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On a vanishing conjecture appearing in the geometric Langlands correspondence

By D. Gaitsgory*

Introduction

0.1. This paper should be regarded as a sequel to [7]. There it was shown that the geometric Langlands conjecture for GL_n follows from a certain vanishing conjecture. The goal of the present paper is to prove this vanishing conjecture.

Let X be a smooth projective curve over a ground field k. Let E be an m-dimensional local system on X, and let Bun_m be the moduli stack of rank m vector bundles on X.

The geometric Langlands conjecture says that to E we can associate a perverse sheaf $\mathcal{F}_E$ on Bun_m, which is a Hecke eigensheaf with respect to E.

The vanishing conjecture of [7] says that for all integers $n < m$, a certain functor Av^d_E, depending on E and a parameter $d \in \mathbb{Z}^+$, which maps the category $\mathrm{D}(\mathrm{Bun}_n)$ to itself, vanishes identically, when d is large enough.

The fact that the vanishing conjecture implies the geometric Langlands conjecture may be regarded as a geometric version of the converse theorem. Moreover, as will be explained in the sequel, the vanishing of the functor Av^d_E is analogous to the condition that the Rankin-Selberg convolution of E, viewed as an m-dimensional Galois representation, and an automorphic form on GL_n with $n < m$ is well-behaved.

Both the geometric Langlands conjecture and the vanishing conjecture can be formulated in any of the sheaf-theoretic situations, e.g., $\overline{\mathbb{Q}}_\ell$-adic sheaves (when $\mathrm{char}(k) \neq \ell$), D-modules (when $\mathrm{char}(k) = 0$), and sheaves with coefficients in a finite field $\mathbb{F}_\ell$ (again, when $\mathrm{char}(k) \neq \ell$).

When the ground field is the finite field $\mathbb{F}_q$ and we are working with ℓ-adic coefficients, it was shown in [7] that the vanishing conjecture can be deduced from Lafforgue's theorem that establishes the full Langlands correspondence for global fields of positive characteristic; cf. [9].

*The author is a prize fellow at the Clay Mathematics Institute.

The proof to be given in this paper treats the cases of various ground fields and coefficients uniformly, and in particular, it will be independent of Lafforgue's results.

However, we will be able to treat only the case of characteristic 0 coefficients, or, more generally, the case of $\mathbb{F}_\ell$-coefficients when ℓ is $> d$, where d is the parameter appearing in the formulation of the vanishing conjecture.

0.2. Let us briefly indicate the main steps of the proof. First, we show that instead of proving that the functor Av^d_E vanishes, it is sufficient to prove that it is exact, i.e., that it maps perverse sheaves to perverse sheaves. The $\{$ exactness $\} \to \{$ vanishing $\}$ implication is achieved by an argument involving the comparison of Euler-Poincaré characteristics of complexes obtained by applying the functor Av^d_E for various local systems E of the same rank.

Secondly, we show that the functor Av^d_E can be expressed in terms of the "elementary" functor Av^1_E using the action of the symmetric group Σ_d. (It is this step that does not allow one to treat the case of $\mathbb{F}_\ell$-coefficients if $\ell \leq d$.)

Thirdly, we define a certain quotient triangulated category $\widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ of $\mathrm{D}(\mathrm{Bun}_n)$ by "killing" objects that one can call degenerate. (This notion of degeneracy is spelled out using what we call Whittaker functors.)

The main properties of the quotient $\widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ are as follows: (0) $\widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ inherits the perverse t-structure from $\mathrm{D}(\mathrm{Bun}_n)$, (1) the Hecke functors defined on $\mathrm{D}(\mathrm{Bun}_n)$ descend to $\widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ and are exact, and (2) the subcategory of objects of $\mathrm{D}(\mathrm{Bun}_n)$ that map to 0 in $\widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ is orthogonal to cuspidal complexes.

Next we show that properties (0) and (1) above and the irreducibility assumption on E formally imply that the elementary functor Av^1_E is exact on the quotient category. From that, we deduce that the functor Av^d_E is also exact modulo the subcategory of degenerate sheaves.

Finally, by induction on n we show that Av^d_E maps $\mathrm{D}(\mathrm{Bun}_n)$ to the subcategory of cuspidal sheaves, and, using property (2) above, we deduce that once Av^d_E is exact modulo degenerate sheaves, it must be exact.

0.3. Let us now explain how the the paper is organized. In Section 1 we recall the formulation of the vanishing conjecture. In addition, we discuss some properties of the Hecke functors.

In Section 2 we outline the proof of the vanishing conjecture, parallel to what we did above. We reduce the proof to two statements: one is Theorem 2.14 which says that the functor Av^1_E is exact on the quotient category, and the other is the existence of the quotient category $\widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ with the desired properties.

In Section 3 we prove Theorem 2.14. Sections 4–8 are devoted to the construction of the quotient category and verification of the required properties. Let us describe the main ideas involved in the construction.

We start with some motivation from the theory of automorphic functions, following [12] and [13].

Let $\mathcal{K}$ be a global field, and $\mathbb{A}$ the ring of adeles. Let P be the mirabolic subgroup of GL_n. It is well-known that there is an isomorphism between the space of cuspidal functions on $P(\mathcal{K})\backslash\mathrm{GL}_n(\mathbb{A})$ and the space of Whittaker functions on $N(\mathcal{K})\backslash\mathrm{GL}_n(\mathbb{A})$, where $N \subset \mathrm{GL}_n$ is the maximal unipotent subgroup. Moreover, this isomorphism can be written as a series of $n-1$ Fourier transforms along the topological group $\mathcal{K}\backslash\mathbb{A}$.

In Sections 4 and 5 we develop the corresponding notions in the geometric context. For us, the space of functions on $P(\mathcal{K})\backslash\mathrm{GL}_n(\mathbb{A})$ is replaced by the category $\mathrm{D}(\mathrm{Bun}'_n)$, and the space of Whittaker functions is replaced by a certain subcategory in $\mathrm{D}(\overline{\mathcal{Q}})$ (cf. Section 4, where the notation is introduced).

The main result of these two sections is that there exists an exact "Whittaker" functor $W : \mathrm{D}(\mathrm{Bun}'_n) \to \mathrm{D}^W(\overline{\mathcal{Q}})$. The exactness is guaranteed by an interpretation of W as a series of Fourier-Deligne transform functors.

In Section 6 we show that the kernel $\ker(W) \subset \mathrm{D}(\mathrm{Bun}'_n)$ is orthogonal to the subcategory $\mathrm{D}_{\mathrm{cusp}}(\mathrm{Bun}'_n)$ of cuspidal sheaves.

In Section 7 we define the action of the Hecke functors on $\mathrm{D}(\mathrm{Bun}'_n)$ and $\mathrm{D}^W(\overline{\mathcal{Q}})$, and show that the Whittaker functor W commutes with the Hecke functors. The key result of this section is Theorem 7.8, which says that the Hecke functor acting on $\mathrm{D}^W(\overline{\mathcal{Q}})$ is right-exact. This fact ultimately leads to the desired property (1) above, that the Hecke functor is exact on the quotient category.

Finally, in Section 8 we define our quotient category $\widetilde{\mathrm{D}}(\mathrm{Bun}_n)$.

0.4 *Conventions*. In the main body of the paper we will be working over a ground field k of positive characteristic p (which can be assumed algebraically closed) and with ℓ-adic sheaves. All the results carry over automatically to the D-module context for schemes over a ground field of characteristic 0, where instead of the Artin-Schreier sheaf we use the corresponding D-module "e^x" on the affine line. This paper allows us to treat the case of $\mathbb{F}_\ell$ coefficients, when $\ell > d$ (cf. below) in exactly the same manner.

We follow the conventions of [7] in everything related to stacks and derived categories on them. In particular, for a stack $\mathcal{Y}$ *of finite type*, we will denote by $\mathrm{D}(\mathcal{Y})$ the corresponding bounded derived category of sheaves on $\mathcal{Y}$. If $\mathcal{Y}$ is of infinite type, but has the form $\mathcal{Y} = \underset{i}{\cup}\mathcal{Y}_i$, where $\mathcal{Y}_i$ is an increasing family of open substacks of finite type (the basic example being Bun_n), $\mathrm{D}(\mathcal{Y})$ is by definition the inverse limit of $\mathrm{D}(\mathcal{Y}_i)$.

Throughout the paper we will be working with the perverse t-structure on $\mathrm{D}(\mathcal{Y})$, and will denote by $\mathrm{P}(\mathcal{Y}) \subset \mathrm{D}(\mathcal{Y})$ the abelian category of perverse sheaves. For $\mathcal{F} \in \mathrm{D}(\mathcal{Y})$, we will denote by $h^i(\mathcal{F})$ its perverse cohomology sheaves.

For a map $\mathcal{Y}_1 \to \mathcal{Y}_2$ and $\mathcal{F} \in \mathrm{D}(\mathcal{Y}_2)$ we will sometimes write $\mathcal{F}|_{\mathcal{Y}_1}$ for the $*$ pull-back of $\mathcal{F}$ on $\mathcal{Y}_1$.

For a group Σ acting on $\mathcal{Y}$ we will denote by $\mathrm{D}^{\Sigma}(\mathcal{Y})$ the corresponding equivariant derived category. In most applications, the group Σ will be finite, which from now on we will assume.

If the action of Σ on $\mathcal{Y}$ is trivial, we have the natural functor of invariants $\mathcal{F} \mapsto (\mathcal{F})^{\Sigma} : \mathrm{D}^{\Sigma}(\mathcal{Y}) \to \mathrm{D}(\mathcal{Y})$. This functor is *exact* when we work with coefficients of characteristic zero, or when the order of Σ is co-prime with the characteristic.

The exactness of this functor is crucial for this paper, and it is the reason why we have to assume that $\ell > d$, since the finite groups in question will be the symmetric groups $\Sigma_{d'}$, $d' \leq d$.

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1. The conjecture

1.1. We will first recall the formulation of the Vanishing Conjecture, as it was stated in [7]. Let Bun_n be the moduli stack of rank n vector bundles on our curve X. Let Mod_n^d denote the stack classifying the data of $(\mathcal{M}, \mathcal{M}', \beta)$, where $\mathcal{M}, \mathcal{M}' \in \mathrm{Bun}_n$, and β is an embedding $\mathcal{M} \hookrightarrow \mathcal{M}'$ as coherent sheaves, and the quotient $\mathcal{M}'/\mathcal{M}$ (which is automatically a torsion sheaf) has length d.

We have the two natural projections

$$\mathrm{Bun}_n \xleftarrow{\overleftarrow{h}} \mathrm{Mod}_n^d \xrightarrow{\overrightarrow{h}} \mathrm{Bun}_n,$$

which remember the data of $\mathcal{M}$ and $\mathcal{M}'$, respectively.

Let $X^{(d)}$ denote the d-th symmetric power of X. We have a natural map $s : \mathrm{Mod}_n^d \to X^{(d)}$, which sends a triple $(\mathcal{M}, \mathcal{M}', \beta)$ to the divisor of the map $\Lambda^n(\mathcal{M}) \to \Lambda^n(\mathcal{M}')$. In addition, we have a smooth map $\mathfrak{s} : \mathrm{Mod}_n^d \to \mathrm{Coh}_0^d$, where Coh_0^d is the stack classifying torsion coherent sheaves of length d. The map $\mathfrak{s}$ sends a triple as above to $\mathcal{M}'/\mathcal{M}$.

Recall that to a local system E on X, Laumon associated a perverse sheaf $\mathcal{L}_E^d \in \mathrm{P}(\mathrm{Coh}_0^d)$. The pull-back $\mathfrak{s}^*(\mathcal{L}_E^d)$ (which is perverse up to a cohomological shift) can be described as follows:

Let $\overset{\circ}{X}{}^d$ denote the complement to the diagonal divisor in $X^{(d)}$. Let $\overset{\circ}{\mathrm{Mod}}{}^d_n$ denote the preimage of $\overset{\circ}{X}{}^d$ under s, and let $\overset{\circ}{s} : \overset{\circ}{\mathrm{Mod}}{}^d_n \to \overset{\circ}{X}{}^d$ be the corresponding map. Unlike s, the map $\overset{\circ}{s}$ is smooth. Finally, let j denote the open embedding of $\overset{\circ}{\mathrm{Mod}}{}^d_n$ into Mod^d_n.

Consider the symmetric power of E as a sheaf $E^{(d)} \in \mathrm{D}(X^{(d)})$, and let $\overset{\circ}{E}{}^{(d)}$ denote its restriction to $\overset{\circ}{X}{}^d$. It is easy to see that $\overset{\circ}{E}{}^{(d)}$ is lisse.

We have:

$$\mathfrak{s}^*(\mathcal{L}^d_E) \simeq j_{!*}\left(\overset{\circ}{s}{}^*(\overset{\circ}{E}{}^{(d)})\right). \tag{1}$$

1.2. We introduce the averaging functor $\mathrm{Av}^d_E : \mathrm{D}(\mathrm{Bun}_n) \to \mathrm{D}(\mathrm{Bun}_n)$ as follows:

$$\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n) \mapsto \overleftarrow{h}_!\left(\overrightarrow{h}^*(\mathcal{F}) \otimes \mathfrak{s}^*(\mathcal{L}^d_E)\right)[nd].$$

Let us note immediately, that this functor is essentially Verdier self-dual, in the sense that

$$\mathbb{D}(\mathrm{Av}^d_E(\mathcal{F})) \simeq \mathrm{Av}^d_{E^*}(\mathbb{D}(\mathcal{F})),$$

where E^* is the dual local system. This follows from the fact that the map $\mathfrak{s} \times \overrightarrow{h} : \mathrm{Mod}^d_n \to \mathrm{Coh}^d_0 \times \mathrm{Bun}_n$ is smooth of relative dimension nd, and the map $\overleftarrow{h}$ is proper.

The following conjecture was proposed in [7]:

Conjecture 1.3. *Assume that E is irreducible, of rank $> n$. Then for d, which is greater than $(2g-2)\cdot n \cdot \mathrm{rk}(E)$, the functor Av^d_E is identically equal to zero.*

1.4. Let us discuss some rather tautological reformulations of Conjecture 1.3. Consider the map $\overleftarrow{h} \times \overrightarrow{h} : \mathrm{Mod}^d_n \to \mathrm{Bun}_n \times \mathrm{Bun}_n$; it is representable, but not proper, and set $\mathcal{K}^d_E := (\overleftarrow{h} \times \overrightarrow{h})_!(\mathfrak{s}^*(\mathcal{L}^d_E)) \in \mathrm{D}(\mathrm{Bun}_n \times \mathrm{Bun}_n)$.

Let $\mathcal{M} \in \mathrm{Bun}_n$ be a geometric point (corresponding to a morphism denoted $\iota_{\mathcal{M}} : \mathrm{Spec}(k) \to \mathrm{Bun}_n$), and let $\delta_{\mathcal{M}} \in \mathrm{D}(\mathrm{Bun}_n)$ be $(\iota_{\mathcal{M}})_!(\overline{\mathbb{Q}}_l)$. Note that since the stack Bun_n is not separated, $\iota_{\mathcal{M}}$ need not be a closed embedding; therefore, $\delta_{\mathcal{M}}$ is *a priori* a complex of sheaves.

Lemma 1.5. *The vanishing of the functor Av^d_E is equivalent to each of the following statements*:

(1) *For every $\mathcal{M} \in \mathrm{Bun}_n$, the object $\mathrm{Av}^d_E(\delta_{\mathcal{M}}) \in \mathrm{D}(\mathrm{Bun}_n)$ vanishes.*

(2) *The object $\mathcal{K}^d_E \in \mathrm{D}(\mathrm{Bun}_n \times \mathrm{Bun}_n)$ vanishes.*

Proof. First, statements (1) and (2) above are equivalent: For $\mathcal{M}$, the stalk of $\mathrm{Av}_E^d(\delta_{\mathcal{M}})$ at $\mathcal{M}' \in \mathrm{Bun}_n$ is isomorphic to the stalk of $\mathcal{K}_E^d$ at $(\mathcal{M} \times \mathcal{M}') \in \mathrm{Bun}_n \times \mathrm{Bun}_n$.

Obviously, Conjecture 1.3 implies statement (1). Conversely, assume that statement (1) above holds. Let $\mathrm{Av}_{E^*}^{-d}$ be the (both left and right) adjoint functor of Av_E^d; explicitly,

$$\mathrm{Av}_{E^*}^{-d}(\mathcal{F}) = \overrightarrow{h}_! \left(\overleftarrow{h}^*(\mathcal{F}) \otimes \mathfrak{s}^*(\mathcal{L}_{E^*}^d) \right) [nd].$$

It is enough to show that $\mathrm{Av}_{E^*}^{-d}$ identically vanishes. However, by adjointness, for an object $\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n)$, the co-stalk of $\mathrm{Av}_{E^*}^{-d}(\mathcal{F})$ at $\mathcal{M} \in \mathrm{Bun}_n$ is isomorphic to $\mathrm{RHom}_{\mathrm{D}(\mathrm{Bun}_n)}(\mathrm{Av}_E^d(\delta_{\mathcal{M}}), \mathcal{F})$. □

1.6. The assertion of the above conjecture is a geometric analog of the statement that the Rankin-Selberg convolution $L(\pi, \sigma)$, where π is an automorphic representation of GL_n and σ is an irreducible m-dimensional Galois representation with $m > n$, has an analytic continuation and satisfies a functional equation.

More precisely, let X be a curve over a finite field, and $\mathcal{K}$ the corresponding global field. Then it is known that the double quotient

$$\mathrm{GL}_n(\mathcal{K}) \backslash \mathrm{GL}_n(\mathbb{A}) / \mathrm{GL}_n(\mathbb{O})$$

can be identified with the set (of isomorphism classes) of points of the stack Bun_n.

By passing to the traces of the Frobenius, we have a function-theoretic version of the averaging functor; let us denote it by $\mathrm{Funct}(\mathrm{Av}_{\mathrm{E}}^{\mathrm{d}})$, which is now an operator from the space of functions on $\mathrm{GL}_n(\mathcal{K}) \backslash \mathrm{GL}_n(\mathbb{A}) / \mathrm{GL}_n(\mathbb{O})$ to itself.

Now, let f_π be a spherical vector in some unramified automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$. One can show that

$$\sum_{d \geq 0} \mathrm{Funct}(\mathrm{Av}_{\mathrm{E}}^{\mathrm{d}})(f_\pi) = L(\pi, E) \cdot f_\pi, \tag{2}$$

where the L-function $L(\pi, E)$ is regarded as a formal series in d.

The assertion of Conjecture 1.3 implies that the above series is a polynomial of degree $\leq m \cdot n \cdot (2g - 2)$. And this is the same estimate as the one following from the functional equation, which $L(\pi, E)$ is supposed to satisfy.

1.7. In the rest of this section we will make several preparatory steps towards the proof of Conjecture 1.3.

Recall that the Hecke functor $\mathrm{H} : \mathrm{D}(\mathrm{Bun}_n) \to \mathrm{D}(X \times \mathrm{Bun}_n)$ is defined using the stack $\mathcal{H} = \mathrm{Mod}_n^1$, as

$$\mathcal{F} \mapsto (s \times \overleftarrow{h})_!(\overrightarrow{h}^*(\mathcal{F}))[n].$$

In the sequel it will be important to introduce parameters in all our constructions. Thus, for a scheme S, we have a similarly defined functor

$$\mathrm{H}_S : \mathrm{D}(S \times \mathrm{Bun}_n) \to \mathrm{D}(S \times X \times \mathrm{Bun}_n).$$

For an integer d let us consider the d-fold iteration $\mathrm{H}_{S\times X^{d-1}} \circ \cdots \circ \mathrm{H}_{S\times X} \circ \mathrm{H}_S$, denoted

$$\mathrm{H}_S^{\boxtimes d} : \mathrm{D}(S \times \mathrm{Bun}_n) \to \mathrm{D}(S \times X^d \times \mathrm{Bun}_n).$$

PROPOSITION 1.8. *The functor* $\mathrm{H}_S^{\boxtimes d}$ *maps* $\mathrm{D}(S\times\mathrm{Bun}_n)$ *to the equivariant derived category* $\mathrm{D}^{\Sigma_d}(S\times X^d\times\mathrm{Bun}_n)$, *where* Σ_d *is the symmetric group acting naturally on* X^d.

