sum_{i\geq 1} \zeta_i x^i$. *Put*

$$\Delta = \Delta(\mathcal{F}_j, D_j; x, y, z);\ \Delta^* = \Delta(\mathcal{F}_j, D_j; x, y^*, z).$$

Then $\mathrm{K}(\Delta) = \mathrm{K}(\Delta^*)$ *and for any vertex* $(\alpha, \beta) \in \mathrm{Vert}(\mathrm{K}(\Delta))$, *up to identifying* $\bar{y}$ *with* $\bar{y}^*$,

$$\mathrm{In}^{(\alpha,\beta)}(\omega; x, y, z) = \mathrm{In}^{(\alpha,\beta)}(\omega; x, y^*, z).$$

Moreover, if the vertex $(\alpha(\Delta), \beta(\Delta))$ *of* Δ *is well prepared, then there is a maximally combinatorial well prepared coordinate system* $(x, \tilde{y}, \tilde{z})$ *at* P_j *such that*

$$(\alpha(\Delta), \beta(\Delta)) = (\alpha(\widetilde{\Delta}), \beta(\widetilde{\Delta})),$$

where $\widetilde{\Delta} = \Delta(\mathcal{F}_j, D_j; x, \tilde{y}, \tilde{z})$.

Proof. The first part is an elementary computation on characteristic polygons. For the second part, choose (x, y^*, z) maximally combinatorial. Then

$$(\alpha(\Delta^*), \beta(\Delta^*)) = (\alpha(\Delta), \beta(\Delta))$$

and it is a well prepared vertex of Δ^* (this depends only on the initial form). Now, the preparation algorithm respects the (well prepared) main vertex. $\square$

PROPOSITION 43. *Assume that* $k \geq 0$ *is a break index for the sequence* $\mathcal{S}$. *Let* (x, y, z) *be any admissible coordinate system at* P_k *such that the corresponding main vertex* $(\alpha(\Delta), \beta(\Delta))$ *of* $\Delta = \Delta(\mathcal{F}_k, D_k; x, y, z)$ *is well prepared. Then there is a maximally combinatorial well prepared coordinate system* $(\tilde{x}, \tilde{y}, \tilde{z})$ *at* P_{k+1} *such that*

$$\beta\left(\widetilde{\Delta}\right) \leq \beta\left(\Delta\right),$$

where $\widetilde{\Delta} = \Delta(\mathcal{F}_{k+1}, D_{k+1}; \tilde{x}, \tilde{y}, \tilde{z})$.

Proof. In view of the above lemma, it is enough to find an admissible coordinate system (x', y', z') at P_{k+1} such that the corresponding main vertex $(\alpha(\Delta'), \beta(\Delta'))$ is well prepared and $\beta(\Delta') \leq \beta(\Delta)$.

Applying the preparation algorithm, we shall assume without loss of generality that (x, y, z) has the following property:

Any vertex (α, β) of Δ such that $\beta \geq \overline{\beta}(\Delta)$ is well prepared.

This does not affect the main vertex. Moreover, the above property and $\mathrm{K}(\Delta)$ are stable under any coordinate change of the type

$$z \mapsto z_1 = z + \sum_{(p,q)\in\mathcal{R}\cap\mathbb{Z}_0^2} \xi_{pq} x^p y^q,$$

where $\mathcal{R} = \{(u, v) \in \mathbb{R}_0^2;\ v \leq \overline{\beta}(\Delta),\ u + v \geq \delta(\Delta)\}$. Let us also note that the directrix is given by $\bar{z} = 0$.

The fact that $e_k = 2$ and $e_{k+1} = 1$ implies that the blowing-up π_{k+1} is necessarily quadratic and thus we have coordinates (x', y', z') at P_{k+1} given by

$$x = x';\ y = x'(y' - \zeta);\ z = x'z',$$

where $\zeta \neq 0$. An elementary computation, in view of the nonresonance $\mathrm{Rs}_k \neq 0$ shows that (x', y', z') is admissible. Hence, in order to prove the theorem it is enough to prove the following statements:

a) $\beta(\Delta') \leq \overline{\beta}(\Delta) \leq \beta(\Delta)$ and $\alpha(\Delta') = \delta(\Delta) - 1$.

b) Applying a coordinate change $z \mapsto z_1$ as above, we can choose (x, y, z) in such a way that the main vertex $(\alpha(\Delta'), \beta(\Delta'))$ is well prepared.

Proof of a). Put $y^* = y + \zeta x$. Then $x = x'$, $y^* = x'y'$, $z = x'z'$. That is, we have a transformation of the type T1 if we start with the, nonadmissible, coordinate system (x, y^*, z). For a local generator ω of $\mathcal{F}_k$, then, $\mathcal{F}_{k+1}$ is locally generated by

$$\omega' = {x'}^{-r}\pi_{k+1}^*\left(\frac{y}{x}\omega\right) = (y' - \zeta){x'}^{-r}\pi_{k+1}^*(\omega).$$

Let us write

$$\omega = f_1\frac{dx}{x} + f_2\frac{dy}{y} + f_3\frac{dz}{z}$$

where $f_i = \sum_{s\geq 0} f_{i,s}(x, y)z^s$, for $i = 1, 2, 3$. Put $(y/x)\omega = (y^*/x)\omega_1 - \zeta\omega_2$, where

$$\omega_1 = f_1\frac{dx}{x} + f_2\frac{dy^*}{y^*} + f_3\frac{dz}{z};\ \omega_2 = (f_1 + f_2)\frac{dx}{x} + f_3\frac{dz}{z}.$$

Now, look at the following family of polygons:

$$\Delta_s^{(1)} = \left(\frac{-1}{r-s}, \frac{1}{r-s}\right) + \Delta^{r-s}\left(\{f_{1,s}, f_{2,s}, f_{3,s}\}; x, y, z\right),$$

$$\Delta_s^{*(1)} = \left(\frac{-1}{r-s}, \frac{1}{r-s}\right) + \Delta^{r-s}\left(\{f_{1,s}, f_{2,s}, f_{3,s}\}; x, y^*, z\right)$$

for $0 \le s \le r-1$. Also, put

$$\Delta^{(1)} = \left[\left[\bigcup_{s=0}^{r-1} \Delta_s^{(1)}\right]\right];\ \Delta^{*(1)} = \left[\left[\bigcup_{s=0}^{r-1} \Delta_s^{*(1)}\right]\right]$$

and

$$\Delta^{(2)} = \Delta^r\left(\{f_1+f_2, f_3\}; x, y, z\right);\ \Delta^{*(2)} = \Delta^r\left(\{f_1+f_2, f_3\}; x, y^*, z\right).$$

Finally, let $\Delta^* = \left[\left[\Delta^{*(1)} \cup \Delta^{*(2)}\right]\right]$. Note that $\Delta^{*(1)}$ and $\Delta^{*(2)}$ can be considered as being the respective *characteristic polygons* of $(y/x)\omega_1$ and ω_2 relatively to the coordinates (x, y^*, z). Moreover, the fact that there are no resonant vertices implies that $\Delta = \Delta^{(2)}$. Now, a tedious but simple computation shows that

$$\Delta' = [[\sigma(\Delta^*)]]$$

where $\sigma(u,v) = (u+v-1, v)$. In particular $(\alpha(\Delta'), \beta(\Delta')) = (\delta(\Delta^*)-1, \underline{\beta}(\Delta^*))$. Let us compute $\delta(\Delta^*) = \min\{\delta(\Delta^{*(1)}), \delta(\Delta^{*(2)})\}$. We have that

$$\delta\left(\Delta_s^{*(1)}\right) = \delta\left(\Delta_s^{(1)}\right) = \delta\left(\Delta^{r-s}\left(\{f_{1,s}, f_{2,s}, f_{3,s}\}; x, y, z\right)\right)$$

and $\delta\left(\Delta^{*(1)}\right) = \min_s\left\{\delta\left(\Delta_s^{*(1)}\right)\right\} = \delta\left(\Delta\right)$, since

$$\Delta = \left[\left[\bigcup_{s=0}^{r-1} \Delta^{r-s}\left(\{f_{1,s}, f_{2,s}, f_{3,s}\}; x, y, z\right)\right]\right].$$

On the other hand, since $\Delta = \Delta^{(2)}$ and $\delta(\Delta^{*(2)}) = \delta(\Delta^{(2)})$, we conclude that $\delta(\Delta^*) = \delta(\Delta)$. This proves the second part of a). Let us estimate $\underline{\beta}(\Delta^*)$. The fact $\Delta = \Delta^{(2)}$ implies that $(\overline{\alpha}(\Delta), \overline{\beta}(\Delta))$ is a vertex of $\mathrm{K}(\Delta^{*(2)}) = \mathrm{K}(\Delta^{(2)})$. In particular $(\overline{\alpha}(\Delta), \overline{\beta}(\Delta)) \in \Delta^*$ and since

$$\overline{\alpha}(\Delta) + \overline{\beta}(\Delta) = \delta(\Delta) = \delta(\Delta^*),$$

we conclude that $\overline{\beta}(\Delta) \ge \underline{\beta}(\Delta^*)$. Hence

$$\beta(\Delta') = \underline{\beta}(\Delta^*) \le \overline{\beta}(\Delta) \le \beta(\Delta).$$

This ends the proof of a).

Proof of b). Consider the region $\mathcal{R}' = \sigma(\mathcal{R})$. We know that $\mathcal{R}'$ is the set of (u,v) such that $u \ge \alpha(\Delta')$ and $v \le \overline{\beta}(\Delta)$. Up to a coordinate change of the type

$$z_1' = z' + \sum_{(p',q') \in \mathcal{R}' \cap \mathbb{Z}_0^2} \xi'_{p'q'} {x'}^{p'} {y'}^{q'}$$

that corresponds to a coordinate change of the type $z \mapsto z_1$ in the initial coordinates (x,y,z), we can assume without loss of generality that each vertex of Δ' in $\mathcal{R}'$ is well prepared. In particular, the main vertex $(\alpha(\Delta'), \beta(\Delta'))$. $\square$

Corollary. *Neither* I *nor* II *occurs for the sequence* $\mathcal{S}$.

Proof. The only possible case is the existence of infinitely many break indices. Let $j \geq 0$ be an index such that $e_j = 1$ and let k be the first break index such that $k > j$. Choose a maximally combinatorial well prepared coordinate system (x_j, y_j, z_j) at P_j. For any index s with $j + 1 \leq s \leq k$ let us denote by (x_s, y_s, z_s) the coordinate system at P_s obtained from $(x_{s-1}, y_{s-1}, z_{s-1})$ as indicated in Remark 16 by one of the transformations T1, T2, T3 or T4. Let us denote by Δ_s the corresponding characteristic polygons. We know that

$$\Delta_s = [[\sigma(\Delta_{s-1})]]$$

where σ is the affine map σ_i if the transformation is Ti, for $i = 1, 2, 3, 4$ (see Proposition 41). In our case, the center selection criterion says that if either $\{x_{s-1} = z_{s-1} = 0\}$ or $\{y_{s-1} = z_{s-1} = 0\}$ is appropriate, we take one of those centers for the blowing-up π_s. This is equivalent to saying that Δ_{s-1} is included in $\{(u, v); u \geq 1\}$ or $\{(u, v); v \geq 1\}$, respectively. Now, following the same combinatorial considerations as Hironaka in the Bowdoin College seminar, we see that necessarily $\beta(\Delta_s) \leq \beta(\Delta_{s-1})$. Moreover, if $e_{s-1} = 1$ and $e_s = 2$ we get the strict inequality, since we necessarily have T2 or T4 (and if T2, then $\alpha(\Delta_{s-1}) < 1$). In particular, we get that

$$\beta(\Delta_k) < \beta(\Delta_j).$$

Now, consider the coordinate system $(x_{k+1}, y_{k+1}, z_{k+1})$ at P_{k+1} given by the above theorem. Then

$$\beta(\Delta_j) > \beta(\Delta_k) \geq \beta(\Delta_{k+1}).$$

We repeat the argument. The value $\beta(\Delta)$ cannot decrease infinitely many times and we get the desired contradiction. □

3.5. *The jumping situation.* Let us consider $\mathcal{S}$, $\mathcal{F}$ and D as in the beginning of Section 3. We shall assume here that the sequence $\mathcal{S}$ corresponds to the m-unstable case. That is, we have that $m^* = r + 1$ and that for any $k \geq 0$ there is an index $j \geq k$ such that $m_j = r$ and $m_{j+1} = r + 1$. *A fortiori,* there is an index $j' > j$ such that $m_{j'} = r + 1$ and $m_{j'+1} = r$. That is why we say also that $\mathcal{S}$ is in *the jumping situation*. Our objective is to get a contradiction if $\mathcal{S}$ is an infinite sequence in the jumping situation. This completes the proof of the local control theorem and hence the main Theorem 1 of reduction to presimple singularities stated in Chapter 2.

