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Unique decomposition of tensor products of irreducible representations of simple algebraic groups

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Abstract

We show that a tensor product of irreducible, finite dimensional representations of a simple Lie algebra over a field of characteristic zero determines the individual constituents uniquely. This is analogous to the uniqueness of prime factorisation of natural numbers.

1. Introduction

1.1. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. The main aim of this paper is to prove the following unique factorisation of tensor products of irreducible, finite dimensional representations of $\mathfrak{g}$:

Theorem 1. *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Let $V_1, \dots, V_n$ and $W_1, \dots, W_m$ be nontrivial, irreducible, finite dimensional $\mathfrak{g}$-modules. Assume that there is an isomorphism of the tensor products,*

$$V_1 \otimes \cdots \otimes V_n \simeq W_1 \otimes \cdots \otimes W_m,$$

as $\mathfrak{g}$-modules. Then $m = n$, and there is a permutation τ of the set $\{1, \dots, n\}$, such that

$$V_i \simeq W_{\tau(i)},$$

as $\mathfrak{g}$-modules.

The particular case which motivated the above theorem is the following corollary:

Corollary 1. *Let V, W be irreducible $\mathfrak{g}$-modules. Assume that*

$$\mathrm{End}(V) \simeq \mathrm{End}(W),$$

as $\mathfrak{g}$-modules. Then V is either isomorphic to W or to the dual $\mathfrak{g}$-module W^*.

When $\mathfrak{g} = \mathfrak{sl}_2$, and the number of components is at most two, the theorem follows by comparing the highest and lowest weights that occur in the tensor product. However, this proof seems difficult to generalize (see Subsection 2.1). The first main step towards a proof of the theorem, is to recast the hypothesis as an equality of the corresponding products of characters of the individual representations occurring in the tensor product. A pleasant, arithmetical proof for $\mathfrak{sl}_2$ (see Proposition 4), indicates that we are on a right route. The proof in the general case depends on the fact that the Dynkin diagram of a simple Lie algebra is connected, and proceeds by induction on the rank of $\mathfrak{g}$, by the fact that any simple Lie algebra of rank l, has a simple subalgebra of rank $l-1$. We analyze the restriction of the numerator of the Weyl character formula of $\mathfrak{g}$ to the centralizer of the simple subalgebra, by expanding along the characters of the central $\mathfrak{gl}_1$.

We compare the coefficients, which are numerators of characters of the simple subalgebra, of the highest and the second highest degrees occurring in the product. The highest degree term is again the character corresponding to a tensor product of irreducible representations. The second highest degee term is a sum of the products of irreducible characters. To understand this sum, we again argue by induction using character expansions. However, instead of leading to further complicated sums, the induction argument stabilizes, and we can formulate and prove a linear independence property of products of characters of a particular type. Combining the information obtained from the highest and the second highest degree terms occurring in the product, we obtain the theorem.

The outline of this paper is as follows: first we recall some preliminaries about representations and characters of semisimple Lie algebras. We then give the proof for $\mathfrak{sl}_2$, and also of an auxiliary result which comes up in the proof by induction. Although not needed for the proof in the general case, we present the proof for GL_n, since the ideas involved in the proof seem a bit more natural. Here the numerator of the Weyl-Schur character formula appears as a determinant, which can be looked upon as a polynomial function on the diagonal torus. The inductive argument arises upon expanding this function in one of the variables, the coefficients of which are given by the numerators occurring in the Weyl-Schur character formula for appropriate representations of GL_{n-1}. We then set up the formalism for general simple $\mathfrak{g}$, so that we can carry over the proof for GL_n to the general case.

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2. Preliminaries

We fix the notation and recall some of the relevant aspects of the representation and structure theory of semisimple Lie algebras. We refer to [H], [S] for further details.

(1) Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and $\Phi \subset \mathfrak{h}^*$ the roots of the pair $(\mathfrak{g}, \mathfrak{h})$.

(2) Denote by $\Phi^+ \subset \Phi$, the subset of positive roots with respect to some ordering of the root system, and by Δ a base for Φ^+.

(3) Let $\Phi^* \subset \mathfrak{h}$, Φ^{*+}, Δ^* be respectively the set of co-roots, positive co-roots and fundamental co-roots. Given a root $\alpha \in \Phi$, α^* will denote the corresponding co-root.

(4) Denote by $\langle .,. \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ the duality pairing. For any root α, we have $\langle \alpha^*, \alpha \rangle = 2$, and the pairing takes values in integers when the arguments consist of roots and co-roots.

(5) Given a root α, by the properties of the root system, there are reflections s_α, s_{α^*} of $\mathfrak{h}^*$, $\mathfrak{h}$ respectively, defined by

$$s_\alpha(u) = u - \langle \alpha^*, u \rangle \alpha \quad \text{and} \quad s_{\alpha^*}(x) = x - \langle x, \alpha \rangle \alpha^*,$$

where $x \in \mathfrak{h}$ and $u \in \mathfrak{h}^*$. We have $s_\alpha(\Phi) \subset \Phi$ and $s_{\alpha^*}(\Phi^*) \subset \Phi^*$.

(6) Let W denote the Weyl group of the root system. The Weyl group W is generated by the reflections s_α for $\alpha \in \Delta$, subject to the relations (see [C, Th. 2.4.3])

$$s_\alpha^2 = 1 \quad \text{and} \quad s_\alpha s_\beta s_\alpha = s_{s_\alpha(\beta)}, \quad \forall\ \alpha,\ \beta \in \Phi. \tag{1}$$

In particular s_α and s_β commute if $s_\alpha(\beta) = \beta$. There is a natural isomorphism between the Weyl groups of the root system and the dual root system, given by $\alpha \mapsto \alpha^*$ and $s_\alpha = {}^t s_{\alpha^*}$ the transpose of s_{α^*}. We identify the two actions of the Weyl group.

(7) Denote by $P \subset \mathfrak{h}^*$ the lattice of integral weights, given by

$$P = \{\mu \in \mathfrak{h}^* \mid \mu(\alpha^*) \in \mathbb{Z}, \forall \alpha \in \Phi^*\}.$$

Dually we have a definition of the lattice of integral co-weights P^*.

(8) Let P_+ be the set of dominant, integral weights with respect to the chosen ordering, defined by

$$P_+ = \{\lambda \in P \mid \lambda(\alpha^*) \geq 0, \forall \alpha \in \Phi^{*+}\}.$$

The irreducible $\mathfrak{g}$-modules are indexed by elements in P_+, given by highest weight theory. To each dominant, integral weight λ, we denote the corresponding irreducible $\mathfrak{g}$-module with highest weight λ by V_λ. Let $l = |\Delta|$ be the rank of $\mathfrak{g}$. Index the collection of fundamental roots by $\alpha_1, \dots, \alpha_l$. Denote by $\omega_1, \dots, \omega_l$ (resp. $\omega_1^*, \dots, \omega_l^*$), the set of fundamental weights (resp. fundamental co-weights) defined by

$$\omega_i(\alpha_j^*) = \delta_{ij} \quad \text{and} \quad \omega_i^*(\alpha_j) = \delta_{ij}, \quad 1 \le i, j \le l.$$

The fundamental weights form a $\mathbb{Z}$-basis for P.

(9) Let $l(w)$ denote the length of an element in the Weyl group, given by the least length of a word in the $s_\alpha,\ \alpha \in \Delta$ defining w. Let $\varepsilon(w) = (-1)^{l(w)}$ be the sign character of W.