Proof. In the proof we will suppress S to simplify the notation. Let ItMod_n^d denote the stack of iterated modifications; i.e., it classifies the data of a pair of vector bundles $\mathcal{M}, \mathcal{M}' \in \mathrm{Bun}_n$ together with a flag

$$\mathcal{M} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_d = \mathcal{M}',$$

where each $\mathcal{M}_i/\mathcal{M}_{i-1}$ is a torsion sheaf of length 1.

Let $\mathfrak{r}$ denote the natural map $\mathrm{ItMod}_n^d \to \mathrm{Mod}_n^d$, and let $\widetilde{\overleftarrow{h}}$ and $\widetilde{\overrightarrow{h}}$ be the two maps from ItMod_n^d to Bun_n equal to $\overleftarrow{h} \circ \mathfrak{r}$ and $\overrightarrow{h} \circ \mathfrak{r}$, respectively. We will denote by $\widetilde{s}$ the map $\mathrm{ItMod}_n^d \to X^d$, which remembers the supports of the successive quotients $\mathcal{M}_i/\mathcal{M}_{i-1}$.

It is easy to see that the functor $\mathcal{F} \mapsto \mathrm{H}^{\boxtimes d}(\mathcal{F})$ can be rewritten as

$$\mathcal{F} \mapsto (\widetilde{s} \times \widetilde{\overleftarrow{h}})_!(\widetilde{\overrightarrow{h}}^*(\mathcal{F}))[nd]. \tag{3}$$

We will now introduce a stack intermediate between Mod_n^d and ItMod_n^d. Consider the Cartesian product

$$\mathrm{IntMod}_n^d := \mathrm{Mod}_n^d \underset{X^{(d)}}{\times} X^d.$$

Note that IntMod_n^d carries a natural action of the symmetric group Σ_d via its action on X^d. Let $\widetilde{\widetilde{\overleftarrow{h}}}, \widetilde{\widetilde{\overrightarrow{h}}}$ be the corresponding projections from IntMod_n^d to Bun_n, and $\widetilde{\widetilde{s}}$ the map $\mathrm{IntMod}_n^d \to X^d$. All these maps are Σ_d-invariant.

We have a natural map $\mathfrak{r}_{\mathrm{Int}} : \mathrm{ItMod}_n^d \to \mathrm{IntMod}_n^d$.

LEMMA 1.9. *The map* $\mathfrak{r}_{\mathrm{Int}}$ *is a small resolution of singularities.*

The proof of this lemma follows from the fact that IntMod_n^d is squeezed between ItMod_n^d and Mod_n^d, and the fact that the map $\mathfrak{r} : \mathrm{ItMod}_n^d \to \mathrm{Mod}_n^d$ is known to be small from the Springer theory, cf. [2].

Hence, the direct image of the constant sheaf on ItMod_n^d under $\mathfrak{r}_{\mathrm{Int}}$ is isomorphic to the intersection cohomology sheaf $\mathrm{IC}_{\mathrm{IntMod}_n^d}$, up to a cohomological shift.

Therefore, by the projection formula, the expression in (3) can be rewritten as

$$(4) \qquad (\widetilde{\widetilde{s}} \times \overset{\approx}{\overleftarrow{h}})_! \left(\overset{\approx}{\overrightarrow{h}}{}^*(\mathcal{F}) \otimes \mathrm{IC}_{\mathrm{IntMod}_n^d} \right) [-\dim(\mathrm{Bun}_n)].$$

However, since the map $\overset{\approx}{\overrightarrow{h}}$ is Σ_d-invariant, and $\mathrm{IC}_{\mathrm{IntMod}_n^d}$ is a Σ_d-equivariant object of $\mathrm{D}(\mathrm{IntMod}_n^d)$, we obtain that $\overset{\approx}{\overrightarrow{h}}{}^*(\mathcal{F}) \otimes \mathrm{IC}_{\mathrm{IntMod}_n^d}$ is naturally an object of $\mathrm{D}^{\Sigma_d}(\mathrm{IntMod}_n^d)$. Similarly, since the map $\overset{\approx}{\overleftarrow{h}}$ is Σ_d-invariant, the expression in (4) is naturally an object of $\mathrm{D}^{\Sigma_d}(\mathrm{Bun}_n)$. □

1.10. Let $\Delta(X) \subset X^i$ be the main diagonal. Obviously, the symmetric group Σ_i acting on X^i stabilizes $\Delta(X)$. Hence, for an object $\mathcal{F} \in \mathrm{D}^{\Sigma_i}(S \times X^i \times \mathrm{Bun}_n)$, it makes sense to consider

$$\mathrm{Hom}_{\Sigma_i}(\rho, \mathcal{F}|_{S \times \Delta(X) \times \mathrm{Bun}_n}) \in \mathrm{D}(S \times X \times \mathrm{Bun}_n)$$

for various representations ρ of Σ_i. In particular, let us consider the following functor $\mathrm{D}(S \times \mathrm{Bun}_n) \to \mathrm{D}(S \times X \times \mathrm{Bun}_n)$ that sends $\mathcal{F}$ to

$$\mathrm{Hom}_{\Sigma_i}(\mathrm{sign}, \mathrm{H}_S^{\boxtimes i}(\mathcal{F})|_{S \times \Delta(X) \times \mathrm{Bun}_n}),$$

where sign is the sign representation of Σ_i.

The following has been established in [7]:

Proposition 1.11. *The functor*

$$\mathcal{F} \mapsto \mathrm{Hom}_{\Sigma_i}(\mathrm{sign}, \mathrm{H}_S^{\boxtimes i}(\mathcal{F})|_{S \times \Delta(X) \times \mathrm{Bun}_n})$$

is zero if $i > n$ and for $i = n$ it is canonically isomorphic to

$$\mathcal{F} \mapsto (\mathrm{id}_S \times m)^*(\mathcal{F})[n],$$

where $m : X \times \mathrm{Bun}_n \to \mathrm{Bun}_n$ is the multiplication map, i.e., $m(x, \mathcal{M}) = \mathcal{M}(x)$.

Proof. Again, to simplify the notation we will suppress the scheme S.

Let $\mathrm{Mod}_n^{i,\Delta}$ denote the preimage of $\Delta(X) \subset X^i$ inside IntMod_n^i. Note that the symmetric group Σ_i acts trivially on $\mathrm{Mod}_n^{i,\Delta}$, and the $*$-restriction $\mathrm{IC}_{\mathrm{IntMod}_n^i}|_{\mathrm{Mod}_n^{i,\Delta}}$ is a Σ_i-equivariant object of $\mathrm{D}(\mathrm{Mod}_n^{i,\Delta})$.

Note also that for $i = n$, $\mathrm{Mod}_n^{i,\Delta}$ contains $X \times \mathrm{Bun}_n$ as a closed subset via

$$(x, \mathcal{M}) \mapsto (\mathcal{M}, \mathcal{M}(x), x^i) \in \mathrm{Mod}_n^i \underset{X^{(i)}}{\times} X^i.$$

The following is also a part of the Springer correspondence; cf. [2, §3]:

LEMMA 1.12. *The object*

$$\mathrm{Hom}_{\Sigma_i}(\mathrm{sign}, \mathrm{IC}_{\mathrm{IntMod}_n^i}|_{\mathrm{Mod}_n^{i,\Delta}})$$

is zero if $i > n$, *and for* $i = n$ *it is isomorphic to the constant sheaf on* $X \times \mathrm{Bun}_n \subset \mathrm{Mod}_n^{i,\Delta}$ *cohomologically shifted by* $[\dim(\mathrm{Bun}_n) + n]$.

This lemma and the projection formula imply the proposition. □

1.13. We will now perform manipulations analogous to the ones of Proposition 1.8 and Proposition 1.11 with the averaging functor Av_E^d.

Let us observe that for $d = 1$, the averaging functor can be described as follows:

$$\mathrm{Av}_E^1(\mathcal{F}) \simeq p_!(\mathrm{H}(\mathcal{F}) \otimes q^*(E)),$$

where p and q are the projections $X \times \mathrm{Bun}_n \to \mathrm{Bun}_n$ and $X \times \mathrm{Bun}_n \to X$, respectively.

We introduce the functor $\mathrm{ItAv}_E^d : \mathrm{D}(\mathrm{Bun}_n) \to \mathrm{D}(\mathrm{Bun}_n)$ as a d-fold iteration of Av_E^1.

PROPOSITION 1.14. *The functor* ItAv_E^d *maps* $\mathrm{D}(\mathrm{Bun}_n)$ *to the equivariant derived category* $\mathrm{D}^{\Sigma_d}(\mathrm{Bun}_n)$.

Proof. First, it is easy to see that $\mathrm{ItAv}_E^d(\mathcal{F})$ can be rewritten as

$$p_!(\mathrm{H}^{\boxtimes d}(\mathcal{F}) \otimes q^*(E^{\boxtimes d})),$$

where p, q are the two projections from $X^d \times \mathrm{Bun}_n$ to Bun_n and X^d, respectively.

Hence, the assertion that $\mathrm{ItAv}_E^d(\mathcal{F})$ naturally lifts to an object of the equivariant derived category $\mathrm{D}^{\Sigma_d}(\mathrm{Bun}_n)$ follows from Proposition 1.8. □

The next assertion allows us to express the functor Av_E^d via Av_E^1. This is the only essential place in the paper where we use characteristic zero coefficients.

PROPOSITION 1.15. *There is a canonical isomorphism of functors*

$$\mathrm{Av}_E^d(\mathcal{F}) \simeq (\mathrm{ItAv}_E^d(\mathcal{F}))^{\Sigma_d}.$$

Proof. The following lemma was proved in the original paper of Laumon (cf. [10]):

LEMMA 1.16. *The direct image* $\mathcal{S}pr_E^d := \mathfrak{r}_!(\widetilde{s}^*(E^{\boxtimes d})) \in \mathrm{D}(\mathrm{Mod}_n^d)$ *is naturally* Σ_d*-equivariant. Moreover,*

$$\mathfrak{s}^*(\mathcal{L}_E^d) \simeq (\mathcal{S}pr_E^d)^{\Sigma_d}.$$

Using the projection formula and the lemma, we can rewrite $\mathrm{ItAv}_E^d(\mathcal{F})$ as

$$\overleftarrow{h}_!(\overrightarrow{h}^*(\mathcal{F}) \otimes \mathcal{S}pr_E^d)[nd].$$

(It is easy to see that the Σ_d-equivariant structure on $\mathrm{ItAv}_E^d(\mathcal{F})$, which arises from the last expression is the same as the one constructed before.)

Using Lemma 1.16 we conclude the proof. □

2. Strategy of the proof

In this section we will reduce the assertion of Conjecture 1.3 to a series of theorems, which will be proved in the subsequent sections.

2.1. By induction we will assume that Conjecture 1.3 holds for all n' with $n' < n$. We will deduce Conjecture 1.3 for n from the following weaker statement:

Theorem 2.2. *Let E, n and d be as in Conjecture* 1.3. *Then the functor* $\mathrm{Av}_E^d : \mathrm{D}(\mathrm{Bun}_n) \to \mathrm{D}(\mathrm{Bun}_n)$ *is exact in the sense of the perverse t-structure.*

First we will prove that Theorem 2.2 implies Conjecture 1.3. In fact, we will give two proofs: the one discussed below is somewhat simpler, but at some point it resorts to some nontrivial results from the classical theory of automorphic functions. The second proof, which is due to A. Braverman, will be given in the appendix.

Thus, let us assume that Theorem 2.2 holds. Using Lemma 1.5(1), to prove Conjecture 1.3, it suffices to show that $\mathrm{Av}_E^d(\mathcal{F}) = 0$, whenever $\mathcal{F}$ is a perverse sheaf, which appears as a constituent in some $\delta_{\mathcal{M}}$ for $\mathcal{M} \in \mathrm{Bun}_n$. Set $\mathcal{F}' = \mathrm{Av}_E^d(\mathcal{F})$. By Theorem 2.2, we know that $\mathcal{F}'$ is perverse.

Lemma 2.3. *To show that a perverse sheaf $\mathcal{F}'$ on a stack $\mathcal{Y}$ vanishes, it is sufficient to show that the Euler-Poincaré characteristics of its stalks $\mathcal{F}'_y$ at all $y \in \mathcal{Y}$ are zero.*

Proof. If $\mathcal{F}' \neq 0$, there exists a locally closed substack $\mathcal{Y}^0 \subset \mathcal{Y}$, such that $\mathcal{F}'|_{\mathcal{Y}^0}$ is a lisse sheaf, up to a cohomological shift. But then the Euler-Poincaré characteristics of $\mathcal{F}'$ on $\mathcal{Y}^0$ are obviously nonzero. □

Now we have the following assertion, which states that the Euler-Poincaré characteristics of $\mathrm{Av}_E^d(\mathcal{F})$ do not depend on the local system.

Lemma 2.4. *Let E' be any other local system on X (irreducible or not) with $\mathrm{rk}(E') = \mathrm{rk}(E)$. Then the pointwise Euler-Poincaré characteristics of $\mathrm{Av}_E^d(\mathcal{F})$ and $\mathrm{Av}_{E'}^d(\mathcal{F})$ are the same for any $\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n)$.*

Proof. We will deduce the lemma from the following theorem of Deligne, cf. [8]:

Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a proper map of schemes, and let $\mathcal{S}$ and $\mathcal{S}'$ be two objects of $\mathrm{D}(\mathcal{Y}_1)$, *which are étale-locally isomorphic. Then the Euler-Poincaré characteristics of $f_!(\mathcal{S})$ and $f_!(\mathcal{S}')$ at all points of $\mathcal{Y}_2$ coincide.*

We apply this theorem in the following situation:

$\mathcal{Y}_2 = \mathrm{Bun}_n$, $\mathcal{Y}_1 = \mathrm{Mod}_n^d$, $\mathcal{S} := \overrightarrow{h}^*(\mathcal{F}) \otimes \mathfrak{s}^*(\mathcal{L}_E^d)$, $\mathcal{S}' := \overrightarrow{h}^*(\mathcal{F}) \otimes \mathfrak{s}^*(\mathcal{L}_{E'}^d)$, and $f = \overleftarrow{h}$.

The assertion of the lemma follows from the fact that $s^*(\mathcal{L}_E^d)$ and $s^*(\mathcal{L}_{E'}^d)$ are étale-locally isomorphic, because E and E' are. $\square$

Thus, it suffices to show that for our $\mathcal{F} \in \mathrm{P}(\mathrm{Bun}_n)$ and *some* local system E' of rank equal to that of E, the Euler-Poincaré characteristics of the stalks of $\mathrm{Av}_{E'}^d(\mathcal{F})$ vanish.

When we are working in the ℓ-adic situation over a finite field, the required fact was established in [7] where we exhibited a local system E', for which the functor $\mathrm{Av}_{E'}^d$ was zero.[1]

In particular, we obtain that in the ℓ-adic situation over a finite field the vanishing of the Euler-Poincaré characteristics takes place when E' is the trivial local system.

Using the fact that our initial perverse sheaf $\mathcal{F}$ was of geometric origin, the standard reduction argument (cf. [1, §6.1.7]) implies the vanishing of the Euler-Poincaré characteristics for the trivial local system in the setting of ℓ-adic sheaves over any ground field, and, when the field equals $\mathbb{C}$, also for constructible sheaves with complex coefficients.

By the Riemann-Hilbert correspondence, this translates to the required vanishing statement in the setting of D-modules over $\mathbb{C}$, and, hence, over any field of characteristic zero.

2.5. Thus, from now on, our goal will be to prove Theorem 2.2. In view of Proposition 1.15, a natural idea would be to show that the "elementary" functor Av_E^1 is exact. The latter, however, is false.

Recall that Av_E^1 is a composition of $\mathrm{H} : \mathrm{D}(\mathrm{Bun}_n) \to \mathrm{D}(X \times \mathrm{Bun}_n)$ followed by the functor $\mathcal{F} \mapsto p_!(q^*(E) \otimes \mathcal{F}) : \mathrm{D}(X \times \mathrm{Bun}_n) \to \mathrm{D}(\mathrm{Bun}_n)$.

As it turns out, the source of the nonexactness of Av_E^1 is the fact that the Hecke functor H is not exact, except when $n = 1$. Therefore, we will first consider the latter case, which would be the prototype of the argument in general.

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[1] This part of the argument will be replaced by a different one in the appendix.

2.6. *The case* $n = 1$. Of course, the assertion of Conjecture 1.3 in this case is known, cf. [5]. However, the proof we give below is completely different.

First, let us note that it is indeed sufficient to show that the functor Av^1_E is exact:

The exactness of Av^1_E implies that the functor ItAv^d_E is exact for any d. Since the coefficients of our sheaves are of characteristic 0, from Proposition 1.15 we obtain that Av^d_E is a direct summand of ItAv^d_E, and, therefore, is exact as well.

To prove that Av^1_E is exact, it is enough to show that for an irreducible perverse sheaf $\mathcal{F}$, $\mathrm{Av}^1_E(\mathcal{F})$ has no cohomologies above 0 (because Av^1_E is essentially Verdier self-dual).

For $n = 1$, Bun_n is the Picard stack Pic, and the Hecke functor can be identified with the pull-back $\mathcal{F} \mapsto m^*(\mathcal{F})[1]$, where $m : X \times \mathrm{Pic} \to \mathrm{Pic}$ is the multiplication map. We have:

$$\mathrm{Av}^1_E(\mathcal{F}) \simeq p_!(m^*(\mathcal{F})[1] \otimes q^*(E)),$$

where p and q are the two projections $X \times \mathrm{Pic}$ to Pic and X, respectively.

Since the map m is smooth, the sheaf $m^*(\mathcal{F})[1]$ is also perverse and irreducible,[2] and $m^*(\mathcal{F})[1] \otimes q^*(E)$ is perverse. Since p is a projection with 1-dimensional fibers, it is enough to show that

$$h^1\left(p_!\left(m^*(\mathcal{F})[1] \otimes q^*(E)\right)\right) = 0.$$

We will argue by contradiction. If $\mathcal{F}^1 = h^1\left(p_!\left(m^*(\mathcal{S})[1] \otimes q^*(E)\right)\right) \neq 0$, by adjunction we have a surjective map

$$m^*(\mathcal{F})[1] \otimes q^*(E) \to p^*(\mathcal{F}^1)[1], \tag{5}$$

which gives rise to a map

$$m^*(\mathcal{F})[1] \to E^*[1] \boxtimes \mathcal{F}^1. \tag{6}$$

Since E was assumed irreducible, sub-objects of the right-hand side of (6) are in bijection with sub-objects of $\mathcal{F}^1$. Therefore, since the map of (5) is surjective, so is the map in (6). By the irreducibility of $\mathcal{F}$, it must, therefore, be an isomorphism.

We claim that this cannot happen if the rank of E is greater than 1.

Indeed, let us consider the pull-back

$$(\mathrm{id} \times m)^*\left(m^*(\mathcal{F})\right)[2] \in \mathrm{P}(X \times X \times \mathrm{Pic}).$$

On the one hand, we know that it is isomorphic to $E^*[1] \boxtimes m^*(\mathcal{F}^1)[1]$. On the other hand, it is equivariant with respect to the permutation group Σ_2 acting on $X \times X$.

[2] The fact that we can control irreducibility under the Hecke functors is another simplification of the $n = 1$ case.

