Annals of Mathematics, **160** (2004), 839–906

Positive extensions, Fejér-Riesz factorization and autoregressive filters in two variables

By Jeffrey S. Geronimo and Hugo J. Woerdeman*

Abstract

In this paper we treat the two-variable positive extension problem for trigonometric polynomials where the extension is required to be the reciprocal of the absolute value squared of a stable polynomial. This problem may also be interpreted as an autoregressive filter design problem for bivariate stochastic processes. We show that the existence of a solution is equivalent to solving a finite positive definite matrix completion problem where the completion is required to satisfy an additional low rank condition. As a corollary of the main result a necessary and sufficient condition for the existence of a spectral Fejér-Riesz factorization of a strictly positive two-variable trigonometric polynomial is given in terms of the Fourier coefficients of its reciprocal.

Tools in the proofs include a specific two-variable Kronecker theorem based on certain elements from algebraic geometry, as well as a two-variable Christoffel-Darboux like formula. The key ingredient is a matrix valued polynomial that appears in a parametrized version of the Schur-Cohn test for stability. The results also have consequences in the theory of two-variable orthogonal polynomials where a spectral matching result is obtained, as well as in the study of inverse formulas for doubly-indexed Toeplitz matrices. Finally, numerical results are presented for both the autoregressive filter problem and the factorization problem.

Contents

*The research of both authors was partially supported by NSF grants DMS-9970613 (JSG) and DMS-9500924 and DMS-9800704 (HJW). In addition, JSG was supported by a Fulbright fellowship and HJW was supported by a Faculty Research Assignment grant from the College of William and Mary.

1. Introduction

The trigonometric moment problem, orthogonal polynomials on the unit circle, predictor polynomials, stable factorizations, etc., have led to a rich and exciting area of mathematics. These problems were considered early in 20th century in the works of Carathéodory, Fejér, Kolomogorov, Riesz, Schur, Szegö, and Toeplitz, and wonderful accounts of this theory may be found in classical books, such as [44], [35], [2], and [1]. The theory is not only rich in its mathematics but also in its applications, most notably in signal processing [36], systems theory [31], [30], prediction theory [23, Ch. XII], and wavelets [16, Ch. 6]. More recently, these problems have been studied in the context of unifying frameworks from which the classical results appear as special cases. We mention here the commutant lifting approach [31], the reproducing kernel Hilbert space approach [25], the Schur parameter approach [15], and the band method approach [28], [40], [66].

About halfway through the 20th century, multivariable variations started to appear. Several questions lead to extensive multivariable generalizations (e.g, [47], [48], [18], [19], [21]), while others lead to counterexamples ([10], [58], [33], [22], [54], [53]). In this paper we solve some of the two-variable problems that heretofore remained unresolved. In particular, we solve the positive extension problem that appears in the design of causal bivariate autoregressive filters. As a result we also solve the spectral matching problem for orthogonal polynomials and the spectral Fejér-Riesz factorization problem for strictly positive trigonometric polynomials of two variables. In the next section we will present these three main results. It may be helpful to first read Section 1.3 in which some terminology and some notational conventions are introduced.

1.1. *The main results.*

1.1.1. *The positive extension problem.* A polynomial $p(z)$ is called *stable* if $p(z) \neq 0$ for $z \in \overline{\mathbb{D}} := \{z \in \mathbb{C} \ : \ |z| \leq 1\}$. For such a polynomial define its *spectral density function* by $f(z) = \frac{1}{p(z)\overline{p(1/\overline{z})}}$. Recall the following classical extension problem: given are complex numbers c_i, $i = 0, \pm 1, \pm 2, \dots, \pm n$, find a stable polynomial of degree n so that its spectral density function f has Fourier coefficients $\widehat{f}(k) = c_k$, $k = -n, \dots, n$. The solution of this problem goes back to the works of Carathéodory, Toeplitz and Szegö, and is as follows: *A solution exists if and only if the Toeplitz matrix $C := (c_{i-j})_{i,j=0}^n$ is positive definite* (*notation*: $C > 0$). *In that case, the stable polynomial $p(z) = p_0 + \cdots + p_n z^n$* (*which is unique when we require $p_0 > 0$*) *may be found via the Yule-Walker equation*

$$\begin{pmatrix} c_0 & \bar{c}_1 & \cdots & \bar{c}_n \\ c_1 & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bar{c}_1 \\ c_n & \cdots & c_1 & c_0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} \frac{1}{p_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This result was later generalized to the matrix-valued case in [17] and [26] and in the operator-valued case in [41]. The spectral density function f of p has in fact a so-called *maximum entropy* property (see [9]), which states that among all positive functions on the unit circle with the prescribed Fourier coefficients c_k, $k = -n, \dots, n$, this particular solution maximizes the entropy integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log(f(e^{i\theta})) d\theta.$$

The elegant proofs of these results in [26] have lead to the band method, which is a general framework for solving positive and contractive extension problems. It was initiated in [28], and pursued in [40], [66], [56], and other papers (see also [37, Ch. XXXV] and references therein).

In this paper we generalize the above result to the two-variable case. Unlike the one-variable case, it does not suffice to write down a single matrix and check whether it is positive definite. In fact, one needs to solve a positive definite completion problem where the to-be-completed matrix is also required to have a certain low rank submatrix. The precise statement is the following.

THEOREM 1.1.1. *Complex numbers $c_{k,l}$, $(k,l) \in \{0, \dots, n\} \times \{0, \dots, m\}$, are given. There exists a stable* (*no roots in $\overline{\mathbb{D}}^2$*) *polynomial*

$$p(z,w) = \sum_{k=0}^{n} \sum_{l=0}^{m} p_{kl} z^k w^l$$

with $p_{00} > 0$ so that its spectral density function

$$f(z,w) := (p(z,w)\overline{p(1/\overline{z}, 1/\overline{w})})^{-1}$$

has Fourier coefficients $\widehat{f}(k,l) = c_{kl}, (k,l) \in \{0,\dots,n\} \times \{0,\dots,m\}$, *if and only if there exist complex numbers* $c_{k,l}$, $(k,l) \in \{1,\dots,n\} \times \{-m,\dots,-1\}$, *so that the* $(n+1)(m+1) \times (n+1)(m+1)$ *doubly indexed Toeplitz matrix*

$$\Gamma = \begin{bmatrix} C_0 & \cdots & C_{-n} \\ \vdots & \ddots & \vdots \\ C_n & \cdots & C_0 \end{bmatrix},$$

where

$$C_j = \begin{bmatrix} c_{j0} & \cdots & c_{j,-m} \\ \vdots & \ddots & \vdots \\ c_{jm} & \cdots & c_{j0} \end{bmatrix}, \qquad j = -n,\dots,n,$$

and $c_{-k,-l} = \bar{c}_{k,l}$, *has the following two properties*:

(1) Γ *is positive definite*;

(2) *The* $(n+1)m \times (m+1)n$ *submatrix of* Γ *obtained by removing scalar rows* $1 + j(m+1)$, $j = 0,\dots,n$, *and scalar columns* $1, 2, \dots, m+1$, *has rank* nm.

In this case one finds the column vector

$$[p_{00}^2 \ p_{00}p_{01} \ \cdots \ p_{00}p_{0m} \ p_{00}p_{10} \cdots \ p_{00}p_{1m} \ p_{00}p_{20} \cdots \ \cdots \ p_{00}p_{nm}]^T$$

as the first column of the inverse of Γ. *Here* ^{T} *denotes a transpose.*

A more general version will appear in Section 2.4. The main motivation for this problem is the bivariate autoregressive filter problem, which we shall discuss in Section 3.2.

1.1.2. *Two-variable orthogonal polynomials.* The theory of one-variable orthogonal polynomials is well-established, beginning with the results of Szegö [61], [62]. The following is well known.

A *positive Borel measure* ρ *with support on the unit circle containing at least* $n+1$ *points is given. Let* $\{\phi_i(z)\}$, $i = 0,\dots,n$, *be the unique sequence of polynomials such that* $\phi_i(z)$ *is a polynomial of degree* i *in* z *with positive leading coefficient and* $\int_{-\pi}^{\pi} \phi_i(e^{i\theta})\overline{\phi_j(e^{i\theta})}d\rho(\theta) = \delta_{i-j}$. *Then* $p_n(z) := \overleftarrow{\phi}_n(z) = z^n\overline{\phi_n(\frac{1}{\bar{z}})}$ *is stable and has spectral matching, i.e.,* $\frac{1}{|p_n(e^{i\theta})|^2}$ *has the same Fourier coefficients* c_i *as* ρ *for* $i = 0, \pm 1, \pm 2 \cdots, \pm n$.

In this paper we explore the two-variable case. In the papers by Delsarte, Genin and Kamp [18], [19] the first steps were made towards a general multivariable theory. We add to this the following spectral matching result.

Theorem 1.1.2. *Given is a positive Borel measure ρ with support on the bitorus $\mathbb{T}^2$, denote the Fourier coefficients of ρ by c_u, $u \in \mathbb{Z}^2$, and suppose that*

$$\det(c_{u-v})_{u,v\in\{0,\dots,n\}\times\{0,\dots,m\}} > 0.$$

Let $\phi(z,w) = \sum_{k=0}^{n}\sum_{l=0}^{m} \phi_{kl} z^k w^l$ be the polynomial so that $\phi_{nm} > 0$,

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi(e^{i\theta}, e^{i\eta}) e^{-ik\theta - il\eta} d\rho(\theta,\eta) = 0,$$

$$(n,m) \neq (k,l) \in \{0,\dots,n\}\times\{0,\dots,m\},$$

and

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi(e^{i\theta}, e^{i\eta}) \overline{\phi(e^{i\theta}, e^{i\eta})} d\rho(\theta,\eta) = 1.$$

Then $p(z,w) = z^n w^m \overline{\phi(1/\overline{z}, 1/\overline{w})}$ is stable (no roots inside $\overline{\mathbb{D}^2}$) and the Fourier coefficients $\widetilde{c}_u$ of $\frac{1}{|p(e^{i\theta},e^{i\eta})|^2}$ satisfy $\widetilde{c}_u = c_u$, $u \in \{0,\dots,n\}\times\{0,\dots,m\}$, if and only if

$$\operatorname{rank}(c_{u-v})_{\substack{u\in\{1,\dots,n\}\times\{0,\dots,m\}\\ v\in\{0,\dots,n\}\times\{1,\dots,m\}}} = nm. \tag{1.1.1}$$

In that case, $\widetilde{c}_u = c_u$, $u \in \{-n,\dots,n\}\times\{-m,\dots,m\}$.

One of the main tools in proving this result is the establishment of a two-variable Christoffel-Darboux-like formula (see Proposition 2.3.3).

1.1.3. *Fejér-Riesz factorization.* The well-known Fejér-Riesz lemma states that a trigonometric polynomial $f(z) = f_{-n}z^{-n} + \cdots + f_n z^n$ that takes on nonnegative values on the unit circle (i.e., $f(z) \geq 0$ for $|z| = 1$) can be written as the modulus squared of a polynomial of the same degree. That is, there exists a polynomial $p(z) = p_0 + \cdots + p_n z^n$ such that

$$f(z) = |p(z)|^2, \qquad |z| = 1.$$

In fact, one may choose $p(z)$ to be *outer*, i.e., $p(z) \neq 0$, $|z| < 1$. In the nonsingular case when $f(z) > 0$, $|z| = 1$, one may choose $p(z)$ to be stable. This factorization result has many applications, among others in H_∞-control (see, e.g., [32]) and in the construction of compactly supported wavelets (see [16, Ch. 6]). A natural question is whether analogs of the Fejér-Riesz lemma exist for functions of several variables. One such variation is the following: let

$$f(z,w) = \sum_{l=-m}^{m}\sum_{k=-n}^{n} f_{kl} z^k w^l, \qquad |z| = |w| = 1,$$

be so that $f(z,w) > 0$ for all $|z| = |w| = 1$. Does there exist a stable polynomial $p(z,w) = \sum_{l=0}^{m}\sum_{k=0}^{n} p_{kl} z^k w^l$ so that

$$f(z,w) = |p(z,w)|^2, \qquad |z| = |w| = 1? \tag{1.1.2}$$

In general, this question has a negative answer, as $f(z,w)$ may not even be written as a sum of square magnitudes of polynomials of the same degree ([10], [58]), let alone as a sum with one term, which necessarily has the same degree. As an aside, we mention that a strictly positive trigonometric polynomial may always be written as a sum of square magnitudes of polynomials that typically will be of higher degree [24, Cor. 5.2]. From a "degree of freedom" argument the general failure of factorization (1.1.2) is not too surprising. Indeed, if $f(z,w)$ is positive on the bitorus, one may perturb the $(n+1)(m+1)+nm$ coefficients $f_{kl} = f^*_{-k,-l}$, $(k,l) \in \{0,\dots,n\} \times \{0,\dots,m\} \cup \{1,\dots,n\} \times \{-m,\dots,-1\}$, independently while remaining positive. If one wants to perturb $p(z,w)$ while maintaining equality in (1.1.2), one only has $(n+1)(m+1)$ coefficients p_{kl}, $(k,l) \in \{0,\dots,n\} \times \{0,\dots,m\}$ to perturb, leading to a generic impossibility. (Note that one may always assume that $p_{00} \in \mathbb{R}$ and that necessarily $f_{00} \in \mathbb{R}$, so that the difference in count is indeed nm complex variables.)

As a consequence of the positive extension result, we arrive at the following characterization for when a stable factorization (1.1.2) exists.

THEOREM 1.1.3. *Suppose that* $f(z,w) = \sum_{k=-n}^{n} \sum_{l=-m}^{m} f_{kl} z^k w^l$ *is positive for* $|z| = |w| = 1$. *Then there exists a polynomial* $p(z,w) = \sum_{k=0}^{n} \sum_{l=0}^{m} p_{kl} z^k w^l$ *with* $p(z,w) \neq 0$ *for* $|z|, |w| \leq 1$, *and* $f(z,w) = |p(z,w)|^2$ *if and only if the matrix* Γ *as in Theorem* 1.1.1 *built from the Fourier coefficients* $c_{k,l} := \widehat{\frac{1}{f}}(k,l)$ *of the reciprocal of* f, *satisfies condition* (2) *of Theorem* 1.1.1. *In that case, the polynomial* p *is unique up to multiplication with a complex number of modulus* 1.

A more general version will appear in Section 3.3.

1.2. *Overall strategy and organization.* There exist many different proofs for the classical one-variable problem described in Subsection 1.1.1. Several of these methods may be generalized to deal with the following two-variable variation: given $c_{kl} = \overline{c}_{-k,-l}$, $k \in \mathbb{Z}$, $l = -m, \dots, m$, find a stable function $p(z,w) = \sum_{k=0}^{\infty} p_{k0} z^k + \sum_{k=-\infty}^{\infty} \sum_{l=1}^{m} p_{kl} z^k w^l$ whose spectral density function f has Fourier coefficients $\widehat{f}(k,l) = c_{kl}$, $k \in \mathbb{Z}$, $l = -m, \dots, m$. We shall refer to this two-variable problem as the "strip" case, because of the shape of the region $S_m := \mathbb{Z} \times \{-m, \dots, m\} \subset \mathbb{Z}^2$. Papers where this case appears include [19], [55] (reflection coefficient approach), [6], [56] (band method approach). In this paper we deal with a *finite* index set in $\mathbb{Z}^2$ where the Fourier coefficients of the sought spectral density function are specified. A standard case we will consider is the set $\Lambda_+ \cup (-\Lambda_+)$ with $\Lambda_+ = \{0, \dots, n\} \times \{0, \dots, m\}$. As it is known how to deal with the strip case one would like to determine the Fourier coefficients in a strip containing $\Lambda_+ \cup (-\Lambda_+)$, and then solve the problem from there. The main question is how to do this. The answer we have found lies in a parametrized version of the Gohberg-Semencul formula [43]. The following simple observation turns out to be crucial.

Observation 1. Let $p(z,w) = \sum_{k=0}^{n}\sum_{l=0}^{m} p_{kl}z^k w^l$ be a stable polynomial, and let $f(z,w) := \frac{1}{p(z,w)\overline{p}(1/z,1/w)}$ be its spectral density function. Write $p(z,w) = \sum_{l=0}^{m} p_l(z)w^l$ and

$$f(z,w) = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} f_{ij}z^i w^j = \sum_{j=-\infty}^{\infty} f_j(z)w^j.$$

Then

$$\begin{aligned}
&[(f_{i-j}(z))_{i,j=0}^{m}]^{-1} \\
&= \begin{bmatrix} p_0(z) & & \bigcirc \\ \vdots & \ddots & \\ p_m(z) & \cdots & p_0(z) \end{bmatrix} \begin{bmatrix} \bar{p}_0(1/z) & \cdots & \bar{p}_m(1/z) \\ & \ddots & \vdots \\ \bigcirc & & \bar{p}_0(1/z) \end{bmatrix} \\
&\quad - \begin{bmatrix} \bar{p}_{m+1}(1/z) & & \bigcirc \\ \vdots & \ddots & \\ \bar{p}_1(1/z) & \cdots & \bar{p}_{m+1}(1/z) \end{bmatrix} \begin{bmatrix} p_{m+1}(z) & \cdots & p_1(z) \\ & \ddots & \vdots \\ \bigcirc & & p_{m+1}(z) \end{bmatrix} := E_m(z),
\end{aligned}$$

where $p_{m+1}(z) \equiv 0$. Moreover, $E_m(z)$ is a matrix-valued trigonometric polynomial in z of degree n.

This last observation implies that $E_m(z)$ is uniquely determined by the Fourier coefficients $F_i = (f_{i,k-l})_{k,l=0}^{m}$, $i = -n, \dots, n$, of the matrix-valued function $(f_{i-j}(z))_{i,j=0}^{m}$. Moreover, it is known exactly [26, §6] how to construct $E_m(z)$ from $F_{-n}, \dots, F_n$. For this construction we need to know f_{ik}, $(i,k) \in \{-n, \dots, n\} \times \{-m, \dots, m\} = \Lambda_+ - \Lambda_+$. Since $\Lambda_+ - \Lambda_+ \neq \Lambda_+ \cup (-\Lambda_+)$ we first need to solve for the unknowns $f_{ik} = \overline{f}_{-i,-k}$, $(i,k) \in \{1, \dots, n\} \times \{-m, \dots, -1\}$. It turns out that for the resolution of this step the particular structure of $E_m(z)$ plays an important role. The crucial observation here is again a simple one, namely:

Observation 2. If $M_{m-1}(z)$ is a stable matrix polynomial so that $E_{m-1}(z) = M_{m-1}(z)M_{m-1}(z)^*$, $z \in \mathbb{T}$, then

$$M_m(z) := \begin{pmatrix} p_0(z) & 0 \\ \operatorname{col}(p_i(z))_{i=1}^{m} & M_{m-1}(z) \end{pmatrix}$$

is a stable matrix polynomial satisfying $E_m(z) = M_m(z)M_m(z)^*$, $z \in \mathbb{T}$.

With the help of this observation we are able to find the conditions the unknowns in f_{jk}, $(j,k) \in \Lambda_+ - \Lambda_+$, need to satisfy in order to lead to a solution. These main observations will appear in Chapter 2 which contains the solution of the positive extension problem.

We now describe the organization of the paper in detail. Chapter 2 contains the main positive extension result and is organized as follows. In Section 2.1 we study matrix polynomials of the form $E_m(z)$ as above, and extract the crucial structure they contain. As a by-product we formulate a test for stability of two-variable polynomials that only uses one-variable root tests. In Section 2.2 we study the Fourier coefficients of the spectral density function corresponding to a stable polynomial, and exhibit their low rank behavior. This low rank behavior ultimately leads to the solution of the positive extension problem. In Section 2.3 we show that the polynomial constructed from the completed data has the desired properties (stability and "spectral matching" = the matching of the Fourier coefficients of its spectral density function). In Section 2.4 we formulate and solve the general positive extension problem for arbitrary given finite data.

Chapter 3 contains several consequences of the main result. The positive extension problem is recast in the settings of two-variable orthogonal polynomials and of bivariate autoregressive filter design. These interpretations of the main results appear in Sections 3.1 and 3.2, respectively. In Section 3.3 we state and prove the spectral Fejér-Riesz factorization result for strictly positive trigonometric polynomials. In Section 3.4 we present what our result means for a possible generalization of the Gohberg-Semencul formula to doubly indexed Toeplitz matrices.

In the appendix, finally, we provide an alternative way to prove one direction of the positive extension result. The method here uses minimal rank completions within the class of doubly indexed Toeplitz matrices.

1.3. *Conventions and notation.* For purposes of easy reference we mention in this section the most important notational conventions used in this paper.

Symbols for several frequently used sets are $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{T}$, $\mathbb{D}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{C}_\infty$, which stand for the sets of positive integers, nonnegative integers, integers, complex numbers of modulus one, complex numbers of modulus less than one, real numbers, complex numbers, and complex numbers including infinity, respectively.

In this paper we shall deal with subsets of $\mathbb{Z}^2$ and with orderings on them. The most frequently used ordering is the *lexicographical ordering* which is defined by

$$(k,l) <_{\text{lex}} (k_1,l_1) \iff k < k_1 \text{ or } (k = k_1 \text{ and } l < l_1).$$

We shall also use the *reverse lexicographical ordering* which is defined by

$$(k,l) <_{\text{revlex}} (k_1,l_1) \iff (l,k) <_{\text{lex}} (l_1,k_1).$$

Both these orderings are linear orders and in addition they satisfy

$$(k,l) < (m,n) \Longrightarrow (k+p, l+q) < (m+p, n+q). \tag{1.3.1}$$

In such a case, one may associate a *halfspace* with the ordering which is defined by $\{(k,l) \ : \ (0,0) < (k,l)\}$. In the case of the lexicographical ordering we shall denote the associated halfspace by H and refer to it as *the standard halfspace*. In the case of the reverse lexicographical ordering we shall denote the associated halfspace by $\tilde{H}$. Instead of starting with the ordering, one may also start with a halfspace $\hat{H}$ of $\mathbb{Z}^2$ (i.e., a set $\hat{H}$ satisfying $\hat{H} + \hat{H} \subset \hat{H}$, $\hat{H} \cap (-\hat{H}) = \emptyset$, $\hat{H} \cup (-\hat{H}) \cup \{(0,0)\} = \mathbb{Z}^2$) and define an ordering via

$$(k, l) <_{\hat{H}} (k_1, l_1) \Longleftrightarrow (k_1 - k, l_1 - l) \in \hat{H}.$$

We shall refer to the order $<_{\hat{H}}$ as *the order associated with* $\hat{H}$.

Throughout the paper we shall use matrices whose rows and columns are indexed by subsets of $\mathbb{Z}^2$. For example, if $I = \{(0,0),(1,0),(0,1)\}$ and $J = \{(2,1),(2,2),(2,3)\}$, then

$$C = (c_{u-v})_{u \in I, v \in J}$$

is the 3×3 matrix

$$C = \begin{pmatrix} c_{-2,-1} & c_{-2,-2} & c_{-2,-3} \\ c_{-1,-1} & c_{-1,-2} & c_{-1,-3} \\ c_{-2,0} & c_{-2,-1} & c_{-2,-2} \end{pmatrix}.$$

The matrix C may be referred to as an $I \times J$ matrix. The first row in this matrix will be referred to as the $(0,0)^{\text{th}}$, while, for instance, the second column will be referred to as the $(2,2)^{\text{th}}$. The entries are referred to according to the row and column index. Thus for example, in this particular matrix, the $((1,0),(2,3))$ entry contains the element $c_{-1,-3}$. The inverse of this matrix has rows and columns that are indexed by J and I, respectively. In other words, C^{-1} is a $J \times I$ matrix. In the case when C is invertible, we may for example have statements of the form: $(C^{-1})_{(2,2),(0,1)} = 0$ if and only if

$$\operatorname{rank} \begin{pmatrix} c_{-2,-1} & c_{-2,-3} \\ c_{-1,-1} & c_{-1,-3} \end{pmatrix} \le 1,$$

which is a true statement by Kramer's rule. In parts of the paper the index sets I and J may be given without an order (e.g., $I = \{1, \dots, n\} \times \{\dots, m-2, m-1, m\}$), in which case any order may be chosen. Clearly, in that case the statements made about the matrices will be independent of the chosen order, such as statements about rank and zeroes in the inverse. When $I = J$ we will always choose the same order for the rows and columns, as in this case we may want to make statements about self-adjointness and positive definiteness. In algebraic manipulations with matrices indexed by subsets of $\mathbb{Z}^2$ common sense rules apply. For example, if C is an $I \times J$ matrix and D a $J \times K$ matrix, then CD is an $I \times K$ matrix whose $(i,k)^{\text{th}}$ entry equals $\sum_{j \in J} c_{ij} d_{jk}$. Quite often we will encounter matrices whose rows and columns are indexed by the particular set $\Lambda_+ = \{0, \dots, n\} \times \{0, \dots, m\}$. It is a useful observation that when we order Λ_+ in the lexicographical ordering, the corresponding matrix is an $(n+1) \times (n+1)$ block Toeplitz matrix whose block entries are themselves $(m+1)\times(m+1)$ Toeplitz matrices. In the reverse lexicographical order we also get such a doubly-indexed Toeplitz matrix, but now the matrix is an $(m+1)\times(m+1)$ block matrix whose blocks are of size $(n+1)\times(n+1)$.

Row and column vectors may be indexed by subsets of $\mathbb{Z}^2$. The notation

$$\mathrm{row}(c_k)_{k\in K}, \mathrm{col}(c_k)_{k\in K}$$

stands for a row and column vector containing the entries $c_k, k\in K$, in some order, respectively. We shall also use the more conventional notation

$$\mathrm{row}(F_i)_{i=1}^n = \begin{pmatrix} F_1 & \cdots & F_n \end{pmatrix}, \mathrm{col}(F_i)_{i=1}^n = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}.$$

Polynomials and pseudopolynomials (negative powers are allowed) in one and two variables will appear. For a one-variable polynomial $p(z)=\sum_{i=0}^n p_i z^i$, we have the notation

$$\overline{p}(z) := \sum_{i=0}^n \overline{p_i} z^n, \overleftarrow{p}(z) := z^n \overline{p}(\frac{1}{z}) = \sum_{i=0}^n \overline{p_{n-i}} z^i.$$

The polynomial $\overleftarrow{p}(z)$ is called the *reverse* of $p(z)$. In this definition it is important to know how many terms (of which some may be zero) $p(z)$ has. We shall use the term "degree" here, so that the polynomial $p(z)$ above has degree n. It is a slight deviation from the standard way of using the term degree as its use usually implies that the coefficient of the highest degree monomial is nonzero. For our two variables we shall use z and w. The monomial $z^i w^j$ will in shorthand be denoted by $\binom{z}{w}^k$ where $k=(i,j)$. When $K\subset\mathbb{Z}^2$ is a finite set and p_k, $k\in K$, are complex numbers, then $p(z,w)=\sum_{k\in K} p_k \binom{z}{w}^k$ is called a *pseudopolynomial*. For this pseudopolynomial we define

$$\overline{p}(z,w) = \sum_{k\in K} \overline{p_k} \begin{pmatrix} z \\ w \end{pmatrix}^k.$$

In addition, we have a notion of "reverse" for a two-variable pseudopolynomial, but in this case the index set K needs to be ordered, say $K=\{k_0,\dots,k_m\}$. In that case,

$$\overleftarrow{p}(z,w) = \begin{pmatrix} z \\ w \end{pmatrix}^{k_m} \overline{p}(\frac{1}{z},\frac{1}{w}).$$

It is a slight abuse of notation not to include the ordering of K in the notation of $\overleftarrow{p}(z,w)$, but in all instances we will make clear what order on K applies (or, at least indicate which element of K appears last in the ordering).