(10) *The Weyl character formula.* All the representations considered will be finite dimensional. Let V be a $\mathfrak{g}$-module. With respect to the action of $\mathfrak{h}$, we have a decomposition,

$$V = \oplus_{\pi \in \mathfrak{h}^*} V^\pi,$$

$$\text{where} \quad V^\pi = \{v \in V \mid hv = \pi(h)v,\ h \in \mathfrak{h}\}.$$

The linear forms π for which the V^π are nonzero belong to the weight lattice P, and these are the weights of V. Let $\mathbb{Z}[P]$ denote the group algebra of P, with basis indexed by e^π for $\pi \in P$. The (formal) character $\chi_V \in \mathbb{Z}[P]$ of V is defined by,

$$\chi_V = \sum_{\pi \in P} m(\pi) e^\pi,$$

where $m(\pi) = \dim(V^\pi)$ is the multiplicity of π. The character is a ring homomorphism from the Grothendieck ring $K[\mathfrak{g}]$ defined by the representations of $\mathfrak{g}$ to the group algebra $\mathbb{Z}[P]$. In particular,

$$\chi_{V \otimes V'} = \chi_V \chi_{V'}.$$

The irreducible $\mathfrak{g}$-modules are indexed by elements in P_+, given by highest weight theory. To each dominant, integral weight λ, we denote the corresponding irreducible $\mathfrak{g}$-module with highest weight λ by V_λ, and the corresponding character by χ_λ. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \omega_1 + \cdots + \omega_l.$$

Define the Weyl denominator D as,

$$D = \sum_{w \in W} \varepsilon(w) e^{w\rho} \in \mathbb{Z}[P].$$

The Weyl character formula for V_λ is given by,

$$\chi_\lambda = \frac{1}{D}\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\rho)}.$$

Let $S_\lambda = \sum_{w\in W}\varepsilon(w)e^{w(\lambda+\rho)}$ denote the numerator occurring in the Weyl character formula. We have $D = S_0$.

We now recast the main theorem. From the theory of characters, the main theorem is equivalent to the following theorem:

THEOREM 2. *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Assume that there are positive integers $n \geq m$, and nonzero dominant weights $\lambda_1, \dots, \lambda_n,\ \mu_1, \dots, \mu_m$ in P_+ satisfying,*

$$S_{\lambda_1}\dots S_{\lambda_n} = S_{\mu_1}\dots S_{\mu_m}(S_0)^{n-m}. \tag{2}$$

Then $m = n$, and there is a permutation τ of the set $\{1, \dots, n\}$, such that

$$\lambda_i = \mu_{\tau(i)}, \quad 1 \leq i \leq n.$$

We adopt a slight change in the notation. Assume $n \geq m$. Then (2) can be rewritten as,

$$S_{\lambda_1}\dots S_{\lambda_n} = S_{\mu_1}\dots S_{\mu_n}, \tag{3}$$

where $\mu_i = 0$ for $m+1 \leq i \leq n$.

2.1. $\mathfrak{sl}_2$ *and* PRV-*components.* Let $\mathfrak{g} = \mathfrak{sl}_2$. Let V_n denote the irreducible representation of $\mathfrak{sl}_2$ of dimension $n+1$, isomorphic to the symmetric n^{th} power $S^n(V_1)$ of the standard representation V_1. Suppose we have an isomorphism of $\mathfrak{sl}_2$-modules,

$$V_{n_1}\otimes V_{n_2} \simeq V_{m_1}\otimes V_{m_2}.$$

For any pair of positive integers $l \geq k$, we have the decompostion,

$$V_k \otimes V_l \simeq V_{l+k}\oplus V_{l+k-2}\oplus\cdots\oplus V_{l-k}.$$

It follows that $n_1 + n_2 = m_1 + m_2$ by comparison of the highest weights. Assuming $n_1 \geq n_2$ and $m_1 \geq m_2$, we have on comparing the lowest weights occurring in the tensor product, that $n_1 - n_2 = m_1 - m_2$. Hence the theorem follows in this special case.

It is immediate from the hypothesis of the theorem, that we have an equality of the sum of the highest weights corresponding to the irreducible modules $V_1, \dots, V_n$ and $W_1, \dots, W_m$ respectively. The above proof for $\mathfrak{sl}_2$ suggests the use of PRV-components: if V_λ and V_μ are highest weight finite dimensional $\mathfrak{g}$-modules with highest weights λ and μ respectively, and w is an element of the Weyl group, then it is known that there is a Weyl group translate $\overline{\lambda + w\mu}$ of the weight $\lambda + w\mu$, which is dominant and such that the corresponding highest weight module $V_{\overline{\lambda+w\mu}}$ is a direct summand in the tensor product module $V_\lambda \otimes V_\mu$ (see [SK1]). These are the generalized Parthasarathy-Ranga Rao-Varadarajan (PRV)-components. The standard PRV-component is obtained by taking $w = w_0$, the longest element in the Weyl group. But the above proof for $\mathfrak{sl}_2$ does not generalize, as the following example for the simple Lie algebra $\mathfrak{sp}_6$ shows that it is not enough to consider just the standard PRV-component:

Example 1.

$$\mathfrak{g} = \mathfrak{sp}_6, \quad \mathfrak{h} = \mathbb{C}\langle e_1, e_2, e_3\rangle, \quad \Delta = \{e_1 - e_2, e_2 - e_3, 2e_3\}, \quad w_0 = -1.$$

Consider the following highest weights on sp_6:

$$\begin{aligned} \lambda_1 &= 6e_1 + 4e_2 + 2e_3 \qquad & \lambda_2 &= 4e_1 + 2e_2 \\ \mu_1 &= 6e_1 + 2e_2 + 2e_3 \qquad & \mu_2 &= 4e_1 + 4e_2. \end{aligned}$$

Clearly $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$. Since the Weyl group contains sign changes, we see that there exists an element of the Weyl group such that $\lambda_1 - \lambda_2 = w(\mu_1 - \mu_2)$.

Thus we are led to consider generalized PRV-components. The problem with this approach is that although the standard PRV-component can be characterised as the component on which the Casimir acts with the smallest eigenvalue, there is no abstract characterisation of the generalized PRV-component inside the tensor product. It is not clear that a generalized PRV-component of one side of the tensor product, is also a PRV-component for the other tensor product. Although the PRV-components occur with 'high' multiplicity [SK2], (greater than or equal to the order of the double coset $W_\lambda \backslash W / W_\mu$, where W_λ and W_μ are the isotropy subgroups of λ and μ respectively), the converse is not true. Even for $\mathfrak{sl}_2$, it does not seem easy to extend the above proof when the number of components involved is more than two.

3. GL(2)

The aim of this and the following section is to prove the main theorem in the context of GL(r):

THEOREM 3. *Let $G = \mathrm{GL}(r)$. Suppose $V \simeq V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ and $W \simeq V_{\mu_1} \otimes \cdots \otimes V_{\mu_m}$ are tensor products of irreducible representations with nonzero highest weights $\lambda_1, \ldots, \mu_m$. Assume that $V \simeq W$ as G-modules. Then $n = m$ and there is a permutation τ of $\{1, \cdots .n\}$ such that for $1 \le i \le n$,*

$$V_{\lambda_i} = V_{\mu_{\tau(i)}} \otimes \det{}^{\alpha_i},$$

for some integers α_i.