We will describe a finite list of (twelve) hierarchized situations or *levels* that we denote as follows:

$$\mathcal{L}_{(0,1)}, \mathcal{L}_{(0,2)}, \mathcal{L}_{(0,3)}, \mathcal{L}_{(1,0)}, \mathcal{L}_{(2,0)}, \mathcal{L}_{(2,1)},$$
$$\mathcal{L}_{(2,2)}, \mathcal{L}_{(3,0)}, \mathcal{L}_{(3,1)}, \mathcal{L}_{(4,0)}, \mathcal{L}_{(4,1)} \text{ and } \mathcal{L}_{(5,0)}.$$

Each level at the step P_j will correspond to one of the possibilities $m_j = r$ or $m_j = r+1$. The hierarchy is given by $\mathcal{L}_{(a,b)} \leq \mathcal{L}_{(a',b')}$ if and only $(a,b) \leq (a',b')$ for the lexicographical ordering. We shall use the notation $P_j \in \mathcal{L}_{(a,b)}$ to indicate that the step j of the sequence $\mathcal{S}$ corresponds to the level $\mathcal{L}_{(a,b)}$.

In this section we shall prove the following propositions:

PROPOSITION 44. *Given an index $j \geq 0$ such that $m_j = r$, there is $a \in \{1,2,3,4,5\}$ such that $P_j \in \mathcal{L}_{(a,0)}$.*

PROPOSITION 45. *Given an index $j \geq 0$, the following statement is true*:

$\mathbf{P}_{(a,b)}$: If $P_j \in \mathcal{L}_{(a,b)}$, then $P_{j+1} \in \mathcal{L}_{(a',b')}$, with $(a,b) \geq (a',b')$.

for any pair (a,b) corresponding to one of the above twelve levels.

Before starting the precise definition of each level and the proof of these propositions, let us show how we get a contradiction to the existence of $\mathcal{S}$. Since the sequence $\mathcal{S}$ corresponds to the m-stable case, there is an index $k \geq 0$ such that $m_k = r$. By Proposition 44, there is a_k such that $P_k \in \mathcal{L}_{a_k}$. Applying Proposition 45, for each $j \geq k$, we get a pair (a_j, b_j) such that

$$(a_k, 0) \geq (a_{k+1}, b_{k+1}) \geq (a_{k+2}, b_{k+2}) \cdots .$$

Moreover, this sequence must decrease strictly infinitely many times, since each level corresponds to one of the possibilities $m_j = r$ or $m_j = r+1$ and, because of the m-unstability, the adapted multiplicity will change infinitely many times. The contradiction follows since we have only finitely many (twelve) levels.

We shall prove first the statement $\mathbf{P}_{(a,b)}$ of Proposition 45 for each pair (a,b) immediately after the precise definition of the corresponding level $\mathcal{L}_{(a,b)}$. The proof of Proposition 44 will be given at the end of the section by a direct remark on the definitions of the levels $\mathcal{L}_{(a,0)}$ for $a = 1,2,3,4,5$.

Let us establish some notational conventions in order to avoid repetitions. We denote the invariants m_j, Rs_j, e_j^*, d_j, l_j and δ_j as usual, always referred to the nondicritical divisor $E_j = \mathrm{Nd}(D_j, \mathcal{F}_j)$. Note that we put $e_j^* = \mathrm{e}(E_j, P_j)$ in order to avoid confusions with $e_j = \mathrm{e}(D_j, P_j)$. We shall always denote by ω , respectively ω', a conveniently chosen local generator of $\mathrm{Sat}(\mathcal{F}_j, E_j)$ at P_j, respectively of $\mathrm{Sat}(\mathcal{F}_{j+1}, E_{j+1})$ at P_{j+1}. Once a local coordinate system (x,y,z) at P_j is fixed we will frequently have a coordinate system (x',y',z') at P_{j+1} given by one of the sets of equations T1, T2, T3 or T4, already used in this paper. Moreover, we shall also use the equations

$$\begin{array}{rl} (\mathrm{T1},\zeta): & x = x',\ y = x'(y'-\zeta),\ z = x'z' \\ (\mathrm{T1},\zeta,\xi): & x = x',\ y = x'(y'-\zeta),\ z = x'(z'-\xi) \end{array}$$

to describe in some cases the quadratic blowing-ups.

THE LEVEL $\mathcal{L}_{(0,1)}$. *$P_j \in \mathcal{L}_{(0,1)}$ if and only if $m_j = r+1$, $e_j^* \leq 1$ and $d_j \geq 2$.*

Proof of $\mathbf{P}_{(0,1)}$. Note that $d_j \geq 2$ implies that $l_j \leq 1$. If $l_j = 0$, we are done. Thus $l_j = 1$ and $d_j = 2$. Choose a local coordinate system (x, y, z) at P_j such that $x = 0$ contains E_j and the directrix is given by $\bar{y} = \bar{z} = 0$. The blowing-up π_{j+1} is necessarily quadratic and given at P_{j+1} by the equations T1; thus $e_{j+1}^* \leq 1$. In particular $m_{j+1} = r+1$. By Proposition 5, we get that $d_{j+1} \geq d_j$. Then $P_{j+1} \in \mathcal{L}_{(0,1)}$. □

THE LEVEL $\mathcal{L}_{(0,2)}$. *We say that $P_j \in \mathcal{L}_{(0,2)}$ if and only if $m_j = r+1$, there are local coordinates (x, y, z) at P_j and a scalar $\alpha \notin \mathbb{Q}_{<0}$ such that $E_j \subset \{x = 0\}$ and*

$$\omega = z^r \left(\alpha y \frac{dx}{x} + dy \right) + x\widetilde{\omega}_1 + \widetilde{\omega}_2$$

where $\widetilde{\omega}_i \in \Omega_{X_j, P_j}[x = 0]$, for $i = 1, 2$ and $\mu(\widetilde{\omega}_2, \{x = 0\}; P_j) \geq r + 2$.

Proof of $\mathbf{P}_{(0,2)}$. Note that the adapted multiplicity $\mu(\widetilde{\omega}_1, \{x = 0\}; P_j) \geq r$, the stated conditions are not affected by any coordinate change of the type $y \mapsto y + \phi(x)$, $z \mapsto z + \psi(x)$ and there is a hyperplane $\bar{z} + \lambda\bar{x} = 0$ that contains the directrix. Let us separate the quadratic and the monoidal cases.

First case: *The blowing-up π_{j+1} is quadratic.* Since $\alpha + 1 \neq 0$, it is nondicritical and, up to a change $y \mapsto y + \zeta x$, $z \mapsto z + \xi x$, it is given at P_{j+1} either by T1 or by T2. Assume first T1. Then $e_{j+1}^* = 1$ and hence $m_{j+1} = r+1$. We can write

$$\omega' = z'^r \left((\alpha + 1)y' \frac{dx'}{x'} + dy' \right) + \left(\frac{1}{x'} \right)^r \pi_{j+1}^* \widetilde{\omega}_1 + x' \widetilde{\omega}_2'.$$

Put $\eta' = x'^{-r} \pi_{j+1}^* \widetilde{\omega}_1 \in \Omega_{X_{j+1}, P_{j+1}}[x' = 0]$. If $\mu(\widetilde{\omega}_1, \{x = 0\}; P_j) = r$, by Proposition 1 the adapted multiplicity of η' is at most equal to r, this contradicts $m_{j+1} = r+1$. Then necessarily $\mu(\widetilde{\omega}_1, \{x = 0\}; P_j) \geq r+1$ and we can write $\eta' = x'\widetilde{\omega}'$, where $\widetilde{\omega}' \in \Omega_{X_{j+1}, P_{j+1}}[x' = 0]$. We get that $P_{j+1} \in \mathcal{L}_{(0,2)}$. Assume T2 now and put $\widetilde{\omega}_1' = y'^{-r} \pi_{j+1}^* \widetilde{\omega}$. Then,

$$\omega' = z'^r \left(\alpha \frac{dx'}{x'} + (\alpha + 1) \frac{dy'}{y'} \right) + x' \widetilde{\omega}_1' + y' \widetilde{\omega}_2'.$$

We get $m_{j+1} = r$ but $\mathrm{Rs}_{j+1} = 0$, a contradiction to $m^* = r+1$.

Second case: *The blowing-up π_{j+1} is monoidal.* No appropriate center is tangent to $x = 0$. Up to a coordinate change not affecting our situation, assume that $Y_j = \{y = z = 0\}$ and the directrix is $\bar{z} = 0$. Then π_{j+1} is given at P_{j+1} by T4. We get

$$\omega' = z'^r \left(\alpha \frac{dx'}{x'} + \frac{dy'}{y'} \right) + x'\widetilde{\omega}'_1 + y'\widetilde{\omega}'_2$$

and $m_{j+1} = r$, but $\mathrm{Rs}_{j+1} = 0$, since $\alpha \notin \mathbb{Q}_{<0}$, a contradiction to $m^* = r + 1$. □

THE LEVEL $\mathcal{L}_{(0,3)}$. *$P_j \in \mathcal{L}_{(0,3)}$ if and only if $m_j = r + 1$, $e_j^* = 0$ and $\delta_j = r$.*

Proof of $\mathbf{P}_{(0,3)}$. If $d_j \geq 2$, then $P_j \in \mathcal{L}_{(0,1)}$ and we are done. Thus $d_j \leq 1$. Since $e_j^* = 0$, then $d_j = 1$ and $l_j = 2$. Choose local coordinates (x, y, z) at P_j such that the directrix is given by $\bar{z} = 0$. The property $\delta_j = r$ means that we can write

$$\omega = z^r (\lambda dx + \mu dy) + \widetilde{\omega}$$

where $\mu(\widetilde{\omega}, \emptyset; P_j) \geq r+2$ and $(\lambda, \mu) \neq (0,0)$. Up to a linear coordinate change in x, y, we get $(\lambda, \mu) = (0,1)$ and then $P_j \in \mathcal{L}_{(0,2)}$. □

Remark 17. This level $\mathcal{L}_{(0,3)}$ is superfluous, since it is covered by $\mathcal{L}_{(0,1)}$ and $\mathcal{L}_{(0,2)}$, but we keep it for simplicity.

