Annals of Mathematics, **160** (2004), 795–837

Extension properties of meromorphic mappings with values in non-Kähler complex manifolds

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0. Introduction

0.1. *Statement of the main result.* Denote by $\Delta(r)$ the disk of radius r in $\mathbb{C}$, $\Delta := \Delta(1)$, and for $0 < r < 1$ denote by $A(r,1) := \Delta \setminus \bar{\Delta}(r)$ an annulus in $\mathbb{C}$. Let $\Delta^n(r)$ denote the polydisk of radius r in $\mathbb{C}^n$ and $\Delta^n := \Delta^n(1)$. Let X be a compact complex manifold and consider a meromorphic mapping f from the ring domain $\Delta^n \times A(r,1)$ into X. In this paper we shall study the following:

Question. Suppose we know that for some nonempty open subset $U \subset \Delta^n$ our map f extends onto $U \times \Delta$. What is the maximal $\hat{U} \supset U$ such that f extends meromorphically onto $\hat{U} \times \Delta$?

This is the so-called Hartogs-type extension problem. If $\hat{U} = \Delta^n$ for any f with values in our X and any initial (nonempty!) U then one says that the Hartogs-type extension theorem holds for meromorphic mappings into this X. For $X = \mathbb{C}$, i.e., for holomorphic functions, the Hartogs-type extension theorem was proved by F. Hartogs in [Ha]. If $X = \mathbb{CP}^1$, i.e., for meromorphic functions, the result is due to E. Levi, see [Lv]. Since then the Hartogs-type extension theorem has been proved in at least two essentially more general cases than just holomorphic or meromorphic functions. Namely, for mappings into Kähler manifolds and into manifolds carrying complete Hermitian metrics of nonpositive holomorphic sectional curvature, see [Gr], [Iv-3], [Si-2], [Sh-1].

The goal of this paper is to initiate the systematic study of extension properties of meromorphic mappings with values in non-Kähler complex manifolds. Let h be some Hermitian metric on a complex manifold X and let ω_h be the associated $(1,1)$-form. We call ω_h (and h itself) *pluriclosed* or dd^c-closed if $dd^c\omega_h = 0$. In the sequel we shall not distinguish between Hermitian metrics and their associated forms. The latter we shall call simply metric forms.

*This research was partially done during the author's stays at MSRI (supported in part by NSF grant DMS-9022140) and at MPIM. I would like to give my thanks to both institutions for their hospitality.

Let A be a subset of Δ^{n+1} of Hausdorff $(2n-1)$-dimensional measure zero. Take a point $a\in A$ and a complex two-dimensional plane $P\ni a$ such that $P\cap A$ is of zero length. A sphere $\mathbb{S}^3=\{x\in P:\|x-a\|=\varepsilon\}$ with ε small will be called a "transversal sphere" if in addition $\mathbb{S}^3\cap A=\emptyset$. Take a nonempty open $U\subset\Delta^n$ and set $H_U^{n+1}(r)=\Delta^n\times A(r,1)\cup U\times\Delta$. We call this set the Hartogs figure over U.

Main Theorem. *Let $f:H_U^{n+1}(r)\to X$ be a meromorphic map into a compact complex manifold X, which admits a Hermitian metric h, such that the associated $(1,1)$-form ω_h is dd^c-closed. Then f extends to a meromorphic map $\hat{f}:\Delta^{n+1}\setminus A\to X$, where A is a complete $(n-1)$-polar, closed subset of Δ^{n+1} of Hausdorff $(2n-1)$-dimensional measure zero. Moreover, if A is the minimal closed subset such that f extends onto $\Delta^{n+1}\setminus A$ and $A\neq\emptyset$, then for every transversal sphere $\mathbb{S}^3\subset\Delta^{n+1}\setminus A$, its image $f(\mathbb{S}^3)$ is not homologous to zero in X.*

Remarks. 1. A (two-dimensional) *spherical shell* in a complex manifold X is the image Σ of the standard sphere $\mathbb{S}^3\subset\mathbb{C}^2$ under a holomorphic map of some neighborhood of $\mathbb{S}^3$ into X such that Σ is not homologous to zero in X. The Main Theorem states that if the singularity set A of our map f is nonempty, then X contains spherical shells.

2. If, again, $A\neq\emptyset$ then, because $A\cap H_U^{n+1}(r)=\emptyset$, the restriction $\pi|_A$: $A\to\Delta^n$ of the natural projection $\pi:\Delta^{n+1}\to\Delta^n$ onto A is proper. Therefore $\pi(A)$ is an $(n-1)$-polar subset in Δ^n of zero $(2n-1)$-dimensional measure. So, returning to our question, we see that $\hat{U}$ is equal to Δ^n minus a "thin" set.

We shall give a considerable number of examples illustrating results of this paper. Let us mention few of them.

Examples 1. Let X be the Hopf surface $X=(\mathbb{C}^2\setminus\{0\})/(z\sim 2z)$ and $f:\mathbb{C}^2\setminus\{0\}\to X$ be the canonical projection. The $(1,1)$-form $\omega=\frac{i}{2}\frac{dz_1\wedge d\bar{z}_1+dz_2\wedge d\bar{z}_2}{\|z\|^2}$ is well defined on X and $dd^c\omega=0$. In this example one easily sees that f is not extendable to zero and that the image of the unit sphere from $\mathbb{C}^2$ is not homologous to zero in X. Note also that $dd^cf^*\omega=dd^c\omega=-c_4\delta_{\{0\}}dz\wedge d\bar{z}$, where c_4 is the volume of the unit ball in $\mathbb{C}^2$ and $\delta_{\{0\}}$ is the delta-function.

2. In Section 3.6 we construct Example 3.7 of a 4-dimensional compact complex manifold X and a holomorphic mapping $f:\mathbb{B}^2\setminus\{a_k\}\to X$, where $\{a_k\}$ is a sequence of points converging to zero, such that f cannot be meromorphically extended to the neighborhood of any a_k.

3. We also construct an Example 3.6 where the singularity set A is of Cantor-type and pluripolar. This shows that the type of singularities described in our Main Theorem may occur. At the same time it should be noticed that we do not know if this X can be endowed with a pluriclosed metric.

4. Consider now the Hopf three-fold $X = (\mathbb{C}^3 \setminus \{0\})/(z \sim 2z)$. The analogous metric form $\omega = \frac{i}{2}\frac{dz_1\wedge d\bar{z}_1 + dz_2\wedge d\bar{z}_2 + dz_3\wedge d\bar{z}_3}{\|z\|^2}$ is no longer pluriclosed but only plurinegative (i.e. $dd^c\omega \leq 0$). Moreover, if we consider ω as a bidimension $(2,2)$ current, then it will provide a natural obstruction for the existence of a pluriclosed metric form on X. Natural projection $f : \mathbb{C}^3 \setminus \{0\} \to X$ has singularity of codimension three and X does not contain spherical shells of dimension two (but does contain a spherical shell of dimension three).

We also prove the Hartogs-type extension result for mappings into (reduced, normal) complex spaces with dd^c-negative metric forms, see Theorem 2.2. More examples, which are useful for the understanding of the extension properties of meromorphic mappings into non-Kähler manifolds are given in the last paragraph. There, also, a general conjecture is formulated.

0.2. *Corollaries.* All compact complex surfaces admit pluriclosed Hermitian metric forms. Therefore we have

Corollary 1. *If X is a compact complex surface, then*:

(a) *Every meromorphic map $f : H_U^{n+1}(r) \to X$ extends onto $\Delta^{n+1} \setminus A$, where A is an analytic set of pure codimension two*;

(b) *If Ω is a Stein surface and $K \Subset \Omega$ is a compact with connected complement, then every meromorphic map $f : \Omega\setminus K \to X$ extends onto $\Omega\setminus\{$finite set $\}$. If this set is not empty* (*respectively, if A from* (a) *is nonempty*), *then X is of class* VII *in the Enriques-Kodaira classification*;

(c) *If $f : \Omega \setminus K \to X$ is as in* (b) *but Ω of dimension at least three, then f extends onto the whole Ω.*

Remarks 1. The fact that in the case of surfaces, A is a genuine analytic subset of pure codimension two requires some additional (not complicated) considerations and is given in Section 3.4, where, also, some other cases when A can be proved to be analytic are discussed.

2. A wide class of complex manifolds without spherical shells is for example the class of such manifolds X where the Hurewicz homomorphism $\pi_3(X) \to H_3(X, Z)$ is trivial.

3. The Main Theorem was proved in [Iv-2] under an additional (very restrictive) assumption: *the manifold X does not contain rational curves.* In this case meromorphic maps into X are *holomorphic* . Also in [Iv-2] nothing was proved about the structure of the singular set A.

4. There is a hope that the surfaces with spherical shells could be classified, as well as surfaces containing at least one rational curve. Therefore the following somewhat surprising speculation, which immediately follows from Corollary 1, could be of some interest:

Corollary 2. *If a compact complex surface X is not "among the known ones" then for every domain D in a Stein surface every meromorphic mapping $f : D \to X$ is in fact holomorphic and extends as a holomorphic mapping $\hat{f} : \hat{D} \to X$ of the envelope of holomorphy $\hat{D}$ of D into X.*

At this point let us note that the notion of a spherical shell, as we understand it here, is different from the notion of *global spherical shell* from [Ka-1].

5. A real two-form ω on a complex manifold X is said to "tame" the complex structure J if for any nonzero tangent vector $v \in TX$ we have $\omega(v, Jv) > 0$. This is equivalent to the property that the $(1,1)$-component $\omega^{1,1}$ of ω is strictly positive. Complex manifolds admitting a *closed* form, which tames the complex structure, are of special interest. The class of such manifolds contains all Kähler manifolds. On the other hand, such metric forms are dd^c-closed. Indeed, if $\omega = \omega^{2,0} + \omega^{1,1} + \bar{\omega}^{2,0}$ and $d\omega = 0$, then $\partial\omega^{1,1} = -\bar{\partial}\omega^{2,0}$. Therefore $dd^c\omega^{1,1} = 2i\partial\bar{\partial}\omega^{1,1} = 0$. So the Main Theorem applies to meromorphic mappings into such manifolds. In fact, the technique of the proof gives more:

Corollary 3. *Suppose that a compact complex manifold X admits a strictly positive $(1,1)$-form, which is the $(1,1)$-component of a closed form. Then every meromorphic map $f : H_U^{n+1}(r) \to X$ extends onto Δ^{n+1}.*

This statement generalizes the Hartogs-type extension theorem for meromorphic mappings into Kähler manifolds from [Iv-3], but this generalization cannot be obtained by the methods of [Iv-3] and result from [Si-2] involved there. The reason is simply that the upper levels of Lelong numbers of pluriclosed (i.e., dd^c-closed) currents are no longer analytic (also integration by parts for dd^c-closed forms does not work as well as for d-closed ones).

It is also natural to consider the extension of meromorphic mappings from *singular* spaces. This is equivalent to considering multi-valued meromorphic correspondences from smooth domains, and this reduces to single-valued maps into symmetric powers of the image space, see Section 3 for details. However, one pays a price for these reductions. In this direction we construct, in Section 3, Example 3.5, which shows that a manifold possessing the Hartogs extension property for single-valued mappings may not possess it for multi-valued ones. The reason is that $\mathrm{Sym}^2(X)$ may contain a spherical shell, even if X contains none.

0.3. *Sketch of the proof.* Let us give a brief outline of the proof of the Main Theorem. We first consider the case of dimension two, i.e., $n = 1$. For $z \in \Delta$ set $\Delta_z := \{z\} \times \Delta$. For a meromorphic map $f : H_U^2(r) \to (X, \omega)$ denote by $a(z) = area_{\omega} f(\Delta_z) = \int_{\Delta} f|_{\Delta_z}^* \omega$ - the area of the image of the disk Δ_z. This is well defined for $z \in U$ after we shrink $A(r, 1)$ if necessary.

Step 1. Using dd^c-closedness of ω (and therefore of $f^*\omega$) we show that for "almost every" sequence $\{z_n\} \subset U$ converging to the boundary, areas $a(z_n)$ are uniformly bounded and converge to the area of $f(\Delta_{z_\infty})$, here $z_\infty \in \partial U \cap \Delta$ is the limit of $\{z_n\}$. This means in particular that $f_{z_\infty} := f|_{\{z_\infty\}\times A(r,1)}$ extends onto Δ_{z_∞}. And then we show that f can be extended holomorphically onto $V \times \Delta$, where V is a neighborhood of z_∞. Therefore if $\hat{U}$ is the maximal open set such that f can be extended onto $H^2_{\hat{U}}(r)$, then $\partial\hat{U} \cap \Delta$ should be "small". In fact we show that $\partial\hat{U} \cap \Delta$ is of harmonic measure zero; see Lemmas 2.3, 2.4.

Step 2. Interchanging coordinates in $\mathbb{C}^2$ and repeating Step 1, we see that f holomorphically extends onto $\Delta^2 \setminus (S_1 \times S_2)$, where S_1 and S_2 are compacts (after shrinking) of harmonic measure zero. We can use shrinking here, because subsets of harmonic measure zero in $\mathbb{C}$ are of Hausdorff dimension zero. Set $S = S_1 \times S_2$. Smooth form $T := f^*\omega$ on $\Delta^2 \setminus S$ has coefficients in $L^1_{\mathrm{loc}}(\Delta^2)$ and therefore has trivial extension $\tilde{T}$ onto Δ^2, see Lemma 3.3 from [Iv-2] and Lemma 2.1. We prove that $\mu := dd^c\tilde{T}$ is a nonpositive measure supported on S.

Step 3. Take a point $s_0 \in S$ and, using the fact that S is of Hausdorff dimension zero, take a small ball B centered at s_0 such that $\partial B \cap S = \emptyset$. Now we have two possibilities. First: $f(\partial B)$ is not homologous to zero in X. Then ∂B represent a spherical shell in X, as said in the remark after the Main Theorem. Second: $f(\partial B) \sim 0$ in X. Then we can prove, see Lemmas 2.5, 2.8, that $\tilde{T}$ is dd^c-closed and consequently can be written in the form $\tilde{T} = i(\partial\bar{\gamma} - \bar{\partial}\gamma)$, where γ is some $(0,1)$-current on B, which is smooth on $B \setminus S$. This allows us to estimate the area function $a(z)$ in the neighborhood of s_0 and extend f.

Step 4. We consider now the case $n \geq 2$. Using case $n = 1$ by sections we extend f onto $\Delta^{n+1} \setminus A$ where A is complete pluripolar of Hausdorff codimension four. Then take a transversal to A at point $a \in A$ complex two-dimensional direction and decompose the neighborhood W of a as $W = B^{n-1} \times B^2$, where $A \cap (B^{n-1} \times \partial B^2) = \emptyset$. If $f(\{a\} \times \partial B^2)$ is homologous to zero then we can repeat Step 3 "with parameters." This will give a uniform bound of the volume of the two-dimensional sections of the graph of f. Now we are in a position to apply the Lemma 1.3 (which is another main ingredient of this paper) to extend f onto W.

Remark. We want to finish this introduction with a brief account of existing methods of extension of meromorphic mappings. The first method, based on Bishop's extension theorem for analytic sets (appearing here as the graphs of mappings) and clever integration by parts was introduced by P. Griffiths in [Gr], developed by B. Shiffman in [Sh-2] and substantially enforced by Y.-T. Siu in [Si-2] (where the Thullen-type extension theorem is proved for mappings into Kähler manifolds), using his celebrated result on the analyticity of upper level sets of Lelong numbers of closed positive currents. The latter was by the way inspired by the extension theorem just mentioned. Finally, in [Iv-3] the Hartogs-type extendibility for the mappings into Kähler manifolds was proved using the result of Siu and a somewhat generalized classical method of "analytic disks". This method works well for mappings into Kähler manifolds.

The second method, based on the Hironaka imbedded resolution of singularities and lower estimates of Lelong numbers was proposed in [Iv-4] together with an example showing the principal difference between Kähler and non-Kähler cases. This method implies the Main Theorem of this paper for $n = 1, 2$ (this was not stated in [Iv-4]). However, further increasing of n meets technical difficulties at least on the level of the full and detailed proof of Hironaka's theorem (plus it should be accomplished with the detailed lower estimates of the Lelong numbers by blowings-up).

The third method is therefore proposed in this paper and is based on the Barlet cycle space theory. It gives definitely stronger and more general results than the previous two and is basically *much more simple*. The key point is Lemma 1.3 from Section 1. An important ingredient of the last two methods is the notion of a meromorphic family of analytic subsets and especially Lemma 2.4.1 from [Iv-4] about such families. The reader is therefore supposed to be familiar with Sections 2.3 and 2.4 of [Iv-4] while reading proofs of both Lemma 1.3 and Main Theorem.

I would like to give my thanks to the referee, who pointed out to me a gap in the proof of the analyticity of the singular set.

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1. Meromorphic mappings and cycle spaces

1.1. *Cycle space associated to a meromorphic map.* We shall freely use the results from the theory of cycle spaces developed by D. Barlet; see [Ba-1]. For the English spelling of Barlet's terminology we refer to [Fj]. Recall that an analytic k-cycle in a complex space Y is a formal sum $Z = \sum_j n_j Z_j$, where $\{Z_j\}$ is a locally finite sequence of analytic subsets (always of pure dimension k) and n_j are positive integers called multiplicities of Z_j. Let $|Z| := \bigcup_j Z_j$ be the support of Z. All complex spaces in this paper are reduced, normal and countable at infinity. All cycles, if the opposite is not stated, are supposed to have *connected* support. Set $A^k(r,1) = \Delta^k \setminus \bar{\Delta}^k(r)$.

Let X be a normal, reduced complex space equipped with some Hermitian metric. Let a holomorphic mapping $f : \bar{\Delta}^n \times \bar{A}^k(r,1) \to X$ be given. We shall start with the following space of cycles related to f. Fix some positive constant C and consider the set $\mathcal{C}_{f,C}$ of all analytic k-cycles Z in $Y := \Delta^{n+k} \times X$ such that:

(a) $Z \cap [\Delta^n \times \bar{A}^k(r,1) \times X] = \Gamma_{f_z} \cap [\bar{A}^k_z(r,1) \times X]$ for some $z \in \Delta^n$, where Γ_{f_z} is the graph of the restriction $f_z := f \mid_{A^k_z(r,1)}$. Here $A^k_z(r,1) := \{z\} \times A^k(r,1)$. This means, in particular, that for this z the mapping f_z extends meromorphically from $\bar{A}^k_z(r,1)$ onto $\bar{\Delta}^k_z := \{z\} \times \bar{\Delta}^k$.