LEMMA 2.7. *Let $\mathcal{S}$ be an irreducible perverse sheaf on a variety of the form $X \times X \times \mathcal{Y}$, which, on the one hand, is Σ_2-equivariant, and on the other hand, has a form $E[1] \boxtimes \mathcal{S}'$, where E is an irreducible local system, and $\mathcal{S}' \in \mathrm{P}(X \times \mathcal{Y})$. Then $\mathcal{S}$ must be of the form $\mathcal{S} \simeq E[1] \boxtimes E[1] \boxtimes \mathcal{S}''$; moreover the Σ_2-equivariant structure on $\mathcal{S}$ is the standard one on $E[1] \boxtimes E[1]$ times some Σ_2-action on $\mathcal{S}''$.*

Proof. Let q and p be the projections from $X \times \mathcal{Y}$ to X and $\mathcal{Y}$, respectively. It is enough to show that

$$h^1\left(p_!(\mathcal{S}' \otimes q^*(E^*))\right) \neq 0.$$

For $i = 1, 2$ let q_i be the projection $X \times X \times \mathcal{Y} \to X$ on the i-th factor, and let p_i be the complementary projection on $X \times \mathcal{Y}$. We have

$$E \boxtimes p_!(\mathcal{S}' \otimes q^*(E^*)) \simeq p_{2!}(\mathcal{S} \otimes q_2^*(E^*)),$$

which, due to the Σ_2-equivariance assumption, is isomorphic to $p_{1!}(\mathcal{S} \otimes q_1^*(E^*))$. The latter has nontrivial cohomology in dimension 1. □

Thus, from the lemma, we obtain that $(\mathrm{id} \times m)^*(m^*(\mathcal{F}))$ has the form $E^* \boxtimes E^* \boxtimes \mathcal{F}''$. Let us restrict $(\mathrm{id} \times m)^*(m^*(\mathcal{F}))$ to the diagonal $(\Delta \times \mathrm{id}) : X \times \mathrm{Pic} \subset X \times X \times \mathrm{Pic}$, and take Σ_2 anti-invariants.

On the one hand, from Proposition 1.11 (which is especially easy in the $n = 1$ case) we know that for any $\mathcal{F} \in \mathrm{D}(\mathrm{Pic})$,

$$\mathrm{Hom}_{\Sigma_2}\left(\mathrm{sign}, (\mathrm{id} \times m)^*(m^*(\mathcal{F}))\,|_{X \times \mathrm{Pic}}\right) = 0.$$

But on the other hand, $(\mathrm{id} \times m)^*(m^*(\mathcal{F}))\,|_{X \times \mathrm{Pic}} \simeq (E^*)^{\otimes 2} \boxtimes \mathcal{F}''$, and taking Σ_2 anti-invariants we obtain

$$\left(\mathrm{Sym}^2(E^*) \boxtimes \mathrm{Hom}_{\Sigma_2}\left(\mathrm{sign}, \mathcal{F}''\right)\right) \oplus \left(\Lambda^2(E^*) \boxtimes (\mathcal{F}'')^{\Sigma_2}\right).$$

Now, since $\mathrm{rk}(E) > 1$, neither $\Lambda^2(E^*)$ nor $\mathrm{Sym}^2(E^*)$ is 0; therefore, the entire expression cannot vanish.

2.8. The key fact used in the above argument was that the Hecke functor, which in this case acts as $\mathcal{F} \mapsto m^*(\mathcal{F})[1]$, is exact.

For $n \geq 1$, our approach will consist of making the Hecke functors exact by passing to a quotient triangulated category.

Recall that if $\mathcal{C}$ is a triangulated category, and $\mathcal{C}' \subset \mathcal{C}$ is a full triangulated subcategory, one can form a quotient $\mathcal{C}/\mathcal{C}'$. This quotient is a triangulated category endowed with a projection functor

$$\mathcal{C} \to \mathcal{C}/\mathcal{C}',$$

which is universal with respect to the property that it makes any arrow $\mathcal{S}_1 \to \mathcal{S}_2$ in $\mathcal{C}$, whose cone belongs to $\mathcal{C}'$, into an isomorphism.

Note that the inclusion of $\mathcal{C}'$ into $\ker(\mathcal{C} \to \mathcal{C}/\mathcal{C}')$ is not necessarily an equivalence. Rather, $\ker(\mathcal{C} \to \mathcal{C}')$ is the full subcategory, consisting of objects, which appear as direct summands of objects of $\mathcal{C}'$.

Suppose now that $\mathcal{C}$ is endowed with a t-structure. Let $\mathrm{P}(\mathcal{C})$ be the corresponding abelian subcategory, and let $\mathcal{C}' \subset \mathcal{C}$ be as above.

Definition 2.9. We say that $\mathcal{C}'$ is *compatible* with the t-structure if

(1) $\mathrm{P}(\mathcal{C}') := \mathrm{P}(\mathcal{C}) \cap \mathcal{C}'$ is a *Serre subcategory* of $\mathrm{P}(\mathcal{C})$.[3]

(2) If an object $\mathcal{S} \in \mathcal{C}$ belongs to $\mathcal{C}'$, then so do its cohomological truncations $\tau^{\leq 0}(\mathcal{S})$ and $\tau^{>0}(\mathcal{S})$.

A typical way of producing categories $\mathcal{C}'$ satisfying this definition is given by the following lemma:

Lemma 2.10. *Let $\mathcal{C}_1$, and $\mathcal{C}_2$ be two triangulated categories endowed with t-structures. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor, which is t-exact. Then $\mathcal{C}'_1 :=$* $\ker(F) \subset \mathcal{C}_1$ *is compatible with the t-structure.*

The following proposition is in some sense a converse to the above lemma:

Proposition 2.11. *Let $\mathcal{C}$ be as above, and $\mathcal{C}' \subset \mathcal{C}$ be compatible with the t-structure. Then the quotient category $\widetilde{\mathcal{C}} := \mathcal{C}/\mathcal{C}'$ carries a canonical t-structure, such that*

(1) *The projection functor $\mathcal{C} \to \widetilde{\mathcal{C}}$ is exact.*

(2) *The abelian category $\mathrm{P}(\widetilde{\mathcal{C}})$ identifies with the Serre quotient $\mathrm{P}(\mathcal{C})/\mathrm{P}(\mathcal{C}')$.*

Proof. Let $\mathcal{S}$ be an object of $\mathcal{C}$, viewed as an object of the quotient category $\mathcal{C}/\mathcal{C}'$. We say that it belongs to $\widetilde{\mathcal{C}}^{\leq 0}$ (resp., $\widetilde{\mathcal{C}}^{>0}$) if $\tau^{>0}(\mathcal{S})$ (resp., $\tau^{\leq 0}(\mathcal{S})$) belongs to $\mathcal{C}'$.

If $\mathcal{S}_1 \to \mathcal{S}_2$ is a morphism, whose cone belongs to $\mathcal{C}'$, it is easy to see that $\mathcal{S}_1$ belongs to $\widetilde{\mathcal{C}}^{\leq 0}$ (resp., $\widetilde{\mathcal{C}}^{>0}$) if and only if $\mathcal{S}_2$ does.

We have to check now that if $\mathcal{S}_1 \in \widetilde{\mathcal{C}}^{\leq 0}$, and $\mathcal{S}_2 \in \widetilde{\mathcal{C}}^{>0}$, then $\mathrm{Hom}_{\widetilde{\mathcal{C}}}(\mathcal{S}_1, \mathcal{S}_2) = 0$.

Indeed, with no restriction of generality, by applying the cohomological truncation functor, we can assume that $\mathcal{S}_1$ is represented by an object of $\mathcal{C}$, which lies in $\mathcal{C}^{\leq 0}$, and $\mathcal{S}_2$ is represented by an object, which belongs to $\mathcal{C}^{>0}$.

Each element of the Hom group can be represented by a diagram

$$\mathcal{S}_1 \leftarrow \mathcal{S}_3 \to \mathcal{S}_2,$$

[3]Recall that a Serre subcategory of an abelian category is a full subcategory stable under taking sub-objects and extensions.

where the cone of $\mathcal{S}_3 \to \mathcal{S}_1$ belongs to $\mathcal{C}'$. Hence, this diagram can be replaced by an equivalent one

$$\mathcal{S}_1 \leftarrow \tau^{\leq 0}(\mathcal{S}_3) \to \mathcal{S}_2,$$

where τ is the cohomological truncation.

But now, any map $\tau^{\leq 0}(\mathcal{S}_3) \to \mathcal{S}_2$ is zero already in $\mathcal{C}$, since $\mathcal{S}_2 \in \mathcal{C}^{>0}$.

The projection $\mathcal{C} \to \widetilde{\mathcal{C}}$ is exact by construction. By the universal property of the Serre quotient, we have a functor $\mathrm{P}(\mathcal{C})/\,\mathrm{P}(\mathcal{C}') \to \mathrm{P}(\widetilde{\mathcal{C}})$. Again, by construction, this functor is surjective on objects, and to prove that it is fully-faithful it is sufficient to show that for $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{P}(\mathcal{C})$ a map $\mathcal{S}_1 \to \mathcal{S}_2$ is an isomorphism in $\mathrm{P}(\widetilde{\mathcal{C}})$ if and only if its kernel and cokernel belong to $\mathrm{P}(\mathcal{C}')$.

Let $\mathcal{S}$ denote the cone of this map, regarded as an object of $\mathcal{C}$. By assumption, it belongs to $\mathcal{C}'$; therefore $h^0(\mathcal{S})$ and $h^1(\mathcal{S})$ both belong to $\mathcal{C}'$, by Definition 2.9. But the above $h^0(\mathcal{S})$ and $h^1(\mathcal{S})$, both of which are objects of $\mathcal{C}' \cap \mathrm{P}(\mathcal{C}) = \mathrm{P}(\mathcal{C}')$, are the kernel and cokernel, respectively, of $\mathcal{S}_1 \to \mathcal{S}_2$. □

2.12. Thus, our strategy will be to find an appropriate quotient category of $\mathrm{D}(\mathrm{Bun}_d)$. More precisely, we will construct for every base S a category $\widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n)$, which is the quotient of $\mathrm{D}(S \times \mathrm{Bun}_n)$ by a triangulated subcategory $\mathrm{D}_{\mathrm{degen}}(S \times \mathrm{Bun}_n)$, such that $\mathrm{D}_{\mathrm{degen}}(S \times \mathrm{Bun}_n)$ is compatible with the perverse t-structure, and such that the following properties will hold:

Property 0. The categories $\mathrm{D}(S \times \mathrm{Bun}_n)$ inherit the standard four functors. In other words, for a map of schemes $f : S_1 \to S_2$ the four direct and inverse image functors $\mathrm{D}(S_1 \times \mathrm{Bun}_n) \rightleftarrows \mathrm{D}(S_2 \times \mathrm{Bun}_n)$ preserve the corresponding subcategories, and thus define the functors $\widetilde{\mathrm{D}}(S_1 \times \mathrm{Bun}_n) \rightleftarrows \widetilde{\mathrm{D}}(S_2 \times \mathrm{Bun}_n)$. Moreover, the same is true for the Verdier duality functor on $\mathrm{D}(S \times \mathrm{Bun}_n)$, and for the functor $\mathrm{D}(S) \times \mathrm{D}(S \times \mathrm{Bun}_n) \to \mathrm{D}(S \times \mathrm{Bun}_n)$, given by the tensor product along S.

Property 1. The Hecke functor $\mathrm{H}_S : \mathrm{D}(S \times \mathrm{Bun}_n) \to \mathrm{D}(S \times X \times \mathrm{Bun}_n)$ preserves the corresponding triangulated subcategories, and the resulting functor

$$\widetilde{\mathrm{H}}_S : \widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(S \times X \times \mathrm{Bun}_n)$$

is exact.

Property 2. There exists an integer d_0 large enough such that the following holds: if $\mathcal{F}_1 \in \mathrm{D}(\mathrm{Bun}_n)$ is supported on the connected component Bun_n^d with $d \geq d_0$ (cf. §7.8) for our conventions regarding the connected components of the stack Bun_n, and is cuspidal (cf. [7] or §6 for the notion of cuspidality), and $\mathcal{F}_2 \in \mathrm{D}_{\mathrm{degen}}(\mathrm{Bun}_n)$, then the Hom group $\mathrm{Hom}_{\mathrm{D}(\mathrm{Bun}_n)}(\mathcal{F}_1, \mathcal{F}_2)$ vanishes.

2.13. Assuming the existence of such a family of quotient categories, we will now derive Theorem 2.2.

First, let us observe that the functor $\mathrm{Av}^1_E : \mathrm{D}(\mathrm{Bun}_n) \to \mathrm{D}(\mathrm{Bun}_n)$ descends to a functor $\widetilde{\mathrm{Av}}^1_E : \widetilde{\mathrm{D}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(\mathrm{Bun}_n)$.

Indeed, according to Section 1.13, the functor Av^1_E is the composition of

$$\widetilde{\mathrm{H}} : \widetilde{\mathrm{D}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(X \times \mathrm{Bun}_n),$$

well-defined according to Property 1 above, followed by a functor $\widetilde{\mathrm{D}}(X \times \mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ that sends $\mathcal{S} \in \mathrm{D}(X \times \mathrm{Bun}_n)$ to $p_!(q^*(E) \otimes \mathcal{S})$, which is well-defined due to Property 0 (the maps p and q here are as in §1.13).

The first step in the proof of Theorem 2.2 is the following theorem, which is the key result of this paper. The proof will be given in the next section, and it mimics the argument for the $n = 1$ case, discussed above.

THEOREM 2.14. *The functor* $\widetilde{\mathrm{Av}}^1_E : \widetilde{\mathrm{D}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ *is exact.*

To state a corollary of Theorem 2.14, which we will actually use in the proof of Theorem 2.2, we need to make some preparations.

Let S be a base and Σ a finite group acting on S. (Here it becomes important that the characteristic of the coefficients of our sheaves is either 0 or coprime with $|\Sigma|$.) We define the category $\widetilde{\mathrm{D}}^\Sigma(S \times \mathrm{Bun}_n)$ as the quotient of the equivariant derived category $\mathrm{D}^\Sigma(S \times \mathrm{Bun}_n)$ by the triangulated subcategory $\mathrm{D}^\Sigma_{\mathrm{degen}}(S \times \mathrm{Bun}_n)$ equal to the preimage of $\mathrm{D}_{\mathrm{degen}}(S \times \mathrm{Bun}_n)$ under the forgetful functor

$$\mathrm{D}^\Sigma(S \times \mathrm{Bun}_n) \to \mathrm{D}(S \times \mathrm{Bun}_n).$$

The quotient acquires a t-structure, according to Lemma 2.10 and Proposition 2.11.

Let $\widetilde{\mathrm{P}}^\Sigma(S \times \mathrm{Bun}_n)$ be the corresponding abelian subcategory in $\widetilde{\mathrm{D}}^\Sigma(S \times \mathrm{Bun}_n)$. By construction, this is the quotient of $\mathrm{P}^\Sigma(S \times \mathrm{Bun}_n)$ by a Serre subcategory consisting of objects, whose image in $\widetilde{\mathrm{P}}(S \times \mathrm{Bun}_n)$ is zero.

In the applications we will take $S = X^d$ and Σ to be the symmetric group Σ_d.

Assume now that the action of Σ on S is actually trivial. Then we have the functor of invariants denoted $\mathcal{F} \mapsto (\mathcal{F})^\Sigma$ from $\mathrm{D}^\Sigma(S \times \mathrm{Bun}_n)$ to $\mathrm{D}(S \times \mathrm{Bun}_n)$. We claim that it descends to a well-defined functor $\widetilde{\mathrm{D}}^\Sigma(S \times \mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n)$. Indeed, for an object $\mathcal{F} \in \mathrm{D}^\Sigma(S \times \mathrm{Bun}_n)$, its image under the forgetful functor $\mathrm{D}^\Sigma(S \times \mathrm{Bun}_n) \to \mathrm{D}(S \times \mathrm{Bun}_n)$ contains $(\mathcal{F})^\Sigma$ as a direct summand. (In fact, in the case of a trivial action, every object of $\widetilde{\mathrm{D}}^\Sigma(S \times \mathrm{Bun}_n)$ can be canonically written as $\bigoplus\limits_\rho \mathcal{S}_\rho \otimes \rho$, where ρ runs over the set of irreducible representations of Σ, and $\mathcal{S}_\rho$ is an object of $\widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n)$.)

That said, first, from Proposition 1.8 we obtain that the functor

$$\mathrm{H}_S^{\boxtimes d} : \mathrm{D}(S \times \mathrm{Bun}_n) \to \mathrm{D}^{\Sigma_d}(S \times X^d \times \mathrm{Bun}_n)$$

gives rise to a functor $\widetilde{\mathrm{H}}_S^{\boxtimes d} : \widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n) \to \widetilde{\mathrm{D}}^{\Sigma_d}(S \times X^d \times \mathrm{Bun}_n)$.

Secondly, from Proposition 1.11 we obtain that the functor

$$(\mathrm{id}_S \times m)^* : \mathrm{D}(S \times \mathrm{Bun}_n) \to \mathrm{D}(S \times X \times \mathrm{Bun}_n)$$

descends to a well-defined functor $\widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(S \times X \times \mathrm{Bun}_n)$, which is isomorphic to

$$\mathcal{S} \mapsto \mathrm{Hom}_{\Sigma_i}\left(\mathrm{sign}, \widetilde{\mathrm{H}}_S^{\boxtimes i}(\mathcal{S})|_{S \times \Delta(X) \times \mathrm{Bun}_n}\right)[-n]; \tag{7}$$

for $i = n$, and for $i > n$ the latter functor is zero.

Thirdly, from Proposition 1.15, we obtain that the functor

$$\mathrm{ItAv}_E^d : \mathrm{D}(\mathrm{Bun}_n) \to \mathrm{D}^{\Sigma_d}(\mathrm{Bun}_n)$$

gives rise to a functor $\widetilde{\mathrm{ItAv}}_E^d : \widetilde{\mathrm{D}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{D}}^{\Sigma_d}(\mathrm{Bun}_n)$.

And finally, we obtain that the functor $\mathrm{Av}_E^d : \mathrm{D}(\mathrm{Bun}_n) \to \mathrm{D}(\mathrm{Bun}_n)$ gives rise to a well-defined functor $\widetilde{\mathrm{Av}}_E^d : \widetilde{\mathrm{D}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ with

$$\widetilde{\mathrm{Av}}_E^d(\mathcal{S}) \simeq (\widetilde{\mathrm{ItAv}}_E^d(\mathcal{S}))^{\Sigma_d}. \tag{8}$$

Now Theorem 2.14 implies the following:

COROLLARY 2.15. *The functor* $\widetilde{\mathrm{Av}}_E^d : \widetilde{\mathrm{D}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ *is exact.*

Proof. Theorem 2.14 readily implies that the functor $\widetilde{\mathrm{ItAv}}_E^d$ is exact.

Since the functor $\mathcal{F} \mapsto (\mathcal{F})^{\Sigma_d} : \mathrm{D}^{\Sigma_d}(\mathrm{Bun}_n) \to \mathrm{D}(\mathrm{Bun}_n)$ is exact (which follows from our assumption on the characteristic of the coefficients), we obtain that the same is true for the corresponding functor $\widetilde{\mathrm{D}}^{\Sigma_d}(\mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(\mathrm{Bun}_n)$.

Hence, the assertion follows from (8). □

2.16. We proceed with the proof of Theorem 2.2 modulo the existence of the categories $\widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n)$ and Theorem 2.14, and the induction hypothesis that Conjecture 1.3 holds for all $n' < n$. The following assertion is essentially borrowed from [7]:

LEMMA 2.17. *For any* $\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n)$, *the object* $\mathrm{Av}_E^d(\mathcal{F})$ *is cuspidal, provided that* $d > (2g-2) \cdot n \cdot \mathrm{rk}(E)$.

Proof. We have to show that the constant term functors $\mathrm{CT}^n_{n_1,n_2}(\mathrm{Av}_E^d(\mathcal{F}))$ all vanish.

However, it was shown in [7], Lemma 9.8, that $\mathrm{CT}^n_{n_1,n_2}(\mathrm{Av}_E^d(\mathcal{F}))$ is an extension of objects of the form

$$(\mathrm{Av}_E^{d_1} \boxtimes \mathrm{Av}_E^{d_2})(\mathrm{CT}^n_{n_1,n_2}(\mathcal{F})),$$

for all possible $d_1, d_2 \geq 0$, $d_1 + d_2 = d$, where $\mathrm{Av}_E^{d_1} \boxtimes \mathrm{Av}_E^{d_2}$ denotes the corresponding functor $\mathrm{D}(\mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2}) \to \mathrm{D}(\mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2})$.

