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Kloosterman identities over a quadratic extension

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Abstract

We prove an identity of Kloosterman integrals which is the fundamental lemma of a relative trace formula for the general linear group in n variables.

1. Introduction

One of the simplest examples of Langlands' principle of functoriality is the quadratic base change. Namely, let E/F be a quadratic extension of global fields and $z \mapsto \overline{z}$ the corresponding Galois conjugation. The base change associates to every automorphic representation π of $\mathrm{GL}(n,F)$ an automorphic representation Π of $\mathrm{GL}(n,E)$. If $n=1$ then π is an idèle class character and $\Pi(z)=\pi(z\overline{z})$. An automorphic representation Π of $\mathrm{GL}(n,E)$ is a base change if and only if it is invariant under the Galois action. The existence of the base change is established by the twisted trace formula [3]. Formally, if f and f' are smooth functions of compact support on $G(E_{\mathbb{A}})$ and $G(F_{\mathbb{A}})$ respectively, then one defines

$$K_f(x,y)=\sum_{\xi\in\mathrm{GL}(n,E)} f(x^{-1}\xi y)\,,\ K_{f'}(x,y)=\sum_{\xi\in\mathrm{GL}(n,F)} f'(x^{-1}\xi y)\,.$$

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The identity of the twisted trace formula is that

$$\int K_f(x,\overline{x})dx = \int K_{f'}(x,x)dx\,,$$

for many pairs of functions (f,f'). The existence of such an identity depends on a simple relation between orbital integrals of the form

$$\int f(x\gamma\overline{x}^{-1})dx\,,\ \int f'(x\gamma' x^{-1})dx\,.$$

In turn, to establish such a relation one needs to compare at almost all places v of F inert in E the orbital integrals of specific functions. This is the *fundamental lemma* [9].

There is another possible characterization of the base change. Indeed, in the case $n=1$, Π is a base change if and only if it is trivial on the group of elements of norm 1, that is, on the unitary group in one variable. Thus it is natural to conjecture that a representation Π is a base change if and only if it is *distinguished* by some unitary group H: this means that there is an element ϕ in the space of H such that the *period integral*

$$\int_{H(F)\backslash H(F_{\mathbb{A}})} \phi(h)dh$$

does not vanish.

To establish this conjecture one is led to consider a *relative trace formula* of the form

$$\int\int K_f(h,u)dh\theta(u\overline{u}) = \int\int K_{f'}({}^tu_1,u_2)\overline{\theta}(u_1)\theta(u_2)du_1du_2; \tag{1}$$

here N_n denotes the group of upper triangular matrices with unit diagonal, $u \in N_n(E)\backslash N_n(E_{\mathbb{A}})$, $u_1,u_2 \in N_n(F)\backslash N_n(F_{\mathbb{A}})$; θ is a character of $N_n(F_{\mathbb{A}})$ trivial on $N_n(F)$ and in general position. One needs to establish this identity for many pairs (f,f'). This depends on the comparison of orbital integrals of the form

$$\int\int f(h\xi u)\theta(u\overline{u})dudh\,,\ \int f'({}^tu_1\xi' u_2)\theta(u_1u_2)du_1u_2)du_1du_2\,.$$

Just as in the case of the standard trace formula, at almost all places v of F inert in E, one needs to establish a certain relation between the orbital integrals of specific functions. The integrals are closely related to Kloosterman sums. The relation is the *fundamental lemma* for the relative trace formula. The purpose of this paper is to prove this fundamental lemma.

Before we describe the result in more details we remark that the same set up should apply to the stabilizer H of an automorphism of order 2 of a reductive group G. However, the information obtained from the conjectural relative trace formula depends on the particular case at hand. In many cases, the period integral is related to the special values of L-functions. For a discussion of the meaning of the period integral here see [6, (6)]. Of course one expects to have in the general situation a fundamental lemma. See for instance [7] and [11] where the proof of the fundamental lemma at hand is conceptual.

In the case at hand, we need to consider all forms of the unitary group simultaneously. Moreover, integrating a function over H produces a function on $H\backslash G$, thus a function of the space $S(n\times n)$ of Hermitian matrices. It is then more convenient to adopt a slightly different point of view. The group $G(E)=\mathrm{GL}(n,E)$ operates on $S(n\times n)$ by $s\mapsto {}^t\overline{g}sg$. If Ψ is in $\mathcal{C}_c^\infty(S(n\times n,F_{\mathbb{A}}))$ we construct a function $\Theta_\Psi(g)$ on $G(E)\backslash G(E_{\mathbb{A}})$ by

$$\Theta_\Psi(g)=\sum_{\xi\in S(n\times n,F)}\Psi({}^t\overline{g}\xi g)\,.$$

The invariant space spanned by the functions Θ_Ψ is the *automorphism spectrum* of the space of Hermitian matrices. The (cuspidal) automorphism representations which appear in the spectrum are exactly the cuspidal representations π which are *distinguished* by the stabilizer H of some point in $S(n\times n,F)$; thus H is indeed a unitary group.

We consider a similarly defined space of functions on $(G(F)\backslash G(F_{\mathbb{A}}))^2$. The group $\mathrm{GL}(n,F)\times\mathrm{GL}(n,F)$ operates on $\mathrm{GL}(n,F)$ by $s\mapsto {}^tg_1sg_2$. To every function Φ in $\mathcal{C}_c^\infty(\mathrm{GL}(n,F_{\mathbb{A}}))$ we associate a function $\Theta_\Phi(g_1,g_2)$ defined by

$$\Theta_\Phi(g_1,g_2)=\sum_{\xi\in\mathrm{GL}(n,F)}\Phi({}^tg_1\xi g_2)\,.$$

We consider the invariant space spanned by these functions. The automorphism cuspidal representations $\pi=\pi_1\otimes\pi_2$ which appear in the space are exactly those distinguished by the twisted diagonal subgroup $\{({}^tg^{-1},g)\}$, that is, those π for which π_1 is contragradient to π_2.

We replace (1) by

$$(2)\quad \int_{N(E)\backslash N(E_{\mathbb{A}})}\Theta_\Psi(n)\theta(n\overline{n})dn=\int_{(N(F)\backslash N(F_{\mathbb{A}}))^2}\Theta_\Phi(n_1,n_2)\theta(n_1n_2)dn_1dn_2\,,$$

and we say that Ψ *matches* Φ if the identity holds. One wants to prove that any Ψ matches a Φ and conversely.

The notion of global matching depends on the notion of *local matching* that we now describe in the context of a quadratic unramified extension of local non Archimedean fields. Thus we let E/F be such an extension. We let η be the corresponding (unramified) quadratic character of $F^\times$. We assume the residual characteristic is not 2. We denote by $v(\bullet)$ the valuation of F. We let q be the cardinality of the residual field of F and set $\mid x\mid_F=q^{-v(x)}$. We let ψ be an additive character of F whose conductor is the ring of integers $\mathcal{O}_F$ of F. We let $\mathcal{P}_F$ be the maximal ideal in $\mathcal{O}_F$ and ϖ a generator of $\mathcal{P}_F$. We denote by dx the self dual Haar measure on F. We let N_n be the group of upper triangular matrices with unit diagonal in $\mathrm{GL}(n)$. We define a character θ_n or simply $\theta : N_n(F) \to \mathbb{C}^\times$ by

$$\theta(u) = \psi\left(\sum_i u_{i,i+1}\right) .$$

Locally, it is best to consider orbital integrals for smooth functions of compact support on $M(n \times n, F)$. Let Φ be a such a function. We define the *Kloosterman integral*

$$\Omega(\Phi, \psi : a) := \int_{N_n(F)\times N_n(F)} \Phi({}^t u_1 a u_2)\theta(u_1 u_2) du_1 du_2$$

where

$$a = \mathrm{diag}(a_1, a_2, a_3, \dots , a_n)$$

is a diagonal matrix with

$$a_i \in F^\times \,,\; 1 \leq i \leq n-1 \;,\; a_n \in F\,,$$

and du is the Haar measure on $N_n(F)$ such that

$$\int_{N_n(\mathcal{O}_F)} du = 1\,.$$

We often write $\Omega(\Phi, \psi : a)$ as a function of n variables:

$$\Omega(\Phi, \psi : a_1, a_2, a_3, \dots , a_n)\,.$$

Likewise, we define a character $u \mapsto \theta(u\overline{u})$ of $N_n(E)$ by

$$\theta(u\overline{u}) = \psi\left(\sum_i (u_{i,i+1} + \overline{u}_{i,i+1})\right) .$$