(b) $\mathrm{vol}(Z) < C$ and the support $|Z|$ of Z is connected.

We put $\mathcal{C}_f := \bigcup_{C>0} \mathcal{C}_{f,C}$ and shall show that $\mathcal{C}_f$ is an analytic space of finite dimension in a neighborhood of each of its points.

Let Z be an analytic cycle of dimension k in a (reduced, normal) complex space Y. In our applications Y will be $\Delta^{n+k} \times X$. By a coordinate chart adapted to Z we shall understand an open set V in Y such that $V \cap |Z| \neq \emptyset$ together with an isomorphism j of V onto a closed subvariety $\tilde{V}$ in the neighborhood of $\bar{\Delta}^k \times \bar{\Delta}^q$ such that $j^{-1}(\bar{\Delta}^k \times \partial\Delta^q) \cap |Z| = \emptyset$. We shall denote such a chart by (V,j). The image $j(Z)$ of cycle Z under isomorphism j is the image of the underlying analytic set together with multiplicities. Sometimes we shall, following Barlet, denote: $\Delta^k = U, \Delta^q = B$ and call the quadruple $E = (V,j,U,B)$ a *scale* adapted to Z.

If $\mathrm{pr} : \mathbb{C}^k \times \mathbb{C}^q \to \mathbb{C}^k$ is the natural projection, then the restriction $\mathrm{pr} \mid_{j(Z)}$: $j(Z) \to \Delta^k$ is a branched covering of degree say d. The number q depends on the imbedding dimension of Y (or X in our case). Sometimes we shall skip j in our notation. The branched covering $\mathrm{pr} \mid_Z$: $Z \cap (\Delta^k \times \Delta^q) \to \Delta^k$ defines in a natural way a mapping $\phi_Z : \Delta^k \to \mathrm{Sym}^d(\Delta^q)$ — the d^{th} symmetric power of Δ^q — by setting $\phi_Z(z) = (\mathrm{pr} \mid_Z)^{-1}(z)$. This allows us to represent a cycle $Z \cap \Delta^{k+q}$ with $|Z| \cap (\bar{\Delta}^k \times \partial\Delta^q) = \emptyset$ as the graph of a d-valued holomorphic map.

Without loss of generality we suppose that our holomorphic mapping f is defined on $\Delta^n(a) \times A^k(r_1,b)$ with $a,b > 1, r_1 < r$. Now, each $Z \in \mathcal{C}_f$ can be covered by a *finite* number of adapted neighborhoods (V_α, j_α). Such covering will be called an adapted covering. Denote the union $\bigcup_\alpha V_\alpha$ by W_Z. Taking this covering $\{(V_\alpha, j_\alpha)\}$ to be small enough, we can further suppose that:

(c) If $V_{\alpha_1} \cap V_{\alpha_2} \neq \emptyset$, then on every irreducible component of the intersection $Z \cap V_{\alpha_1} \cap V_{\alpha_2}$ a point x_1 is fixed so that: (c_1) either there exists a polycylindrical neighborhood $\Delta^k_1 \subset \Delta^k$ of $pr(j_{\alpha_1}(x_1))$ such that the chart $V_{12} = j^{-1}_{\alpha_1}(\Delta^k_1 \times \Delta^q)$

is adapted to Z and is contained in V_{α_2}, where V_{12} is given the same imbedding j_{α_1}, (c_2) or this is fulfilled for V_{α_2} instead of V_{α_1};

(d) If $V_\alpha \ni y$ with $p(y) \in \bar{\Delta}^n(c) \times A^k(\frac{r+1}{2}, 1)$, then $p(\bar{V}_\alpha) \subset \bar{\Delta}^n(\frac{c+1}{2}) \times A^k(r,1)$.

Here we denote by $p : \Delta^{n+k} \times X \to \Delta^{n+k}$ the natural projection. Case (c_1) can be realized when the imbedding dimension of V_{α_1} is smaller or equal to that of V_{α_2}, and (c_2) in the opposite case; see [Ba-1, pp. 91–92].

Let $E = (V, j, U, B)$ be a scale on the complex space Y. Denote by $H_Y(\bar{U}, \mathrm{sym}^d(B)) := \mathrm{Hol}_Y(\bar{U}, \mathrm{sym}^d(B))$ the Banach analytic set of all d-sheeted analytic subsets on $\bar{U} \times B$, contained in $j(Y)$. The subsets W_Z together with the topology of uniform convergence on $H_Y(\bar{U}, \mathrm{sym}^d(B))$ define a (metrizable) topology on our cycle space $\mathcal{C}_f$, which is equivalent to the topology of currents; see [Fj], [H-S].

We refer the reader to [Ba-1] for the definition of the isotropicity of the family of elements from $H_Y(\bar{U}, \mathrm{sym}^d(B))$ parametrized by some Banach analytic set $\mathcal{S}$. Space $H_Y(\bar{U}, \mathrm{sym}^d(B))$ can be endowed by another (more rich) analytic structure. This new analytic space will be denoted by $\hat{H}_Y(\bar{U}, \mathrm{sym}^d(B))$. The crucial property of this new structure is that the tautological family $\hat{H}_Y(\bar{U}, \mathrm{sym}^d(B)) \times U' \to \mathrm{sym}^d(B)$ is isotropic in $H_Y(\bar{U}', \mathrm{sym}^d(B))$ for any relatively compact polydisk $U' \Subset U$, see [Ba-1]. In fact for isotropic families $\{Z_s : s \in \mathcal{S}\}$ parametrized by Banach analytic sets the following projection changing theorem of Barlet holds.

Theorem (Barlet). *If the family $\{Z_s : s \in \mathcal{S}\} \subset H_Y(\bar{U}, \mathrm{sym}^d(B))$ is isotropic, then for any scale $E_1 = (V_1, j_1, U_1, B_1)$ in $U \times B$ adapted to some Z_{s_0}, there exists a neighborhood U_{s_0} of s_0 in $\mathcal{S}$ such that $\{Z_s : s \in U_{s_0}\}$ is again isotropic in V_1.*

This means, in particular, that the mapping

$$s \to Z_s \cap V_1 \subset H_Y(\bar{U}_1, \mathrm{sym}^d(B_1))$$

is analytic, i.e., can be extended to a neighborhood of any $s \in U_{s_0}$. Neighborhood means here a neighborhood in some complex Banach space where $\mathcal{S}$ is defined as an analytic subset.

This leads naturally to the following

Definition 1.1. A family $\mathcal{Z}$ of analytic cycles in an open set $W \subset Y$, parametrized by a Banach analytic space $\mathcal{S}$, is called analytic in a neighborhood of $s_0 \in \mathcal{S}$ if for any scale E adapted to Z_{s_0} there exists a neighborhood $U \ni s_0$ such that the family $\{\mathcal{Z}_s : s \in U\}$ is isotropic.

1.2. *Analyticity of $\mathcal{C}_f$ and construction of $\mathcal{G}_f$.* Let $f : \bar{\Delta}^n \times \bar{A}^k(r,1) \to X$ be our map. Take a cycle $Z \in \mathcal{C}_f$ and a finite covering (V_α, j_α) satisfying conditions (c) and (d). As above, put $W_Z = \bigcup V_\alpha$. We want to show now that $\mathcal{C}_f$ is an analytic space of finite dimension in a neighborhood of Z. We divide V_α's into two types.

Type 1. These are V_α as in (d). For them put

$$H_\alpha := \bigcup_z \{[\Gamma_{f_z} \cap \bar{A}_z^k(r,1) \times X] \cap V_\alpha\} \subset H_Y(\bar{U}_\alpha, \mathrm{Sym}^{d_\alpha}(B_\alpha)). \tag{1.2.1}$$

The union is taken over all $z \in \Delta^n$ such that V_α is adapted to Γ_{f_z}.

Type 2. These are all others. For V_α of this type we put $H_\alpha := \hat{H}_Y(\bar{U}_\alpha, \mathrm{Sym}^{d_\alpha}(B_\alpha))$.

All H_α are open sets in complex Banach analytic subsets and for V_α of the first type they are of dimension n and smooth. The latter follows from the Barlet-Mazet theorem, which says that if $h : A \to \mathcal{S}$ is a holomorphic injection of a finite dimensional analytic set A into a Banach analytic set $\mathcal{S}$, then $h(A)$ is also a Banach analytic set of finite dimension; see [Mz].

For every irreducible component of $V_\alpha \cap V_\beta \cap Z_l$ we fix a point $x_{\alpha\beta l}$ on this component (the subscript l indicates the component), and a chart $V_\alpha \cap V_\beta \supset (V_{\alpha\beta l}, \phi_{\alpha\beta l}) \ni x_{\alpha\beta l}$ adapted to this component as in (c). Put $H_{\alpha\beta l} := \hat{H}(\Delta^k, \mathrm{Sym}^{d_{\alpha\beta l}}(\Delta^p))$. In the sequel it will be convenient to introduce an order on our finite covering $\{V_\alpha\}$ and write $\{V_\alpha\}_{\alpha=1}^N$.

Consider finite products $\Pi_{(\alpha)} H_\alpha$ and $\Pi_{(\alpha\beta l)} H_{\alpha\beta l}$. In the second product we take only triples with $\alpha < \beta$. These are Banach analytic spaces and by the projection changing theorem of Barlet, for each pair $\alpha < \beta$ we have two holomorphic mappings $\Phi_{\alpha\beta} : H_\alpha \to \Pi_{(l)} H_{(\alpha\beta l)}$ and $\Psi_{\alpha\beta} : H_\beta \to \Pi_{(l)} H_{\alpha\beta l}$. This defines two holomorphic maps $\Phi, \Psi : \Pi_{(\alpha)} H_\alpha \to \Pi_{\alpha<\beta,l} H_{\alpha\beta l}$. The kernel $\mathcal{A}$ of this pair, i.e., the set of $h = \{h_\alpha\}$ with $\Phi(h) = \Psi(h)$, consists exactly analytic cycles in the neighborhood W_Z of Z. This kernel is a Banach analytic set, and moreover the family $\mathcal{A}$ is an analytic family in W_Z in the sense of Definition 1.1.

LEMMA 1.1. *$\mathcal{A}$ is of finite dimension.*

Proof. Take a smaller covering $\{V_\alpha^{'}, j_\alpha\}$ of Z. Namely, $V_\alpha^{'} = V_\alpha$ for V_α of the first type and $V_\alpha^{'} = j_\alpha^{-1}(\Delta_{1-\varepsilon} \times \Delta^p)$ for the second. In the same manner define $H_\alpha^{'}$ and $H^{'} := \Pi_\alpha H_\alpha^{'}$. Repeating the same construction as above we obtain a Banach analytic set $\mathcal{A}^{'}$. We have a holomorphic mapping $K : \mathcal{A} \to \mathcal{A}^{'}$ defined by the restrictions. The differential $dK \equiv K$ of this map is a compact operator.

Let us show that we also have an inverse *analytic* map $F : \mathcal{A}^{'} \to \mathcal{A}$. The analyticity of F means, more precisely, that it should be defined in some neighborhood of $\mathcal{A}^{'}$ in $H^{'}$. For scales $E_\alpha = (V_\alpha, U_\alpha, B_\alpha, j_\alpha)$ of the second type the mapping $F_\alpha : \mathcal{A}^{'} \to H_Y(\bar{U}_\alpha, \mathrm{Sym}^{k_\alpha} B_\alpha)$ is defined by the isotropicity of the family $\mathcal{A}^{'}$ as in [Ba-1]. In particular, this F_α extends analytically to a neighborhood in $H^{'}$(!) of each point of $\mathcal{A}^{'}$.

For scales $E_\alpha = (V_\alpha, U_\alpha = U_\alpha^{'}, B_\alpha, j_\alpha)$ of the first type define F_α as follows. Let $Y = (Y_\alpha)$ be some point in $H^{'}$. Using the fact that $H_\alpha = H_\alpha^{'}$ in this case, we can correctly define $F_\alpha(Y) := Y_\alpha$ viewed as an element of H_α. This directly defines F_α on the whole $H^{'}$. Analyticity is also obvious.

Put $F := \Pi_\alpha F_\alpha : \mathcal{A}^{'} \to \mathcal{A}$. F is defined and analytic in a neighborhood of each point of $\mathcal{A}^{'}$. Observe further that $\mathrm{id} - dK \circ dF$ is Fredholm. Since $\mathcal{A}^{'} \subset \{h \in \Pi_{(i)} H_i^{'} : (\mathrm{id} - K \circ F)(h) = 0\}$, we obtain that $\mathcal{A}^{'}$ is an analytic subset in a complex manifold of finite dimension. □

Therefore $\mathcal{C}_f$ is an analytic space of finite dimension in a neighborhood of each of its points. The $\mathcal{C}_{f,C}$ are open subsets of $\mathcal{C}_f$. Note further that for $C_1 < C_2$ the set $\mathcal{C}_{f,C_1}$ is an open subset of $\mathcal{C}_{f,C_2}$. This implies that for each irreducible component $\mathcal{K}_C$ of $\mathcal{C}_{f,C}$ there is a unique irreducible component $\mathcal{K}$ of $\mathcal{C}_f$ containing $\mathcal{K}_C$ and moreover $\mathcal{K}_C$ is an open subset of $\mathcal{K}$. Of course, in general the dimension of irreducible components of $\mathcal{C}_f$ is not bounded, and in fact the space $\mathcal{C}_f$ is too big. Let us denote by $\mathcal{G}_f$ the union of irreducible components of $\mathcal{C}_f$ that contain at least one irreducible cycle or, in other words, a cycle of the form Γ_{f_z} for some $z \in \Delta^n$.

Denote by $\mathcal{Z}_f := \{Z_a : a \in \mathcal{C}_f\}$ the universal family. In the sequel $\mathcal{B}_k(X)$ will denote the Barlet space of compact analytic k-cycles in normal, reduced complex space X.

Lemma 1.2. 1. *Irreducible cycles form an open dense subset $\mathcal{G}_f^0$ in $\mathcal{G}_f$.*

2. *The dimension of $\mathcal{G}_f$ is not greater than n.*

3. *If $k = 1$, then all compact irreducible components of cycles in $\mathcal{G}_f$ are rational.*

Proof. 1. $\mathcal{G}_f^0$ is clearly open, this follows immediately from (4) and (6) of Lemma 2.3.1 in [Iv-4]. Denote by $\hat{\mathcal{C}}_f$ the normalization of $\mathcal{C}_f$ and denote by $\hat{\mathcal{Z}}_f$ the pull-back of the universal family under the normalization map $\mathcal{N} : \hat{\mathcal{C}}_f \to \mathcal{C}_f$. Consider the following "forgetting of extra compact components" mapping $\Pi : \hat{\mathcal{C}}_f \to \hat{\mathcal{C}}_f$. Note that each cycle $Z \in \hat{\mathcal{C}}_f$ can be uniquely represented as $Z = \Gamma_{f_s} + \Sigma_{j=1}^N B_s^j$, where each B_s^j is a compact analytic k-cycle in $\Delta_s^k(r) \times X$ with connected support. Mark those B_s^j which possess the following property: there is a neighborhood in V of Z in $\hat{\mathcal{C}}_f$ such that every cycle $Z_1 \in V$ decomposes as $Z_1 = \hat{Z}_1 + B_1$, where B_1 is a compact cycle in a neighborhood of B_s^j in the Barlet space $\mathcal{B}_k(X)$. Our mapping $\Pi : \hat{\mathcal{C}}_f \to \hat{\mathcal{C}}_f$ sends each cycle Z to the cycle obtained from this Z by deleting all the marked components. This is clearly an analytic map. Every irreducible cycle is clearly a fixed point of Π. Thus the set of fixed points is open in $\hat{\mathcal{G}}_f \subset \hat{\mathcal{C}}_f$ and so contains the whole $\hat{\mathcal{G}}_f$.

Now we shall prove that every fixed point Z of Π is a limit of irreducible cycles. For the sequel note that the compositions $\psi := p \circ \mathbf{ev} : \mathcal{Z}_f \to \Delta^{n+k}$ and $\phi := p_1 \circ \mathbf{ev} \circ \pi^{-1} : \mathcal{C}_f \to \Delta^n$ are well defined. Here $p_1 : \Delta^{n+k} \times X \to \Delta^n$ is one more natural projection and $\mathbf{ev} : \mathcal{Z}_f \to \Delta^{n+k} \times X$ is the natural evaluation map. Let $\phi(Z) = s \in \Delta^n$ and $Z = \Gamma_{f_s} + \Sigma_{j=1}^N B_s^j$. Next, Z i being a fixed point of Π means that in any neighborhood of Z one can find a cycle Z_1 such that $Z_1 = \Gamma_{f_{s_1}} + \Sigma_{j=2}^N B_{s_1}^j$, where $B_{s_1}^j$ are compact cycles close to B_s^j. Observe that every cycle in a neighborhood of Z_1 has the same form, i.e., in its decomposition $j \geq 2$, which follows from Lemma 2.3.1 from [Iv-4]. Since Z_1 is also a fixed point for Π, we can repeat this procedure N times to obtain finally an irreducible cycle in a given neighborhood of Z.

We conclude that $\mathcal{G}_f^0$ is dense in $\mathcal{G}_f$.

2. Take an irreducible $Z \in \mathcal{G}_f^0 \cap \mathrm{Reg}(\mathcal{G}_f)$. Take a neighborhood $Z \in V \subset \mathrm{Reg}(\mathcal{G}_f)$ that consists from irreducible cycles only. Then $\phi|_V : V \to \Delta^n$ is injective and holomorphic. Thus $\dim \mathcal{G}_f \leq n$.

3. This part follows from Lemma 7 in [Iv-5] because every cycle from $\mathcal{G}_f$ is a limit of analytic disks. □

Definition 1.2. We shall call the space $\mathcal{G}_f$ the *cycle space* associated to a meromorphic map f.