For polynomials of one or two variables we shall allow ∞ as a root. In one variable, we say that $a(z)=\sum_{i=0}^n a_n z^n$ has a root at infinity when $a_n=0$. Equivalently, ∞ is a root of $a(z)$ if and only if 0 is a root of $\overleftarrow{a}(z)$. As a consequence, we get the following interpretation of ∞ as a root for polynomials of two variables. Let

$$p(z,w) = \sum_{i=0}^{n}\sum_{j=0}^{m} p_{ij} z^i w^j = \sum_{j=0}^{m} p_j(z) w^j = \sum_{i=0}^{n} \tilde{p}_i(w) z^i$$

be a polynomial of degree (n,m). Then $p(z,\infty)=0$ corresponds to the statement $p_m(z)=0$, while $p(\infty,w)=0$ corresponds to the statement $\tilde{p}_n(w)=0$. The statement $p(\infty,\infty)=0$ corresponds to $p_{nm}=0$. Finally, for an $r\times r$ matrix polynomial $G(z)=\sum_{i=0}^{n} G_i z^i$ of degree n, we say that ∞ is in the spectrum of G if $\det G_n = 0$. This is equivalent to the statement that the polynomial $\det(G(z))$ of degree rn has a root at ∞.

We will need the notions of left and right stable factorizations of matrix-valued trigonometric polynomials. We say that a polynomial $a(z)$ is *stable* if $a(z)\neq 0$, $z\in\overline{\mathbb{D}}$. A square matrix polynomial $G(z)$ is called *stable* if $\det G(z)$ is stable. Let $A(z)=\sum_{i=-n}^{n} A_i z^i$ be a matrix-valued trigonometric polynomial that is positive definite on $\mathbb{T}$, i.e., $A(z)>0$ for $|z|=1$. In particular, since the values of $A(z)$ on the unit circle are Hermitian, we have $A_i = A_{-i}^*$, $i=0,\dots,n$. The positive matrix function $A(z)$ allows a *left stable factorization*, that is, we may write

$$A(z) = M(z)M(1/\overline{z})^*, z\in\mathbb{C}\setminus\{0\},$$

with $M(z)$ a stable matrix polynomial of degree n. In the scalar case, this is the well-known Fejér-Riesz factorization and goes back to the early 1900's. For the matrix case the result goes back to [57] and [46]. When we require that $M(0)$ is lower triangular with positive diagonal entries, the stable factorization is unique. We shall refer to this unique factor $M(z)$ as *the left stable factor* of $A(z)$. Similarly, we define right variations of the above notions. In particular, if $N(z)$ is so that $A(z)=N(1/\overline{z})^*N(z)$, $z\in\mathbb{C}\setminus\{0\}$, $N(z)$ is stable and $N(0)$ is lower triangular with positive diagonal elements, then $N(z)$ is called the *right stable factor* of $A(z)$. For scalar functions f of two variables, stability is defined as $f(z,w)\neq 0$ for $(z,w)\in\overline{\mathbb{D}}\times\mathbb{T}\cup\{0\}\times\overline{\mathbb{D}}$. As we shall see in Proposition 2.1.1, when f is a polynomial stability is equivalent to $f(z,w)\neq 0$, $(z,w)\in\overline{\mathbb{D}}^2$.

Cholesky factorizations of positive definite matrices will play an important role as well. Given a positive definite matrix M, we say that L is its *lower Cholesky factor* when L is lower triangular, has positive entries on the diagonal and satisfies $M=LL^*$. We say that U is the *upper Cholesky factor* of M when U is upper triangular, has positive entries on the diagonal and satisfies $M=UU^*$.

We also mention the notation $\widehat{f}(k)$ which stands for the k^{th} Fourier coefficient of f. In the case when $k\in\mathbb{Z}$ we are considering a function on $\mathbb{T}$, while in the case when $k\in\mathbb{Z}^2$ we are considering a function on $\mathbb{T}^2$. The *support* of $\widehat{f}$ is the set $\{k : \widehat{f}(k)\neq 0\}$. Finally, we will frequently use the *Kronecker delta*, which is defined as $\delta_u = 1$ when $u = 0$ and $\delta_u = 0$ otherwise. Here u typically ranges in a subset of $\mathbb{Z}$ or $\mathbb{Z}^2$.

1.4. *Acknowledgments.* JSG was partially supported by NSF grant DMS-9970613 and a Fulbright fellowship. JSG gratefully acknowledges the hospitality of Professor Pierre Moussa and other members at the SPHT, Sacley France. HJW was partially supported by NSF grants DMS-9500924 and DMS-9800704, a Faculty Research Assignment of the College of William and Mary and by the Ecole Nationale Supérieure de Techniques Avancées (ENSTA). HJW gratefully acknowledges the hospitality of Professor Laurent El Ghaoui and other members of the Unité de Mathématiques Appliquées at ENSTA during his stay in Paris where this work originated. Both authors also thank Professor Albert Cohen (Paris VII) for his hospitality and for introducing them to one another. We would like to thank Dr. Glaysar Castro for useful discussions, especially on the subject of factorization. Finally, we wish to thank Professors M. A. Kaashoek and J. C. Willems for useful discussions during the revision process.

2. Stable polynomials and positive extensions

In this chapter we treat the positive extension problem where, given a finite number of Fourier coefficients, a stable polynomial is sought whose spectral density function has the prescribed Fourier coefficients. We will show that the required positive extension exists if and only if a structured partial matrix has a positive definite structured completion satisfying a certain low rank condition. In order to show the necessity we shall study stable polynomials and their density functions. In particular, we shall find expressions for the Fourier coefficients of the corresponding spectral density function in terms of realizations of a one-variable matrix polynomial that we associate with the stable polynomial. This matrix polynomial may be viewed as a parametrized Schur-Cohn expression. The sufficiency proof is achieved by showing that a completed matrix as described above has an associated predictor polynomial that is stable and that has the spectral matching property. For this latter part, we first prove a useful formula that may be interpreted as a two-variable Christoffel-Darboux like formula. Along the way we will also obtain a stability test for two-variable polynomials that consists of two one-variable root tests and a single matrix positive definiteness test.

2.1. *Stability via one-variable root tests.* The classical Schur-Cohn test states that a polynomial $a(z) = a_0 + \cdots + a_n z^n$ is stable if and only if

(2.1.1)
$$\begin{bmatrix} a_0 & & \bigcirc \\ \vdots & \ddots & \\ a_{n-1} & \cdots & a_0 \end{bmatrix} \begin{bmatrix} \bar{a}_0 & \cdots & \bar{a}_{n-1} \\ & \ddots & \vdots \\ \bigcirc & & \bar{a}_0 \end{bmatrix} - \begin{bmatrix} \bar{a}_n & & \bigcirc \\ \vdots & \ddots & \\ \bar{a}_1 & \cdots & \bar{a}_n \end{bmatrix} \begin{bmatrix} a_n & \cdots & a_1 \\ & \ddots & \vdots \\ \bigcirc & & a_n \end{bmatrix} > 0.$$

In this section we study two-variable stable polynomials. By definition $p(z,w)$ is *stable* if $p(z,w) \neq 0$ for $(z,w) \in \overline{\mathbb{D}} \times \mathbb{T} \cup \{0\} \times \overline{\mathbb{D}}$. Consequently, one may write $p(z,w) = \sum_{i=0}^{n} a_i(w) z^i$ and require that (2.1.1) holds for $a_i = a_i(w)$ for all $w \in \mathbb{T}$. It is therefore natural in this context to study matrix-valued trigonometric polynomials of the type (2.1.1) where a_i are polynomials. We will do this in this section and obtain a stability test for two-variable polynomials that only requires one-variable root tests. More importantly, we develop the basic results needed to solve the positive extension problem. We start with some preliminary material.

Let f be a complex valued continuous function of two variables whose domain includes $\overline{\mathbb{D}} \times \mathbb{T} \cup \{0\} \times \overline{\mathbb{D}}$. We say that f is *stable* if $f(z,w) \neq 0$ for $(z,w) \in \overline{\mathbb{D}} \times \mathbb{T} \cup \{0\} \times \overline{\mathbb{D}}$. Note that stability of f implies that f is invertible as a function on the bitorus $\mathbb{T}^2$. We have the following equivalent statements for the stability of *polynomials* p *of degree* (n,m), that is, polynomials of the form

$$p(z,w) = \sum_{i=0}^{n} \sum_{j=0}^{m} p_{ij} z^i w^j. \tag{2.1.2}$$

Note that we do not have any nonzero requirements on the coefficients of p, so that the degree has to be specified along with the polynomial. The $(k,l)^{\text{th}}$ Fourier coefficient of a function $q(z,w)$ is denoted by $\widehat{q}(k,l)$.

PROPOSITION 2.1.1. *Let $p(z,w)$ be a polynomial of degree (n,m). The following are equivalent*:

(i) *p is stable*,

(ii) $\widehat{p^{-1}}(k,l) = 0$ *for all* $(k,l) \in \{(k,l) : k < 0$ *or* $(k = 0$ *and* $l < 0)\}$,

(iii) $\widehat{p^{-1}}(k,l) = 0$ *for all* $(k,l) \in \{(k,l) : k < 0$ *or* $l < 0\}$,

(iv) $p(z,w) \neq 0$ *for all* $|z| \leq 1$ *and* $|w| \leq 1$.

The equivalence of (i) and (ii) holds for all stable functions and actually provides the motivation for its definition.

Proof. For (i) $\Rightarrow$ (ii) use [29] to see that stability implies that $\widehat{p^{-1}}(k,l) = 0$ for $k < 0$. In addition, it follows from $p(0,w) \neq 0$ for $|w| \leq 1$ that $\widehat{p^{-1}}(0,l) = 0$ for $l < 0$. For (iii) $\Rightarrow$ (iv) use the fact that (iii) implies p^{-1} has an absolutely summable Fourier expansion of the form

$$p^{-1}(z,w) = \sum_{k,l \geq 0} \widehat{p^{-1}}(k,l) z^k w^l, \qquad |z| = |w| = 1.$$

Thus p^{-1} can be extended for values of z and w inside the unit disk, proving (iv). The implication (iv) $\Rightarrow$ (i) is trivial.

It remains to show (ii) $\Rightarrow$ (iii). For this write

$$p(z,w)=\sum_{j=0}^{n}\tilde{p}_j(w)z^j,$$

and

$$p^{-1}(z,w)=\sum_{k=0}^{\infty}q_k(w)z^k.$$

Thus $\widehat{q_k}(l)=\widehat{p^{-1}}(k,l)$. Note that $\widehat{q_0}(l)=0$, $l<0$. Since $p(z,w)p^{-1}(z,w)\equiv 1$,

$$\tilde{p}_0(w)q_0(w)\equiv 1$$

and

$$\sum_{l=0}^{j}\tilde{p}_{j-l}(w)q_l(w)\equiv 0, \qquad j\geq 1.$$

We proceed by induction. Suppose that for $j\leq k$, with $k\geq 0$, we have shown that $\widehat{q_j}(s)=0$, $s<0$. Then

$$\begin{aligned} q_{k+1}(w) &= \tfrac{-1}{\tilde{p}_0(w)}\left(\textstyle\sum_{l=0}^{k}\tilde{p}_{k+1-l}(w)q_l(w)\right)\\ &= -q_0(w)\left(\textstyle\sum_{l=0}^{k}\tilde{p}_{k+1-l}(w)q_l(w)\right)\end{aligned}$$

contains only nonnegative powers of w. Thus $\widehat{q_{k+1}}(s)=0$, $s<0$. $\square$

We introduce the notion of intersecting zeros. We will allow for roots to be at ∞ as explained in Section 1.3. Given a polynomial $p(z,w)$ of degree (n,m), we say that a pair $(z,w)\in\mathbb{C}^2_\infty$ is an *intersecting zero* of p if

$$p(z,w)=0=\overleftarrow{p}(z,w). \tag{2.1.3}$$

In general a polynomial could have continua of intersecting zeros. We will see that when p is stable, it only has a finite number of them. In fact, the intersecting roots will play a crucial role in the stability test we develop. This is because they appear in the description of the spectrum of matrix trigonometric polynomials constructed from a parametrized Schur-Cohn-type test. This is part of the content of the following proposition.

For a stable polynomial $p(z,w)$ we define its *spectral density function* by

$$f(z,w)=1/(p(z,w)\overline{p}(z^{-1},w^{-1})),$$

where for p as in (2.1.2) we let $\overline{p}(z,w)=\sum_{i=0}^{n}\sum_{j=0}^{m}\overline{p}_{ij}z^iw^j$. Note that when $p(z_0,w_0)\neq 0$ for some $|z_0|=|w_0|=1$, then $f(z_0,w_0)>0$. In particular, if p is stable, then $f>0$ on $\mathbb{T}^2$. In addition, for a square matrix-valued function $G(z)$ we define its *spectrum* by $\Sigma(G)=\{z:\ \det G(z)=0\}$. In case $G(z)$ is a matrix polynomial we allow for ∞ to be in the spectrum of G as explained in Section 1.3. So in this case $\Sigma(G)\subset\mathbb{C}_\infty$. We remind the reader that the definition of left stable factor may be found in Section 1.3.

Proposition 2.1.2. *Let $p(z,w)$ be a stable polynomial of degree (n,m) with $p(0,0)>0$, and let $f(z,w)$ be its spectral density function. Write*

$$p(z,w)=\sum_{i=0}^{m} p_i(z)w^i, f(z,w)=\sum_{i=-\infty}^{\infty} f_i(z)w^i.$$

Put $p_i(z)\equiv 0$ for $i>m$. Then the following hold:

(i) $T_k(z):=(f_{i-j}(z))_{i,j=0}^{k}>0$ *for all $k\in\mathbb{N}_0$ and all $z\in\mathbb{T}$.*

(ii) *For all $k\geq m-1$ and for all z in the domain of T_k with $z\notin\Sigma(T_k)$:*

(2.1.4)
$$\begin{aligned} T_k(z)^{-1} &= \begin{bmatrix} p_0(z) & & \bigcirc \\ \vdots & \ddots & \\ p_k(z) & \cdots & p_0(z)\end{bmatrix}\begin{bmatrix} \bar{p}_0(1/z) & \cdots & \bar{p}_k(1/z) \\ & \ddots & \vdots \\ \bigcirc & & \bar{p}_0(1/z)\end{bmatrix} \\ &\quad - \begin{bmatrix} \bar{p}_{k+1}(1/z) & & \bigcirc \\ \vdots & \ddots & \\ \bar{p}_1(1/z) & \cdots & \bar{p}_{k+1}(1/z)\end{bmatrix}\begin{bmatrix} p_{k+1}(z) & \cdots & p_1(z) \\ & \ddots & \vdots \\ \bigcirc & & p_{k+1}(z)\end{bmatrix} \\ &=: E_k(z). \end{aligned}$$

(iii) *For $k\geq m-1$, the left stable factors $M_k(z)$ and $M_{k+1}(z)$ of the positive trigonometric matrix polynomials $E_k(z)$ and $E_{k+1}(z)$, respectively, satisfy*

$$M_{k+1}(z)=\begin{bmatrix} p_0(z) & 0 \\ \mathrm{col}(p_l(z))_{l=1}^{k+1} & M_k(z)\end{bmatrix}. \tag{2.1.5}$$

(iv) *The spectra of M_{m-1}, $\overleftarrow{M}_{m-1}$ and z^nE_{m-1} are given by*

$$\begin{aligned} \Sigma(M_{m-1}) &= \{z\in\mathbb{C}_\infty\setminus\overline{\mathbb{D}} : \exists w \text{ such that } (z,w) \text{ is an intersecting zero of } p\}, \\ \Sigma(\overleftarrow{M}_{m-1}) &= \{z\in\mathbb{D} : \exists w \text{ such that } (z,w) \text{ is an intersecting zero of } p\}, \\ \Sigma(z^nE_{m-1}) &= \{z\in\mathbb{C}_\infty : \exists w \text{ such that } (z,w) \text{ is an intersecting zero of } p\} \\ &\subset \mathbb{C}_\infty\setminus\mathbb{T}. \end{aligned}$$

In particular, p has only a finite number of intersecting zeros. In addition, for $k\geq m$, $\Sigma(M_k)=\Sigma(M_{m-1})\cup\{z\in\mathbb{C}_\infty : p_0(z)=0\}$, $\Sigma(\overleftarrow{M}_k)=\Sigma(\overleftarrow{M}_{m-1})\cup\{z\in\mathbb{C}_\infty : \overleftarrow{p_0}(z)=0\}$, $\Sigma(z^nE_k)=\Sigma(M_k)\cup\Sigma(\overleftarrow{M}_k)$.

Note that the statement above shows that $E_k(z)>0$, $z\in\mathbb{T}$, as $E_k(z)=T_k(z)^{-1}$. One may also see this by using the Schur-Cohn test for stability.

We shall use the following lemma.

Lemma 2.1.3. *Let $p(z,w)=\sum_{i=0}^{m} p_i(z)w^i$ be a polynomial of degree (n,m), and let $E_{m-1}(z)$ be defined by (2.1.4). Then*

$$\Sigma(z^nE_{m-1})=\{z\in\mathbb{C}_\infty \ : \ \exists w \text{ such that } (z,w) \text{ is an intersecting zero of } p\}$$

The ideas in the proof below appeared earlier in the context of Bezoutians (see, e.g., the proof of Theorem 1 in Section 13.3 of [52]).

Proof. Write $\overleftarrow{p}(z,w) = \sum_{i=0}^{m} q_i(z)w^i$, or equivalently, set $q_j(z) = z^n\overline{p}_{m-j}(1/z)$. First suppose that $p_m(z) \equiv 0$ and $q_m(z) \equiv 0$. Then (z,∞) is an intersecting root for every $z \in \mathbb{C}_\infty$. Moreover, it is easy to see that the first column of $z^nE_{m-1}(z)$ is the constant zero column, and consequently $\Sigma(z^nE_{m-1}) = \mathbb{C}_\infty$. Thus the result follows in this case.

Suppose now that $q_m(z) \not\equiv 0$. Consider the Sylvester matrix

$$S(z) = \begin{pmatrix} p_0(z) & & \bigcirc & q_0(z) & & \bigcirc \\ \vdots & \ddots & & \vdots & \ddots & \\ p_{m-1}(z) & \cdots & p_0(z) & q_{m-1}(z) & \dots & q_0(z) \\ p_m(z) & \dots & p_1(z) & q_m(z) & \dots & q_1(z) \\ & \ddots & \vdots & & \ddots & \vdots \\ \bigcirc & & p_m(z) & \bigcirc & & q_m(z) \end{pmatrix} \tag{2.1.6}$$

corresponding to $p(z,w)$ and $\overleftarrow{p}(z,w)$ viewed as polynomials in w. Since the determinant of $S(z)$ is the resultant of these two polynomials, we obtain that there exists a w so that (2.1.3) holds if and only if $S(z)$ is singular. Notice that if we write $S(z)$ as

$$S(z) = \begin{pmatrix} \alpha(z) & z^n\beta(z) \\ \gamma(z) & z^n\delta(z) \end{pmatrix}, \tag{2.1.7}$$

with all blocks of size $m \times m$, then $\alpha(z)$ and $\beta(z)$ are lower triangular Toeplitz, and therefore they commute. The matrices $\gamma(z)$ and $\delta(z)$ are upper triangular Toeplitz and commute as well. Moreover, by (2.1.4), $E_{m-1}(z) = \alpha(z)\delta(z) - \beta(z)\gamma(z)$. By using Schur complements we have for $z \notin \Sigma(\delta)$ that

$$\det S(z) = \det(\alpha(z) - \beta(z)\delta(z)^{-1}\gamma(z))\det(z^n\delta(z)) = \det(z^nE_{m-1}(z)),$$

where in the last step we used the product rule for determinants and the fact that $\gamma(z)$ and $\delta(z)$ commute. Since $\Sigma(\delta)$ is finite (due to $q_m(z) \not\equiv 0$) , $\det S(z) = \det(z^nE_{m-1}(z))$ for all z, and thus it follows that z is a zero of $\det(z^nE_{m-1}(z))$ if and only if $S(z)$ is singular. This yields the description of $\Sigma(z^nE_{m-1})$.

The case when $p_m(z) \not\equiv 0$ is similar. $\square$

Proof of Proposition 2.1.2. (i). Fix $|z| = 1$. Since $f(z,w) > 0$ for all $|w| = 1$, the multiplication operator $g(w) \to f(z,w)g(w)$ is a positive definite operator on the Lebesgue space $L_2(\mathbb{T})$. But then so is its restriction to the linear span of $\{1, w, \dots, w^k\}$. This yields (i).

(ii). Fix $|z| = 1$. Since $f(z,w)p(z,w) = 1/\overline{p}(\overline{z}, 1/w)$ is analytic for $w \in \mathbb{C}_\infty \setminus \mathbb{D}$, the $0, \dots, k$ Fourier coefficients of $f(z,w)p(z,w)$ viewed as a function of w are $1/\overline{p_0(z)}, 0, \dots, 0$. In other words,

$$T_k(z)\begin{bmatrix} p_0(z) \\ p_1(z) \\ \vdots \\ p_k(z) \end{bmatrix} = \begin{bmatrix} 1/\overline{p_0(z)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, k \geq m.$$

Equation (2.1.4) for $|z| = 1$ now follows directly from the celebrated Gohberg-Semencul formulas [43]. Since both sides of (2.1.4) are rational, we get that (2.1.4) holds for all z in the domain of T_k with $z \notin \Sigma(T_k)$.

(iii). Let $M_k(z)$ be the stable factor of $E_k(z)$. Define $M_{k+1}(z)$ via (2.1.5). Writing out the product $M_{k+1}(z)M_{k+1}(1/\bar{z})^*$ and comparing it to $E_{k+1}(z)$, it is straightforward to see that $M_{k+1}(z)M_{k+1}(1/\bar{z})^* = E_{k+1}(z)$. Since both $p_0(z)$ and $M_k(z)$ are stable, $M_{k+1}(z)$ is stable as well. Moreover, since $p_0(0) > 0$ and $M_k(0)$ is lower triangular with positive diagonal entries, the same holds for $M_{k+1}(0)$. Thus $M_{k+1}(z)$ must be the stable factor of $E_{k+1}(z)$.

(iv). By Lemma 2.1.3 the description of $\Sigma(z^n E_{m-1})$ follows. But then it also follows that z is a zero of the stable factor $M_{m-1}(z)$ of $E_{m-1}(z)$ if and only if $z \in \mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ and (z, w) is an intersecting zero of p for some w. The description of $\Sigma(\overleftarrow{M}_{m-1})$ follows by symmetry. The expressions for $\Sigma(\overleftarrow{M}_k)$, $\Sigma(M_k)$, and $\Sigma(z^n E_k)$, $k \geq m$, follow directly from (iii). $\square$

One can state several variations of the above result. We state the following one. It may be proven by using the above result (with the roles of z and w reversed) together with the observation that if A is a Toeplitz matrix then $JA^TJ = A$ where J is the matrix with 1's on the anti-diagonal and zeros elsewhere. The latter implies, for instance, that the right and left spectral factors N_k and M_k, respectively, of E_k are related by $N_k = JM_k^TJ$. The proposition may also be proven directly. The details are omitted.

Proposition 2.1.4. *Let $p(z, w)$ be a stable polynomial of degree (n, m) with $p(0, 0) > 0$, and let $f(z, w)$ be its spectral density function. Write*

$$p(z, w) = \sum_{i=0}^{n} \tilde{p}_i(w)z^i, f(z, w) = \sum_{i=-\infty}^{\infty} \tilde{f}_i(w)z^i.$$

Put $\tilde{p}_i(w) \equiv 0$ for $i > n$. Then the following hold:

(i) *$\tilde{T}_k(w) := (\tilde{f}_{i-j}(w))_{i,j=0}^k > 0$ for all $k \in \mathbb{N}_0$ and all $w \in \mathbb{T}$.*

(ii) *For all $k \geq n - 1$ and for all w in the domain of $\tilde{T}_K$ with $w \notin \Sigma(\tilde{T}_k)$:*

(2.1.8)

$$\begin{aligned}\tilde{T}_k(w)^{-1} = &\begin{bmatrix} \overline{\tilde{p}}_0(1/w) & \cdots & \overline{\tilde{p}}_k(1/w) \\ & \ddots & \vdots \\ \bigcirc & & \overline{\tilde{p}}_0(1/w) \end{bmatrix} \begin{bmatrix} \tilde{p}_0(w) & & \bigcirc \\ \vdots & \ddots & \\ \tilde{p}_k(w) & \cdots & \tilde{p}_0(w) \end{bmatrix} \\ &- \begin{bmatrix} \tilde{p}_{k+1}(w) & \cdots & \tilde{p}_1(w) \\ & \ddots & \vdots \\ \bigcirc & & \tilde{p}_{k+1}(w) \end{bmatrix} \begin{bmatrix} \overline{\tilde{p}}_{k+1}(1/w) & & \bigcirc \\ \vdots & \ddots & \\ \overline{\tilde{p}}_1(1/w) & \cdots & \overline{\tilde{p}}_{k+1}(1/w) \end{bmatrix} \\ &=: \tilde{E}_k(w).\end{aligned}$$

(iii) *For $k \geq n-1$, the right stable factors $\tilde{M}_k(w)$ and $\tilde{M}_{k+1}(w)$ of the positive trigonometric matrix polynomials $\tilde{E}_k(w)$ and $\tilde{E}_{k+1}(w)$, respectively, satisfy*

$$\tilde{M}_{k+1}(w) = \begin{bmatrix} \tilde{M}_k(w) & 0 \\ \operatorname{row}(\tilde{p}_{k+1-l}(w))_{l=0}^{k} & \tilde{p}_0(w) \end{bmatrix}. \tag{2.1.9}$$

(iv) *The spectra of $\tilde{M}_{n-1}$, $\overleftarrow{\tilde{M}}_{n-1}$ and $w^m \tilde{E}_{n-1}$ are given by*

$$\begin{aligned} \Sigma(\tilde{M}_{n-1}) &= \{w \in \mathbb{C}_\infty \setminus \overline{\mathbb{D}} : \exists z \text{ such that } (z,w) \text{ is an intersecting zero of } p\}, \\ \Sigma(\overleftarrow{\tilde{M}}_{n-1}) &= \{w \in \mathbb{D} : \exists z \text{ such that } (z,w) \text{ is an intersecting zero of } p\}, \\ \Sigma(w^m \tilde{E}_{n-1}) &= \{w \in \mathbb{C}_\infty : \exists z \text{ such that } (z,w) \text{ is an intersecting zero of } p\} \\ &\quad \subset \mathbb{C}_\infty \setminus \mathbb{T}. \end{aligned}$$

In particular, p has only a finite number of intersecting zeros. In addition, for $k \geq n$, $\Sigma(\tilde{M}_k) = \Sigma(\tilde{M}_{n-1}) \cup \{w \in \mathbb{C}_\infty : \tilde{p}_0(w) = 0\}$, $\Sigma(\overleftarrow{\tilde{M}}_k) = \Sigma(\overleftarrow{\tilde{M}}_{n-1}) \cup \{w \in \mathbb{C}_\infty : \overleftarrow{\tilde{p}_0}(w) = 0\}$, $\Sigma(w^m \tilde{E}_k) = \Sigma(\tilde{M}_k) \cup \Sigma(\overleftarrow{\tilde{M}}_k)$.

We now obtain a criterion for stability in terms of intersecting zeros.