Up to twisting by a power of the determinant, we can assume that the highest weight representations V_λ of $\mathrm{GL}(r)$ are parametrized by their 'normalized' highest weights,

$$\lambda = (a_1, \cdots a_r), \quad a_1 \geq a_2 \geq \cdots \geq a_r = 0,$$

and a_i are nonnegative integers. It is enough then to show under the hypothesis of the theorem, that the normalized highest weights coincide. Let $x = (x_1, \ldots, x_r)$ be a multivariable. We have the symmetric functions (Schur functions), defined as the quotient of two determinants,

$$\chi_\lambda(x) = \frac{|x_j^{a_i+r-i}|}{|x_j^{r-i}|} = \frac{S_\lambda}{D},$$

where S_λ denotes the determinant appearing in the numerator and D the standard Vandermonde determinant appearing in the denominator. It is known that on the set of regular diagonal matrices the Schur function χ_λ is equal to the character of V_λ. Since we have assumed $a_n = 0$, we have that the polynomials S_λ and x_1 are coprime, for any highest weight λ.

Hence by character theory, the hypothesis of the theorem can be recast as

$$S_{\lambda_1} \ldots S_{\lambda_n} = S_{\mu_1} \ldots S_{\mu_n}, \tag{4}$$

and where $\mu_i = 0$ for $m + 1 \leq i \leq n$.

Write for $1 \leq i \leq n$,

$$\begin{aligned} \lambda_i &= (a_{i1}, a_{i2}, \ldots, a_{i(r-1)}, 0), \\ \mu_i &= (b_{i1}, b_{i2}, \ldots, b_{i(r-1)}, 0). \end{aligned}$$

2.2. $\mathrm{GL}(2)$. We present now the proof of the theorem for $\mathrm{GL}(2)$.

PROPOSITION 4. *Theorem* 3 *is true for* $\mathrm{GL}(2)$.

Proof. Specializing Equation 3 to the case of $\mathrm{GL}(2)$, we obtain

$$(x_1^{a_1+1} - x_2^{a_1+1}) \cdots (x_1^{a_n+1} - x_2^{a_n+1}) = (x_1^{b_1+1} - x_2^{b_1+1}) \cdots (x_1^{b_n+1} - x_2^{b_n+1}),$$

where for the sake of simplicity we drop one of the indices in the weights. Specialising the equation to $x_2 = 1$, and letting $x = x_1$, we obtain an equality of the product of polynomials,

$$(x^{a_1+1} - 1)(x^{a_2+1} - 1) \cdots (x^{a_n+1} - 1) = (x^{b_1+1} - 1)(x^{b_2+1} - 1) \cdots (x^{b_n+1} - 1).$$

Assume that $a_1 = \max\{a_1, \ldots, a_n\}$ and $b_1 = \max\{b_1, \ldots, b_n\}$. For any positive integer m, let ζ_m denote a primitive m^{th} root of unity. The left-hand side polynomial has a zero at $x = \zeta_{a_1+1}$, and the equality forces the right side polynomial to vanish at ζ_{a_1+1}. Hence we obtain that $a_1 \leq b_1$, and by symmetry $b_1 \leq a_1$. Thus $a_1 = b_1$ and $\chi_{\lambda_1} = \chi_{\mu_1}$. Cancelling the first factor from both sides, we are left with an equality of a product of characters involving fewer numbers of factors than the equation we started with, and by induction we have proved the theorem for GL(2). □

Remark 1. It would be interesting to know the arithmetical properties of the varieties defined by the polynomials S_λ for general semisimple Lie algebras $\mathfrak{g}$. It seems difficult to generalize the above arithmetical proof to general simple Lie algebras. The proof in the general case proceeds by induction on the rank, finally reducing to the case of $\mathfrak{sl}_2$.

3.2. *A linear independence result.* We now prove an auxiliary result for GL(2), which arises in the inductive proof of Theorem 3.

LEMMA 1. *Let $\lambda_1, \dots, \lambda_n$ be a set of normalized weights in P_+. Let c be a positive integer and ω_1 denote the fundamental weight. Then the set*

$$\{S_{\lambda_1} \cdots S_{\lambda_{i-1}} S_{\lambda_i + c\omega_1} S_{\lambda_{i+1}} \cdots S_{\lambda_n} \mid 1 \le i \le n\},$$

is linearly independent. In particular, suppose that there are subsets $I,\ J \subset \{1, \dots, n\}$ satisfying the following:

$$\begin{aligned} \sum_{i\in I} & S_{\lambda_1} \cdots S_{\lambda_{i-1}} S_{\lambda_i + c\omega_1} S_{\lambda_{i+1}} \cdots S_{\lambda_n} \\ & = \sum_{j \in J} S_{\lambda_1} \cdots S_{\lambda_{j-1}} S_{\lambda_j + c\omega_1} S_{\lambda_{j+1}} \cdots S_{\lambda_n}. \end{aligned} \tag{5}$$

Then there is a bijection $\theta : I \to J$, such that $\lambda_i = \lambda_{\theta(i)}$.

An equivalent statement can be made in the Grothendieck ring $K[\mathfrak{g}]$ or with characters in place of S_λ.

Proof. Suppose we have a relation

$$\sum_{1\le i\le n} z_i S_{\lambda_1} \cdots S_{\lambda_{i-1}} S_{\lambda_i + c\omega_1} S_{\lambda_{i+1}} \cdots S_{\lambda_n} = 0,$$

for some collection of complex numbers z_i. For any index i, let

$$E(i) = \{j \mid \lambda_j = \lambda_i\}.$$

To show the linear independence, we have to show that for any index i, we have

$$\sum_{j \in E(i)} z_j = 0.$$

Dividing by $\prod_{l=1}^n S_{\lambda_l}$ on both sides and equating, we are left with the equation,

$$\sum_{1 \le i \le n} z_i \frac{S_{\lambda_i + c\omega_1}}{S_{\lambda_i}} = 0.$$

Specialising $x_1 = 1$ and writing t instead of x_2, we obtain

$$\sum_{1\leq i\leq n} z_i \frac{1-t^{a_i+c}}{1-t^{a_i}} = 0.$$

Expand this now as a power series in t. Consider the collection of indices i, for which a_i attains the minimum value, say for $i = 1$. Equating the coefficient of t^{a_1}, we see that $\sum_{j\in E(1)} z_j = 0$. Hence these terms can be removed from the relation, and we can proceed by induction to complete the proof of the lemma.

□

Remark 2. In retrospect, both Proposition 4 and Lemma 1, can be proved by comparing the coefficient of the second highest power of x_1 occurring on both sides of the equation (3), as in the proofs occurring in the next section. But we have included the proofs here, since it lays emphasis on the arithmetical properties of the varieties defined by these characters.

4. Tensor products of GL(r)-modules

We now come to the proof of Theorem 3 for arbitrary r. The proof will proceed by induction on r and the maximum number of components n. We assume that the theorem is true for GL(s) with $s < r$, and for GL(r) with the number of components fewer than n. Associated to the highest weight $\lambda = (a_1, a_2, \ldots, a_{r-1}, 0)$ of a GL(r)-irreducible module, define

$$\begin{aligned}\lambda' &= (a_2, a_3, \ldots, a_{r-1}, 0),\\ \lambda'' &= (a_1+1, a_3, \ldots, a_{r-1}, 0).\end{aligned}$$

We can rewrite

$$\lambda'' = \lambda' + c(\lambda)\omega_1,$$

where $\omega_1 = (1, 0, \ldots, 0)$ is the highest weight of the standard representation of GL($r-1$), and

$$c(\lambda) = 1 + (a_1 - a_2). \tag{6}$$

Both λ' and λ'' are the highest weights of some GL($r-1$) irreducible modules. (Note: $0'' \neq 0$.)