Denote by $\mathcal{G}_{f,C}$ the open subset of $\mathcal{G}_f$ consisting of Z with $\mathrm{vol}(Z) < C$.

1.3. *Proof of the Main Lemma.* Now we are ready to state and prove the main lemma of this paragraph, i.e. Lemma 1.3. From now on we restrict our universal family $\mathcal{Z}_f$ onto $\mathcal{G}_f$ without changing notation. That is, now $\mathcal{Z}_{f,C} := \{Z_a : a \in \mathcal{G}_{f,C}\}$, $\mathcal{Z}_f := \bigcup_{C>0} \mathcal{Z}_{f,C}$ and $\pi : \mathcal{Z}_f \to \mathcal{G}_f$ is the natural projection. Also, $\mathcal{Z}_f$ is a complex space of finite dimension. We have an evaluation map

$$\mathbf{ev} : \mathcal{Z}_f \to \Delta^{n+k} \times X, \tag{1.3.1}$$

defined by $Z_a \in \mathcal{Z}_f \to Z_a \subset \Delta^{n+k} \times X$, which will be used in the proof of the Lemma 1.3.

Recall that we suppose that our complex space X is equipped with some Hermitian metric h.

LEMMA 1.3. *Let a holomorphic map $f : \bar{\Delta}^n \times \bar{A}^k(r,1) \to X$ (a complex space) be given. Suppose that:*

1) *For every $z \in \bar{\Delta}^n$ the restriction f_z extends meromorphically onto the whole k-disk $\bar{\Delta}_z^k$;*

2) *The volumes of graphs of these extensions are uniformly bounded*;

3) *There exists a compact* $K \Subset X$ *which contains* $f(\bar{\Delta}^n \times \bar{A}^k(r,1))$ *and* $f(\bar{\Delta}^k_z)$ *for all* $z \in \bar{\Delta}^n$.

Then f *extends meromorphically onto* Δ^{n+k}.

Proof. Denote by $\nu = \nu(K)$ the minimal volume of a compact k-dimensional analytic subset in K, $\nu > 0$ by Lemma 2.3.1 from [Iv-4]. Denote by W the maximal open subset of Δ^n such that f extends meromorphically onto $\Delta^n \times A^k(r,1) \cup W \times \Delta^k$. Set $S = \Delta^n \setminus W$. Let

$$S_l = \{z \in S : \mathrm{vol}(\Gamma_{f_z}) \leq l \cdot \frac{\nu}{2}\}. \tag{1.3.2}$$

The maximality of W (and thus the minimality of S) and Lemma 2.4.1 from [Iv-4] imply that $S_{l+1} \setminus S_l$ are pluripolar and by the Josefson theorem so is S. In particular, $W \neq \emptyset$.

Consider the analytic space

$$\mathcal{G}_{f,2C_0,c} := \{Z \in \mathcal{G}_{f,2C_0} : \|\phi(Z)\| < c\}, \tag{1.3.3}$$

where $0 < c \leq 1$ is fixed. Here C_0 such that $\mathrm{vol}(\Gamma_{f_z}) \leq C_0$ for all $z \in \bar{\Delta}^n$. Since, by Lemma 1.2 cycles of the form Γ_{f_z} are dense in $\mathcal{G}_{f,2C_0,1}$, we have that for every $Z \in \mathcal{G}_{f,2C_0,1}$ $\mathrm{vol}(\mathbf{ev}(Z)) \leq C_0$. Therefore we see that $\bar{\mathcal{G}}_{f,C_0,1} \cap \phi^{-1}(\Delta^n(1))$ is closed and open in $\mathcal{G}_{f,2C_0,1}$ and in fact coincides with $\mathcal{G}_{f,2C_0,1}$. Closures are in the cycle space $\mathcal{G}_f$.

For any $c < 1$ the set $\bar{\mathcal{G}}_{f,C_0,c} = \phi^{-1}(\bar{\Delta}^n(c))$ is compact by the Harvey-Shiffman generalization of Bishop's theorem. Therefore $\phi : \mathcal{G}_{f,2C_0,1} \to \Delta^n$ is proper and $\mathbf{ev} : \mathcal{Z}_f \to \Delta^{n+k} \times X$ is also proper and by the Remmert proper mapping theorem its image is an analytic set extending the graph of f. The latter follows from the fact that $\phi(\mathcal{G}_{f,2C_0,1}) \supset W$ and therefore in fact $\phi(\mathcal{G}_{f,2C_0,1}) = \Delta^n(1)$. □

Definition 1.3. A complex space X is disk-convex in dimension k if for every compact $K \Subset X$ there exists a compact $\hat{K}$ such that for every meromorphic mapping $\phi : \bar{\Delta}^k \to X$ with $\phi(\partial\Delta^k) \subset K$ one has $\phi(\bar{\Delta}^k) \subset \hat{K}$.

Remarks. 1. For $k = 1$ we say simply that X is disk-convex.

2. Recall that a complex space X is called k-convex (in the sense of Grauert) if there is an exhaustion function $\phi : X \to [0, +\infty[$ which is k-convex at all points outside some compact K, i.e., its Levi form has at least $\dim X - k + 1$ positive eigenvalues. By an appropriate version of the maximum principle for k-convex functions k-convexity implies disk-convexity in dimension k.

3. Condition (3) of Lemma 1.3 (as well as of Theorems 1.4 and 1.5 below) is automatically satisfied if X is disk-convex in dimension k.

1.4. *The Levi-type extension theorem.* In the proof of the Main Theorem we will deal with the situation where a holomorphic map $f : \Delta^n \times A^k(r,1) \to X$ extends from $A^k_z(r,1)$ to Δ^k_z not for all $z \in \Delta^n$ but only for z in some "thick" set S.

Definition 1.4. A subset $S \subset \Delta^n$ is called thick at the origin if for any neighborhood U of zero $U \cap S$ is not contained in a proper analytic subset of U.

The case of dimension two, where $n = 1$, is somewhat special. Let us consider this case separately. Here S is thick at the origin if and only if S contains a sequence $\{s_n\}$ which converges to zero.

THEOREM 1.4. *Let $f : \Delta \times A(r,1) \to X$ be a holomorphic map into a normal, reduced complex space X. Suppose that for a sequence $\{s_n\}$ of points in Δ, converging to the origin the restrictions $f_{s_n} := f|_{A_{s_n}}$ extend holomorphically onto Δ_{s_n}. Suppose in addition that*:

1) *There exists a compact $K \Subset X$ such that $\left[\bigcup_{n=1}^{\infty} f(\Delta_{s_n})\right] \cup f(\Delta \times A(r,1)) \subset K$;*

2) *Areas of images $f(\Delta_{s_n})$ are uniformly bounded.*

Then there exists an $\varepsilon > 0$ such that f extends as a meromorphic map onto $\Delta(\varepsilon) \times \Delta$.

In dimensions bigger than two the situation becomes more complicated; see Examples 3.2 and 3.3 in Section 3. Let us give a condition on X sufficient to maintain the conclusion of Theorem 1.4. Denote by $\mathbf{ev} : \mathcal{Z} \to X$ the natural evaluation map from the universal space $\mathcal{Z}$ over $\mathcal{B}_k(X)$ to X.

Definition 1.5. Let us say that X has unbounded cycle geometry in dimension k if there exists a path $\gamma : [0,1[\to \mathcal{B}_k(X)$ with $\mathrm{vol}_{2k}(\mathbf{ev}(Z_{\gamma(t)})) \to \infty$ as $t \to \infty$ and $\mathbf{ev}(Z_{\gamma(t)}) \subset K$ for all t, where K is some compact in X.

Now we can state the following

THEOREM 1.5. *Let $f : \Delta^n \times A^k(r,1) \to X$ be a holomorphic mapping into a normal, reduced complex space X. Suppose that there are a constant $C_0 < \infty$ and a compact $K \Subset X$ such that for s in some subset $S \subset \Delta^n$, which is thick at the origin the following holds*:

(a) *The restrictions $f_s := f|_{A^k_s(r,1)}$ extend meromorphically onto the polydisk Δ^k_s, and $\mathrm{vol}(\Gamma_{f_s}) \leq C_0$ for all $s \in S$;*

(b) $f(\Delta^n \times A^k(r,1)) \subset K$ *and* $f_s(\Delta^k) \subset K$ *for all* $s \in S$.

If X *has bounded cycle geometry in dimension* k, *then there exist a neighborhood* $U \ni 0$ *in* Δ^n *and a meromorphic extension of* f *onto* $U \times \Delta^k$.

We shall use the Theorem 1.5 when $k = 1$. In this case it admits a nice refinement. A 1-cycle $Z = \Sigma_j n_j Z_j$ is called rational if all Z_j are rational curves, i.e., images of the Riemann sphere $\mathbb{CP}^1$ in X under nonconstant holomorphic mappings. Considering the space of rational cycles $\mathcal{R}(X)$ instead of Barlet space $\mathcal{B}_1(X)$ we can define as in Definition 1.5 the notion of bounded rational cycle geometry.

Corollary 1.6. *Suppose that in the conditions of Theorem* 1.5 *one has additionally that* $k = 1$. *Then the conclusion of this theorem holds provided* X *has bounded rational cycle geometry.*

Proof of Theorems 1.4, 1.5 *and Corollary* 1.6.

Case $n = 1$. Define $\mathcal{G}_0$ as the set of all limits $\{\Gamma_{f_{s_n}}, s_n \in S, s_n \to 0\}$. Consider the union $\hat{\mathcal{G}}_0$ of those components of $\mathcal{G}_{f,2C_0}$ that intersect $\mathcal{G}_0$. At least one of these components, say $\mathcal{K}$, contains two points a_1 and a_2 such that Z_{a_1} projects onto Δ^k_0 and Z_{a_2} projects onto Δ^k_s with $s \neq 0$. This is so because S contains a sequence converging to zero. Consider the restriction $\mathcal{Z}_f \mid_{\mathcal{K}}$ of the universal family onto $\mathcal{K}$. This is a complex space of finite dimension. Join the points a_1 and a_2 by an analytic disk $h : \Delta \to \mathcal{K}$, $h(0) = a_1, h(1/2) = a_2$. Then the composition $\psi = \phi \circ h : \Delta \to \Delta$ is not degenerate because $\psi(0) = 0 \neq s = \psi(1/2)$. Here $\phi := p_1 \circ \mathbf{ev} \circ \pi^{-1} : \mathcal{C}_f \to \Delta^n$ is as defined in the proof of Lemma 1.2. Map ϕ restricted to $\mathcal{G}_f$ will be denoted also as ϕ. Thus ψ is proper and obviously so is the map $\mathbf{ev} : \mathcal{Z} \mid_{\psi(\Delta)} \to F(\mathcal{Z} \mid_{\psi(\Delta)}) \subset \Delta^{1+k} \times X$. Therefore $\mathbf{ev}(\mathcal{Z} \mid_{\psi(\Delta)})$ is an analytic set in $U \times \Delta^k \times X$ for small enough U extending Γ_f by the reason of dimension.

This proves Theorem 1.4.

Case $n \geq 2$. We shall treat this case in two steps.

Step 1. Fix a point $z \in \Delta^n$ such that $\phi(\mathcal{G}_f) \ni z$. Then there exists a relatively compact open $W \subset \mathcal{G}_f$, which contains $\mathcal{G}_{f,C_0}$ such that $\phi(W)$ is an analytic variety in some neighborhood V of z.

Consider the analytic subset $\phi^{-1}(z)$ in $\mathcal{G}_f$. Every Z_a with $a \in \phi^{-1}(z)$ has the form $B_a + \Gamma_{f_z}$, where B is a compact cycle in $\Delta^k_z \times X$. Thus connected components of $\phi^{-1}(z)$ parametrize connected and closed subvarieties in $\mathcal{B}_k(\Delta^k \times X)$. Holomorphicity of f on $\Delta^n \times A^k(r,1)$ and condition (b) of Theorem 1.5 imply that $B_a \subset \bar{\Delta}^k_z \times K$. So, if $\phi^{-1}(z)$ had non compact connected components, this would imply the unboundness of cycle geometry of X.

Thus, all connected components of $\phi^{-1}(z)$ should be compact. Let $\mathcal{K}$ denote the union of connected components of $\phi^{-1}(z)$ intersecting $\mathcal{G}_{f,C_0}$. Since $\mathcal{K}$ is compact, there obviously exist a relatively compact open $W \Subset \mathcal{G}_f$ containing $\mathcal{G}_{f,C_0}$ and $\mathcal{K}$, and a neighborhood $V \ni z$ such that $\phi\,|_W : W \to V$ is proper. By Remmert's proper mapping theorem. $\phi(W) \subset V$ is an analytic subset of V.

Step 2. If S is thick at z then there exists a neighborhood $V \ni z$ such that f meromorphically extends onto $V \times \Delta^k$.

Since $\phi(W) \supset S \cap V$ and S is thick at the origin, the first step implies that $\phi(W) \cap V = V$. Since $W \Subset \mathcal{G}_f$ there exist a constant C such that $\mathrm{vol}\{Z_s : s \in W\} \leq C$. This allows us to apply Lemma 1.3 and obtain the extension of f onto $V \times \Delta^k$.

This proves the Theorem 1.5.

Case $k = 1$. The limit of a sequence of analytic disks of bounded area is an analytic disk plus a rational cycle, see for example [Iv-1]. Therefore we need to consider only the space of rational cycles in this case. The rest is obvious. This gives Corollary 1.6. □

Step 1 in the proof of Theorem 1.5 gives the following statement, which will be used later.

Corollary 1.7. *Let $f : \Delta^n \times A^k(r,1) \to X$ be a holomorphic mapping into a normal, reduced complex space X which has bounded cycle geometry in dimension k. Suppose that there are a constant $C_0 < \infty$ and a compact $K \Subset X$ such that for s in some subset $S \subset \Delta^n$, the following hold*:

(a) *The restrictions $f_s := f\,|_{A^k_s(r,1)}$ extend meromorphically onto the polydisk Δ^k_s, and $\mathrm{vol}(\Gamma_{f_s}) \leq C_0$ for all $s \in S$;*

(b) *$f(\Delta^n \times A^k(r,1)) \subset K$ and $f_s(\Delta^k) \subset K$ for all $s \in S$.*

Then there exists a neighborhood $V \ni 0$ and an analytic subvariety W of V such that $W \supset S \cap V$ and such that for every $z \in W$, f_z meromorphically extends onto Δ^k_z with $\mathrm{vol}(\Gamma_{f_z}) \leq C_0$.

In the same spirit one obtains the following:

Corollary 1.8. *Let a meromorphic mapping $f : \Delta^n \times A(r,1) \to X$ be given, where X is a compact complex manifold with bounded rational cycle geometry. Let S be a subset of Δ^n consisting of such points s that f_s is well defined and extends holomorphically onto Δ_s. If S is not contained in a countable union of locally closed proper analytic subvarieties of Δ^n, then there exist an open nonempty $U \subset \Delta^n$ and a meromorphic extension of f onto $U \times \Delta$.*

Indeed, one easily deduces the existence of a point $p \in \Delta^n$, that can play the role of the origin in Theorem 1.5.

1.5. *A remark about spaces with bounded cycle geometry.* To apply Theorem 1.5 in the proof of the Main Theorem we need to check the boundedness of cycle geometry of the manifold X which carries a pluriclosed metric form. We shall do this in Proposition 1.9 below. We start from the following simple observation:

> *Every compact complex manifold of dimension* $k+1$ *carries a strictly positive* (k,k)*-form* Ω^k *with* $dd^c\Omega^k = 0$.

Indeed, either a compact complex manifold carries a dd^c-closed strictly positive (k,k)-form or it carries a bidimension $(k+1,k+1)$-current T with $dd^cT \geq 0$ but $\not\equiv 0$. In the case of $\dim X = k+1$ such a current is nothing but a nonconstant plurisubharmonic function, which does not exist on compact X.

Let us introduce the class $\mathcal{G}_k$ of normal complex spaces, carrying a nondegenerate positive dd^c-closed strictly positive (k,k)-form. Note that the sequence $\{\mathcal{G}_k\}$ is rather exhaustive: $\mathcal{G}_k$ contains all compact complex manifolds of dimension $k+1$.

Introduce furthermore the class of normal complex spaces $\mathcal{P}_k^-$ which carry a strictly positive (k,k)-form $\Omega^{k,k}$ with $dd^c\Omega^{k,k} \leq 0$. Note that $\mathcal{P}_k^- \supset \mathcal{G}_k$. As was mentioned in the introduction a Hopf three-fold $X^3 = \mathbb{C}^3 \setminus \{0\}/(z \sim 2z)$ belongs to $\mathcal{P}_1^-$ but not to $\mathcal{G}_1$.

Proposition 1.9. *Let* $X \in \mathcal{P}_k^-$ *and let* $\mathcal{K}$ *be an irreducible component of* $\mathcal{B}_k(X)$ *such that* $\mathbf{ev}(\mathcal{Z}\,|_{\mathcal{K}})$ *is relatively compact in* X. *Then*:

1) $\mathcal{K}$ *is compact.*
2) *If* $\Omega^{k,k}$ *is a* dd^c*-negative* (k,k)*-form on* X, *then* $\int_{Z_s}\Omega^{k,k} \equiv \mathrm{const}.$ *for* $s \in \mathcal{K}$.
3) X *has bounded cycle geometry in dimension* k.

Proof. Let $\mathbf{ev}: \mathcal{Z}\,|_{\mathcal{K}} \to X$ be the evaluation map, and let $\Omega^{k,k}$ be a strictly positive dd^c-negative (k,k)-form on X. Then $\int_{Z_s}\Omega^{k,k}$ measures the volume of Z_s. Let us prove that the function $v(s) = \int_{Z_s}\Omega^{k,k}$ is plurisuperharmonic on $\mathcal{K}$. Take an analytic disk $\phi: \Delta \to \mathcal{K}$. Then for any nonnegative test function ψ on Δ by Stokes's theorem and reasons of bidegree we have

$$\begin{aligned}\langle \psi, \Delta\phi^*(v)\rangle &= \int_{\Delta}\Delta\psi \cdot \int_{Z_{\phi(s)}}\Omega^{k,k} = \int_{\mathcal{Z}|_{\phi(\Delta)}} dd^c(\pi^*\psi)\wedge\Omega^{k,k} \\ &= \int_{\mathcal{Z}|_{\phi(\Delta)}} \pi^*\psi \wedge dd^c\Omega^{k,k} \leq 0.\end{aligned}$$

Here $\pi : \mathcal{Z} |_{\mathcal{K}} \to \mathcal{K}$ is the natural projection. So $\Delta\phi^*(v) \le 0$ for any analytic disk in $\mathcal{K}$ in the sense of distributions. Therefore v is plurisuperharmonic.