We let $H(n \times n, E/F)$ be the space of Hermitian matrices. Let Ψ be a smooth function of compact support on $H(n \times n, E/F)$. We define the *relative Kloosterman integral*

$$\Omega(\Psi, E/F, \psi : a) := \int_{N_n(E)} \Psi({}^t\overline{u} a u)\theta(u\overline{u}) du$$

where a is as above and du is the Haar measure on $N_n(E)$ such that

$$\int_{N_n(\mathcal{O}_E)} du = 1\,.$$

We say that Φ *matches* Ψ for ψ (see [6]) and we write $\Phi \overset{\psi}{\leftrightarrow} \Psi$ if

$$\Omega(\Phi, \psi : a) = \gamma(a)\Omega(\Psi, E/F, \psi : a)\,,$$

where

$$\gamma(a) = \eta(a_1)\eta(a_1 a_2)\cdots\eta(a_1 a_2 \cdots a_{n-1})\,.$$

By the results of [6, (3), (4), (5)] this identity implies similar identities for the other orbital integrals.

The fundamental lemma takes then the following form.

THEOREM 1 (The Fundamental Lemma). *Let* Φ_n *be the characteristic function of* $M(n \times n, \mathcal{O}_F)$ *and* Ψ_n *be the characteristic function of*

$$M(n \times n, \mathcal{O}_E) \cap H(n \times n, E/F).$$

Then $\Phi_n \overset{\psi}{\leftrightarrow} \Psi_n$; *that is,*

$$\Omega(\Phi_n, \psi : a) = \gamma(a)\Omega(\Psi_n, E/F, \psi : a)\,.$$

Ngo [12, (1)] formulates the identity in terms of trigonometric sums rather than integrals. Indeed (*loc. cit.*)

$$\Omega(\Phi_n, \psi : a) := \sum \theta(u_1 u_2)\,,$$

where the sum is over

$$(u_1, u_2) \in (N_n(F)/N_n(\mathcal{O}_F))^2,\ {}^t u_1 a u_2 \in M(m \times m, \mathcal{O}_F)\,,$$

and

$$\Omega(\Psi_n, E/F, \psi : a) = \sum \theta(u\overline{u})\,,$$

where the sum is over

$$u \in N_n(E)/N_n(\mathcal{O}_E)\,,\ {}^t\overline{u} a u \in M(n \times n, \mathcal{O}_E)\,.$$

As Ngo observes, the above result appears then as a generalization of the following classical identity. Let k be a finite field, k' its quadratic extension, $\psi : k \to \mathbb{C}^\times$ a nontrivial character. Then, for $c \in k^\times$,

$$\sum_{\substack{x_1, x_2 \in k \\ x_1 x_2 = c}} \psi(x_1 + x_2) = -\sum_{\substack{x \in k' \\ \overline{x}x = c}} \psi(x + \overline{x})\,. \tag{3}$$

It is a striking fact that our proof is ultimately based on this identity, or rather, on the slightly more general Weil formula that we now recall. Define the Fourier transform of $\Phi \in \mathcal{C}^\infty(E)$ by

$$\hat{\Phi}(z) = \int_E \Phi(u)\psi(-uz - \overline{uz})du\,.$$

Then, for $a \in F^\times$,

$$\int_E \hat{\Phi}(z)\psi(az\overline{z})dz = |a|_F^{-1}\eta(a)\int_E \Phi(z)\psi\left(-\frac{z\overline{z}}{a}\right)dz\,.$$

The sophisticated cohomological interpretation of the fundamental lemma of [2] is not needed.

Our purpose is to prove the above fundamental lemma. Originally, the fundamental lemma conjectured by the author and Ye was that the respective characteristic functions of the sets

$$\mathrm{GL}(n, \mathcal{O}_F)\,,\ \mathrm{GL}(n, \mathcal{O}_E) \cap H(n \times n, E/F)$$

match. Ngo [12] stated and proved the fundamental lemma in the above form in the case of positive characteristic. As will be apparent in the proof, it is essential to use Ngo's formulation. The proof is based on the fact, previously proved by the author [6, (4)] that the orbital integrals at hand are invariant under an integral transform. The proof of the fundamental lemma is based on the fact that the invariance property and support conditions characterize the orbital integrals. The author takes this opportunity to thank one of the referees of [6, (4)] for a crucial comment on the case of GL(2).

We first recall the results in question. We define the *normalized* orbital integrals

$$\tilde{\Omega}(\Phi, \psi : a) := |a_1||a_1 a_2| \cdots |a_1 a_2 \cdots a_{n-1}| \\ \times \Omega(\Phi, \psi : a).$$

We note that for $n = 1$

$$\tilde{\Omega}(\Phi, \psi : a) = \Omega(\Phi, \psi : a) = \Phi(a)\,.$$

Then, for $\Phi \in \mathcal{S}(M(n \times n, F))$,

$$\tag{4} \tilde{\Omega}(\check{\Phi}, \overline{\psi} : a_1, a_2, \ldots, a_n) \\ = \int \tilde{\Omega}(\Phi, \psi : p_1, p_2, \cdots, p_n) \\ \times \psi\left(-\sum_{i=1}^{i=n} p_i a_{n+1-i} + \sum_{i=1}^{i=n-1} \frac{1}{p_i a_{n-i}}\right) dp_n dp_{n-1} \cdots dp_1\,.$$

The multiple integral is only an iterated integral. Here $\check{\Phi}$ is the Fourier transform of Φ (suitably defined). We note that Φ_n is its own Fourier transform and that it is invariant under conjugation by the diagonal matrix

$$\tag{5} \mathrm{diag}(1, -1, 1, -1, \dots)\,.$$

It follows that

$$\tilde{\Omega}(\Phi_n, \psi : a) = \tilde{\Omega}(\Phi_n, \overline{\psi} : a)\,,$$

and the function $\tilde{\Omega}(\Phi_n, \psi : a)$ satisfies the following functional equation:

(6)

$$\omega(a_1, a_2, \ldots, a_m) = \int \omega(p_1, p_2, \cdots, p_m) \\ \times \psi\left(-\sum_{i=1}^{i=m} p_i a_{m+1-i} + \sum_{i=1}^{i=m-1} \frac{1}{p_i a_{m-i}}\right) dp_m dp_{m-1} \cdots dp_1\,.$$

If g is an $n \times n$ matrix, then we let g_i be the submatrix formed with the first i rows and the first i columns of g. We set $\Delta_i(g) = \det g_i$. The functions Δ_i are constant on the orbits. It follows that the function $\tilde{\Omega}(\Phi_n, \psi : a)$ is supported on the set defined by

$$(7) \qquad |a_1| \le 1\,,\ |a_1 a_2| \le 1\,,\ \ |a_1 a_2 \cdots a_{n-1}| \le 1\,,\ |a_1 a_2 \cdots a_{n-1} a_n| \le 1\,.$$

Finally, the following result is well known in the context of Kloosterman sums.