Expanding S_λ as a polynomial in x_1 we obtain,

$$\begin{aligned}S_\lambda(x_1, \ldots, x_r) =&(-1)^{r+1}x_1^{a_1+r-1}S_{\lambda'}(x_2, \ldots, x_r)\\ &+(-1)^r x_1^{a_2+r-2}S_{\lambda''}(x_2, \ldots, x_r) + Q,\end{aligned}$$

where Q is a polynomial whose x_1 degree is less than $a_2 + r - 2$. Substituting in (3), and equating the top degree term, we have an equality

$$x_1^{\sum_j a_{j1}+n(r-1)} S_{\lambda'_1}\cdots S_{\lambda'_n} = x_1^{\sum_j b_{j1}+n(r-1)} S_{\mu'_1}\cdots S_{\mu'_n}.$$

Since S_λ and x_1 are coprime polynomials for normalized λ, we have in particular

$$S_{\lambda'_1} \cdots S_{\lambda'_n} = S_{\mu'_1} \cdots S_{\mu'_n}. \tag{7}$$

Hence by induction it follows that there exists a permutation τ' of the set $\{1, \dots, n\}$ such that,

$$\lambda'_i = \mu'_{\tau'(i)} \quad 1 \leq i \leq n.$$

Remark 3. At this point, with a little bit of extra work, a proof of the main theorem can be given when the number of components n is at most two. Substituting $x_i = t^{i-1}$ for $1 \leq i \leq n$, the determinants S_λ can be evaluated as Vandermonde determinants. Arguing as in the proof of the main theorem for GL(2), it can be seen that we have an equality,

$$\{a_{11}, a_{21}\} = \{b_{11}, b_{21}\}.$$

If we look at the constant coefficients, we obtain an equality,

$$\{a_{11} - a_{12}, a_{21} - a_{22}\} = \{b_{11} - b_{12}, b_{21} - b_{22}\}.$$

These two inequalities combine to prove the main theorem when the number of components is at most two.

Remark 4. The proof does not proceed by restricting to various possible SL(2) mapping to GL(r), and using the theorem for SL(2). For instance, it is not even possible to distinguish between a representation and its dual by restricting to various possible SL(2)'s mapping to GL(r). Morever we have the following example:

Example 2. Consider the following triples of highest weights on GL(3):

$$\begin{array}{lll} \lambda_1 = (3,1,0), & \lambda_2 = (2,2,0), & \lambda_3 = (1,0,0), \\ \mu_1 = (3,2,0), & \mu_2 = (2,0,0), & \mu_3 = (1,1,0). \end{array}$$

The calculations in the foregoing remark give in particular that for any co-root α^*, the sets of integers

$$\{\langle \alpha^*, \lambda_i + \rho \rangle \mid 1 \leq i \leq 3\} = \{\langle \alpha^*, \mu_i + \rho \rangle \mid 1 \leq i \leq 3\}$$

are equal, and it is not possible to differentiate the corresponding sets of highest weights. A calculation with the characters of the tensor product indicates that the term corresponding to the second highest coefficient of x_1 in the corresponding product of the characters is different. This observation motivates the rest of the proof of Theorem 1, with which we continue. Let

$$c_1 = \min\{c(\lambda_i) \mid 1 \leq i \leq n\} \quad \text{and} \quad c_2 = \min\{c(\mu_j) \mid 1 \leq j \leq n\}.$$

Equating the coefficient of the second highest power of x_1 in equation 3, we obtain that $c_1 = c_2 = c$. Let $I, J \subset \{1, \ldots, n\}$ denote the sets where for $i \in I$ and $j \in J$, $c(\lambda_i)$ (resp. $c(\mu_{\tau'(j)})$) attains the minimum value. We obtain on equating the coefficient of the second highest power of x_1 in equation 3:

$$\sum_{i\in I} S_{\lambda'_1} \cdots S_{\lambda'_{i-1}} S_{\lambda'_i + c\omega'_2} S_{\lambda'_{i+1}} \cdots S_{\lambda'_n} = \sum_{j\in J} S_{\lambda'_1} \cdots S_{\lambda'_{j-1}} S_{\lambda'_j + c\omega'_2} S_{\lambda'_{j+1}} \cdots S_{\lambda'_n}.$$

Suppose there are indices $i \in I$ and $j \in J$ such that $\lambda'_i = \mu'_j$ and $a_{i1} - a_{i2} = b_{j1} - b_{j2} = c - 1$. It follows that $\lambda_i = \mu_j$. Cancelling these two factors from the hypothesis of Theorem 3, we can proceed by induction on the number of components to prove Theorem 3. Thus, the proof of Theorem 3 reduces now to the proof of the following key auxiliary lemma (applied to $\mathrm{GL}(r-1)$), generalizing Lemma 1.

LEMMA 2. *Let $\lambda_1, \ldots, \lambda_n$ be a set of normalized weights in P_+. Let d be a positive integer and ω_1 denote the fundamental weight. For any index i, let $E(i) = \{j \mid \lambda_j = \lambda_i\}$. Suppose there is a relation*

$$\sum_{1\le i\le n} z_i S_{\lambda_1} \cdots S_{\lambda_{i-1}} S_{\lambda_i + d\omega_1} S_{\lambda_{i+1}} \cdots S_{\lambda_n} = 0,$$

for some collection of complex numbers z_i. Then for any index i,

$$\sum_{j\in E(i)} z_i = 0.$$

Proof. By Lemma 1, the lemma has been proved for $\mathrm{GL}(2)$. We argue by induction on the number of components n and on r. Let

$$c = \min\{c(\lambda_i) \mid 1 \le i \le n\}.$$

Let M be the subset of $\{1, \ldots, n\}$ consisting of those indices i such that $c = 1 + a_{i1} - a_{i2}$. We expand both sides as a polynomial in x_1, and compare the coefficient of the second highest power of x_1. Since $d > 0$, the term $S_{\lambda_i + d\omega_1}$ in the product $S_{\lambda_1} \cdots S_{\lambda_i + d\omega_1} \cdots S_{\lambda_n}$ does not contribute to the coefficient of the second highest degree term in x_1. Further we observe that $(\lambda_i + d\omega_1)' = \lambda'_i$. Hence the contribution to the coefficient of the second highest degree term in x_1 of $S_{\lambda_1} \cdots S_{\lambda_i + d\omega_1} \cdots S_{\lambda_n}$ is given by,

$$\sum_{k\in M\setminus M\cap\{i\}} S_{\lambda'_1} \cdots S_{\lambda'_k + cf_1} \cdots S_{\lambda'_n},$$

where $f_1 = (1, 0, \ldots, 0)$ is a vector in $\mathbb{Z}^{r-1}$. Hence we obtain,

$$\sum_{i=1}^{n} z_i \sum_{k\in M\setminus M\cap\{i\}} S_{\lambda'_1} \cdots S_{\lambda'_k + cf_1} \cdots S_{\lambda'_n} = 0.$$