Note that by Harvey-Shiffman generalization of Bishop's theorem $v(s) \to \infty$ as $s \to \partial\mathcal{K}$. So by the minimum principle $v \equiv$ const. and $\mathcal{K}$ is compact again by Bishop's theorem.

2) The same computation shows that $\int_{Z_s} \Omega^{k,k}$ is plurisuperharmonic for any dd^c-negative (k,k)-form. Since $\mathcal{K}$ is proved to be compact, we obtain the statement.

3) Let $\mathcal{R}$ be any connected component of $\mathcal{B}_k(X)$. Write $\mathcal{R} = \bigcup_j \mathcal{K}_j$, where $\mathcal{K}_j$ are irreducible components. From (1) we have that v is constant on $\mathcal{R}$. So if $\{\mathcal{K}_j\}$ is not finite then $\mathcal{R}$ has an accumulation point $s = \lim s_j$ by Bishop's theorem, where all s_j belong to different components $\mathcal{K}_j$ of $\mathcal{R}$. This contradicts the fact that $\mathcal{B}_k(X)$ is a complex space. $\square$

2. Hartogs-type extension and spherical shells

2.1. *Generalities on pluripotential theory.* For the standard facts from pluripotential theory we refer to [Kl]. Denote by $\mathcal{D}^{k,k}(\Omega)$ the space of C^∞-forms of bidegree (k,k) with compact support on a complex manifold Ω. Note that $\phi \in \mathcal{D}^{k,k}(\Omega)$ is real if $\bar{\phi} = \phi$. The dual space $\mathcal{D}_{k,k}(\Omega)$ is the space of currents of bidimension (k,k) (bidegree $(n-k,n-k)$, $n = \dim_{\mathbb{C}}\Omega$). Also, $T \in \mathcal{D}_{k,k}(\Omega)$ is real if $\langle T, \bar{\phi}\rangle = \overline{\langle T, \phi\rangle}$ for all $\phi \in \mathcal{D}^{k,k}(\Omega)$.

Definition 2.1. A current $T \in \mathcal{D}_{k,k}(\Omega)$ is called positive if for all $\phi_1, \dots, \phi_k \in \mathcal{D}^{1,0}(\Omega)$

$$\langle T, \frac{i}{2}\phi_1 \wedge \bar{\phi}_1 \wedge \dots \wedge \frac{i}{2}\phi_k \wedge \bar{\phi}_k\rangle \ge 0.$$

T is negative if $-T$ is positive.

Definition 2.2. A current $T \in \mathcal{D}_{k,k}(\Omega)$ is *pluripositive* (*-negative*) if T is positive and dd^cT is positive (-negative). Also, T is *pluridefinite* if it is either pluripositive or plurinegative. A current T (not necessarily positive) is *pluriclosed* if $dd^cT = 0$.

If K is a complete pluripolar compact in strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and T is a closed, positive current on $\Omega \setminus K$, then T has locally finite mass in a neighborhood of K; see [Iv-2, Lemma 2.1]. For a current T, which has locally finite mass in a neighborhood of K, one denotes by $\tilde{T}$ its trivial extension onto Ω; see [Lg].

LEMMA 2.1. (a) *Let K be a complete pluripolar compact in a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and T be a pluridefinite current of bidegree* $(1,1)$ *on*

$\Omega \setminus K$ of locally finite mass in a neighborhood of K and such that dT has coefficient measures in $\Omega \setminus K$. Then $dd^c\tilde{T}$ has coefficient measures on Ω.

(b) *If $n = 2$ and K is of Hausdorff dimension zero, then $\chi_K \cdot dd^c\tilde{T}$ is negative, where χ_K is the characteristic function of K.*

Proof. Part (a) of this lemma was proved in [Iv-2, Prop. 2.3] for currents of bidimension (1,1) (the condition on dT was forgotten there). If T is of bidegree (1,1), then consider $T \wedge (dd^c\|z\|^2)^{n-2}$ to get the same conclusion.

(b) Let $\{u_k\}$ be a sequence of smooth plurisubharmonic functions in Ω, equal to zero in a neighborhood of K, $0 \le u_k \le 1$ and such that $u_k \nearrow \chi_{\Omega\setminus K}$ uniformly on compacts in $\Omega \setminus K$; see Lemma 1.2 from [Sb]. Put $v_k = u_k - 1$.

Let $\widetilde{dd^cT}$ be a negative measure on Ω, be denoted as μ_0. According to part (a) the distribution $\mu := dd^c\tilde{T}$ is a measure. Write

$$\mu = \chi_K \cdot \mu + \chi_{\Omega\setminus K} \cdot \mu, \tag{2.1.1}$$

where obviously $\chi_{\Omega\setminus K} \cdot \mu = \mu_0$. Denote the measure $\chi_K \cdot \mu$ by μ_s. We shall prove that the measure μ_s is nonpositive. Take a ball B in $\mathbb{C}^2$ centered at $s_0 \in K$ such that $\partial B \cap K = \emptyset$. One has

$$\mu_s(B \cap K) = -\lim_{k\to\infty} \int_B v_k \cdot \mu = -\lim_{k\to\infty} \langle v_k, dd^c\tilde{T}\rangle = -\lim_{k\to\infty} \langle dd^c v_k, \tilde{T}\rangle \le 0, \tag{2.1.2}$$

because $\tilde{T}$ is positive and $dd^c v_k \ge 0$. So for any such ball we have

$$\mu_s(B \cap K) \le 0. \tag{2.1.3}$$

All that is left, is to use the following Vitali-type theorem for general measures; see [Fd, p. 151]. Let D be an open set in $\mathbb{C}^2$ and σ a finite positive Borel measure on D. Further let $\mathcal{B}$ be a family of closed balls of positive radii such that for any point $x \in D$ the family $\mathcal{B}$ contains balls of arbitrarily small radii centered at x. Then one can find a countable subfamily $\{B_i\}$ of pairwise disjoint balls in $\mathcal{B}$ such that

$$\sigma(D \setminus \bigcup_{(i)} B_i) = 0. \tag{2.1.4}$$

Represent our measure μ_s as a difference $\mu_s = \mu_s^+ - \mu_s^-$ of two nonnegative measures. Fix a relatively compact open subset $D \subset \Omega$. Let $\mathcal{B}$ represent the family of all balls such that $\partial B \cap K = \emptyset$. Since K is of dimension zero this is a Vitali-type covering. Let $\{B_i\}$ be pairwise disjoint and such that $\mu_s^+(D \setminus \bigcup_{(i)} B_i) = 0$. Then $\mu_s^+(D) = \mu_s^+(D \setminus \bigcup_{(i)} B_i) + \sum_{(i)} \mu_s^+(B_i) = \sum_{(i)} \mu_s^+(B_i)$. Consequently,

$$\begin{aligned} \mu_s(D) &= \mu_s^+(D) - \mu_s^-(D) \le \mu_s^+(\bigcup_{(i)} B_i) - \mu_s^-(\bigcup_{(i)} B_i) \\ &= \sum_i \mu_s^+(B_i) - \sum_i \mu_s^-(B_i) = \sum_i \mu_s(B_i) \le 0 \end{aligned} \tag{2.1.5}$$

by (2.1.3). Thus $\mu_s(D) \le 0$ for any relatively compact open set D in Ω. So the measure μ_s is negative. $\square$

Together with the Main Theorem we shall prove a somewhat more general result. A metric form ω on X we call plurinegative if $dd^c\omega \le 0$. But first recall the following:

Definition 2.3. Recall that a subset $K \subset \Omega$ is called (*complete*) *p-polar* if for any $a \in \Omega$ there exist a neighborhood $V \ni a$ and coordinates $z_1, \dots, z_n$ in V such that the sets $K_{z_I^0} = K \cap \{z_{i_1} = z_{i_1^0}, \dots, z_{i_p} = z_{i_p^0}\}$ are (*complete*) *pluripolar* in the subspaces $V_{z_i^0} := \{z \in V : z_{i_1} = z_{i_1^0}, \dots, z_{i_p} = z_{i_p^0}\}$ for almost all $z_I^0 = (z_{i_1}^0, \dots, z_{i_p}^0) \in \pi^I(V)$, where I runs over a finite set of multi-indices with $|I| = p$, such that $\{(\pi^I)^* w_e^I\}_I$ generates the space of (p,p)-forms. Here $\pi^I(z_1, \dots, z_n) = (z_{i_1}, \dots, z_{i_p})$ denotes the projection onto the space of variables $(z_{i_1}, \dots, z_{i_p})$ and $w_e^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$; see [Sb].

Now we can state the main result in most general form.

THEOREM 2.2. *Let $f : H_U^{n+1}(r) \to X$ be a meromorphic map into a disk-convex complex space X that admits a plurinegative Hermitian metric form ω. Then*:

(1) *f extends to a meromorphic map $\hat{f} : \Delta^{n+1} \setminus A \to X$, where A is a closed, complete $(n-1)$-polar subset of Δ^{n+1} of Hausdorff $(2n-1)$-dimensional measure zero.*

(2) *If, in addition, ω is pluriclosed and if $A \neq \emptyset$ is the minimal subset such that f extends onto $\Delta^{n+1} \setminus A$, then for every transversal sphere $\mathbb{S}^3 \subset \Delta^{n+1} \setminus A$, its image $f(\mathbb{S}^3)$ is not homologous to zero in X.*

We would like to turn attention to the difference between plurinegative and pluriclosed cases. Example of the Hopf three-fold, given in the introduction, shows that when X admits only plurinegative metric form the singular set A can have "components" of Hausdorff codimension higher than four and that the homological characterization of A is also not valid in general.

2.2. *Proof in dimension two.* Let a meromorphic mapping $f : H_U^2(1-r) \to X$ from the two-dimensional Hartogs figure into a disk-convex complex space be given. Since the indeterminancy set $I(f)$ of f is discrete, we can suppose after shrinking $A(1-r,1)$ and Δ if necessary, that f is holomorphic in the neighborhood of $\bar{\Delta} \times \bar{A}(1-r,1)$. Let ω be a plurinegative metric form on X. Denote by W the maximal open subset of the unit disk Δ such that f extends holomorphically onto $H_W^2(1-r) := W \times \Delta \cup \Delta \times A(1-r,1)$. Note that W contains U except possibly a discrete set. Let $I(f)$ be the fundamental set of f and denote by $\hat{f}$ the mapping $\hat{f}(z)=(z,f(z))$ into the graph. For $z\in W$ define

$$a(z)=\mathrm{area}\hat{f}(\Delta_z)=\int_{\Delta_z}(dd^c|\lambda|^2+f\,|_{\Delta_z}^*\,\omega). \tag{2.2.1}$$

Here $\Delta_z=\{(z,\lambda):|\lambda|<1\}$. We start with the following simple observation. Denote by $\nu_1=\nu_1(K)$ the infimum of areas of compact complex curves contained in a compact $K\Subset X$. Then $\nu_1>0$; see Lemma 2.3.1 in [Iv-4].

LEMMA 2.3. *Let $f:\bar{\Delta}\times\bar{A}(1-r,1)\to X$ be a holomorphic mapping into a disk-convex complex space X. Suppose that for some sequence of points $\{s_n\}\subset\Delta$, $s_n\to 0$, the following hold*:

(a) $f_{s_n}:=f\,|_{\{s_n\}\times A(1-r,1)}$ *extends holomorphically onto* $\Delta_{s_n}:=\{s_n\}\times\Delta$;

(b) $\mathrm{area}\hat{f}(\Delta_{s_n})\le C$ *for all* n.

Then $f_0:=f|_{\{s_0\}\times A(1-r,1)}$ *extends holomorphically onto* Δ_0.

If moreover,

(c) *for a compact K in X containing the set*

$$f\left[\left(\Delta\left(\frac{1}{2}\right)\times A\left(1-\frac{2}{3}\cdot r,1-\frac{1}{3}\cdot r\right)\right)\cup\bigcup_{(n)}\{s_n\}\times\Delta\left(1-\frac{1}{3}\cdot r\right)\right],$$

one has

$$\left|\mathrm{area}\hat{f}\left(\Delta_{s_n}\left(1-\frac{1}{3}\cdot r\right)\right)-\mathrm{area}\hat{f}\left(\Delta_0\left(1-\frac{1}{3}\cdot r\right)\right)\right|\le\frac{1}{2}\cdot\nu_1(K), \tag{2.2.2}$$

for $n\gg 1$, then f extends holomorphically onto $V\times\Delta$ for some open $V\ni 0$.

Proof. The first statement is standard. Let us prove the second one. First of all we show that $\mathcal{H}-\lim_{n\to\infty}\hat{f}(\bar{\Delta}_{s_n}(1-\frac{1}{3}\cdot r))=\hat{f}(\bar{\Delta}_0(1-\frac{1}{3}\cdot r))$, i.e., the sequence of graphs $\{\hat{f}(\bar{\Delta}_{s_n}(1-\frac{1}{3}\cdot r))\}$ converges in the Hausdorff metric to the graph of the limit. If not, there would be a subsequence (still denoted by $\{\hat{f}(\bar{\Delta}_{s_n}(1-\frac{1}{3}\cdot r))\}$) such that

$$\mathcal{H}-\lim_{n\to\infty}\hat{f}\left(\bar{\Delta}_{s_n}\left(1-\frac{1}{3}\cdot r\right)\right)=\hat{f}\left(\bar{\Delta}_0\left(1-\frac{1}{3}\cdot r\right)\right)\cup\bigcup_{j=1}^{N}\{p_j\}\times C_j,$$

where $\{C_j\}$ are compact curves; see Lemma 2.3.1 in [Iv-4]. Thus by (2.3.2) from [Iv-4] we have

$$\mathrm{area}\hat{f}\left(\bar{\Delta}_{s_n}\left(1-\frac{1}{3}\cdot r\right)\right)\ge\mathrm{area}\hat{f}\left(\bar{\Delta}_0\left(1-\frac{1}{3}\cdot r\right)\right)+N\cdot\nu_1(K).$$

This contradicts (2.2.2).

Take a Stein neighborhood V of $\hat{f}(\bar{\Delta}_0(1-\frac{1}{3}\cdot r))$, see [Si-1]. Then for $\delta>0$ small enough we have $f(\Delta_\delta\times A_{1-\frac{1}{3}r-\delta,1-\frac{1}{3}r+\delta})\subset V$ and $f(\Delta_{s_n}(1-\frac{1}{3}r))\subset V$ if $s_n\in\Delta_\delta$. From Hartogs theorem for holomorphic functions we see that f extends to a holomorphic map from $\Delta_\delta\times\Delta_{1-\frac{1}{3}r-\delta}$ to V. $\square$

LEMMA 2.4. *If the metric form ω on a disk-convex complex space X is plurinegative and W is maximal, then $\partial W \cap \Delta$ is complete polar in Δ.*

Proof. Take a point $z_0 \in \partial W \cap \Delta$. Choose a relatively compact neighborhood V of z_0 in Δ. Denote by $T = \frac{i}{2} t^{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$ the current $f^*\omega + dd^c\|z\|^2$. The area function from (2.2.1) can be now written as

$$a(z_1) = \frac{i}{2} \cdot \int_{|z_2|\leq 1} t^{2\bar{2}}(z_1,z_2) dz_2 \wedge d\bar{z}_2. \tag{2.2.3}$$

The condition that dd^cT is negative means that

$$\frac{\partial^2 t^{1\bar{1}}}{\partial z_2 \partial \bar{z}_2} + \frac{\partial^2 t^{2\bar{2}}}{\partial z_1 \partial \bar{z}_1} - \frac{\partial^2 t^{1\bar{2}}}{\partial z_2 \partial \bar{z}_1} - \frac{\partial^2 t^{2\bar{1}}}{\partial z_1 \partial \bar{z}_2} \leq 0 \tag{2.2.4}$$

on $H^2_W(1-r)$. Now we can estimate the Laplacian of a:

$$\begin{aligned} \Delta a(z_1) &= i \int_{|z_2|\leq 1} \frac{\partial^2 t^{2\bar{2}}}{\partial z_1 \partial \bar{z}_1} dz_2 \wedge d\bar{z}_2 \\ &\leq i \int_{|z_2|\leq 1} \left(-\frac{\partial^2 t^{1\bar{1}}}{\partial z_2 \partial \bar{z}_2} + \frac{\partial^2 t^{1\bar{2}}}{\partial z_2 \partial \bar{z}_1} + \frac{\partial^2 t^{2\bar{1}}}{\partial z_1 \partial \bar{z}_2} \right) dz_2 \wedge d\bar{z}_2 \\ &= i \int_{|z_2|=1} \frac{\partial t^{1\bar{1}}}{\partial z_2} dz_2 + i \int_{|z_2|=1} \frac{\partial t^{1\bar{2}}}{\partial \bar{z}_1} d\bar{z}_2 - i \int_{|z_2|=1} \frac{\partial t^{2\bar{1}}}{\partial z_1} dz_2 = \psi(z_1). \end{aligned} \tag{2.2.5}$$

Inequality (2.2.5) holds for $z_1 \in V \cap W$. But the right-hand side ψ is smooth in all of V. Let Ψ be a smooth solution of $\Delta\Psi = \psi$ in V. Put $\hat{a}(z) = a(z) - \Psi(z)$. Then $\hat{a}$ is superharmonic and bounded from below in $V \cap W$, maybe after V is shrunk.

Denote further by E the set of points $z_1 \in \partial W \cap V$ such that $a(z) \to +\infty$ as $z \in W, z \to z_1$. Note that $\hat{a}(z)$ also tends to $+\infty$ in this case. For any point $z_\infty \in [\partial W \cap V] \setminus E$ we can find a sequence $\{z_n\} \subset W, z_n \to z_\infty$ such that $a_t(z_n) \leq C$. Thus by Lemma 2.3 $f|_{\Delta_{z_\infty} \setminus \Delta_{z_\infty}(1-r)}$ extends onto Δ_{z_∞}.