THEOREM 2.1.5. *Let $p(z,w)$ be a polynomial of degree (n,m) of two variables. The following conditions are equivalent*:

(i) *$p(z,w)$ is stable,*

(ii) *$p(z,a) \neq 0$ for all $|z| \leq 1$ and some $|a| = 1$, $p(b,w) \neq 0$ for all $|w| \leq 1$ and some $|b| \leq 1$, and the intersecting zeros of p lie in $\mathbb{D} \times (\mathbb{C}_\infty \setminus \overline{\mathbb{D}}) \cup (\mathbb{C}_\infty \setminus \overline{\mathbb{D}}) \times \mathbb{D}$.*

(iii) *$p(z,a) \neq 0$ for all $|z| \leq 1$ and some $|a| = 1$, $p(b,w) \neq 0$ for all $|w| \leq 1$ and some $|b| \leq 1$, and every intersecting zero (z,w) of p satisfies $|z| \neq 1$ or $|w| \neq 1$.*

(iv) *$p(b,w) \neq 0$ for all $|w| \leq 1$ and some $|b| \leq 1$, $\tilde{E}_{n-1}(a) > 0$ for some $|a| = 1$, and* $\det \tilde{E}_{n-1}(w) \neq 0$ *for all $|w| = 1$.*

Clearly, one may reverse the roles of z and w, and obtain additional equivalences.

Proof. That (i) implies (ii) follows directly from Proposition 2.1.1(iv). For (ii) $\to$ (iv) note that the stability of $p(z,a)$ is equivalent to $\tilde{E}_{n-1}(a) > 0$. Moreover, $\Sigma(w^m \tilde{E}_{n-1}) = \{w : \exists z$ such that (z,w) is an intersecting zero of $p\}$ does not contain any elements from $\mathbb{T}$.

For (iv) $\to$ (iii) notice that $\tilde{E}_{n-1}(a) > 0$ is equivalent to $p(z,a)$ being stable. In addition, since $\Sigma(w^m \tilde{E}_{n-1}) \cap \mathbb{T} = \emptyset$, we have by the variation of Lemma 2.1.3 with the roles of z and w interchanged, that all intersecting zeros of $p(z,w)$ satisfy $|w| \neq 1$.

Finally, in order to see that (iii) implies (i) suppose that (iii) is satisfied. We claim that $p(z,w) \neq 0$ for $|z| = |w| = 1$. Indeed, suppose by contradiction that $p(z_0,w_0) = 0$, for some $|z_0| = |w_0| = 1$. Then, by taking complex conjugates, we get $0 = \sum_{i=0}^n \sum_{j=0}^m \overline{p_{ij}} \frac{1}{z_0^i} \frac{1}{w_0^j} = \frac{\overleftarrow{p}(z_0,w_0)}{z_0^n w_0^m}$, and thus $\overleftarrow{p}(z_0,w_0) = 0$ as well. This contradicts (iii). The result now follows from Theorem 2 in [60] (see also Theorem 3 in [20]). $\square$

It should be observed that checking stability via Theorem 2.1.5(iv) may be done by two single variable polynomial root tests (e.g., check that $p(0,w)$ is stable and that $\det \tilde{E}_{n-1}(w) \neq 0$, $|w| = 1$) and a positive definiteness test (e.g., $\tilde{E}_{n-1}(1) > 0$). We note that in [8] a test of this type was alluded to, but a proof is not present there.

2.2. *Fourier coefficients of spectral density functions.* In the following we show that the spectral density function of a stable polynomial of degree (n,m) has an associated Hankel operator of rank nm. This is done by developing formulas for the Fourier coefficients appearing in the Hankel operator. The spectrum (= the set of eigenvalues) of a constant square matrix A is denoted by $\sigma(A)$. Further, denote $\delta_u = 0$ for $u \neq (0,0)$ and $\delta_{(0,0)} = 1$.

THEOREM 2.2.1. *Let $p(z,w) = \sum_{i=0}^n \sum_{j=0}^m p_{ij} z^i w^j$ be a stable polynomial of degree (n,m), and let $f(z,w)$ be its spectral density function. Then there exists a row vector $x \in \mathbb{C}^{nm}$, a column vector $y \in \mathbb{C}^{nm}$ and commuting matrices $S, \tilde{S} \in \mathbb{C}^{nm \times nm}$ such that*

$$\begin{aligned} &(2.2.1) \quad \sigma(S) = \{z \in \mathbb{D} \ : \ \exists w \text{ such that } (z,w) \text{ is an intersecting zero of } p\}, \\ &\qquad\qquad \sigma(\tilde{S}) = \{w \in \mathbb{D} \ : \ \exists z \text{ such that } (z,\overline{w}) \text{ is an intersecting zero of } p\}, \end{aligned}$$

and

$$\widehat{f}(k,j) = x \tilde{S}^{m+j-1} S^{n-1-k} y, \qquad k \leq n-1, \quad j \geq -m+1. \tag{2.2.2}$$

Choose x, y, S and $\tilde{S}$ as follows:

$$x = \mathrm{row}(\widehat{f}((n-1,0)-u))_{u\in\Delta},$$
$$y = \mathrm{col}(\delta_{u+(0,-m+1)})_{u\in\Delta}, S = \Phi^{-1}\Phi_1, \tilde{S} = \Phi^{-1}\Phi_2, \tag{2.2.3}$$

where

$$\Phi = (\widehat{f}(u-v))_{u,v\in\Delta}, \qquad \Phi_1 = (\widehat{f}(u-v-(1,0)))_{u,v\in\Delta},$$
$$\Phi_2 = (\widehat{f}(u-v+(0,1))_{u,v\in\Delta}$$

and $\Delta = \{0,\dots,n-1\}\times\{0,\dots,m-1\}$. *In particular the matrix*

$$(\widehat{f}(u-v))_{\substack{u\in\{\dots,n-2,n-1\}\times\{0,1,\dots\}\\ v\in\{0,1,\dots\}\times\{\dots,m-2,m-1\}}} \tag{2.2.4}$$

has rank equal to nm.

In case $n = m = 2$ and the lexicographical ordering is used, equation (2.2.3) yields the choice

$$x = \begin{pmatrix}\widehat{f}(1,0) & \widehat{f}(1,-1) & \widehat{f}(0,0) & \widehat{f}(0,-1)\end{pmatrix}, y = \begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}^T,$$

$$\Phi = \begin{pmatrix}\widehat{f}(0,0) & \widehat{f}(0,-1) & \widehat{f}(-1,0) & \widehat{f}(-1,-1)\\ \widehat{f}(0,1) & \widehat{f}(0,0) & \widehat{f}(-1,1) & \widehat{f}(-1,0)\\ \widehat{f}(1,0) & \widehat{f}(1,-1) & \widehat{f}(0,0) & \widehat{f}(0,-1)\\ \widehat{f}(1,1) & \widehat{f}(1,0) & \widehat{f}(0,1) & \widehat{f}(0,0)\end{pmatrix},$$

$$\Phi_1 = \begin{pmatrix}\widehat{f}(-1,0) & \widehat{f}(-1,-1) & \widehat{f}(-2,0) & \widehat{f}(-2,-1)\\ \widehat{f}(-1,1) & \widehat{f}(-1,0) & \widehat{f}(-2,1) & \widehat{f}(-2,0)\\ \widehat{f}(0,0) & \widehat{f}(0,-1) & \widehat{f}(-1,0) & \widehat{f}(-1,-1)\\ \widehat{f}(0,1) & \widehat{f}(0,0) & \widehat{f}(-1,1) & \widehat{f}(-1,0)\end{pmatrix}$$

and

$$\Phi_2 = \begin{pmatrix}\widehat{f}(0,1) & \widehat{f}(0,0) & \widehat{f}(-1,1) & \widehat{f}(-1,0)\\ \widehat{f}(0,2) & \widehat{f}(0,1) & \widehat{f}(-1,2) & \widehat{f}(-1,1)\\ \widehat{f}(1,1) & \widehat{f}(1,0) & \widehat{f}(0,1) & \widehat{f}(0,0)\\ \widehat{f}(1,2) & \widehat{f}(1,1) & \widehat{f}(0,2) & \widehat{f}(0,1)\end{pmatrix}.$$

Notice that the above result is reminiscent of (one direction of) the classical Kronecker Theorem (see, e.g., [69]) which relates functions with a finite number of poles in $\mathbb{D}$ with a low rank Hankel operator. In addition, the choice of the matrices (2.2.3) has the flavor of a two-variable version of Silverman's algorithm [59, Proof of Th. 11] for finding realizations.

Clearly, the matrix (2.2.4) may be interpreted as a restriction of the multiplication operator M_f on the Lebesgue space $L^2(\mathbb{T}^2)$ with symbol f. Indeed, if for $\Lambda \subseteq \mathbb{Z}^2$ we denote by P_Λ the orthogonal projector on $L^2(\mathbb{T}^2)$ given by

$$P_\Lambda \left(\sum_{(k,l)\in\mathbb{Z}^2} c_{kl} z^k w^l \right) = \sum_{(k,l)\in\Lambda} c_{kl} z^k w^l, \tag{2.2.5}$$

then $P_I M_f P_J : \mathrm{Im} P_J \to \mathrm{Im} P_I$ has a matrix representation (with respect to the canonical basis $\{z^k w^l\}_{k,l}$)

$$(\widehat{f}(u-v))_{u\in I, v\in J}.$$

Proof of Theorem 2.2.1. We shall use the notation of Propositions 2.1.2 and 2.1.4. The strategy of the proof is as follows. The matrix-valued functions $\tilde{T}_l(w)$ and $T_k(z)$ both have inverses that are matrix-valued trigonometric polynomials (use part (ii) of Propositions 2.1.2 and 2.1.4). Therefore, their Fourier coefficients may be represented as CA^iB, $i \geq 0$, for appropriately chosen *finite* matrices A, B, and C. Since the matrix valued functions $\tilde{T}_l(w)$ and $T_k(z)$ are closely related, the representations of their Fourier coefficients are closely related as well. Using this the desired representation of the Fourier coefficients of f are found. Let us start.

For $k \geq m-1$, consider the equality $T_k(z) = M_k(1/\overline{z})^{*-1} M_k(z)^{-1}$. Notice that $M_k(z)$ is a $(k+1)\times(k+1)$ matrix polynomial of degree n, and that $M_k(0)$ is invertible. Thus $\overleftarrow{M_k}(z) = z^n M_k(1/\overline{z})^*$ is a polynomial of degree n with an invertible leading term $M_k(0)^*$. As M_k is stable and $\overleftarrow{M}_k$ is anti-stable (all spectra inside the unit circle), they do not have a common spectrum. Since, in addition $\overleftarrow{M}_k$ has an invertible leading term, there exist by Theorem 3.5 in [39] matrix polynomials $P_k(z)$ and $Q_k(z)$ of degree at most $n-1$ so that

$$\overleftarrow{M_k}(z) P_k(z) + Q_k(z) M_k(z) \equiv I_{k+1}.$$

Moreover, $Q_k(z)$ is given by

$$Q_k(z) = -\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\overleftarrow{M_k}(z) - \overleftarrow{M_k}(\lambda)}{z-\lambda} \overleftarrow{M_k}(\lambda)^{-1} M_k(\lambda)^{-1} d\lambda.$$

Notice that by the particular structure of $M_k(z)$, as described in Proposition 2.1.2(iii),

$$Q_k(z) = \begin{pmatrix} * & * \\ * & Q_{k-1}(z) \end{pmatrix}, k \geq m,$$

and also

$$M_k(0)^{*-1} Q_k(z) = \begin{pmatrix} * & * \\ * & M_{k-1}(0)^{*-1} Q_{k-1}(z) \end{pmatrix}, k \geq m. \tag{2.2.6}$$

Now

$$\begin{aligned} T_k(z) &= M_k(1/\overline{z})^{*-1} (\overleftarrow{M_k}(z) P_k(z) + Q_k(z) M_k(z)) M_k(z)^{-1} \\ &= z^n P_k(z) M_k(z)^{-1} + z^n \overleftarrow{M_k}(z)^{-1} Q_k(z). \end{aligned}$$

As $P_k(z)M_k(z)^{-1}$ is analytic in $\overline{\mathbb{D}}$,

$$T_k(z) = z^n \overleftarrow{M_k}(z)^{-1} Q_k(z) + O(z^n). \tag{2.2.7}$$

Next, we write $\overleftarrow{M_k}(z)^{-1}Q_k(z)$ in realization form, as follows. Write

$$\begin{aligned} M_k(0)^{*-1}\overleftarrow{M_k}(z) &= z^n I + L_{n-1}^{(k)} z^{n-1} + \cdots + L_0^{(k)}, \\ M_k(0)^{*-1} Q_k(z) &= Q_{n-1}^{(k)} z^{n-1} + \cdots + Q_0^{(k)}. \end{aligned}$$

Note that by Proposition 2.1.2(iii) ,

$$L_j^{(k)} = \begin{pmatrix} \overline{(\frac{p_{n-j,0}}{p_{00}})} & * \\ 0 & L_j^{(k-1)} \end{pmatrix}, k \geq m, j = 0, \dots, n. \tag{2.2.8}$$

By repeatedly using (2.2.8) we obtain,

$$L_j^{(k)} = \begin{pmatrix} \overline{(\frac{p_{n-j,0}}{p_{00}})} & & * & \\ & \ddots & & \\ & & \overline{(\frac{p_{n-j,0}}{p_{00}})} & \\ & \bigcirc & & L_j^{(m-1)} \end{pmatrix}, k \geq m, j = 0, \dots, n, \tag{2.2.9}$$

where $\overline{(\frac{p_{n-j,0}}{p_{00}})}$ appears $k-m+1$ times. By [7, Th. II.2.3] (we transpose twice to apply the result directly), we have

$$\overleftarrow{M_k}(z)^{-1} Q_k(z) = \hat{C}(zI - \hat{A})^{-1}\hat{B}, z \notin \Sigma(\overleftarrow{M_k}), \tag{2.2.10}$$

where

$$\hat{C} = \begin{pmatrix} 0 & \cdots 0 & I_{k+1} \end{pmatrix}, \hat{B} = \operatorname{col}(Q_j^{(k)})_{j=0}^{n-1}, \hat{A} = \begin{pmatrix} 0 & \cdots & 0 & -L_0^{(k)} \\ I & & 0 & -L_1^{(k)} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & -L_{n-1}^{(k)} \end{pmatrix},$$

which are of size $(k+1) \times n(k+1)$, $n(k+1) \times (k+1)$ and $n(k+1) \times n(k+1)$, respectively. The representation (2.2.10) is called a realization of the rational matrix function $\overleftarrow{M_k}^{-1} Q_k$ (see, e.g., [7]). Due to (2.2.9) we may apply a permutation $\hat{\pi}_k$ to $\hat{A}$ so that we obtain the following block upper triangular form

$$A := \hat{\pi}_k \hat{A} \hat{\pi}_k^{-1} = \begin{pmatrix} T & & * & \\ & \ddots & & \\ & & T & \\ & \bigcirc & & S' \end{pmatrix},$$

where

$$T = \begin{pmatrix} 0 & \cdots & 0 & -\overline{(\frac{p_{n0}}{p_{00}})} \\ 1 & & 0 & -\overline{(\frac{p_{n-1,0}}{p_{00}})} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -\overline{(\frac{p_{1,0}}{p_{00}})} \end{pmatrix}, \quad S' = \begin{pmatrix} 0 & \cdots & 0 & -L_0^{(m-1)} \\ I & & 0 & -L_1^{(m-1)} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & -L_{n-1}^{(m-1)} \end{pmatrix},$$

and the matrix T appears $k-m+1$ times in A. Notice that $\sigma(S') = \Sigma(\overleftarrow{M}_{m-1}) \subset \mathbb{D}$. The permutation $\hat{\pi}_k$ transforms $\hat{C}$ and $\hat{B}$ into

$$C := \hat{C}\hat{\pi}_k^{-1} = \begin{pmatrix} E_1 & & \bigcirc & \\ & \ddots & & \\ & & E_1 & \\ & \bigcirc & & E_m \end{pmatrix}, \quad B := \hat{\pi}_k \hat{B} = \begin{pmatrix} * \\ \vdots \\ * \\ W_k' \end{pmatrix},$$

where

$$E_l = \begin{pmatrix} 0 & \cdots & 0 & I_l \end{pmatrix}$$

is of size $l \times nl$ and $W_k' = \operatorname{col}(PQ_j^{(k)})_{j=0}^{n-1}$ with P the $m \times (k+1)$ matrix $P = [0 \; I_m]$. With the help of (2.2.6) it is straightforward to check that

$$W_k' = \begin{pmatrix} * & W_{k-1}' \end{pmatrix}, k \geq m. \tag{2.2.11}$$

Expanding (2.2.10), we now obtain from (2.2.7) and the definition of $T_k(z)$ that

$$(f_{i-l}(z))_{i,l=0}^{k} = \sum_{i=0}^{\infty} z^{n-i-1} CA^i B + O(z^n).$$

By taking the j^{th} Fourier coefficient on both sides, and writing only the last m rows, we get

$$\begin{aligned} H_{jk} &:= \begin{pmatrix} f_{j,k-m+1} & \cdots & f_{j0} & \cdots & f_{j,-m+1} \\ \vdots & & \vdots & \ddots & \vdots \\ f_{jk} & \cdots & f_{j,m-1} & \cdots & f_{j0} \end{pmatrix} \\ &= E_m (S')^{n-j-1} W_k', \qquad j \leq n-1, k \geq m-1. \end{aligned} \tag{2.2.12}$$

In a similar way, but now using Proposition 2.1.4, we obtain

$$\begin{aligned} \tilde{H}_{lj} &:= \begin{pmatrix} f_{0j} & \cdots & f_{-n+1,j} & \cdots & f_{-l,j} \\ \vdots & \ddots & \vdots & & \vdots \\ f_{n-1,j} & \cdots & f_{0j} & \cdots & f_{-l+n-1,j} \end{pmatrix} \\ &= F_n (\hat{S}^*)^{m+j-1} \hat{W}_l, \qquad j \geq -m+1, l \geq n-1, \end{aligned} \tag{2.2.13}$$

where $\sigma(\hat{S}) = \Sigma(\overleftarrow{\tilde{M}}_{n-1}) \subset \mathbb{D}$,

$$F_n = \begin{pmatrix} 0 & \cdots & 0 & I_n \end{pmatrix}$$

is of size $n \times nm$, and $\hat{W}_l$ is a matrix of size $nm \times l$ with the property that

$$\hat{W}_l = \begin{pmatrix} \hat{W}_{l-1} & * \end{pmatrix}, l \geq n.$$

Notice that H_{jk} defined in (2.2.12) and $\tilde{H}_{lj}$ defined in (2.2.13) are related in the following way

$$(H_{i-j,k})_{i=0,\ j=0}^{n-1\ \ l} = \pi_1[(\tilde{H}_{l,i-j})_{i=-m+1,\ j=-l}^{0\ \ \ \ 0}]\pi_2,$$

where π_1 and π_2^{-1} are appropriately chosen permutations (that convert reverse lexicographical ordering to lexicographical ordering). Notice that π_2 depends on k and l, but we will suppress this dependency. Combining (2.2.12) and (2.2.13) we therefore get

$$\begin{aligned} &\operatorname{col}(E_m(S')^{n-1-j})_{j=0}^{n-1}\operatorname{row}((S')^j W'_k)_{j=0}^{l} \\ &\qquad = \pi_1\operatorname{col}(F_n(\hat{S}^*)^j)_{j=0}^{m-1}\operatorname{row}((\hat{S}^*)^{k-j}\hat{W}_l)_{j=0}^{k}\pi_2. \end{aligned} \tag{2.2.14}$$

When $k = m-1$ and $l = n-1$, (2.2.14) equals the invertible $nm \times nm$ matrix $\Phi = (f_{u-v})_{u,v \in \{0,\dots,n-1\}\times\{0,\dots,m-1\}}$; i.e.,

$$\begin{aligned} \Phi &= \operatorname{col}(E_m(S')^{n-1-j})_{j=0}^{n-1}\operatorname{row}((S')^j W'_{m-1})_{j=0}^{n-1} \\ &= \pi_1\operatorname{col}(F_n(\hat{S}^*)^j)_{j=0}^{m-1}\operatorname{row}((\hat{S}^*)^{m-1-j}\hat{W}_{n-1})_{j=0}^{m-1}\pi_2. \end{aligned} \tag{2.2.15}$$

Thus the $nm \times nm$ matrices

$$\operatorname{col}(E_m(S')^{n-1-j})_{j=0}^{n-1}, \quad \operatorname{col}(F_n(\hat{S}^*)^j)_{j=0}^{m-1}, \quad \operatorname{row}((S')^j W'_{m-1})_{j=0}^{n-1}$$

and

$$\operatorname{row}((\hat{S}^*)^{m-1-j}\hat{W}_{n-1})_{j=0}^{m-1}$$

are all invertible. We now let

$$K = \operatorname{row}((S')^j W'_{m-1})_{j=0}^{n-1}\ ,\ L = \operatorname{row}((\hat{S}^*)^{m-1-j}\hat{W}_{n-1})_{j=0}^{m-1}\pi_2,$$

and put

$$E = E_m K\ ,\ S = K^{-1}S'K\ , \tilde{F} = F_n L\ ,\ \tilde{S} = L^{-1}\hat{S}^* L.$$

Then (2.2.15) yields

$$\Phi = \operatorname{col}(ES^{n-1-j})_{j=0}^{n-1} = \pi_1\operatorname{col}(\tilde{F}\tilde{S}^j)_{j=0}^{m-1}. \tag{2.2.16}$$

Let x denote the first row of E, which by (2.2.16) equals the $((n-1)m+1)^{\text{th}}$ row of Φ. As π_1 picks out the j^{th} scalar row from each block to make the j^{th} block, we have by (2.2.16) that x equals the last row of $\tilde{F}$. In fact, we obtain from (2.2.16) that

$$\tilde{F} = \operatorname{col}(xS^{n-1-j})_{j=0}^{n-1}, \quad E = \operatorname{col}(x\tilde{S}^j)_{j=0}^{m-1}, \tag{2.2.17}$$

and, more generally,

$$\tilde{F}\tilde{S}^i = \mathrm{col}(x\tilde{S}^i S^{n-1-j})_{j=0}^{n-1},$$
$$ES^r = \mathrm{col}(xS^r\tilde{S}^j)_{j=0}^{m-1}, i = 0,\dots,m-1; r = 0,\dots,n-1.$$

Also, let

$$W_k = K^{-1}W_k' \ , \ \tilde{W}_l = L^{-1}\hat{W}_l.$$

Then the definitions of K and L yield

$$I_{nm} = \mathrm{row}(S^jW_{m-1})_{j=0}^{n-1} = \mathrm{row}(\tilde{S}^{m-1-j}\tilde{W}_{n-1})_{j=0}^{m-1}\pi_2, \tag{2.2.18}$$

and, by (2.2.14) and (2.2.16),

$$\mathrm{row}(S^jW_k)_{j=0}^{l} = \mathrm{row}(\tilde{S}^{k-j}\tilde{W}_l)_{j=0}^{k}\pi_2, k \geq m-1, l \geq n-1. \tag{2.2.19}$$

Denoting the last column of W_k by y (which by (2.2.11) is independent of k), we get from (2.2.18) that y is the m^{th} column of I_{nm} and also equals the first column of $\tilde{W}_l$. In addition, from (2.2.19),

(2.2.20)
$$S^jW_k = \mathrm{row}(\tilde{S}^{k-r}S^jy)_{r=0}^{k}, \quad \tilde{S}^i\tilde{W}_l = \mathrm{row}(S^r\tilde{S}^iy)_{r=0}^{l}, \quad k \geq m-1, l \geq n-1.$$

In particular, $\tilde{W}_l = [y \ \cdots \ S^ly]$, and thus $\tilde{S}^i\tilde{W}_l = [\tilde{S}^iy \ \cdots \ \tilde{S}^iS^ly]$. Comparing this with the representation of $\tilde{S}^i\tilde{W}_l$ in (2.2.20) we obtain

$$S^j\tilde{S}^ky = \tilde{S}^kS^jy, k \geq 0, j \geq 0. \tag{2.2.21}$$

Since

$$\begin{aligned} I_{nm} &= \mathrm{row}(S^jW_{m-1})_{j=0}^{n-1} = \mathrm{row}(S^j\mathrm{row}(\tilde{S}^{m-1-r}y)_{r=0}^{m-1})_{j=0}^{n-1} \\ &= \mathrm{row}(\tilde{S}^{m-1-j}\tilde{W}_{n-1})_{j=0}^{m-1}\pi_2 = \mathrm{row}(\tilde{S}^{m-1-j}\mathrm{row}(S^ry)_{r=0}^{n-1})_{j=0}^{m-1}\pi_2, \end{aligned} \tag{2.2.22}$$

we have by (2.2.21) that $S\tilde{S} = \tilde{S}S$, and thus S and $\tilde{S}$ commute. It follows now from (2.2.12) that

$$H_{jk} = ES^{n-j-1}W_k = \mathrm{col}(x\tilde{S}^j)_{j=0}^{m-1}S^{n-j-1}\mathrm{row}(\tilde{S}^{k-r}y)_{r=0}^{k}.$$

By inspection (2.2.2) follows directly.

Moreover, using equation (2.2.16) we obtain

$$\begin{aligned} \Phi S &= (\mathrm{col}(ES^{n-j-1})_{j=0}^{n-1})S = \mathrm{col}(E_mS^{n-j-1})_{j=-1}^{n-2} \\ &= \mathrm{col}(E_mS^{n-j-1})_{j=-1}^{n-2}\mathrm{row}(S^jW_{m-1})_{j=0}^{n-1} = (H_{i-j,m-1})_{i=-1,\ j=0}^{n-2,\ n-1} = \Phi_1. \end{aligned}$$

Thus S is as in (2.2.3). Similarly, we obtain that $\tilde{S}$ is given by (2.2.3).

Finally, the fact that the infinite matrix (2.2.4) has rank nm follows from the observation that

(2.2.23)
$$(\widehat{f}(u-v))_{\substack{u\in\{\dots,n-2,n-1\}\times\{0,1,\dots\}\\ v\in\{0,1,\dots\}\times\{\dots,m-2,m-1\}}} = \mathrm{col}(xS^{n-1-k}\tilde{S}^j)_{(k,j)\in\{\dots,n-2,n-1\}\times\{0,1,\dots\}}$$
$$\times \mathrm{row}(S^k\tilde{S}^{m-1-j}y)_{(k,j)\in\{0,1,\dots\}\times\{\dots,m-2,m-1\}}.$$

□

It should be noticed that the proof of Theorem 2.2.1 also gives a way to derive formulas for the other Fourier coefficients of f. These now also involve the matrices

$$T = \begin{pmatrix} 0 & \cdots & 0 & -\overline{(\frac{p_{n0}}{p_{00}})} \\ 1 & & 0 & -\overline{(\frac{p_{n-1,0}}{p_{00}})} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -\overline{(\frac{p_{1,0}}{p_{00}})} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\overline{(\frac{p_{0m}}{p_{00}})} & -\overline{(\frac{p_{0,m-1}}{p_{00}})} & \cdots & -\overline{(\frac{p_{01}}{p_{00}})} \end{pmatrix}.$$

As those formulas do not play a critical role in the positive extension result, we do not pursue this here.

2.3. *Stability and spectral matching of a predictor polynomial.* Before we come to the positive extension result, we would first like to address the following question. Let $\Lambda_+ = \{0,\dots,n\}\times\{0,\dots,m\}$ and let complex numbers c_u, $u \in \Lambda_+ - \Lambda_+ = \{-n,\dots,n\}\times\{-m,\dots,m\}$ be given so that $(c_{u-v})_{u,v\in\Lambda_+} > 0$. Then we can define an inner product on the finite-dimensional space $\{\binom{z}{w}^v : v \in \Lambda_+\}$ by setting

$$\langle \begin{pmatrix} z \\ w \end{pmatrix}^v, \begin{pmatrix} z \\ w \end{pmatrix}^u \rangle_c = c_{v-u}.$$

When we perform a Gram-Schmidt orthogonalization procedure on the basis $\{\binom{z}{w}^v : v \in \Lambda_+\}$, we obtain polynomials $\phi_v(z,w)$, $v \in \{0,\dots,n\}\times\{0,\dots,m\}$. It is well known that in the one-variable case the reverses of these polynomials are stable and have a spectral matching property (see also Subsection 1.1.2). The following result states that under an additional condition on the numbers c_u the polynomial ϕ_{nm} has similar properties. As we shall see in the next section, the polynomial $\overleftarrow{\phi}_{nm}(z,w)$ yields exactly the solution to the positive extension result.