PROPOSITION 1. *Suppose that*

$$1 \le i \le n-1$$

and

$$|a_1 a_2 \cdots a_i| = 1\,.$$

Then

$$(8) \qquad \tilde{\Omega}(\Phi_n, \psi : a) = \tilde{\Omega}(\Phi_i, \psi : a_1, a_2, \dots, a_i)\tilde{\Omega}(\Phi_{n-i}, \psi : a_{i+1}, a_{i+2}, \dots, a_n)\,.$$

Similarly we define

$$\begin{aligned}&\tilde{\Omega}(\Psi, E/F, \psi : a)\\ &\quad := \eta(a_1)|a_1|\eta(a_1 a_2)|a_1 a_2| \cdots \eta(a_1 a_2 \cdots a_{n-1})|a_1 a_2 \cdots a_{n-1}|\\ &\qquad \times \Omega(\Psi, E/F, \psi : a)\,.\end{aligned}$$

The condition $\Phi \overset{\psi}{\leftrightarrow} \Psi$ is equivalent to

$$\tilde{\Omega}(\Phi, \psi : a) = \tilde{\Omega}(\Psi, E/F, \psi : a)\,.$$

The function $\tilde{\Omega}(\Psi_n, E/F, \psi : a)$ has properties analogous to the properties of $\tilde{\Omega}(\Phi_n, \psi : a)$.

Now we set

$$(9) \qquad \omega(a) := \tilde{\Omega}(\Phi_n, \psi : a) - \tilde{\Omega}(\Psi_n, E/F, \psi : a)\,.$$

We note that by the results of [6, (4)],

$$\omega(a) = \tilde{\Omega}(\Phi, \psi : a)$$

for some function Φ. The fundamental lemma amounts to saying that the function (9) vanishes identically.

The function (9) satisfies (6) and is supported on the set defined by (7). The case $n = 1$ being vacuous, we may assume $n > 1$ and the fundamental lemma true for $m \le n-1$. From Proposition (1) which is valid for Ψ_n as well, we see that ω is supported on the set defined by

$$(10) \qquad |a_1 a_2 \cdots a_i| \le |\varpi|\,,\ 1 \le i \le n-1\,,\ |a_1 a_2 \cdots a_n| \le 1\,.$$

We will use this to prove that $\omega = 0$. As a matter of fact, we will only use the fact that ω is supported on the set defined by

$$|a_1 a_2 \cdots a_i| \leq 1\,,\ 2 \leq i \leq n\,,\ |a_1| \leq |\varpi|\,. \tag{11}$$

We state this as a proposition.

Proposition 2. *Suppose that ω is the normalized orbital integral of some function. Suppose further that ω satisfies the functional equation* (6) *and is supported on the set* (11). *Then ω vanishes identically*

In the next section, for the sake of completeness, we verify Proposition 1. The rest of the paper is devoted to the proof of Proposition 2.

2. Proof of Proposition 1

With the notation of the proposition, it amounts to the same to prove the corresponding identity for the unnormalized orbital integrals:

$$\Omega(\Phi_n, \psi : a) = \Omega(\Phi_i, \psi : a_1, a_2, \ldots, a_i)\Omega(\Phi_{n-i}, \psi : a_{i+1}, a_{i+2}, \ldots, a_n)\,. \tag{12}$$

To see this is true we introduce the following partial orbital integral, as a function on $\mathrm{GL}(i, F) \times M((n-i) \times (n-i), F)$:

(13)

$$\Omega^i_{n-i}\,[\Phi, \psi : A_i, B_{n-i}] := \int \Phi\left[\begin{pmatrix} 1_i & 0 \\ {}^tY & 1_{n-i} \end{pmatrix}\begin{pmatrix} A_i & 0 \\ 0 & B_{n-i} \end{pmatrix}\begin{pmatrix} 1_i & X \\ 0 & 1_{n-i} \end{pmatrix}\right] \times\theta\left[\begin{pmatrix} 1_i & X \\ 0 & 1_{n-i} \end{pmatrix}\begin{pmatrix} 1_i & Y \\ 0 & 1_{n-i} \end{pmatrix}\right] dXdY\,.$$

If $\Phi = \Phi_n$ and $|\det A_i| = 1$ then in the above integral X and Y range over the set of matrices with integral entries. Then

$$\Omega^i_{n-i}\,[\Phi_n, \psi : A_i, B_{n-i}] = \Phi_i(A_i)\Phi_{n-i}(B_{n-i})\,. \tag{14}$$

On the other hand, the orbital integral of a given function Φ can be computed in stages as

(15) $\Omega(\Phi, \psi : a)$

$$= \int\int \Omega^i_{n-i}\left[\Phi, \psi :\, {}^tu_1 a^i u_2,\, {}^tv_1 a^{n-i} v_2\right] \theta_i(u_1u_2)du_1du_2\theta_{n-i}(v_1v_2)dv_1dv_2,$$

where

$$a^i = \mathrm{diag}(a_1, a_2, \ldots a_i)\,,\ a^{n-i} = \mathrm{diag}(a_{i+1}, a_{i+2}, \ldots a_n)\,.$$

If $|a_1 a_2 \cdots a_i| = 1$ then $|\det{}^t u_1 a^i u_2| = 1$ and the identity (13) follows from (14) and (15).

3. The Kloosterman transform

We will denote by $\mathcal{I}_n$ the space of functions ω on $(F^\times)^{n-1} \times F$ of the form

$$\omega(a_1, a_2, \dots, a_n) = \tilde{\Omega}(\Phi, \psi : a)\,.$$

By conjugating by the diagonal matrix (5), we see that the space does not change if we replace ψ by $\overline{\psi}$. If ω is in this space we denote by $\mathcal{K}_{n,\psi}(\omega)$ the right-hand side of (6). We call it the Kloosterman transform of ω. It is an element of $\mathcal{I}_n$.

To make the definition of the Kloosterman transform more precise, we define inductively two sequences of functions. First we set

$$\sigma_0(a_1, a_2, \dots, a_n) := \mu_0(a_1, a_2, \dots, a_n) := \omega(a_1, a_2, \dots, a_n)\,.$$

Then we set

$$\mu_1(a_1, a_2, \dots, a_{n-1}, b_1) := \int \sigma_0(a_1, a_2, \dots, a_{n-1}, a_n)\psi(-a_n b_1) da_n\,,$$

$$\sigma_1(a_1, a_2, \dots, a_{n-1}, b_1) = \mu_1(a_1, a_2, \dots, a_{n-1}, b_1)\psi\left(\frac{1}{a_{n-1}b_1}\right)\,.$$

Inductively, if $1 \le i \le n-1$ and we have defined

$$\sigma_i(a_1, a_2, \dots, a_{n-i}, b_i, b_{i-1}, \dots, b_1)\,,$$

then we define

$$\mu_{i+1}(a_1, a_2, \dots, a_{n-i-1}, b_{i+1}, b_i, \dots, b_1) := \int \sigma_i(a_1, a_2, \dots, a_{n-i}, b_i, b_{i-1}, \dots, b_1)\psi(-a_{n-i}b_{i+1}) da_{n-i}$$

and

$$\sigma_{i+1}(a_1, a_2, \dots, a_{n-i-1}, b_{i+1}, b_i, \dots, b_1) := \mu_{i+1}(a_1, a_2, \dots, a_{n-i-1}, b_{i+1}, b_i, \dots, b_1)\psi\left(\frac{1}{a_{n-i-1}b_{i+1}}\right)\,.$$

In particular

$$\sigma_n(b_n, b_{n-1}, \dots, b_1) := \mu_n(b_n, b_{n-1}, \dots, b_1)\,.$$

Note that our definition of σ_n and σ_0 is in accordance with the convention that an empty product has the value 1. We emphasize that the integral defining μ_{i+1} is *absolutely convergent*. Moreover, for fixed i, the functions σ_i and μ_i have the *same support*. We have then

$$\mathcal{K}_{n,\psi}(b_1, b_2, \dots b_n) = \mu_n(b_n, b_{n-1}, \dots, b_1)\,.$$