Let ν_1 be as in Lemma 2.3 above for an appropriate $K \Subset X$. This compact K should be taken to contain $f(\bar{V} \times \bar{A}(1-r,t) \cup (W \cap \bar{V}) \times \bar{\Delta}_t)$. It exists because of disk-convexity of X. Set $E_j = \{z \in \partial W \cap V : a(z) \leq \frac{j}{2}\nu_1\}$ for $j = 1, 2, \ldots$. From Lemma 2.3 we see that E_j are closed subsets of $\partial W \cap V$, $E_j \subset E_{j+1}$, and we have $\partial W \cap V = E \cup \bigcup_{j=1}^{\infty} E_j$.

Furthermore from Lemma 2.3 we see that $E_{j+1} \setminus E_j$ is a discrete subset of $V \setminus E_j$, say $E_{j+1} \setminus E_j = \{a_{ij}\}$. Now put

$$u_1(z) = -\sum_{i,j} c_{ij} \log |z - a_{ji}|. \tag{2.2.6}$$

Here positive constants c_{ij} are chosen in such a manner that $\sum_{i,j} c_{ij} < +\infty$. Then $u_1(z)$ is superharmonic in V, $u_1(z) \to +\infty$ as $z \to \bigcup_{j=1}^{\infty} E_j$ and $u_1(z) \neq$

$+\infty$ for all $z \in V \cap W$. Now put $u_2(z) = \hat{a}(z) + u_1(z)$. Note that u_2 is superharmonic in $W \cap V$ and $u_2(z) \to +\infty$ as $z \to \partial W \cap V$. Define

$$u_n(z) = \min\{n, u_2(z)\} \tag{2.2.7}$$

for $n \geq 3$. Note that u_n are superharmonic in V, because $u_n \equiv n$ in the neighborhood of $\partial W \cap V$. Put now $u(z) = \lim_{n\to\infty} u_n(z)$. Then u is superharmonic in V as a nondecreasing limit of superharmonic functions. Using the fact that $\hat{a}$ is finite on W, we obtain that $u(z) = u_2(z) \neq +\infty$ for any $z \in V \cap W$ and $u\,|_{V\setminus W} \equiv +\infty$; i.e. $\partial W \cap \Delta$ is *complete* polar in Δ. So the lemma is proved. □

In what follows we shall use the fact that a closed set of zero harmonic measure in the plane has zero Hausdorff dimension; see [Gl]. Put $S_1 = \Delta \setminus W$, where W is the maximal domain in Δ such that our map f extends holomorphically onto $H^2_W(1-r)$. We have proved that S_1 is polar i.e. of harmonic measure zero. In particular, S_1 is zero-dimensional. For any $\delta > 0$ we can find $0 < \delta_1 < \delta$ such that $\partial\Delta_{1-\delta_1} \cap S_1 = \emptyset$. Now we can change coordinates z_1, z_2 and consider the Hartogs figure $H = \{(z_1,z_2) \in \Delta^2 : 1-r < |z_2| < 1, |z_1| < 1$ or $|z_2| < 1, 1-\delta_1-\varepsilon < |z_1| < 1-\delta_1+\varepsilon\}$, where ε is small enough. Applying Lemma 2.4 again we extend f onto $\Delta \times (\Delta \setminus S_2)$ where S_2 is of harmonic measure zero. Therefore we obtain a holomorphic extension of f onto $\Delta^2 \setminus S$, where S is a product of two complete polar sets in Δ. So S is complete polar itself and has Hausdorff dimension zero. This proves Part 1 of Theorem 2.2 in dimension two.

Denote by T the positive $(1,1)$-current (in fact the smooth form) $f^*\omega$ on $\Delta^2 \setminus S$. By Lemma 3.3 from [Iv-2] we have that T has locally summable coefficients on the whole Δ^2 and from Lemma 2.1 above we see that $dd^c\tilde{T}$ is a negative measure with singular support contained in S. We write $dd^c\tilde{T} = \mu$. We set furthermore $\mu_s := \chi_S \cdot \mu$ and $\widetilde{dd^cT} = \mu_0$. All μ, μ_s and μ_0 are negative measures, in fact μ_0 is an L^1-function and $\mu = \mu_0 + \mu_s$.

Let us suppose now that the metric form ω on X is pluriclosed. Shrinking, if necessary we shall suppose that S is compact.

Lemma 2.5. *Suppose that the metric form ω is pluriclosed and take a ball $B \subset\subset \Delta^2$ such that $\partial B \cap S = \emptyset$.*

(i) *If $f(\partial B)$ is homologous to zero in X then $dd^c\tilde{T} = 0$ on B.*

(ii) *If $dd^c\tilde{T} = 0$ then f extends meromorphically onto B.*

In [Iv-2, Lemma 4.4], this statement was proved for the case when $S \cap B = \{0\}$. One can easily check that the same proof goes through for the case when $S \cap B$ is closed zero-dimensional. In fact in Lemmas 2.8 and 2.9 we will prove this statement "with parameters".

So, statement (2) of Theorem 2.2 and thus Main Theorem are proved in the case $n = 1$, i.e. in dimension two.

2.3. *Proof in higher dimensions*: *plurinegative metrics.* Let us turn to the proof of these theorems in higher dimensions. First we suppose that the metric form ω is plurinegative.

Let $f : H^{n+1}_U(1-r) \to X$ be our map. It will be convenient to set $U = \Delta^n(r)$.

Step 1. f extends to a holomorphic map of $\bigcup_{z' \in \Delta^{n-1}_r \setminus R_1} (\Delta^2_{z'} \setminus S_{z'})$ into X, where R_1 is contained in a locally finite union of locally closed proper subvarieties of Δ^{n-1}_r and $S_{z'}$ is zero-dimensional and pluripolar in $\Delta^2_{z'}$.

Proof of Step 1. For $z' = (z_1, \dots, z_{n-1}) \in \Delta^{n-1}_r$ denote by $H^2_{z'} = H^2_{z'}(1-r)$ the two-dimensional Hartogs domain $\{z'\} \times H^2(1-r)$ in the bidisk $\Delta^2_{z'} = \{z'\} \times \Delta^2 \in \mathbb{C}^{n+1}$. Shrinking $H^{n+1}(1-r)$ if necessary, we can suppose that $I(f)$ consists of finitely many irreducible components. Denote by R_1 the set of $z' \in \Delta^{n-1}_r$ such that $\dim[H^2_{z'} \cap I(f)] > 0$. R_1 is clearly contained in a finite union of locally closed proper analytic subsets of Δ^{n-1}_r. For $z' \in \Delta^{n-1}_r \setminus R_1$, by the results of Section 2.2 the map $f|_{H^2_{z'}}$ extends to a holomorphic map $f_{z'} : \Delta^2_{z'} \setminus S_{z'} \to X$, where $S_{z'}$ is zero-dimensional and complete pluripolar in $\Delta^2_{z'}$. Note also that $S_{z'} \supset \Delta^2_{z'} \cap I(f)$.

Take a point $z' \in \Delta^{n-1}_r \setminus R_1$ and a point $z_n \in \Delta \setminus \pi_n(S_{z'})$. Here $\pi_n : \{z'\} \times \Delta \times \Delta \to \{z'\} \times \Delta$ is the projection onto the variable z_n. Take a domain $U \subset\subset \{z'\} \times \Delta \times \{0\}$ that is biholomorphic to the unit disk, does not contain points from $\pi_n(S_{z'})$ and contains the points $u := (z', 0, 0)$ and $v := (z', z_n, 0)$. We also take U intersecting $A(1-r, 1)$. If $\{z'\} \times \{0\}$ is in $\pi_n(S_{z'})$ then take as u some point close to $(z', 0, 0)$ in $\{z'\} \times \Delta$. Find a Stein neighborhood V of the graph $\Gamma_{f|_{\{z'\} \times \bar{U} \times \Delta}}$. Let $w \in \partial U \cap A(1-r, 1)$ be some point. We have $f(\{z', w\} \times \Delta) \subset V$ and $f(\{z'\} \times \partial U \times \Delta) \subset V$. So the usual continuity principle for holomorphic functions gives us a *holomorphic* extension of f to the neighborhood of $\{z'\} \times \bar{U} \times \Delta$ in Δ^{n+1}. Changing a little the slope of the z_{n+1}-axis and repeating the arguments as above we obtain a holomorphic extension of f onto the neighborhood of $\{z'\} \times (\Delta \setminus S_{z'})$ for each $z' \in \Delta^{n-1}_r \setminus R_1$.

Step 2. f extends holomorphically onto $(\Delta^{n-1}_r \times \Delta^2) \setminus R$, where R is a closed subset of $\Delta^{n-1}_r \times \Delta^2$ of Hausdorff codimension 4.

Proof of Step 2. Consider a subset $R_2 \subset R_1$ consisting of such $z' \in \Delta^{n-1}_r$ that $\dim[H^2_{z'} \cap I(f)] = 2$, i.e. $H^2_{z'} \subset I(f)$. This is a finite union of locally closed subvarieties of Δ^{n-1}_r of complex codimension at least two. Thus $\bigcup_{z' \in R_2} \Delta^2_{z'}$ has Hausdorff codimension at least four.

For $z^{'} \in R_1 \setminus R_2 = \{z^{'} \in \Delta_r^{n-1} : \dim[H^2_{z'}(1-r) \cap I(f)] = 1\}$, using Section 2.2 we can extend $f_{z'}$ holomorphically onto $\Delta^2_{z'}$ minus a zero-dimensional polar set. Repeating the arguments from Step 1 we can extend f holomorphically to a neighborhood of $\Delta^2_{z'} \setminus C_{z'}$ in $\Delta_r^{n-1} \times \Delta^2$. Here $C_{z'}$ is a complex curve containing all one-dimensional components of $H^2_{z'}(1-r) \cap I(f)$.

$\bigcup_{z' \in R_1 \setminus R_2} C_{z'}$ has Hausdorff codimension at least four. Thus the proof of Step 2 is completed by setting $R = \bigcup_{z' \in R_1 \setminus R_2} C_{z'} \cup \bigcup_{z' \in R_2} \Delta^2_{z'}$.

Step 3. We shall state this step in the form of a lemma.

LEMMA 2.6. *There exists a closed, complete $(n-1)$-polar subset $A \subset R$ and a holomorphic extension of f onto $(\Delta_r^{n-1} \times \Delta^2) \setminus A$ such that the current $T := f^*\omega$ has locally summable coefficients in a neighborhood of A. Moreover, $dd^c\tilde{T}$ is negative, where $\tilde{T}$ is the trivial extension of T.*

Take a point $z_0 \in R$ and using the fact that R is of Hausdorff codimension four in $\mathbb{C}^{n+1}$, find a neighborhood $V \ni z_0$ with a coordinate system $(z_1, \dots, z_{n+1})$ such that $V = \Delta^{n-1} \times \Delta^2$ in these coordinates and for all $z' \in \Delta^{n-1}$ one has $R \cap \partial\Delta^2_{z'} = 0$. By Section 2.2 the restrictions $f_{z'}$ extend holomorphically onto $\Delta^2_{z'} \setminus A(z')$, where $A(z')$ are closed complete pluripolar subsets in $\Delta^2_{z'}$ of Hausdorff dimension zero. By the arguments similar to those used in Step 1, f extends holomorphically to a neighborhood of $V \setminus A$, $A := \bigcup_{z' \in \Delta^{n-1}} A(z')$.

Consider now the current $T = f^*\omega$ defined on $(\Delta^{n-1} \times \Delta^2) \setminus R$. Note that T is smooth, positive and $dd^cT \leq 0$ there. By Lemma 3.3 from [Iv-2] every restriction $T_{z'} := T \mid_{\Delta^2_{z'}} \in L^1_{\mathrm{loc}}(\Delta^2_{z'})$, $z' \in \Delta^{n-1}$. We shall use the following Oka-type inequality for plurinegative currents proved in [F-Sb]:

There is a constant C_ρ such that for any plurinegative current T in Δ^2,

$$\|T\|(\Delta^2) + \|dd^cT\|(\Delta^2) \leq C_\rho \|T\|(\Delta^2 \setminus \bar{\Delta}^2_\rho). \tag{2.3.1}$$

Here $0 < \rho < 1$.

Apply (2.3.1) to the the trivial extensions $\tilde{T}_{z'}$ of $T_{z'}$, which are plurinegative by (b) of Lemma 2.1, to obtain that the masses $\|\tilde{T}_{z'}\|(\Delta^2)$ are uniformly bounded on z' on compacts in Δ^{n-1}. On L^1 the mass norm coincides with the L^1-norm. So taking the second factor in $\Delta^{n-1} \times \Delta^2$ with different slopes and using Fubini's theorem we obtain that $T \in L^1_{\mathrm{loc}}(\Delta^{n-1} \times \Delta^2)$.

All that is left to prove is that $dd^c\tilde{T}$ is negative. It is enough to show that for any collection L of $(n-1)$ linear functions $\{l_1, \dots, l_{n-1}\}$ the measure $dd^c\tilde{T} \wedge \frac{i}{2}\partial l_1 \wedge \overline{\partial l_1} \wedge \dots \wedge \frac{i}{2}\partial l_{n-1} \wedge \overline{\partial l_{n-1}}$ is nonpositive; see [Hm]. Complete these functions to a coordinate system $\{z_1 = l_1, \dots, z_{n-1} = l_{n-1}, z_n, z_{n+1}\}$ and note that for almost all collections L the set $\Delta^2_{z'} \cap A$ is of Hausdorff dimension zero for all $z' \in \Delta^{n-1}$. Thus $\tilde{T} \mid_{z'}$ is plurinegative for all such z'. Taking a nonnegative function, $\phi \in \mathcal{D}(\Delta^{n+1})$, we have

$$(n-1)!\langle dd^c\tilde{T}\wedge\frac{i}{2}\partial l_1\wedge\overline{\partial l_1}\wedge\cdots\wedge\frac{i}{2}\partial l_{n-1}\wedge\overline{\partial l_{n-1}},\phi\rangle$$
$$=\int_{\Delta^{n+1}}\tilde{T}\wedge(dd^c\|z'\|^2)^{n-1}\wedge dd^c\phi$$
$$=\int_{\Delta^{n-1}}(dd^c\|z'\|^2)^{n-1}\int_{\Delta^2}(\tilde{T})_{z'}\wedge dd^c\phi=\int_{\Delta^{n-1}}(dd^c\|z'\|^2)^{n-1}\int_{\Delta^2}\tilde{T}_{z'}\wedge dd^c\phi$$
$$=\int_{\Delta^{n-1}}(dd^c\|z'\|^2)^{n-1}\int_{\Delta^2}dd^c(\tilde{T})_{z'}\wedge\phi\le 0.$$

We used here Fubini's theorem for L^1-functions, the fact that $(\tilde{T})_{z'}=\tilde{T}_{z'}$ for currents from L^1_{loc} that are smooth outside of a suitably situated set A, and finally the plurinegativity of $\tilde{T}_{z'}$.

Therefore $\tilde{T}$ is plurinegative. We got an extension of f onto $\Delta_r^{n-1}\times\Delta^2\setminus A$, but this obviously implies an extension (holomorphic) onto $\Delta^{n+1}\setminus A$, where A has zero Hausdorff $(2n-2)$-dimensional measure and is complete $(n-1)$-polar. Therefore the first part of the Theorem 2.2 is proved. □

The following statement is interesting by itself, but will not be used later. Let W be the maximal open subset of Δ^{n+1} such that f meromorphically extends onto W. Denote by $\theta(x_0,dd^c\tilde{T})$, the Lelong number of the closed negative current $dd^c\tilde{T}$.

Lemma 2.7. *Under the conditions above, if $x_0\in W$ then $\theta(x_0,dd^c\tilde{T})=0$.*

Proof. Find an orthonormal coordinate system $(z_1,\dots,z_{n-1})=z'$, $(z_n,z_{n+1})=z''$ with center in x_0 and $r_0>0$ such that for every $x'\in\Delta^{n-1}(r_0)$ the intersection $\Delta^2_{x'}\cap\mathcal{I}(f)$ is finite. Here $\mathcal{I}(f)$ is the indeterminancy set of f. For $r_0>r>0$ set $\tilde{\Gamma}_f(r):=p^{-1}(\Delta^{n+1}(x_0,r))$ and $\tilde{\Gamma}_{f,x'}(r):=p^{-1}(\Delta^2_{x'}(r))$. Note that by the geometric flattening theorem, see [Ba-2], $\mathrm{vol}[\tilde{\Gamma}_{f,x'}(r_0)]\le C$ for all $x'\in\Delta^2(r_0)$. Now, because $\tilde{T}=T$ on W we see that

$$\theta(x_0,dd^c\tilde{T})=\lim_{r\to 0}\frac{1}{r^{2(n-1)}}\int_{\Delta^{n+1}(x_0,r)}dd^cT\wedge(dd^c\|z\|^2)^{(n-1)}. \tag{2.3.6}$$

This integral (and limit) can be estimated by the sum of integrals of the type

$$\lim_{r\to 0}\frac{1}{r^{2(n-1)}}\int_{\Delta^{n+1}(x_0,r)}dd^cT\wedge(dd^c\|z'\|^2)^{(n-1)}, \tag{2.3.7}$$

with sufficiently many orthonormal coordinate systems centered at x_0. So let us prove that the last limit is zero. First note that

$$(2.3.8)\quad \int_{\Delta^{n+1}(0,r)} dd^cT\wedge(dd^c\|z'\|^2)^{(n-1)} \le \int_{\Delta^{n-1}(0,r)}(dd^c\|z'\|^2)^{n-1}\int_{\Delta^2_{z'}(0,r)} dd^cT$$
$$\le r^{2(n-1)}\sup_{z'\in\Delta^{(n-1)}(0,r)}\int_{\Delta^2_{z'}(0,r)} dd^cT.$$

Now we need to prove that $\sup_{z'\in\Delta^{(n-1)}(0,r)}\int_{\Delta^2_{z'}(0,r)} dd^cT \to 0$ as $r\to 0$. This will be done in two steps.