THE LEVEL $\mathcal{L}_{(1,0)}$. *$P_j \in \mathcal{L}_{(1,0)}$ if and only if $m_j = r$, $e_j^* = 2$ and $d_j \geq 1$.*

Proof of $\mathbf{P}_{(1,0)}$. If $\mathrm{Rs}_j \neq 1$, the stability of the vertical invariants assures that $m_{j+1} = r$ and $d_{j+1} \geq d_j \geq 1$. Note that $d_{j+1} \geq 1$ implies $e_{j+1}^* \leq 2$. Then $P_{j+1} \in \mathcal{L}_{(1,0)}$. Assume now that we are in a radially dicritical situation $\mathrm{Rs}_j = 1$. Choose local coordinates (x, y, z) at P_j such that $E_j = \{xy = 0\}$, the hyperplane $\bar{z} = 0$ contains the directrix and the center Y_j of π_{j+1} is either the point P_j or the curve $\{x = z = 0\}$. Also, by symmetry on the coordinates x, y we can assume that π_{j+1} is given at P_{j+1} either by T1, (T1, ζ) or T3. Write

$$\omega = \Phi_r(x, y, z) \left[\frac{dx}{x} - \frac{dy}{y} \right] + \widetilde{\omega}$$

with $\mu(\widetilde{\omega}, \{xy = 0\}; P_j) \geq r + 1$ and $\Phi_r(x, y, z)$ homogeneous of degree r. The fact $d_j \geq 1$ means that the variable z appears effectively in the expression of Φ_r.

First case: T1. Since π_{j+1} is dicritical, $E_{j+1} = \{y' = 0\}$. Now,

$$\omega' = -\Phi_r(1, y', z') \frac{dy'}{y'} + x'\widetilde{\omega}'$$

where $\widetilde{\omega}' \in \Omega_{X_{j+1}, P_{j+1}}[x'y' = 0]$. Then $m_{j+1} = r$, $e_{j+1}^* = 1$ and hence $\mathrm{Rs}_{j+1} = 0$, which contradicts $m^* = r + 1$.

Second case: $(\mathrm{T1}, \zeta)$. Then $e^*_{j+1} = 0$ and

$$\omega' = -(y' - \zeta)^{-1}\Phi_r(1, y' - \zeta, z')dy' + x'\widetilde{\omega}',$$

where $\widetilde{\omega}' \in \Omega_{X_{j+1},P_{j+1}}[x' = 0]$. If $l_{j+1} \leq 1$, then $d_{j+1} \geq 2$ and $P_{j+1} \in \mathcal{L}_{(0,1)}$. Otherwise $l_{j+1} = 2$ and Φ_r is a power of a linear form of the type $z + \lambda x + \mu y$. Up to an initial coordinate change, assume $\Phi_r = z^r$. The directrix at P_{j+1} is given by $\bar{z}' + \mu'\bar{x}' = 0$. Then up to a coordinate change of the type $z' \mapsto z' + \mu' x'$ and to multiply ω' by a unit we get

$$\omega' = z'^r\left(\alpha' dx' + dy'\right) + \widetilde{\omega}''$$

where $\mu(\widetilde{\omega}'', \emptyset; P_{j+1}) \geq r + 2$. Then $P_{j+1} \in \mathcal{L}_{(0,3)}$.

Third case: T3. Necessarily $l_j = 2$ and $\Phi_r = z^r$, since the blowing-up is monoidal and we have selected the equations T3. We have

$$\omega' = z'^r\left(\frac{dx'}{x'} - \frac{dy'}{y'}\right) + x'\widetilde{\omega}'_1 + y'\widetilde{\omega}'_2$$

where $\widetilde{\omega}'_i \in \Omega_{X_{j+1},P_{j+1}}[x'y' = 0]$, for $i = 1, 2$. In particular, the blowing-up is nondicritical and $E_{j+1} = \{x'y' = 0\}$. From the above expression we get that $d_{j+1} \geq 1$ and thus $P_{j+1} \in \mathcal{L}_{(1,0)}$. $\square$

THE LEVEL $\mathcal{L}_{(2,0)}$. *$P_j \in \mathcal{L}_{(2,0)}$ if and only if $m_j = r$, $e^*_j \geq 2$, $d_j = 0$, $l_j \leq 1$ and moreover, in the case $e^*_j = 3$, $l_j = 1$*

$$\omega = \Phi_r(y + \zeta x, z + \xi x)\left[s(1 - \lambda)\frac{dx}{x} - \lambda\frac{dy}{y} + \frac{dz}{z}\right] + \widetilde{\omega}$$

where $E_j = \{xyz = 0\}$, $\mu(\widetilde{\omega}, E_j; P_j) \geq r + 1$, $s \in \mathbb{Z}_{\geq 0}$, $\lambda \neq 1$ and Φ_r is a homogeneous polynomial of degree r.

Proof of $\mathbf{P}_{(2,0)}$. Assume that $l_j = 1$. Then π_{j+1} is quadratic. Consider first the case $e^*_j = 3$. The directrix is then given by $\bar{y} + \zeta\bar{x} = \bar{z} + \xi\bar{x} = 0$ and the blowing-up is nondicritical, since $\lambda \neq 1$. If $\zeta = \xi = 0$, the blowing-up is given by T1 at P_{j+1} and

$$\omega' = \Phi_r(y', z')\left[(s + 1)(1 - \lambda)\frac{dx'}{x'} - \lambda\frac{dy'}{y'} + \frac{dz'}{z'}\right] + x'\widetilde{\omega}'.$$

Then $e^*_{j+1} = 3$ and $l_{j+1} \leq 1$. If $l_{j+1} = 0$, then $P_{j+1} \in \mathcal{L}_{(2,0)}$. If $l_{j+1} = 1$ the directrix at P_{j+1} is necessarily given by $\bar{y}' + \zeta'\bar{x}' = \bar{z}' + \xi'\bar{x}' = 0$ and we can write the coefficients of degree r of ω' in the variables $y' + \zeta' x'$ and $z' + \xi' x'$. Then

$$\omega' = \Phi_r(y' + \zeta' x', z' + \xi' x')\left[(s + 1)(1 - \lambda)\frac{dx'}{x'} - \lambda\frac{dy'}{y'} + \frac{dz'}{z'}\right] + \widetilde{\omega}''$$

where $\mu(\widetilde{\omega}'', \{x'y'z'=0\}; P_{j+1}) \geq r+1$. Thus $P_{j+1} \in \mathcal{L}_{(2,0)}$. If $\zeta \neq 0 = \xi$, then π_{j+1} is given at P_{j+1} by $(\mathrm{T1}, \zeta)$. We get

$$\omega' = \Phi_r(y', z') \left[(s+1)(1-\lambda) \frac{dx'}{x'} - \lambda \frac{1}{y'-\zeta} dy' + \frac{dz'}{z'} \right] + x'\widetilde{\omega}'.$$

Then $e^*_{j+1} = 2$, $m_{j+1} = r$, $d_{j+1} \geq 1$ and hence $P_{j+1} \in \mathcal{L}_{(1,0)}$. The case when $\zeta = 0 \neq \xi$ is proved in a symmetric way. If $\zeta \neq 0 \neq \xi$, similar computations to those above show that $e^*_{j+1} = 1$, $m_{j+1} = r$ and thus $\mathrm{Rs}_{j+1} = 0$, a contradiction to $m^* = r+1$.

Consider now the case $e^*_j = 2$. Choose coordinates (x, y, z) at P_j such that $E_j = \{yz = 0\}$. The directrix is given by $\bar{y} = \bar{z} = 0$, since $d_j = 0$ and

$$\omega = \Phi_r(y, z) \left[\frac{-p}{q} \frac{dy}{y} + \frac{dz}{z} \right] + \widetilde{\omega}.$$

The blowing-up is given by T1. If $p \neq q$, it is nondicritical and

$$\omega' = \Phi_r(y', z') \left[(1 - \frac{p}{q}) \frac{dx'}{x'} - \frac{p}{q} \frac{dy'}{y'} + \frac{dz'}{z'} \right] + x'\widetilde{\omega}'.$$

Reasoning as above, we get that $P_{j+1} \in \mathcal{L}_{(2,0)}$. If $p = q$, the blowing-up is dicritical, we have $E_{j+1} = \{y'z' = 0\}$ and

$$\omega' = \Phi_r(y', z') \left[-\frac{dy'}{y'} + \frac{dz'}{z'} \right] + x'\widetilde{\omega}',$$

where $\widetilde{\omega}' \in \Omega_{X_{j+1}, P_{j+1}}[x'y'z' = 0]$. Then $m_{j+1} = r$ and $l_{j+1} \leq 1$. Thus either $d_{j+1} \geq 1$ and $P_{j+1} \in \mathcal{L}_{(1,0)}$ or $d_{j+1} = 0$ and $P_{j+1} \in \mathcal{L}_{(2,0)}$. □

THE LEVEL $\mathcal{L}_{(2,1)}$. *$P_j \in \mathcal{L}_{(2,1)}$ if and only if $m_j = r+1$, there are coordinates (x, y, z) such that $E_j \subset \{xy = 0\}$ and*

$$\omega = z^{r+1} \left[p\frac{dx}{x} + q\frac{dy}{y} + \frac{dz}{z} \right] + x\omega_1 + y\omega_2$$

where $p, q \in \mathbb{Z}_{\geq 0}$ and $\omega_i \in \Omega_{X_j, P_j}[xyz = 0]$, for $i = 1, 2$.

Proof of $\mathbf{P}_{(2,1)}$. Up to a coordinate change of the type $z \mapsto z + \xi(x, y)$ and $(x, y) \mapsto (\phi(x, y), \psi(x, y))$ not affecting the stated conditions, we may assume that $\bar{z} = 0$ contains the directrix and the blowing-up is given at P_{j+1} either by T1, $(\mathrm{T1}, \zeta)$ or T3. Up to elimination of superfluous parts of $\widetilde{\omega}_i$, for $i = 1, 2$, we may assume that $\mu(\widetilde{\omega}_i, \{xyz = 0\}; P_j) \geq r$ and, in the case T3, we also have that $\mu(\widetilde{\omega}_i, \{xyz = 0\}; Y_j) \geq r$, since $Y_j = \{x = z = 0\}$ is appropriate. Let us see that ω' can be written as follows

$$\omega' = z'^{r+1} \left[p'\frac{dx'}{x'} + q'\frac{dy'}{y'} + \frac{dz'}{z'} \right] + \widetilde{\omega}'_1 + y'\widetilde{\omega}'_2$$

where $p', q' \in \mathbb{Z}_{\geq 0}$, $\widetilde{\omega}'_2 \in \Omega_{X_{j+1},P_{j+1}}[x'y'z'=0]$ and $E_{j+1} \subset \{x'y'=0\}$. This is evident if either T1 or T3 holds. If $(\mathrm{T1}, \zeta)$, write

$$\omega' = {z'}^{r+1}\left[(p+q+1)\frac{dx'}{x'} + \frac{dz'}{z'}\right] + \widetilde{\omega}'_1 + y'\left(\omega'_2 + q\frac{{z'}^{r+1}}{y'-\zeta}\frac{dy'}{y'}\right)$$

that gives the desired structure for ω'. Now, we have two possibilities for $\widetilde{\omega}'_1$:

a) $\widetilde{\omega}'_1 = x'\widetilde{\omega}''_1 + y'\widetilde{\omega}''_2$, for $\widetilde{\omega}''_i \in \Omega_{X_{j+1},P_{j+1}}[x'y'z'=0]$.

b) The condition a) does not hold. Then, since ω' comes from the blowing-up of ω, we have that

$$\widetilde{\omega}'_1 = {z'}^{r}\left[\alpha\frac{dx'}{x'} + \beta\frac{dy'}{y'} + \gamma\frac{dz'}{z'}\right] + x'\widetilde{\omega}''_1 + y'\widetilde{\omega}''_2$$

where $(\alpha,\beta,\gamma) \neq (0,0,0)$. We get $m_{j+1} = r$, $d_{j+1} \geq 1$ and thus $P_{j+1} \in \mathcal{L}_{(1,0)}$.