For $n = 1$ the Kloosterman transform is just the Fourier transform. Just as for the ordinary Fourier transform, there is an inversion formula: the composition $\mathcal{K}_{n,\overline{\psi}} \circ \mathcal{K}_{n,\psi}$ is the identity. More precisely, let us set

$$\check{\mu}_{n-i}(b_1, b_2, \dots, b_i, a_{n-i}, a_{n-i-1}, \dots, a_1) := \sigma_i(a_1, a_2, \dots, a_{n-i}, b_i, b_{i-1}, \dots, b_1)$$

$\check{\sigma}_{n-i}(b_1, b_2, \dots, b_i, a_{n-i}, a_{n-i-1}, \dots, a_1) := \mu_i(a_1, a_2, \dots, a_{n-i}, b_i, b_{i-1}, \dots, b_1).$

Then

$$\int \check{\sigma}_{n-i-1}(b_1, b_2, \dots, b_i, b_{i+1}, a_{n-i-1}, \dots a_1)\psi(b_{i+1}a_{n-i})db_{i+1}$$

$$= \check{\mu}_{n-i}(b_1, b_2, \dots, b_i, a_{n-i}, a_{n-i-1}, \dots, a_1)$$

and

$$\check{\sigma}_{n-i}(b_1, b_2, \dots, b_i, a_{n-i}, a_{n-i-1}, \dots, a_1) = \check{\mu}_{n-i}(b_1, b_2, \dots, b_i, a_{n-i}, a_{n-i-1}, \dots, a_1)\overline{\psi}\left(\frac{1}{b_i a_{n-i}}\right).$$

In particular,

$$\check{\sigma}_n(a_n, a_{n-1}, \dots, a_1) = \mu_0(a_1, a_2 \dots, a_n)$$

is just the inversion formula. We then have a principle of symmetry: we can exchange the variables (a_*) and (b_*), the left and the right, and the character ψ and the character $\overline{\psi}$.

Proposition 2 is a consequence of the following more precise result.

PROPOSITION 3. *Suppose that $\omega \in \mathcal{I}_n$, that ω is supported on the set* (11) *and its Kloosterman transform $\mathcal{K}_{n,\psi}(\omega)$ is supported on the set* (7). *Then $\omega = 0$.*

We now give several consequences of the above definitions. In what follows ω is in $\mathcal{I}_n$ and we define functions μ_i and σ_i as above. It will be convenient to express the results and assumptions on the support of the functions at hand in terms of diagrams. Each diagram has two rows consisting of indexed boxes such as

$$\overset{k}{\boxed{r}}, \overset{k}{\boxed{=r}}$$

where $r \geq 0$ is an integer. In the bottom row, the boxes are a shorthand notation for

$$|a_1 a_2 \cdots a_k| \leq |\varpi^r|, \; |a_1 a_2 \cdots a_k| = |\varpi^r|$$

respectively. In the top row they are a shorthand notation for

$$|b_k b_{k-1} \cdots b_1| \leq |\varpi^r|, \; |b_k b_{k-1} \cdots b_1| = |\varpi^r|$$

respectively. In each diagram the indices are increasing in the bottom row and decreasing in the top row. Indices in boxes in the same column add up to $n+1$. For consistency we introduce dummy boxes

$$\overset{0}{\boxed{=0}}.$$

Thus, in the bottom row say, a diagram

$$\begin{array}{cc} k & k+1 \\ \boxed{=r} & \boxed{s} \end{array}$$

stands for

$$|a_1 a_2 \cdots a_k| = |\varpi^r|\,,\ |a_{k+1}| \leq |\varpi^{s-r}|\,.$$

The assumptions in Proposition 3 can be described in terms of the following diagram:

$$\begin{array}{ccccccccc}
 & n & n-1 & n-2 & & 3 & 2 & 1 & 0 \\
 & \boxed{0} & \boxed{0} & \boxed{0} & \ldots & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{=0} \\
0 & 1 & 2 & 3 & & n-2 & n-1 & n & \\
\boxed{=0} & \boxed{1} & \boxed{0} & \boxed{0} & \ldots & \boxed{0} & \boxed{0} & \boxed{0} &
\end{array} \tag{16}$$

The bottom row is a shorthand notation for the conditions

$$|a_1 a_2 a_3 \cdots a_i| \leq 1\,,\ 2 \leq i \leq n\,,\ |a_1| \leq |\varpi|\,.$$

The assumption of Proposition 3 is that they are satisfied on the support of μ_0. Likewise, the top row is a shorthand notation for the conditions

$$|b_i b_{i-1} \cdots b_1| \leq 1\,,\ 1 \leq i \leq n\,.$$

The assumption of Proposition 3 is that they are satisfied on the support of μ_n.

In a diagram if the highest index in the top row (the first on the left) is k then the top row indicates conditions on the support of

$$\mu_k(a_1, a_2 \ldots a_{n-k}, b_k, b_{k-1}, \ldots b_1)$$

or what amounts to the same

$$\sigma_k(a_1, a_2 \ldots a_{n-k}, b_k, b_{k-1}, \ldots b_1)\,.$$

Likewise, if the highest index in the bottom row (the first on the right) is k then the bottom row indicates conditions on the support of

$$\mu_{n-k}(a_1, a_2 \ldots, a_k, b_{n-k}, b_{n-k-1}, \ldots b_1)$$

or

$$\sigma_{n-k}(a_1, a_2 \ldots, a_k, b_{n-k}, b_{n-k-1}, \ldots b_1)\,.$$

All diagrams have the general form

$$\begin{array}{ccccccc}
 & n-u & n-u-1 & & v+2 & v+1 & v \\
 & \boxed{s_{n-u}} & \boxed{s_{n-u-1}} & \ldots & \boxed{s_{v+2}} & \boxed{s_{v+1}} & \boxed{=\mu} \\
u & u+1 & u+2 & & n-v-1 & n-v & \\
\boxed{=\nu} & \boxed{r_{u+1}} & \boxed{r_{u+2}} & \ldots & \boxed{r_{n-v-1}} & \boxed{r_{n-v}} &
\end{array} \tag{17}$$

where all the indices are ≥ 0 and $\leq n$. In general all entries in boxes are implicitly assumed to be integers ≥ 0. The entry in a box indexed by n or 0 is always 0. Sometimes we drop the indices from the notation if this does not create ambiguities.

It will be convenient to use the statement that the diagram (17) *holds*. This means the following. We are given (sometimes only implicitly)

$$a_1, a_2, \dots, a_u,$$

such that

$$|a_1 a_2 \cdots a_u| = |\varpi^\nu| \tag{18}$$

and

$$b_v, b_{v-1}, \dots, b_1,$$

such that

$$|b_v b_{v-1} \cdots b_1| = |\varpi^\mu| \,. \tag{19}$$

The conditions indicated by the diagram hold on the support of the functions

$$\mu_{n-u}(a_1, a_2, \dots, a_u, b_{n-u}, b_{n-u-1}, \dots b_{v+1}, b_v, \dots b_1)$$

and

$$\mu_v(a_1, a_2, \dots, a_u, a_{u+1}, a_{u+2}, \dots, a_{n-v}, b_v, b_{v-1}, \dots, b_1)$$

Note our use of quantifiers. The variables

$$a_1, a_2, \dots, a_u, \, , \, b_v, b_{v-1}, \dots, b_1, \tag{20}$$

have fixed values. The other variables are free.

We say that the diagram (17) *holds trivially* if in fact

$$\mu_{n-u}(a_1, a_2, \dots, a_u, b_{n-u}, b_{n-u-1}, \dots b_{v+1}, b_v, \dots b_1) = 0$$

and

$$\mu_v(a_1, a_2, \dots, a_u, a_{u+1}, a_{u+2}, \dots, a_{n-v}, b_v, b_{v-1}, \dots, b_1) = 0.$$

Again in this equality the variables (20) have given values satisfying (18) and (19) and the equalities hold for all values of the remaining variables. Since μ_{n-u} is obtained from μ_v by repeatedly multiplying by a nonzero factor and taking a Fourier transform, it is clear that each equality is equivalent to the other. Our assumption is that the diagram (16) holds and our goal is to prove that the diagram (16) holds trivially.