Step 1. $\int_{\Delta^2_{0'}(0,r)} dd^cT \to 0$ as $r\to 0$. Take the irreducible component $\Gamma_{f,0}(r_0)$ of $\tilde\Gamma_{f,0'}(r_0)$ which projects onto $\Delta^2_{0'}(0,r)$ surjectively. This is the graph of the restriction $f|_{\Delta^2_{0'}(0,r_0)}$. Do the same for all $r<r_0$. Notice that $\Gamma_{f,0}(r)\subset\Gamma_{f,0}(r_0)$. As $r\to 0$, $\Gamma_{f,0}(r)$ contracts to a finite union of curves - the fiber of $f|_{\Delta_{0'}(0,r_0)}$ over zero. In particular $vol[\Gamma_{f,0}(r)]\to 0$. Since $dd^cT = dd^c\pi^*\omega$ is a smooth 4-form on $\Gamma_{f,0}(r_0)$ it is straightforward that $\int_{\Delta_{0'}(0,r)} dd^cT\to 0$.

Step 2. $\sup_{z'\in\Delta^{(n-1)}(0,r)}\int_{\Delta^2_{z'}(0,r)} dd^cT\to 0$ as $r\to 0$. Otherwise we would find a sequence $z_n'\to 0$ and $r_n\to 0$ such that $\int_{\Delta_{z_n'}(0,r_n)} dd^cT\le\varepsilon_0<0$. Take any $r_0>\rho>0$ such that $\partial\Delta^2_{(0',\rho)}\cap\mathcal{I}(f)=\emptyset$ and remark that $\partial\Delta^2_{(z',\rho)}\cap\mathcal{I}(f)=\emptyset$ for z' close to zero. Then

$$\varepsilon_0\ge\int_{\Delta^2_{z_n'}(0,r_n)} dd^cT\ge\int_{\Delta^2_{z_n'}(0,\rho)} dd^cT$$
$$=\int_{\partial\Delta^2_{z_n'}(0,\rho)} d^cT\to\int_{\partial\Delta^2_{0'}(0,\rho)} d^cT=\int_{\Delta^2_{0'}(0,\rho)} dd^cT.$$

But as $\rho\to 0$ the last integral tends to zero by Step 1. This contradiction proves the lemma. □

2.4. *Proof in higher dimensions*: *pluriclosed case.* Fix a point $a\in A$ and suppose that there is a transversal sphere $\mathbb{S}^3=\{x\in P:\|x-a\|=\varepsilon\}$ on some two-plane through a such that $f(\mathbb{S}^3)$ is homologous to zero in X. We shall prove that in this case f meromorphically extends to the neighborhood of a. Write $W=B^{n-1}\times B^2$ for some neighborhood of this point $a\in A$ such that $(\bar B^{n+1}\times\partial B^2)\cap A=\emptyset$ and for every $z^{'}\in B^{n-1}$ one has $f(\partial B^2_{z'})\sim 0$. First we prove the following:

Lemma 2.8. *Suppose that the metric form w on X is pluriclosed and for all $z'\in B^{n-1}$, $f(\partial B^2_{z'})\sim 0$ in X. Then*:

(i) $dd^c\tilde T=0$ *in the sense of distributions.*

(ii) *There exists a* (1,0)*-current γ in W, smooth in $W\setminus A$, such that* $\tilde T = i(\partial\bar\gamma-\bar\partial\gamma)$.

Proof. (i) Let $\tilde{T}_\varepsilon$ be smoothings of $\tilde{T}$ by convolution. Then $\tilde{T}_\varepsilon$ are plurinegative and $\tilde{T}_\varepsilon \to \tilde{T}$ in $\mathcal{D}_{n,n}(W)$. We have that

$$\text{(2.4.1)} \quad \int_W dd^c\tilde{T}_\varepsilon \wedge (dd^c\|z'\|^2)^{n-1} = \int_{\partial W} d^c\tilde{T}_\varepsilon \wedge (dd^c\|z'\|^2)^{n-1}$$
$$= \int_{\partial B^{n-1}\times B^2} d^c\tilde{T}_\varepsilon \wedge (dd^c\|z'\|^2)^{n-1} + \int_{B^{n-1}\times \partial B^2} d^c\tilde{T}_\varepsilon \wedge (dd^c\|z'\|^2)^{n-1}.$$

The first integral vanishes by degree considerations. Thus

$$\text{(2.4.2)} \qquad \|dd^c\tilde{T}_\varepsilon \wedge (dd^c\|z'\|^2)^{n-1}\|(W) = -\int_{B^{n-1}} (dd^c\|z'\|^2)^{n-1} \int_{\partial B^2_{z'}} d^c\tilde{T}_\varepsilon.$$

Observe now that $\int_{\partial B^2_{z'}} d^c\tilde{T}_\varepsilon \to \int_{\partial B^2_{z'}} d^c\tilde{T} = \int_{f(\partial B^2_{z'})} d^c w = 0$, because $f(\partial B^2_{z'}) \sim 0$ in X. So the right-hand side of (2.4.2) tends to zero as $\varepsilon \searrow 0$. We obtain that

$$\text{(2.4.3)} \quad \|dd^c\tilde{T} \wedge (dd^c\|z'\|^2)^{n-1}\|(W) = \lim_{\varepsilon\searrow 0} \|dd^c\tilde{T}_\varepsilon \wedge (dd^c\|z'\|^2)^{n-1}\|(W) = 0.$$

Taking sufficiently many such coordinate systems we see that $\|dd^c\tilde{T}\|(W) = 0$.

(ii) $\partial\tilde{T}$ is a $\bar\partial$-closed and ∂-closed $(2,1)$-current. So, if $\phi \in \mathcal{D}_{n-1,n+1}(W)$ is ∂-closed and such that $\bar\partial\phi = \partial\tilde{T}$ then ϕ is smooth on $W \setminus A$ by elliptic regularity of $\bar\partial$. We have now $d\tilde{T} = \partial\tilde{T} + \bar\partial\tilde{T} = \bar\partial\phi + \partial\bar\phi$. Thus $d(\tilde{T} - \phi - \bar\phi) = 0$. So $\tilde{T} - \phi - \bar\phi$ is a d-closed current of degree two on W. Consider the following elliptic system in W:

$$\text{(2.4.4)} \qquad d\gamma = \tilde{T} - \phi - \bar\phi \ , \ d^*\gamma = 0.$$

Then (2.4.4) has a solution in W. Indeed, let γ_1 be any solution of the first equation. Find a distribution η on W with $*d*d\eta = \Delta\eta = *d*\gamma_1$ and put $\gamma_2 = \gamma_1 - d\eta$. Now γ_2 is smooth on $W \setminus A$ because $\Delta\gamma_2 = d^*d\gamma_2 + dd^*\gamma_2 = d^*(\tilde{T} - \phi - \bar\phi)$. Write $\gamma_2 = i(\gamma^{1,0} - \bar\gamma^{1,0})$ – the general form of a real 1-form. We have $i\partial\gamma^{1,0} = -\phi$ and $i\bar\partial\bar\gamma^{1,0} = +\bar\phi$, so that

$$\text{(2.4.5)} \quad \tilde{T} = d\gamma_2 + \phi + \bar\phi = d(i\gamma^{1,0} - i\bar\gamma^{1,0}) - i\partial\gamma^{1,0} + i\bar\partial\bar\gamma^{1,0} = i(-\partial\bar\gamma^{1,0} + \bar\partial\gamma^{1,0})$$

where $\gamma^{1,0}$ has the required regularity. Now $\gamma = -\gamma^{1,0}$ satisfies (ii). □

LEMMA 2.9. *If $\tilde{T}$ is pluriclosed, then the volumes $\Gamma_{f'_z} \cap B^2_z \times X$ are uniformly bounded for $z \in B^{n-1}_r$ and f extends meromorphically onto W.*

Proof. Set $S := T + dd^c\|z\|^2$, where $z = (z_n, z_{n+1})$. Note that S is pluriclosed if T is. Find $\gamma^{1,0}$ for S as in Lemma 2.8; i.e., $\gamma^{1,0}$ is a $(0,1)$-current on W, smooth on $W \setminus A$, such that $S = i(\partial\bar\gamma^{1,0} - \bar\partial\gamma^{1,0})$. Smoothing by convolutions we still have $\tilde{S}_\varepsilon = i(\partial\bar\gamma^{1,0}_\varepsilon - \bar\partial\gamma^{1,0}_\varepsilon)$. Then for $z' \in B^{n-1}$ and $A_{z'} := A \cap B^2_{z'}$ we have:

$$
\begin{aligned}
\text{(2.4.6)} \qquad \operatorname{vol}(\Gamma_{f_{z'}}) &= \int_{B^2_{z'}\setminus A_{z'}} S^2 = \lim_{\varepsilon\searrow 0}\int_{B^2_{z'}\setminus A_{z'}} \tilde{S}^2_\varepsilon \\
&\leq \lim_{\varepsilon\searrow 0}\int_{B^2_{z'}} \tilde{S}^2_\varepsilon = \lim_{\varepsilon\searrow 0}\int_{B^2_{z'}} i^2(\partial\bar{\gamma}^{1,0}_\varepsilon - \bar{\partial}\gamma^{1,0}_\varepsilon)^2 \\
&\leq \lim_{\varepsilon\searrow 0}\int_{B^2_{z'}} i^2 d(\bar{\gamma}^{1,0}_\varepsilon - \gamma^{1,0}_\varepsilon)\wedge d(\bar{\gamma}^{1,0}_\varepsilon - \gamma^{1,0}_\varepsilon) \\
&= \lim_{\varepsilon\searrow 0}\int_{\partial B^2_{z'}} i^2(\bar{\gamma}^{1,0}_\varepsilon - \gamma^{1,0}_\varepsilon)\wedge d(\bar{\gamma}^{1,0}_\varepsilon - \gamma^{1,0}_\varepsilon) \\
&= \int_{\partial B^2_{z'}} i^2(\bar{\gamma}^{1,0} - \gamma^{1,0})\wedge d(\bar{\gamma}^{1,0} - \gamma^{1,0}) \leq \text{const.}
\end{aligned}
$$

In the first inequality we used the positivity of T. In the second, the fact that $-i^2\bar{\partial}\bar{\gamma}^{1,0}_\varepsilon \wedge \partial\gamma^{1,0}_\varepsilon$ is positive and $\bar{\partial}\bar{\gamma}^{1,0}_\varepsilon \wedge \bar{\partial}\bar{\gamma}^{1,0}_\varepsilon = 0$. Finally $\gamma^{1,0}_\varepsilon \to \gamma^{1,0}$ on $\bar{B}^{n-1}\times\partial B^2$, since $\gamma^{1,0}$ is smooth there. This gives the required bound for $\operatorname{vol}(\Gamma_{f_{z'}}) = \int_{B^2_{z'}\setminus A_{z'}} S^2$.

Lemma 1.3 (with $k = 2$) gives us now the extension of f onto $W \cong B^{n-1}\times B^2$. Lemma and Theorem 2.2 are proved. □

We end up with two remarks about the structure of the singularity set A of our mapping in the presence of a pluriclosed metric form.

Consider two natural projections $\pi^1 : \Delta^{n+1} \to \Delta^{n-1}\times\Delta_{z_n}$ and $\pi^2 : \Delta^{n+1} \to \Delta^{n-1}\times\Delta_{z_{n+1}}$. Observe that $\pi^j|_A$ are proper, $j = 1,2$. Set $A_j = \pi^j(A)$. We shall prove that each A_j is pseudoconcave in Δ^n and admits a Sadullaev potential. We start with

Lemma 2.10. *The A_j are complete pluripolar and moreover admit a Sadullaev potential.*

Proof. Recall (see [Lv-Sl]) that a Sadullaev potential for a closed complete pluripolar set $A_j \subset \Delta^n$ is a plurisubharmonic function ψ_j in Δ^n such that ψ_j is pluriharmonic on $\Delta^n\setminus A_j$ and $A_j = \{z\in\Delta^n : \psi_j(z) = -\infty\}$.

For $z' = (z_1,\dots,z_n)\in\Delta^n\setminus A_j$ define the area function $a_j(z')$ as in (2.2.3). The proof of Lemma 2.4 shows without essential changes that $dd^c a_j$ is a smooth $(1,1)$-form in Δ^n. We claim that $a_j(z')\to+\infty$ as $z'\to A_j$.

Suppose not, i.e. there is a sequence $\{p_k\}\subset\Delta^n\setminus A_j$ such that $p_k\to p_0\in A_j$ and $\psi_j(p_k)\leq C_0$ for some C_0 and all k. Corollary 1.7 in this case provides a complex curve W in the neighborhood of p_0 such that f meromorphically extends onto $W\times\Delta$. From this fact it is straightforward to produce a transversal sphere $\mathbb{S}^3\subset\Delta^{n+1}\setminus A$ such that $f(\mathbb{S}^3)\sim 0$ in X. This is a contradiction with the Main Theorem.

Now, $a_j(z')\to\infty$ as $z'\to A_j$. Find next some smooth function $h_j(z')$ such that $dd^c h_j = dd^c a_j$ and put $\psi_j = h_j - a_j$. □

Recall that a closed subset $A_j \subset \Delta^n$ is pseudoconcave if $\Delta^n \setminus A_j$ is pseudoconvex.

COROLLARY 2.11. *The A_j are pseudoconcave.*

Proof. It is sufficient to prove that for a holomorphic map $\phi : \Delta^2(r) \subset \Delta^n$ if $\phi : H^2(r) \to \Delta^n \setminus A_j$ then $\phi(\Delta^2) \subset \Delta^n \setminus A_j$.

Let ψ_j be a Sadullaev potential of A_j. Then $\psi_j \circ \phi$ is pluriharmonic on $H^2(r)$ and therefore extends pluriharmonically onto the Δ^2. Therefore $\phi(\Delta^2) \cap A_j = \emptyset$. □

Remark. In [Lv-Sl] an example of a closed subset $A \subset \Delta \times A(r,1)$ is constructed, which is pluripolar, pseudoconcave and admits a Sadullaev potential. At the same time for every $z \in \Delta$ the set $A_z = \Delta_z \cap A$ is of Cantor-type and A admits no analytic structure.

Some cases when A possesses an analytic structure will be considered in Sections 3.4 and 3.5.

2.5. *The case of tamed structures.* Let us now prove Corollary 3 from the introduction. Namely, we suppose that the metric form w on our space X is the $(1,1)$-component of some closed real two-form w_0, i.e., that there is a $(2,0)$-form $w^{2,0}$ such that $w_0 = w^{2,0} + w + \bar{w}^{2,0}$ and $dw_0 = 0$.

As remarked in the introduction such a w is obviously dd^c-closed. Thus the machinery of the proof of the Main Theorem applies to this case. Therefore our mapping f can be extended meromorphically onto $\Delta^{n+1} \setminus A$, where A is either empty or is analytic of pure codimension two.

Suppose $A \neq \emptyset$. Take a point $a \in A$ with a neighborhood $W \ni a$ biholomorphic to $B^{n-1} \times B^2$ and such that $\pi |_{\hat{A} \cap W} : B^{n-1} \times B^2 \to B^{n-1}$ is proper. Here $\hat{A} = A \cup I(f)$ is a union of A with the set of points of indeterminancy of f. Let us prove that $dd^c\tilde{T} = 0$ in W, where $T = f^*w$ on $\Delta^{n+1} \setminus \hat{A}$.

From Lemma 2.15 we see that all we must prove is that $\int_{\partial B^2_{z'}} d^c\tilde{T}_\varepsilon = 0$ for all $z' \in B^{n-1}$. Indeed, let $T^0 = f^*w_0$ and $T^{2,0} = f^*w^{2,0}$ on $\Delta^{n+1} \setminus \hat{A}$. Then, since $dT^0 = d^cT^0 = 0$, one has:

$$\int_{\partial B^2_{z'}} d^c\tilde{T}_\varepsilon = \int_{\partial B^2_{z'}} d^c(\tilde{T}_\varepsilon - T^0_\varepsilon) = \int_{\partial B^2_{z'}} d^c(-\tilde{T}^{2,0}_\varepsilon - \bar{T}^{2,0}_\varepsilon). \tag{2.5.1}$$

Take a cut-off function η with support in a neighborhood of $B^2_{z'}$. Then

$$\int_{\partial B^2_{z'}} d^c\tilde{T}^{2,0}_\varepsilon = \int_{\partial B^2_{z'}} d^c(\eta\tilde{T}^{2,0})_\varepsilon = \int_{B^2_{z'}} dd^c(\eta\tilde{T}^{2,0})_\varepsilon = 0 \tag{2.5.2}$$

by the reasons of bidegree.

So $\tilde{T}$ is pluriclosed on W and we can extend f onto the whole W using Lemma 2.9. □

3. Examples and open questions

3.1. *General conjecture.* Here we shall propose a general conjecture about extension properties of meromorphic mappings. In Section 1.5 we introduced the class $\mathcal{G}_k$ of reduced complex spaces possessing a strictly positive dd^c-closed (k,k)-form.

We conjecture that meromorphic mappings into the spaces of class $\mathcal{G}_k$ are "almost Hartogs-extendable" in bidimension (n,k) for all $n \geq 1$. That is, we let

$$H_n^k(r) := \Delta_{1-r}^n \times \Delta^k \cup \Delta^n \times A^k(r,1)$$

be the k-concave Hartogs figure.

Conjecture. *Every meromorphic map $f : H_n^k(r) \to X$, where $X \in \mathcal{G}_k$ and is disk-convex in dimension k, extends to a meromorphic map from $\Delta^{n+k} \setminus A$ to X, where A is a closed $(n-1)$-polar subset of Hausdorff $(2n-1)$-dimensional measure zero. Moreover, if $A \neq \emptyset$, then for every transversal sphere $\mathbb{S}^{2k+1}$ in $\Delta^{n+k} \setminus A$ its image $f(\mathbb{S}^{2k+1})$ is not homologous to zero in X. That is, if $A \neq \emptyset$ then X should contain a $(k+1)$-dimensional spherical shell.*

In this paper we proved this conjecture in the case $k = 1$.

3.2. *Examples to the Hartogs-type theorem.* We start with an example due to M. Kato; see [Ka-4].