Let us consider the case a). If $m_{j+1} = r+1$, obviously $P_{j+1} \in \mathcal{L}_{(2,1)}$. Assume that $m_{j+1} = r$ and thus $e^*_{j+1} = 2$. If we are not in the levels $\mathcal{L}_{(1,0)}$ or $\mathcal{L}_{(2,0)}$, we necessarily have that $d_{j+1} = 0$ and $l_{j+1} = 2$. This means that ω' can be written as follows:

$$\omega' = (\lambda x' + \mu y')^r\left[\frac{dx'}{x'} - \frac{s}{t}\frac{dy'}{y'}\right] + {z'}^{r+1}\left[p'\frac{dx'}{x'} + q'\frac{dy'}{y'} + \frac{dz'}{z'}\right] + x'\omega'_1 + y'\omega'_1,$$

where $p', q', s, t \in \mathbb{Z}_{\geq 0}$, $s \neq 0 \neq t$, $(\lambda,\mu) \neq (0,0)$ and $\mu(\omega'_i, \{x'y'z'=0\}; P_{j+1}) \geq r$, for $i = 1,2$. This structure of ω' contradicts the integrability property $\omega' \wedge d\omega' = 0$. To see it, assume by symmetry that $\lambda \neq 0$ and weight the variables x, y, z by $1, 2, r/(r+1)$. The initial form

$$W = \lambda X^r\left[\frac{dX}{X} - \frac{s}{t}\frac{dY}{Y}\right] + Z^{r+1}\left[p'\frac{dX}{X} + q'\frac{dY}{Y} + \frac{dZ}{Z}\right]$$

should be integrable, but the coefficient of the monomial X^rZ^{r+1} in $W \wedge dW$ is $-\lambda(r+1)\left(q' + \frac{s}{t}\left(p' + \frac{r}{r+1}\right)\right) \neq 0$. □

THE LEVEL $\mathcal{L}_{(2,2)}$. *$P_j \in \mathcal{L}_{(2,2)}$ if and only if $m_j = r+1$, $e^*_j = 0$ and $\delta_j = r+1$.*

Proof of $\mathbf{P}_{(2,2)}$. If $l_j \leq 1$, then $d_j \geq 2$ and $P_j \in \mathcal{L}_{(0,1)}$. Assume $l_j = 2$. Take coordinates (x,y,z) such that the directrix is given by $\bar{z} = 0$. Let ω be a local generator of $\mathrm{Sat}(\mathcal{F}_j, D_j)$ at P_j. Since $\delta_j = r+1$, we have that

$$\omega = z^r u(z)dz + a(x,y,z)dx + b(x,y,z)dy + (xc_1(x,y,z) + yc_2(x,y,z))\,dz$$

where $\nu(a, b, xc_1, yc_2) \geq r+1$ and $u(0) \neq 0$. Up to dividing by $u(z)$, we can write

$$\omega = z^{r+1}\frac{dz}{z} + x\omega_1 + y\omega_2$$

for $\omega_i \in \Omega_{X_j,P_j}[xyz=0]$, $i = 1,2$. Then $P_j \in \mathcal{L}_{(2,1)}$. □

Remark 18. This level $\mathcal{L}_{(2,2)}$ is superfluous, since it is covered by $\mathcal{L}_{(0,1)}$ and $\mathcal{L}_{(2,1)}$, but we keep it for simplicity. Note that the case $e_j^*=0$ is completely covered by $\mathcal{L}_{(0,1)}$, $\mathcal{L}_{(0,3)}$ and $\mathcal{L}_{(2,2)}$.

THE LEVEL $\mathcal{L}_{(3,0)}$. *$P_j \in \mathcal{L}_{(3,0)}$ if and only if $m_j = r$, $e_j^* \geq 2$, $d_j = 0$ and there is a contact component of E_j.*

Proof of $\mathbf{P}_{(3,0)}$. To say that $P_j \in \mathcal{L}_{(3,0)}$ is equivalent to saying that $m_j = r$, $d_j = 0$ and there are coordinates (x,y,z) at P_j such that $\{z=0\} \subset E_j \subset \{xyz=0\}$ and

$$\omega = z^r\left(\alpha\frac{dx}{x} + \beta\frac{dy}{y} + \frac{dz}{z}\right) + x\omega_1 + y\omega_2 + \omega_3$$

where $\omega_i \in \Omega_{X_j,P_j}[xyz=0]$, $i=1,2,3$ and $\mu(\omega_3, \{xyz=0\}; P_j) \geq r+1$.

If $\mathrm{Rs}_j \neq 1$, by the stability of the vertical invariants, we get that $m_{j+1} = r$ and if $d_{j+1} = 0$ then there is a contact component. Then P_{j+1} is in one of the levels $\mathcal{L}_{(1,0)}$ or $\mathcal{L}_{(3,0)}$. Assume that $\mathrm{Rs}_j = 1$. Let us separate the cases corresponding to the monoidal and quadratic blowing-up.

First case: *The blowing-up π_{j+1} is monoidal.* In view of the Local Criteria of Blowing-up, we have $e_j^* = 2$, otherwise we contradict rule LR1. Also the dimension of the directrix is two. So, we may assume that $E_j = \{xz = 0\}$; hence $\alpha = -1$, $\beta = 0$, and the directrix is given by $\bar{z} + \xi\bar{x} = 0$ (recall $d_j = 0$). Up to a convenient coordinate change the center Y_j is either $y = z = 0$ or $x = z = 0$. Assume first that $Y_j = \{y = z = 0\}$. Then $\xi = 0$ and π_{j+1} is given at P_{j+1} by T4. We have

$$\omega' = {z'}^r\left(-\frac{dx'}{x'} + \frac{dy'}{y'} + \frac{dz'}{z'}\right) + x'\omega_1' + y'\omega_2'$$

with $E_{j+1} = \{x'y'z' = 0\}$. Then $P_{j+1} \in \mathcal{L}_{(3,0)}$. Assume now that $Y_j = \{x = z = 0\}$. If $\xi = 0$, the blowing-up is given by T3 and

$$\omega' = {z'}^r\frac{dz'}{z'} + x'\omega_1' + y'\omega_2'.$$

Then $m_{j+1} = r$ and either π_{j+1} is dicritical with $e_{j+1}^* = 1$ or it is nondicritical and $e_{j+1}^* = 2$. In both cases $\mathrm{Rs}_{j+1} = 0$, a contradiction to $m^* = r+1$. If $\xi \neq 0$ and π_{j+1} is dicritical, we get $e_{j+1}^* = 0$ and then P_{j+1} is in one of the levels $\mathcal{L}_{(0,1)}$, $\mathcal{L}_{(0,3)}$ or $\mathcal{L}_{(2,2)}$. If π_{j+1} is nondicritical, we have equations $x = x'$, $y = y'$, $z = x'(z' - \xi)$ and

$$\omega' = {z'}^{r+1}\left(\frac{1}{z'-\xi}\frac{dz'}{z'}\right) + x'\omega_1' + y'\omega_2',$$

with $E_{j+1} = \{x' = 0\}$. Then $P_{j+1} \in \mathcal{L}_{(2,1)}$.

Second case: *The blowing-up π_{j+1} is quadratic.* Then π_{j+1} is dicritical. If P_{j+1} is in the strict transform of $z = 0$ then $m_{j+1} = r$ and either $d_{j+1} \geq 1$ and thus $P_{j+1} \in \mathcal{L}_{(1,0)}$ or $d_{j+1} = 0$ and this strict transform is a contact component, hence $P_{j+1} \in \mathcal{L}_{(3,0)}$. Assume that P_{j+1} is not in the strict transform of $z = 0$. If $e^*_j = 2$, then $e^*_{j+1} = 0$ and we are done. Then $e^*_j = 3$ and $e^*_{j+1} = 1$. Up to a reordering in x, y, we have equations given by $x = x'$, $y = x'y'$, $z = x'(z' - \xi)$. We immediately see that $\beta = 0$; otherwise $m_{j+1} = r$ and $\mathrm{Rs}_{j+1} = 0$, since $e^*_{j+1} = 1$. More precisely, we have that $\nu(\omega(y\partial/\partial y)) \geq r + 1$. Then, since $\mathrm{Rs}_j = 1$, we can write ω as follows:

$$\omega = (z^r + x\Phi_{r-1}(x, z)) \left(-\frac{dx}{x} + \frac{dz}{z}\right) + y\omega_2 + \omega_3,$$

where Φ_{r-1} is homogeneous of degree $r - 1$. After the blowing-up,

$$\omega' = \left((z' - \xi)^r + \Phi_{r-1}(1, z' - \xi)\right) \frac{1}{z' - \xi} dz' + +y'\omega'_2 + x'\omega'_1,$$

where $\omega'_i \in \Omega_{X_{j+1},P_{j+1}}[x'y' = 0]$, for $i = 1, 2$. Necessarily we have that $(z' - \xi)^r + \Phi_{r-1}(1, z' - \xi) = {z'}^r$. Up to multiplying ω' by a unit, we have

$$\omega' = {z'}^{r+1} \frac{dz'}{z'} + x'\omega'_1 + y'\omega'_2$$

and $E_{j+1} = \{y' = 0\}$. Then $P_j \in \mathcal{L}_{(2,1)}$. □

Remark 19. The case $m_j = r$ and $e^*_j = 2$ is covered by the levels $\mathcal{L}_{(1,0)}$, $\mathcal{L}_{(2,0)}$ and $\mathcal{L}_{(3,0)}$. To see this, it is enough to note that if $e^*_j = 2$, $l_j = 2$ and $d_j = 0$, we always have a contact component in view of the resonance property.

The level $\mathcal{L}_{(3,1)}$. *$P_j \in \mathcal{L}_{(3,1)}$ if and only if $m_j = r + 1$ and $d_j \geq 1$.*

Proof of $\mathbf{P}_{(3,1)}$. If $m_{j+1} = r + 1$, in view of the stability of the vertical invariants, we have $d_{j+1} \geq 1$ and then $P_{j+1} \in \mathcal{L}_{(3,1)}$. If $m_{j+1} = r$, the fact $d_j \geq 1$ implies that $e^*_{j+1} \leq 2$ and P_{j+1} is in one of the levels $\mathcal{L}_{(1,0)}$, $\mathcal{L}_{(2,0)}$ or $\mathcal{L}_{(3,0)}$. □

The level $\mathcal{L}_{(4,0)}$. *$P_j \in \mathcal{L}_{(4,0)}$ if and only if $m_j = r$, $e^*_j = 3$ and $l_j \leq 1$.*

Proof of $\mathbf{P}_{(4,0)}$. The blowing-up is always quadratic. If $e^*_{j+1} = 3$, then $\mathrm{Rs}_j \neq 1$ (otherwise $e^*_{j+1} = 2$) and $l_{j+1} \leq 1$ by Proposition 6. Then $P_{j+1} \in \mathcal{L}_{(4,0)}$. If $e^*_{j+1} \leq 2$ and $m_{j+1} = r$, we are done according to the remark. If $m_{j+1} = r + 1$, then $\mathrm{Rs}_j = 1$, the blowing-up is dicritical and a computation shows that $d_{j+1} \geq 1$. Then $P_{j+1} \in \mathcal{L}_{(3,1)}$. □

THE LEVEL $\mathcal{L}_{(4,1)}$. *$P_j \in \mathcal{L}_{(4,1)}$ if and only if $m_j = r+1$, $e_j^* \geq 1$ and there are local coordinates (x, y, z) at P_j such that $\{z = 0\} \subset E_j \subset \{xz = 0\}$ and*

$$\omega = z^r y \left(p\frac{dx}{x} + \frac{dy}{y} \right) + x\widetilde{w}$$

where $p \in \mathbb{Z}_{\geq 0}$ and $\widetilde{w} \in \Omega_{X_j, P_j}[xz = 0]$.