Often we will not display the full diagram but only its front end. Thus instead of displaying the full diagram (17) we may display only the part

$$
\begin{array}{cccccc}
 & \overset{n-u}{\boxed{s_{n-u}}} & \overset{n-u-1}{\boxed{s_{n-u-1}}} & \cdots & \overset{n-k+1}{\boxed{s_{n-k+1}}} & \overset{n-k}{\boxed{s_{n-k}}} \\
\overset{u}{\boxed{=\nu}} & \overset{u+1}{\boxed{r_{u+1}}} & \overset{u+2}{\boxed{r_{u+2}}} & \cdots & \overset{k}{\boxed{r_k}} &
\end{array}
\tag{21}
$$

and we will say that the (full) diagram holds. It is understood that there are in fact more unwritten boxes on the right and in particular a last box of the form

$$\boxed{=\mu}$$

in the top row. The box may be a dummy box. The partial diagram (21) reminds us that

$$
\begin{aligned}
&\mu_{n-k}(a_1, a_2, \dots, a_u, a_{u+1}, \dots, a_k, b_{n-k}, \dots b_1) \neq 0 \\
&\qquad \Rightarrow |a_1 a_2 \cdots a_u a_{u+1} \cdots a_i| \leq |\varpi^{r_i}|\,, u+1 \leq i \leq k\,.
\end{aligned}
$$

Again this follows from the fact that μ_{n-k} is obtained from μ_v by repeatedly multiplying by a nonzero factor and taking a Fourier transform. Suppose that the diagram (17) holds. If we choose

$$b_{n-k}, b_{n-k-1}, \dots , b_{v+1},$$

such that

$$|b_{n-k} b_{n-k-1} \cdots b_{v+1} b_v b_{v-1} \cdots b_1| = |\varpi^{s_{n-k}}|$$

then we can say that the following diagram holds

$$
\begin{array}{cccccc}
 & \overset{n-u}{\boxed{s_{n-u}}} & \overset{n-u-1}{\boxed{s_{n-u-1}}} & \cdots & \overset{n-k+1}{\boxed{s_{n-k+1}}} & \overset{n-k}{\boxed{=s_{n-k}}} \\
\overset{u}{\boxed{=\nu}} & \overset{u+1}{\boxed{r_{u+1}}} & \overset{u+2}{\boxed{r_{u+2}}} & \cdots & \overset{k}{\boxed{r_k}} &
\end{array}
\tag{22}
$$

In particular, if the diagram (22) holds trivially for all such choices of

$$b_{n-k}, b_{n-k-1}, \dots , b_{v+1}\,,$$

then the original diagram (17) holds with s_{n-k} replaced by $s_{n-k}+1$ (17).

Finally, we have a principle of symmetry. Any result or statement implies a similar result or statement where the variables (a_i) and (b_j) are interchanged and the diagrams are replaced by the diagrams obtained after reflections about the horizontal axis and the vertical axis.

Our starting point is the following principle, which is valid even for $n = i+1$, $r = 0$.

LEMMA 1 (Uncertainty Principle 1). *If the diagram*

$$\begin{array}{cccc} & & \overset{i+1}{\boxed{s}} & \overset{i}{\boxed{*}} \\[2em] \overset{n-i-1}{\boxed{=r}} & \overset{n-i}{\boxed{r+m}} & & \end{array}$$

holds, then the following diagram holds

$$\begin{array}{cccc} & & \overset{i+1}{\boxed{s}} & \overset{i}{\boxed{s+m}} \\[2em] \overset{n-i-1}{\boxed{=r}} & \overset{n-i}{\boxed{r+m}} & & \end{array}$$

Proof. Before proving the assertion of the lemma, we explain our notation. We are given

$$a_1, a_2, \dots, a_{n-i-1}$$

such that

$$|a_1 a_2 \cdots a_{n-i-1}| = |\varpi^r|\,.$$

We assume that

$$\mu_i(a_1, a_2, \dots, a_{n-i-1}, a_{n-i}, b_i, \dots, b_1) \neq 0 \Rightarrow |a_{n-i}| \leq |\varpi^m|$$
$$\mu_{i+1}(a_1, a_2, \dots, a_{n-i-1}, b_{i+1}, b_i, \dots, b_1) \neq 0 \Rightarrow |b_{i+1} b_i \cdots b_1| \leq |\varpi^s|\,.$$

We want to show that in fact

$$\mu_{i+1}(a_1, a_2, \dots, a_{n-i-1}, b_{i+1}, b_i, \dots, b_1) \neq 0 \Rightarrow |b_i \cdots b_1| \leq |\varpi^{s+m}|$$

Indeed the function

$$b_{i+1} \mapsto \mu_{i+1}(a_1, a_2, \dots, a_{n-i-1}, b_{i+1}, b_i, \dots, b_1)$$

has support contained in the set

$$|b_{i+1}| \leq |\varpi^s (b_i \cdots b_1)^{-1}|\,.$$

It is the Fourier transform of the function

$$a_{n-i} \mapsto \sigma_i(a_1, a_2, \dots, a_{n-i-1}, a_{n-i}, b_i, \dots, b_1)$$

with support contained in the set

$$|a_{n-i}| \leq |\varpi^m|\,.$$

By the uncertainty principle we get

$$|\varpi^s (b_i \cdots b_1)^{-1} \varpi^m| \geq 1$$

or

$$|b_i \cdots b_1| \leq |\varpi^{s+m}|$$

which is the assertion of the lemma. □

LEMMA 2 (Uncertainty Principle 2). *If the following diagram holds*

$$\begin{array}{cccc} & & \overset{i+1}{\boxed{s}} & \overset{i}{\boxed{=s+m}} \\ \overset{n-i-1}{\boxed{=r}} & \overset{n-i}{\boxed{r+m}} & & \end{array}$$

then the functions

$$b_{i+1} \mapsto \mu_{i+1}(a_1, a_2, \dots , a_{n-i-1}, b_{i+1}, b_i, \dots , b_1) \tag{23}$$

$$a_{n-i} \mapsto \sigma_i(a_1, a_2, \dots , a_{n-i-1}, a_{n-i}, b_i, \dots , b_1) \tag{24}$$

are constant on their respective supports.

Proof. Again we are given

$$a_1, a_2, \dots , a_{n-i-1}$$

such that

$$|a_1 a_2 \cdots a_{n-i-1}| = |\varpi^r|$$

and

$$b_i, b_{i-1}, \dots , b_1$$

such that

$$|b_i b_{i-1} \cdots b_1| = |\varpi^{s+m}| .$$

The assumption is that

$$\mu_{i+1}(a_1, a_2, \dots , a_{n-i-1}, b_{i+1}, b_i, \dots , b_1) \neq 0 \Rightarrow |b_{i+1}| \leq |\varpi^{-m}|$$

and

$$\sigma_i(a_1, a_2, \dots , a_{n-i-1}, a_{n-i}, b_i, \dots , b_1) \neq 0 \Rightarrow |a_{n-i}| \leq |\varpi^m| .$$

Since these functions form a Fourier pair

$$\mu_{i+1}(a_1, a_2, \dots , a_{n-i-1}, b_{i+1}, b_i, \dots , b_1)$$

takes a constant value for $|b_{i+1}| \leq |\varpi^{-m}|$ and is 0 otherwise while

$$\sigma_i(a_1, a_2, \dots , a_{n-i-1}, a_{n-i}, b_i, \dots , b_1)$$

takes a constant value for $|a_{n-i}| \leq |\varpi^m|$ and is 0 otherwise. □

So far we have not used the fact that the ratio of σ_i and μ_i is an oscillatory factor. We do in the following lemma.