If $(c_{v,w})_{v\in M,w\in N}$ is a matrix whose entries are indexed by the sets M and N ($\subset \mathbb{Z}^2$, in our case), then

$$[(c_{v,w})_{v\in M,w\in N}]^{-1}_{\substack{A\\B}}$$

denotes the submatrix in its inverse that corresponds to the rows indexed by $A \subset N$ and columns indexed by $B \subset M$. When no specific statement is made about the ordering of the elements of M and N, one may choose any ordering. When $M = N$ we give the rows and the columns the same ordering.

THEOREM 2.3.1. *Let* $\Lambda_+ = \{0, \dots, n\} \times \{0, \dots, m\}$ *and* c_u, $u \in \Lambda_+ - \Lambda_+$, *be given so that* $(c_{u-v})_{u,v \in \Lambda_+} > 0$. *Put*

$$q(z,w) = \mathrm{row}\left(\begin{pmatrix} z \\ w \end{pmatrix}^u\right)_{u \in \Lambda_+} \left[(c_{u-v})_{u,v \in \Lambda_+}\right]^{-1} \mathrm{col}(\delta_u)_{u \in \Lambda_+}, \tag{2.3.1}$$

and let $p(z,w) = q(z,w)/\sqrt{q(0,0)}$. *The predictor polynomial* $p(z,w)$ *is stable and satisfies*

$$c_u = \frac{1}{(2\pi i)^2} \iint_{\mathbb{T}^2} \begin{pmatrix} z \\ w \end{pmatrix}^{-u} \frac{1}{|p(z,w)|^2} \frac{dz}{z} \frac{dw}{w}, u \in \Lambda_+ - \Lambda_+, \tag{2.3.2}$$

if and only if

$$\left[(c_{u-v})_{u,v \in \Lambda_+ \setminus \{(0,0)\}}\right]^{-1}_{\substack{\{1,\dots,n\} \times \{0\} \\ \{0\} \times \{1,\dots,m\}}} = 0. \tag{2.3.3}$$

It should be noted that it may happen that $p(z,w)$ is stable without condition (2.3.3) being satisfied (after all, the set of stable pseudopolynomials is open). However, in that case (2.3.2) does not hold. The following example illustrates this.

Example 2.3.2. Let $\Lambda_+ = \{0,1\} \times \{0,1\}$, and $c_{00} = 1, c_{01} = \frac{1}{4} = c_{1,-1}, c_{10} = 0 = c_{11}$. Then $(c_{u-v})_{u,v \in \Lambda_+} > 0$ and,

$$p(z,w) = (224 - 60w - 16z - 4zw)/\sqrt{46816}.$$

It is easy to see that $p(z,w)$ is stable. Computing the Fourier coefficients of $f(z,w) = 1/(p(z,w)\overline{p}(1/z,1/w)$ yields

$$\begin{array}{rcl} \widehat{f}(0,0) & \approx & 1.0104,\ \widehat{f}(0,1) \approx 0.2702,\ \widehat{f}(1,0) \approx -0.0725, \\ \widehat{f}(1,1) & \approx & -0.2007,\ \widehat{f}(1,-1) \approx -0.0194. \end{array}$$

The proof of the above theorem depends heavily on the theory of matrix polynomials orthogonal on the unit circle, therefore we recall some results from [17]. As usual, we denote the halfspaces associated with the lexicographical ordering and reverse lexicographical ordering by H and $\tilde{H}$, respectively. Let

$$\Gamma_n^k = \begin{bmatrix} C_0^k & C_{-1}^k & \cdots & C_{-n}^k \\ C_1^k & C_0^k & \cdots & C_{1-n}^k \\ \vdots & \vdots & \ddots & \vdots \\ C_n^k & C_{n-1}^k & \cdots & C_0^k \end{bmatrix},$$

where $C^k_{-i} = (C^k_i)^*$ is the $(k+1) \times (k+1)$ Toeplitz matrix given by

$$C^k_i = \begin{bmatrix} c_{i,0} & \cdots & c_{i,-k} \\ \vdots & \ddots & \vdots \\ c_{i,k} & \cdots & c_{i,0} \end{bmatrix}, \qquad i = -n, \dots, n.$$

Likewise, in reverse lexicographic order, set

$$\tilde{\Gamma}^k_m = \begin{bmatrix} \tilde{C}^k_0 & \tilde{C}^k_{-1} & \cdots & \tilde{C}^k_{-m} \\ \tilde{C}^k_1 & \tilde{C}^k_0 & \cdots & \tilde{C}^k_{1-m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{C}^k_m & \tilde{C}^k_{m-1} & \cdots & \tilde{C}^k_0 \end{bmatrix},$$

where $\tilde{C}^k_{-i} = (\tilde{C}^k_i)^*$ is the $(k+1) \times (k+1)$ Toeplitz matrix given by

$$\tilde{C}^k_i = \begin{bmatrix} c_{0,i} & \cdots & c_{-k,i} \\ \vdots & \ddots & \vdots \\ c_{k,i} & \cdots & c_{0,i} \end{bmatrix}, \qquad i = -m, \dots, m.$$

Observe that in the lexicographical ordering $(c_{u-v})_{u,v\in\Lambda_+} = \Gamma^m_n$ while in the reverse lexicographical ordering $(c_{u-v})_{u,v\in\Lambda_+} = \tilde{\Gamma}^n_m$.

Given $\Gamma_s = (C_{i-j})^s_{i,j=0} > 0$ with C_l being matrices of size $r \times r$, we set

$$A_s(x) = [I_r \, xI_r \, \cdots \, x^s \, I_r]\Gamma_s^{-1}[I_r \, 0 \, \cdots \, 0]^T$$

and

$$B_s(x) = [0 \, \cdots 0 \, I_r]\Gamma_s^{-1}[x^s I_r \cdots xI_r \, I_r]^T,$$

where I_r is the $r \times r$ identity matrix. Then one of the versions of the matrix Christoffel-Darboux formula (formula (66) of Theorem 13 in [17]) yields

$$\begin{aligned} (2.3.4) \quad & (1 - x\overline{x}_1)[I_r \; xI_r \cdots x^s I_r]\Gamma_s^{-1}[I_r \; x_1 I_r \cdots x_1^s I_r]^* \\ &= A_s(x)A_s(0)^{-1}A_s(x_1)^* - (x\overline{x}_1)^{s+1}B_s\left(\frac{1}{\overline{x}}\right)^* B_s(0)^{-1}B_s\left(\frac{1}{\overline{x}_1}\right). \end{aligned}$$

If we let U_s denote the upper Cholesky factor of Γ_s^{-1}, then

$$U_s = \begin{pmatrix} U_{s-1} & * \\ 0 & X_{ss} \end{pmatrix}$$

for some matrix X_{ss}, and

$$B_s(x) = X_{ss}[0 \, \cdots 0 \, I_r]U_s^*[x^s I_r \cdots xI_r \, I_r]^T.$$

Using this we see easily that

$$\begin{aligned} (2.3.5) \qquad & [I_r \; xI_r \cdots x^s I_r]\Gamma_s^{-1}[I_r \; x_1 I_r \cdots x_1^s I_r]^* \\ &= [I_r \; xI_r \cdots x^{s-1} I_r]\Gamma_{s-1}^{-1}[I_r \; x_1 I_r \cdots x_1^{s-1} I_r]^* \\ &\quad + (x\bar{x}_1)^s B_s(\frac{1}{\bar{x}})^* B_s(0)^{-1} B_s(\frac{1}{\bar{x}_1}). \end{aligned}$$

But then (2.3.4) and (2.3.5) give the useful variation of the matrix Christoffel-Darboux formula:

$$(2.3.6)\quad (1-x\overline{x}_1)[I_r\ xI_r \cdots x^{s-1}I_r]\Gamma_{s-1}^{-1}[I_r\ x_1I_r\cdots x_1^{s-1}I_r]^* \\ = A_s(x)A_s(0)^{-1}A_s(x_1)^* - (x\overline{x}_1)^s B_s\left(\frac{1}{\overline{x}}\right)^* B_s(0)^{-1}B_s\left(\frac{1}{\overline{x}_1}\right).$$

An important property given by [17, Th. 6] is that if Γ_k is positive then $A(x)$ is stable. If the matrices C_l are themselves Toeplitz matrices, they satisfy $C_l = J_{r-1}C_l^TJ_{r-1}$, where $J_r = (\delta_{i+j-r})_{i,j=0}^r$. This yields that $B(x) = J_{r-1}A(x)^TJ_{r-1}$, as was also observed in [18, after Th. 9] . We will apply the above result to the cases when $C_l = C_l^m$ and when $C_l = C_l^{m-1}$. Equivalently, these are the cases when $\Gamma_s = \Gamma_n^m$ and when $\Gamma_s = \Gamma_n^{m-1}$, respectively. We therefore define for $i = m-1, m$,

$$(2.3.7)\quad A_n^i(z) = [I_{i+1}\ zI_{i+1}\ \cdots\ z^nI_{i+1}](\Gamma_n^i)^{-1}[I_{i+1}\ 0\ \cdots\ 0]^T,$$
$$B_n^i(z) = [0\ \cdots\ 0\ I_{i+1}](\Gamma_n^i)^{-1}[z^nI_{i+1}\ z^{n-1}I_{i+1}\ \cdots\ I_{i+1}]^T.$$

Likewise, for the reverse lexicographical order, we define for $i = n-1, n$,

$$(2.3.8)\quad \tilde{A}_m^i(w) = [I_{i+1}\ wI_{i+1}\ \cdots\ w^mI_{i+1}](\tilde{\Gamma}_m^i)^{-1}[I_{i+1}\ 0\ \cdots\ 0]^T,$$
$$\tilde{B}_m^i(w) = [0\ \cdots\ 0\ I_{i+1}](\tilde{\Gamma}_m^i)^{-1}[w^mI_{i+1}\ w^{n-1}I_{i+1}\ \cdots\ I_{i+1}]^T.$$

The matrices $B_n^i(z)$ and $\tilde{B}_m^i(w)$ satisfy $B_n^i(z) = J_iA_n^i(z)^TJ_i$ and $\tilde{B}_m^i(w) = J_i\tilde{A}_m^i(w)^TJ_i$. Let L_n^i be the lower Cholesky factor of $(\Gamma_n^i)^{-1}, i = m-1, m$. We then define

$$(2.3.9)$$
$$P^i(z,w) := [1\ w\ \cdots\ w^i][I_{i+1}\ zI_{i+1}\ \cdots\ z^nI_{i+1}]L_n^i[I_{i+1}\ 0\ \cdots\ 0]^T$$
$$= [1\ w\ \cdots\ w^i][I_{i+1}\ zI_{i+1}\ \cdots\ z^nI_{i+1}](\Gamma_n^i)^{-1}[((Y_n^i)^{-1})^T\ 0\ \cdots\ 0]^T$$
$$= [1\ w \cdots w^i]A_n^i(z)(Y_n^i)^{-1},$$

where $A_n^i(z)$ is given by (2.3.7) and $(Y_n^i)^*$ is the lower Cholesky factor of $A_n^i(0)$. From the relation between $A_n^i(z)$ and $B_n^i(z)$, and from $B_n^i(z) = J_iA_n^i(z)^TJ_i$ we see that for $i = m-1, m$,

$$(2.3.10)$$
$$[\overleftarrow{P}^i(z,w)]^T := z^nw^i[P^i(1/\bar{z}, 1/\bar{w})^*]^T = [1\ w\ \cdots\ w^i]z^nB_n^i(1/\bar{z})^*(X_n^i)^{*-1}J_i,$$

where $X_n^i (= J_i(Y_n^i)^TJ_i)$ is the upper Cholesky factor of $B_n^i(0)$. It follows from the definition of $p(z,w)$ in Theorem 2.3.1 that the first column of P^m is $p(z,w)$. Thus we shall write

$$(2.3.11)\quad P^m(z,w) = [p(z,w)\quad wP^{(1)}(z,w)],$$

where $P^{(1)}(z,w)$ is some row-valued polynomial in z and w. From the definition for P^m we find

(2.3.12)
$$[\overleftarrow{P}^m(z,w)]^T = z^n w^m [\overline{p}(\frac{1}{z},\frac{1}{w}) \quad \frac{1}{w}(P^{(1)}(\frac{1}{\overline{z}},\frac{1}{\overline{w}})^*)^T] = [\overleftarrow{p}(z,w) \quad \overleftarrow{P^{(1)}}(z,w)^T].$$

Likewise for $i = n-1, n$ set

(2.3.13)
$$\begin{aligned}\tilde{P}^i(z,w) &:= [1\, z \,\cdots\, z^i][I_{i+1}\, wI_{i+1} \,\cdots\, w^m\, I_{i+1}]\tilde{L}^i_m[I_{i+1}\, 0 \,\cdots\, 0]^T \\ &= [1\, z \,\cdots\, z^i][I_{i+1}\, wI_{i+1} \,\cdots\, w^m\, I_{i+1}](\tilde{\Gamma}^i_m)^{-1}[((\tilde{Y}^i_m)^{-1})^T\, 0 \,\cdots\, 0]^T \\ &= [1\, z \cdots z^i]\tilde{A}^i_m(w)(\tilde{Y}^i_m)^{-1},\end{aligned}$$

where $\tilde{L}^i_m$ is the lower Cholesky factor of $(\tilde{\Gamma}^i_m)^{-1}$ and $(\tilde{Y}^i_m)^*$ is the lower Cholesky factor of $\tilde{A}^i_m(0)$. Also

(2.3.14)
$$[\overleftarrow{\tilde{P}}^i(z,w)]^T := z^i w^m [\tilde{P}^i(1/\bar{z},1/\bar{w})^*]^T = [1\, z \,\cdots\, z^i] w^m \tilde{B}^i_m(1/\bar{w})^*(\tilde{X}^i_w)^{*-1}\tilde{J}_i.$$

Similarly, as above,

$$\tilde{P}^n(z,w) = [p(z,w) \quad w\tilde{P}^{(1)}(z,w)], \tag{2.3.15}$$

for some row-valued polynomial $\tilde{P}^{(1)}(z,w)$.

We now state a Christoffel-Darboux-like formula.

Proposition 2.3.3. *Let $\Lambda_+ = \{0,\ldots,n\} \times \{0,\ldots,m\}$ and c_v, $v \in \Lambda_+ - \Lambda_+$, be given so that $(c_{u-t})_{u,t\in\Lambda_+} > 0$ and*

$$\left[(c_{u-v})_{u,v\in\Lambda_+\setminus\{(0,0)\}}\right]^{-1}_{\substack{\{1,\ldots,n\}\times\{0\} \\ \{0\}\times\{1,\ldots,m\}}} = 0 \tag{2.3.16}$$

holds. Then

$$\begin{aligned}p(z,w)\overline{p(z_1,w_1)} - \overleftarrow{p}(z,w)\overline{\overleftarrow{p}(z_1,w_1)} &= (1-w\overline{w}_1)P^{m-1}(z,w)P^{m-1}(z_1,w_1)^* \\ &\quad +(1-z\overline{z}_1)\overleftarrow{\tilde{P}^{n-1}}(z,w)^T\overleftarrow{\tilde{P}^{n-1}}(z_1,w_1)^{*T}.\end{aligned} \tag{2.3.17}$$

We need the following observation regarding Cholesky factors.

Lemma 2.3.4. *Let A be a positive definite $r \times r$ matrix and suppose that for some $1 \le j < k \le r$ $(A^{-1})_{kl} = 0$, $l = 1,\ldots,j$. Then the lower Cholesky factor L of A^{-1} satisfies $L_{kl} = 0$, $l = 1,\ldots,j$. Moreover, if $\tilde{A}$ is the $(r-1)\times(r-1)$ matrix obtained from A by removing the k^{th} row and column, and $\tilde{L}$ is the lower Cholesky factor of $\tilde{A}^{-1}$, then*

$$L_{il} = \tilde{L}_{il}, \; i = 1,\ldots,k-1; l = 1,\ldots,j, \tag{2.3.18}$$

and

$$L_{i+1,l} = \tilde{L}_{il}, \; i = k, \dots, r-1; l = 1, \dots, j. \tag{2.3.19}$$

In other words, the first j columns of L and $\tilde{L}$ coincide after the k^{th} row (which contains zeros in columns $1, \dots, j$) in L has been removed.

Proof. Since the first j columns of a lower Cholesky factor of a matrix M are linear combinations of the first j columns of M, the first statement follows. The second part follows from the above observation and the following general rule: if $M = (M_{ij})_{i,j=1}^3$ is an invertible block matrix with square diagonal entries, $(M_{ij})_{i,j=1}^2$ is invertible, and $(N_{ij})_{i,j=1}^3 = M^{-1}$ satisfies $N_{13} = 0$, then

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}^{-1} = \begin{pmatrix} N_{11} & * \\ N_{21} & * \end{pmatrix}.$$

To see this, write out the first two rows of the product $MN = I$ to see that

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} N_{11} \\ N_{21} \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}. \qquad \square$$

Proof of Proposition 2.3.3. We use the notation introduced in this section. We first show that condition (2.3.16) and a repeated use of Lemma 2.3.4 imply the following equalities:

$$P^{(1)}(z,w) = P^{m-1}(z,w), \qquad \tilde{P}^{(1)}(z,w) = \tilde{P}^{n-1}(z,w), \tag{2.3.20}$$

where $P^{(1)}$ and $\tilde{P}^{(1)}$ are as introduced in (2.3.11) and (2.3.15), and P^{m-1} and $\tilde{P}^{m-1}$ are as defined in (2.3.9) and (2.3.14), respectively. Indeed, for the first equality in (2.3.20) observe that (2.3.11) and (2.3.9) yield

$$\begin{aligned} P^m(z,w) &= [p(z,w) \;\; wP^{(1)}(z,w)] \\ &= [1 \cdots w^m][I_{m+1} \; zI_{m+1} \cdots \; z^n I_{m+1}] L_n^m [I_{m+1} \, 0 \cdots 0]^T. \end{aligned}$$

Denoting by $\hat{L}$ the matrix obtained from L_n^m by removing its first row and column, we find that

$$P^{(1)}(z,w) = [1 \cdots w^{m-1} \, \frac{z}{w} \cdots zw^{m-1} \cdots\cdots \frac{z^n}{w} \cdots z^n w^{m-1}] \hat{L} [I_m \, 0 \cdots 0]^T.$$

By (2.3.16) the matrix $\hat{L}$ contains zeros in the first m columns at rows $mj+1$, $j = 1, \dots, n$. Repeated use of Lemma 2.3.4 now gives that

$$P^{(1)}(z,w) = [1 \cdots w^{m-1}][I_m \; zI_m \cdots \; z^n I_m] L_n^{m-1} [I_m \, 0 \cdots 0]^T = P^{m-1}(z,w).$$

This yields the first equality in (2.3.20). The second equality follows analogously.

We now prove (2.3.17). Apply (2.3.6) with $\Gamma_{s-1} = \Gamma_{n-1}^m$ and multiply (2.3.6) with $[1\, w\, \cdots\, w^m]$ on the left and $[1\, w_1\, \cdots\, w_1^m]^*$ on the right to obtain

$$\begin{aligned}(2.3.21)\quad &(1-z\overline{z_1})[1\, w\, \cdots\, w^m][I_{m+1}\cdots\, z^{n-1}I_{m+1}]\\ &\quad\times(\Gamma_{n-1}^m)^{-1}[I_{m+1}\cdots z_1^{n-1}I_{m+1}]^*[1\, w_1\, \cdots\, w_1^m]^*\\ &= [1\, w\, \cdots\, w^m](A_n^m(z)(Y_n^m)^{-1}(Y_n^{m*})^{-1}A_n^m(z_1)^*\\ &\quad -(z\overline{z_1})^n B_n^m(\frac{1}{\overline{z}})^*(X_n^{m*})^{-1}(X_n^m)^{-1}B_n^m(\frac{1}{\overline{z_1}}))[1\, w_1\, \cdots\, w_1^m]^*.\end{aligned}$$

Next, use (2.3.9), (2.3.10), (2.3.11), (2.3.12) and (2.3.20) to obtain

$$\begin{aligned}(2.3.22)\quad &(1-z\overline{z}_1)[1\cdots w^m][I_{m+1}\cdots z^{n-1}I_{m+1}]\\ &\quad\times(\Gamma_{n-1}^m)^{-1}[I_{m+1}\cdots z_1^{n-1}I_{m+1}]^*[1\cdots w_1^m]^*\\ &\quad = P^m(z,w)P^m(z_1,w_1)^* - [\overleftarrow{P^m}(z,w)]^T[\overleftarrow{P^m}(z_1,w_1)^*]^T\\ &\quad = p(z,w)\overline{p(z_1,w_1)} + w\overline{w}_1P^{m-1}(z,w)P^{m-1}(z_1,w_1)^*\\ &\quad -\overleftarrow{p}(z,w))\overline{\overleftarrow{p}(z,w)} - \overleftarrow{P}^{m-1}(z,w)^T\overleftarrow{P}^{m-1}(z_1,w_1)^{*T}.\end{aligned}$$

Applying now (2.3.6) with $\Gamma_{s-1} = \Gamma_{n-1}^{m-1}$, multiplying with $[1\, w\, \cdots\, w^{m-1}]$ on the right and $[1\, w_1\, \cdots\, w_1^{m-1}]^*$ on the left gives

(2.3.23)

$$\begin{aligned}&P^{m-1}(z,w)P^{m-1}(z_1,w_1)^* - \overleftarrow{P}^{m-1}(z,w)^T\overleftarrow{P}^{m-1}(z_1,w_1)^{*T}\\ &\quad = (1-z\overline{z}_1)[1\cdots w^{m-1}][I_m\cdots z^{n-1}I_m](\Gamma_{n-1}^{m-1})^{-1}[I_m\cdots z_1^{n-1}I_m]^*[1\cdots w_1^{m-1}]^*.\end{aligned}$$

Subtracting (2.3.23) from (2.3.22) yields

(2.3.24)

$$\begin{aligned}&p(z,w)\overline{p(z,w)} - \overleftarrow{p}(z,w))\overline{\overleftarrow{p}(z,w)}\\ &\quad = (1-w\overline{w}_1)P^{m-1}(z,w)P^{m-1}(z_1,w_1)^*\\ &\qquad +(1-z\overline{z}_1)([1\, w\, \cdots\, w^m][I_{m+1}\cdots z^{n-1}I_{m+1}]\\ &\qquad \times(\Gamma_{n-1}^m)^{-1}[I_{m+1}\cdots z_1^{n-1}I_{m+1}]^*[1\cdots w_1^m]^*\\ &\qquad -[1\cdots w^{m-1}][I_m\cdots z^{n-1}I_m](\Gamma_{n-1}^{m-1})^{-1}[I_m\cdots z_1^{n-1}I_m]^*[1\cdots w_1^{m-1}]^*).\end{aligned}$$

Next we put the rows and columns of Γ_{n-1}^m in reverse lexicographical order and note that Γ_{n-1}^m becomes $\tilde{\Gamma}_m^{n-1}$. Thus

(2.3.25)

$$\begin{aligned}&[1\, w\, \cdots\, w^m][I_{m+1}\cdots z^{n-1}I_{m+1}](\Gamma_{n-1}^m)^{-1}[I_{m+1}\cdots z_1^{n-1}I_{m+1}]^*[1\cdots w_1^m]^*\\ &\quad = [1\, z\, \cdots\, z^{n-1}][I_n\cdots w^mI_n](\tilde{\Gamma}_m^{n-1})^{-1}[I_n\cdots w_1^mI_n]^*[1\cdots z_1^{n-1}]^*\\ &\quad = \overleftarrow{\tilde{P}^{n-1}}(z,w)^T\overleftarrow{\tilde{P}^{n-1}}(z,w)^{*T}\\ &\qquad +[1\, z\, \cdots\, z^{n-1}][I_n\cdots w^{m-1}I_n](\tilde{\Gamma}_{m-1}^{n-1})^{-1}[I_n\cdots w_1^{m-1}I_n]^*[1\cdots z_1^{n-1}]^*,\end{aligned}$$

where in the last equality we use an observation as in (2.3.5). Since $\tilde{\Gamma}^{n-1}_{m-1}$ and Γ^{m-1}_{n-1} are just reorderings of each other we finally obtain by combining (2.3.24) and (2.3.25)

$$\begin{aligned} p(z,w)\overline{p(z,w)} - \overleftarrow{p}(z,w)\overline{\overleftarrow{p}(z,w)} \\ &= (1-w\overline{w}_1)P^{m-1}(z,w)P^{m-1}(z_1,w_1)^* \\ &\quad +(1-z\overline{z}_1)\overleftarrow{\tilde{P}^{n-1}}(z,w)^T\overleftarrow{\tilde{P}^{n-1}}(z_1,w_1)^{*T}, \end{aligned}$$

which is the desired resulting equation. □

With the above result we can now prove Theorem 2.3.1. First we remind the reader of the following useful well known fact (see [45]; see also Theorem 2.5 in [67]).

Lemma 2.3.5. *Let A be a matrix of size $p \times q$ and D be a matrix of size $(n-p)\times(n-q)$ and let B, C, P, Q, R, S be matrices of appropriate sizes so that*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}.$$

Then

$$q - \operatorname{rank} C = p - \operatorname{rank} R.$$

In particular, $R=0$ *if and only if* $\operatorname{rank} C = q-p$.

For the sake of completeness we shall provide a proof for this lemma.

Proof. Since $CP = -DR$, $P[\ker R] \subseteq \ker C$. Likewise, since $RA = -SC$, we get $A[\ker C] \subseteq \ker R$. Consequently,

$$AP[\ker R] \subseteq A[\ker C] \subseteq \ker R.$$

Since $AP + BR = I$, $AP[\ker R] = \ker R$, and thus

$$A[\ker C] = \ker R.$$

This yields $\dim\ker C \geq \dim\ker R$. By reversing the roles of C and R one obtains also that $\dim\ker R \geq \dim\ker C$. This gives $\dim\ker R = \dim\ker C$, yielding the lemma. □

Proof of Theorem 2.3.1. Let $\Lambda_+ = \{0,\dots,n\}\times\{0,\dots,m\}$ and c_u, $u \in \Lambda_+ - \Lambda_+$, be given so that $(c_{u-v})_{u,v\in\Lambda_+} > 0$ and (2.3.3) holds. First we show that $p(z,w)$ is stable. Set $z_1 = z$ and $w_1 = w$, $|w| = 1$ in (2.3.17), to obtain

$$|p(z,w)|^2 - |\overleftarrow{p}(z,w)|^2 = (1-|z|^2)\overleftarrow{\tilde{P}^{n-1}}(z,w)\overleftarrow{\tilde{P}^{n-1}}(z,w)^*.$$

If $p(z_0,w_0)=0$ in the region $|z|<1$ and $|w|=1$ then the above equation and equation (2.3.14) imply that $\tilde{B}_m^{n-1}(w_0)^*$ must have a left eigenvector with eigenvalue zero. However, this leads to a contradiction since $\det(\tilde{B}_m^{n-1}(w_0)^*) \neq 0$ for $|w|=1$. A similar argument also applies for the region $|w|<1, |z|=1$. If $p(z_0,w_0)=0$ with $|z_0|=1=|w_0|$ then so does $\overleftarrow{p}(z_0,w_0)$. From (2.3.17) with $z_1=z_0$ we find that this would imply that $P^{m-1}(z_0,w_0)P^{m-1}(z_0,w_1)^*=0$ for arbitrary $|w_1|<1$. However from (2.3.9) with $z=z_0$ we see this cannot happen since $\det(A_n^{m-1}(z_0))\neq 0$. It now follows from Theorem 2.1.5(iii) that $p(z,w)$ is stable.