Proof of $\mathbf{P}_{(4,1)}$. If $d_j \geq 1$, then $P_j \in \mathcal{L}_{(3,1)}$. Assume $d_j = 0$. There is no appropriate center transversal to $z = 0$, hence $Y_j \subset \{z = 0\}$. Also, no appropriate center is tangent to $x = 0$. Now, up to a coordinate change $y \mapsto y + \phi(x)$ we assume that if $\dim Y_j = 1$ then $Y_j = \{y = z = 0\}$. Moreover, if π_{j+1} is quadratic, it is nondicritical. Let us separate two cases: $e_j^* = 1$ and $e_j^* = 2$.

First case: $e_j^* = 1$. Here $E_j = \{z = 0\}$ and $p = 0$. The directrix is given by $\bar{z} = 0$, since $d_j = 0$. Up to a linear change $y \mapsto y + \zeta x$ in the quadratic case, the blowing-up is given at P_{j+1} by one of the sets of equations T1, T2 or T4. In particular $e_{j+1}^* \leq 2$. Then we can assume that $m_{j+1} = r+1$, otherwise we get one of the levels $\mathcal{L}_{(1,0)}$, $\mathcal{L}_{(2,0)}$ or $\mathcal{L}_{(3,0)}$. If T1, then

$$\omega' = {z'}^r y' \left(\frac{dx'}{x'} + \frac{dy'}{y'} \right) + \left(\frac{1}{x'} \right)^r \pi_{j+1}^* \widetilde{\omega},$$

where $\eta' = {x'}^{-r} \pi_{j+1}^* \widetilde{\omega} \in \Omega_{X_{j+1}, P_{j+1}}[x'z' = 0]$. We necessarily have that

$$\mu(\eta', \{x'z' = 0\}; P_{j+1}) \geq r + 1$$

and hence $\mu(\widetilde{\omega}, \{z = 0\}; P_j) \geq r + 1$ by Proposition 1. Then we can write $\eta' = x'\widetilde{\omega}'$. Hence $P_{j+1} \in \mathcal{L}_{(4,1)}$. If we have either T2 or T4, we get $m_{j+1} = r$ and we are done.

Second case: $e_j^* = 2$. Here $E_j = \{xz = 0\}$ and the directrix is contained in $\bar{z} + \xi\bar{x} = 0$. If $\xi \neq 0$, then π_{j+1} is necessarily quadratic and, up to a change $y \mapsto y + \zeta x$, it is given either by T2 or by $(\mathrm{T}1, 0, \xi)$. If T2, then $e_{j+1}^* = 3$, but $\mathrm{Rs}_{j+1} = 0$, a contradiction to $m^* = r + 1$. If $(\mathrm{T}1, 0, \xi)$, then $e_{j+1}^* = 1$ and if $m_{j+1} = r + 1$ we get $d_{j+1} \geq 1$ and $P_{j+1} \in \mathcal{L}_{(3,1)}$. If $\xi = 0$, reasoning as above, we have either T1, T2 or T4. If T1, then $e_{j+1}^* = 2$ and

$$\omega' = {z'}^r y' \left((p+1)\frac{dx'}{x'} + \frac{dy'}{y'} \right) + x'\widetilde{\omega}',$$

as in the preceding case. If T2, T4, then $e_{j+1}^* = 3$, but $\mathrm{Rs}_{j+1} = 0$, a contradiction to $m^* = r + 1$. □

THE LEVEL $\mathcal{L}_{(5,0)}$. *$P_j \in \mathcal{L}_{(5,0)}$ if and only if $m_j = r$, $e_j^* \geq 2$, $d_j = 0$, $l_j = 2$ and there are no contact components of E_j.*

Proof of $\mathbf{P}_{(5,0)}$. Necessarily $e_j = 3$. All the cases with $m_j = r$ are covered by the levels $\mathcal{L}_{(1,0)}$, $\mathcal{L}_{(2,0)}$, $\mathcal{L}_{(3,0)}$, $\mathcal{L}_{(4,0)}$ and $\mathcal{L}_{(5,0)}$. Now, assume that $m_{j+1} = r$.

This implies that $\mathrm{Rs}_j = 1$, in view of the stability of the vertical invariants. The stated conditions mean now that we can write

$$\omega = z^r \left(\frac{dx}{x} - \frac{dy}{y} \right) + \widetilde{\omega}$$

where $E_j = \{xyz = 0\}$ and $\mu(\widetilde{\omega}, \{xyz = 0\}; P_j) \geq r + 1$. In particular, we have equations given either by T1, $(\mathrm{T1}, \zeta)$, T2, T3 or T4. The only one that possibly gives $m_{j+1} = r + 1$ is $(\mathrm{T1}, \zeta)$. After the blowing-up, we get

$$\omega' = \frac{1}{y' - \zeta} \left({z'}^r y' \frac{dy'}{y'} + x' \widetilde{\omega}' \right)$$

where $\widetilde{\omega}' \in \Omega_{X_{j+1}, P_{j+1}}[x'z' = 0]$. Then $P_{j+1} \in \mathcal{L}_{(4,1)}$. □

Proof of Proposition 44. Let us consider an index $j \geq 0$ such that $m_j = r$. By definition of the levels $\mathcal{L}_{(a,0)}$, for $a = 1, 2, 3, 4, 5$, we get the following properties:

(1) If we have the conditions $e_j^* = 2$, $d_j \geq 1$, then $P_j \in \mathcal{L}_{(1,0)}$.

(2) If we have the conditions $e_j^* = 2$, $d_j = 0$, $l_j \leq 1$, then $P_j \in \mathcal{L}_{(2,0)}$.

(3) If we have the conditions $e_j^* \geq 2$, $d_j = 0$, $l_j = 2$ and there is a contact component, then $P_j \in \mathcal{L}_{(3,0)}$.

(4) If we have the conditions $e_j^* = 3$, $d_j = 0$, $l_j \leq 1$, then $P_j \in \mathcal{L}_{(4,0)}$.

(5) If we have the conditions $e_j^* \geq 2$, $d_j = 0$, $l_j = 2$ and there is no contact component, then $P_j \in \mathcal{L}_{(5,0)}$.

The above conditions cover all the possibilities for $e_j^* \geq 2$. But the case $e_j^* \leq 1$ does not occur, since the fact that $m_j = r$ and $m^* = r + 1$ implies that we have a resonant situation $\mathrm{Rs}_j \neq 0$ and we necessarily have that $e_j^* \geq 2$.

4. Getting simple singularities

This chapter is devoted to the study of the passage from presimple to simple singularities in the case of an ambient space of dimension three. The definition of simple singularity is given in [9] but we recall it here for completeness. However, the passage from presimple to simple is considerably easier if we add a nondicriticalness assumption as in [9]. The main difference is the procedure of destruction of the resonances. To be precise, in this chapter we give a proof of the following theorem.

Theorem 3. *Let $\mathcal{F}$ be a singular foliation over an ambient space X and let D be a normal crossings divisor on X. Assume that $\dim X = 3$, that X is a germ along a compact analytic subset Z and that all the singularities of $\mathcal{F}$ are presimple adapted to $E = \mathrm{Nd}(\mathcal{F}, D)$. Then there is a morphism $\pi : X' \to X$ such that*

(1) *The morphism π is the composition of a finite sequence of blowing-ups with nonsingular closed analytic centers. Moreover, each center is invariant for $\mathcal{F}$ and has normal crossings with the total transform of D.*

(2) *If $\mathcal{F}'$ is the strict transform of $\mathcal{F}$ and $E' = \mathrm{Nd}(D', \mathcal{F}')$, where D' is the total transform of D in X', then $\mathcal{F}'$ has at most simple singularities adapted to E'. Moreover D' has* normal crossings *with $\mathcal{F}'$.*

(The definition of simple singularities and the normal crossings property between a divisor and a foliation will be introduced in Section 4.1.)

This result, jointly with Theorem 1 of reduction to presimple singularities, gives the reduction of the singularities of a singular foliation $\mathcal{F}$ over a three-dimensional ambient space X that is a germ along a compact core. Two important examples of such ambient spaces are given by $(\mathbb{C}^3, 0)$ and any compact X.

4.1. *Jordanization.* The proof of the following result is essentially the same one as in [9], that was inspired by Martinet's Bourbaki seminar [23]. We recall it for completeness.

Proposition 46. *Let $\mathcal{F}$ be a singular foliation over X, with $\dim X = n$ and let D be a normal crossings divisor on X. Put $D^* = \mathrm{Nd}(D, \mathcal{F})$. Let P be a presimple singularity for $\mathcal{F}$ adapted to D^* of dimensional type $s = \tau(\mathcal{F}, P)$. Then, there are formal coordinates $(x_1, \ldots, x_n)$ at P, where $(x_{s+1}, \ldots, x_n)$ are convergent, such that D^* is formally contained in $\prod_{i=1}^{s} x_i = 0$ and $\mathcal{F}$ is given at P by a 1-form ω of one of the following types*:

A. *There are $\lambda_i \in \mathbb{C}^*$, for $i = 1, \ldots, s$, such that*

$$\omega = \sum_{i=1}^{s} \lambda_i \frac{dx_i}{x_i}.$$

B. *There is an integer k, with $1 \le k \le s$ such that*

$$\omega = \sum_{i=1}^{k} p_i \frac{dx_i}{x_i} + \psi\left(x_1^{p_1} \cdots x_k^{p_k}\right) \sum_{i=2}^{s} \alpha_i \frac{dx_i}{x_i}$$

where $p_1, \ldots, p_k \in \mathbb{Z}_{>0}$ are without common factor, the formal series $\psi(t)$ is not a unit, that is $\psi(0) = 0$, and $(\alpha_2, \ldots, \alpha_s) \in \mathbb{C}^{s-1} - \{0\}$, with $\alpha_i \neq 0$ for $i = k+1, \ldots, s$.

C. *There is an integer k, with $2 \leq k \leq s$ such that*

$$\omega = dx_1 - x_1 \sum_{i=2}^{k} p_i \frac{dx_i}{x_i} + x_2^{p_2} \dots x_k^{p_k} \sum_{i=2}^{s} \alpha_i \frac{dx_i}{x_i}$$

where $(p_2, \dots, p_k) \in \mathbb{Z}_{>0}^{k-1}$ and $(\alpha_2, \dots, \alpha_s) \in \mathbb{C}^{s-1} - \{0\}$, with $\alpha_i \neq 0$ for $i = k+1, \dots, s$.