LEMMA 3 (adjacent variables). *Let*

$$a_1, a_2, \dots, a_{n-i-1}\,,\ b_{i-1}, \dots, b_2, b_1$$

be given. Suppose that for $|a_{n-i}| = |\varpi^s|$, *and* $|b_i| = |\varpi^t|$ *the function*

$$\sigma_i(\bullet, a_{n-i}, b_i, \bullet)$$

does not depend on a_{n-i} *while the function*

$$\mu_i(\bullet, a_{n-i}, b_i, \bullet)$$

does not depend on b_i. *If* $s+t>0$ *then in fact, for* $|a_{n-i}| = |\varpi^s|$ *and* $|b_i| = |\varpi^t|$

$$\sigma_i(\bullet, a_{n-i}, b_i, \bullet) = 0\,,\ \mu_i(\bullet, a_{n-i}, b_i, \bullet) = 0\,.$$

Proof. We assume this is not true. We fix a_{n-i} with $|a_{n-i}| = |\varpi^s|$. Next, we choose ε such that

$$|\varepsilon| = |\varepsilon + a_{n-i}| = |\varpi^s|\,.$$

This is always possible if the residual characteristic is not 2. Then

$$\sigma_i(\bullet, a_{n-i}+\varepsilon, b_i, \bullet) = \sigma_i(\bullet, a_{n-i}, b_i, \bullet)\,.$$

This implies the relation

$$\psi\left(\frac{\varepsilon}{a_{n-i}(a_{n-i}+\varepsilon)b_i}\right) = \frac{\mu_i(\bullet, a_{n-i}+\varepsilon, b_i, \bullet)}{\mu_i(\bullet, a_{n-i}, b_i, \bullet)}\,.$$

Now the right-hand side does not depend on b_i for $|b_i| = |\varpi^t|$. Thus the same is true of the left-hand side. Since

$$\left|\frac{\varepsilon}{a_{n-i}(a_{n-i}+\varepsilon)}\right| = |\varpi^{-s}|\,.$$

This amounts to saying that $\psi(u)$ is constant on the shell $\{u : |u| = |\varpi^{-s-t}|\}$. Since $s+t>0$, this is a contradiction which proves the lemma. □

4. Key lemmas

In this section $\mu \geq 0$ is an integer. We study diagrams ending in following pattern:

$$\boxed{=\mu}$$

$$\boxed{\mu}$$

Lemma 4. *If the diagram*

$$\begin{array}{cccc} & \boxed{m} & \boxed{=\mu} & \\ \boxed{*} & \boxed{\mu} & & \end{array}$$

holds, then the following diagram holds

$$\begin{array}{cccc} & \boxed{m} & \boxed{=\mu} & \\ \boxed{m} & \boxed{\mu} & & \end{array}$$

If, furthermore,

$$\begin{array}{cccc} & \overset{k}{\boxed{m}} & \overset{k-1}{\boxed{=\mu}} & \\ \overset{n-k}{\boxed{=m}} & \overset{n-k+1}{\boxed{\mu}} & & \end{array}$$

holds, then the function

$$b_k \mapsto \mu_k(\bullet, b_k, \bullet)$$

does not depend on b_k on its support, in particular, on the shell

$$\{b_k : |b_k| = |\varpi^{m-\mu}|\}\,.$$

Proof. In view of the principle of symmetry this is a restatement of Lemmas 1 and 2. □

Lemma 5. *Suppose $2m_0 > m_1 + \mu$. Then if the diagram*

$$\begin{array}{cccc} & \overset{k+1}{\boxed{m_1}} & \overset{k}{\boxed{m_0}} & \overset{k-1}{\boxed{=\mu}} \\ \overset{n-k-1}{\boxed{=m_1}} & \overset{n-k}{\boxed{m_0}} & \overset{n-k+1}{\boxed{\mu}} & \end{array}$$

holds, it holds trivially.

Proof. We are given

$$b_{k-1}, b_{k-2}, \dots, b_1$$

such that

$$|b_{k-1}b_{k-2}\cdots b_1| = |\varpi^{\mu}|$$

and

$$a_1, a_2, \dots, a_{n-k-1}$$

such that

$$|a_1 a_2 \cdots a_{n-k-1}| = |\varpi^{m_1}| \,.$$

We let $s \geq 0$ be an integer such that the following diagram holds:

$$\begin{array}{cccc} & \boxed{m_1} & \boxed{m_0+s} & \boxed{=\mu} \\ \boxed{=m_1} & \boxed{m_0} & \boxed{\mu} & \end{array}$$

By Lemma 4 the following diagram holds

$$\begin{array}{cccc} & \boxed{m_1} & \boxed{m_0+s} & \boxed{=\mu} \\ \boxed{=m_1} & \boxed{m_0+s} & \boxed{\mu} & \end{array}$$

We then show that the diagram

$$\begin{array}{ccc} & \overset{k+1}{\boxed{m_1}} & \overset{k}{\boxed{=m_0+s}} \\ \overset{n-k-1}{\boxed{=m_1}} & \overset{n-k}{\boxed{m_0+s}} & \end{array}$$

holds trivially. To that end, we let b_k be such that

$$|b_k| = |\varpi^{m_0+s-\mu}| \,.$$

We have to prove that

$$\mu_k(\bullet, a_{n-k}, b_k, \bullet) = 0 \,,\; \sigma_k(\bullet, a_{n-k}, b_k, \bullet) = 0 \,. \tag{25}$$

As before,

$$\sigma_k(\bullet, a_{n-k}, b_k, \bullet) \neq 0 \Rightarrow |a_{n-k}| \leq |\varpi^{m_0+s-m_1}|$$

$$\mu_{k+1}(\bullet, b_{k+1}, b_k, \bullet) \neq 0 \Rightarrow |b_{k+1}| \leq |\varpi^{m_1-m_0-s}|.$$

Thus $\sigma_k(\bullet, a_{n-k}, b_k, \bullet)$ does not depend on a_{n-k} on its support; thus it suffices to prove (25) for

$$|a_{n-k}| = |\varpi^{m_0+s-m_1}| \,.$$

Then $|a_1 a_2 \cdots a_{n-k}| = |\varpi^{m_0+s}|$ and so by Lemma 4, $\mu_k(\bullet, a_{n-k}, b_k, \bullet)$ does not depend on b_k. Since

$$m_0 + s - \mu + m_0 + s - m_1 = 2s + 2m_0 - \mu - m_1 > 0$$

we can use the pair (a_{n-k}, b_k) as the adjacent variables of Lemma 3 to obtain (25). Thus we may replace s by $s+1$. Hence the diagram

$$\begin{array}{cccc} & \overset{k+1}{\boxed{m_1}} & \overset{k}{\boxed{t}} & \overset{k-1}{\boxed{=\mu}} \\ \overset{n-k-1}{\boxed{=m_1}} & \overset{n-k}{\boxed{m_0}} & \overset{n-k+1}{\boxed{\mu}} & \end{array}$$

holds, then for any $t > 0$ and our conclusion follows. $\square$

Lemma 6. *Suppose $k \geq 1$, $r \geq 0$, $k+r \leq n-1$. Let $\mu \geq 0$. Then if the diagram*

$$\begin{array}{ccccccc} & \overset{k+r}{\boxed{m_r}} & \overset{k+r-1}{\boxed{m_{r-1}}} & \cdots\ \cdots & \overset{k+1}{\boxed{m_1}} & \overset{k}{\boxed{m_0}} & \overset{k-1}{\boxed{=\mu}} \\ \overset{n-k-r}{\boxed{*}} & \overset{n-k-r-1}{\boxed{*}} & \cdots & \cdots & \overset{n-k-1}{\boxed{*}} & \overset{n-k}{\boxed{*}} & \overset{n-k+1}{\boxed{\mu}} & \end{array}$$

holds, then the following diagram holds

$$\begin{array}{ccccccc} & \overset{k+r}{\boxed{m_r}} & \overset{k+r-1}{\boxed{m_{r-1}}} & \cdots\ \cdots & \overset{k+1}{\boxed{m_1}} & \overset{k}{\boxed{m_0}} & \overset{k-1}{\boxed{=\mu}} \\ \overset{n-k-r}{\boxed{m_r}} & \overset{n-k-r-1}{\boxed{m_{r-1}}} & \cdots & \cdots & \overset{n-k-1}{\boxed{m_1}} & \overset{n-k}{\boxed{m_0}} & \overset{n-k+1}{\boxed{\mu}} & \end{array}$$