Next we show that $p(z,w)$ satisfies equation (2.3.2). We begin by writing $p(z,w)=\sum_{i=0}^m p_i(z)w^i$. Then straightforward algebraic manipulations (or, alternatively, see [51, §4]) show that

(2.3.26)

$$\frac{p(z,w)\overline{p(1/\bar{z},w_1)}-w\bar{w}_1\overleftarrow{p}(z,w)\overline{\overleftarrow{p}(1/\bar{z},w_1)}}{1-w\bar{w}_1}$$
$$=(1,\dots,w^m)\left(\begin{bmatrix} p_0(z) & & \bigcirc \\ \vdots & \ddots & \\ p_m(z) & \cdots & p_0(z)\end{bmatrix}\begin{bmatrix}\bar{p}_0(1/z) & \cdots & \bar{p}_m(1/z)\\ & \ddots & \vdots \\ \bigcirc & & \bar{p}_0(1/z)\end{bmatrix}\right.$$
$$\left.-\begin{bmatrix}\bar{p}_{m+1}(1/z) & & \bigcirc \\ \vdots & \ddots & \\ \bar{p}_1(1/z) & \cdots & \bar{p}_{m+1}(1/z)\end{bmatrix}\begin{bmatrix} p_{m+1}(z) & \cdots & p_1(z)\\ & \ddots & \vdots \\ \bigcirc & & p_{m+1}(z)\end{bmatrix}\right)\begin{pmatrix}1\\ \vdots \\ \bar{w}_1^m\end{pmatrix},$$

where $p_{m+1}(z)\equiv 0$. Furthermore, by (2.3.17) with $z_1=1/\bar{z}$,

$$\frac{p(z,w)\overline{p(1/\bar{z},w_1)}-\overleftarrow{p}(z,w)\overline{\overleftarrow{p}(1/\bar{z},w_1)}}{1-w\bar{w}_1}=P^{m-1}(z,w)P^{m-1}(1/\bar{z},w_1)^*.$$

Multiplication of both sides by $w\bar{w}_1$ and addition of $p(z,w)\overline{p(1/\bar{z},w_1)}$ to both sides yields

(2.3.27)

$$\frac{p(z,w)\overline{p(1/\bar{z},w_1)}-w\bar{w}_1\overleftarrow{p}(z,w)\overline{\overleftarrow{p}(1/\bar{z},w_1)}}{1-w\bar{w}_1}=P^{m}(z,w)P^{m}(1/\bar{z},w_1)^*,$$

where we used that

$$P^m(z,w)=[p(z,w)\ wP^{m-1}(z,w)].$$

Combining (2.3.26), (2.3.27), and (2.3.9) we find

(2.3.28)
$$\begin{aligned} E_m(z) &= \begin{bmatrix} p_0(z) & & \bigcirc \\ \vdots & \ddots & \\ p_m(z) & \cdots & p_0(z) \end{bmatrix} \begin{bmatrix} \bar{p}_0(1/z) & \cdots & \bar{p}_m(1/z) \\ & \ddots & \vdots \\ \bigcirc & & \bar{p}_0(1/z) \end{bmatrix} \\ &\quad - \begin{bmatrix} \bar{p}_{m+1}(1/z) & & \bigcirc \\ \vdots & \ddots & \\ \bar{p}_1(1/z) & \cdots & \bar{p}_{m+1}(1/z) \end{bmatrix} \begin{bmatrix} p_{m+1}(z) & \cdots & p_1(z) \\ & \ddots & \vdots \\ \bigcirc & & p_{m+1}(z) \end{bmatrix} \\ &= A_n^m(z) A_n^m(0)^{-1} A_n^m(1/\bar{z})^*. \end{aligned}$$

Recall that $A_n^m(z)$ is stable [17, Th. 6]. Therefore, on the unit circle we find that $E_m(z) > 0$. Let $F(z) = E_m(z)^{-1}$ and write

$$F(z) = \sum_{-\infty}^{\infty} F_n z^n.$$

Note that by the Gohberg-Semencul formula $F(z)$ is Toeplitz for every z. Furthermore, we get, using the stability of $A_n^m(z)$, that

$$F(z)A_n^m(z) = A_n^m(1/\bar{z})^{*-1} A_n^m(0) = I + O(1/z).$$

Comparing the $0, \dots, n$ Fourier coefficients on both sides yields the equation

$$\begin{pmatrix} F_0 & \cdots & F_{-n} \\ \vdots & \ddots & \vdots \\ F_n & \cdots & F_0 \end{pmatrix} \begin{pmatrix} A_0 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} I \\ \vdots \\ 0 \end{pmatrix}, \tag{2.3.29}$$

where $A_n^m(z) = \sum_{i=0}^n A_i z^i$. On the other hand, by the definition (2.3.7) of $A_n^m(z)$

$$\begin{pmatrix} C_0^m & \cdots & C_{-n}^m \\ \vdots & \ddots & \vdots \\ C_n^m & \cdots & C_0^m \end{pmatrix} \begin{pmatrix} A_0 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} I \\ \vdots \\ 0 \end{pmatrix}. \tag{2.3.30}$$

By the matrix version of the Gohberg-Semencul formula (see [38]) a positive definite block Toeplitz matrix is uniquely determined by the first block column of its inverse. It therefore follows that the equations (2.3.29) and (2.3.30) are the same, or in other words,

$$C_l^m = F_l, l = -n, \dots, n. \tag{2.3.31}$$

Since $F(z)$ is Toeplitz we may write $F(z) = (f_{i-j}(z))_{i,j=0}^m$. Fix $z \in \mathbb{T}$. By (2.3.28) we may view $p(z,w) = \sum_{i=0}^m p_i(z)w^i$ as the polynomial in w formed from taking the first column of the lower Cholesky factor of $F(z)^{-1}(= E_m(z))$. But then the one-variable theory (see subsection 1.1.2) gives that

$$f_l(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-il\theta}}{|p(z,e^{i\theta})|^2}d\theta, \qquad l=-m,\dots,m.$$

Now (2.3.31) yields that for $l=-m,\dots,m$,

$$\begin{aligned} c_{kl} = \hat{f}_l(k) &= \frac{1}{2\pi}\int_0^{2\pi} f_l(e^{i\eta})e^{-ik\eta}d\eta \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \frac{e^{-il\theta-ik\eta}}{|p(e^{i\eta},e^{i\theta})|^2}d\theta d\eta, k=-n,\dots,n. \end{aligned}$$

This proves (2.3.2)

For the converse, let $p(z,w)$ be stable. Observe that c_u defined in (2.3.2) is the u^{th} Fourier coefficient of the spectral density function associated with $p(z,w)$. But then it follows directly from Theorem 2.2.1 that

$$\begin{aligned} nm &= \operatorname{rank}\,\Phi \le \operatorname{rank}(c_{u-v})_{\substack{u\in\{1,\dots,n\}\times\{0,\dots,m\} \\ v\in\{0,\dots,n\}\times\{1,\dots,m\}}} \\ &= \operatorname{rank}(c_{v-u})_{\substack{v\in\{-1,\dots,n-1\}\times\{0,\dots,m-1\} \\ u\in\{0,\dots,n-1\}\times\{-1,\dots,m-1\}}} \le nm. \end{aligned}$$

Now, by Lemma 2.3.5 we obtain (2.3.3). □

2.4. *Positive extensions.* Let $H=\{(n,m): n>0 \text{ or } (n=0 \text{ and } m>0)\}$ be the standard halfspace in $\mathbb{Z}^2$, and let Λ_+ be a finite set in $H\cup\{(0,0)\}$ containing $(0,0)$. We consider the following problem which arises in the design of autoregressive filters. For given complex numbers c_{kl}, $(k,l)\in\Lambda_+$, find if possible a pseudopolynomial

$$p(z,w) = \sum_{(k,l)\in\Lambda_+} c_{kl}z^k w^l, \qquad |z|=|w|=1,$$

so that

(i) $p(z,w)$ is stable

(ii) $\frac{1}{|p(z,w)|^2}$ has Fourier coefficients $c_{k,l}$ for $(k,l)\in\Lambda_+$.

In the one-variable case where $\Lambda_+ = \{0,1,2,\dots,n\}$ the necessary and sufficient condition is that the finite Toeplitz matrix

$$C = \begin{bmatrix} c_0 & \cdots & c_{-n} \\ \vdots & \ddots & \vdots \\ c_n & \cdots & c_0 \end{bmatrix}$$

is positive definite, where $c_{-k}=\bar{c}_k$ for $k\in\{1,\dots,n\}$. In that case, the desired polynomial equals

$$p(z) = p_0^{-1/2}(p_0+p_1z+\cdots+p_nz^n), \qquad |z|=1,$$

where

$$\begin{bmatrix} p_0 \\ \vdots \\ p_n \end{bmatrix} = C^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

In this section we shall give necessary and sufficient conditions for the two-variable problem in terms of positive definite matrix completions. We start with the case when

$$\Lambda_+ = \{0, \dots, n\} \times \{0, \dots, m\}.$$

As usual, we denote by δ_u the Kronecker delta on $\mathbb{Z}^2$, i.e., $\delta_u = 0$ for $u \neq (0,0)$ and $\delta_{(0,0)} = 1$.

THEOREM 2.4.1. *Let* $\Lambda_+ = \{0, \dots, n\} \times \{0, \dots, m\}$, *and let* c_u, $u \in \Lambda_+$, *be given complex numbers. Put* $c_{-u} = \bar{c}_u$, $u \in \Lambda_+$. *The following are equivalent*:

(i) *There exists a stable polynomial* p *with support* $(\widehat{p}) \subseteq \Lambda_+$ *such that* $\frac{1}{|p|^2}$ *has Fourier coefficients* $\widehat{\frac{1}{|p|^2}}(u) = c_u$, $u \in \Lambda_+$;

(ii) *There exist complex numbers* c_u, $u \in (\Lambda_+ - \Lambda_+) \backslash (\Lambda_+ \cup -\Lambda_+)$ *so that*

$$(c_{u-v})_{u,v \in \Lambda_+} > 0$$

and

$$\operatorname{rank}(c_{u-v})_{\substack{u \in \{1,\dots,n\} \times \{0,\dots,m\} \\ v \in \{0,\dots,n\} \times \{1,\dots,m\}}} = nm; \tag{2.4.1}$$

(iii) *There exist complex numbers* c_u, $u \in (\Lambda_+ - \Lambda_+) \backslash (\Lambda_+ \cup -\Lambda_+)$ *so that*

$$(c_{u-v})_{u,v \in \Lambda_+} > 0$$

and

$$\left[(c_{u-v})_{u,v \in \Lambda_+ \backslash \{(0,0)\}}\right]^{-1}_{\substack{\{1,\dots,n\} \times \{0\} \\ \{0\} \times \{1,\dots,m\}}} = 0.$$

(iv) *For all pairs of sets* S_1 *and* S_2 *with*

$$\begin{aligned} &\{1, \dots, n\} \times \{0, \dots, m\} \subseteq S_1 \subseteq \{1, 2, \dots\} \times \{\dots, m-1, m\}, \\ &\{0, \dots, n\} \times \{1, \dots, m\} \subseteq S_2 \subseteq \{\dots, n-1, n\} \times \{1, 2, \dots\}, \end{aligned} \tag{2.4.2}$$

there exist c_u, $u \in (S - S) \backslash (\Lambda_+ \cup (-\Lambda_+))$, *where* $S = \{(0,0)\} \cup S_1 \cup S_2$, *such that*

$$\sum_{u \in S - S} |c_u| < \infty,$$

$$(c_{u-v})_{u,v \in S} > 0 \qquad (\textit{acting on } l_2(S)),$$

and

$$\operatorname{rank}(c_{u-v})_{\substack{u\in S_1\\ v\in S_2}} = nm; \tag{2.4.3}$$

(v) *For some pair of sets* S_1 *and* S_2 *satisfying* (2.4.2) *there exist* c_u, $u \in (S-S)\backslash(\Lambda_+ \cup (-\Lambda_+))$, *where* $S = \{(0,0)\} \cup S_1 \cup S_2$, *such that*

$$\sum_{u\in S-S} |c_u| < \infty$$

$$(c_{u-v})_{u,v\in S} > 0 \qquad (\text{acting on } l_2(S)),$$

and

$$\operatorname{rank}(c_{u-v})_{\substack{u\in S_1\\ v\in S_2}} = nm;$$

(vi) *For all pairs of finite sets* S_1 *and* S_2 *satisfying* (2.4.2) *there exist* c_u, $u \in (S-S)\backslash(\Lambda_+ \cup (-\Lambda_+))$, *where* $S = \{(0,0)\} \cup S_1 \cup S_2$, *such that*

$$(c_{u-v})_{u,v\in S} > 0$$

and

$$[(c_{u-v})_{u,v\in S_1\cup S_2}]^{-1}_{\substack{S_2\backslash S_1\\ S_1\backslash S_2}} = 0.$$

(vii) *For some pair of finite sets* S_1 *and* S_2 *satisfying* (2.4.2) *there exist* c_u, $u \in (S-S)\backslash(\Lambda_+ \cup (-\Lambda_+))$, *where* $S = \{(0,0)\} \cup S_1 \cup S_2$, *such that*

$$(c_{u-v})_{u,v\in S} > 0$$

and

$$[(c_{u-v})_{u,v\in S_1\cup S_2}]^{-1}_{\substack{S_2\backslash S_1\\ S_1\backslash S_2}} = 0.$$

In case one of (i)–(vii) (*and thus all of* (i)–(vii)) *hold, put*

$$(q_u)_{u\in\Lambda_+} = \left[(c_{u-v})_{u,v\in\Lambda_+}\right]^{-1} (\delta_u)_{u\in\Lambda_+} \tag{2.4.4}$$

and let

$$p(z,w) = q_{00}^{-1/2} \left(\sum_{(k,l)\in\Lambda_+} q_{kl} z^k w^l \right). \tag{2.4.5}$$

Then $p(z,w)$ *is a polynomial satisfying* (i), *and* $p(z,w)$ *is unique up to multiplication with a constant of modulus* 1.

Proof. The equivalence of (ii) and (iii) follows directly from Lemma 2.3.5. The implications (iv) → (vi) and (vii) → (v) also follow from Lemma 2.3.5. The implications (ii) → (v), (iv) → (v), (iv) → (ii), (iii) → (vii), (vi) → (vii), (vi) → (iii) are tautologies. The implication (v) → (ii) follows from the observation that the matrices appearing in (ii) are submatrices of the matrices appearing in (v), and the fact that $(c_{u-v})_{u,v\in\Lambda_+} > 0$ implies that

$$\operatorname{rank}(c_{u-v})_{\substack{u\in\{1,\dots,n\}\times\{0,\dots,m\}\\ v\in\{0,\dots,n\}\times\{1,\dots,m\}}} \geq nm. \tag{2.4.6}$$

For the equivalence of (i)–(vii) it remains to prove the implications (i) → (iv) and (iii) → (i).

Assume that a stable polynomial $p(z,w)$ as in (i) exists. Let $f(z,w)$ be the spectral density function of $p(z,w)$ and put

$$c_k = \hat{f}(k), \qquad k \in \mathbb{Z}^2.$$

Then, because of (i), for $k \in \Lambda_+$ this definition of c_k coincides with the prescribed c_k's. In addition, f is in the Wiener class, so that $\sum_{u\in\mathbb{Z}^2} |c_u| < \infty$. Moreover, since $f(z,w) > 0$ for $|z| = |w| = 1$, the multiplication operator $M_f : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ defined by $M_f(g)(z,w) = f(z,w)g(z,w)$ is positive definite. Letting S_1, S_2 and S be as in (iv), we get that the restriction of M_f to $P_S(L^2(\mathbb{T}^2))$ is positive definite. Here, for $K \subset \mathbb{Z}^2$, the projection P_K is the orthogonal projection of $L^2(\mathbb{T}^2)$ onto the subspace of functions with Fourier support in K. That is, $P_K(\sum a_v \binom{z}{w}) = \sum_{v\in K} a_v \binom{z}{w}$. Thus we obtain the positive definiteness of $(c_{u-v})_{u,v\in S}$. In addition, since the matrix in (2.4.3) is the adjoint of a submatrix of the matrix in (2.2.4), we get by Theorem 2.2.1 that

$$\operatorname{rank}(c_{u-v})_{\substack{u\in S_1\\ v\in S_2}} \leq nm.$$

This together with observation (2.4.6) which is valid in this case, we obtain (2.4.3). This proves (i) → (iv).

Assume now that (iii) holds. Define $p(z,w)$ as in (2.4.5). By Theorem 2.3.1, p is stable, and moreover, $c_u = \widehat{\frac{1}{|p|^2}}(u), u \in \Lambda_+ - \Lambda_+$. This proves (i).

Suppose now that (i)–(vii) are valid, and let $p(z,w)$ be as under (i). By multiplying with a constant of modulus one we may choose $p(z,w)$ so that $p(0,0) = p_{00} > 0$. Let $f(z,w)$ be the spectral density function corresponding to $p(z,w)$. Then $f(z,w)p(z,w) = \frac{1}{\bar{p}(1/z,1/w)}$. Since $p(z,w)$ is stable,

$$P_{H\cup\{(0,0)\}}(fp) = P_{H\cup\{(0,0)\}}\left(\frac{1}{\bar{p}}\right) = \frac{1}{p_{00}}\,,$$

where in the last step we used the stability and H is the standard halfspace in $\mathbb{Z}^2$. Thus, in particular,

$$P_{\Lambda_+}(fp) = \frac{1}{p_{00}}\,,$$

which in matrix notation gives that

$$(c_{u-v})_{u,v\in\Lambda_+}(p_u)_{u\in\Lambda_+} = \left(\frac{1}{p_{00}}\,\delta_u\right)_{u\in\Lambda_+}.$$

By multiplying both sides with p_{00} it follows that $p(z,w)$ is given by (2.4.5) where q_u, $u \in \Lambda_+$, is given by (2.4.4). $\square$

Remark 2.4.2. Note that in fact the proof shows that $\widehat{\frac{1}{|p|^2}}(u) = c_u$, $u \in S - S$, for all applicable S.

In the appendix we shall provide an alternative proof of (ii) $\to$ (i) based on minimal rank completions, and the full strip positive extension problem (see [5], [6]).

Note that the proof of Theorem 2.4.1 yields that the polynomial $p(z,w)$ with $p_{00} > 0$ is uniquely determined by the matrix $(c_{u-v})_{u,v\in\Lambda_+}$. One may ask whether in turn all unknown entries c_u, $u \in (\Lambda_+ - \Lambda_+)\backslash(\Lambda_+ \cup -\Lambda_+)$ in this matrix are determined by the conditions in Theorem 2.4.1(ii). When $n = 1$ or $m = 1$, it is not hard to see that the rank condition (2.4.1) determines c_u, $u \in (\Lambda_+ - \Lambda_+)\backslash(\Lambda_+ \cup -\Lambda_+)$ uniquely. For example, when $n = 1$ the coefficients $c_{1,-1}, \ldots, c_{1,-m}$ are determined uniquely by the equations

(2.4.7)
$$c_{1,-j} = [c_{0,-1} \cdots c_{0,-m}][(c_{0,i-k})_{i,k=0}^{m-1}]^{-1}[c_{1,-j+1} \cdots c_{1,-j+m}]^T, j = 1, \ldots, m.$$

It is still an open problem whether the coefficients c_u, $u \in (\Lambda_+ - \Lambda_+)\backslash(\Lambda_+ \cup -\Lambda_+)$ are determined uniquely in general by the conditions in Theorem 2.4.1(ii). If not, it would mean that there are cases in which there are multiple solutions p to the problem. Our computations so far have led us to believe, however, that this cannot occur.

Another natural question is whether the existence of c_u, $u \in (\Lambda_+ - \Lambda_+)\backslash(\Lambda_+ \cup -\Lambda_+)$ so that $(c_{u-v})_{u,v\in\Lambda_+} > 0$, automatically implies the existence of a choice for c_u, $u \in (\Lambda_+ - \Lambda_+)\backslash(\Lambda_+ \cup -\Lambda_+)$ so that in addition condition (2.4.1) is satisfied. This is false. For example, one may take $n = 1$, $m = 3$, $c_{00} = 7.7$, $c_{01} = 6.3$, $c_{02} = 4.5$, $c_{03} = 2.5$, $c_{10} = 3$, $c_{11} = 1.5$, $c_{12} = 2$ and $c_{13} = 1.6$. By setting $c_{1,-1} = 4.9301$, $c_{1,-2} = 7.2776$ and $c_{1,-3} = 7.0593$ (which we determined using the software of [3]), one may check that one obtains a positive definite matrix $(c_{u-v})_{u,v\in\Lambda_+}$ (its smallest eigenvalue is 0.0099). However, equation (2.4.7) forces $c_{1,-1} = 2.4372$, $c_{1,-2} = 1.9405$ and $c_{1,-3} = 1.1570$, which does not give a positive definite matrix (it has an eigenvalue equal to -0.5228; even the submatrix obtained by deleting the $(0,0)$ column and row has a negative eigenvalue -0.3535).

Theorem 1.1.1 follows directly from Theorem 2.4.1.

Proof of Theorem 1.1.1. Let c_u, $u \in \Lambda_+$, be given so that c_u, $u \in (\Lambda_+ - \Lambda_+)\backslash(\Lambda_+ \cup -\Lambda_+)$ exist satisfying (1) and (2) in the statement of Theorem 1.1.1. Thus Theorem 2.4.1(ii) is satisfied, yielding the existence of a stable polynomial $p(z,w) = \sum_{k=0}^n \sum_{l=0}^m p_{k,l} z^k w^l$ with $p_{00} > 0$ as in (i).

Conversely, given a stable polynomial satisfying Theorem 2.4.1(i), Theorem 2.4.1(ii) is valid, yielding (1) and (2) in Theorem 1.1.1. □

We shall now build up to the general case of a finite set $\Lambda_+ \subseteq H$. We first consider the case when $\{(0,0)\} \subseteq \Lambda_+ \subseteq \{0,\dots,n\} \times \{0,\dots,m\}$.

THEOREM 2.4.3. *Let* $\{(0,0)\} \subseteq \Lambda_+ \subseteq \{0,\dots,n\} \times \{0,\dots,m\}$, *and let* c_u, $u \in \Lambda_+$, *be given complex numbers. Put* $c_{-u} = \bar{c}_u$, $u \in \Lambda_+$. *The following are equivalent*:

(i) *There exists a stable polynomial* p *with support* $(\hat{p}) \subseteq \Lambda_+$ *such that* $\frac{1}{|p|^2}$ *has Fourier coefficients* $\widehat{\frac{1}{|p|^2}}(u) = c_u$, $u \in \Lambda_+$;

(ii) *There exist complex numbers* c_u, $u \in \{-n,\dots,n\} \times \{-m,\dots,m\} \setminus (\Lambda_+ \cup -\Lambda_+)$ *so that*

$$(c_{u-v})_{u,v \in \{0,\dots,n\}\times\{0,\dots,m\}} > 0, \tag{2.4.8}$$

$$\left[(c_{u-v})_{u,v \in \{0,\dots,n\}\times\{0,\dots,m\}\setminus\{(0,0)\}}\right]^{-1}_{\substack{\{1,\dots,n\}\times\{0\} \\ \{0\}\times\{1,\dots,m\}}} = 0, \tag{2.4.9}$$

and

$$\left[(c_{u-v})_{u,v \in \{0,\dots,n\}\times\{0,\dots,m\}}\right]^{-1}_{\substack{\{0,\dots,n\}\times\{0,\dots,m\}\setminus\Lambda_+ \\ \{0\}\times\{0\}}} = 0. \tag{2.4.10}$$

In case (i) (*and* (ii)) *holds, a solution* p *is given by* (2.4.4) *and* (2.4.5).

Note that (ii) in this theorem reduces to Theorem 2.4.1(iii) in the case when $\Lambda_+ = \{0,\dots,n\} \times \{0,\dots,m\}$. One may also formulate analogs of Theorem 2.4.1 (ii), (iv)–(vii) but we leave this to the interested reader.

Proof. Suppose (i) is valid. Let $f(z,w)$ be the spectral density function of $p(z,w)$ and put

$$c_u = \hat{f}(u), \qquad u \in \mathbb{Z}^2.$$

Now the polynomial $p(z,w)$ satisfies Theorem 2.4.1(i) for the collection of numbers $\{c_u, u \in \{0,\dots,n\}\times\{0,\dots,m\}\}$. Thus Theorem 2.4.1(iii) and (2.4.4) and (2.4.5) are valid. Theorem 2.4.1(iii) implies the first two conditions in (ii). Since p is given by (2.4.4) and (2.4.5) (up to a constant) we have that support$(\hat{p}) \subseteq \Lambda_+$ implies (2.4.10). This shows that (ii) is valid.

Next, assume that (ii) is valid. The first two properties in (ii) give that Theorem 2.4.1(iii) is satisfied. Thus Theorem 2.4.1(i) is valid, yielding that there exists a stable polynomial given by (2.4.4) and (2.4.5) so that $\widehat{\frac{1}{|p|^2}}(u) = c_u$, $u \in \{0,\dots,n\} \times \{0,\dots,m\}$. Thus, in particular, this polynomial has the right match,

$$\widehat{\frac{1}{|p|^2}}(u) = c_u, \qquad u \in \Lambda_+,$$

and, moreover, by the construction of p by (2.4.4) and (2.4.5) one sees that condition (2.4.10) yields that support $(\hat{p}) \subseteq \Lambda_+$. This shows that (i) is valid. $\square$

Next consider an index set of the following type:

$$J(n,m,q) = \bigcup_{i=0}^{n} \{i\} \times \{-iq, \ldots, m-iq\}, \qquad n,m \geq 0, \quad q \in \mathbb{Z}.$$

Thus, $J(n,m,0) = \{0,\ldots,n\} \times \{0,\ldots,m\}$. We have the following proposition.

PROPOSITION 2.4.4. *Let n,m be nonnegative integers and $q \in \mathbb{Z}$, and let $\{(0,0)\} \subseteq \Delta_+ \subseteq J(n,m,q)$. Let d_u, $u \in \Delta_+$, be given complex numbers. Put $\Lambda_+ = \{(k,l+kq) : (k,l) \in \Delta_+\}$ and*

$$c_{(r,s)} = d_{(r,s-rq)}, \qquad (r,s) \in \Lambda_+.$$

Then $\Lambda_+ \subseteq J(n,m,0)$. Moreover, the following are equivalent.

(i) *There exists a stable pseudopolynomial $q(z,w)$ with support $(\hat{q}) \subseteq \Delta_+$ such that $\widehat{\frac{1}{|q|^2}}(u) = d_u$, $u \in \Delta_+$.*

(ii) *There exists a stable polynomial $p(z,w)$ with support $(\hat{p}) \subseteq \Lambda_+$ such that $\widehat{\frac{1}{|p|^2}}(u) = c_u$, $u \in \Lambda_+$.*

Proof. Use the correspondence $q(z,w) = p\left(\frac{z}{w^q}, w\right)$, $|z| = |w| = 1$. $\square$

It remains to observe that any finite $\{(0,0)\} \subset \Lambda_+ \subseteq H \cup \{(0,0)\}$ is a subset of some $J(n,m,q)$. Indeed, let

$$n = \max\{k : (k,l) \in \Lambda_+\} \quad (\geq 0),$$
$$q = -\min\left\{ \left\lfloor \frac{l}{k} \right\rfloor : (k,l) \in \Lambda_+, k \geq 1 \right\},$$

and

$$m = \max\{l + kq : (k,l) \in \Lambda_+\} \quad (\geq 0).$$

Then $\Lambda_+ \subseteq J(n,m,q)$. Consequently, we have, by applying a combination of Proposition 2.4.4 and Theorem 2.4.3, the problem introduced in the beginning of this section reduced to a finite positive definite matrix completion problem where the completion is required to be block Toeplitz with Toeplitz matrix entries satisfying certain inverse constraints. As established in [64], finding such completions (if they exist) is numerically feasible. We shall give some numerical results in Section 4.3. Interesting open questions remain regarding the d-variable case (when $d \geq 3$), and also whether, for instance, if $\{c_u, u \in \Lambda_+\}$ and $\{d_k, u \in \Lambda_+\}$ satisfy the conditions of Theorem 2.4.1 the sum sequence $\{c_u + d_u, u \in \Lambda_+\}$ also satisfies these conditions.