Fix an index $i_0 \in M$, and count the number of times $\lambda'_{i_0} + cf_1$ terms occur as a component in the product. For this, any index $i \neq i_0$ will contribute. Hence the above sum can be rewritten as,

$$(n-1) \sum_{i \in M} z_i S_{\lambda'_1} \cdots S_{\lambda'_i + cf_1} \cdots S_{\lambda'_n} = 0. \tag{8}$$

Cancelling those polynomials S'_{λ_l} for $l \notin M$, we have by induction for any index $i_0 \in M$,

$$\sum_{j \in E'(i_0)} z_j = 0,$$

where $E'(i_0) = \{j \in M \mid \lambda'_j = \lambda'_{i_0}\}$. We have assumed that $j \in M$, since we have cancelled those extraneous terms with indices not in M. But since j and i_0 belong to M, we conclude that $\lambda_j = \lambda_{i_0}$. Hence $E'(i_0) = E(i_0)$, and we have proved the lemma. □

Remark 5. The surprising fact is that the induction step, rather than giving expressions where the character sums are spiked at more than one index, actually yields back Equation 8, which is again of the same type as that in the hypothesis of the lemma.

It is not clear whether there is a more general context in which the above result can be placed. For example, fix a weight Λ. Consider the collection of characters

$$\left\{ \prod_{i \in I} \chi_{\lambda_i} \mid I \subset P_+, \sum_{i \in I} \lambda_i = \Lambda \right\}.$$

It is not true that this set of characters is linearly independent, since for a fixed Λ the cardinality of this set grows exponentially (since it is given by the partition function), whereas the dimension of the space of homogeneous polynomials in two variables of fixed degree depends polynomially (in fact linearly) on the degree.

5. Proof of the main theorem in the general case

We now revert to the notation of Section 2. Our aim is to set up the correct formalism in the general case, so that we can carry over the inductive proof for $\mathrm{GL}(n)$ given above. Let $\mathfrak{g}$ be a simple Lie algebra of rank greater than one. Choose a fundamental root $\alpha_1 \in \Delta$. Let $\Delta' = \Delta \backslash \{\alpha_1\}$, and let $\Phi' \subset \Phi$ be the subset of roots lying in the span of the roots generated by Δ'. Let

$$\mathfrak{h}' = \sum_{\alpha \in \Delta'} \mathbb{C}\alpha^*.$$

It is known that Δ' is a base for the semisimple Lie algebra $\mathfrak{g}'$ defined by,

$$\mathfrak{g}' := \mathfrak{h}' \oplus \sum_{\alpha \in \Phi'} \mathfrak{g}^{\alpha},$$

where $\mathfrak{g}^{\alpha}$ is the weight space of α corresponding to the adjoint action of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}'$ is a semisimple Lie algebra of rank $l-1$, and the roots of $(\mathfrak{g}', \mathfrak{h}')$ can be identified with Φ'. A special case is when α_1 corresponds to a corner vertex in the Dynkin diagram of $\mathfrak{g}$. In this case, $\mathfrak{g}'$ will be a simple Lie algebra.

We now want to find a suitable $\mathfrak{gl}_1$ complement to $\mathfrak{g}'$ inside $\mathfrak{g}$. This is given by the following lemma:

LEMMA 3. $\omega_1^* \notin \mathfrak{h}'$.

Proof. Suppose we can write,

$$\omega_1^* = \sum_{i=2}^{l} a_i \alpha_i^*,$$

for some complex numbers a_i. By definition of ω_1^*, we obtain on pairing with α_j for $2 \leq j \leq l$, the following system of $l-1$ linear equations, in the unknown a_i:

$$\sum_{i=2}^{l} \langle \alpha_i^*, \alpha_j \rangle a_i = 0.$$

But the matrix $(\langle \alpha_i^*, \alpha_j \rangle)_{2 \leq i,\ j \leq l}$ is the Cartan matrix of the semisimple Lie algebra $\mathfrak{g}'$, and hence is nonsingular. Thus $a_i = 0$ for $i = 2, \ldots, l$, and that is a contradiction as ω_1^* is nonzero. $\square$

Let W' denote the Weyl group of $(\mathfrak{g}', \mathfrak{h}')$, which can be identified with the subgroup of W generated by the fundamental reflections s_α for $\alpha \in \Delta'$. For such an α, we have

$$\langle s_\alpha(\omega_1^*), \alpha_j \rangle = \langle \omega_1^*, \alpha_j \rangle - \langle \omega_1^*, \alpha \rangle \langle \alpha^*, \alpha_j \rangle = \delta_{1j}.$$

Hence W' fixes ω_1^*. Conversely, it follows from the fact that ω_1^* is orthogonal to all the roots $\alpha_2, \ldots, \alpha_l$, that any element of the Weyl group fixing ω_1^* lies in the subgroup of W generated by the simple reflections s_{α_i}, $2 \leq i \leq l$, and hence lies in W' [C, Lemma 2.5.3].

Our next step is to study the restriction of the character χ_λ to $\mathfrak{g}' \oplus \mathfrak{gl}_1 \subset \mathfrak{g}$. Let P' denote the lattice of weights of $\mathfrak{g}'$. We consider P' as a subgroup of P, consisting of those weights which vanish when evaluated on ω_1^*.

Choose a natural number m such that $\pi(\omega_1^*) \in \frac{1}{m}\mathbb{Z}$ for all weights $\pi \in P$. Let Z_1 be the subgroup, isomorphic to the integers, of linear forms 1 on $\mathfrak{h}$, which are trivial on $\mathfrak{h}'$ and such that $1(\omega_1^*) \in \frac{1}{m}\mathbb{Z}$. We have

$$P \subset Z_1 \oplus P',$$

and we decompose the character χ_λ with respect to this direct sum decomposition. Given a weight $\pi \in P$, denote by $\pi' \in P'$ its restriction to $\mathfrak{h}'$. Let l_1 be the weight in P, vanishing on $\mathfrak{h}'$ and taking the value 1 on ω_1^*. We define

$$d_1(\pi) = \pi(\omega_1^*)$$

as the *degree* of π along l_1. Write any weight π with respect to the above decomposition as,

$$\pi = d_1(\pi)l_1 + \pi' \quad \text{so that} \quad e^\pi = e^{d_1(\pi)l_1}e^{\pi'}.$$

The numerator of the Weyl character formula decomposes as,

$$S_\lambda = \sum_{d \in \frac{1}{m}\mathbb{Z}} e^{dl_1}\left(\sum_{w \in W_d} \varepsilon(w)e^{w(\lambda+\rho)'}\right), \tag{9}$$
$$\text{where} \quad W_d = \{w \in W \mid (w(\lambda+\rho))(\omega_1^*) = d\}.$$

We refer to the inner sum as the coefficient of the degree d component along l_1, or as the coefficient of e^{dl_1}. Given a dominant integral weight $\lambda \in P_+$, define

$$a_1(\lambda) = \max\{w\lambda(\omega_1^*) \mid w \in W\},$$
$$a_2(\lambda) = \max\{w\lambda(\omega_1^*) \mid w \in W \text{ and } w\lambda(\omega_1^*) \neq a_1(\lambda)\}.$$

The formalism that we require in order to carry over the proof for $\mathrm{GL}(n)$ to the general case, is given by the following lemma:

Lemma 4. *Let λ be a regular weight in P_+.*

(1) *The largest value $a_1(\lambda)$ of $(w\lambda)(\omega_1^*)$ for $w \in W$, is attained precisely for $w \in W'$. In particular,*

$$a_1(\lambda) = \lambda(\omega_1^*).$$

(2) *The second highest value $a_2(\lambda)$ is attained precisely for w in $W's_{\alpha_1}$, and the value is given by*

$$a_2(\lambda) = s_{\alpha_1}\lambda(\omega_1^*) = a_1(\lambda) - \lambda(\alpha_1^*).$$

Proof. 1) By [C, Lemma 2.5.3], we have to show that if $w\lambda(\omega_1^*)$ attains the maximum value, then w fixes ω_1^*. Since ω_1^* is a fundamental co-weight, we have

$$\omega_1^* - w\omega_1^* = \sum_{i=1}^{l} n_i\alpha_i^*,$$

for some nonnegative natural numbers n_i. Hence,

$$\lambda(\omega_1^* - w\omega_1^*) = \sum_{i=1}^{l} n_i\lambda(\alpha_i^*),$$

and the latter expression is strictly positive, if some $n_i > 0$, since λ is regular and dominant. This proves the first part.

2) We prove the second part by induction on the length $l(w)$ of w. The statement is clear for the fundamental reflections, which are of length one. Consider an element of the form ws_β such that $ws_\beta\lambda(\omega_1^*) = a_2(\lambda)$, β is a fundamental root and $l(ws_\beta) = l(w) + 1$. By [C, Lemma 2.2.1 and Th. 2.2.2], it follows that,

$$w(\beta) \in \Phi^+.$$

We have

$$\begin{aligned}\lambda - ws_\beta\lambda &= (\lambda - w\lambda) + w(\lambda - s_\beta\lambda)\\ &= (\lambda - w\lambda) + \langle\beta^*, \lambda\rangle w\beta.\end{aligned}$$

Since $w\beta \in \Phi^+$, and $\langle\beta^*, \lambda\rangle$ is positive, it follows that

$$ws_\beta\lambda(\omega_1^*) \leq w\lambda(\omega_1^*),$$

with strict inequality if $\langle\omega_1^*, w\beta\rangle$ is positive. Hence by induction we can assume that w either belongs to W', or is of the form $w_0 s_{\alpha_1}$, with $w_0 \in W'$. We only have to consider the second possiblity. We obtain,

$$(\lambda - w\lambda)(\omega_1^*) = (\lambda - s_{\alpha_1}\lambda)(\omega_1^*) = \langle\alpha_1^*, \lambda\rangle > 0.$$

Assuming the hypothesis of Lemma 4 for ws_β, we see that $ws_\beta(\omega_1^*) = w(\omega_1^*)$, and hence obtain that $s_{\alpha_1}(\beta)$ has no α_1 component when we expand it as a linear combination of the fundamental roots. But

$$s_{\alpha_1}(\beta) = \beta - \langle\alpha_1^*, \beta\rangle\alpha_1,$$

and it follows that $\langle\alpha_1^*, \beta\rangle = 0$. From the relations defining the Weyl group, it follows that s_{α_1} and s_β commute, and hence the element ws_β is of the form $w_1 s_{\alpha_1}$ for some element $w_1 \in W'$; this concludes the proof of the lemma. □

The restriction of the fundamental weights $\omega_2, \dots, \omega_l$ to $\mathfrak{h}'$ are the fundamental weights (with the indexing set ranging from $2, \dots, l$, instead of $1, \dots, l-1$) of $\mathfrak{g}'$. In particular, the restriction ρ' of ρ is the sum of the fundamental weights of $\mathfrak{g}'$. For $\lambda \in P_+$, define $\lambda'' \in P'_+$ by,

$$\lambda'' = (s_{\alpha_1}(\lambda + \rho))' - \rho'. \tag{10}$$

We have the following corollary, giving the character expansion for the first two terms along l_1:

Corollary 2. *With notation as in the character expansion given by equation* (9),

$$S_\lambda = e^{a_1(\lambda+\rho)l_1} S_{\lambda'} - e^{a_2(\lambda+\rho)l_1} S_{\lambda''} + L(\lambda), \tag{11}$$

where $L(\lambda)$ denotes the terms of degree along l_1 less than the second highest degree.

Proof. The proof is immediate from Lemma 4 and equation 9, when $\lambda+\rho$ is regular. The second term has the opposite sign, since $l(ws_{\alpha_1}) = l(w)+1$, for $w \in W'$, and the length function of W restricts to the length function of W', taken with respect to Δ and Δ' respectively. $\square$

We write these facts down explicitly in terms of the fundamental weights.

$$\lambda = n_1(\lambda)\omega_1 + \cdots + n_l(\lambda)\omega_l,$$

in terms of the fundamental weights, so that $\lambda+\rho = (n_1(\lambda)+1)\omega_1 + \cdots + (n_l(\lambda)+1)\omega_l$. Now,

$$(\lambda+\rho)' = (n_2(\lambda)+1)\omega_2' + \cdots + (n_l(\lambda)+1)\omega_l'. \tag{12}$$

$$\text{Let} \quad \mathfrak{g}' \simeq \oplus_{s\in S}\mathfrak{g}_s',$$

be the decomposition of $\mathfrak{g}'$ into simple Lie algebras. For each simple component $\mathfrak{g}_s'$ of $\mathfrak{g}'$, let α_s be the unique simple root connected to α_1 in the Dynkin diagram of $\mathfrak{g}$. Then

$$-\langle \alpha_1^*, \alpha_s\rangle = m_{1s},$$

is positive for each s. This is possible since we have assumed that $\mathfrak{g}$ is a *simple* Lie algebra of larger rank. A calculation yields,

$$s_{\alpha_1}(\lambda+\rho)' = \sum_{s\in S}(n_s(\lambda)+1+m_{1s}(n_1(\lambda)+1))\omega_s' + \sum_{t\in\Delta\setminus S} n_t\omega_t'. \tag{13}$$

For example, if α_1 is a corner root, then $\mathfrak{g}'$ is simple. Let α_2 be the root adjacent to α_1. In this particular case, we have

$$\lambda'' = \lambda' + m_{12}(n_1(\lambda)+1)\omega_2'. \tag{14}$$

We are now in a position to prove the main theorem, the proof of which is along the same lines as the proof for $\mathrm{GL}(n)$. We assume that there is an equality as in equation (3):

$$S_{\lambda_1}\dots S_{\lambda_n} = S_{\mu_1}\dots S_{\mu_n}. \tag{15}$$

We now choose a corner root α_1, and from equation (2), we obtain,

$$\prod_{i=1}^{n}(e^{a_1(\lambda_i+\rho)l_1}S_{\lambda_i'} - e^{a_2(\lambda_i+\rho)l_1}S_{\lambda_i''} + L(\lambda_i))$$
$$= \prod_{i=1}^{n}(e^{a_1(\mu_i+\rho)l_1}S_{\mu_i'} - e^{a_2(\mu_i+\rho)l_1}S_{\mu_i''} + L(\mu_i)).$$