Example 3.1. In $\mathbb{CP}^3$ with homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ consider the domain $D = \{z \in \mathbb{CP}^3 : |z_0|^2 + |z_1|^2 > |z_2|^2 + |z_3|^2\}$. The natural action of $\mathrm{Sp}(1,1)$ on $\mathbb{CP}^3$ preserves D, i.e., $g(D) = D$ for all $g \in \mathrm{Sp}(1,1)$. This action is transitive on D and Kato proved, using result of Vinberg, that there exists a discrete subgroup $\Gamma \subset \mathrm{Sp}(1,1)$, acting properly and discontinuously on D, and such that $D/\Gamma = X^3$ is a compact complex manifold, see [Ka-4] for details.

The projective plane $\mathbb{CP}^2 = \{z_3 = 0\}$ intersects D by the complement to the closed unit ball $\bar{B}^4 \subset \mathbb{CP}^2$, namely, by $\mathbb{CP}^2 \setminus \bar{B}^4 = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 : |z_2|^2 < |z_0|^2 + |z_1|^2\}$. If $\pi : D \to X^3$ is the natural projection, then its restriction $\pi \mid_{\mathbb{CP}^2 \cap D} : \mathbb{CP}^2 \setminus B^4 \to X^3$ defines a holomorphic map from the complement to the closed unit ball to X^3, which has a singularity at each point of ∂B^4!

However, it is not difficult to see that X does not contain either two-dimensional spherical shells no three-dimensional ones.

This example shows also that one cannot hope for better than is conjectured in Section 3.1. Following two examples illustrate how the bounded cycle space geometric condition can be violated.

Example 3.2. There exists a holomorphic mapping $f:\Delta\times\Delta_{\frac{1}{2}}\times A(\frac{1}{2},1)\to X^3$ such that:

(1) For any $s\in S=\{(z_0,z_2)\in\Delta\times\Delta_{\frac{1}{2}}:|z_0|^2>|z_2|^2\}$, the restriction $f_s=f\mid_{A_s(r,1)}$ extends holomorphically onto Δ_s;

(2) For any $t>1$ there is a constant $C_t<\infty$ such that for all $s\in S_t=\{(z_0,z_2)\in\Delta\times\Delta_{\frac{1}{2}}:|z_0|^2>t\cdot|z_2|^2\}$ one has $\mathrm{area}(\Gamma_{f_s})\le C_t$;

(3) But for all $z\in\Delta^2\setminus\bar{S}=\{(z_0,z_2)\in\Delta\times\Delta_{\frac{1}{2}}:|z_0|^2<|z_2|^2\}$ the inner circle of the annulus $A_z^1(r,1):=\{z_1\in\Delta_z:1>|z_1|^2>r^2\}$ consists of essentially singular points of $f_z:A_z(r,1)\to X$, here $r^2=|z_2|^2-|z_0|^2$.

Blow up $\mathbb{CP}^4$ at the origin of its affine part. Denote by $\mathbb{CP}^4_0$ the resulting manifold. There exists the natural holomorphic projection $p:\mathbb{CP}^4_0\to\mathbb{CP}^3$, where $\mathbb{CP}^3$ is considered as the exceptional divisor. Now Γ, being a group of 4×4 matrices, acts naturally on the affine part $\mathbb{C}^4$ of $\mathbb{CP}^4$. This action obviously extends onto $\mathbb{CP}^4$ and lifts onto $\mathbb{CP}^4_0$. Moreover, the actions of Γ on $\mathbb{CP}^3$ and $\mathbb{CP}^4_0$ are equivariant with respect to the projection p. Put $\hat{D}:=p^{-1}(D)$ and $\hat{X}:=\hat{D}/\Gamma$. Note that p descends to a holomorphic map (in fact, a $\mathbb{CP}^1$-fibration) $p:\hat{X}\to X^3$. If we take $\hat{f}:\{z\in\mathbb{C}^4:|z_0|^2+|z_1|^2>|z_2|^2,z_3=0\}\to\hat{X}$ to be the restriction of a quotient map, we get a mapping into $\hat{X}$ with properties (1),(2),(3). Taking composition $f:=p\circ\hat{f}:\Delta\times\Delta_{\frac{1}{2}}\times A(\frac{1}{2},1)\to X^3$ we get an example of the same type with a 3-dimensional image manifold.

Remark. The rational cycle space in this example has a noncompact irreducible component of dimension four.

For detailed construction of the following example we refer the reader to [Iv-4], where another property of this example was studied. Here we only list the properties related to the cycle space geometry. Namely, denote by $z=(z_1,z_2)$ the coordinates in $\mathbb{C}^2\times\{0\}\subset\mathbb{C}^3$:

Example 3.3. There exists a compact complex manifold X^3 of dimension three and a meromorphic map $f:\Delta^3\setminus\{0\}\to X^3$ such that:

(1) For every cone $K_n:=\{z=(z_1,z_2)\in\Delta^2:|z_2|>|z_1|^n\}$ there is a constant C_n such that $\mathrm{area}(\Gamma_{f_z})\le C_n$;

(2) $\mathrm{area}(\Gamma_{f_z})\to\infty$ where $z=(z_1,0)$ and $z_1\to 0$;

(3) f_0 extends from $\Delta_0\setminus\{0\}$ onto Δ_0;

(4) For every $t\in\mathbb{CP}^1$ $\lim_{z\to 0}\Gamma_{f_z}$ (where $z=(z_1,tz_1^n)$) is equal to Γ_{f_0} plus a rational cycle $Z_{n,t}$ which consists of n components (counted with multiplicities);

(5) $\{Z_{n,t} : t \in \mathbb{CP}^1\}$ *forms an irreducible component* A_n *of* $\mathcal{R}(X^3)$ *and* $\{A_n\}_{n=1}^{\infty}$ *is a connected chain of irreducible components of* $\mathcal{R}(X^3)$.

Remark. We see that the space $\mathcal{R}(X)$ can contain an infinite connected chain of compact irreducible components having nonbounded rational cycle geometry.

We propose that the interested reader check the details. Of course, f is the same map that was constructed and studied in detail in [Iv-4].

A statement similar to Lemma 1.3 (in fact a bit weaker) was implicitly used in [Sb, Théorème 5.2], and was claimed to follow from Fubini's theorem. The claim was that if one can bound the volumes of a meromorphic graph along sufficiently many two-dimensional directions then one can bound the total volume. If this were true, this would give a new proof of Siu's theorem of the removability of codimension-two singularities for meromorphic mappings into compact Kähler manifolds. It would also replace our Lemma 1.3 in the proof of the Main Theorem.

However, statements of this type cannot be derived from Fubini's theorem, because the measure on the graph is not the product measure of the measures on the slices. Moreover, we see that an analogous statement is not valid in the real case (and thus there is no "formula" to prove such things).

Of course, it should be made clear that this small gap does not touch any of the main results of [Sb].

Example 3.4. There exists a sequence of smooth mappings f_n from the square $\Pi = [0,1] \times [0,1] \subset \mathbb{R}^2$ to $\mathbb{R}^6$ such that:

(a) the lengths of the curves $f_n(x,\cdot) : [0,1] \to \mathbb{R}^6$ and $f_n(\cdot,y) : [0,1] \to \mathbb{R}^6$ are uniformly bounded for all $x,y \in [0,1]$, but

(b) the areas of $f_n(\Pi)$ turn to infinity.

First of all one easily constructs a sequence of smooth strictly positive functions ϕ_n on the square $\Pi = [0,1] \times [0,1]$, which:

a) are equal to 1 in some fixed neighborhood of $\partial\Pi$;

b) for all $x,y \in [0,1]$

$$\int_0^1 \phi_n(x,t)dt \le 2 \text{ and } \int_0^1 \phi_n(t,y)dt \le 2;$$

c) $\int_0^1 \int_0^1 \phi_n^2(x,y)dt \ge n$.

Consider Riemannian metrics $ds_n^2 = \phi_n^2 dx \otimes dx + \phi_n^2 dy \otimes dy$ on the square. The length of any segment parallel to the axis in this metric is

$$\int_0^1 \sqrt{\phi_n^2(x,t) + \phi_n^2(x,t)}dt \leq 2\sqrt{2},$$

while the area is $\int_0^1 \int_0^1 \sqrt{\phi_n^2(x,y) \cdot \phi_n^2(x,y)}dt \geq n$.

Now one can isometrically imbed (Π, ds_n^2) into $\mathbb{R}^6$; see [G]. This gives the required example.

The author should say at this point that he does not know such an example with meromorphic f_n-s (and thus it may be that the implicit statement from [Sb] is nevertheless correct), and moreover he thinks that in a complex analytic setting such "area-volume" estimates could be true. The proof might follow from cycle space techniques like those used in the present paper.

3.3. *Meromorphic correspondences.* Let D be a domain in complex space Ω and $x_0 \in \partial D$ be a boundary point. D is said to be q-concave at x_0 if there are a neighborhood $U \supset x_0$ and a smooth function $\rho : U \to \mathbb{R}$ such that

1) $D \cap U = \{x \in U : \rho(x) < 0\}$;

2) the Levi form of ρ at x_0 has at least $n - q + 1$ negative eigenvalues.

Here $n = \dim \Omega$. By the projection lemma of Siu, see [Si-T], if x_0 is a q-concave boundary point of D, $q \leq n-1$, one can find neighborhoods $U \ni x_0$ and $V \ni 0 \in \mathbb{C}^n$ and a proper holomorphic map $\pi : (U, x_0) \to (V, 0)$ such that $\pi(D \cap V)$ will contain a Hartogs figure H, whose associated polydisk P contains the origin. Let d be the branching number of π.

Now suppose that a meromorphic map $f : D \to X$ is given, where X is another complex space. $f \circ \pi^{-1}$ defines a d-valued meromorphic correspondence between H and X.

Definition 3.1. A d-valued meromorphic correspondence between complex spaces H and X is an irreducible analytic subset $Z \subset H \times X$ such that the restriction $p_1|_Z$ of the natural projection onto the first factor on Z is proper, surjective and generically d-to-one.

Thus the extension of f onto the neighborhood of x_0 is equivalent to the extension of Z from H to P. It is clear that if f was also a correspondence it would produce no additional complications. Thus we should discuss how far the problem of extension of correspondences goes from the extension of mappings.

Let Z be a d-valued meromorphic correspondence between the Hartogs figure H and X. Note that Z defines in a natural way a mapping $f_Z : H \to \mathrm{Sym}^d(X)$ - the symmetric power of X of degree d. Clearly the extension of Z

onto P is equivalent to the extension of f_Z onto P. If X was, for example, a Kähler manifold, then $\mathrm{Sym}^d(X)$ would be a Kähler space by [V]. So, meromorphic correspondences with values in Kähler manifolds are extendable through pseudo-concave boundary points.

For the manifolds from class $\mathcal{G}_1$ this is no longer the case, even if they do not contain spherical shells.

Example 3.5. There exists a compact complex (elliptic) surface X such that:

(a) Every meromorphic map $f : H^2(r) \to X$ extends meromorphically onto Δ^2, but

(b) There exists a two-valued meromorphic correspondence Z between $\mathbb{C}^2_*$ and X that cannot be extended to the origin.

Consider the standard Hopf surface $H = \mathbb{C}^2 \setminus \{0\}/(z \sim 2 \cdot z)$. Denote by $\pi : H \to \mathbb{CP}^1$ the standard projection. Let $\phi : C \to \mathbb{CP}^1$ be a nonconstant meromorphic function on the Riemann surface C of positive genus. Let ϕ be a d-sheeted ramified covering of $\mathbb{CP}^1$ by C. If we take C to be a torus we can have such ϕ with $d = 2$. Following Kodaira we shall construct an elliptic surface over C in the following way. Put

$$X_1 = \{(z, y) \in C \times H : \phi(z) = \pi(y)\}. \tag{3.3.1}$$

Elliptic structure on X_1 is given by the restriction onto X_1 of the natural projection $p_1 : C \times H \to C$. Note that the restriction of the natural projection $p_2 : C \times H \to H$ onto X_1 gives us a d-sheeted covering $p_2 |_{X_1}$ of H by X_1 preserving the elliptic structure. Let $n : X \to X_1$ be the normalization of X_1. Then X is a smooth elliptic surface over C with elliptic fibration $p := p_1 |_{X_1} \circ n : X \to C$, and $F := p_2 |_{X_1} \circ n : X \to H$ is a d-sheeted covering.

$Z := F^{-1} \circ \pi : \mathbb{C}^2_* \to X$ is the d-valued meromorphic correspondence between $\mathbb{C}^2_*$ and X, which cannot be extended to the origin, because the projection $\pi : \mathbb{C}^2_* \to H$ cannot be extended meromorphically to zero.

On the other hand in [Iv-1] it was proved that meromorphic mappings from $H^2(r)$ to an elliptic surface over a Riemann surface of positive genus are extendable onto Δ^2.

One can interpret this example in the way that $\mathrm{Sym}^2(X)$ may have a spherical shell even when X has not.

In view of the discussion above we can restate our results for meromorphic correspondences.

Definition 3.2. By a *branched spherical shell of degree d* in a complex space X we shall mean the image Σ of $\mathbb{S}^3 \subset \mathbb{C}^2$ under a d-valued meromorphic correspondence between some neighborhood of $\mathbb{S}^3$ and X such that $\Sigma \not\sim 0$ in X.

COROLLARY 3.1. *Let Z be a meromorphic correspondence from the domain D in a complex space Ω into a disk-convex complex space $X \in \mathcal{G}_1$ and let x_0 be a concave boundary point of D. Then Z extends onto some neighborhood of x_0 in Ω minus (possibly empty) complex variety A of pure codimension two. If X does not contain branched spherical shells then $A = \emptyset$.*

Remark. We would like to point out here that a branched shell could be a much more irregular object than a nonbranched one. To see this, consider a smooth complex curve C in a neighborhood of $\bar{\mathbb{B}}^2(2)\setminus\mathbb{B}^2(1/2)$ that does not extend to $\mathbb{B}^2(1)$. Let $\pi : W \to \mathbb{B}^2(2)\setminus\bar{\mathbb{B}}^2(1/2)$ be a covering branched along C. The images of $\hat{S} := \pi^{-1}(\mathbb{S}^3)$ in complex spaces could be branched shells that would not bound an abstract Stein domain.

3.4. *Mappings into compact complex surfaces*. Here we shall prove Corollary 1 from the introduction. Let X denote now a compact complex surface. First of all we note that if X is Kähler then by [Iv-3] every meromorphic map from $H_U^{n+1}(r)$ to X extends onto Δ^{n+1}. In particular the singularity set A is empty. From Enriques-Kodaira classification we are left with two classes: elliptic surfaces and surfaces of class VII without meromorphic functions. Since every compact complex surface carries a pluriclosed metric form the main theorem applies.

Elliptic surfaces. Let our map be extended onto $\Delta^{n+1}\setminus A$ as in the main theorem. Take a point $a \in A$ and choose coordinates in a neighborhood V of a such that $V \cong \Delta^{n-1}\times\Delta^2$ and f is holomorphic in the neighborhood of $\Delta^{n-1}\times\partial\Delta^2$. Now for every $a \in \Delta^{n-1}$ the 2-disk $\Delta_a^2 := \{a\}\times\Delta^2$ is "transversal" to A; i.e., $A\cap\Delta_a^2$ is compact of dimension zero. Consider the restriction f_a of f onto B_a^2, where f_a is defined and meromorphic outside of a compact set. Lemma 5 from [Iv-1] shows that f_a extends to the complement of finitely many points $\{b_k(a)\}$. Moreover, the set $A_a := \{b_k(a)\}$ is not empty only in the case when X is an elliptic surface over $\mathbb{CP}^1$ with projection $\pi : X \to \mathbb{CP}^1$ and $\{b_k(a)\}$ is exactly the set of points of indeterminancy $I(\pi\circ f_a)$ of $\pi\circ f_a$. Since $I(\pi\circ f_a)\subset I(\pi\circ f)$ and the latter is analytic, the Corollary 1 for elliptic surfaces follows.

Surfaces of class VII. Toruses and $K3$ surfaces are Kähler, and therefore for them A is empty as above. We shall crucially use that for all others surfaces from the class VII $b_1 = 1$. First we prove that any meromorphic map $f : \mathbb{B}^2(1)\setminus\mathbb{B}^2(r) \to X$ from the spherical shell into X extends onto $\mathbb{B}^2\setminus\{\text{finite set}\}$ and the number of points in this set is bounded by some number N. We shall show that $N \le |\int_{f(\partial B^2)} d^c\omega|/|\int_M d^c\omega|$, where M is the generator of the torsion free part of $H_3(X,\mathbb{Z})$.

By the Main Theorem, f extends into $\mathbb{B}^2$ minus a compact A of Hausdorff dimension zero. For any sphere S around $b \in A$ such that $S\cap A = \emptyset$ its image $f(S)$ is not homologous to zero. Since the first Betti number of X is one by the Poincaré duality the third is also one. Let $\varepsilon_1 = \int_M d^c\omega$, where ω is a pluriclosed metric form on X. Then for every S there is a nonzero integer n such that $f(S)$ is homologous to nM modulo torsion. The image $f(S)$ is not contained in the torsion part of $H_3(X,\mathbb{Z})$ because $\int_{f(S)} d^c\omega \neq 0$. Therefore $\int_{f(S)} d^c\omega = n\varepsilon_1$.

Case 1. $\varepsilon_1 = 0$. This implies that all $\int_{f(S)} d^c\omega$ are equal to zero and by Lemma 2.5, f extends meromorphically onto every such b. Thus A is empty in this case.

Case 2. $\varepsilon_1 \neq 0$. Since $\int_{f(S)} d^c\omega = n\varepsilon_1$ for nonzero integer n this implies that $|\int_{f(S)} d^c\omega|$ are separate from zero. On the other hand $\Sigma_{b\in A}\int_{f(S(b))} d^c\omega = \int_{\partial\mathbb{B}^2} d^c\omega$ and therefore is finite. Therefore the set A is finite and $|A| \leq |\int_{f(\partial B^2)} d^c\omega|/|\int_M d^c\omega|$.