Partial necessary conditions for the autoregressive filter problem appear in [11] (see also [12]), where it was shown that if Theorem 2.4.3(i) holds then

$$\left[(c_{u-v})_{u,v\in\Lambda_+-\Lambda_+}\right]^{-1}_{\substack{(\Lambda_+-\Lambda_+)\setminus\Lambda_+\\ \{0\}\times\{0\}}}=0. \tag{2.4.11}$$

That this condition is not sufficient is shown by the following example. Let $\Lambda_+=\{(0,0),(1,0),(0,1)\}$, and $c_{00}=1, c_{01}=.25, c_{10}=.25$. If we choose $c_{1,-1}=.125$ and $c_{1,-2}=5/16$, then (2.4.11) is satisfied. Computing for p we find $p(z,w)=\frac{9}{8}-\frac{1}{4}z-\frac{1}{4}w$, which is stable (since $|p(z,w)|\geq\frac{9}{8}-\frac{1}{4}-\frac{1}{4}>0$ when $|z|\leq 1$ and $|w|\leq 1$). However, the function $\frac{1}{|p|^2}$ does not have the prescribed Fourier coefficients, as

$$\widehat{\frac{1}{|p|^2}}(0,0)=0.9923,\quad \widehat{\frac{1}{|p|^2}}(0,1)=\widehat{\frac{1}{|p|^2}}(1,0)=0.0545.$$

The correct choice is given by $c_{1,-1}=0.0625$ and $c_{1,-2}=0.0156$, yielding the stable polynomial $p(z,w)=1.1333-0.2667z-0.2667w$ satisfying

$$\widehat{\frac{1}{|p|^2}}(0,0)=1,\quad \widehat{\frac{1}{|p|^2}}(0,1)=\widehat{\frac{1}{|p|^2}}(1,0)=0.25.$$

Let us end this section with a comparison to the extension problem for positive-definite functions as considered in [58]. There, a pattern $\Lambda\subseteq\mathbb{Z}^2$ is said to have *the extension property* if every sequence $(c_u)_{u\in\Lambda-\Lambda}$ which satisfies the positivity requirement

$$(c_{u-v})_{u,v\in\Lambda}\geq 0, \tag{2.4.12}$$

admits the existence of a positive Borel measure μ on $\mathbb{T}^2$ so that

$$c_{k,l}=\int_{\mathbb{T}^2} z^k w^l d\mu(z,w),\ (k,l)\in\Lambda-\Lambda.$$

Note that in our terminology, we would let $\Lambda_+=(\Lambda-\Lambda)\cap(H\cup\{(0,0)\})$. Moreover, we study the strictly positive definite case and look for a measure of the special form

$$d\mu(z,w)=\frac{1}{|p(z,w)|^2}\frac{dzdw}{(2\pi i)^2 zw}, \tag{2.4.13}$$

where $p(z,w)$ is a stable polynomial with Fourier support in Λ_+. Following [58] a construction of a positive extension is given in [4] in the case that $\Lambda=\{0,1\}\times\{0,\dots,m\}$, which in our terminology corresponds to the case when $\Lambda_+=\{0\}\times\{0,\dots,m\}\cup\{1\}\times\{-m,\dots,m\}$. We remark that their construction does not yield a measure of the form (2.4.13) (see formula (3) in [4]), and indeed one cannot expect that strict positive definiteness in (2.4.12) yields a measure of this special form as the rank condition (2.4.1) also needs to be satisfied.

3. Applications of the extension problem

In this chapter we treat four applications of the extension results. They concern two-variable orthogonal polynomials, two-variable stable autoregressive filters, Fejér-Riesz factorization for two-variable trigonometric functions, and inverse formulas for doubly indexed Toeplitz matrices.

3.1. *Orthogonal and minimizing pseudopolynomials.* We fix $H = \{(n,m) : n \geq 1 \text{ or } (n = 0 \text{ and } m > 0)\} \subseteq \mathbb{Z}^2$ to be the standard halfspace in $\mathbb{Z}^2$. Let ρ be a positive Borel measure on $\mathbb{T}^2$ and $L^2(\rho, \mathbb{T}^2)$ be the space of functions square integrable with respect to ρ, i.e. $\int_{\mathbb{T}^2} |f(\theta,\phi)|^2 d\rho < \infty$. On this space there is a natural inner product given by

$$\langle f, g\rangle_\rho = \int_{\mathbb{T}^2} f(\theta,\phi)\bar{g}(\theta,\phi)d\rho, \tag{3.1.1}$$

for all $f, g \in L^2(\rho, \mathbb{T}^2)$. We denote the Fourier coefficients of ρ by c_{kl}, $(k,l) \in \mathbb{Z}^2$, which are given by

$$c_{k,l} = \int_{\mathbb{T}^2} e^{-ik\theta}e^{-ik\phi}d\rho(\theta,\phi).$$

Let Λ_+ be a finite subset of $H \cup \{(0,0)\}$ containing $(0,0)$, and suppose that ρ is such that

$$(c_{u-v})_{u,v\in\Lambda_+} > 0. \tag{3.1.2}$$

As mentioned before, for $v = (k,l) \in \mathbb{Z}^2$ we denote by $\binom{z}{w}^v$ the monomial $\binom{z}{w}^v = z^k w^l$. For an ordered set $\{v_0, \ldots, v_m\}$ we let $C(v_0, \ldots, v_m)$ denote the $(m+1) \times (m+1)$ matrix

$$C(v_0, \ldots, v_m) := (c_{v_i - v_j})_{i,j=0}^m.$$

Definition 3.1.1. For an ordered subset $\{v_0, \ldots, v_m\}$ of Λ_+ with $v_0 = (0,0)$, we define the *orthogonal pseudopolynomials* [33] $\phi\left(v_0, \ldots, v_i; \binom{z}{w}\right), i = 0, \ldots, m$, by the relations,

$$\phi\left(v_0, \ldots, v_i; \begin{pmatrix} z \\ w \end{pmatrix}\right) = \sum_{j=0}^{i} a_{i,j} \begin{pmatrix} z \\ w \end{pmatrix}^{v_j}, \tag{3.1.3}$$

with $a_{i,i} > 0$, and

$$\langle \phi(v_0, \ldots, v_i), \phi(v_0, \ldots, v_j)\rangle_\rho = \delta_{v_i - v_j} \ i,j = 0, \cdots, m. \tag{3.1.4}$$

Here $\delta_v = 0$ if $v \neq (0,0)$ and $\delta_{(0,0)} = 1$. For the construction of $\phi(v_0, \ldots, v_i; \binom{z}{w})$

the above orthogonality equations are equivalent to

$$\langle \phi(v_0,\ldots,v_i), \begin{pmatrix} z \\ w \end{pmatrix}^{v_j} \rangle_\rho = \frac{1}{a_{i,i}} \delta_{v_i - v_j} \; j = 0,\cdots,i.$$

Thus,

$$\phi\left(v_0,\ldots,v_i;\tbinom{z}{w}\right) = \frac{\det \begin{pmatrix} c_{v_0-v_0} & \cdots & c_{v_0-v_i} \\ \vdots & & \vdots \\ c_{v_{i-1}-v_0} & \cdots & c_{v_{i-1}-v_i} \\ \begin{pmatrix} z \\ w \end{pmatrix}^{v_0} & \cdots & \begin{pmatrix} z \\ w \end{pmatrix}^{v_i} \end{pmatrix}}{\sqrt{\det C(v_0,\ldots,v_i) \det C(v_0,\ldots,v_{i-1})}}. \tag{3.1.5}$$

They are called pseudopolynomials since negative powers of z and w may arise. From the above equations we see that the orthogonal pseudopolynomials $\phi\left(v_0,\ldots,v_i;\tbinom{z}{w}\right)$, $i = 0,\ldots,m$, form a basis for the space spanned by the monomials $\{\tbinom{z}{w}^{v_0},\cdots,\tbinom{z}{w}^{v_m}\}$.

As usual the monic orthogonal pseudopolynomials solve the following minimization problem: Let $\Pi(v_0,\cdots,v_m)$ be the set of polynomials with exponents taken from $\{v_0,\cdots,v_m\}$ with the coefficient of $\tbinom{z}{w}^{v_m}$ equal to one. Then $a_{mm}\phi\left(v_0,\ldots,v_m;\tbinom{z}{w}\right)$ is the solution to the minimization problem

$$\min_{\pi\in\Pi(v_0,\cdots,v_m)} \int_{\mathbb{T}^2} |\pi(\theta,\phi)|^2 d\rho(\theta,\phi).$$

Another important set of polynomials called minimizing pseudopolynomials studied in [18] can be characterized as follows.

Definition 3.1.2. For an ordered subset $\{v_0,\ldots,v_m\}$ of Λ_+ with $v_0 = (0,0)$, we define the *minimizing pseudopolynomial* $p\left(v_0,\ldots,v_m;\tbinom{z}{w}\right)$ by

$$p\left(v_0,\ldots,v_m;\begin{pmatrix} z \\ w \end{pmatrix}\right) = \frac{1}{k(v_0,\ldots,v_m)} \times \left(\begin{pmatrix} z \\ w \end{pmatrix}^{v_0} \cdots \begin{pmatrix} z \\ w \end{pmatrix}^{v_m}\right) C(v_0\cdots v_m)^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.1.6}$$

where

$$k(v_0,\ldots,v_m) = \sqrt{\frac{\det C(v_1\cdots v_m)}{\det C(v_0\cdots v_m)}}.$$

Alternative formulas for the minimizing polynomials are given by

$$p\left(v_0,\ldots,v_m;\begin{pmatrix} z\\ w\end{pmatrix}\right)=\frac{\det\begin{pmatrix}\begin{pmatrix} z\\ w\end{pmatrix}^{v_0} & \cdots & \begin{pmatrix} z\\ w\end{pmatrix}^{v_m}\\ c_{v_1-v_0} & \cdots & c_{v_1-v_m}\\ \vdots & \vdots & \vdots\\ c_{v_m-v_0} & \cdots & c_{v_m-v_m}\end{pmatrix}}{\sqrt{\det C(v_0,\ldots,v_m)\det C(v_1,\ldots,v_m)}}$$

and

$$p\left(v_0,\ldots,v_m;\begin{pmatrix} z\\ w\end{pmatrix}\right)=\left(\begin{pmatrix} z\\ w\end{pmatrix}^{v_0}\cdots\begin{pmatrix} z\\ w\end{pmatrix}^{v_m}\right)L_1,$$

where L_1 is the first column of the lower triangular Cholesky factor L of $C(v_0\cdots v_m)^{-1}$ $(=LL^*)$. It should be noted that in [18] the normalization constant $\frac{1}{k(v_0,\cdots,v_m)}$ does not appear in the definition of the minimizing pseudopolynomial. For our purposes it is convenient to include this factor in the definition. In the definition above the 2-tuples $v_0,\ldots,v_m$ are ordered, however it is easy to check that for any permutation π on $\{0,\ldots,m\}$ with $\pi(0)=0$

$$p\left(v_{\pi(0)},\ldots,v_{\pi(m)};\begin{pmatrix} z\\ w\end{pmatrix}\right)=p\left(v_0,\ldots,v_m;\begin{pmatrix} z\\ w\end{pmatrix}\right).$$

Thus, on occasion we shall also write $p\left(\Delta;\binom{z}{w}\right)$ where Δ is the set $\{v_0,\ldots,v_m\}$, and it is understood that $\binom{0}{0}$ is first in the ordering. Minimizing pseudopolynomials appear naturally in the following context. Let

$$\Phi_\rho:\mathrm{span}\{\begin{pmatrix} z\\ w\end{pmatrix}^{v_k}\ :\ k=0,\ldots,m\}\to\mathbb{R}$$

be given by

$$\Phi_\rho(g)=\langle g,g\rangle_\rho-2\,\mathrm{Re}(g_{00}).$$

Then (see [18],[19]) Φ_ρ is minimized by $k(v_0,\ldots,v_m)p\left(v_0,\ldots,v_m;\binom{z}{w}\right)$. In taking the reverse polynomial of $p\left(v_0,\ldots,v_m;\binom{z}{w}\right)$ the term of $\binom{z}{w}^{v_m}$ is taken to appear last. In other words, if $p\left(v_0,\ldots,v_m;\binom{z}{w}\right)=\sum_{i=0}^m a_i\binom{z}{w}^{v_i}$, then $\overleftarrow{p}\left(v_0,\cdots,v_m;\binom{z}{w}\right)=\binom{z}{w}^{v_m}\overline{p}\left(\frac{1}{z},\frac{1}{w}\right)$.

There is a close relationship between the two sets of pseudopolynomials introduced in this section, namely:

$$\begin{aligned}\overleftarrow{p}&\left(v_0,\ldots,v_m;\begin{pmatrix} z\\ w\end{pmatrix}\right)\\ &=\begin{pmatrix} z\\ w\end{pmatrix}^{v_m}\overline{p}\left(v_m-v_m,v_m-v_{m-1},\ldots,v_m-v_0;\begin{pmatrix} 1/z\\ 1/w\end{pmatrix}\right)\\ &=\phi\left(v_m-v_m,v_m-v_{m-1},\ldots,v_m-v_0;\begin{pmatrix} z\\ w\end{pmatrix}\right).\end{aligned}\tag{3.1.7}$$

Both sets of polynomials appear also in a prediction context. In Section 3 of [42] there is an eloquent explanation of the one-variable prediction theory. One easily adjusts this to the bivariate context and sees that $p(v_0, \dots, v_m; \binom{z}{w})$ appears in backward prediction, while the pseudopolynomial $\phi(v_0, \dots, v_m; \binom{z}{w})$ plays a role in forward prediction. We will not further pursue this here.

In Lemmas 3.1.3, 3.1.4, 3.1.5 and Theorem 3.1.6 we recall some familiar properties of the minimizing pseudopolynomials and their reverses. Using the connection (3.1.7), one may state comparable properties of the orthogonal pseudopolynomials. We will focus our attention mostly on the minimizing pseudopolynomials, following the lead of [18] and [19].

The first lemma follows from the two determinantal formulas above, and describes their orthogonal properties.

LEMMA 3.1.3 ([18, Cor. of Th. 1]). *Let ρ be a positive Borel measure on $\mathbb{T}^2$ with Fourier coefficients c_u, $u \in \mathbb{Z}^2$. Let $\{(0,0)\} \subset \Lambda_+ \subset H \cup \{(0,0)\}$ be a finite set and assume that* (3.1.2) *holds. Further, let $\{v_0, \dots, v_m\}$ be an ordered subset of Λ_+ with $v_0 = (0,0)$. Denote $p(z,w) = p\left(v_0, \dots, v_m; \binom{z}{w}\right)$. Then p satisfies, and up to an overall complex constant of modulus one is determined by the orthonormal relations*

$$\langle p, p \rangle_\rho = 1 \tag{3.1.8}$$

and

$$\langle p, \begin{pmatrix} z \\ w \end{pmatrix}^{v_i} \rangle_\rho = 0, \qquad 0 < i \le m, \tag{3.1.9}$$

with the inner product defined as in (3.1.1)*. The above undetermined complex constant is uniquely fixed by requiring the trailing coefficient of p to be positive.*

Note that equation (3.1.7) and the definition of $\phi(v_0, \dots, v_m; \binom{z}{w})$ imply that

$$\langle \overleftarrow{p}, \begin{pmatrix} z \\ w \end{pmatrix}^{v_m - v_i} \rangle_\rho = \beta_m \delta_i, \qquad 0 \le i \le m,$$

where $\beta_m = \sqrt{\frac{\det C(v_m \cdots v_0)}{\det C(v_m \cdots v_1)}} \neq 0$.

Next we will see that there is a recurrence relation among the minimizing pseudopolynomials. To this end let

$$C(v_0 \cdots v_m \mid w_0 \cdots w_m) = (c_{v_i - w_j})_{i,j=0,\cdots,m}$$

be the matrix with rows indexed by v_i and columns indexed by w_i.

Lemma 3.1.4. *The minimizing pseudopolynomial $p\left(v_0\cdots v_m; \binom{z}{w}\right)$ satisfies the relation*

(3.1.10)
$$p\left(v_0\cdots v_m; \begin{pmatrix} z \\ w \end{pmatrix}\right) = \frac{k(v_0\cdots v_m)}{k(v_0\cdots v_{m-1})}\left(p\left(v_0\cdots v_{m-1}; \begin{pmatrix} z \\ w \end{pmatrix}\right)\right.$$
$$\left. + \alpha(v_0\cdots v_m)\begin{pmatrix} z \\ w \end{pmatrix}^{v_1} \overleftarrow{p}\left(v_m - v_m\cdots v_m - v_1; \begin{pmatrix} z \\ w \end{pmatrix}\right)\right),$$

where
$$\alpha(v_0\cdots v_m) = \frac{(-1)^m \det C(v_1\cdots v_m \mid v_0\cdots v_{m-1})}{\sqrt{\det C(v_0\cdots v_{m-1})\det C(v_1\cdots v_m)}}.$$

Furthermore,

(3.1.11)
$$\overleftarrow{p}\left(v_0\cdots v_m; \begin{pmatrix} z \\ w \end{pmatrix}\right) = \frac{k(v_0\cdots v_m)}{k(v_0\cdots v_{m-1})}\left(\begin{pmatrix} z \\ w \end{pmatrix}^{v_m - v_{m-1}} \overleftarrow{p}\left(v_0\cdots v_{m-1}; \begin{pmatrix} z \\ w \end{pmatrix}\right)\right.$$
$$\left. + \overline{\alpha(v_0\cdots v_m)} p\left(v_m - v_m, v_m - v_{m-1}\cdots v_m - v_1; \begin{pmatrix} z \\ w \end{pmatrix}\right)\right).$$

We remark that equation (3.1.10) is given in Theorem 2 of [18].

Proof. $\overleftarrow{p}\left(v_m - v_m, v_m - v_{m-1}, \cdots v_m - v_1; \binom{z}{w}\right)$ is characterized up to multiplication by a constant by its orthogonality to $\binom{z}{w}^{-v_i+v_1}$, $i = 0, 1, \ldots, m-1$. Now
$$\begin{pmatrix} z \\ w \end{pmatrix}^{-v_1}\left(p\left(v_0\cdots v_m; \begin{pmatrix} z \\ w \end{pmatrix}\right) - \frac{k(v_0\cdots v_m)}{k(v_0\cdots v_{m-1})} p\left(v_0\cdots v_{m-1}; \begin{pmatrix} z \\ w \end{pmatrix}\right)\right)$$
is orthogonal to $\begin{pmatrix} z \\ w \end{pmatrix}^{-v_i+v_1}$, $i = 0, \ldots, m-1$, which gives (3.1.10) up to a scalar factor. By comparing coefficients of $\begin{pmatrix} z \\ w \end{pmatrix}^{v_m}$ on both sides of the recurrence relation we find that

$$\alpha(v_0\cdots v_m)$$
$$= \frac{(-1)^m \det C(v_1\cdots v_m \mid v_0\cdots v_{m-1})\sqrt{\det C(v_1\cdots v_{m-1})}}{\sqrt{\det C(v_0\cdots v_{m-1})\det C(v_{m-1} - v_{m-1}\cdots v_{m-1} - v_1)\det C(v_1\cdots v_m)}}.$$

Equation (3.1.10) now follows since
$$\det C(v_1\cdots v_i) = \det C(v_i - v_{i-1}\cdots v_i - v_1).$$

Equation (3.1.11) is obtained by taking reversals in equation (3.1.10). $\square$

From the definition of k and α in terms of determinants it is easy to see that the following are true. Let $w_j = v_m - v_{m-j}$, $j = 0, \dots, m$, then $\alpha(w_0 \cdots w_m) = \alpha(v_0 \cdots v_m) = \overline{\alpha(-v_0 \cdots - v_m)}$. Moreover, the Jacobi identity implies that

$$\frac{k^2(v_0 \cdots v_m)}{k^2(v_0 \cdots v_{m-1})} \left(1 - |\alpha(v_0 \cdots v_m)|^2\right) = 1.$$

LEMMA 3.1.5. *The minimizing pseudopolynomial* $p\left(v_0 \cdots v_m; \binom{z}{w}\right)$ *satisfies the relation*

(3.1.12)

$$\begin{aligned}
&p\left(v_0 \cdots v_m; \binom{z}{w}\right) \overline{p\left(v_0 \cdots v_m; \binom{z_1}{w_1}\right)} \\
&\quad - \overleftarrow{p}\left(v_m - v_m \cdots v_m - v_0; \binom{z}{w}\right) \overline{\overleftarrow{p}\left(v_m - v_m \cdots v_m - v_0; \binom{z_1}{w_1}\right)} \\
&= p\left(v_0 \cdots v_{m-1}; \binom{z}{w}\right) \overline{p\left(v_0 \cdots v_{m-1}; \binom{z_1}{w_1}\right)} \\
&\quad - \binom{z}{w}^{v_1} \overleftarrow{p}\left(v_m - v_m \cdots v_m - v_1; \binom{z}{w}\right) \\
&\quad \times \overline{\binom{z_1}{w_1}^{v_1} \overleftarrow{p}\left(v_m - v_m \cdots v_m - v_1; \binom{z_1}{w_1}\right)}.
\end{aligned}$$

Proof. Set

$$p_m\binom{z}{w} = p\left(v_0 \cdots v_m; \binom{z}{w}\right)$$

and

$$p_m^i\binom{z}{w} = \overleftarrow{p}\left(v_m - v_m \cdots v_m - v_i; \binom{z}{w}\right),$$

for $i = 0, 1$. From the recurrence relation we find

$$\begin{aligned}
p_m\binom{z}{w} \overline{p_m\binom{z_1}{w_1}} &= \frac{k^2(v_0 \cdots v_m)}{k^2(v_0 \cdots v_{m-1})} \Bigg[p_{m-1}\binom{z}{w} \overline{p_{m-1}\binom{z_1}{w_1}} \\
&\quad + \alpha(v_0 \cdots v_m) \binom{z}{w}^{v_1} \overleftarrow{p}^{\,1}_{m-1}\binom{z}{w} \overline{p_{m-1}\binom{z_1}{w_1}} \\
&\quad + \overline{\alpha(v_0 \cdots v_m)} \overline{\binom{z_1}{w_1}^{v_1}} p_{m-1}\binom{z}{w} \overline{\overleftarrow{p}^{\,1}_{m-1}\binom{z_1}{w_1}} \\
&\quad + \alpha(v_0 \cdots v_m) \overline{\alpha(v_0 \cdots v_m)} \binom{z}{w}^{v_1} \overline{\binom{z_1}{w_1}^{v_1}} \overleftarrow{p}^{\,1}_{m-1}\binom{z}{w} \overline{\overleftarrow{p}^{\,1}_{m-1}\binom{z_1}{w_1}} \Bigg].
\end{aligned}$$

Also,

$$
\begin{aligned}
\overleftarrow{p}^{\,0}_{m}\binom{z}{w}\overline{\overleftarrow{p}^{\,0}_{m}\binom{z_1}{w_1}} &= \frac{k^2(v_m-v_m\cdots v_m-v_0)}{k^2(v_m-v_m\cdots v_m-v_1)}\\
&\times\Bigg[\binom{z}{w}^{v_1}\overline{\binom{z_1}{w_1}^{v_1}}\overleftarrow{p}^{\,1}_{m-1}\binom{z}{w}\overline{\overleftarrow{p}^{\,1}_{m-1}\binom{z_1}{w_1}}\\
&+\overline{\binom{z_1}{w_1}^{v_1}}\,\overline{\alpha(v_m-v_m\cdots v_m-v_0)}p_{m-1}\binom{z}{w}\overline{\overleftarrow{p}^{\,1}_{m-1}\binom{z_1}{w_1}}\\
&+\binom{z}{w}^{v_1}\alpha(v_m-v_m\cdots v_m-v_0)\overline{p_{m-1}\binom{z_1}{w_1}}\overleftarrow{p}^{\,1}_{m-1}\binom{z}{w}\\
&+\alpha(v_m-v_m\cdots v_m-v_0)\overline{\alpha(v_m-v_m\cdots v_m-v_0)}p_{m-1}\binom{z}{w}\overline{p_{m-1}\binom{z_1}{w_1}}\Bigg].
\end{aligned}
$$

Now using the relations between $\alpha(v_0\cdots v_m)$ and $\alpha(v_m-v_m\cdots v_m-v_0)$ and $k(v_0\cdots v_m)$ and $k(v_m-v_m\cdots v_0-v_1)$, and then subtracting the lower equation from the upper gives the result. □

The theorem below in the case of reverse lexicographical ordering is Theorem 8 in [18].

THEOREM 3.1.6. *Let ρ be a positive Borel measure on $\mathbb{T}^2$ with Fourier coefficients c_u, $u\in\mathbb{Z}^2$. Let $\{(0,0)\}\subset\Lambda_+\subset H\cup\{(0,0)\}$ be a finite set and assume that* (3.1.2) *holds. Further, order Λ_+ as $\Lambda_+=\{v_0,\dots,v_m\}$. The pseudopolynomials $\left\{\binom{z}{w}^{v_i}p\left(v_i-v_i,v_{i+1}-v_i\cdots v_m-v_i;\binom{z}{w}\right)\ :\ i=0,\dots,m\right\}$ form an orthonormal basis of the space $\{\binom{z}{w}^{v}\ :\ v\in\Lambda_+\}$ endowed with the inner product $\langle,\rangle_\rho$. Furthermore, if*

$$
\begin{aligned}
&P(z,w)=\\
&\quad\Bigg[p\left(v_0-v_0,\dots,v_m-v_0;\binom{z}{w}\right),\binom{z}{w}^{v_1}p\left(v_1-v_1,\dots,v_m-v_1;\binom{z}{w}\right),\\
&\qquad\qquad\dots,\binom{z}{w}^{v_m}p\left(v_m-v_m;\binom{z}{w}\right)\Bigg],
\end{aligned}
$$

then $P=\left[\binom{z}{w}^{v_0}\cdots\binom{z}{w}^{v_m}\right]L$, where L is the lower triangular Cholesky factor of $C(v_0,\dots,v_m)^{-1}$ i.e., $C(v_0,\dots,v_m)^{-1}$.

Note that in this theorem the order of the rows and columns in $C(v_0,\dots,v_m)$ is important. Furthermore, the indices arising in the l^{th} pseudopolynomial above can be read off from the lower triangular part of the l^{th} column of the matrix C in the ordering chosen.