Proof. We have already established this assertion for $r=0$ (Lemma 4). Thus we may assume $r \geq 1$ and our assertion is established for $r-1$. Thus we already know that the following diagram holds:

$$\begin{array}{ccccccc} & \overset{k+r}{\boxed{m_r}} & \overset{k+r-1}{\boxed{m_{r-1}}} & \cdots\ \cdots & \overset{k+1}{\boxed{m_1}} & \overset{k}{\boxed{m_0}} & \overset{k-1}{\boxed{=\mu}} \\ \overset{n-k-r}{\boxed{*}} & \overset{n-k-r-1}{\boxed{m_{r-1}}} & \cdots & \cdots & \overset{n-k-1}{\boxed{m_1}} & \overset{n-k}{\boxed{m_0}} & \overset{n-k+1}{\boxed{\mu}} & \end{array}$$

Thus it is a matter of proving that if

$$\begin{array}{ccccccc} & \overset{k+r}{\boxed{m_r}} & \overset{k+r-1}{\boxed{m_{r-1}}} & \cdots\ \cdots & \overset{k+1}{\boxed{m_1}} & \overset{k}{\boxed{m_0}} & \overset{k-1}{\boxed{=\mu}} \\ \overset{n-k-r}{\boxed{=\theta}} & \overset{n-k-r-1}{\boxed{m_{r-1}}} & \cdots & \cdots & \overset{n-k-1}{\boxed{m_1}} & \overset{n-k}{\boxed{m_0}} & \overset{n-k+1}{\boxed{\mu}} & \end{array}$$

holds, then it holds trivially unless $\theta \geq m_r$. Thus assume $\theta < m_r$ and the diagram holds. Let $s \geq 0$ be such that the following diagram holds:

$$\begin{array}{ccccccc} & \overset{k+r}{\boxed{m_r}} & \overset{k+r-1}{\boxed{m_{r-1}+s}} & \cdots\ \cdots & \overset{k+1}{\boxed{m_1}} & \overset{k}{\boxed{m_0}} & \overset{k-1}{\boxed{=\mu}} \\ \overset{n-k-r}{\boxed{=\theta}} & \overset{n-k-r-1}{\boxed{m_{r-1}}} & \cdots & \cdots & \overset{n-k-1}{\boxed{m_1}} & \overset{n-k}{\boxed{m_0}} & \overset{n-k+1}{\boxed{\mu}} & \end{array}$$

By the induction hypothesis the following diagram holds

$$\begin{array}{ccccccc} & \overset{k+r}{\boxed{m_r}} & \overset{k+r-1}{\boxed{m_{r-1}+s}} & \cdots\ \cdots & \overset{k+1}{\boxed{m_1}} & \overset{k}{\boxed{m_0}} & \overset{k-1}{\boxed{=\mu}} \\ \overset{n-k-r}{\boxed{=\theta}} & \overset{n-k-r-1}{\boxed{m_{r-1}+s}} & \cdots & \cdots & \overset{n-k-1}{\boxed{m_1}} & \overset{n-k}{\boxed{m_0}} & \overset{n-k+1}{\boxed{\mu}} & \end{array}$$

By the uncertainty principle the following diagram holds as well

$$\begin{array}{ccccccc} \overset{k+r}{\boxed{m_r}} & \overset{k+r-1}{\boxed{m_{r-1}+s+m_r-\theta}} & \dots & \dots & \overset{k+1}{\boxed{m_1}} & \overset{k}{\boxed{m_0}} & \overset{k-1}{\boxed{=\mu}} \end{array}$$

$$\begin{array}{ccccccc} \overset{n-k-r}{\boxed{=\theta}} & \overset{n-k-r-1}{\boxed{m_{r-1}+s}} & \dots & \dots & \overset{n-k-1}{\boxed{m_1}} & \overset{n-k}{\boxed{m_0}} & \overset{n-k+1}{\boxed{\mu}} \end{array}$$

So we have replaced s by $s+m_r-\theta > s$. Thus, for any $t>0$, the following diagram holds

$$\begin{array}{ccccccc} \overset{k+r}{\boxed{m_r}} & \overset{k+r-1}{\boxed{t}} & \dots & \dots & \overset{k+1}{\boxed{m_1}} & \overset{k}{\boxed{m_0}} & \overset{k-1}{\boxed{=\mu}} \end{array}$$

$$\begin{array}{ccccccc} \overset{n-k-r}{\boxed{=\theta}} & \overset{n-k-r-1}{\boxed{m_{r-1}}} & \dots & \dots & \overset{n-k-1}{\boxed{m_1}} & \overset{n-k}{\boxed{m_0}} & \overset{n-k+1}{\boxed{\mu}} \end{array}$$

and our conclusion follows. □

5. Proof of Proposition 3

In this section, we consider diagrams of the following form

$$\begin{array}{cccccccc} \boxed{\nu} & \boxed{m_k} & \boxed{m_{k-1}} & \dots & \dots & \boxed{m_2} & \boxed{m_1} & \boxed{=\mu} \end{array}$$

$$\begin{array}{cccccccc} \boxed{=\nu} & \boxed{m_k} & \boxed{m_{k-1}} & \dots & \dots & \boxed{m_2} & \boxed{m_1} & \boxed{\mu} \end{array}$$

where $k \geq 1$. We call the double sum

$$2(m_1+m_2+\cdots m_k)$$

the *weight* of the diagram. We will prove the following result.

PROPOSITION 4. *Suppose $k \geq 1$ and the diagram*

$$\begin{array}{cccccccc} \overset{n-u}{\boxed{\nu}} & \boxed{m_k} & \boxed{m_{k-1}} & \dots & \dots & \boxed{m_2} & \boxed{m_1} & \overset{v}{\boxed{=\mu}} \end{array}$$

$$\begin{array}{cccccccc} \overset{u}{\boxed{=\nu}} & \boxed{m_k} & \boxed{m_{k-1}} & \dots & \dots & \boxed{m_2} & \boxed{m_1} & \overset{n-v}{\boxed{\mu}} \end{array}$$

holds. Let w be the weight of the diagram. If

$$w > k(\mu+\nu)\,,$$

then the diagram holds trivially.

In more detail, we fix $a_1, a_2, \dots, a_u$ such that

$$|a_1 a_2 \cdots a_u| = |\varpi^\nu|$$

and $b_v, b_{v-1}, \ldots, b_1$ such that

$$|b_v b_{v-1} \cdots b_1| = |\varpi^\mu|\,.$$

We assume that the conditions indicated by the diagram hold on the supports of the functions

$$\mu_{n-u}(a_1, a_2, \ldots , a_u, b_{n-u}, b_{n-u-1}, \ldots , b_{v+1}, b_v, \ldots b_1)$$

and

$$\mu_v(a_1, a_2, \ldots , a_u, a_{u+1}, \ldots , a_{n-v}, b_v, \ldots , b_1)\,.$$

The conclusion is that the supports are in fact empty; that is, the functions vanish. In particular, we may use this result to prove Proposition 3. Indeed, the diagram (16) holds. By the uncertainty principle the following diagram holds as well:

(26)
$$\begin{array}{cccccccc}
n & n-1 & n-2 & & 3 & 2 & 1 & 0 \\
\boxed{0} & \boxed{1} & \boxed{0} & \ldots & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{=0} \\
\\
0 & 1 & 2 & 3 & & n-2 & n-1 & n \\
\boxed{=0} & \boxed{1} & \boxed{0} & \boxed{0} & \ldots & \boxed{0} & \boxed{0} & \boxed{0}
\end{array}$$

From Proposition 4 it follows that the diagram (26) holds trivially which is the conclusion of Proposition 3. It remains to prove Proposition 4.