Proof. If $s < n$ there is a nonsingular vector field ξ tangent to $\mathcal{F}$. Then, both $\mathcal{F}$ and D^* are the pullback by the projection given by the flow of ξ of a pair foliation-divisor in an ambient space of dimension $n-1$. Then we are done (in the case where $s < n$) by working inductively on the dimension n. Thus it is enough to consider the case $s = n$. Recall that D^* has either $s-1$ or s components at P. After adding in a unique way a formal hypersurface to D^* (in the case I below) as in Proposition 13, we can write ω in one of the following ways:

I. $\omega = \frac{dx_1}{x_1} + \sum_{i=2}^{n} a_i(x_1, \dots, x_n) \frac{dx_i}{x_i}$.

II. $\omega = dx_1 + \sum_{i=2}^{n} a_i(x_1, \dots, x_n) \frac{dx_i}{x_i}$, with $a_i(0) = 0$, for $i = 2, \dots, n$.

Thus, we have a family of $n-1$ commuting vector fields $\mathcal{X}_i$ given by $a_i x_1 \partial/\partial x_1 - x_i \partial/\partial x_i$ if I and by $a_i \partial/\partial x_1 - x_i \partial/\partial x_i$ if II, such that $\omega(\mathcal{X}_i) = 0$. Hence, we recover $\mathcal{F}$ from the $\mathcal{X}_i$. Consider as in [9] the simultaneous jordanization of this vector field and choose formal coordinates $(x_1, \dots, x_n)$ that linearize simultaneously the semisimple parts of the $\mathcal{X}_i$. Moreover, we get these coordinates by means of a change of the type $x_1 \mapsto \text{unit} \cdot x_1$ in the case I or $x_1 \mapsto x_1'$ in the case II. This change respects the equation of the divisor and the above form of the vector fields. Hence, assume that $(x_1, \dots, x_n)$ is already a linearizing system for the semisimple parts of the vector fields $\mathcal{X}_i$ and write

$$\mathcal{X}_i = \lambda_i x_1 \frac{\partial}{\partial x_1} - x_i \frac{\partial}{\partial x_i} + b^{(i)}(x_1, \dots, x_n) \frac{\partial}{\partial x_1}.$$

where $b^{(i)}(x_1, \dots, x_n) \partial/\partial x_1$ is the nilpotent part. Let Λ be the set of multi-indices $M = (m_1, \dots, m_n)$ such that $M \neq 0$, $m \neq (1, 0, \dots, 0)$ and $\lambda_i(m_1 - 1) = m_i$, for $i = 2, \dots, n$. The commutativity, between semisimple and nilpotent parts of the $\mathcal{X}_i$, implies that

$$b^{(i)}(x_1, \dots, x_n) \frac{\partial}{\partial x_1} = \sum_{M \in \Lambda} b_M^{(i)} x_1^{m_1} \dots x_n^{m_n}.$$

If $\Lambda = \emptyset$, we get A. If $\Lambda \neq \emptyset$, we have two possible cases:

b) There are $p_1, \dots, p_n \in \mathbb{Z}_{\geq 0}$, without common factor and with $p_1 \neq 0$ such that $p_1 \lambda_i = p_i$ for $i = 2, \dots, n$. Then Λ is the set of the n-tuples of the form $(mp_1 + 1, mp_2, \dots, mp_n)$, where $m \in \mathbb{Z}_{>0}$.

c) There is $(p_2,\dots,p_n)\in\mathbb{Z}_{\geq 0}-\{0\}$ such that $\lambda_i=-p_i$, for $i=2,\dots,n$. Then Λ is a singleton whose unique element is $(0,p_2,\dots,p_n)$.

This allows us to write ω in one of the following ways:

A) $\omega=\dfrac{dx_1}{x_1}+\sum_{i=2}^n\lambda_i\dfrac{dx_i}{x_i}$.

B) $\omega=\dfrac{dx_1}{x_1}+\sum_{i=2}^n\left(\dfrac{p_i}{p_1}+\psi_i\left(x_1^{p_1}\dots x_n^{p_n}\right)\right)\dfrac{dx_i}{x_i}$.

C) $\omega=dx_1-x_1\sum_{i=2}^n p_i\dfrac{dx_i}{x_i}+x_2^{p_2}\dots x_n^{p_n}\sum_{i=2}^n\alpha_i\dfrac{dx_i}{x_i}$.

Up to reordering $x_2,\dots,x_n$ and recalling that $\nu(\mathcal{F},\emptyset;P)=n-1$, we get the corresponding statements for A) and C). The form in B) is the pullback of

$$\eta=\frac{dt}{t}+\sum_{i=2}^n\psi_i(t)\frac{dx_i}{x_i}$$

under the maximal generic rank map $t=x_1^{p_1}\dots x_n^{p_n}$. Hence $\eta\wedge d\eta=0$ and from this we get $\alpha_i\in\mathbb{C}$ and a formal series $\psi(t)$ such that $\psi_i(t)=\alpha_i\psi(t)$ for $i=2,\dots,n$. We obtain in this way the statement for B). □

Remark 20. The above types A, B and C are intrinsic and we use them to classify the presimple singularities. Also the invariant k in B, C is intrinsic: note that type A can be viewed as a type B with invariant $k=0$. Consider the $(s-k)$-tuple $(\alpha_{k+1},\dots,\alpha_s)$ in cases B, C or $(\lambda_1,\dots,\lambda_s)$ in case A. It is invariant up to reordering and multiplication by a nonzero constant. We call it the *residual spectrum of the singularity.*

Before defining simple singularities, let us consider two types of resonances for a n-tuple $\beta=(\beta_1,\dots,\beta_n)\in{\mathbb{C}^*}^n$. We say that β is *nonresonant* if for any function $\Phi:\{1,\dots,n\}\to\mathbb{Z}_{>0}$ we have that $\sum_{i=1}^n\Phi(i)\beta_i\neq 0$. We say that β is *strongly nonresonant* if for any nonzero function $\Phi:\{1,\dots,n\}\to\mathbb{Z}_{\geq 0}$ we have that $\sum_{i=1}^n\Phi(i)\beta_i\neq 0$. Note that β is strongly nonresonant if and only if for any subset $S\subset\{1,\dots,n\}$, $\beta_S=(\beta_i)_{i\in S}$ is nonresonant.

Definition 13. Let P be a presimple singularity for $\mathcal{F}$ adapted to D^*. We say that P is a *simple singularity* for $\mathcal{F}$ adapted to D^* if and only if it is not of the type C above and the residual spectrum is strongly nonresonant. If the residual spectrum is nonresonant but not necessarily strongly nonresonant, we call it a *quasi-simple* singularity.

Let us end this section with the definition of normal crossings between a normal crossings divisor D on X and a singular foliation $\mathcal{F}$ at a point $P\in X$. Denote as usual $D^*=\mathrm{Nd}(D,\mathcal{F})$. Let E be the union of the dicritical components of D; that is, E is the closure of $D-D^*$.

Definition 14. In the above situation, D and $\mathcal{F}$ have *normal crossings* at P if and only if there are local coordinates $(x_1, \ldots, x_n)$ at P and an integer s, with $0 \leq s \leq n$ such that, locally at P, $E = \{\prod_{i=s+1}^{n} x_i = 0\}$ and $\mathcal{F}$ is given by a 1-form of the type $\sum_{i=1}^{s} a_i(x_1, \ldots, x_s) dx_i$. (*Note that* $s \geq \tau(\mathcal{F}, P)$.)

4.2. *The singular locus.* Assume that the ambient space has dimension three, is a germ along a compact core Z and the singular foliation $\mathcal{F}$ has at most presimple singularities adapted to the nondicritical part $E = \mathrm{Nd}(D, \mathcal{F})$ of the normal crossings divisor D. Let us give a description of the singular locus $\mathrm{Sing}\mathcal{F}$.

Let us denote by $\mathcal{T}_s(\mathcal{F})$ the set of points P of dimensional type $\tau(\mathcal{F}, P) = s$. The singular locus is then

$$\mathrm{Sing}\mathcal{F} = \mathcal{T}_3(\mathcal{F}) \cup \mathcal{T}_2(\mathcal{F}).$$

Moreover, since the dimensional type of P is equal to $\nu(\mathcal{F}, \emptyset; P) - 1$ for presimple singularities, we see from the formal expressions A, B and C that $\mathcal{T}_3(\mathcal{F})$ is a set of isolated points (hence a finite set by compactness). These points are not isolated in the singular locus; more precisely, in formal coordinates the singular locus $\mathrm{Sing}\mathcal{F}$ is given by

$$\begin{array}{ll} \{x_1 = x_2 = 0\} \cup \{x_1 = x_3 = 0\} \cup \{x_2 = x_3 = 0\}, & \text{if A, B or } (\mathrm{C}, k = 3). \\ \{x_1 = x_2 = 0\} \cup \{x_2 = x_3 = 0\}, & \text{if } (\mathrm{C}, k = 2). \end{array}$$

Note that each of the local irreducible components of $\mathrm{Sing}\mathcal{F}$ is contained in a different set of components of the divisor E. At the points in $\mathcal{T}_2(\mathcal{F})$ the singular locus is of dimension one, nonsingular, it is contained in all (one or two) the components of E passing through the point and it locally coincides with $\mathcal{T}_2(\mathcal{F})$. We deduce from these properties that $\mathrm{Sing}\mathcal{F}$ is a *finite union of nonsingular curves* and it has normal crossings with the divisor E.

Let Γ be an irreducible curve contained in $\mathrm{Sing}\mathcal{F}$ (in particular Γ is nonsingular). At any point $P \in \Gamma - \mathcal{T}_3(\mathcal{F})$ the curve Γ is invariant under a nonsingular vector field ξ tangent to $\mathcal{F}$. Moreover, the foliation $\mathcal{F}$ is locally determined by its restriction to an $(n-1)$-dimensional transversal section to Γ. By connectedness of $\Gamma - \mathcal{T}_3(\mathcal{F})$ and following locally the flow of ξ, the germs of the singular foliation $\mathcal{F}$ at two points $P_1, P_2 \in \Gamma - \mathcal{T}_3(\mathcal{F})$ are isomorphic. In particular the formal type A, B or C, the invariant k and the residual spectra coincide. Call these data *generic formal types* along Γ and denote them by $\mathrm{GFT}(\mathcal{F}, D; \Gamma)$.

Now, consider a point $P \in \Gamma \cap \mathcal{T}_3(\mathcal{F})$. Then Γ corresponds to a formal branch $\widehat{\Gamma}$ of the germ of the singular locus at P. Using the formal equations A, B or C and working as if they were convergent we deduce generic data along the axis $\widehat{\Gamma}$: that is, a type A, B, or C, an invariant k and a residual spectrum. Denote it by $\widehat{\mathrm{GFT}}(\mathcal{F}, D; \Gamma, P)$. The difference between $\mathrm{GFT}(\mathcal{F}, D; \Gamma)$ and $\widehat{\mathrm{GFT}}(\mathcal{F}, D; \Gamma, P)$ is that the first one is obtained by looking at any point of Γ

outside $\mathcal{T}_3(\mathcal{F})$ and the second one is obtained from the formal equations at a fixed point $P \in \Gamma$ possibly in $\mathcal{T}_3(\mathcal{F})$. A natural question is to ask if we have the equality

$$\mathrm{GFT}(\mathcal{F}, D; \Gamma) = \widehat{\mathrm{GFT}}(\mathcal{F}, D; \Gamma, P).$$

The answer is obviously positive if the formal coordinates are convergent. The property is true in all cases, but we do not prove it here. We just give the results that we need to get simple singularities.

PROPOSITION 47. *Let* Γ *be an irreducible curve contained in* $\mathrm{Sing}\mathcal{F}$ *and consider a point* $P \in \Gamma \cap \mathcal{T}_3(\mathcal{F})$. *Then*:

(1) *The type of* $\mathrm{GFT}(\mathcal{F}, D; \Gamma)$ *is* C *if and only if it is so for* $\widehat{\mathrm{GFT}}(\mathcal{F}, D; \Gamma, P)$.

(2) *The type of* $\mathrm{GFT}(\mathcal{F}, D; \Gamma)$ *is* A *with resonant residual spectrum if and only if it is so for* $\widehat{\mathrm{GFT}}(\mathcal{F}, D; \Gamma, P)$. *Moreover, in this case the residual spectra are the same.*

Proof. These statements may be proved by means of blowing-ups with center Γ, that are compatible with the use of formal coordinates. The fact that we get the type C is characterized by the following property: after finitely many blowing-ups centered in the singular locus over Γ there is exactly one step in which we get only one component of the singular locus etale over the preceding one, instead of two ones. The fact that we get the type A is characterized by the following property: after finitely many blowing-ups determined by Euclid's algorithm (corresponding to the resonance (p, q) associated to the residual spectrum $(-q, p)$) we find a dicritical component. □

Remark 21. The situation in the above proof is classical in dimension two [8]. The case C corresponds to a nondicritical presimple singularity that produces exactly one saddle-node after reduction of singularities. The case A with resonant residual spectrum corresponds to a dicritical presimple singularity.