Proof. For $0 \le j \le i \le m$ we need to show that

$$\langle \begin{pmatrix} z \\ w \end{pmatrix}^{v_j} p\left(v_j - v_j, \ldots, v_m - v_j; \begin{pmatrix} z \\ w \end{pmatrix}\right), \begin{pmatrix} z \\ w \end{pmatrix}^{v_i} p\left(v_i - v_i, \ldots, v_m - v_i; \begin{pmatrix} z \\ w \end{pmatrix}\right)\rangle_\rho = \delta_{i,j}. \tag{3.1.13}$$

The result for $i = j$ follows from equation (3.1.8) with

$$p\begin{pmatrix} z \\ w \end{pmatrix} = p\big(v_j - v_j, v_{j+1} - v_j, \ldots, v_m - v_j; \begin{pmatrix} z \\ w \end{pmatrix}\big).$$

For $i > j$ the above result will follow if it can be shown that

$$\langle p\left(v_j - v_j, \ldots, v_m - v_j; \begin{pmatrix} z \\ w \end{pmatrix}\right), \begin{pmatrix} z \\ w \end{pmatrix}^{(v_i - v_j)} \rangle_\rho = 0,$$

for $i = j+1, \ldots, m$. But this is exactly the content of equation (3.1.9). Consequently we see that the polynomials $\binom{z}{w}^{v_i} p\left(v_i - v_i, \ldots, v_m - v_i; \binom{z}{w}\right)$, $i = 0, \ldots, m$, are linearly independent and thus they form a basis for $\{\binom{z}{w}^v : v \in \Lambda_+\}$.

In matrix form we see that (3.1.13) can be rewritten as $L^* C(v_0, \ldots, v_m) L = I$ which implies that $C(v_0, \ldots, v_m)^{-1} = LL^*$. Since L has positive diagonal elements we see that each pseudopolynomial must have a positive trailing coefficient which uniquely specifies the pseudopolynomial. □

Up until this point ordering on the monomials has not played any special role. In the results that follow the ordering will be important.

As noted in [18, Th. 7], Theorem 3.1.6 allows us to connect certain minimizing pseudopolynomials with the matrix orthogonal polynomials in (2.3.7) and (2.3.8), as follows. From Theorem 3.1.6, and equation (2.3.9) with $i = m$ it follows that,

$$P^m(z,w) = [p^{(0)}(z,w) \; wp^{(1)}(z,w) \cdots w^m p^{(m)}(z,w)],$$

where

$$p^{(j)}(z,w) = p\left(\{0\} \times \{0, \ldots, m-j\} \cup \{1, \ldots, n\} \times \{-j, \ldots, m-j\}; \tbinom{z}{w}\right), \quad j = 0, \ldots, m.$$

This coupled with (2.3.20) in Section 2.3 implies that

$$P^m(z,w) = [p(z,w) \quad wP^{m-1}(z,w)], \tag{3.1.14}$$

where $P^{(m-1)}$ has the following representation in terms of pseudopolynomials,

$$P^{m-1}(z,w) = [p^{(1)}(z,w) \; wp^{(2)}(z,w) \cdots w^{m-1} p^{(m-1)}(z,w)].$$

Analogous formulas for $\tilde{P}^i, i = n, n-1$, also hold. With this we can recast Proposition 2.3.3 as follows.

THEOREM 3.1.7. *Let ρ be a positive Borel measure on $\mathbb{T}^2$ with Fourier coefficients c_u, $u \in \mathbb{Z}^2$. Let $\Lambda_+ = \{0, \dots, n\} \times \{0, \dots, m\}$ and assume that* (3.1.2) *holds. In addition, assume that*

$$\left[(c_{u-v})_{u,v\in\Lambda_+\setminus\{(0,0)\}}\right]^{-1}_{\substack{\{1,\dots,n\}\times\{0\}\\ \{0\}\times\{1,\dots,m\}}} = 0. \tag{3.1.15}$$

Then

$$\begin{aligned} p\left(\Lambda_+; \begin{pmatrix} z \\ w \end{pmatrix}\right)\overline{p\left(\Lambda_+; \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}\right)} - \overleftarrow{p}\left(\Lambda_+; \begin{pmatrix} z \\ w \end{pmatrix}\right)\overline{\overleftarrow{p}\left(\Lambda_+; \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}\right)} \\ = (1 - w\overline{w}_1)\sum_{k=1}^{m}(w\overline{w}_1)^{k-1} p\left(Q_k; \begin{pmatrix} z \\ w \end{pmatrix}\right)\overline{p\left(Q_k; \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}\right)} \\ + (1 - z\overline{z}_1)\sum_{k=1}^{n}\overleftarrow{p}\left(\tilde{Q}_k; \begin{pmatrix} z \\ w \end{pmatrix}\right)\overline{\overleftarrow{p}\left(\tilde{Q}_k; \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}\right)}, \end{aligned} \tag{3.1.16}$$

where

$$Q_k = \{0\} \times \{0, \dots, m-k\} \cup \{1, \dots, n\} \times \{-k+1, \dots, m-k\}, \qquad k = 1, \dots, m,$$

and

$$\tilde{Q}_k = \{0, \dots, n-k\} \times \{0\} \cup \{-k+1, \dots, n-k\} \times \{1, \dots, m\},$$

and Q_k and $\tilde{Q}_k$ are ordered so that $(n, m-k)$ and $(n-k, m)$ appear last, respectively.

In addition, we may recast Theorem 2.3.1 in the current context as follows.

THEOREM 3.1.8. *Let $\Lambda_+ = \{0, \dots, n\} \times \{0, \dots, m\}$ be ordered lexicographically, and let ρ be a positive Borel measure on $\mathbb{T}^2$ so that its Fourier coefficients c_u, $u \in \mathbb{Z}^2$ satisfy $(c_{u-v})_{u,v\in\Lambda_+} > 0$. Then the polynomial $p(\Lambda_+; \binom{z}{w})$ is stable and satisfies*

$$c_u = \frac{1}{(2\pi i)^2}\iint_{\mathbb{T}^2}\begin{pmatrix} z \\ w \end{pmatrix}^{-u}\frac{1}{|p(\Lambda_+; \binom{z}{w})|^2}\frac{dz}{z}\frac{dw}{w}, u \in \Lambda_+ - \Lambda_+, \tag{3.1.17}$$

if and only if

$$\left[(c_{u-v})_{u,v\in\Lambda_+\setminus\{(0,0)\}}\right]^{-1}_{\substack{\{1,\dots,n\}\times\{0\}\\ \{0\}\times\{1,\dots,m\}}} = 0. \tag{3.1.18}$$

Similarly, the orthogonal polynomial $\phi(\Lambda_+; \binom{z}{w})$ is anti-stable (i.e., $\phi(\Lambda_+; \binom{z}{w}) \neq 0$ for $(z, w) \in (\mathbb{C}_\infty \setminus \mathbb{D})^2$) and satisfies

$$c_u = \frac{1}{(2\pi i)^2}\iint_{\mathbb{T}^2}\begin{pmatrix} z \\ w \end{pmatrix}^{-u}\frac{1}{|\phi(\Lambda_+; \binom{z}{w})|^2}\frac{dz}{z}\frac{dw}{w}, \quad u \in \Lambda_+ - \Lambda_+, \tag{3.1.19}$$

if and only if (3.1.18) *holds.*

Proof. The first part is exactly the statement in Theorem 2.3.1. For the second part, use the connection (3.1.7) and the fact that $(n, m) - \Lambda_+ = \Lambda_+$. □

Proof of Theorem 1.1.2 follows directly from Theorem 3.1.8. □

3.2. *Stable autoregressive filters.* Two-dimensional signal processing has been an important field of study in the last decades. Early influential papers in this area are the ones by Whittle [63], and Helson and Lowdenslager [47], [48], where many of the one-dimensional results were generalized to the two-dimensional situation after introduction of a notion of causality based on halfspaces.

In this section we shall show how the positive extension results may be interpreted in the context of autoregressive filters. We consider stochastic processes $X = (x_u)_{u \in \mathbb{Z}^2}$ depending on two discrete variables defined on a fixed probability space $(\Omega, \mathcal{A}, P)$. We shall consider *zero mean* processes $X = (x_u)_{u \in \mathbb{Z}^2}$; i.e., $E(x_u) = 0$ for all u. Recall that the space $L^2(\Omega, \mathcal{A}, P)$ of square integrable random variables endowed with the inner product

$$\langle x, y \rangle := E(y^* x)$$

is a Hilbert space. A stochastic process $X = (x_u)_{u \in \mathbb{Z}^2}$ is called a (*wide sense*) *stationary process* on $\mathbb{Z}^2$ if for $u, v \in \mathbb{Z}^2$ we have that

$$E(x_u^* x_v) = E(x_{u+p}^* x_{v+p}) =: R_X(u - v), \text{ for all } p \in \mathbb{Z}^2.$$

It is known that the function R_X, termed the *covariance function* of X, defines a *positive semi-definite function* on $\mathbb{Z}^2$; i.e.,

$$\sum_{i,j=1}^{p} \alpha_i \bar{\alpha}_j R_X(u_i - u_j) \geq 0,$$

for all $p \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_p \in \mathbb{C}, u_1, \ldots, u_p \in \mathbb{Z}^2$. The theorem of Herglotz, Bochner and Weil (see, e.g., [49, Ch. 8]) on positive definite functions states that for such a function R_X there is a positive regular bounded measure μ_X defined for Borel sets on the torus $[0, 2\pi]^2$ such that

$$R_X(u) = \int e^{-i\langle u, t \rangle} d\mu_X(t),$$

for all two tuples of integers u. The measure μ_X is referred to as the *spectral distribution measure* of the process X. The *spectral density* $f_X(t)$ of the process X is the spectral density of the absolutely continuous part of μ_X, i.e., the absolutely continuous part of μ_X equals

$$f_X(t_1, t_2) \frac{dt_1 dt_2}{(2\pi)^2}.$$

Let H be the standard halfspace in $\mathbb{Z}^2$, and let $(0, 0) \in \Lambda_+ \subset H \cup \{(0, 0)\}$ be a finite set. A zero-mean stationary stochastic process $X = (x_u)_{u \in \mathbb{Z}^2}$

is said to be AR(Λ_+) (autoregressive), if there exist complex numbers a_k, $k \in \Lambda_+ \setminus \{(0,0)\}$, so that for every u

$$x_u + \sum_{\substack{v\in\Lambda_+\\ v\neq(0,0)}} a_v x_{u-v} = e_u, \qquad u \in \mathbb{Z}^2, \tag{3.2.1}$$

where $\{e_u \; ; u \in \mathbb{Z}^2\}$ is a white noise zero mean process with variance σ^2, for some positive σ. The AR(Λ_+) process is said to be *causal* if there is a solution to equations (3.2.1) of the form

$$x_u = \sum_{v\in H\cup\{(0,0)\}} \phi_v e_{u-v}, u \in \mathbb{Z}^2,$$

with $\sum\limits_{v\in H\cup\{(0,0)\}} |\phi_v| < \infty$. The bivariate (AR) model problem concerns the following. Given autocorrelation elements

$$c_u = E(x_u \bar{x}_0), \; u \in \Lambda_+,$$

determine, if possible, the coefficients $a_v, v \in \Lambda_+ \setminus \{(0,0)\}$, and the variance σ^2 of a causal autoregressive filter representation (3.2.1). It is well known that if (3.2.1) is causal then

$$p(z,w) := \frac{1}{\sigma}\left(1 + \sum_{0\neq v\in\Lambda_+} \overline{a_v} \binom{z}{w}^v\right)$$

is stable and its spectral density function has Fourier coefficients equal to $E(x_u \bar{x}_0)$. Conversely, a solution $p(z,w) = \sum_{u\in\Lambda_+} p_u \binom{z}{w}^u$ to the positive extension problem with given data $c_u, u \in \Lambda_+$, yields a solution to the stable bivariate autoregressive filter problem by putting $\sigma = \frac{1}{p_{00}}$, and $a_u = \frac{\overline{p_u}}{p_{00}}$. We may therefore interpret the results of Section 2.4 in terms of autoregressive filters. Below is this interpretation for the case when $\Lambda_+ = \{0,\ldots,n\} \times \{0,\ldots,m\}$.

THEOREM 3.2.1. *There exists a causal solution to* (3.2.1) *for the given autocorrelation elements* $c_{k,l}$, $(k,l) \in \{0,\ldots,n\} \times \{0,\ldots,m\}$ *if and only if there exist complex numbers* $c_{k,l}$, $(k,l) \in \{1,\ldots,n\} \times \{-m,\ldots,-1\}$, *so that the* $(n+1)(m+1) \times (n+1)(m+1)$ *doubly indexed Toeplitz matrix*

$$\Gamma = \begin{bmatrix} C_0 & \cdots & C_{-n} \\ \vdots & \ddots & \vdots \\ C_n & \cdots & C_0 \end{bmatrix},$$

where

$$C_j = \begin{bmatrix} c_{j0} & \cdots & c_{j,-m} \\ \vdots & \ddots & \vdots \\ c_{jm} & \cdots & c_{j0} \end{bmatrix}, \qquad j = -n,\ldots,n,$$

and $c_{-k,-l} = \bar{c}_{k,l}$, *has the following two properties*:

(1) Γ *is positive definite*;

(2) *The* $(n+1)m \times (m+1)n$ *submatrix of* Γ *obtained by removing scalar rows* $1 + j(m+1)$, $j = 0, \ldots, n$, *and scalar columns* $1, 2, \ldots, m+1$, *has rank* nm.

In this case the vector

$$\frac{1}{\sigma^2}[a_{nm} \cdots a_{n0} \ \cdots \ a_{0m} \cdots a_{01} \ 1]$$

is the last row of the inverse of Γ.

Proof. Let c_u, $u \in \Lambda_+$, be given so that c_u, $u \in (\Lambda_+ - \Lambda_+) \backslash (\Lambda_+ \cup -\Lambda_+)$ exist satisfying (1) and (2) in the statement of the theorem. Thus Theorem 2.4.1(ii) is satisfied, yielding the existence of a stable polynomial $p(z,w) = \sum_{k=0}^{n} \sum_{l=0}^{m} p_{k,l} z^k w^l$ with $p_{00} > 0$ as in (i) of Theorem 2.4.1. Put now, $\sigma = \frac{1}{p_{00}}$ and $a_{kl} = \overline{p_{kl}} p_{00}$, $(k,l) \neq (0,0)$. These choices for σ and $a_{k,l}$ provide the desired AR representation (3.2.1). That the solution is causal follows from Proposition 2.1.1.

Conversely, when a causal solution to the AR representation (3.2.1) is given, one may set $p_{00} = \frac{1}{\sigma}$ and $p_{k,l} = \frac{\overline{a_{kl}}}{\sigma}$, $(k,l) \neq (0,0)$, and obtain a stable polynomial satisfying Theorem 2.4.1(i). Thus Theorem 2.4.1(ii) is valid, yielding (1) and (2) in Theorem 3.2.1. $\square$

For other sets Λ_+ one needs to use the appropriate result of Section 2.4.

Based on characterization Theorem 2.4.3(ii) for the existence of a causal solution to the AR model problem, a numerical algorithm was developed in [64] for computing the solution. The algorithm has been implemented in MATLAB and several experiments have been executed. We cite here two experiments.

Experiment 1. For the given data

$$c_{00} = 8, c_{01} = 4, c_{02} = 1, c_{03} = .25, c_{04} = 0.01, c_{12} = 2, c_{13} = 0.5,$$
$$c_{14} = 0.03, c_{15} = 0.006, c_{24} = 1, c_{25} = 0.1, c_{26} = 0.01, c_{27} = 0.001,$$

the program arrives at the pseudopolynomial (in MATLAB short format)

$$\begin{aligned} p(z,w) = \frac{1}{\sqrt{0.1925}} (&0.1925 - 0.1215w + 0.0450w^2 - 0.0158w^3 + 0.0049w^4 \\ &-0.0521zw^2 + 0.0486zw^3 - 0.0239zw^4 + 0.0083zw^5 - 0.0157z^2w^4 \\ &+0.0157z^2w^5 - 0.0089z^2w^6 + 0.0034z^2w^7). \end{aligned}$$

After computing the Fourier coefficients of $1/|p(w,z)|^2$ (by using 2D-fft and 2D-ifft with grid size 64) we arrive at an error of 1.1026e-09. The error is the Euclidian norm of the vector of differences of the given and the obtained Fourier coefficients.

Experiment 2. For the data

$$c_{00} = 1, \quad c_{01} = .4, \quad c_{02} = .1, \quad c_{03} = .04, \quad c_{10} = .2,$$
$$c_{11} = .05, \quad c_{12} = .02, \quad c_{13} = .005, \quad c_{20} = .1, \quad c_{21} = .05, \quad c_{22} = .01,$$
$$c_{23} = .003, \quad c_{30} = .04, \quad c_{31} = .015, \quad c_{32} = .002, \quad c_{33} = .0005,$$

we find the pseudopolynomial

$$\frac{1}{\sqrt{1.2646}}(1.2646 - .5572w + .1171w^2 - .0429w^3 - .2612z + .1791zw$$
$$-.0791zw^2 + .0324zw^3 - .0607z^2 - .0171z^2w + .0336z^2w^2 - .0143z^2w^3$$
$$-.0132z^3 + .0107z^3w - .0058z^3w^2 + .0037z^3w^3).$$

The error here is 2.0926e-11.

3.3. *Fejér-Riesz factorization.* The well-known Fejér-Riesz lemma, in the nonsingular case, states that a trigonometric polynomial $f(z) = f_{-n}z^{-n} + \cdots + f_nz^n$ that takes on positive values on the circle (i.e., $f(z) > 0$ for $|z| = 1$) can be written as the modulus squared of a stable polynomial of the same degree. That is, there exists a stable polynomial $p(z) = p_0 + \cdots + p_nz^n$ such that

$$f(z) = |p(z)|^2, \qquad |z| = 1.$$

In this section we obtain a two-variable variation of this result.

Let H be the standard halfspace in $\mathbb{Z}^2$, and let Λ_+ be a subset of $H \cup \{(0,0)\}$ containing $(0,0)$. Let $f(z,w)$ be a Wiener function with Fourier support in $\Lambda_+ - \Lambda_+$. Thus

$$f(z,w) = \sum_{(k,l)\in\Lambda_+-\Lambda_+} f_{kl}z^kw^l \ , \quad \sum_{(k,l)\in\Lambda_+-\Lambda_+} |f_{kl}| < \infty.$$

Supposing that $f(z,w) > 0$ for $|z| = |w| = 1$, we ask the question whether there exists a stable Wiener function $p(z,w)$ with Fourier support in Λ_+ so that $f(z,w) = |p(z,w)|^2$, $(z,w) \in \mathbb{T}^2$. For the case when Λ_+ is the strip $\Lambda_+ = \{(n,m) : 0 < n \leq r \text{ or } (n = 0 \text{ and } m \geq 0)\}$ this question was answered affirmatively in [5], [6]. Also, for the truncated strip $\Lambda_+ = \{(n,m) : 0 < n < r$ or $(n = 0$ and $m \geq 0)$ or $(n = r$ and $m \leq s)\}$ the answer is affirmative, as was observed in [56]. It needs to be noted that in both these two cases (as well as in the classical one-variable case) $\Lambda_+ - \Lambda_+ = \Lambda_+ \cup (-\Lambda_+)$, which was conjectured by A. Seghier to be crucial for a direct factorization result to exist. In the following theorem we shall deal with the case when Λ_+ is a finite subset of $\mathbb{Z}^2$. In that case we always have that $\Lambda_+ - \Lambda_+ \neq \Lambda_+ \cup (-\Lambda_+)$ (unless Λ_+ lies on a line, reducing it to the one-variable case). One may of course consider algebras of functions other than the Wiener algebra (e.g., continuous functions, essentially bounded functions); however, for the case when $|\Lambda_+| < \infty$

the problem is independent of the choice of any reasonable algebra. Recall that

$$J(n,m,q)=\bigcup_{i=0}^{n}\{i\}\times\{-iq,\ldots,m-iq\},\qquad n,m\geq 0,\quad q\in\mathbb{Z}.$$

THEOREM 3.3.1. *Let $(0,0)\in\Lambda_+\subset H$ be a finite set, and suppose that*

$$f(z,w)=\sum_{(k,l)\in\Lambda_+-\Lambda_+} f_{kl}z^kw^l,$$

is positive on the bitorus. Let c_{rs}, $(r,s)\in\mathbb{Z}^2$, denote the Fourier coefficients of $\frac{1}{f(z,w)}$. The following are equivalent:

(i) *There exists a stable pseudopolynomial $p(z,w)$ with support $(\hat{p})\subseteq\Lambda_+$ such that $f(z,w)=|p(z,w)|^2$, $|z|=|w|=1$;*

(ii) *For some $J(n,m,q)$ with $\Lambda_+\subseteq J(n,m,q)$*

$$\left[(c_{u-v})_{u,v\in J(n,m,q)\setminus\{(0,0)\}}\right]^{-1}_{\substack{\{(1,-q),(2,-2q),\ldots,(n,-nq)\}\\ \{0\}\times\{1,\ldots,m\}}}=0 \tag{3.3.1}$$

and

$$\left[(c_{u-v})_{u,v\in J(n,m,q)}\right]^{-1}_{\substack{J(n,m,q)\setminus\Lambda_+\\ \{0\}\times\{0\}}}=0. \tag{3.3.2}$$

(iii) *For all $J(n,m,q)$ with $\Lambda_+\subseteq J(n,m,q)$ (3.3.1) and (3.3.2) hold.*

In the case one of (i)–(iii) (*and thus all of* (i)–(iii)) *hold, there exists*

$$p(z,w)=q_{00}^{-1/2}\left(\sum_{(k,l)\in\Lambda_+} q_{kl}z^kw^l\right), \tag{3.3.3}$$

where

$$(q_u)_{u\in\Lambda_+}=\left[(c_{u-v})_{u,v\in\Lambda_+}\right]^{-1}(\delta_u)_{u\in\Lambda_+}. \tag{3.3.4}$$

Proof. Choose $J(n,m,q)$ so that $\Lambda_+\subseteq J(n,m,q)$. Using the change of variables $\tilde{f}(z,w):=f(zw^q,w)=|p(zw^q,w)|^2=:|\tilde{p}(z,w)|^2$, we get that the Fourier coefficients $\tilde{c}_{kl}$ of $\frac{1}{\tilde{f}}$ satisfy $\tilde{c}_{kl}=c_{k,l+kq}$, so that the corresponding Fourier support is $J(n,m,0)$. We may therefore without loss of generality assume that $q=0$.

(i) $\to$ (iii). Consider the set of Fourier coefficients $\{c_{kl},\ (k,l)\in\Lambda_+\}$. This collection satisfies the conditions in Theorem 2.4.3(i), and therefore we may find complex numbers $c_u\in(J(n,m,0)-J(n,m,0))\setminus(\Lambda_+\cup(-\Lambda_+))$ so that (2.4.8), (2.4.9) and (2.4.10) are satisfied. Moreover, they are obtained in the proof of Theorem 2.4.3 by letting $c_u = \widehat{\frac{1}{|p|^2}}(u)$. Note that conditions (2.4.9) and (2.4.10) coincide with conditions (3.3.1) and (3.3.2), finishing the proof of (i) $\to$ (iii).

The implication (iii) $\to$ (ii) is trivial.

For (ii) $\to$ (i), observe that the coefficients c_u satisfy (2.4.8), (2.4.9) and (2.4.10). Indeed, (2.4.9) and (2.4.10) follow directly from (3.3.1) and (3.3.2), while (2.4.8) follows from the positivity of f. Introduce now the stable $p(z,w)$ as in (3.3.3) and (3.3.4), obtaining that $\widehat{\frac{1}{|p|^2}}(u) = c_u = \widehat{\frac{1}{f}}(u)$, $u \in \Lambda_+ - \Lambda_+$ (see Remark 2.4.2). Consequently, $\frac{1}{|p|^2}$ and $\frac{1}{f}$ are both, in the terminology of [4], positive extensions of $\{c_u\}_{u\in\Lambda_+-\Lambda_+}$ whose reciprocal has Fourier support in $\Lambda_+ - \Lambda_+$. By the uniqueness result of the maximum entropy extension (see [68] or Theorem 3.1 in [4]) , $\frac{1}{|p|^2} = \frac{1}{f}$, yielding (i). $\square$

Proof of Theorem 1.1.3. Follows directly from Theorem 3.3.1 with $\Lambda_+ = J(n,m,0)$, and Proposition 2.1.1. $\square$

Note that in terms of inner/outer factorizations Theorem 3.3.1 gives a criterion for when an invertible pseudopolynomial P has an outer factor with the same Fourier support. Indeed, one lets $f = |P|^2$ and checks whether conditions (3.3.1) and (3.3.2) hold. If so, p as in (3.3.3) gives the outer factor (since support$(\widehat{p^{\pm 1}}) \subseteq H \cup \{(0,0)\}$) and $\frac{P}{p}$ has modulus constant equal to 1.

The criterion in Theorem 3.3.1 allows for a numerical algorithm to obtain the factor p, when it exists. Let us illustrate this in the following example.

Example 3.3.2. Let

$$f(z,w) = \sum_{i=-2}^{2}\sum_{j=-2}^{2} z^i w^j \Big(\sum_{r=0}^{2-|i|}\sum_{s=0}^{2-|j|} 2^{-2(r+s)-|i|-|j|}\Big).$$

Computing the Fourier coefficients of the reciprocal of f (using MATLAB; truncating the Fourier series at index 64), we get:

$$c_{0,0} = 1.6125, c_{0,1} = c_{1,0} = -0.6450, c_{0,2} = c_{2,0} = -0.0806, c_{1,-2} = 0.0322,$$

$$c_{1,-1} = 0.2580, c_{1,1} = 0.2580, c_{1,2} = c_{2,1} = 0.0322, c_{2,-2} = 0.0040,$$

$$c_{2,-1} = 0.0322, c_{2,2} = 0.0040,$$

where only the first four decimal digits show. In order to check (3.3.1) (where $n = m = 2, q = 0$) we compute

$$\begin{pmatrix} c_{0,0} & c_{0,-1} & c_{-1,1} & c_{-1,0} & c_{-1,-1} & c_{-2,1} & c_{-2,0} & c_{-2,-1} \\ c_{0,1} & c_{0,0} & c_{-1,2} & c_{-1,1} & c_{-1,0} & c_{-2,2} & c_{-2,1} & c_{-2,0} \\ c_{1,-1} & c_{1,-2} & c_{0,0} & c_{0,-1} & c_{0,-2} & c_{-1,0} & c_{-1,-1} & c_{-1,-2} \\ c_{1,0} & c_{1,-1} & c_{0,1} & c_{0,0} & c_{0,-1} & c_{-1,1} & c_{-1,0} & c_{-1,-1} \\ c_{1,1} & c_{1,0} & c_{0,2} & c_{0,1} & c_{0,0} & c_{-1,2} & c_{-1,1} & c_{-1,0} \\ c_{2,-1} & c_{2,-2} & c_{1,0} & c_{1,-1} & c_{1,-2} & c_{0,0} & c_{0,-1} & c_{0,-2} \\ c_{2,0} & c_{2,-1} & c_{1,1} & c_{1,0} & c_{1,-1} & c_{0,1} & c_{0,0} & c_{0,-1} \\ c_{2,1} & c_{2,0} & c_{1,2} & c_{1,1} & c_{1,0} & c_{0,2} & c_{0,1} & c_{0,0} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0.9375 & 0.3750 & 0.0000 & 0.4688 & 0.1875 & 0.0000 & 0.2344 & 0.0938 \\ 0.3750 & 0.9375 & 0.0000 & 0.1875 & 0.4688 & 0.0000 & 0.0938 & 0.2344 \\ 0.0000 & 0.0000 & 0.9375 & 0.4688 & 0.2344 & 0.3750 & 0.1875 & 0.0938 \\ 0.4688 & 0.1875 & 0.4688 & 1.3477 & 0.5625 & 0.1875 & 0.5625 & 0.2344 \\ 0.1875 & 0.4688 & 0.2344 & 0.5625 & 1.1719 & 0.0938 & 0.2344 & 0.4922 \\ 0.0000 & 0.0000 & 0.3750 & 0.1875 & 0.0938 & 0.9375 & 0.4688 & 0.2344 \\ 0.2344 & 0.0938 & 0.1875 & 0.5625 & 0.2344 & 0.4688 & 1.1719 & 0.4922 \\ 0.0938 & 0.2344 & 0.0938 & 0.2344 & 0.4922 & 0.2344 & 0.4922 & 0.9961 \end{pmatrix},$$

which has zeroes in the required positions. Since $\Lambda_+ = J(2,2,0)$ the condition (3.3.2) is void. Computing $p(z,w)$ one finds $p(z,w) = \sum_{k,l=0}^{2} 2^{-k-l} z^k w^l$.