In the case of a higher dimension take a point $a\in A$ and choose a neighborhood $V \cong \Delta^{n-1}\times\Delta^2$ of a such that $A\cap(\Delta^{n-1}\times\partial\Delta^2) = \emptyset$. Note that $A\cap V$ is a graph of an N- valued continuous mapping of Δ^{n-1} to Δ^2, where N-valued means that $|A_{z'}| \leq N$ for every $z' \in \bar{\Delta}^{n-1}$ and there exists $z_0^{'}$ such that $|A_{z'}^0| = N$. A multi-valued mapping is continuous if the set A is closed, which is obviously our case. Note that the trivial extension $\tilde{T}$ of $T = f^*\omega$ is an L^1-current on V with $dd^c\tilde{T} \leq 0$ supported on A.

LEMMA 3.2. *Let A be the graph of an N-valued continuous mapping of $\bar{\Delta}^k$ to Δ^l and let R be a closed positive current in Δ^{k+l} of bidimension (k,k) supported on A. Then A is a pure k-dimensional analytic variety in Δ^{k+l}.*

Proof. Write $R = R_{K,\bar{J}}(\frac{i}{2})^k\frac{\partial}{\partial z^K}\wedge\frac{\partial}{\partial\bar{z}^J}$, where K and J are multi-indices of length k. Consider the measure $R_{K,\bar{J}}$, denote by $\mu_{K,\bar{J}} = \pi_*(R_{K,\bar{J}})$ their direct images and disintegrate this measure with respect to the natural projection $\pi : \Delta^k\times\Delta^l \to \Delta^k$. Disintegration means that one has probability measures $\nu_{K,\bar{J},z'}$ on $\Delta_{z'}^l := \{z'\}\times\Delta^l$ with the property that for every continuous function h in Δ^{k+l}

$$\langle R_{K,\bar{J}}, h\rangle = \int_{\Delta^k}\Big(\int_{\Delta_{z'}^l}\bar{h}\,|_{\Delta_{z'}^l}\, d\nu_{K,\bar{J},z'}\Big)d\mu_{K,\bar{J}}; \tag{3.4.1}$$

see [D-M].

Let Ω be the maximal open subset of Δ^k such that the multi-valued map s, which is given by its graph A, takes on exactly N different values (and N is maximal). First we shall prove that $A\cap(\Omega\times\Delta^l)$ is analytic.

Further, let Ω_1 be some simply connected open subset of Ω. Then $s\,|_{\Omega_1}$ decomposes to N well-defined single-valued maps $s^1,\dots,s^N$. So it is enough to consider the case when s is single-valued. Put $s(z') = (s_1(z'),\dots,s_l(z'))$. Note that in this case $\nu_{K,\bar{J},z'} = \delta_{\{z''-s(z')\}}$. Therefore for the coefficients $R_{K,\bar{J}}$ of our current R and for $\phi \in C^\infty(\Omega_1 \times \Delta^l)$ such that $\pi(\mathrm{supp}\phi) \subset\subset \Omega_1$ we can write that

$$\langle R_{K,\bar{J}}, \phi\rangle = \int_{\Omega_1} \bar{\phi}(z', s(z'))d\mu_{K,\bar{J}}(z'). \tag{3.4.2}$$

If we choose ϕ not depending on $z'' := (z_{k+1}, \dots, z_{k+l})$ then (3.4.2) gives

$$\langle R_{K,\bar{J}}, \phi\rangle = \int_{\Omega_1} \bar{\phi}(z')d\mu_{K,\bar{J}}(z'). \tag{3.4.3}$$

From the closedness of R we obtain that

$$\begin{aligned} 0 &= \left\langle R, d\left[\left(\frac{i}{2}\right)^k \bar{\phi}(z')dz_1 \wedge \dots dz_{p-1} \wedge dz_{p+1} \wedge \dots \wedge dz_k \wedge d\bar{z}_J\right]\right\rangle \\ &= \left\langle R_{1\dots k,\bar{J}}, \frac{\partial\bar{\phi}}{\partial z_p}\right\rangle = \int_{\Omega_1} \frac{\partial\phi}{\partial\bar{z}_p} d\mu_{1\dots k,\bar{J}}. \end{aligned}$$

So $\mu_{1\dots k,\bar{J}}(z') = c_{1\dots k,\bar{J}}(z')\cdot(\frac{i}{2})^k dz' \wedge d\bar{z}'$, where $c_{1\dots k,\bar{J}}$ are holomorphic for all J. In particular $c_{1\dots k,\bar{1}\dots\bar{k}}$ is constant. Now take the $(k-1,k)$- forms $\psi_{q\bar{p}} = \bar{\phi}(z')\cdot\bar{z}_p\cdot(\frac{1}{2})^k dz_1 \wedge \dots \wedge dz_{q-1} \wedge dz_{q+1} \wedge \dots \wedge dz_k \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_k$. We have

$$\begin{aligned} 0 = \langle R, d\psi_{q,\bar{p}}\rangle &= \left\langle R, \frac{\partial\bar{\phi}}{\partial z_q}\cdot\bar{z}_p\left(\frac{i}{2}\right)^k dz' \wedge d\bar{z}'v\right\rangle = \left\langle R_{1\dots k,\bar{1}\dots\bar{k}}, \frac{\partial\bar{\phi}}{\partial z_q}\bar{z}_{\bar{p}}\right\rangle \\ &= c_{1\dots k,\bar{1}\dots\bar{k}} \int_{\Omega_1} \frac{\partial\phi}{\partial\bar{z}_q}(z')\cdot s_p(z')\left(\frac{i}{2}\right)^k dz' \wedge d\bar{z}'; \end{aligned}$$

i.e., the s_p are holomorphic.

Thus we have proved that s is an N-valued analytic map of Ω into Δ^l. Considering appropriate discriminants and using Rado's theorem, we obtain analyticity of s on the whole Δ^k. □

3.5. *More remarks about the structure of the singularity set.* One can slightly generalize considerations above and prove in several other cases that the singularity set A of a meromorphic map into a complex manifold with pluriclosed metric form is analytic of pure codimension two. We shall prove the following:

THEOREM 3.3. *Suppose that a disk-convex complex space X admits a pluriclosed metric form ω such that $d^c\omega \in H^3(X,\mathbb{Z})$. Then every meromorphic map $f : H_U^{n+1}(r,1) \to X$ extends onto $\Delta^{n+1} \setminus A$, where A is an analytic subset of pure codimension two.*

We start with the case $n = 1$.

Lemma 3.4. *For X as in Theorem 3.3, $f : H_U^2(r,1) \to X$ can be meromorphically extended onto $\Delta^2 \setminus A$, where A is discrete.*

Proof. By the Main Theorem we know that f extends onto $\Delta^2 \setminus A$, where A is pluripolar of Hausdorff dimension zero. We suppose that A is a minimal subset of Δ^2 such that f extends onto $\Delta^2 \setminus A$. Take a relatively compact open subset $P \subset \Delta^2$ such that $\partial P \cap A = \emptyset$ and choose a finite subcomplex K of CW-complex X to contain the $f(\bar{P} \setminus A)$, which is a compact subset of X due to the disk convexity of the latter. All we need to prove is that $P \cap A$ is finite. Let $\theta_1, \dots, \theta_N$ be the generators of the free part of $H^3(K,\mathbb{Z})$ and $\psi_1, \dots, \psi_L$ be the generators of the free part of $H_3(K,\mathbb{Z})$. Take integer numbers $r_1, \dots, r_N$ such that

$$d^c\omega = r_1\theta_1 + \cdots + r_N\theta_N. \tag{3.5.1}$$

Take a ball $B \subset\subset \Delta^2$ with $\partial B \cap A = \emptyset$. Then there are integers $z_1, \dots, z_L$ such that

$$f(\partial B) = z_1\psi_1 + \cdots + z_L\psi_L \tag{3.5.2}$$

in $H_3(K,\mathbb{Z})$ modulo torsion. For the measure μ defined from $dd^c\tilde{T} = \mu \cdot \omega_e^2$ we have that

$$\mu(B \cap A) = \int_B dd^c\tilde{T} = \int_{\partial B} d^cT. \tag{3.5.3}$$

Using that, we can write

$$\mu(B \cap A) = \int_{f(\partial B)} d^c\omega = \sum_{k=1}^{N}\sum_{i=1}^{L} z_i r_k \int_{\psi_i} \theta_k. \tag{3.5.4}$$

Put $c^{ik} = \int_{\psi_i} \theta_k \in \mathbb{Z}$. Now if we put $\tilde{z}^k = \sum_{i=1}^{L} z_i c^{ik} \in \mathbb{Z}$ then

$$\mu(B \cap A) = \sum_{k=1}^{N} \tilde{z}^k r_k. \tag{3.5.5}$$

The right-hand side of (3.5.5) is a negative integer and therefore is separated from zero; i.e., there exists an $\varepsilon_0 < 0$ such that $\mu(B \cap A) < \varepsilon_0$ if $B \cap A \neq \emptyset$. Therefore, if A is not discrete there is a sequence $\{s_n\} \subset A$ converging to $s_0 \in A$, and therefore if we take nonintersecting $B_{\varepsilon_n}(s_n)$ we obtain $\mu(S) \leq \Sigma_n \mu(B_{\varepsilon_n}(s_n)) = -\infty$. This is a contradiction. □

Further increasing of the dimension is now an obvious repetition of the case of surfaces. □

3.6. *More examples of a singularity set.* Arguments from the previous sections will not work if $d^c\omega$ is not in $H^3(X,\mathbb{Z})$. For example, if there are two 3-cycles (holomorphic images of $\mathbb{S}^3 \subset \mathbb{C}^2$) M_1, M_2 such that $\int_{M_1} d^c\omega = 1$ and say

$\int_{M_1} d^c\omega$ is irrational, then there could be a sequence of points $\{a_k\} \subset A \subset B^2$ such that a meromorphic mapping $f : B^2 \setminus A \to X$ has the property that for sufficiently small spheres S_k around a_k, $f(S_k) = n_k M_1 + m_k M_2$ with $\{m_k\}$ unbounded. Then one could have that $\int_{f(S_k)} \omega \to 0$ and arguments from the two previous sections will fail. A can be now the the set of accumulation points of the sequence $\{a_k\}$ and can be a Cantor-type set. In this section we give such examples, both with $A = \{a_k\}$ and A uncountable. In the Example 3.6 below a map will behave exactly as described above. Unfortunately it is not clear if that manifold X can be endowed with a pluriclosed (or at least plurinegative) metric form.

Example 3.6. There exists a compact complex threefold X with $H^3(X,\mathbb{Z})$ generated by two cycles M_1, M_2, which are holomorphic images of $\mathbb{S}^3 \subset \mathbb{C}^2$ (i.e. spherical shells) and a holomorphic mapping $f : B^2 \setminus A :\to X$ such that:

a) A is an uncountable Cantor-type complete pluripolar compact subset of B^2;

b) All points of A are singular for f;

c) There is a dense, in A, sequence $\{a_k\} \subset A$ and spheres S_k around a_k such that $f(S_k) = n_k M_1 + m_k M_2$ with $\{(n_k, m_k)\}$ unbounded.

Take $\mathbb{CP}^3$ with homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ and take two lines $l_1 = \{z_0 = z_1 = 0\}$ and $h_1 = \{z_2 = z_3 = 0\}$. Consider the following function $\phi_1(z) = \frac{|z_0|^2+|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2+|z_3|^2}$. Observe that $l_1 = \{z : \phi_1(z) = 0\}$ and $h_1 = \{z : \phi_1(z) = 1\}$. For a complex number α such that $|\alpha| < 1$ consider a domain $D^1_\alpha \subset \mathbb{CP}^3$ defined by $D^1_\alpha = \{z \in \mathbb{CP}^3 : |\alpha| < \phi_1(z) < |\frac{1}{\alpha}|\}$. The group of automorphisms of $\mathbb{CP}^3$ generated by $g_1(z) = [\frac{1}{\alpha^2} z_0 : \frac{1}{\alpha^2} z_1 : z_2 : z_3]$ has D^1_α as its fundamental domain.

Take new coordinates

$$w_0 = \frac{1}{\sqrt{2}}(z_0 + z_2), w_2 = \frac{1}{\sqrt{2}}(z_1 + z_3),$$
$$w_1 = \frac{1}{\sqrt{2}}(z_0 - z_2), w_3 = \frac{1}{\sqrt{2}}(z_1 - z_3),$$

and repeat the above considerations: Define lines $l_2 = \{w_0 = w_1 = 0\}$ and $h_2 = \{w_2 = w_3 = 0\}$, function $\phi_2(z) = \frac{|w_0|^2+|w_1|^2}{|w_0|^2+|w_1|^2+|w_2|^2+|w_3|^2}$, take the domain $D^2_\alpha = \{w \in \mathbb{CP}^3 : |\alpha| < \phi_2(w) < |\frac{1}{\alpha}|\}$, and finally take an automorphism $g_2(w) = [\frac{1}{\alpha^2} w_0 : \frac{1}{\alpha^2} w_1 : w_2 : w_3]$.

Now the domain $D_\alpha := D^1_\alpha \cap D^2_\alpha$ will be a fundamental domain for the group G of biholomorphic automorphisms of $\mathbb{CP}^3$ generated by g_1, g_2. Set $\Omega_\alpha \bigcup_{g\in G} g(D_\alpha)$. Complementary to Ω_α in $\mathbb{CP}^3$ is a Cantor set of complex lines A_α. If $|\alpha|$ was taken sufficiently small the set A_α will be 2-polar and of Hausdorff 3-dimensional measure zero.

Set $X = \Omega_\alpha/G$; this is our manifold. $H_3(X,\mathbb{Z})$ is generated by 3-cycles $M_1 = \{z_3 = 0, |z_0|^2 + |z_1|^2 = |z_2|^2\}$ and $M_2 = \{w_3 = 0, |w_0|^2 + |w_1|^2 = |w_2|^2\}$. Both are holomorphic images of the standard sphere from $\mathbb{C}^2$.

Take the plane $P = \{z_3 = 0\}$ and the ball $B := \{\phi_1 |_P (z) < |\alpha|\}$ on this plane. The the natural map $f : B \setminus A \to X$ has all needed properties, where $A = B \cap A_\alpha$.

It should be noted that the author does not know whether or not this X admits a plurinegative metric form.

Remark. Manifolds of this type where first constructed by M. Kato in dimension 3; see [Ka-2], [Ka-3] and by M. Nori in higher dimensions; see [No].

Let us give one more example with an interesting singular set.

Example 3.7. There exist a compact complex manifold X of dimension four and a holomorphic map $f : \mathbb{B}^2_* \setminus \{(\frac{1}{2^{k-1}} \cdot \frac{3}{4}, 0)\}_{k=1}^{\infty} \to X$ such that each point $a_k = (\frac{1}{2^{k-1}} \cdot \frac{3}{4}, 0)\}$ is an essential singularity of f. This means that f does not extend to the neighborhood of such a_k even meromorphically.

Construction of X. Take a Hopf surface $H = \mathbb{C}^2 \setminus \{0\}/(z \sim 2z)$. By $\pi_1 : \mathbb{C}^2 \setminus \{0\} \to H$ denote the canonical projection. Fix the point $a_1 = (\frac{3}{4}, 0)$ in $\mathbb{B}^2_* := \mathbb{B}^2 \setminus \{0\}$ and its image $b := \pi_1(a_1)$ in H. Recall (see [Gr-Ha, p. 726]), that on a compact complex surface H with $h^{0,2} = 0$ there exists a holomorphic rank-two vector bundle $p : E \to H$ having a holomorphic section $s : H \to E$ with b being its only zero point. The second Betti number of the Hopf surface is zero and therefore from $b^2 = h^{2,0} + h^{1,1} + h^{0,2}$ we see that in our case the condition $h^{0,2} = 0$ is satisfied.

Denote by Z the zero section of E. Let $g : E \setminus Z \to E \setminus Z$ be multiplication by 2. The desired four-manifold X is the quotient of $E \setminus Z$ by the group of biholomorphisms $G := \{g^n\}_{n\in\mathbb{Z}}$. Denote by $\pi_2 : E \setminus Z \to X$ the canonical projection.

Construction of the map f. Denote by Γ_s the graph of the section s in the total space of the bundle E and by $\Gamma'_s := \Gamma_s \cap (E \setminus Z)$. Our map f is defined to be the composition $f := \pi_2 \circ s \circ \pi_1 : \mathbb{B}^2_* \setminus \{(\frac{1}{2^{k-1}} \cdot \frac{3}{4}, 0)\}_{k=1}^{\infty} \to X$.

Since this map satisfies $f(\frac{1}{2}z) = f(z)$ the set $\{(\frac{1}{2^{k-1}} \cdot \frac{3}{4}, 0)\}_{k=1}^{\infty}$ is precisely the set of all points where f is not defined. On $\mathbb{B}^2_* \setminus \{(\frac{1}{2^{k-1}} \cdot \frac{3}{4}, 0)\}_{k=1}^{\infty}$ the map f is well defined and holomorphic.

To see that f does not extend meromorphically to the neighborhood of any a_k it is sufficient to show that for a small sphere S_k around a_k its image $f(S_k)$ is not homologous to zero in X. Take S_k to be the boundary of the

Euclidean ball B_k with center at a_k small enough to satisfy the following two conditions:

1) E can be trivialized in the neighborhood U_k of $\pi_1(B_k)$;
2) B_k does not contain any a_j for $j \neq k$ and moreover B_k is contained in the spherical region $\{z \in \mathbb{C}^2 : \frac{1}{2^k} < \|z\| < \frac{1}{2^{k-1}}\}$.

The image $\pi_2(U_k \times \mathbb{C}^2_*)$ is a product $U_k \times H$. Now it is clear that $f(S_k)$ is homologous to the generator of $H_3(U_k \times H, \mathbb{Z}) = H_3(H, \mathbb{Z}) = \mathbb{Z}$.

Remark. If any meromorphic mapping $f : H^2(r) \to X$ (with X admitting a pluriclosed metric form) extends onto Δ^2 minus a countable set (as in the example above), then using the theorem of Nishino, see [Ni], one can prove that every meromorphic map $f : H^n(r) \to X$ extends onto $\Delta^n \setminus A$, where A is at most a countable union of locally closed analytic sets of pure codimension two.

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(Received May 4, 1999)