However, for every pair d_1, d_2 as above, at least one of the parameters satisfies $d_i > (2g-2) \cdot n_i \cdot \mathrm{rk}(E)$. Hence, the corresponding functor $\mathrm{Av}_E^{d_i}$: $\mathrm{D}(\mathrm{Bun}_{n_i}) \to \mathrm{D}(\mathrm{Bun}_{n_i})$ vanishes by the induction hypothesis. $\square$

Now we are ready to finish the proof of Theorem 2.2. Since Av_E^d is essentially Verdier self-dual, it is enough to show that Av_E^d is right-exact.

We can assume that we start with a perverse sheaf $\mathcal{F}$ supported on $\mathrm{Bun}_n^{d'}$ with $d' \geq d + d_0$, and we have to show that $\mathrm{Av}_E^d(\mathcal{F}) \in \mathrm{P}(\mathrm{Bun}_n^{d'-d})$ has no cohomologies in degrees > 0.

Suppose not, and let

$$\mathrm{Av}_E^d(\mathcal{F}) \to \tau^{>0}(\mathrm{Av}_E^d(\mathcal{F}))$$

be the truncation map. This map cannot be zero, unless $\tau^{>0}(\mathrm{Av}_E^d(\mathcal{F}))$ vanishes.

By Lemma 2.17, we know that $\mathrm{Av}_E^d(\mathcal{F})$ is cuspidal. On the other hand, by Corollary 2.15, we know that $\tau^{>0}(\mathrm{Av}_E^d(\mathcal{F}))$ projects to zero in $\widetilde{\mathrm{D}}(\mathrm{Bun}_n)$. This is a contradiction in view of Property 2 of $\widetilde{\mathrm{D}}(\mathrm{Bun}_n)$.

3. The symmetric group argument

The goal of this section is to prove Theorem 2.14, assuming the existence of the quotient categories $\widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n)$ which satisfy Properties 0 and 1 of Section 2.12.

3.1. Since the situation is essentially Verdier self-dual, it would be sufficient to prove that the functor $\widetilde{\mathrm{Av}}_E^1 : \widetilde{\mathrm{D}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ is right-exact.

Let us suppose that it is not in order to arrive at a contradiction.

By definition, the functor $\widetilde{\mathrm{Av}}_E^1$ is a composition of an exact functor

$$\widetilde{\mathrm{H}} : \widetilde{\mathrm{D}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(X \times \mathrm{Bun}_n)$$

followed by the functor

$$\mathcal{S} \mapsto p_!(q^*(E) \otimes \mathcal{S}) : \widetilde{\mathrm{D}}(X \times \mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(\mathrm{Bun}_n)$$

of cohomological amplitude $[-1, 1]$. Thus, $\mathcal{S} \mapsto h^1(\widetilde{\mathrm{Av}}_E^1(\mathcal{S}))$ is a right-exact functor $\widetilde{\mathrm{P}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{P}}(\mathrm{Bun}_n)$.

Similarly, the amplitude of $\widetilde{\mathrm{ItAv}}_E^i : \widetilde{\mathrm{D}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ is at most $[-i, i]$, and $\mathcal{S} \mapsto h^i(\widetilde{\mathrm{ItAv}}_E^i(\mathcal{S}))$ is a right-exact functor $\widetilde{\mathrm{P}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{P}}^{\Sigma_i}(\mathrm{Bun}_n)$.

Proposition-Construction 3.2. *For $\mathcal{S} \in \widetilde{\mathrm{P}}(\mathrm{Bun}_n)$, there is a natural map in $\widetilde{\mathrm{P}}^{\Sigma_i}(X^i \times \mathrm{Bun}_n)$:*

$$\widetilde{\mathrm{H}}^{\boxtimes i}(\mathcal{S}) \to (E^*)^{\boxtimes i}[i] \boxtimes h^i(\widetilde{\mathrm{ItAv}}_E^i(\mathcal{S})).$$

When E is irreducible, the above map is surjective.

Proof. The adjointness of the functors $p^!, p_! : \mathrm{D}(X^i \times \mathrm{Bun}_n) \rightleftarrows \mathrm{D}(\mathrm{Bun}_n)$ gives rise to a pair of mutually adjoint functors $\mathrm{P}(X^i \times \mathrm{Bun}_n) \rightleftarrows \mathrm{P}(\mathrm{Bun}_n)$ given by

$$\mathcal{F}' \mapsto h^i(p_!(\mathcal{F}')) \text{ and } \mathcal{F} \mapsto p^*(\mathcal{F})[i],$$

(the former being the left adjoint of the latter). Since X^i is smooth and connected, for $\mathcal{F} \in \mathrm{P}(\mathrm{Bun}_n)$, every sub-quotient of $p^*(\mathcal{F})[-i]$ is of the form $p^*('\mathcal{F})[-i]$, where $'\mathcal{F}$ is a sub-quotient of $\mathcal{F}$. This implies that for any $\mathcal{F}' \in \mathrm{P}(X^i \times \mathrm{Bun}_n)$, the adjunction morphism $\mathcal{F}' \to p^*\left(h^i(p_!(\mathcal{F}'))\right)[i]$ is surjective.

We have another pair of mutually adjoint functors between the same categories:

$$\mathcal{F}' \mapsto h^i\left(p_!\left(q^*(E^{\boxtimes i}) \otimes \mathcal{F}'\right)\right) \text{ and } \mathcal{F} \mapsto (E^*)^{\boxtimes i}[i] \boxtimes \mathcal{F},$$

and, when E is irreducible, the adjunction map

$$\mathcal{F}' \mapsto (E^*)^{\boxtimes i}[i] \boxtimes h^i\left(p_!\left(q^*(E^{\boxtimes i}) \otimes \mathcal{F}'\right)\right)$$

is also surjective which follows from the next lemma:

Lemma 3.3. *If for $\mathcal{F}' \in \mathrm{P}(X^i \times \mathrm{Bun}_n)$ and $\mathcal{F}'' \in \mathrm{P}(\mathrm{Bun}_n)$ there is a surjective map $q^*(E^{\boxtimes i}) \otimes (\mathcal{F}') \to p^*(\mathcal{F}'')[i]$, and E is irreducible, then the adjoint map*

$$\mathcal{F}' \to (E^*)^{\boxtimes i}[i] \boxtimes \mathcal{F}''$$

is also surjective.

Moreover, the same assertions remain true for the corresponding functors that act on the level of equivariant categories: $\mathrm{P}^{\Sigma_i}(X^i \times \mathrm{Bun}_n) \rightleftarrows \mathrm{P}^{\Sigma_i}(\mathrm{Bun}_n)$.

By passing to the quotient $\widetilde{\mathrm{D}}^{\Sigma_i}(X^i \times \mathrm{Bun}_n)$, and using Property 0 of the quotient categories, for every $\mathcal{S}' \in \widetilde{\mathrm{D}}^{\Sigma_i}(X^i \times \mathrm{Bun}_n)$ we obtain a functorial map

$$\mathcal{S}' \to (E^*)^{\boxtimes i}[i] \boxtimes h^i\left(p_!\left(q^*(E^{\boxtimes i}) \otimes \mathcal{S}'\right)\right),$$

which is surjective if E is irreducible.

By now setting $\mathcal{S}' = \widetilde{\mathrm{H}}^{\boxtimes i}(\mathcal{S})$ we arrive to the assertion of the proposition. □

3.4. *The case $i = n+1$.* Note that for $\mathcal{S} \in \widetilde{\mathrm{P}}(\mathrm{Bun}_n)$ we have

$$h^i(\widetilde{\mathrm{ItAv}}^i_E(\mathcal{S})) \simeq h^1(\widetilde{\mathrm{Av}}^1_E) \circ \cdots \circ h^1(\widetilde{\mathrm{Av}}^1_E)(\mathcal{S}),$$

as functors $\widetilde{\mathrm{P}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{P}}(\mathrm{Bun}_n)$.

Therefore, if for some $\mathcal{S} \in \widetilde{\mathrm{P}}(\mathrm{Bun}_n)$, $h^i(\widetilde{\mathrm{ItAv}}^i_E(\mathcal{S})) \neq 0$, then $h^j(\widetilde{\mathrm{ItAv}}^j_E(\mathcal{S})) \neq 0$ for all $j \leq i$.

Our first step will be to show that for all $\mathcal{S} \in \widetilde{\mathrm{P}}(\mathrm{Bun}_n)$, $h^i(\widetilde{\mathrm{ItAv}}^i_E(\mathcal{S})) = 0$ for $i = n+1$, which would imply that the same is true for all $i \geq n+1$.

Consider the restriction of the surjection of Proposition-Construction 3.2 to the diagonal $X \times \mathrm{Bun}_n \overset{\Delta\times\mathrm{id}}{\longrightarrow} X^i \times \mathrm{Bun}_n$. That is, there exists the following map in $\widetilde{\mathrm{D}}^{\Sigma_i}(X \times \mathrm{Bun}_n)$:

(9)
$$(\Delta \times \mathrm{id})^* \left(\widetilde{\mathrm{H}}^{\boxtimes i}(\mathcal{S})\right)[1-i] \to (\Delta \times \mathrm{id})^* \Big((E^*)^{\boxtimes i}[i] \boxtimes h^i \left(\widetilde{\mathrm{ItAv}}^i_E(\mathcal{S})\right)\Big)[1-i].$$

A key technical result, that we will need, states that both sides of (9) belong in fact to $\widetilde{\mathrm{P}}^{\Sigma_i}(X \times \mathrm{Bun}_n)$ and that the map in (9) is still surjective. In fact, we will prove the following:

Proposition 3.5. *Let $\mathcal{K} \in \mathrm{P}(X^i \times \mathrm{Bun}_n)$ be a perverse sheaf, which appears as a sub-quotient of some $h^k(\mathrm{H}^{\boxtimes i}(\mathcal{S}))$ for some object $\mathcal{S} \in \mathrm{D}(\mathrm{Bun}_n)$. Then for any smooth sub-variety $X' \subset X^i$, the $*$-restriction $\mathcal{K}|_{X'\times \mathrm{Bun}_n}$ lives in the cohomological dimension $-\mathrm{codim}(X', X^i)$.*

This proposition will be proved in Section 3.6. Let us now explain how it implies what we need about (9).

Indeed, $\widetilde{\mathrm{H}}^{\boxtimes i}(\mathcal{S})$ can be represented by a sub-quotient $\mathcal{K}$ of $h^0(\mathrm{H}^{\boxtimes i}(\mathcal{F}))$ for some $\mathcal{F} \in \mathrm{P}(\mathrm{Bun}_n)$. Hence, the left-hand side of (9) can be represented by $(\Delta\times\mathrm{id})^*(\mathcal{K})[1-i]$, which belongs to $\mathrm{P}(X\times\mathrm{Bun}_n)$ according to Proposition 3.5.

The fact that the right-hand side of (9) belongs to $\widetilde{\mathrm{P}}(X\times\mathrm{Bun}_n)$ is obvious. In fact, it is isomorphic to $(E^*)^{\otimes i}[1] \boxtimes h^i(\widetilde{\mathrm{ItAv}}^i_E(\mathcal{S}))$.

Finally, the map of Proposition-Construction 3.2 can be represented by a surjective map of perverse sheaves $\mathcal{K} \to \mathcal{K}'$, where $\mathcal{K}$ is as above. By the long exact sequence, the cokernel of (9) injects into

$$h^1\Big((\Delta \times \mathrm{id})^*(\ker(\mathcal{K} \to \mathcal{K}'))[1-i]\Big),$$

which vanishes according to Proposition 3.5.

Now we are ready to prove that $h^i(\widetilde{\mathrm{ItAv}}^i_E(\mathcal{S})) = 0$ for $i = n+1$.

Since the functor of taking Σ_{n+1}-invariants is exact, the map in (9) will continue to be surjective when we pass to the sign-isotypic components on both sides; i.e., we have:

$$\mathrm{Hom}_{\Sigma_{n+1}}\Big(\mathrm{sign}, (\Delta \times \mathrm{id})^* \left(\widetilde{\mathrm{H}}^{\boxtimes n+1}(\mathcal{S})\right)[-n]\Big)$$
$$\twoheadrightarrow \mathrm{Hom}_{\Sigma_{n+1}}\Big(\mathrm{sign}, (E^*)^{\otimes n+1}[1] \boxtimes h^{n+1}(\widetilde{\mathrm{ItAv}}^{n+1}_E(\mathcal{S}))\Big).$$

Now, by (7), the left-hand side in the above formula is zero. By surjectivity, the right-hand side must also be zero. But we claim that this can only happen if $h^{n+1}(\widetilde{\mathrm{ItAv}}^{n+1}_E(\mathcal{S})) = 0$.

Indeed, let ρ be an irreducible Σ_{n+1}-representation, which has a non-trivial isotypic component in $h^{n+1}(\widetilde{\mathrm{ItAv}}_E^{n+1}(\mathcal{S}))$. However, since $\mathrm{rk}(E) \geq n+1$, by the Schur-Weyl theory, $\rho^* \otimes sign$ appears with a nonzero multiplicity in $(E^*)^{\otimes n+1}$. Hence, $sign$ appears with a nonzero multiplicity in $(E^*)^{\otimes n+1}[1] \boxtimes h^{n+1}(\widetilde{\mathrm{ItAv}}_E^{n+1}(\mathcal{S}))$.

3.6. *Proof of Proposition* 3.5. Recall the notion of universal local acyclicity in the situation when we have an object $\mathcal{F} \in \mathrm{D}(Z)$ on a scheme (or stack) Z over a smooth base $\mathcal{Y}$ (cf. [4] or [3]). In our case $Z = X^i \times \mathrm{Bun}_n$, $\mathcal{Y} = X^i$.

The first observation is:

Lemma 3.7. *For any $\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n)$, the object $\mathcal{F}' = \mathrm{H}^{\boxtimes i}(\mathcal{F})$ is* ULA *with respect to the projection $q : X^i \times \mathrm{Bun}_n \to X^i$.*

Proof. The lemma is proved by induction. Supposing the validity for an integer $i = j$, let us deduce the corresponding assertion for $i = j+1$. In other words, it suffices to show that if $\mathcal{F}'' \in \mathrm{D}(X^i \times \mathrm{Bun}_n)$ is ULA with respect to $X^i \times \mathrm{Bun}_n \to X^i$, then $\mathrm{H}_{X^i}(\mathcal{F}'') \in \mathrm{D}(X^{i+1} \times \mathrm{Bun}_n)$ is ULA with respect to $X^{i+1} \times \mathrm{Bun}_n \to X^{i+1}$.

Consider the diagram

$$X^i \times X \times \mathrm{Bun}_n \overset{\mathrm{id}_{X^i} \times s \times \overleftarrow{h}}{\longleftarrow} X^i \times \mathrm{Mod}_n^1 \overset{\mathrm{id}_{X^i} \times s \times \overrightarrow{h}}{\longrightarrow} X^i \times X \times \mathrm{Bun}_n .$$

By definition, $\mathrm{H}_{X^i}(\mathcal{F}'') = (\mathrm{id}_{X^i} \times s \times \overleftarrow{h})_! \circ (\mathrm{id}_{X^i} \times \overrightarrow{h})^*(\mathcal{F}'')[i \cdot n]$.

The ULA property is stable under direct images under proper morphisms. Since the map $\mathrm{id}_{X^i} \times s \times \overleftarrow{h}$ is proper, it is enough to show that $(\mathrm{id}_{X^i} \times \overrightarrow{h})^*(\mathcal{F}'') \in \mathrm{D}(X^i \times \mathrm{Mod}_n^1)$ is ULA with respect to $X^i \times \mathrm{Mod}_n^1 \overset{\mathrm{id} \times s}{\to} X^{i+1}$. However, this follows from the assumption on $\mathcal{F}''$, since the map $\mathrm{id}_{X^i} \times s \times \overleftarrow{h} : X^i \times \mathrm{Mod}_n^1 \to X^i \times X \times \mathrm{Bun}_n$ is smooth. $\square$

The proposition will now follow from the next general observation:

Let $\mathcal{F} \in \mathrm{D}(Z)$ be a complex, which is ULA with respect to a projection $Z \to \mathcal{Y}$, where $\mathcal{Y}$ is smooth. Let $\mathcal{K}$ be a sub-quotient of $h^k(\mathcal{F})$ for some k, and let $\mathcal{Y}' \subset \mathcal{Y}$ be a smooth sub-variety. Denote by Z' the preimage of $\mathcal{Y}'$ in Z. In the above circumstances we have:

Lemma 3.8. *The $*$-restriction $\mathcal{K}|_{Z'}$ lives in the cohomological dimension $-d$, where $d := \mathrm{codim}(\mathcal{Y}', \mathcal{Y})$.*

Proof. Note that the assertion of the lemma implies that $\mathcal{K}|_{Z'}[-d]$ is a sub-quotient of $h^{k-d}(\mathcal{F}|_{Z'})$.

Therefore, to prove the lemma, we can assume by induction that $d = 1$, and that, moreover, $\mathcal{Y}'$ is cut by the equation of a function with a nonvanishing differential.

Let Ψ, Φ be the corresponding near-by and vanishing cycles functors: $\mathrm{D}(Z) \to \mathrm{D}(Z')$. By assumption, we have $\Phi(\mathcal{F}) = 0$. The exactness of Φ implies that $\Phi(\mathcal{K}) = 0$ as well. Therefore, $\mathcal{K}|_{Z'} \simeq \Psi(\mathcal{K})[1]$, which is what we had to prove. $\square$

3.9. Now let i be the maximal integer, for which the functor $\mathcal{S} \mapsto h^i(\widetilde{\mathrm{ItAv}}^i_E(\mathcal{S})) : \widetilde{\mathrm{D}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ is non-identically zero. We know already that $i \leq n$. We are assuming that $i \geq 1$ and we want to arrive at a contradiction.

For $\mathcal{S} \in \widetilde{\mathrm{P}}(\mathrm{Bun}_n)$, we denote by $\mathcal{S}^i$ the object $h^i(\widetilde{\mathrm{ItAv}}^i_E(\mathcal{S})) \in \widetilde{\mathrm{P}}^{\Sigma_i}(\mathrm{Bun}_n)$ and consider the canonical surjection of Proposition-Construction 3.2

$$\widetilde{\mathrm{H}}^{\boxtimes i}(\mathcal{S}) \to (E^*)^{\boxtimes i}[i] \boxtimes \mathcal{S}^i.$$

We now apply the functor $\widetilde{\mathrm{H}}^{\boxtimes n}_{X^i} : \widetilde{\mathrm{P}}(X^i \times \mathrm{Bun}_n) \to \widetilde{\mathrm{P}}(X^{i+n} \times \mathrm{Bun}_n)$ to both sides. This functor maps $\widetilde{\mathrm{P}}^{\Sigma_i}(X^i \times \mathrm{Bun}_n)$ to $\widetilde{\mathrm{P}}^{\Sigma_i \times \Sigma_n}(X^{i+n} \times \mathrm{Bun}_n)$, and obtain a morphism

$$\widetilde{\mathrm{H}}^{\boxtimes i+n}(\mathcal{S}) \to (E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i), \tag{10}$$

which is still surjective, by the right exactness of $\widetilde{\mathrm{H}}^{\boxtimes n}_{X^i}$.

Note that the left-hand side of (10) is in fact an object of $\widetilde{\mathrm{P}}^{\Sigma_{i+n}}(X^{i+n} \times \mathrm{Bun}_n)$. We have a natural induction functor

$$\mathrm{Ind}^{\Sigma_{i+n}}_{\Sigma_i \times \Sigma_n} : \mathrm{P}^{\Sigma_i \times \Sigma_n}(X^{i+n} \times \mathrm{Bun}_n) \to \mathrm{P}^{\Sigma_{i+n}}(X^{i+n} \times \mathrm{Bun}_n),$$

which is the left (and right) adjoint to the forgetful functor. By passing to the quotient we obtain the corresponding induction functor from $\widetilde{\mathrm{P}}^{\Sigma_i \times \Sigma_n}(X^{i+n} \times \mathrm{Bun}_n)$ to $\widetilde{\mathrm{P}}^{\Sigma_{i+n}}(X^{i+n} \times \mathrm{Bun}_n)$.