Our result is quite different from results regarding writing positive trigonometric polynomials as sums of squares of (pseudo-)polynomials (see, e.g., [10], [58], [4]), again stressing the fact that we are considering functions of more than one variable. For example, the positive function $|z-4|^2 + |w-2|^2$ cannot be written as $|p(z,w)|^2$ where p is a pseudopolynomial (i.e., p has finite Fourier support). One may, however, write $|z-4|^2 + |w-2|^2 = |p(z,w)|^2$ when one allows p to be a Wiener function with infinite Fourier support $\{0\} \times \{0,1,2,\dots\} \cup \{1\} \times \{\dots,-2,-1,0\}$ and in that case p can be chosen to be stable as well (see [56]).

3.4. *Inverses of doubly-indexed Toeplitz matrices.* Due to the results developed in Section 2.3, we may formulate the following procedure for finding the inverse of a doubly indexed positive definite Toeplitz matrix that satisfies a low rank condition. In particular, it shows that in this case the matrix is fully determined by the first column of its inverse. Recall that the notion of a left stable factor is defined in Section 1.3.

THEOREM 3.4.1. *Let C be a positive definite block Toeplitz matrix $C = (C_{i-j})_{i,j=0}^n$ whose blocks $C_j = (c_{j,k-l})_{k,l=0}^m$ are also Toeplitz. Suppose in addition that*

$$\operatorname{rank}(c_{u-v})_{\substack{u\in\{0,\dots,n\}\times\{1,\dots,m\} \\ v\in\{1,\dots,n\}\times\{0,\dots,m\}}} = nm.$$

Let the $(i(m+1)+j)$th *entry of the first column of* C^{-1} *be denoted by* q_{ij}, $i=0,\dots,n$, $j=0,\dots,m$. *Then* $p(z,w) := \frac{1}{\sqrt{q_{00}}}\sum_{i=0}^{n}\sum_{j=0}^{m} q_{ij}z^iw^j$ *is stable. Furthermore, let*

$$E_{m-1}(z) := \begin{bmatrix} p_0(z) & & \bigcirc \\ \vdots & \ddots & \\ p_{m-1}(z) & \cdots & p_0(z) \end{bmatrix} \begin{bmatrix} \bar p_0(1/z) & \cdots & \bar p_{m-1}(1/z) \\ & \ddots & \vdots \\ \bigcirc & & \bar p_0(1/z) \end{bmatrix} - \begin{bmatrix} \bar p_m(1/z) & & \bigcirc \\ \vdots & \ddots & \\ \bar p_1(1/z) & \cdots & \bar p_m(1/z) \end{bmatrix} \begin{bmatrix} p_m(z) & \cdots & p_1(z) \\ & \ddots & \vdots \\ \bigcirc & & p_m(z) \end{bmatrix},$$

where $p(z,w)=\sum_{i=0}^{m} p_i(z)w^i$. *Then the following formula for* C^{-1} *holds*:

$$C^{-1} = \begin{pmatrix} P_0 & & & \\ P_1 & P_0 & & \\ \vdots & \vdots & \ddots & \\ P_n & P_{n-1} & \cdots & P_0 \end{pmatrix} \begin{pmatrix} P_0^* & P_1^* & \cdots & P_n^* \\ & P_0^* & \cdots & P_{n-1}^* \\ & & \ddots & \vdots \\ & & & P_0^* \end{pmatrix}$$
$$- \begin{pmatrix} 0 & & & \\ J_m(P_n^*)^TJ_m & 0 & & \\ \vdots & \ddots & \ddots & \\ J_m(P_1^*)^TJ_m & \cdots & J_m(P_n^*)^TJ_m & 0 \end{pmatrix} \begin{pmatrix} 0 & J_mP_n^TJ_m & \cdots & J_mP_1^TJ_m \\ & 0 & \ddots & \vdots \\ & & \ddots & J_mP_n^TJ_m \\ & & & 0 \end{pmatrix},$$

where

$$P_i = \begin{pmatrix} p_{i0} & 0 \\ \mathrm{col}(p_{ij})_{j=1}^m & F_i \end{pmatrix},$$

and $F(z)=\sum_{i=0}^{n} F_iz^i$ *is the left stable factor of* $E_{m-1}(z)$.

Proof. Let $p(z,w)$ be as above. It follows from Theorem 2.3.1 in Chapter 2 that $p(z,w)$ is stable. In addition, it is straightforward to check that

(3.4.1)
$$\frac{p(z,w)\overline{p(1/\bar z,w_1)} - \overleftarrow{p}(z,w)\overline{\overleftarrow{p}(1/\bar z,w_1)}}{1-w\bar w_1} = (1,\dots,w^{m-1})E_{m-1}(z)\begin{pmatrix} 1 \\ \vdots \\ \bar w_1^{m-1}\end{pmatrix}.$$

We have used a similar observation in the proof of Theorem 2.3.1. Furthermore, by (2.3.17) with $z_1 = 1/\bar z$,

(3.4.2)
$$\frac{p(z,w)\overline{p(1/\bar z,w_1)} - \overleftarrow{p}(z,w)\overline{\overleftarrow{p}(1/\bar z,w_1)}}{1-w\bar w_1} = P^{m-1}(z,w)P^{m-1}(1/\bar z,w_1)^*.$$

Combining (3.4.1), (3.4.2), and (2.3.9) of Section 2.3 we find

$$E_{m-1}(z) = A_n^{m-1}(z)A_n^{m-1}(0)^{-1}A_n^{m-1}(1/\bar{z})^*, \tag{3.4.3}$$

where $A_n^{m-1}(z)$ is as defined in (2.3.7). Since $A_n^{m-1}(z)(Y_n^{m-1})^{-1}$ is stable (use [17, Th. 6] where $(Y_n^{m-1})^*$ is the lower Cholesky factor of $A_n^{m-1}(0)$), and is lower triangular at 0 with positive diagonal entries, we must have that $A_n^{m-1}(z)(Y_n^{m-1})^{-1}$ is the left stable factor $F(z)$ of $E(z)$. Thus $F(z) = A_n^{m-1}(z)(Y_n^{m-1})^{-1}$. By Proposition 2.1.2(iii) we now have that $P(z)$ is the left stable factor of $E_m(z)$. By equation (2.3.28) $P(z) = A_n^m(z)(Y_n^m)^{-1}$. By the definition (2.3.7) of $A_n^m(z)$, this yields that $\mathrm{col}(P_i)_{i=0}^n$ is the first column of the lower Cholesky factor of C^{-1}. The result now follows from the matrix version of the Gohberg-Semencul formula (see [38]). □

Though the above result gives a way to construct C^{-1} based solely on its first column, the formula does not have the simple algebraic form as the classical Gohberg-Semencul [43] formula does. When $n = m = 1$ the formula for C^{-1} is as follows:

$$C^{-1} = \begin{pmatrix} p_{00} & \overline{p_{01}} & \overline{p_{10}} & \overline{p_{11}} \\ p_{01} & f & \frac{p_{01}\overline{p_{10}}}{p_{00}} & \overline{p_{10}} \\ p_{10} & \frac{p_{10}\overline{p_{01}}}{p_{00}} & f & \overline{p_{01}} \\ p_{11} & p_{10} & p_{01} & p_{00} \end{pmatrix},$$

where

$$\begin{aligned} f = \frac{1}{2p_{00}}(&p_{00}^2 + |p_{10}|^2 + |p_{01}|^2 - |p_{11}|^2 + (p_{00}^4 - 2|p_{10}|^2p_{00}^2 - 2p_{00}^2|p_{01}|^2 \\ &-2p_{00}^2|p_{11}|^2 + |p_{10}|^4 - 2|p_{01}|^2|p_{10}|^2 - 2|p_{10}|^2|p_{11}|^2 + |p_{01}|^4 \\ &-2|p_{01}|^2|p_{11}|^2 + |p_{11}|^4 + 4p_{11}\overline{p_{10}}\overline{p_{01}}p_{00} + 4p_{10}\overline{p_{11}}p_{01}p_{00})^{1/2}). \end{aligned}$$

Here it was assumed that $c_{1,-1} = \frac{\overline{c_{01}}c_{10}}{c_{00}}$. Clearly, the formula for the (2,2) entry (or, to be more precise, the $((0,1),(0,1))$ entry) of C^{-1} is uniquely determined by the first column of C^{-1}, but the formula also involves taking square roots, a feature that is not present in the classical Gohberg-Semencul formula. This suggests that an algebraic formula as simple the classical Gohberg-Semencul formula may not exist for doubly-indexed Toeplitz matrices.

Appendix

In this appendix we present an alternative proof of Theorem 2.4.1 (ii) → (i). Assume that c_u, $u \in \{-n, \ldots, n\} \times \{-m, \ldots, m\}$ are given so that

$$(c_{u-v})_{u,v\in\Lambda_+} > 0 \tag{A.1}$$

and

$$\operatorname{rank}(c_{u-v})_{\substack{u\in\{1,\ldots,n\}\times\{0,\ldots,m\}\\ v\in\{0,\ldots,n\}\times\{1,\ldots,m\}}} = nm. \tag{A.2}$$

Let C_j be the $(m+1)\times(m+1)$ Toeplitz matrix

$$C_j = \begin{bmatrix} c_{j0} & \cdots & c_{j,-m} \\ \vdots & \ddots & \vdots \\ c_{jm} & \cdots & c_{j0} \end{bmatrix}, \qquad j \in \{-n,\ldots,n\},$$

and Γ_k the $(k+1)\times(k+1)$ block Toeplitz matrix

$$\Gamma_k = \begin{bmatrix} C_0 & \cdots & C_{-k} \\ \vdots & \ddots & \vdots \\ C_k & \cdots & C_0 \end{bmatrix}, \qquad k \in \{0,\ldots,n\}.$$

By (A.1) , $\Gamma_n > 0$. Introduce the matrix-valued trigonometric polynomial

$$F(\lambda) = \sum_{j=-n}^{n} \lambda^j C_j, \qquad |\lambda| = 1.$$

By the results in Section 6 of [26] (see also [34], Section III.2 in [66] or Section II.3 in [40]) there exist unique $(m+1)\times(m+1)$ matrices C_j, $|j|>n$, so that

$$\sum_{j=-\infty}^{\infty} \|C_j\| < \infty,$$

and $F_{\mathrm{ext}}(\lambda) := \sum_{j=-\infty}^{\infty} \lambda^j C_j$ satisfies

$$\begin{aligned} F_{\mathrm{ext}}(\lambda) &> 0, \qquad |\lambda| = 1, \\ \widehat{F_{\mathrm{ext}}^{-1}}(k) &= 0, \qquad |k| > n. \end{aligned}$$

These matrices $C_j = C_{-j}^*$, $j>n$, are given inductively by

$$C_{n+j} = \begin{bmatrix} C_{n+j-1} \cdots C_j \end{bmatrix} \Gamma_{n-1}^{-1} \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}, \qquad j = 1,2,\ldots, \tag{A.3}$$

(see e.g., [27], [13], [14], [15]). We claim that because of (A.2), the matrices C_j, $j>n$, are Toeplitz.

LEMMA A.1. *The matrices C_j, $|j|>n$, are Toeplitz matrices.*

Proof. Let P and Q be the $(m+1)\times m$ matrices

$$P = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & & \bigcirc \\ \bigcirc & \ddots & \\ & & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & & \bigcirc \\ \bigcirc & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{pmatrix}.$$

Note that an $(m+1)\times(m+1)$ matrix M is Toeplitz if and only if

$$P^*MP = Q^*MQ.$$

Condition (A.2) tells us that

$$\text{(A.4)} \qquad \operatorname{rank}\begin{pmatrix} C_1P & C_0P & \cdots & C_{-n+1}P \\ C_2P & C_1P & \cdots & C_{-n+2}P \\ \vdots & & & \\ C_nP & C_{n-1}P & \cdots & C_0P \end{pmatrix} = nm.$$

We also have that

$$\text{(A.5)} \qquad \operatorname{rank}\begin{pmatrix} P^*C_0P & \cdots & P^*C_{-n+1}P \\ \vdots & & \\ P^*C_{n-1}P & \cdots & P^*C_0P \end{pmatrix} = nm,$$

since this matrix is a principal submatrix of size $nm \times nm$ of the positive definite matrix Γ_n.

Consider now the partial matrices

$$\text{(A.6)} \qquad \left(\bigoplus_{i=1}^{n+1} J_1\right) \begin{bmatrix} C_1 & C_0 & \cdots & C_{-n+1} \\ C_2 & C_1 & \cdots & C_{-n+2} \\ \vdots & \vdots & & \vdots \\ C_n & C_{n-1} & \cdots & C_0 \\ ? & C_n & \cdots & C_1 \end{bmatrix} \left(\bigoplus_{i=1}^{n+1} J_2\right)$$

where

$$(J_1, J_2) \in \{(I_{m+1}, P), (Q^*, I_{m+1}), (P^*, P), (Q^*, Q)\}.$$

Recall from [50] (see also [65] or Section IV.2 [66]) that

$$\begin{bmatrix} A & B \\ ? & C \end{bmatrix}$$

has a unique minimal rank completion if and only if

$$\operatorname{rank}\begin{bmatrix} A & B \end{bmatrix} = \operatorname{rank} B = \operatorname{rank}\begin{bmatrix} B \\ C \end{bmatrix}$$

and in that case

$$\begin{bmatrix} A & B \\ CB^{(-1)}A & C \end{bmatrix}$$

is the minimal rank completion, where $B^{(-1)}$ is a generalized inverse of B. The rank of this unique minimal rank completion equals $\operatorname{rank}(B)$.

From (A.4) and (A.5) and the Toeplitz structure it is not hard to see that all four partial matrices in (A.6) satisfy this uniqueness condition, and that the unique minimal rank completion of (A.6) is given by completing with

$$J_1 C_{n+1} J_2,$$

where C_{n+1} is as given by (A.3). We next note that, due to the Toeplitz structure of $C_{-n+1}, \ldots, C_n$, we have that the partial matrices in (A.6) with $(J_1, J_2) = (P^*, P)$ and $(J_1, J_2) = (Q^*, Q)$ are the same. Therefore, they have the same unique minimal rank completion, and thus

$$P^* C_{n+1} P = Q^* C_{n+1} Q,$$

giving that C_{n+1} is Toeplitz. In addition,

$$\operatorname{Im}\left(\bigoplus_{i=1}^{n+2} J_1\right)\begin{bmatrix} C_1 \\ \vdots \\ C_{n+1}\end{bmatrix} \subseteq \operatorname{Im}\left(\left(\bigoplus_{i=1}^{n+2} J_1\right)\begin{bmatrix} C_0 & \cdots & C_{-n+1} \\ \vdots & & \vdots \\ C_n & \cdots & C_1\end{bmatrix}\left(\bigoplus_{i=1}^{n} J_2\right)\right)$$

for all four possibilities of (J_1, J_2).

By repeating the same arguments for taller matrices (A.6) (i.e., block rows are added) one may show that $C_{n+2}, C_{n+3}, \ldots$ are Toeplitz as well. $\square$

Since C_j, $|j| > n$, are Toeplitz, we may define c_{jk}, $|j| > n$, $|k| \leq m$, via setting

$$C_j = \begin{pmatrix} c_{j0} & \cdots & c_{j,-m} \\ \vdots & & \vdots \\ c_{jm} & \cdots & c_{j0}\end{pmatrix}, \qquad |j| > n.$$

Let now

$$f_C(z, w) = \sum_{\substack{j \in \mathbb{Z} \\ |k| \leq m}} c_{jk} z^j w^k, \qquad |z| = |w| = 1.$$

We may now apply Theorem 1.1 in [6], where the positive definiteness of the Toeplitz operator follows from $F_{\text{ext}}(\lambda) > 0$, $|\lambda| = 1$. It is not hard to see (because of the construction of C_j, $|j| > n$) that the function $xD(x)^{-1/2}$ in Theorem 1.1 of [6] corresponds exactly to $p(z, w)$ in (2.4.5) of Theorem 2.4.1. Thus by Theorem 1.1 in [6] , p is stable, and $\widehat{\frac{1}{|p|^2}}(u) = c_u$, $u \in \mathbb{Z} \times \{-m, \ldots, m\}$. Thus, we have established Theorem 2.4.1(i).

Georgia Institute of Technology, Atlanta, GA
E-mail address: geronimo@math.gatech.edu

Drexel University, Philadelphia, PA
E-mail address: Hugo.Woerdeman@drexel.edu

References

[1] N. I. Akheizer and M. Krein, *Some Questions in the Theory of Moments*, A. M. S., Providence, R.I., 1962.

[2] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner Publ. Co., New York, 1965.

[3] F. Alizadeh, J.-P. A. Haeberly, M. V. Nayakkankuppam, and M. L. Overton, Sdppack version 0.8 beta for matlab 4.2: semidefinite programs (1997); http://www.cs.nyu.edu/overton/sdppack/sdppack.html.

[4] M. Bakonyi and G. Naevdal, On the matrix completion method for multidimensional moment problems, *Acta Sci. Math.* (Szeged) **64** (1998), 547–558.

[5] M. Bakonyi, L. Rodman, I. M. Spitkovsky, and H. J. Woerdeman, Positive extensions of matrix functions of two variables with support in an infinite band, *C. R. Acad. Sci. Paris Sér. I Math.* **323** (1996), 859–863.

[6] ———, Positive matrix functions on the bitorus with prescribed Fourier coefficients in a band, *J. Fourier Anal. Appl.* **5** (1999), 21–44.

[7] H. Bart, I. Gohberg, and M. A. Kaashoek, *Minimal Factorization of Matrix and Operator Functions*, Birkhäuser Verlag, Basel, 1979.

[8] N. K. Bose and Eliahu I. Jury, Positivity and stability test for multidimensional filters (discrete-continuous), *IEEE Trans. Acoust. Speech Signal Processing* ASSP-**22** (1974), 174–180.

[9] J. P. Burg, Maximum entropy spectral analysis, Ph.D. thesis, Department of Geophysics, Stanford University, Stanford, California, 1975.

[10] A. Calderon and R. Pepinsky, On the phases of Fourier coefficients for positive real periodic functions, in *Computing Methods and the Phase Problem in X-Ray Crystal Analysis* (R. Pepinsky, ed.) (1950), 339–346.

[11] G. Castro, Coefficient de réflexion généralisés, extension de covariance multidimensionelles et autres applications, Ph.D. thesis, Université de Paris-Sud Centre d'Orsay, Orsay, France, 1997.

[12] G. Castro and A. Seghier, Positive definite extension in the multidimensional case: New results, unpublished manuscript.

[13] T. Constantinescu, On the structure of positive Toeplitz forms, in *Dilation Theory, Toeplitz Operators, and Other Topics* (Timişoara/Herculane, 1982), Birkhäuser, Basel, 1983, pp. 127–149.

[14] ———, Schur analysis of positive block-matrices. I, in *Schur Methods in Operator Theory and Signal Processing*, Birkhäuser, Basel, 1986, pp. 191–206.

[15] ———, *Schur Parameters, Factorization and Dilation Problems*, Birkhäuser Verlag, Basel, 1996.

[16] I. Daubechies, Ten lectures on wavelets, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[17] P. Delsarte, Y. V. Genin, and Y. G. Kamp, Orthogonal polynomial matrices on the unit circle, *IEEE Trans. Circuits and Systems* CAS-**25** (1978), 149–160.

[18] ———, Planar least squares inverse polynomials. I. Algebraic properties, *IEEE Trans. Circuits and Systems* **26** (1979), 59–66.

[19] ———, Half-plane Toeplitz systems, *IEEE Trans. Inform. Theory* **26** (1980), 465–474.

[20] ———, A simple proof of Rudin's multivariable stability theorem, *IEEE Trans. Acoust. Speech Signal Process* **28** (1980), 701–705.

[21] ———, Half-plane minimization of matrix-valued quadratic functionals, *SIAM J. Algebraic Discrete Methods* **2** (1981), 192–211.

[22] B. W. Dickinson, Two-dimensional Markov spectrum estimates need not exist, *IEEE Trans. Inform. Theory* **26** (1980), 120–121.

[23] J. L. Doob, *Stochastic Processes*, John Wiley & Sons Inc., New York, 1953.

[24] M. A. Dritschel, On factorization of trigonometric polynomials, *Integral Equations Operator Theory* **49** (2004), 11–42.

[25] H. Dym, J contractive matrix functions, reproducing kernel Hilbert spaces and interpolation, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1989.

[26] H. Dym and I. Gohberg, Extensions of matrix valued functions with rational polynomial inverses, *Integral Equations Operator Theory* **2** (1979), 503–528.

[27] ———, Extensions of band matrices with band inverses, *Linear Algebra Appl.* **36** (1981), 1–24.

[28] ———, Extensions of kernels of Fredholm operators, *J. Analyse Math.* **42** (1982/83), 51–97.

[29] M. P. Ekstrom and J. W. Woods, Two-dimensional spectral factorization with applications in recursive digital filtering, *IEEE Trans. Acoust. Speech Signal Processing* ASSP-**24** (1976), 115–128.

[30] C. Foias, A. E. Frazho, I. Gohberg, and M. A. Kaashoek, *Metric Constrained Interpolation*, *Commutant Lifting and Systems*, Birkhäuser Verlag, Boston, Mass., 1998.

[31] C. Foias and A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, Birkhäuser Verlag, Boston, Mass., 1990.

[32] B. A. Francis, *A Course in H_∞ Control Theory*, Springer-Verlag, New York, 1987.

[33] Y. Genin and Y. Kamp, Counterexample in the least-squares inverse stabilization of 2d recursive filters, *Electron Lett.* **11** (1975), 330–331.

[34] J. S. Geronimo, Matrix orthogonal polynomials on the unit circle, *J. Math. Phys.* **22** (1981), 1359–1365.

[35] L. Ya. Geronimus, *Orthogonal Polynomials*: *Estimates*, *Asymptotic Formulas*, *and Series of Polynomials Orthogonal on the Unit Circle and on an Interval*, Consultants Bureau, New York, 1961.

[36] I. Gohberg (ed.), *I. Schur Methods in Operator Theory and Signal Processing*, Birkhäuser Verlag, Boston, Mass., 1986.

[37] I. Gohberg, S. Goldberg, and M. A. Kaashoek, *Classes of Linear Operators*. *Vol. I*, *II*, Birkhäuser Verlag, Boston, Mass., 1993.

[38] I. Gohberg and G. Heinig, Inversion of finite Toeplitz matrices consisting of elements of a noncommutative algebra, *Rev. Roumaine Math. Pures Appl.* **19** (1974), 623–663.

[39] I. Gohberg, M. A. Kaashoek, and D. C. Lay, Equivalence, linearization, and decomposition of holomorphic operator functions, *J. Funct. Anal.* **28** (1978), 102–144.

[40] I. Gohberg, M. A. Kaashoek, and H. J. Woerdeman, The band method for positive and contractive extension problems, *J. Operator Theory* **22** (1989), 109–155.

[41] ———, The band method for positive and strictly contractive extension problems: an alternative version and new applications, *Integral Equations Operator Theory* **12** (1989), 343–382.

[42] I. Gohberg and H. J. Landau, Prediction and the inverse of Toeplitz matrices, in *Approximation and Computation* (West Lafayette, IN, 1993), Birkhäuser Boston, Boston, Mass., 1994, pp. 219–229.

[43] I. Gohberg and A. A. Semencul, The inversion of finite Toeplitz matrices and their continual analogues, *Mat. Issled.* **7** (1972), 201–223, 290.

[44] U. Grenander and G. Szegö, *Toeplitz Forms and their Applications*, Univ. of California Press, Berkeley, 1958.

[45] W. H. Gustafson, A note on matrix inversion, *Linear Algebra Appl.* **57** (1984), 71–73.

[46] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

[47] H. Helson and D. Lowdenslager, Prediction theory and Fourier series in several variables, *Acta Math.* **99** (1958), 165–202.

[48] ———, Prediction theory and Fourier series in several variables. II, *Acta Math.* **106** (1961), 175–213.

[49] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Vol.* II: *Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups*, Springer-Verlag, New York, 1970.

[50] M. A. Kaashoek and H. J. Woerdeman, Unique minimal rank extensions of triangular operators, *J. Math. Anal. Appl.* **131** (1988), 501–516.

[51] T. Kailath, A. Vieira, and M. Morf, Inverses of Toeplitz operators, innovations, and orthogonal polynomials, *SIAM Rev.* **20** (1978), 106–119.

[52] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, second edition, Academic Press Inc., Orlando, FL, 1985.

[53] H. Lev-Ari, S. R. Parker, and T. Kailath, Multidimensional maximum-entropy covariance extension, *IEEE Trans. Inform. Theory* **35** (1989), 497–508.

[54] J. S. Lim and N. A. Malik, A new algorithm for two-dimensional maximum entropy power spectrum estimation, *IEEE Trans. Acoust.*, *Speech, Signal Processin* **29** (1981), 401–412.

[55] T. L. Marzetta, Two-dimensional linear prediction: autocorrelation arrays, minimum-phase prediction error filters, and reflection coefficient arrays, *IEEE Trans. Acoust. Speech Signal Process* **28** (1980), 725–733.

[56] L. Rodman, I. M. Spitkovsky, and H. J. Woerdeman, *Abstract Band Method via Factorization*, *Positive and Band Extensions of Multivariable Almost Periodic Matrix Functions*, *and Spectral Estimation*, *Memoirs Amer. Math. Soc.* **160** (2002), A.M.S., Providence, RI.

[57] M. Rosenblatt, A multi-dimensional prediction problem, *Ark. Mat.* **3** (1958), 407–424.

[58] W. Rudin, The extension problem for positive-definite functions, *Illinois J. Math.* **7** (1963), 532–539.

[59] L. M. Silverman, Realization of linear dynamical systems, *IEEE Trans. Automatic Control* **AC-16** (1971), 554–567.

[60] M. G. Strintzis, Tests of stability of multidimensional filters, *IEEE Trans. Circuits and Systems* **CAS-24** (1977), 432–437.

[61] G. Szegő, Beiträge zur Theorie der Toeplitzschen Formen (erste mitteilung), *Math. Z.* **6** (1920), 167–202.

[62] ———, Uber die Randwerte einer analytischen Funktion, *Math. Ann.* **84** (1921), 232–244.

[63] P. Whittle, On stationary processes in the plane, *Biometrika* **41** (1954), 434–449.

[64] H. J. Woerdeman, J. S. Geronimo, and G. Castro, A numerical algorithm for stable 2d autoregressive filter design, *Signal Processing* **83** (2003), 1299–1308.

[65] H. J. Woerdeman, The lower order of lower triangular operators and minimal rank extensions, *Integral Equations Operator Theory* **10** (1987), 859–879.

[66] ———, Matrix and operator extensions, Stichting Mathematisch Centrum voor Wiskunde en Informatica, Amsterdam, 1989.

[67] H. J. Woerdeman, Minimal rank completions of partial banded matrices, *Linear and Multilinear Algebra* **36** (1993), 59–68.

[68] J. W. Woods, Two-dimensional Markov spectral estimation, *IEEE Trans. Information Theory* IT-**22** (1976), 552–559.

[69] N. Young, *An Introduction to Hilbert Space*, Cambridge Univ. Press, Cambridge, 1988.

(Received January 24, 2000)