COROLLARY. *Assume that there are no singularities of type* C *and that all the singularities of type* A *are quasi-simple singularities. Then* $\mathcal{F}$ *has at most simple singularities adapted to* E.

Proof. Let $P \in \mathrm{Sing}\mathcal{F}$. Assume first that $\tau(\mathcal{F}, P) = 2$. If P is of type B, it is simple, since the residual spectrum has at most one entry. If P is of type A, the residual spectrum has two entries and to be simple is equivalent to being quasi-simple. Assume that $\tau(\mathcal{F}, P) = 3$. If P is of the type B and $k \geq 2$, then P is simple. If $k = 1$, the residual spectrum (α_1, α_2) corresponds to a curve $x_2 = x_3 = 0$ whose generic formal type is simple of type A. In particular (α_1, α_2) is strongly nonresonant. If P is of type A but not simple, up to reordering the residual spectrum $(\lambda_1, \lambda_2, \lambda_3)$ we have a resonant (λ_2, λ_3) that contradicts the generic formal type of $x_2 = x_3 = 0$. □

4.3. *Elimination of the Jordan blocks.* In this section we will show how to eliminate the singularities of the type C, in order to start the proof of Theorem 3. Let us take the same hypotheses and notation as in the statement of the theorem.

We denote by $\mathrm{Sing}^C\mathcal{F}$ the singularities of type C, and call this set *the* C-*locus*. Looking at the formal expressions and in view of the local compatibility of the generic formal type, we deduce that $\mathrm{Sing}^C\mathcal{F}$ is a closed analytic subset of $\mathrm{Sing}\mathcal{F}$, which is the union of finitely many nonsingular curves. We shall eliminate the C-locus by blowing-up points or curves contained in $\mathrm{Sing}^C\mathcal{F}$, always having normal crossings with the divisor D. Such blowing-ups are nondicritical. We divide the procedure in two steps: first we get the property that any dicritical component H of D has empty intersection with the C-locus; second we eliminate the C-locus by blowing-up curves contained in it.

Let H be a dicritical component of D. Consider the normal crossings divisor L on H induced by E and let $\mathcal{H}$ be the restriction of $\mathcal{F}$ to H. Note that $\mathcal{H}$ is well defined because H is a dicritical component, but it is not necessarily saturated. By means of finitely many blowing-ups of points in $H \cap \mathrm{Sing}^C\mathcal{F}$, we get the following properties:

(1) The singular locus $\mathrm{Sing}\mathcal{H}$ has normal crossings with L at each point of $H \cap \mathrm{Sing}^C\mathcal{F}$. Note that the singular locus of $\mathcal{H}$ may have dimension one, because of the nonsaturation.

(2) The saturation $\mathcal{G} = \mathrm{Sat}(\mathcal{H}, \emptyset)$ has at most simple singularities adapted to $L^* = \mathrm{Nd}(L, \mathcal{G})$ at each point of $H \cap \mathrm{Sing}^C\mathcal{F}$.

(3) $\mathrm{Sing}\mathcal{H} \subset L$, locally at each point of $H \cap \mathrm{Sing}^C\mathcal{F}$.

We do not detail how to get the first two properties. Just let us note that they are stable under blowing-up of points in the C-locus. Let us show how to get the last one. Let P be a point in $H \cap \mathrm{Sing}^C\mathcal{F}$ and assume that

$$\Gamma \subset \mathrm{Sing}\mathcal{H} \subset \Gamma \cup L$$

locally at P, where Γ is a nonsingular curve transversal to L (L is nonsingular). In particular we have that $e(E, P) = 1$ and then the dimensional type of $\mathcal{F}$ at P is two. Note that Γ defines a curve in the ambient space X that is transversal to E. Also $\mathrm{Sing}\mathcal{F}$ is nonsingular at P, the two-dimensional generic type along the singular locus is of type C and it has an invariant $p = p_2$, according to the formal expression of the singularity (note that $k = 2$). Blow-up the point P and consider the infinitely near point P' corresponding to the strict transform of Γ. If we get a point of type C, it is of dimensional type two and the invariant p drops exactly one unit. This cannot be repeated infinitely many times. In this way we get that $\mathrm{Sing}\mathcal{H} \subset L$, as stated in the third property.

Assume now that the three above properties hold. Consider an irreducible curve Γ contained in $\mathrm{Sing}\mathcal{H}$ and such that there is a point $P \in \Gamma \cap \mathrm{Sing}^C\mathcal{F}$. Then necessarily $\Gamma \subset \mathrm{Sing}^C\mathcal{F}$. The facts that H has normal crossings with E and $\Gamma \subset E$ imply that $\Gamma \subset \mathrm{Sing}\mathcal{F}$, since the restriction of $\mathcal{F}$ to H at the points of E nonsingular for $\mathcal{F}$ is also nonsingular by transversality. Moreover, such a Γ is the intersection of H and one component of E, which implies that $\Gamma \subset \mathrm{Sing}^C\mathcal{F}$, since Γ cannot be the intersection of two components of E. Blow-up Γ. Nothing moves in H except that we divide one time $\mathcal{H}$ by the reduced equation of Γ. Repeat. After finitely many times Γ disappears from the singular locus of $\mathcal{H}$. In this way we get that after finitely many operations $\mathcal{H}$ has only isolated singularities locally at the points of the C-locus. In particular $\mathcal{H}$ is saturated and hence $\mathcal{G} = \mathcal{H}$. Since the singularities are simple, we know (see [24]) that the dimensional type of $\mathcal{F}$ at that points is two and the generic type is given by the section over H. This is a contradiction, since we are in the C-locus and thus our singularity is not simple. This means that there are no points of the C-locus in H. Repeat with the other dicritical components and we get that the dicritical components do not touch the C-locus.

Now, consider a curve Γ contained in the C-locus and let $p = p_2$ be the invariant corresponding to the generic type of Γ. First, blow-up finitely many points not in the C-locus to get that Γ and D have normal crossings; second, blow-up Γ. We create at most one new curve in the C-locus and the corresponding invariant is $p-1$. In this way we make disappear all the curves in the C-locus. This implies that the C-locus has disappeared.

4.4. *Killing the resonances.* We continue here the proof of Theorem 3. Let us take the same hypotheses and notation as in the statement of the theorem and assume, in addition, that there are no singularities of the type C. This last property is stable under blowing-ups with centers that are singular points or curves in the singular locus having normal crossings with the divisor D. In this section we will eliminate the nonsimple singularities by performing blowing-ups of that kind. This consists mainly in destroying the resonances of the residual spectrum. As in the preceding paragraph, we shall divide our procedure in two steps: first we get that the divisor D has normal crossings with the foliation $\mathcal{F}$ at all singular points which are not simple points, second we eliminate the nonsimple points following a certain criterion of global blowing-up.

Normal crossings with the foliation. The property of normal crossings between $\mathcal{F}$ and D is stable in our case under any blowing-up with center a singular point or a curve contained in the singular locus. Then, new dicritical components may appear, but they are always in *good* position, since the dimensional type drops in that case. At the initial step, we shall write the divisor D as a union of the nondicritical part D^* and the *initial* dicritical components $H_1, \ldots, H_s$. We shall do blowing-ups in order to get that the strict transforms of H_i finally do not intersect the nonsimple points. This is enough, since the new dicritical components we introduce already have the desired normal crossings property. Put $H = H_1$. Under the blowing-ups we are going to do, replace H_i by its strict transform and the divisor D, as usual, by the total transform. Also denote by $\widetilde{D}$ the total transform of the initial nondicritical D^*. Note that $\widetilde{D}$ contains the *new* dicritical components of D and hence $\widetilde{D}$ has normal crossings with the foliation. Denote by $\mathcal{H}$ the restriction of $\mathcal{F}$ to H and let $\mathcal{G}$ be the saturation of $\mathcal{H}$. Let L be the divisor induced in H by $\widetilde{D}$. Now, by means of finitely many blowing-ups of nonsimple singular points in $H \cap \mathrm{Sing}\mathcal{H}$ we get the following properties:

(1) The singular locus $\mathrm{Sing}\mathcal{H}$ has normal crossings with L at each nonsimple point of $H \cap \mathrm{Sing}\mathcal{F}$.

(2) The foliation $\mathcal{G}$ has at most simple singularities adapted to $L^* = \mathrm{Nd}(L, \mathcal{G})$ at each nonsimple point of $H \cap \mathrm{Sing}\mathcal{F}$.

(3) $\mathrm{Sing}\mathcal{H} \subset L$, locally at each nonsimple point of $H \cap \mathrm{Sing}\mathcal{F}$.

This statement is parallel to the corresponding one in Section 4.3 and the proof is similar. Anyway, let us detail the third property. Note first that following the curve Γ we obtain either a simple singularity or a nonsingular point, by reasoning as in the two-dimensional case. Now, we can eliminate the curves contained in $\mathrm{Sing}\mathcal{H}$ and then $\mathcal{H}$ is saturated locally at each nonsimple singular point in $H \cap \mathrm{Sing}\mathcal{F}$. Hence $\mathcal{H} = \mathcal{G}$ and the point is a simple point for $\mathcal{H}$. This means that the singularity is also a simple singularity for $\mathcal{F}$ in the three-dimensional ambient space. Hence H is out of the nonsimple singular points of $\mathcal{F}$.

We repeat this procedure with the other components $H_2, \ldots, H_s$ and we get the desired property of normal crossings between $\mathcal{F}$ and D at the nonsimple singular points of the foliation.

Elimination of the resonances. Assume now that the divisor D has normal crossings with the foliation $\mathcal{F}$ at all singular points that are nonsimple singularities. This property is stable under any blowing-up centered either at a singular point or at a curve contained in the singular locus of $\mathcal{F}$. Note that simple singularities are stable under these blowing-ups. Now let us do a sequence of blowing-ups respecting the following global criteria:

I. Assume that there is a curve Γ contained in $\mathrm{Sing}\mathcal{F}$ of generic type A with resonant residual spectrum. Then blow-up such a curve Γ.

II. Assume that there is no curve Γ of generic type A with resonant residual spectrum and that there is a point $P \in \mathrm{Sing}\mathcal{F}$ of type A which is nonquasisimple. Then blow-up such a point P.

Note first that in case I, the curve Γ has necessarily normal crossings with the divisor D, since all the points in Γ, of dimensional type either two or three, are necessarily nonsimple singular points for $\mathcal{F}$. Also let us remark that either I or II holds if not all the singularities are simple ones. To see this, note that if there is no curve Γ of generic type A with resonant residual spectrum, then all the nonsimple singularities are necessarily of dimensional type three, of type A and not quasi-simple.

We have to prove that the sequence of global blowing-ups that we obtain in this way necessarily stops. That is, we get that all the singularities are simple ones. To do this we consider two cases: a) All the singularities of type A of dimensional type three are quasi simple. b) The general case.

In case a) we necessarily blow-up a curve Γ and looking at the formal expressions, we deduce that the property a) persists. Moreover, there is at most a single curve Γ' that projects inside Γ and is resonant of generic type A. If $(-q,p)$, with $p<q$, is the residual spectrum corresponding to the generic type along Γ, then $(p-q,p)$ is the corresponding residual spectrum along Γ'. The sum $p+q$ strictly decreases and Γ disappears after finitely many steps. In this way we make disappear all the nonsimple singularities.

Assume that we are in case b) and that we do not get case a) at any step of our infinite sequence. By general arguments of local global control, already used in our main strategy, we get an infinite sequence of points $P_1, P_2, \dots$ such that:

(1) Each point is obtained from the preceding one by means of a local blowing-up that follows the rule: blow-up a curve if there is one with resonant generic residual spectrum, otherwise, blow-up the point.