Thus, we obtain a map in $\widetilde{\mathrm{P}}^{\Sigma_{i+n}}(X^{i+n} \times \mathrm{Bun}_n)$:

$$\widetilde{\mathrm{H}}^{\boxtimes i+n}(\mathcal{S}) \to \mathrm{Ind}^{\Sigma_{i+n}}_{\Sigma_i \times \Sigma_n}\left((E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i)\right).$$

The assumption that i was maximal will yield the following:

Proposition 3.10. *The above map*

$$\widetilde{\mathrm{H}}^{\boxtimes i+n}(\mathcal{S}) \to \mathrm{Ind}^{\Sigma_{i+n}}_{\Sigma_i \times \Sigma_n}\left((E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i)\right)$$

is surjective.

We conclude the proof of Theorem 2.14 using this proposition. Let $\Delta_i : X \to X^i$, $\Delta_n : X \to X^n$, $\Delta_2 : X \to X \times X$, and $\Delta_{i+n} : X \to X^{i+n}$ be the corresponding diagonal embeddings. According to Proposition 3.5, in the formula

$$(\Delta_{i+n} \times \mathrm{id})^* \left(\widetilde{\mathrm{H}}^{\boxtimes i+n}(\mathcal{S})\right)[1-i-n] \\ \to (\Delta_{i+n} \times \mathrm{id})^* \Big(\mathrm{Ind}_{\Sigma_i \times \Sigma_n}^{\Sigma_{i+n}} \left((E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i)\right)\Big)[1-i-n]$$

both sides belong to $\widetilde{\mathrm{P}}^{\Sigma_{i+n}}(X \times \mathrm{Bun}_n)$, and the map is surjective. Therefore, the above map will still be surjective when we pass to the *sign*-isotypic component on both sides with respect to Σ_{i+n}.

By (7), the left-hand side, i.e.,

$$\mathrm{Hom}_{\Sigma_{i+n}}\Big(\mathrm{sign}, (\Delta_{i+n} \times \mathrm{id})^* \left(\mathrm{H}^{\boxtimes i+n}(\mathcal{S})\right)[1-i-n]\Big)$$

vanishes. Therefore, so must the right-hand side.

Since the induction functors commute with the restriction functor $(\Delta_{i+n} \times \mathrm{id})^*$, by adjunction we obtain that

$$\mathrm{Hom}_{\Sigma_{i+n}}\Big(\mathrm{sign}, (\Delta_{i+n} \times \mathrm{id})^* \Big(\mathrm{Ind}_{\Sigma_i \times \Sigma_n}^{\Sigma_{i+n}} \left((E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i)\right)\Big)[1-i-n]\Big) \\ \simeq \mathrm{Hom}_{\Sigma_i \times \Sigma_n}\Big(\mathrm{Res}_{\Sigma_i \times \Sigma_n}^{\Sigma_{i+n}}(\mathrm{sign}), (\Delta_{i+n} \times \mathrm{id})^* \Big((E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i))[1-i-n]\Big)\Big).$$

We have: $\mathrm{Res}_{\Sigma_i \times \Sigma_n}^{\Sigma_{i+n}}(\mathrm{sign}) \simeq \mathrm{sign} \times \mathrm{sign}$, and $\Delta_{i+n} = \Delta_2 \circ (\Delta_i \times \Delta_n)$. Let us, therefore, rewrite the last expression as

(11)

$$(\Delta_2 \times \mathrm{id})^* \Big(\mathrm{Hom}_{\Sigma_i}\Big(\mathrm{sign}, (E^*)^{\otimes i} \boxtimes \mathrm{Hom}_{\Sigma_n}\Big(\mathrm{sign}, (\Delta_n \times \mathrm{id})^* (\widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i))[1-n]\Big)\Big)\Big).$$

Recall the multiplication map $m : X \times \mathrm{Bun}_n \to \mathrm{Bun}_n$, and recall also from (7), that for $\mathcal{S} \in \widetilde{\mathrm{D}}(\mathrm{Bun}_n)$

$$\mathrm{Hom}_{\Sigma_n}\left(\mathrm{sign}, (\Delta_n \times \mathrm{id})^* (\widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}))[1-n]\right) \simeq m^*(\mathcal{S})[1].$$

Therefore, (11) can be rewritten as

$$(\Delta_2 \times \mathrm{id})^* \Big(\mathrm{Hom}_{\Sigma_i}\left(\mathrm{sign}, (E^*)^{\otimes i} \boxtimes m^*(\mathcal{S}^i)[1]\right)\Big) \\ \simeq \mathrm{Hom}_{\Sigma_i}\Big(\mathrm{sign}, q^* \left((E^*)^{\otimes i}\right) \otimes m^*(\mathcal{S}^i)[1]\Big).$$

As in Section 3.4, since $i \leq \mathrm{rk}(E)$, we see that the vanishing of the latter expression implies that $m^*(\mathcal{S}^i) = 0$. Therefore, the functor

$$\mathcal{S} \mapsto m^*\big(h^i\left(\widetilde{\mathrm{ItAv}}^i_E(\mathcal{S})\right)\big)$$

vanishes identically.

We claim that this implies that the functor $\mathcal{S} \to h^i\left(\widetilde{\mathrm{ItAv}}^i_E(\mathcal{S})\right)$ vanishes. Indeed, for any fixed $x \in X$, consider the pull-back map $m_x^* : \mathrm{D}(\mathrm{Bun}_n) \to \mathrm{D}(\mathrm{Bun}_n)$, which is the composition of m^* and the restriction to $x \times \mathrm{Bun}_n \subset X \times \mathrm{Bun}_n$.

Obviously,

$$m_x^* \circ h^i\left(\mathrm{ItAv}^i_E(\mathcal{S})\right) \simeq h^i\left(\mathrm{ItAv}^i_E(m_x^*(\mathcal{S}))\right).$$

Hence, the corresponding functors on the level of $\widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ are also isomorphic.

Thus, we obtain that the functor $\mathcal{S} \mapsto h^i\left(\widetilde{\mathrm{ItAv}}^i_E(\mathcal{S})\right)$ "kills" the image of $m_x^* : \widetilde{\mathrm{P}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{P}}(\mathrm{Bun}_n)$.

However, since $m_x^* : \mathrm{D}(\mathrm{Bun}_n) \to \mathrm{D}(\mathrm{Bun}_n)$ is essentially surjective (i.e., surjective on objects), the same is true for $m_x^* : \widetilde{\mathrm{P}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{P}}(\mathrm{Bun}_n)$; in other words, $h^i\left(\widetilde{\mathrm{ItAv}}^i_E(\mathcal{S})\right)$ vanishes on the entire $\widetilde{\mathrm{P}}(\mathrm{Bun}_n)$.

3.11. *Proof of Proposition* 3.10. Observe that as an object of $\widetilde{\mathrm{P}}(X^{i+n} \times \mathrm{Bun}_n)$, $\mathrm{Ind}^{\Sigma_{i+n}}_{\Sigma_i \times \Sigma_n}((E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i))$ can be written as

$$\bigoplus_{\sigma \in \Sigma_{i+n}} \sigma^*((E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i)), \tag{12}$$

where the sum is taken over the coset representatives of $\Sigma_{i+n}/\Sigma_i \times \Sigma_n$.

The proof of the proposition is based on the following observation:

Lemma 3.12. *Let $\mathcal{K} \to \bigoplus_i \mathcal{K}_i$ be a map of objects of an Artinian abelian category, such that each of the maps $\mathcal{K} \to \mathcal{K}_i$ is surjective. Assume that for $i \neq j$, $\mathcal{K}_i$ and $\mathcal{K}_j$ have no isomorphic quotients. Then the map $\mathcal{K} \to \bigoplus_i \mathcal{K}_i$ is surjective as well.*

We know that the map $\widetilde{\mathrm{H}}^{\boxtimes i+n}(\mathcal{S}) \to (E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i)$ is surjective. By the Σ_{i+n}-equivariance of $\widetilde{\mathrm{H}}^{\boxtimes i+n}(\mathcal{S})$, we obtain that each

$$\widetilde{\mathrm{H}}^{\boxtimes i+n}(\mathcal{S}) \to \sigma^*\left((E^*)^{\boxtimes i}[i] \boxtimes \mathrm{H}^{\boxtimes n}(\mathcal{S}^i)\right)$$

is surjective as well.

To apply this lemma we need to verify that for $\sigma_1, \sigma_2 \in \Sigma_{i+n}$, which belong to different cosets, the objects $\sigma_1^*\left((E^*)^{\boxtimes i}[i] \boxtimes \mathrm{H}^{\boxtimes n}(\mathcal{S}^i)\right)$ and $\sigma_2^*\left((E^*)^{\boxtimes i}[i] \boxtimes \mathrm{H}^{\boxtimes n}(\mathcal{S}^i)\right)$ of $\widetilde{\mathrm{P}}(X^i \times X^n \times \mathrm{Bun}_n)$ have no isomorphic quotients.

Again, by Σ_{i+n}-equivariance, we can assume that σ_1 is the unit element, and $\sigma = \sigma_2$ is such that the permutation that it defines on the set $\{1, \dots, i+n\}$ satisfies $\sigma(1) = i+1$.

For any $j \in \{1, \dots, k\}$, let $q_{k,j}$ denote the projection on the j-th factor $X^k \times \mathrm{Bun}_n \to X$, and $p_{k,j}$ the complementary projection on $X^{k-1} \times \mathrm{Bun}_n$.

Let $\mathcal{S}'$ be a quotient common to

$$(E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i) \text{ and } \sigma^*\left((E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i)\right).$$

Since E is irreducible, every (sub)-quotient of $(E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i)$ is of the form $(E^*)^{\boxtimes i}[i] \boxtimes \mathcal{S}''$, where $\mathcal{S}'' \in \widetilde{\mathrm{P}}(X^n \times \mathrm{Bun}_n)$ is a (sub)-quotient of $\widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i)$. Therefore,

$$h^1\left(p_{i+n,1!}(q^*_{i+n,1}(E) \otimes \mathcal{S}')\right) \neq 0. \tag{13}$$

As in Proposition 3.2, this implies:

$$h^1\left(p_{i+n,1!}\left(q^*_{i+n,1}(E) \otimes \sigma^*\left((E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i)\right)\right)\right) \neq 0,$$

which is equivalent to

$$h^1\left(p_{i+n,i+1!}\left(q^*_{i+n,i+1}(E) \otimes \left((E^*)^{\boxtimes i}[i] \boxtimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i)\right)\right)\right) \neq 0,$$

and hence $h^1\left(p_{n,1!}\left(q^*_{n,1}(E) \otimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i)\right)\right) \neq 0$.

A simple diagram chase shows:

LEMMA 3.13. *For any $\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n)$,*

$$p_{k,1!}\left(q^*_{k,1}(E) \otimes \mathrm{H}^{\boxtimes k}(\mathcal{F})\right) \simeq \mathrm{H}^{\boxtimes k-1}(\mathrm{Av}^1_E(\mathcal{F})).$$

The lemma implies that we also have an isomorphism

$$p_{k,1!}\left(q^*_{k,1}(E) \otimes \widetilde{\mathrm{H}}^{\boxtimes k}(\mathcal{S})\right) \simeq \widetilde{\mathrm{H}}^{\boxtimes k-1}\left(\widetilde{\mathrm{Av}}^1_E(\mathcal{S})\right)$$

as functors $\widetilde{\mathrm{D}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(X^{k-1} \times \mathrm{Bun}_n)$.

Therefore, the fact that $h^1\left(p_{n,1!}\left(q^*_{n,1}(E) \otimes \widetilde{\mathrm{H}}^{\boxtimes n}(\mathcal{S}^i)\right)\right)$ is nonzero as an object of $\widetilde{\mathrm{P}}(X^{n-1} \times \mathrm{Bun}_n)$ implies that $h^1\left(\widetilde{\mathrm{H}}^{\boxtimes n-1}\left(\widetilde{\mathrm{Av}}^1_E(\mathcal{S}^i)\right)\right) \neq 0$. The exactness of $\widetilde{\mathrm{H}}^{\boxtimes n-1} : \widetilde{\mathrm{P}}(\mathrm{Bun}_n) \to \widetilde{\mathrm{P}}(X^{n-1} \times \mathrm{Bun}_n)$ forces $h^1\left(\widetilde{\mathrm{Av}}^1_E(\mathcal{S}^i)\right) \simeq h^{i+1}(\widetilde{\mathrm{ItAv}}^{i+1}_E(\mathcal{S})) \neq 0$.

However, this contradicts the assumption that i was the maximal integer for which $h^i(\widetilde{\mathrm{ItAv}}^i_E(\mathcal{S})) \neq 0$.

4. Whittaker categories

From this moment on we will be occupied with construction of the quotient categories $\widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n)$. This will be done using the formalism of Whittaker categories and functors between them. The first step, i.e., the definition of the appropriate categories, is the goal of the present section, which we carry out in a way similar to the definition of Whittaker categories in Section 6 of [6].

4.1. *Drinfeld's compactifications.* Let Bun_n' be the stack classifying pairs $(\mathcal{M}, \kappa_1)$, where κ_1 is a nonzero map $\Omega^{n-1} \to \mathcal{M}$. Let π denote the natural projection $\mathrm{Bun}_n' \to \mathrm{Bun}_n$.

Recall also the stack $\overline{\mathcal{Q}}$ introduced in [7]. We will now introduce a series of stacks $\overline{\mathcal{Q}}_1, \dots, \overline{\mathcal{Q}}_n$ with $\overline{\mathcal{Q}}_1 = \mathrm{Bun}_n'$, $\overline{\mathcal{Q}}_n = \overline{\mathcal{Q}}$, which interpolate between the two.

Namely, $\overline{\mathcal{Q}}_k$ classifies the data of a rank n bundle $\mathcal{M}$ and a collection of nonzero maps

$$\kappa_i : \Omega^{n-1+\cdots+n-i} \to \Lambda^i(\mathcal{M}),\ i = 1, \dots, k,$$

which satisfy the Plücker relations in the sense of [7].

Let, in addition, $\overline{\mathcal{Q}}_{k,ex}$ be the stack classifying the same data $(\mathcal{M}, \kappa_1, \dots, \kappa_k)$ as above, but where we allow the last map, i.e., κ_k, to vanish. In particular, $\overline{\mathcal{Q}}_k$ is an open substack in $\overline{\mathcal{Q}}_{k,ex}$.

We have the natural forgetful maps $\pi_{k+1,k} : \overline{\mathcal{Q}}_{k+1} \to \overline{\mathcal{Q}}_k$, and $\pi_{k+1,ex,k} : \overline{\mathcal{Q}}_{k+1,ex} \to \overline{\mathcal{Q}}_k$.

We will introduce certain triangulated categories $\mathrm{D}^W(\overline{\mathcal{Q}}_k)$ (resp., $\mathrm{D}^W(\overline{\mathcal{Q}}_{k,ex})$) of sheaves on $\overline{\mathcal{Q}}_k$ (resp., $\mathrm{D}^W(\overline{\mathcal{Q}}_{k,ex})$), that we will call the Whittaker categories.

Each $\mathrm{D}^W(\overline{\mathcal{Q}}_k)$ will be a full triangulated subcategory of $\mathrm{D}(\overline{\mathcal{Q}}_k)$ defined by the condition that $\mathcal{F} \in \mathrm{D}^W(\overline{\mathcal{Q}}_k)$ if its perverse cohomologies belong to a certain Serre subcategory $\mathrm{P}^W(\overline{\mathcal{Q}}_k) \subset \mathrm{P}(\overline{\mathcal{Q}}_k)$, singled out by some equivariance condition; and similarly for $\mathrm{D}^W(\overline{\mathcal{Q}}_{k,ex})$.

By definition, for $k = 1$, $\mathrm{D}^W(\overline{\mathcal{Q}}_k)$ is the entire $\mathrm{D}(\overline{\mathcal{Q}}_1) = \mathrm{D}(\mathrm{Bun}_n')$, i.e., the equivariance condition in this case is vacuous.

4.2. For a fixed point $y \in X$, let $\overline{\mathcal{Q}}_k^y$ be an open substack that corresponds to the condition that neither of the maps $\kappa_1, \dots, \kappa_k$ has a zero at y.

If $(\mathcal{M}, \kappa_1, \dots, \kappa_k)$ is a point of $\overline{\mathcal{Q}}_k^y$, on the formal disk $\mathcal{D}_y$ around y we obtain a flag

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_k \subset \mathcal{M}|_{\mathcal{D}_y}$$

with $\mathcal{M}_j/\mathcal{M}_{j-1} \simeq \Omega^{n-j}|_{\mathcal{D}_y}$.

Let $N_{k,\mathcal{D}_y}$ be the group-scheme (of infinite type) over $\overline{\mathcal{Q}}_k^y$ defined as follows: its fiber over a point $(\mathcal{M}, \kappa_1, \dots, \kappa_k)$ as above consists of all automorphisms of $\mathcal{M}|_{\mathcal{D}_y}$, which are strictly upper-triangular with respect to the flag of the $\mathcal{M}_i$'s.

In addition, we have a group-indscheme $N_{k,\mathcal{D}_y^*}$, which contains $N_{k,\mathcal{D}_y}$ as a group-subscheme, and whose fiber over $(\mathcal{M},\kappa_1,\dots,\kappa_k)$ consists of all automorphisms, strictly upper-triangular with respect to the given flag, of $\mathcal{M}$ over the formal punctured disk $\mathcal{D}_y^*$.

As in [6], one can show that $N_{k,\mathcal{D}_y^*}$ is in fact an ind-groupscheme. More precisely, $N_{k,\mathcal{D}_y^*}$ can be represented as a union of group-schemes $N^i_{k,\mathcal{D}_y^*}$, $i\in\mathbb{N}$ with $N^i_{k,\mathcal{D}_y^*}\supset N_{k,\mathcal{D}_y}$ and $N^i_{k,\mathcal{D}_y^*}/N_{k,\mathcal{D}_y}$ finite-dimensional.

The quotient $\mathcal{H}_{N_k^y}:=N_{k,\mathcal{D}_y^*}/N_{k,\mathcal{D}_y}$ is an ind-scheme over the stack $\overline{\mathcal{Q}}^y_k$ and is a version of the Hecke stack for the corresponding unipotent group. We have: $\mathcal{H}_{N_k^y}=\bigcup\limits_{i\in\mathbb{N}}\mathcal{H}^i_{N_k^y}$, where $\mathcal{H}^i_{N_k^y}:=N^i_{k,\mathcal{D}_y^*}/N_{k,\mathcal{D}_y}$; the latter is isomorphic to a tower of fibrations into affine spaces over $\overline{\mathcal{Q}}^y_k$.

We let pr_k (resp., pr^i_k) denote the natural projection $\mathcal{H}_{N_k^y}\to\overline{\mathcal{Q}}^y_k$ (resp., $\mathcal{H}^i_{N_k^y}\to\overline{\mathcal{Q}}^y_k$).

4.3. *The groupoids.* We claim that $\mathcal{H}_{N_k^y}$ carries a natural structure of a groupoid over $\overline{\mathcal{Q}}^y_k$. This is the standard procedure that makes the Hecke stack a groupoid over the moduli space of bundles; cf. [6].

Namely, we define the second projection $\mathrm{act}_k:\mathcal{H}_{N_k^y}\to\overline{\mathcal{Q}}^y_k$ as follows:

Given a point $(\mathcal{M},\kappa_1,\dots,\kappa_k)\in\overline{\mathcal{Q}}^y_k$ and an automorphism $g:\mathcal{M}|_{\mathcal{D}_y^*}\to\mathcal{M}|_{\mathcal{D}_y^*}$, the new bundle $\mathcal{M}'$ is defined to be equal to $\mathcal{M}$ on $X-y$, and a meromorphic section $m'\in\Gamma(X-y,\mathcal{M}')$ is regular if $g(m')$, viewed as an element of $\Gamma(\mathcal{D}_y^*,\mathcal{M})$, belongs to $\Gamma(\mathcal{D}_y,\mathcal{M})$.

The condition that g is strictly upper-triangular means that $\mathcal{M}'|_{\mathcal{D}_y}$ is still endowed with a filtration

$$0=\mathcal{M}'_0\subset\mathcal{M}'_1\subset\cdots\subset\mathcal{M}'_k\subset\mathcal{M}'_n:=\mathcal{M}'|_{\mathcal{D}_y}$$

with $\mathcal{M}'_j/\mathcal{M}'_{j-1}\simeq\Omega^{n-j}|_{\mathcal{D}_y}$.