Proof of Proposition 4. If $k = 1$, then the assertion is Lemma 5.

Now we assume $k > 1$ and our assertion established for $1 \leq i < k$. We assume

(27) $$w = 2(m_1 + m_2 + \cdots + m_k)\,,\ w > k(\mu + \nu).$$

We will show then in fact a diagram of the form

$$\begin{array}{cccccccc}
\boxed{\nu} & \boxed{m_k^1} & \boxed{m_{k-1}^1} & \ldots & \ldots & \boxed{m_2^1} & \boxed{m_1^1} & \boxed{=\mu} \\
\boxed{=\nu} & \boxed{m_k^1} & \boxed{m_{k-1}^1} & \ldots & \ldots & \boxed{m_2^1} & \boxed{m_1^1} & \boxed{\mu}
\end{array}$$

holds with a weight w_1 which verifies $w_1 > w$ and thus $w_1 > k(\mu+\nu)$. This will prove our assertion. Indeed, by induction, we find then sequences of integers (m_j^i) such that the following diagrams hold

$$\begin{array}{cccccccc}
\boxed{\nu} & \boxed{m_k^i} & \boxed{m_{k-1}^i} & \ldots & \ldots & \boxed{m_2^i} & \boxed{m_i^i} & \boxed{=\mu} \\
\boxed{=\nu} & \boxed{m_k^i} & \boxed{m_{k-1}^i} & \ldots & \ldots & \boxed{m_2^i} & \boxed{m_1^i} & \boxed{\mu}
\end{array}$$

and the sequence

$$w_i := 2(m_1^i + m_2^i + \cdots + m_k^i)$$

tends to infinity. Thus there is at least one index j such that the sequence m_j^i tends to infinity and that shows that the supports of the relevant functions are empty.

We will show that there is at least one index i, $2 \le i \le k$, such that in the bottom row the box

$$\boxed{m_i}$$

can replaced by

$$\boxed{m_i + 1}$$

or there is an index i, $k-1 \ge i \ge 1$ such that in the top row the box

$$\boxed{m_{k-i}}$$

can replaced by

$$\boxed{m_{k-i} + 1}$$

Note that by Lemma 6, in the first case, we can then replace m_i by $m_i + 1$ in the top row as well, likewise for the second case. Correspondingly, the weight w is increased by 2 and we are done. To prove the existence of the index in the bottom or the top row, we proceed again by contradiction. We assume there is no such index. In the bottom row, this means that for every i with $2 \le i \le k$, the following diagram holds nontrivially

$$\begin{array}{ccccccccc} & \boxed{m_i} & \boxed{m_{i-1}} & \boxed{m_{i-2}} & \cdots & \cdots & \boxed{m_2} & \boxed{m_1} & \boxed{=\mu} \\ \boxed{=m_i} & \boxed{m_{i-1}} & \boxed{m_{i-1}} & \cdots & \cdots & \boxed{m_2} & \boxed{m_1} & \boxed{\mu} & \end{array}$$

By the induction hypothesis, this implies

$$2(m_1 + m_2 + \cdots + m_{i-1}) \le (i-1)(m_i + \mu)\,. \tag{28}$$

This inequality is trivially true for $i = 1$ as both sides are then 0. Adding up these inequalities together we get

$$\sum_{i=1}^{i=k} 2(k-i)m_i \le \sum_{i=1}^{k} (i-1)m_i + \frac{k(k-1)}{2}\mu$$

or

$$\sum_{i=1}^{k} (2k - 3i + 1)m_i \le \frac{k(k-1)}{2}\mu\,. \tag{29}$$

Likewise, considering the top row we obtain

$$\sum_{i=1}^{k} (2k - 3(k-i+1) + 1)m_i \le \frac{k(k-1)}{2}\nu$$

or

$$\sum_{i=1}^{k}(-k+3i-2)m_i \leq \frac{k(k-1)}{2}\nu\,. \tag{30}$$

Adding up the inequalities (29) and (30) we obtain

$$(k-1)\sum_{i=1}^{k} m_i \leq \frac{k(k-1)}{2}(\mu+\nu)$$

or

$$2(m_1+m_2+\cdots+m_k) \leq k(\mu+\nu)$$

which contradicts (27). This concludes the proof. □

6. Complement

We can also characterize the function

$$\Omega_n(a) := \Omega(\Phi_n, \psi : a)$$

by properties of support. The function Ω_n is not zero. Indeed by Proposition 1 we see that

$$|a_1| = |a_2| = \cdots = |a_n| = 1 \Rightarrow \Omega_n(a) = 1\,.$$

PROPOSITION 5. *Suppose* $\omega \in \mathcal{I}_n$ *is supported on the set* (7) *and its Kloosterman transform is supported on the same set. Then* $\omega = c\Omega_n$ *for a suitable constant* c*.*

Proof. Our assertion is trivial for $n = 1$. Thus we may assume $n > 1$ and our assertion proved for $n-1$. Fix a_1 with $|a_1| = 1$. Then the function

$$(a_2, a_3, \ldots, a_n) \mapsto \omega(a_1, a_2, \ldots, a_n)$$

has for Kloosterman transform (in $n-1$ variables) the function

$$(b_1, b_2, \ldots, b_{n-1}) \mapsto \mu_{n-1}(a_1, b_{n-1}, b_{n-2}, \ldots, b_1)\,.$$

It satisfies the assumption of the proposition for $n-1$; thus

$$\omega(a_1, a_2 \ldots, a_n) = \theta(a_1)\Omega_{n-1}(a_2, a_2 \ldots, a_n)\,,$$

where θ is a function on $\mathcal{O}_F^\times$. Since Ω_{n-1} is invariant under the Kloosterman transform, we get

$$\mu_{n-1}(a_1, b_{n-1}, b_{n-2}, \ldots, b_1) = \theta(a_1)\Omega_{n-1}(b_1, b_2, \ldots, b_{n-1})\,.$$

To determine the function θ we choose $b_1, b_2, \ldots, b_{n-1}$ such that

$$|b_1| = |b_2| = \cdots = |b_{n-1}|\,.$$

We get then

$$\mu_{n-1}(a_1, b_{n-1}, b_{n-2}, \dots, b_1) = \theta(a_1)\,.$$

On the other hand,

$$|b_{n-1}b_{n-2}\cdots b_1| = 1\,.$$

Considering the diagram

$$\begin{array}{cccc} & & \overset{n}{\boxed{0}} & \overset{n-1}{\boxed{=0}} \\ \\ & \overset{0}{\boxed{=0}} & \overset{1}{\boxed{0}} & \end{array}$$

and using Lemma (2) we see that

$$\mu_{n-1}(a_1, b_{n-1}, b_{n-2}, \dots, b_1)$$

does not depend on a_1 with $|a_1| = 1$. Thus θ has a constant value c. We have found that

$$\mu_{n-1}(a_1, b_{n-1}, b_{n-2}, \dots, b_1) = c\Omega_{n-1}(b_1, b_2, \dots, b_{n-1})$$

for $|a_1| = 1$ and all b_i. Using the fact that Ω_{n-1} is invariant under $\mathcal{K}_{n-1,\overline{\psi}}$ we get

$$\omega(a_1, a_2, \dots, a_n) = c\Omega_{n-1}(a_2, \dots, a_n)$$

for $|a_1| = 1$. On the other hand, for $|a_1| = 1$,

$$\Omega_n(a_1, a_2, \dots, a_n) = \Omega_{n-1}(a_2, \dots, a_n)\,.$$

It follows that

$$\omega - c\Omega_n$$

vanishes for $|a_1| = 1$. By Proposition 3 it vanishes identically and we are done. □

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