On taking products and comparing the coefficients of the topmost degree, we get,

$$S_{\lambda_1'}\dots S_{\lambda_n'} = S_{\mu_1'}\dots S_{\mu_n'}.$$

By induction, we can thus assume that up to a permutation there is an equality,

$$(\lambda'_1, \dots, \lambda'_n) = (\mu'_1, \dots, \mu'_n). \tag{16}$$

Now we compare the term contributing to the second highest degree in the product. Let I (resp. J) consist of those indices in the set $\{1, \dots, n\}$, for which $n_1(\lambda_i) = \lambda_i(\alpha_1^*)$ (resp. $n_1(\mu_i)$) is minimum. By Part (2) of Lemma 4 and the character expansion as given by Corollary 2, the second highest degree along l_1 in the product is of degree $n_1(\lambda + \rho) = n_1(\lambda) + 1$ less than the total degree. In particular the minimum of $n_1(\lambda_i)$ and the minimum of $n_1(\mu_j)$ coincide, as we vary over the indices. We have the following equality of the second highest degree terms along l_1:

$$\sum_{i \in I} S_{\lambda'_1} \cdots S_{\lambda'_{i-1}} S_{\lambda''_i} S_{\lambda'_{i+1}} \cdots S_{\lambda'_n} = \sum_{j \in J} S_{\mu'_1} \cdots S_{\mu'_{j-1}} S_{\mu''_j} S_{\mu'_{j+1}} \cdots S_{\mu'_n}.$$

From equations (14), and (16), we can recast this equality as,

$$\begin{aligned} &\sum_{i \in I} S_{\lambda'_1} \cdots S_{\lambda'_{i-1}} S_{\lambda'_i + d\omega'_2} S_{\lambda'_{i+1}} \cdots S_{\lambda'_n} \\ &\quad = \sum_{j \in J} S_{\lambda'_1} \cdots S_{\lambda'_{j-1}} S_{\lambda'_j + d\omega'_2} S_{\lambda'_{j+1}} \cdots S_{\lambda'_n} \end{aligned} \tag{17}$$

where $d = m_{12}(n_1(\lambda_i) + 1)$ is a positive integer, since $\mathfrak{g}$ has been assumed to be simple.

Granting Lemma 5 given below, the main theorem follows, since we have indices $i_0,\ j_0$ such that,

$$\lambda'_{i_0} = \mu'_{j_0} \quad \text{and} \quad n_1(\lambda_{i_0}) = n_1(\mu_{j_0}),$$

where the indices $i_0,\ j_0$ are such that the minimum of $n_1(\lambda_i)$ and $n_1(\mu_j)$ is attained. Hence we have,

$$\lambda_{i_0} = \mu_{j_0}.$$

Cancelling these terms from equation (3), we are left with an equality where the number of components occurring in the tensor product in the hypothesis of the main theorem is less than the one we started with, and an induction on the number n of components in the tensor product proves the main theorem.

Remark 6. When the number of components is at most two, we do not need Lemma 5. In the above equality (17), we can assume that $I \cap J$ is empty, and so can take for example $I = \{1\}$ and $J = \{2\}$, to obtain,

$$S_{\lambda'_1 + d\omega'_2} S_{\lambda'_2} = S_{\lambda'_1} S_{\lambda'_2 + d\omega'_2}.$$

By induction on the rank, assuming that the main theorem is true with number of components at most two, we obtain the main theorem for all $\mathfrak{g}$.

To complete the proof of the theorem, we have to state the auxiliary linear independence property, generalizing Lemmas 1 and 2.

LEMMA 5. *Let $\mathfrak{g}$ be a simple Lie algebra, and let $\lambda_1, \dots, \lambda_n$ be a set of dominant, integral weights in P_+. Let d be a positive integer and ω_p denote a fundamental weight corresponding to the root α_p. For any index i, let $E(i) = \{j \mid \lambda_j = \lambda_i\}$. Suppose there is a relation*

$$\sum_{1\le i\le n} z_i S_{\lambda_1} \cdots S_{\lambda_{i-1}} S_{\lambda_i + d\omega_p} S_{\lambda_{i+1}} \cdots S_{\lambda_n} = 0,$$

for some collection of complex numbers z_i. Then for any index i,

$$\sum_{j\in E(i)} z_i = 0.$$

Remark 7. Instead of ω_p, we can spike up the equation with any nonzero highest weight λ, but the proof is essentially the same.

The proof of this lemma will be by induction on the rank. For simple Lie algebras not of type D or E, and if ω_p is a fundamental weight corresponding to a corner root in the Dynkin diagram of $\mathfrak{g}$, the proof follows along the same lines as in the proof of Lemma 2, and that is sufficient to prove the main theorem in these cases. For Lie algebras of type D and E, the proof becomes complicated, due to the fact that the root adjacent to a corner root α_1 in the Dynkin diagram of $\mathfrak{g}$ need not be a corner root in the Dynkin diagram associated to $\Delta\backslash\{\alpha_1\}$. Before embarking on a proof of this lemma, we will need a preliminary lemma.

LEMMA 6. *Assume that Lemma* 5 *holds for all simple Lie algebras of rank at most l. Let $\oplus_{s\in S}\mathfrak{g}_s$ be a direct sum of simple Lie algebras of $\mathfrak{g}_s$ of rank at most l. For each $s \in S$, assume that we are given dominant, integral weights $\lambda_{s1}, \dots, \lambda_{sn}$ of $\mathfrak{g}_s$, a positive integer d_s, and a fundamental weight ω_s of $\mathfrak{g}_s$. Suppose that we have a relation,*

$$\sum_{1\le i\le n} z_i S_{\Lambda_1} \cdots S_{\Lambda_{i-1}} S_{\hat{\Lambda}_i} S_{\Lambda_{i+1}} \cdots S_{\Lambda_n} = 0, \tag{18}$$

for some collection of complex numbers z_i, where for $1 \le i \le n$

$$S_{\Lambda_i} = \prod_{s\in S} S_{\lambda_{si}},$$

$$\textit{and}\quad S_{\hat{\Lambda}_i} = \prod_{s\in S} S_{\lambda_{si}+d_s\omega_s}.$$

Then for any index i,

$$\sum_{j\in E(i)} z_j = 0,$$

where $E(i) = \{j \mid (\lambda_{sj})_{s\in S} = (\lambda_{si})_{s\in S}\}$.

Proof. The proof proceeds by induction on the cardinality of S. Consider equation (18) as an equation with respect to one of the simple Lie algebras, say $\mathfrak{g}_1$. The linear independence property reduces to the case when the number of simple Lie algebras involved is one less, and we are through by induction. $\square$

Now we get back to the proof of Lemma 5.

Proof. By Lemma 1, the lemma has been proved for GL(2). We argue by induction on the number of components n and on the rank l of $\mathfrak{g}$. We assume that the lemma has been proved for all simple Lie algebras of rank less than the rank of $\mathfrak{g}$. We use the character expansion given by Corollary 2, where we denote by l_p the linear functional corresponding to ω_p^*. Let

$$\mathfrak{g}' \simeq \oplus_{s\in S}\mathfrak{g}_s,$$

be the decomposition of $\mathfrak{g}'$ into simple Lie algebras (we remove the fundamental root α_p corresponding to the fundamental weight ω_p from the Dynkin diagram). For each $s \in S$, let $\alpha_s \in \Delta$ be the root adjacent to α_p.