Again, from the construction, the "old" maps $\kappa_i:\Omega^{n-1+\dots+n-i}\to\Lambda^i(\mathcal{M})$, which are *a priori* meromorphic as maps $\Omega^{n-1+\dots+n-i}\to\Lambda^i(\mathcal{M}')$, are in fact regular, and thus define the data κ'_i for $\mathcal{M}'$.

Let $\overline{\mathcal{Q}}^y_{k+1,ex}:=\overline{\mathcal{Q}}^y_k\underset{\overline{\mathcal{Q}}_k}{\times}\overline{\mathcal{Q}}_{k+1,ex}$ be the preimage of $\overline{\mathcal{Q}}^y_k$ in $\overline{\mathcal{Q}}_{k+1,ex}$. (Note that $\overline{\mathcal{Q}}^y_{k+1}$ denotes a completely different stack; we have inclusions $\overline{\mathcal{Q}}^y_{k+1}\subset\overline{\mathcal{Q}}_{k+1}\subset\overline{\mathcal{Q}}_{k+1,ex}\supset\overline{\mathcal{Q}}^y_{k+1,ex}$.)

Consider the pull-back $\mathcal{H}_{N_k^y}\underset{\overline{\mathcal{Q}}^y_k}{\times}\overline{\mathcal{Q}}^y_{k+1,ex}$ as an ind-scheme over $\overline{\mathcal{Q}}^y_{k+1,ex}$. The next assertion follows from the construction:

LEMMA 4.4. *The fiber product $\mathcal{H}_{N_k^y}\underset{\overline{\mathcal{Q}}^y_k}{\times}\overline{\mathcal{Q}}^y_{k+1,ex}$ has a natural structure of a groupoid over $\overline{\mathcal{Q}}^y_{k+1,ex}$, i.e., there exists a naturally defined map* $\mathrm{act}_{k,ex}$:

$\mathcal{H}_{N_k^y} \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}^y_{k+1,ex} \to \overline{\mathcal{Q}}^y_{k+1,ex}$, *which makes the following diagram Cartesian*

$$\begin{CD}
\overline{\mathcal{Q}}^y_{k+1,ex} @<{\mathrm{act}_{k,ex}}<< \mathcal{H}_{N_k^y} \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}^y_{k+1,ex} \\
@V{\pi_{k+1,ex,k}}VV @VV{\mathrm{id}\times\pi_{k+1,ex,k}}V \\
\overline{\mathcal{Q}}^y_k @<{\mathrm{act}_k}<< \mathcal{H}_{N_k^y}.
\end{CD}$$

We will denote by $\mathrm{act}^i_{k,ex}$ the restriction of $\mathrm{act}_{k,ex}$ to the sub-groupoid $\mathcal{H}^i_{N_k^y} \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}^y_{k+1,ex}$, and by $\mathrm{pr}_{k,ex}$ (resp., $\mathrm{pr}^i_{k,ex}$) the natural projection from $\mathcal{H}_{N_k^y} \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}^y_{k+1,ex}$ (resp., from $\mathcal{H}^i_{N_k^y} \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}^y_{k+1,ex}$) to $\overline{\mathcal{Q}}^y_{k+1,ex}$.

4.5. *The characters.* One more observation we need to make before introducing the categories of interest is the following:

We claim that there exists a natural morphism

$$\chi_k : \mathcal{H}_{N_k^y} \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}^y_{k+1,ex} \to \mathbb{A}^1.$$

Indeed, a point of $\mathcal{H}_{N_k^y} \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}^y_{k+1,ex}$ is the data of

$$(\mathcal{M}, \kappa_1, \ldots, \kappa_k) \in \overline{\mathcal{Q}}^y_k,\ \kappa_{k+1} : \Omega^{n-1+\cdots+n-k-1} \to \Lambda^{k+1}(\mathcal{M}),\ g \in \mathrm{Aut}(\mathcal{M}|_{\mathcal{D}_y^*}).$$

The endomorphism $(g-\mathrm{Id})$ defines for every $i = 1, \ldots, k$ a map $(\mathcal{M}/\mathcal{M}_i)|_{\mathcal{D}_y^*} \to (\mathcal{M}_i)_{\mathcal{D}_y^*}$, which we compose with

$$\Omega^{n-i-1}|_{\mathcal{D}_y^*} \to (\mathcal{M}/\mathcal{M}_i)|_{\mathcal{D}_y^*} \text{ and } (\mathcal{M}_i)_{\mathcal{D}_y^*} \to (\mathcal{M}_i/\mathcal{M}_{i-1})_{\mathcal{D}_y^*} \simeq \Omega^{n-i}|_{\mathcal{D}_y^*}.$$

As a result, for every $i = 1, \ldots, k$ we obtain a map $\Omega^{n-i-1}|_{\mathcal{D}_y^*} \to \Omega^{n-i}|_{\mathcal{D}_y^*}$, well-defined up to a map regular on $\mathcal{D}_y$, due to the corresponding ambiguity in g. By taking residues at y we obtain k points of $\mathbb{A}^1$, i.e., we obtain well-defined maps ${}^i\chi_k : \mathcal{H}_{N_k^y} \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}^y_{k+1,ex} \to \mathbb{A}^1$ for $i = 1, \ldots, k$.

The map χ_k is defined as a composition

$$\mathcal{H}_{N_k^y} \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}^y_{k+1,ex} \overset{{}^i\chi_k}{\to} \prod_i (\mathbb{A}^1) \overset{\mathrm{sum}}{\to} \mathbb{A}^1.$$

We will denote by χ^i_k the restriction of χ_k to $\mathcal{H}^i_{N_k^y} \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}^y_{k+1,ex} \subset \mathcal{H}_{N_k^y} \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}^y_{k+1,ex}$.

In what follows A-Sch will denote the Artin-Schreier sheaf on $\mathbb{A}^1$.

4.6. Everything said above can be generalized in a straightforward way when one point y is replaced by a finite collection of pairwise distinct points $\overline{y} := y_1, \ldots, y_m$.

Namely, we have the open substack

$$\overline{\mathcal{Q}}_k^{\overline{y}} := \bigcap_j \overline{\mathcal{Q}}_k^{y_j} \subset \overline{\mathcal{Q}}_k,$$

and the groupoid $\mathcal{H}_{N_k^{\overline{y}}}$ over it. In fact,

$$\mathcal{H}_{N_k^{\overline{y}}} \simeq \mathcal{H}_{N_k^{y_1}}|_{\overline{\mathcal{Q}}_k^{\overline{y}}} \underset{\overline{\mathcal{Q}}_k^{\overline{y}}}{\times} \dots \underset{\overline{\mathcal{Q}}_k^{\overline{y}}}{\times} \mathcal{H}_{N_k^{y_m}}|_{\overline{\mathcal{Q}}_k^{\overline{y}}}.$$

In other words, the groupoids $\mathcal{H}_{N_k^{y_j}}|_{\overline{\mathcal{Q}}_k^{\overline{y}}}$, $j = 1, \dots, m$, acting on $\overline{\mathcal{Q}}_k^{\overline{y}}$ pairwise commute in the natural sense, hence we can form the product groupoid, which can be identified with $\mathcal{H}_{N_k^{\overline{y}}}$.

4.7. *The categories on* $\overline{\mathcal{Q}}_{k+1,ex}^{y}$. We define the category $\mathrm{P}^W(\overline{\mathcal{Q}}_{k+1,ex}^{y}) \subset \mathrm{P}(\overline{\mathcal{Q}}_{k+1,ex}^{y})$ to consist of all perverse sheaves $\mathcal{F} \in \mathrm{P}(\overline{\mathcal{Q}}_{k+1,ex}^{y})$, for which the following holds:

For each $i \in \mathbb{N}$, there exists an isomorphism between the following two sheaves on $\mathcal{H}_{N_k}^i \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}_{k+1,ex}^{y}$:

$$\mathrm{act}_{k,ex}^{i}{}^*(\mathcal{F}) \text{ and } \mathrm{pr}_{k,ex}^{i}{}^*(\mathcal{F}) \otimes \chi_k^{i}{}^*(\text{A-Sch}) \tag{14}$$

such that the restriction of this isomorphism to the unit section $\overline{\mathcal{Q}}_{k+1,ex}^{y} \subset \mathcal{H}_{N_k}^i \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}_{k+1,ex}^{y}$ is the identity map.

Note that both sides of (14) are objects of $\mathrm{D}(\mathcal{H}_{N_k}^i \underset{\overline{\mathcal{Q}}_k^y}{\times} \overline{\mathcal{Q}}_{k+1,ex}^{y})$, which become perverse after the cohomological shift by $\dim.\mathrm{rel.}(\mathcal{H}_{N_k}^i, \overline{\mathcal{Q}}_k^y)$, since both maps $\mathrm{pr}_{k,ex}^i$ and $\mathrm{act}_{k,ex}^i$ are smooth of that relative dimension.

Since A-Sch is a 1-dimensional lisse sheaf and the fibers of $\mathrm{pr}_{k,ex}^i$ are connected, if an isomorphism of (14) exists, it is unique. Moreover a family of such isomorphisms for $i \in \mathbb{N}$ is necessarily compatible. All this follows from the next general lemma:

LEMMA 4.8. *Let* $p : \mathcal{Y}_1 \to \mathcal{Y}_2$ *be a smooth surjective map between schemes* (*or stacks*) *of relative dimension* d, *which has connected fibers. Then*

(1) $\mathcal{F} \mapsto p^*(\mathcal{F})[d]$ *is a full embedding of* $\mathrm{P}(\mathcal{Y}_2)$ *into* $\mathrm{P}(\mathcal{Y}_1)$; *its image is stable under sub-quotients.*

(2) *If, moreover, the fibers of* p *are contractible, then the same is true when* $\mathrm{P}(\mathcal{Y}_i)$ *is replaced by* $\mathrm{D}(\mathcal{Y}_i)$, *i.e.*, $\mathrm{D}(\mathcal{Y}_2)$ *is a full triangulated subcategory of* $\mathrm{D}(\mathcal{Y}_1)$. *In particular*, $\mathrm{P}(\mathcal{Y}_2) \subset \mathrm{P}(\mathcal{Y}_1)$ *is stable under extensions, and is therefore a Serre subcategory.*

Since $\mathrm{pr}^i_{k,ex} : \mathcal{H}^i_{N_k} \underset{\overline{\mathcal{Q}}^y_k}{\times} \overline{\mathcal{Q}}^y_{k+1,ex} \to \overline{\mathcal{Q}}^y_{k+1,ex}$ is a tower of affine fibrations, from Lemma 4.8 above, we obtain that $\mathrm{P}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$ is indeed a Serre subcategory of $\mathrm{P}(\overline{\mathcal{Q}}^y_{k+1,ex})$.

We define $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$ as the full triangulated subcategory of $\mathrm{D}(\overline{\mathcal{Q}}^y_{k+1,ex})$, consisting of objects whose perverse cohomologies belong to $\mathrm{P}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$.

From Lemma 4.8 it follows that for any $\mathcal{F} \in \mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$ there exists a unique isomorphism

$$\mathrm{act}^i_{k,ex}{}^*(\mathcal{F}) \simeq \mathrm{pr}^i_{k,ex}{}^*(\mathcal{F}) \otimes \chi^i_k{}^*(\text{A-Sch}),$$

compatible with the restrictions of both sides to the unit section.

In the same way, for a collection of pairwise distinct points $\overline{y} = y_1, \ldots, y_m$, one defines the categories $\mathrm{P}^W(\overline{\mathcal{Q}}^{\overline{y}}_{k+1,ex})$ and $\mathrm{D}^W(\overline{\mathcal{Q}}^{\overline{y}}_{k+1,ex})$, the former being a Serre subcategory of $\mathrm{P}(\overline{\mathcal{Q}}^{\overline{y}}_{k+1,ex})$, and the latter a full triangulated subcategory of $\mathrm{D}(\overline{\mathcal{Q}}^{\overline{y}}_{k+1,ex})$.

4.9. To proceed, we need to recall a natural stratification defined on the stacks $\overline{\mathcal{Q}}_k$.

For a string of nonnegative integers $\overline{d} := (d_1, \ldots, d_k)$ let $X^{(\overline{d})}$ be the corresponding partially symmetrized power of the curve X:

$$X^{(\overline{d})} = \prod_{j=1,\ldots,k} X^{(d_j)}.$$

Let ${}^{\overline{d}}\overline{\mathcal{Q}}_k$ be the stack that classifies the data of $(\mathcal{M}, \kappa_1, \ldots, \kappa_k, D_1, \ldots, D_k)$, where $\mathcal{M}$ is as before, $D_i \in X^{(d_j)}$, and each κ_i is an injective bundle map

$$\Omega^{n-1+\cdots+n-i}(D_i) \to \Lambda^i(\mathcal{M}),$$

such that $\{\kappa_1, \ldots, \kappa_k\}$ satisfies the Plücker relations.

We have a natural map ${}^{\overline{d}}\overline{\mathcal{Q}}_k \to \overline{\mathcal{Q}}_k$. It was shown in [3] that each ${}^{\overline{d}}\overline{\mathcal{Q}}_k$ becomes a locally closed substack of $\overline{\mathcal{Q}}_k$, and that, moreover, these substacks for various $\overline{d}$ define a locally finite decomposition of $\overline{\mathcal{Q}}_k$ into locally closed pieces.

Observe that each ${}^{\overline{d}}\overline{\mathcal{Q}}_k$ can be alternatively viewed as a stack classifying the data of a vector bundle $\mathcal{M}$ endowed with a filtration

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_k \subset \mathcal{M},$$

and identifications $\mathcal{M}_i/\mathcal{M}_{i-1} \simeq \Omega^{n-i}(D_i - D_{i-1})$ for $(D_1, \ldots, D_k) \in X^{(\overline{d})}$.

4.10. For a string of integers $\overline{d}$ as above, let ${}^{\overline{d}}\overline{\mathcal{Q}}^y_k$ denote the intersection ${}^{\overline{d}}\overline{\mathcal{Q}}_k \cap \overline{\mathcal{Q}}^y_k$. Let ${}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex} \subset {}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}$ be the preimages of ${}^{\overline{d}}\overline{\mathcal{Q}}^y_k$ and ${}^{\overline{d}}\overline{\mathcal{Q}}_k$, respectively, in $\overline{\mathcal{Q}}_{k+1,ex}$.

Note that ${}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex}$ is the stack that classifies the data of $(D_1,\dots,D_k)\in (X-y)^{(\overline{d})}$, a vector bundle $\mathcal{M}$, a filtration

$$0=\mathcal{M}_0\subset\mathcal{M}_1\subset\cdots\subset\mathcal{M}_k\subset\mathcal{M}$$

with $\mathcal{M}_i/\mathcal{M}_{i-1}\simeq\Omega^{n-i}(D_i-D_{i-1})$ and, finally, a map $\widetilde{\kappa}_{k+1}:\Omega^{n-k-1}(-D_k)\to\mathcal{M}/\mathcal{M}_k$.

Note that each ${}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex}$ is stable under the action of the groupoid $\mathcal{H}^i_{N_k}\underset{\overline{\mathcal{Q}}^y_k}{\times}\overline{\mathcal{Q}}^y_{k+1,ex}$. Therefore, in the same way as above, we can introduce the category $\mathrm{P}^W({}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex})$ (resp., $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex})$), which is a Serre subcategory (resp., a full triangulated subcategory) of $\mathrm{P}({}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex})$ (resp., $\mathrm{D}({}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex})$).

From Lemma 4.8 we obtain:

LEMMA 4.11. (1) *The $*$ and ! restrictions* $\mathrm{D}(\overline{\mathcal{Q}}^y_{k+1,ex})\to\mathrm{D}({}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex})$ *map the category* $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$ *into* $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex})$.

(2) *The $*$ and ! direct image functors map* $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex})$ *to* $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})\subset\mathrm{D}(\overline{\mathcal{Q}}^y_{k+1,ex})$.

(3) *An object* $\mathcal{F}\in\mathrm{D}(\overline{\mathcal{Q}}^y_{k+1,ex})$ *belongs to* $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$ *if and only if its $*$-restrictions (or, equivalently, !-restrictions) to* ${}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex}$ *belong to* $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex})$ *for all* $\overline{d}$.

Of course, the same assertion holds when we replace one point y by a finite collection of points $\overline{y}$.

Proof. The fact that each ${}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex}$ is stable under the action of $\mathcal{H}^i_{N_k}\underset{\overline{\mathcal{Q}}^y_k}{\times}\overline{\mathcal{Q}}^y_{k+1,ex}$ means that we have a commutative diagram

$$\begin{tikzcd}
{}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex} \arrow[d] & \mathcal{H}^i_{N_k}\underset{\overline{\mathcal{Q}}^y_k}{\times}{}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex} \arrow[l] \arrow[r] \arrow[d] & {}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex} \arrow[d] \\
\overline{\mathcal{Q}}^y_{k+1,ex} & \mathcal{H}^i_{N_k}\underset{\overline{\mathcal{Q}}^y_k}{\times}\overline{\mathcal{Q}}^y_{k+1,ex} \arrow[l, "\mathrm{act}^i_{k,ex}"'] \arrow[r, "\mathrm{pr}^i_{k,ex}"] & \overline{\mathcal{Q}}^y_{k+1,ex},
\end{tikzcd}$$

in which both squares are Cartesian. As remarked above, an object $\mathcal{F}\in\mathrm{D}(\overline{\mathcal{Q}}^y_{k+1,ex})$ belongs to $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$ if and only if we have a compatible system of isomorphisms $\mathrm{act}^i_{k,ex}{}^*(\mathcal{F})\simeq\mathrm{pr}^i_{k,ex}{}^*(\mathcal{F})\otimes\chi^i_k{}^*(\text{A-Sch})$, and similarly for ${}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex}$. This implies assertions (1) and (2) of the lemma.

Assertion (3) follows from (1) and (2), since the decomposition of $\overline{\mathcal{Q}}^y_{k+1,ex}$ into the strata ${}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex}$ is locally finite. $\square$

4.12. We will now analyze how objects of $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$ can look like when restricted to ${}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex}$.

For $\overline{d}$ as above, let ${}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}' \subset {}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}$ denote the closed substack that corresponds to the condition that for $i = 1, \ldots, k$, each $D'_i := D_i - D_{i-1}$ is an effective divisor (by definition, $D'_1 = D_1$) and that moreover for $i = 1, \ldots, k-1$ each $D'_{i+1} - D'_i$ is effective, and $\widetilde{\kappa}_{k+1} : \Omega^{n-k-1}(-D_k) \to \mathcal{M}/\mathcal{M}_k$ factors as $\Omega^{n-k-1}(-D_k) \to \Omega^{n-k-1}(D'_k) \overset{\widetilde{\kappa}'_{k+1}}{\to} \mathcal{M}/\mathcal{M}_k$.

Note that we have a natural map ${}^{\overline{d}}\chi'_k : {}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}' \to \mathbb{A}^1$, defined in a way similar to how the map χ_k was defined. Namely, we have to sum up the classes of the successive extensions

$$0 \to \Omega^{n-i}(D'_i) \to \mathcal{M}_{i+1}/\mathcal{M}_{i-1} \to \Omega^{n-i-1}(D'_{i+1}) \to 0$$

in

$$\mathrm{Ext}^1\left(\Omega^{n-i-1}(D'_{i+1}), \Omega^{n-i}(D'_i)\right) \simeq H^1(X, \Omega(D'_i - D'_{i+1})) \to H^1(X, \Omega) \simeq \mathbb{A}^1$$

for $i = 1, \ldots, k-1$ and the class of the induced extension of $\Omega^{n-k-1}(D'_k)$ by $\Omega^{n-k}(D'_k)$ by means of $\widetilde{\kappa}'_{k+1}$.

Let ${}^{\overline{d}}\mathcal{P}_k$ denote the stack classifying the data of $(D_1, \ldots, D_k) \in X^{(\overline{d})}$ such that each $D'_i := D_i - D_{i-1}$ and $D'_{i+1} - D'_i$ is effective, a vector bundle $\mathcal{M}'$ of rank $n-k$ with a map $\Omega^{n-k-1}(D'_k) \to \mathcal{M}'$.