(2) Each point is of dimensional type three, type A and is non-quasi-simple.

Now let us show that this infinite sequence of points P_i leads to a contradiction. Denote by

$$\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \lambda_3^{(i)})$$

the residual spectrum at the point P_i. We know that each $\lambda^{(i)}$ is resonant; that is, there is a resonance $M^{(i)} = (M_1^{(i)}, M_2^{(i)}, M_3^{(i)})$ in $\mathbb{Z}^3_{>0}$ such that $\sum_{j=1}^3 M_j^{(i)} \lambda_j^{(i)} = 0$. Consider the $\mathbb{Q}$-vector spaces V_i and W_i defined by

$$V_i = \{r \in \mathbb{Q}^3;\ \sum_{j=1}^{3} r_j \lambda_j^{(i)} = 0\}$$
$$W_i = \{t \in \mathbb{Q}^3;\ \sum_{j=1}^{3} r_j t_j = 0,\ \text{ for all } r \in V_i\}.$$

Putting $f_i = \dim V_i$, note that $1 \le f_i \le 2$ and $\dim W_i = 3 - f_i$. Moreover, in view of the evolution of the residual locus, we get that this dimension is constant, that is, $f = f_i$ for all i. We consider two cases: i) $f = 1$. ii) $f = 2$.

When $f = 1$, then $\{M^{(i)}\}$ is a basis for V_i. This implies that each passage $P_i \mapsto P_{i+1}$ is given by a quadratic blowing-up. In fact, two entries of $\lambda^{(i)}$ cannot be resonant, since this would give an element of V_i independent of $M^{(i)}$. In particular, there is an index s such that

$$\lambda_s^{(i+1)} = \lambda_1^{(i)} + \lambda_2^{(i)} + \lambda_3^{(i)}$$

and $\lambda_j^{(i+1)} = \lambda_j^{(i)}$ for $j \neq s$. Now, we can take $M^{(i+1)}$ such that $M_j^{(i+1)} = M_j^{(i)} - M_s^{(i)}$ for $j \neq s$ and $M_s^{(i+1)} = M_s^{(i)}$. The sum of the entries of $M^{(i)}$ decreases strictly. This cannot be repeated infinitely many times.

Assume now that $f = 2$. Then $\dim W_i = 1$ and, after multiplying $\lambda^{(i)}$ by a nonzero constant and reordering the entries, we can assume that

$$\lambda^{(i)} = (p, -q, -n)$$

where $p, q, n \in \mathbb{Z}_{>0}$. The local rule says that we have to blow-up either $x_1 = x_2 = 0$ or $x_1 = x_3 = 0$. Assume, by symmetry, that the center is $x_1 = x_2 = 0$. Then $\lambda^{(i+1)}$ is given either by $(p-q, -q, -n)$ or by $(p, p-q, -n)$. In any case the sum $p + q + n$ strictly drops. The situation cannot be repeated infinitely many times.

4.5. *The final normal crossings.* Let us end here the proof of Theorem 3 and thus the proof of the reduction of the singularities of a singular foliation $\mathcal{F}$ over a three-dimensional ambient space X that is a germ along a compact core.

We shall assume here that all the singularities of $\mathcal{F}$ are simple, adapted to $D^* = \mathrm{Nd}(D, \mathcal{F})$. We shall use blowing-ups with centers (single points or nonsingular curves) having normal crossings with D and invariant for $\mathcal{F}$, so that $\mathcal{F}$ and D have normal crossings after finitely many steps. Such centers give nondicritical blowing-ups and the property that all the singularities are simple ones is preserved. Here is the only part of our procedure where we use invariant centers not necessarily contained in the singular locus $\mathrm{Sing}\mathcal{F}$.

Let us write $D = D^* \cup H$, where $H = H_1 \cup \cdots \cup H_s$ is the union of the dicritical components.

First reduction: *we can assume that $e(H, P) \le 2$ at any point P of the ambient space X.* To get this property it is enough to blow-up the finitely many points P such that $e(H, P) = 3$. Moreover the property is stable under the kind of blowing-ups to be done below.

Second reduction: *we can assume that the foliation $\mathcal{F}$ and the divisor D have normal crossings at each point P such that $e(H, P) = 2$.* This property is also stable under the kind of blowing-ups to be done below. Let us prove that we can get it. First, blow-up any invariant curve of the form $H_i \cap H_j$. We get that the curves of the type $H_i \cap H_j$ are not invariant curves for $\mathcal{F}$ and this is stable under our blowing-ups. Now, there are finitely many points P such that $e(H, P) = 2$ and $\mathcal{F}$ and D do not have normal crossings at P. Blow them up repeatedly. If they do not disappear, by arguments already used in our general strategy, we get an infinite sequence of infinitely near points $P_1, P_2, \ldots$ corresponding to one of the curves $\Gamma = H_i \cap H_j$. Since Γ is noninvariant, P_k is nonsingular to $\mathcal{F}$ for large k. Then the foliation is locally given at P_k by the levels of a nonsingular first integral. We get the normal crossings property by using the contact between Γ and the corresponding nonsingular surface, which decreases strictly unless it is one.

Now, assume that the above two reductions hold. For any index $j = 1, 2 \ldots, s$, let Σ_j be the set of points $P \in H_j$ such that $\mathcal{F}$ and D do not have the normal crossings property at P. Obviously, we end our proof if we get that $\Sigma_j = \emptyset$ for all $j = 1, \ldots, s$. Note that $e(H, P) = 1$ for any $P \in \Sigma_j$.

Fix an index j such that $\Sigma_i = \emptyset$ for any $i < j$. This is stable under our blowing-ups. Let us get that $\Sigma_j = \emptyset$ after finitely many blowing-ups. We will end the proof in this way. Let $\mathcal{H}$ be the restriction of $\mathcal{F}$ to H_j, put $\mathcal{G} = \mathrm{Sat}(\mathcal{H}, \emptyset)$ and $L = H_j \cap D^*$. After finitely many blowing-ups of points,

(1) The singular locus $\mathrm{Sing}\mathcal{H}$ has normal crossings with L. Moreover, each irreducible component of $\mathrm{Sing}\mathcal{H}$ is nonsingular and has normal crossings with D in the ambient space X.

(2) The foliation $\mathcal{G}$ has at most simple singularities adapted to $L^* = \mathrm{Nd}(L, \mathcal{G})$.

Note that any irreducible component Γ of $\mathrm{Sing}\mathcal{H}$ is invariant for $\mathcal{F}$. By blowing-up these components one by one and doing intermediate quadratic blowing-ups if necessary, we get the additional property that

$$\mathrm{Sing}\mathcal{H} \subset L.$$

In particular each irreducible curve Γ contained in $\mathrm{Sing}\mathcal{H}$ is the intersection of two components of D. Moreover, the fact that $\Gamma \subset D^*$ implies that $\Gamma \subset \mathrm{Sing}\mathcal{F}$. Blow-up such Γ. Nothing occurs for $\mathcal{G}$, but $\mathcal{H}$ is divided by one times the local equation of Γ. In this way we get that $\mathcal{H}$ has only isolated singularities, that is $\mathcal{H} = \mathcal{G}$. Since these singularities are simple ones, we know (see [24]) that the dimensional type of $\mathcal{F}$ is at most two at each point in H_j and H_j is everywhere transversal to a local trivializing vector field for $\mathcal{F}$. This is enough to assure the normal crossings property between $\mathcal{F}$ and D at the points P such that $e(H, P) = 1$. Then $\Sigma_j = \emptyset$ and the proof is complete.

Appendix: About simple singularities

Let us recall here in a quick way the characterization in Section 4.1 of simple singularities for a singular foliation $\mathcal{F}$ over an n-dimensional ambient space X adapted to a normal crossings divisor D. Fix a point $P \in \mathrm{Sing}\mathcal{F}$ and let E be the divisor of nondicritical components of D. To say that P is simple for the pair $(\mathcal{F}, D)$ is equivalent to saying that it is simple for $(\mathcal{F}, E)$. Thus we assume that D is a normal crossings divisor without dicritical components: that is, each irreducible component of D is invariant by $\mathcal{F}$.

Let ω be an integrable differential 1-form such that $\mathcal{F}$ is given by $\omega = 0$ locally at P. Consider also local coordinates $(x_1, \ldots, x_n)$ at the point P such that $D = (\prod_{i=1}^{e} x_i = 0)$. We can write ω as follows:

$$\omega = \sum_{i=1}^{n} a_i dx_i = \left(\prod_{i=1}^{e} x_i\right)\left(\sum_{i=1}^{e} b_i \frac{dx_i}{x_i} + \sum_{i=e+1}^{n} b_i dx_i\right),$$

where $(a_i; i = 1, \ldots, n)$ and $(b_i; i = 1, \ldots, n)$ are both vectors of germs of holomorphic functions without a common factor.

Definition 15. We say that P is a *simple singularity* for $(\mathcal{F}, D)$ if and only if there are formal coordinates $(\hat{x}_1, \ldots, \hat{x}_n)$ at P, a formal unit $u(\hat{x}_1, \ldots, \hat{x}_n)$ and an index s, $e-1 \le s \le e$, such that one of the following properties holds for the formal differential 1-form $\widehat{\omega} = u\omega$:

A. There are $\lambda_i \in \mathbb{C}^*$, for $i = 1, \ldots, s$, such that
$$\widehat{\omega} = \left(\prod_{i=1}^{s} \hat{x}_i\right)\left(\sum_{i=1}^{s} \lambda_i \frac{d\hat{x}_i}{\hat{x}_i}\right)$$
and for any nonzero map $\phi : \{1, \ldots, s\} \to \mathbb{Z}_{\ge 0}$ we have that $\sum_{i=1}^{s} \phi(i)\lambda_i \ne 0$.

B. There is an integer k, with $1 \le k \le s$ such that
$$\widehat{\omega} = \left(\prod_{i=1}^{s} \hat{x}_i\right)\left(\sum_{i=1}^{k} p_i \frac{d\hat{x}_i}{\hat{x}_i} + \psi\left(\hat{x}_1^{p_1} \cdots \hat{x}_k^{p_k}\right) \sum_{i=2}^{s} \alpha_i \frac{d\hat{x}_i}{\hat{x}_i}\right)$$
where $p_1, \ldots, p_k \in \mathbb{Z}_{>0}$ are without common factor, the formal series $\psi(t)$ is not a unit, $\alpha_i \in \mathbb{C}$ and for any nonzero map $\phi : \{k+1, \ldots, s\} \to \mathbb{Z}_{\ge 0}$ we have that $\sum_{i=k+1}^{s} \phi(i)\alpha_i \ne 0$.

This definition is equivalent to Definition 13 of Section 4.1. In view of the above formal normal forms, it is possible to prove that the irreducible components of $\prod_{i=1}^{s} \hat{x}_i = 0$ are the only formal invariant hypersurfaces for $\mathcal{F}$ (see [9]). In particular we get that

$$D \subset \left(\prod_{i=1}^{s} \hat{x}_i = 0\right).$$

The equality holds when $e = s$ (*corner singularities*). The *trace singularities* are those corresponding to $e = s - 1$.

Note that in the case of a foliation $df = 0$ given by the levels of a function f, the fact that we have a simple singularity is equivalent to saying that f is locally a monomial $f = \prod_{i=1}^{s} x_i^{\lambda_i}$, where $\lambda_i \in \mathbb{Z}_{>0}$ (hence the nonresonance property is obvious). More generally, if the foliation is given by the levels of a multivalued function we also get a monomial as above, with the nonresonance condition. Type B in the definition corresponds to more general foliations like saddle-nodes and ramifications of saddle-nodes.

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