Let M be the subset of $\{1,\dots,n\}$ consisting of those indices i such that $n_p(\lambda_i)$ attains the minimum. For each $s \in S$, let

$$c_s = \min\{m_{ps}(n_p(\lambda_i)+1) \mid i \in M\}.$$

We expand both sides using Corollary (2), and compare the coefficient of the second highest degree along l_p. Since $d > 0$, the term $S_{\lambda_i+d\omega_p}$ in the product $S_{\lambda_1}\cdots S_{\lambda_i+d\omega_p}\cdots S_{\lambda_n}$ does not contribute to the coefficient of the second highest degree term in l_p. Further we observe that $(\lambda_i+d\omega_1)' = \lambda_i'$. Hence the contribution to the coefficient of the second highest degree term in l_p of $S_{\lambda_1}\cdots S_{\lambda_i+d\omega_p}\cdots S_{\lambda_n}$ is given by,

$$\sum_{k\in M\setminus M\cap\{i\}} S_{\lambda_1'}\cdots S_{\lambda_{k-1}'}S_{\lambda_k''}S_{\lambda_{k+1}'}\cdots S_{\lambda_n'},$$

where λ_k'' is given by equation (13) as follows:

$$\lambda_k''+\rho' = \sum_{s\in S}(n_s(\lambda_k)+1+m_{ps}(n_p(\lambda_k)+1))\omega_s' + \sum_{t\in\Delta\setminus S} n_t\omega_t'.$$

Since $\mathfrak{g}$ is simple and $\lambda_k+\rho$ is regular, we notice that the term $m_{ps}(n_p(\lambda_k+\rho))$ is always positive. On rearranging the sum, we obtain,

$$\sum_{i=1}^{n} z_i \sum_{k\in M\setminus M\cap\{i\}} S_{\lambda_1'}\cdots S_{\lambda_k''}\cdots S_{\lambda_n'} = 0.$$

Fix an index $i_0 \in M$, and count the number of times a given index λ_{i_0}'' occurs as a component in the product. For this, any index $i \neq i_0$ will contribute. Hence the above sum can be rewritten as,

$$(n-1)\sum_{i\in M} z_i S_{\lambda_1'}\cdots S_{\lambda_i''}\cdots S_{\lambda_n'} = 0.$$

Cancel those polynomials S'_{λ_l} for which $l \notin M$. We have by Lemma 6, for any index $i_0 \in M$,

$$\sum_{j \in E'(i_0)} z_j = 0,$$

where $E'(i_0) = \{j \in M \mid \lambda'_j = \lambda'_{i_0}\}$ where we can assume that $j \in M$, since we have cancelled those extraneous terms with indices not in M. But since j and i_0 belong to M and j is in $E(i_0)$, we conclude that $\lambda_j = \lambda_{i_0}$. Hence $E'(i_0) = E(i_0)$, and we have proved the lemma. □

Remark 8. The main theorem indicates the presence of an 'irreduciblity property' for the characters of irreducible representations of simple algebraic groups. However the naive feeling that the characters of irreducible representations are irreducible is false. This can be seen easily for $\mathfrak{sl}_2$. For $\mathrm{GL}(n)$, consider a pair of highest weights of the form,

$$\mu = ((n-1)a, (n-2)a, \dots, a, 0) \quad \text{and} \quad \lambda = ((n-1)b, (n-2)b, \dots, b, 0),$$

for some positive integers k, a, b. Then the characters can be expanded as Vandermonde determinants and we have,

$$\begin{aligned} S_\mu &= \prod_{i<j} (x_i^{a+1} - x_j^{a+1}) \\ \text{and} \quad S_\lambda &= \prod_{i<j} (x_i^{b+1} - x_j^{b+1}). \end{aligned}$$

Thus we see that S_μ divides S_λ if $(a+1)|(b+1)$.

It would be of interest to give necessary and sufficient criteria on the highest weights μ and λ to ensure that S_μ divides S_λ.

6. An arithmetical application

We present here an arithmetical application to recovering l-adic representations. Corollary 1 was motivated by the question of knowing the relationship between two l-adic representations given that their adjoint representations are isomorphic. On the other hand, the application to generalised Asai representations given below was suggested by the work of D. Ramakrishnan. We refer to [R] for more details.

Let K be a global field and let G_K denote the Galois group over K of an algebraic closure $\bar{K}$ of K. Let F be a non-archimedean local field of characteristic zero. Suppose

$$\rho_i : G_K \to \mathrm{GL}_n(F), \quad i = 1, 2$$

are continuous, semisimple representations of the Galois group G_K into $\mathrm{GL}_n(F)$, unramified outside a finite set S of places containing the archimedean places of K. Given ρ, let χ_ρ denote the character of ρ. For each finite place v of K, we choose a place $\bar{v}$ of $\bar{K}$ dividing v, and let $\sigma_{\bar{v}} \in G_K$ be the corresponding Frobenius element. If v is unramified, then the value $\chi_\rho(\sigma_{\bar{v}})$ depends only on v and not on the choice of $\bar{v}$, and we will denote this value by $\chi_\rho(\sigma_v)$.

Given an l-adic representation ρ of G_K, we can construct other naturally associated l-adic representations. We consider here two such constructions: the first one, is given by the adjoint representation

$$\mathrm{Ad}(\rho) = \rho \otimes \rho^* : G_K \to \mathrm{GL}_{n^2}(F),$$

where ρ^* denotes the contragredient representation of ρ.

The second construction is a generalisation of Asai representations. Let K/k be a Galois extension with Galois group $G(K/k)$. Given ρ, we can associate the *pre-Asai representation*,

$$\mathrm{As}(\rho) = \otimes_{g \in G(K/k)} \rho^g,$$

where $\rho^g(\sigma) = \rho(\tilde{g}\sigma\tilde{g}^{-1})$, $\sigma \in G_K$, and where $\tilde{g} \in G_k$ is a lift of $g \in G(K/k)$. At an unramified place v of K, which is split completely over a place u of k, the Asai character is given by,

$$\chi_{\mathrm{As}(\rho)}(\sigma_v) = \prod_{v|u} \chi_\rho(\sigma_v).$$

Hence, up to isomorphism, $\mathrm{As}(\rho)$ does not depend on the choice of the lifts $\tilde{g}$. If further $\mathrm{As}(\rho)$ is irreducible, and K/k is cyclic, then $\mathrm{As}(\rho)$ extends to a representation of G_k (called the Asai representation associated to ρ when $n = 2$ and K/k is quadratic).

THEOREM 5. *Let*

$$\rho_i : G_K \to \mathrm{GL}_n(F), \quad i = 1, 2,$$

be continuous, irreducible representations of the Galois group G_K into $\mathrm{GL}_n(F)$. Let R be the representation $\mathrm{Ad}(\rho_i)$ (adjoint case) or $\mathrm{As}(\rho_i)$ (Asai case) associated to ρ_i, $i = 1, 2$.

Suppose that the set of places v of K not in S, where

$$\mathrm{Tr}(R \circ \rho_1(\sigma_v)) = \mathrm{Tr}(R \circ \rho_2(\sigma_v)),$$

is a set of places of positive density. Assume further that the algebraic envelope of the image of ρ_1 and ρ_2 is connected and that the derived group is absolutely almost simple. Then the following holds:

(1) *(Adjoint case) There is a character $\chi : G_K \to F^*$ such that ρ_2 is isomorphic to either $\rho_1 \otimes \chi$ or to $\rho_1^* \otimes \chi$.*

(2) *(Asai case) There are a character $\chi : G_K \to F^*$, and an element $g \in G(K/k)$ such that ρ_2 is isomorphic to $\rho_1^g \otimes \chi$.*

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