Note that we have a natural projection $\phi_k : {}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}' \to {}^{\overline{d}}\mathcal{P}_k$, with $\mathcal{M}' := \mathcal{M}/\mathcal{M}_k$ in the above notation. The map ϕ_k is smooth and has contractible fibers.

Let ${}^{\overline{d}}\mathcal{P}^y_k \subset {}^{\overline{d}}\mathcal{P}_k$ be the open substack that corresponds to the condition that the divisors D_i avoid y. Let ${}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex}{}' := {}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}' \cap \overline{\mathcal{Q}}^y_{k+1,ex}$.

The following proposition is a version of Lemma 6.2.8 of [6].

PROPOSITION 4.13. (1) *Every object of* $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex})$ *is supported on* $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex}{}')$.

(2) *The functor* $\mathcal{F} \mapsto \phi_k{}^*(\mathcal{F}) \otimes \chi'_k{}^*(\text{A-Sch})$ *defines an equivalence of categories* $\mathrm{D}({}^{\overline{d}}\mathcal{P}^y_k) \to \mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex})$.

A similar assertion holds when a single point y is replaced by a finite collection $\overline{y}$.

COROLLARY 4.14. *Let* $\overline{y} = y_1, \ldots, y_m$ *be a collection of points with* y *being one of them. Then the restriction functor* $\mathrm{D}(\overline{\mathcal{Q}}^y_{k+1,ex}) \to \mathrm{D}(\overline{\mathcal{Q}}^{\overline{y}}_{k+1,ex})$ *maps the category* $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$ *to* $\mathrm{D}^W(\overline{\mathcal{Q}}^{\overline{y}}_{k+1,ex})$.

Proof. Let $\mathcal{F}$ be an object of $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$. According to Lemma 4.11, it suffices to check that the $*$-restrictions $\mathcal{F}|_{{}^{\overline{d}}\overline{\mathcal{Q}}^{\overline{y}}_{k+1,ex}}$ all belong to $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}^{\overline{y}}_{k+1,ex})$.

We know that $\mathcal{F}|_{{}^{\overline{d}}\overline{\mathcal{Q}}^{y}_{k+1,ex}}$ can be described as a pull-back from ${}^{\overline{d}}\mathcal{P}^{y}_{k}$ as in Proposition 4.13, tensored by the pull-back of the Artin-Schreier sheaf, and this description obviously survives the further restriction to ${}^{\overline{d}}\overline{\mathcal{Q}}^{\overline{y}}_{k+1,ex}$. $\square$

4.15. Finally, we define the category $\mathrm{P}^{W}(\overline{\mathcal{Q}}_{k+1,ex})$ to consist of all perverse sheaves $\mathcal{F} \in \mathrm{P}(\overline{\mathcal{Q}}_{k+1,ex})$ for which $\mathcal{F}|_{\overline{\mathcal{Q}}^{\overline{y}}_{k+1,ex}}$ belongs to $\mathrm{P}^{W}(\overline{\mathcal{Q}}^{\overline{y}}_{k+1,ex})$ for all finite collections $\overline{y}$.

According to Lemma 4.8, $\mathrm{P}^{W}(\overline{\mathcal{Q}}_{k+1,ex})$ is a Serre subcategory of $\mathrm{P}(\overline{\mathcal{Q}}_{k+1,ex})$. We let $\mathrm{D}^{W}(\overline{\mathcal{Q}}_{k+1,ex})$ be the full triangulated subcategory of $\mathrm{D}(\overline{\mathcal{Q}}_{k+1,ex})$ generated by $\mathrm{P}^{W}(\overline{\mathcal{Q}}_{k+1,ex})$. In other words, $\mathcal{F} \in \mathrm{D}^{W}(\overline{\mathcal{Q}}_{k+1,ex})$ if and only if all of its perverse cohomologies belong to $\mathrm{P}^{W}(\overline{\mathcal{Q}}_{k+1,ex})$.

According to Corollary 4.14, in order to check that $\mathcal{F} \in \mathrm{D}(\overline{\mathcal{Q}}_{k+1,ex})$ belongs to $\mathrm{D}^{W}(\overline{\mathcal{Q}}_{k+1,ex})$, it is sufficient to check that $\mathcal{F}|_{\overline{\mathcal{Q}}^{y}_{k+1,ex}} \in \mathrm{D}^{W}(\overline{\mathcal{Q}}^{y}_{k+1,ex})$ for all points $y \in X$; i.e., it is enough to consider 1-element sets.

Consider now $\overline{\mathcal{Q}}_{k+1} \subset \overline{\mathcal{Q}}_{k+1,ex}$. Since this open substack is stable under the action of the groupoids used in the definition of $\mathrm{P}^{W}(\overline{\mathcal{Q}}_{k+1,ex})$, the categories $\mathrm{P}^{W}(\overline{\mathcal{Q}}_{k+1}) \subset \mathrm{P}(\overline{\mathcal{Q}}_{k+1})$ and $\mathrm{D}^{W}(\overline{\mathcal{Q}}_{k+1}) \subset \mathrm{D}(\overline{\mathcal{Q}}_{k+1})$ are well-defined.

By Lemma 4.8, the direct and inverse image functors $\mathrm{D}(\overline{\mathcal{Q}}_{k+1}) \rightleftarrows \mathrm{D}(\overline{\mathcal{Q}}_{k+1,ex})$ map the subcategories $\mathrm{P}^{W}(\overline{\mathcal{Q}}_{k+1})$ and $\mathrm{D}^{W}(\overline{\mathcal{Q}}^{y}_{k+1,ex})$ to one another.

We emphasize that by definition, for $k=0$, $\mathrm{D}^{W}(\overline{\mathcal{Q}}_{k+1}) = \mathrm{D}(\overline{\mathcal{Q}}_{k+1})$.

4.16. Now let S be an arbitrary "base" scheme. All the constructions of this section go through when we replace $\overline{\mathcal{Q}}_k$ by the product $S \times \overline{\mathcal{Q}}_k$. In other words, we have well-defined categories $\mathrm{P}^{W}(S \times \overline{\mathcal{Q}}_{k+1,ex})$, $\mathrm{D}^{W}(S \times \overline{\mathcal{Q}}_{k+1,ex})$, $\mathrm{P}^{W}(S \times \overline{\mathcal{Q}}_k)$, and $\mathrm{D}^{W}(S \times \overline{\mathcal{Q}}_k)$. Moreover, for a morphism $S_1 \to S_2$, the two pairs of direct and inverse image functors $\mathrm{D}(S_1 \times \overline{\mathcal{Q}}_k) \rightleftarrows \mathrm{D}(S_2 \times \overline{\mathcal{Q}}_k)$ map the categories $\mathrm{D}^{W}(S_1 \times \overline{\mathcal{Q}}_k)$ and $\mathrm{D}^{W}(S_2 \times \overline{\mathcal{Q}}_k)$ to one another; and similarly for the "*ex*"-version.

5. Whittaker functors

The goal of this section is to prove the following theorem:

THEOREM 5.1. *For each $k = 1, \ldots, n-1$ there is an equivalence of categories $W_{k,k+1,ex} : \mathrm{D}^{W}(\overline{\mathcal{Q}}_k) \to \mathrm{D}^{W}(\overline{\mathcal{Q}}_{k+1,ex})$, which maps $\mathrm{P}^{W}(\overline{\mathcal{Q}}_k)$ to $\mathrm{P}^{W}(\overline{\mathcal{Q}}_{k+1,ex})$. The quasi-inverse functor is given by $\mathcal{F}' \mapsto \pi_{k+1,ex,k!}(\mathcal{F}')$, which in this case is isomorphic to $\pi_{k+1,ex,k*}(\mathcal{F}')$.*

5.2. As a first step, we will describe the functor $W_{k,k+1,ex}$ on the strata ${}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}$ for $\overline{d} = d_1, \ldots, d_k$. Namely, let $\mathrm{D}^{W}({}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex})$ (resp., $\mathrm{D}^{W}({}^{\overline{d}}\overline{\mathcal{Q}}_k)$) be the corresponding subcategory of $\mathrm{D}({}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex})$ (resp., $\mathrm{D}({}^{\overline{d}}\overline{\mathcal{Q}}_k)$), and let us describe the functor

$$ {}^{\overline{d}}W_{k,k+1,ex} : \mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_k) \to \mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}). $$

Recall the substack ${}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}' \subset {}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}$, cf. Section 4.12, and define the corresponding substack ${}^{\overline{d}}\overline{\mathcal{Q}}_k{}' \subset {}^{\overline{d}}\overline{\mathcal{Q}}_k$ as the intersection ${}^{\overline{d}}\overline{\mathcal{Q}}_{k,ex}{}' \cap \overline{\mathcal{Q}}_k$. Let $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}')$ (resp., $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_k{}')$) be the corresponding subcategory of $\mathrm{D}({}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}')$ (resp., $\mathrm{D}({}^{\overline{d}}\overline{\mathcal{Q}}_k{}')$).

As in Proposition 4.13(1), every object of $\mathrm{D}({}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex})$ is supported on the closed substack ${}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}'$, and similarly for ${}^{\overline{d}}\overline{\mathcal{Q}}_k{}'$. Therefore, the required functor ${}^{\overline{d}}W_{k,k+1,ex}$ amounts to a functor $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_k{}') \to \mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}')$.

As in Proposition 4.13(2), the category $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}')$ is equivalent to the category $\mathrm{D}({}^{\overline{d}}\mathcal{P}_k)$ by means of

$$ \mathcal{F} \mapsto \phi_k{}^*(\mathcal{F}) \otimes \chi_k'{}^*(\text{A-Sch}). $$

We will now give a similar explicit description of $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_k{}')$.

For $\overline{d} = d_1, \dots, d_k$ let ${}^{\overline{d}}\mathcal{P}'_{k-1}$ denote the stack classifying the data of $(D_1, \dots, D_k) \in X^{(\overline{d})}$ with $D'_i = D_i - D_{i-1}$ effective for $i = 1, \dots, k$, and $D'_{i+1} - D'_i$ effective for $i = 1, \dots, k-1$, a vector bundle $\mathcal{M}''$ of rank $n-k+1$ with an *injective bundle map* $\widetilde{\kappa}_k : \Omega^{n-k}(D'_k) \to \mathcal{M}''$.

We have a natural projection $\phi'_{k-1} : {}^{\overline{d}}\overline{\mathcal{Q}}_k{}' \to {}^{\overline{d}}\mathcal{P}'_{k-1}$ that sends a point

$$ \left((D_1, \dots, D_k) \in X^{(\overline{d})};\ 0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_k \subset \mathcal{M};\ \mathcal{M}_i/\mathcal{M}_{i-1} \simeq \Omega^{n-i}(D'_i)\right) \in {}^{\overline{d}}\overline{\mathcal{Q}}_k{}' $$

to $\mathcal{M}'' := \mathcal{M}/\mathcal{M}_{k-1}$ and

$$ \Omega^{n-k}(D'_k) \simeq \mathcal{M}_k/\mathcal{M}_{k-1} \hookrightarrow \mathcal{M}/\mathcal{M}_{k-1} = \mathcal{M}''. $$

Again, as in Proposition 4.13, the category $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_k{}')$ is equivalent to $\mathrm{D}({}^{\overline{d}}\mathcal{P}'_{k-1})$, by means of $\mathcal{F} \mapsto \phi'_{k-1}{}^*(\mathcal{F}) \otimes \chi'_{k-1}{}^*(\text{A-Sch})$.

Note that the last two pieces of data in the definition of ${}^{\overline{d}}\mathcal{P}'_{k-1}$, i.e., $(\mathcal{M}'', \widetilde{\kappa}_k)$, can be rewritten as a short exact sequence

$$ 0 \to \Omega^{n-k}(D'_k) \to \mathcal{M}'' \to \mathcal{M}' \to 0, $$

where $\mathcal{M}'$ is a vector bundle of rank $n-k$. From this it is easy to see that the stacks ${}^{\overline{d}}\mathcal{P}'_{k-1}$ and ${}^{\overline{d}}\mathcal{P}_k$ form a pair of mutually dual vector bundles over the base classifying $(\mathcal{M}', D_1, \dots, D_k)$, which is isomorphic to the product $\mathrm{Bun}_{n-k} \times \prod_{i=1,\dots,k} X^{(d'_i - d'_{i-1})}$, where $\overline{d} = d_1, \dots, d_k$ and $d'_i = d_i - d_{i-1}$.

We define the functor ${}^{\overline{d}}W_{k,k+1,ex} : \mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_k{}') \to \mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}')$ as a composition

$$ \mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_k{}') \simeq \mathrm{D}({}^{\overline{d}}\mathcal{P}'_{k-1}) \xrightarrow{\mathrm{Four}'} \mathrm{D}({}^{\overline{d}}\mathcal{P}_k) \simeq \mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}'), $$

where Four' is the Fourier transform functor $\mathrm{D}({}^{\overline{d}}\mathcal{P}'_{k-1}) \to \mathrm{D}({}^{\overline{d}}\mathcal{P}_k)$ followed by the cohomological shift by $\dim.\mathrm{rel.}({}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}', {}^{\overline{d}}\mathcal{P}_k) - \dim.\mathrm{rel.}({}^{\overline{d}}\overline{\mathcal{Q}}_k{}', {}^{\overline{d}}\mathcal{P}'_{k-1})$.

The functor ${}^{\overline{d}}W_{k,k+1,ex}$ is an equivalence of categories mapping perverse sheaves to perverse sheaves, because the same it true for the Fourier transform functor.

Let ${}^{\overline{d}}\pi_{k+1,ex,k} : {}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex} \to {}^{\overline{d}}\overline{\mathcal{Q}}_k$ be the restriction of $\pi_{k+1,ex,k}$ to the corresponding stratum.

Lemma 5.3. *The functor* $\mathcal{F}' \mapsto {}^{\overline{d}}\pi_{k+1,ex,k!}(\mathcal{F}')$ *maps* $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex})$ *to* $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}_k)$ *and induces a functor quasi-inverse to* ${}^{\overline{d}}W_{k,k+1,ex}$. *Moreover, in the above formula the* $!$*-direct image coincides with the* $*$ *one.*

Proof. We have the following Cartesian diagram:

$$\begin{tikzcd}
{}^{\overline{d}}\overline{\mathcal{Q}}_k{}' \arrow[d, "\phi'_{k-1}"'] & & {}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}' \arrow[ll, "{}^{\overline{d}}\pi_{k+1,ex,k}"'] \arrow[d, "\phi_k"] \\
{}^{\overline{d}}\mathcal{P}'_{k-1} & & {}^{\overline{d}}\mathcal{P}'_{k-1} \underset{\mathrm{Bun}_{n-k} \times \prod\limits_{i=1,\dots,k} X^{(d'_i - d'_{i-1})}}{\times} {}^{\overline{d}}\mathcal{P}_k. \arrow[ll]
\end{tikzcd}$$

Therefore, the assertion of the lemma can be translated to the following general situation:

Let $\varphi : \mathcal{E} \to \mathcal{Y}$ be a vector bundle, and $\check{\varphi} : \check{\mathcal{E}} \to \mathcal{Y}$ its dual. Consider the functor $W_{\mathcal{E}} : \mathrm{D}(\mathcal{E}) \to \mathrm{D}(\mathcal{E} \underset{\mathcal{Y}}{\times} \check{\mathcal{E}})$ given by

$$\mathcal{F} \mapsto (\varphi \times \mathrm{id})^*(\mathrm{Four}(\mathcal{F})) \otimes \mathrm{ev}^*(\text{A-Sch})[\dim.\mathrm{rel.}(\mathcal{E}, \mathcal{Y})],$$

where $\varphi \times \mathrm{id}$ is the natural projection $\mathcal{E} \underset{\mathcal{Y}}{\times} \check{\mathcal{E}} \to \check{\mathcal{E}}$, and $\mathrm{ev} : \mathcal{E} \underset{\mathcal{Y}}{\times} \check{\mathcal{E}} \to \mathbb{A}^1$ is the evaluation map. Then for $\mathrm{id} \times \check{\varphi} : \mathcal{E} \underset{\mathcal{Y}}{\times} \check{\mathcal{E}} \to \mathcal{E}$ we have:

$$(\mathrm{id} \times \check{\varphi})_!(W_{\mathcal{E}}(\mathcal{F})) \simeq (\mathrm{id} \times \check{\varphi})_*(W_{\mathcal{E}}(\mathcal{F})) \simeq \mathcal{F},$$

and this follows from the standard properties of the Fourier transform functor. $\square$

5.4. We are now going to extend the above stratum-by-stratum definition of ${}^{\overline{d}}W_{k,k+1,ex}$ to a globally defined functor $W_{k,k+1,ex} : \mathrm{D}^W(\overline{\mathcal{Q}}_k) \to \mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})$.

We will first construct the functor $W^y_{k,k+1,ex} : \mathrm{D}^W(\overline{\mathcal{Q}}^y_k) \to \mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$. (The same definition works for y replaced by a finite collection of points $\overline{y}$.)

For that we will single out two sub-groupoids in the groupoid $\mathcal{H}_{N^y_k}$ over $\overline{\mathcal{Q}}^y_k$, denoted ${}'\mathcal{H}_{N^y_k}$ and ${}''\mathcal{H}_{N^y_k}$, respectively. Both these groupoids correspond to certain group sub-schemes ${}'N_{k,\mathcal{D}_y}, {}''N_{k,\mathcal{D}_y}$ of $N_{k,\mathcal{D}_y}$ (resp., ${}'N_{k,\mathcal{D}^*_y}, {}''N_{k,\mathcal{D}^*_y} \subset N_{k,\mathcal{D}^*_y}$).

Recall that $N_{k,\mathcal{D}_y}$ (resp., $N_{k,\mathcal{D}_y^*}$) consists of automorphisms of $\mathcal{M}|_{\mathcal{D}_y}$ (resp., $\mathcal{M}|_{\mathcal{D}_y^*}$), which are strictly upper-triangular with respect to the filtration $0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_k \subset \mathcal{M}|_{\mathcal{D}_y}$, defined by our point of $\overline{\mathcal{Q}}^y_k$.

The group $'N_{k,\mathcal{D}_y}$ (resp., $'N_{k,\mathcal{D}_y^*}$) consists of those automorphisms, which induce the identity map on $\mathcal{M}_k$. The group $''N_{k,\mathcal{D}_y}$ (resp., $''N_{k,\mathcal{D}_y^*}$) consists of those automorphisms, which induce the identity map on $\mathcal{M}/\mathcal{M}_{k-1}$.

Note that the fiber of $'\mathcal{H}_{N^y_k}$ over a point $(\mathcal{M}, \kappa_1, \ldots, \kappa_k) \in \overline{\mathcal{Q}}^y_k$ is the vector space $\mathrm{Hom}_{\mathcal{D}_y^*}(\mathcal{M}/\mathcal{M}_k, \mathcal{M}_k)/\,\mathrm{Hom}_{\mathcal{D}_y}(\mathcal{M}/\mathcal{M}_k, \mathcal{M}_k)$. Without restricting the generality we can assume that the filtration $\mathcal{H}_{N^y_k} = \bigcup_{i \in \mathbb{N}} \mathcal{H}^i_{N^y_k}$ induces on $'N_{k,\mathcal{D}_y}$ the standard filtration:

$$'\mathcal{H}^i_{N^y_k} = \mathrm{Hom}_{\mathcal{D}_y}\left(\mathcal{M}/\mathcal{M}_k, \mathcal{M}_k(i \cdot y)\right) / \,\mathrm{Hom}_{\mathcal{D}_y}\left(\mathcal{M}/\mathcal{M}_k, \mathcal{M}_k\right).$$

Let $'\,\mathrm{pr}_k$ and $'\,\mathrm{act}_k$ (resp., $'\,\mathrm{pr}^i_k$, $'\,\mathrm{act}^i_k$) be the restrictions to $'\mathcal{H}_{N^y_k}$ (resp., $'\mathcal{H}^i_{N^y_k}$) of the maps $\mathrm{pr}_k, \mathrm{act}_k : \mathcal{H}_{N^y_k} \to \overline{\mathcal{Q}}^y_k$, respectively. We will denote $'\mathcal{H}^i_{N^y_k}$ also by $\mathcal{E}^i_k$ and think of it as a vector bundle over $\overline{\mathcal{Q}}^y_k$. Let $\check{\mathcal{E}}^i_k$ denote the dual vector bundle, and $'\check{\mathrm{pr}}^i_k$ its projection to $\overline{\mathcal{Q}}^y_k$.

By Serre's duality, the fiber of $\check{\mathcal{E}}^i_k$ over $(\mathcal{M}, \kappa_1, \ldots, \kappa_k) \in \overline{\mathcal{Q}}^y_k$ can be identified with the vector space

$$\mathrm{Hom}_{\mathcal{D}_y}\Big(\mathcal{M}_k, ((\mathcal{M}/\mathcal{M}_k)/(\mathcal{M}/\mathcal{M}_k)(-iy)) \otimes \Omega\Big).$$

For $i' \geq i$ we have a natural map $\mathrm{pr}_{i',i} : \check{\mathcal{E}}^{i'}_k \to \check{\mathcal{E}}^i_k$.

Proposition 5.5. *There exists a natural map* $f_i : \overline{\mathcal{Q}}^y_{k+1,ex} \to \check{\mathcal{E}}^i_k$ *for any* i. *Moreover*,

(1) *For* $i' \geq i$ *the composition* $\overline{\mathcal{Q}}^y_{k+1,ex} \xrightarrow{f_{i'}} \check{\mathcal{E}}^{i'}_k \xrightarrow{\mathrm{pr}_{i',i}} \check{\mathcal{E}}^i_k$ *equals* f_i.

(2) *For each open substack* $U \subset \overline{\mathcal{Q}}^y_k$ *of finite type, there exists an integer* $i(U)$ *large enough such that over* U, *the map* $f_i : \overline{\mathcal{Q}}^y_{k+1,ex} \to \check{\mathcal{E}}^i_k$ *is a closed embedding for every* $i \geq i(U)$.

Proof. Let $\check{\mathcal{E}}^{i\,\prime}_k \subset \check{\mathcal{E}}^i_k$ be a vector sub-bundle, whose fiber over a point $(\mathcal{M}, \kappa_1, \ldots, \kappa_k) \in \overline{\mathcal{Q}}^y_k$ is the vector space

$$\mathrm{Hom}_{\mathcal{D}_y}\Big(\Omega^{n-k}, ((\mathcal{M}/\mathcal{M}_k)/(\mathcal{M}/\mathcal{M}_k)(-iy)) \otimes \Omega\Big), \tag{15}$$

which maps to $\mathrm{Hom}_{\mathcal{D}_y}\Big(\mathcal{M}_k, ((\mathcal{M}/\mathcal{M}_k)/(\mathcal{M}/\mathcal{M}_k)(-iy)) \otimes \Omega\Big)$ by means of the projection $\mathcal{M}_k \to \Omega^{n-k}|_{\mathcal{D}^y}$.

Note that given a filtration

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_k \subset \mathcal{M}$$

with $\mathcal{M}_j/\mathcal{M}_{j-1} \simeq \Omega^{n-j}$, specifying a map $\Omega^{n-k} \to ((\mathcal{M}/\mathcal{M}_k)/(\mathcal{M}/\mathcal{M}_k)(-iy)) \otimes \Omega$ is the same as specifying a map $\Omega^{n-1+\cdots+n-k-1} \to \Lambda^{k+1}(\mathcal{M})/\Lambda^{k+1}(\mathcal{M})(-iy)$, which satisfies the Plücker relations with all the maps $\kappa_j : \Omega^{n-1+\cdots+n-j} \to \Lambda^j(\mathcal{M})$, $j = 1, \dots, k$.

The map $\overline{\mathcal{Q}}^y_{k+1,ex} \to \check{\mathcal{E}}^i$ is defined now as follows: having $(\mathcal{M}, \kappa_1, \dots, \kappa_k) \in \overline{\mathcal{Q}}^y_k$, to the data of $\kappa_{k+1} : \Omega^{n-1+\cdots+n-k-1} \to \Lambda^{k+1}(\mathcal{M})$ we attach the corresponding map $\Omega^{n-1+\cdots+n-k-1} \to \Lambda^{k+1}(\mathcal{M})/\Lambda^{k+1}(\mathcal{M})(-iy)$ over $\mathcal{D}_y$. According to the above discussion, this defines a point of $\check{\mathcal{E}}^{i}_k{}'$, and hence of $\check{\mathcal{E}}^i_k$.

Thus, the map f_i has been constructed. Point (1) of the proposition is straightforward from the construction.

For an open substack U of finite type, let $i(U)$ be such that the vector space $\mathrm{Hom}(\Omega^{n-1+\cdots+n-k-1}, \Lambda^{k+1}(\mathcal{M})(-iy))$ is zero for $(\mathcal{M}, \kappa_1, \dots, \kappa_k) \in U$.

Let $\check{\mathcal{E}}^{i}_k{}''$ be the vector bundle over $\overline{\mathcal{Q}}^y_k$, whose fiber over a point as above is the vector space $\mathrm{Hom}\left(\Omega^{n-1+\cdots+n-k-1}, \Lambda^{k+1}(\mathcal{M})/\Lambda^{k+1}(\mathcal{M})(-iy)\right)$. For $i \geq U(i)$ the natural map $\overline{\mathcal{Q}}^y_{k+1,ex} \to \check{\mathcal{E}}^{i}_k{}''$ is a closed embedding over U.

Then for $i \geq i(U)$, we have a sequence of maps

$$\overline{\mathcal{Q}}^y_{k+1,ex} \to \check{\mathcal{E}}^{i}_k{}' \to \check{\mathcal{E}}^{i}_k{}''.$$

We know that the second arrow is a closed embedding, being the set of those sections that satisfy the Plücker relations. We also know that the composed map is a closed embedding. Hence, so is the first map. □

5.6. Consider now the action map ${}'\mathrm{act}^i_k : {}'\mathcal{H}^i_{N^y_k} \to \overline{\mathcal{Q}}^y_k$. Since the projection ${}'\mathrm{pr}^i_k : {}'\mathcal{H}^i_{N^y_k} \to \overline{\mathcal{Q}}^y_k$ is a smooth map and ${}'\mathcal{H}^i_{N^y_k}$ is a groupoid, the map ${}'\mathrm{act}^i_k$ is smooth as well; let $\dim(i,k)$ denote its relative dimension.

We define the functor $W^{y,i}_{k,k+1,ex} : \mathrm{D}^W(\overline{\mathcal{Q}}^y_k) \to \mathrm{D}(\check{\mathcal{E}}^i_k)$ as

$$\mathcal{F} \mapsto \mathrm{Four}\left({}'\mathrm{act}^i_k{}^*(\mathcal{F})[\dim(i,k)]\right),$$

where $\mathrm{Four} : \mathrm{D}(\mathcal{E}^i_k) \to \mathrm{D}(\check{\mathcal{E}}^i_k)$ is the Fourier transform functor. Evidently, this functor is exact. For $i' \geq i$ we have:

$$\mathrm{pr}_{i',i\,!}(W^{y,i'}_{k,k+1,ex}(\mathcal{F})) \simeq W^{y,i}_{k,k+1,ex}(\mathcal{F}).$$

Proposition 5.7. *For an open substack $U \subset \overline{\mathcal{Q}}^y_k$ of finite type and any integer i which is large enough (and in particular $i \geq i(U)$ of Proposition* 5.5*), over the preimage of U, any object of the form $W^{y,i}_{k,k+1,ex}(\mathcal{F})$ for $\mathcal{F} \in \mathrm{D}^W(\overline{\mathcal{Q}}^y_k)$ is supported on $\overline{\mathcal{Q}}^y_{k+1,ex} \subset \check{\mathcal{E}}^i_k$.*

Proof. Recall that to a string of nonnegative integers $\overline{d} = d_1, \dots, d_k$ we attached a locally closed substack ${}^{\overline{d}}\overline{\mathcal{Q}}^y_k \subset \overline{\mathcal{Q}}^y_k$. Let $|\overline{d}|$ be $\sum_i d_i$.

It is easy to see that for every open substack $U \subset \overline{\mathcal{Q}}_k$ of finite type there exists an integer d such that

$$U \subset \bigcup_{|\overline{d}| \le d} {}^{\overline{d}}\overline{\mathcal{Q}}_k. \tag{16}$$

Thus, for a given U there exists an integer $i'(U)$ such that for any $\overline{d}$ and $\mathcal{M} \in U \cap {}^{\overline{d}}\overline{\mathcal{Q}}^y_k$ we have: $\mathrm{Hom}\,(\mathcal{M}_k, \mathcal{M}/\mathcal{M}_k \otimes \Omega(-iy)) = 0$, for $i \ge i'(U)$ where

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_k \subset \mathcal{M}$$

is the filtration with $\mathcal{M}_i/\mathcal{M}_{i-1} \simeq \Omega^{n-i}(D'_i)$, $(D_1, \ldots, D_k) \in (X - y)^{(\overline{d})}$.

Let now $\mathcal{F}$ be an object of $\mathrm{D}^W({}^{\overline{d}}\overline{\mathcal{Q}}^y_k)$ with $|\overline{d}| \le d$. Then $W^{y,i}_{k,k+1,ex}(\mathcal{F})$ yields an object of $\mathrm{D}\left(({}'\check{\mathrm{pr}}^i_k)^{-1}({}^{\overline{d}}\overline{\mathcal{Q}}^y_k)\right)$, where $({}'\check{\mathrm{pr}}^i_k)^{-1}({}^{\overline{d}}\overline{\mathcal{Q}}^y_k)$ is the preimage of ${}^{\overline{d}}\overline{\mathcal{Q}}^y_k$ in $\check{\mathcal{E}}^i_k$.

Lemma 5.8. *For $i \ge i'(U), i(U)$, over the preimage of U, the object $W^{y,i}_{k,k+1,ex}(\mathcal{F})$ is supported on ${}^{\overline{d}}\overline{\mathcal{Q}}^y_{k+1,ex}$ and is isomorphic to ${}^{\overline{d}}W_{k,k+1,ex}(\mathcal{F})$ of Section* 5.2.

Proof. Recall the stacks ${}^{\overline{d}}\mathcal{P}_k$ and ${}^{\overline{d}}\mathcal{P}'_{k-1}$, which form a pair of mutually dual (generalized) vector bundles over the base $\mathrm{Bun}_{n-k} \times \prod_{i=1,\ldots,k} X^{(d'_i - d'_{i-1})}$, where the latter classifies the data of a rank $n-k$ vector bundle $\mathcal{M}'$ and a collection of divisors $D_1, \ldots, D_k$ with $D'_i = D_i - D_{i-1}$ effective and $D'_i - D'_{i-1}$ effective as well. To simplify the notation, let us temporarily denote this base by $\mathcal{Y}$, ${}^{\overline{d}}\mathcal{P}'_{k-1}$ by $\mathcal{E}$ and ${}^{\overline{d}}\mathcal{P}'_{k-1}$ by $\check{\mathcal{E}}$; let φ and $\check{\varphi}$ denote the projections of $\mathcal{E}$ and $\check{\mathcal{E}}$, respectively, on $\mathcal{Y}$.

Consider the fiber product $\mathcal{E} \underset{\mathcal{Y}}{\times} \check{\mathcal{E}}$, and let ev be the natural evaluation map from it to $\mathbb{A}^1$. The assertion of the lemma amounts to a description of the functor $\mathrm{D}(\mathcal{E}) \to \mathrm{D}(\mathcal{E} \underset{\mathcal{Y}}{\times} \check{\mathcal{E}})$ given by

$$\mathcal{F} \mapsto (\varphi \times \mathrm{id})^*(\mathrm{Four}(\mathcal{F})) \otimes \mathrm{ev}^*(\text{A-Sch})[\mathrm{dim.\,rel.}(\mathcal{E}, \mathcal{Y})] \tag{17}$$

in terms of an action of a certain groupoid on $\mathcal{E}$.

Namely, for an integer i consider another vector bundle over $\mathcal{Y}$, denoted $\mathcal{E}^i$, whose fiber over $(\mathcal{M}', D_1, \ldots, D_k) \in \mathcal{Y}$ is the vector space $\mathrm{Hom}(\mathcal{M}', \Omega^{n-k}(i \cdot y)/\Omega^{n-k})$. Let $\check{\mathcal{E}}^i$ be the dual vector bundle, whose fiber, by Serre's duality can be identified with $\mathrm{Hom}(\Omega^{n-k-1}, \mathcal{M}'/\mathcal{M}'(-i \cdot y))$. We have a natural map $\mathcal{E}^i \to \mathcal{E}$. When working over an open substack of finite type in Bun_{n-k}, for a large enough integer i, the dual map $\check{\mathcal{E}} \to \check{\mathcal{E}}^i$ is a closed embedding.

Using the group-scheme structure, we can think of $\mathcal{E}^i$ as a groupoid acting on $\mathcal{E}$. Let a and p denote the corresponding maps $\mathcal{E}_i \underset{\mathcal{Y}}{\times} \mathcal{E} \to \mathcal{E}$. Thus, we can consider the functor $\mathrm{D}(\mathcal{E}) \to \mathrm{D}(\check{\mathcal{E}}_i \underset{\mathcal{Y}}{\times} \mathcal{E})$ given by

$$\mathcal{F} \mapsto \mathrm{Four}\,(a^*(\mathcal{F}))\,[\dim.\mathrm{rel.}(\mathcal{E}^i, \mathcal{Y})], \tag{18}$$

where Four is the relative Fourier transform functor $\mathrm{D}(\mathcal{E}_i \underset{\mathcal{Y}}{\times} \mathcal{E}) \to \mathrm{D}(\check{\mathcal{E}}_i \underset{\mathcal{Y}}{\times} \mathcal{E})$.

The assertion of the lemma follows from the fact that the functor in (18) is isomorphic to the composition of the functor of (17), followed by the direct image under the closed embedding $\mathcal{E} \underset{\mathcal{Y}}{\times} \check{\mathcal{E}} \to \mathcal{E} \underset{\mathcal{Y}}{\times} \check{\mathcal{E}}^i$. □

This lemma implies the proposition. Indeed for a given $\mathcal{F} \in \mathrm{D}^W(\overline{\mathcal{Q}}^y_k)$ to show that, over the preimage of U, $W^{y,i}_{k,k+1,ex}(\mathcal{F})$ is supported on $\overline{\mathcal{Q}}^y_{k+1,ex}$, it is enough to do so over the preimage of each stratum ${}^{d}\overline{\mathcal{Q}}^y_k \subset \overline{\mathcal{Q}}^y_k$. The latter support property is insured by Lemma 5.8. □

5.9. Since $\mathrm{pr}_{i',i\,!}(W^{y,i'}_{k,k+1,ex}(\mathcal{F})) \simeq W^{y,i}_{k,k+1,ex}(\mathcal{F})$, the above proposition implies that we obtain a well-defined functor $W^y_{k,k+1,ex} : \mathrm{D}^W(\overline{\mathcal{Q}}^y_k) \to \mathrm{D}(\overline{\mathcal{Q}}^y_{k+1,ex})$.[4] Moreover, by combining Lemma 5.8 and Lemma 4.11(3) we obtain that the image of $W^y_{k,k+1,ex}$ lies in $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$.

Proposition 5.10. *The direct image functor $\mathcal{F} \mapsto \pi_{k+1,ex,k!}(\mathcal{F})$ maps $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$ to $\mathrm{D}^W(\overline{\mathcal{Q}}^y_k)$ and is a quasi-inverse to $W^y_{k,k+1,ex}$. Moreover, for $\mathcal{F} \in \mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$, $\pi_{k+1,ex,k!}(\mathcal{F}) \to \pi_{k+1,ex,k*}(\mathcal{F})$ is an isomorphism.*

Proof. First, let us show that for $\mathcal{F} \in \mathrm{D}^W(\overline{\mathcal{Q}}^y_k)$, $\pi_{k+1,ex,k!}(W^y_{k,k+1,ex}(\mathcal{F})) \simeq \mathcal{F}$. Indeed, by working over a fixed stack U of finite type and a large enough integer i, we are reduced to showing that

$$'\check{\mathrm{pr}}^i_{k!}\left(\mathrm{Four}\left('\,\mathrm{act}^{i\,*}_k(\mathcal{F})[\dim(i,k)]\right)\right) \simeq \mathcal{F},$$

where $'\check{\mathrm{pr}}^i_k$ is the projection $\check{\mathcal{E}}^i_k \to \overline{\mathcal{Q}}^y_k$.

However, by the general properties of the Fourier transform functor we obtain that the left-hand side of the above expression is isomorphic to the restriction of $'\,\mathrm{act}^{i\,*}_k(\mathcal{F})$ to the unit section of $\mathcal{E}^i_k \simeq {}'\mathcal{H}^y_{N_k}$, i.e., to $\mathcal{F}$ itself.

Now let us show that $\pi_{k+1,ex,k!}$ indeed maps $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$ to $\mathrm{D}^W(\overline{\mathcal{Q}}^y_k)$. However, this follows immediately from the definitions:

By unfolding the definition of $\mathrm{D}^W(\overline{\mathcal{Q}}^y_k)$, we see that it is defined by means of an equivariance property with respect to the groupoid $''\mathcal{H}_{N^y_k}$ (cf. Section 5.4). However, $''\mathcal{H}_{N^y_k}$ acts on $\overline{\mathcal{Q}}^y_{k+1,ex}$, being a part of $\mathcal{H}_{N^y_k}$; i.e., we have a Cartesian

[4]This definition of $W^y_{k,k+1,ex}$ was inspired by a certain construction of V. Drinfeld in the $n = 2$ case, one incarnation of which is explained in Section 5.16.

diagram

$$\begin{CD}
\overline{\mathcal{Q}}^y_{k+1,ex} @<{\mathrm{act}_{k,ex}}<< {}''\mathcal{H}_{N^y_k} \underset{\overline{\mathcal{Q}}^y_k}{\times} \overline{\mathcal{Q}}^y_{k+1,ex} \\
@V{\pi_{k+1,ex,k}}VV @VV{\mathrm{id}\times\pi_{k+1,ex,k}}V \\
\overline{\mathcal{Q}}^y_k @<{\mathrm{act}_k}<< {}''\mathcal{H}_{N^y_k}.
\end{CD}$$

Finally,

$$\chi_{k-1}|_{''\mathcal{H}_{N^y_k} \underset{\overline{\mathcal{Q}}^y_k}{\times} \overline{\mathcal{Q}}^y_{k+1,ex}} = \chi_k|_{''\mathcal{H}_{N^y_k} \underset{\overline{\mathcal{Q}}^y_k}{\times} \overline{\mathcal{Q}}^y_{k+1,ex}}.$$

Now, let us show that for $\mathcal{F}' \in \mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$, the object

$$W^y_{k,k+1,ex}(\pi_{k+1,ex,k!}(\mathcal{F}'))$$

is canonically isomorphic to $\mathcal{F}'$.

Note that as in Lemma 4.4, the groupoid ${}'\mathcal{H}^y_{N_k}$ "lifts" to $\check{\mathcal{E}}^i_k$; i.e., we have a Cartesian diagram

$$\begin{CD}
\check{\mathcal{E}}^i_k @<{\mathrm{act}_{\mathcal{E}^i_k}}<< \mathcal{E}^i_k \underset{\overline{\mathcal{Q}}^y_k}{\times} \check{\mathcal{E}}^i_k \\
@V{'\check{\mathrm{pr}}^i_k}VV @VV{\mathrm{id}\times{}'\check{\mathrm{pr}}^i_k}V \\
\overline{\mathcal{Q}}^y_k @<{'\,\mathrm{act}^i_k}<< \mathcal{E}^i_k.
\end{CD}$$

Thus we may assume that we start with $\mathcal{F}' \in \mathrm{D}(\check{\mathcal{E}}^i_k)$, which satisfies

$$\mathrm{act}^*_{\mathcal{E}^i_k}(\mathcal{F}') \simeq ({}'\mathrm{pr}^i_k \times \mathrm{id})^*(\mathcal{F}') \otimes \mathrm{ev}^*(\text{A-Sch}), \tag{19}$$

where ${}'\,\mathrm{pr}^i_k \times \mathrm{id}$ is the map $\mathcal{E}^i_k \underset{\overline{\mathcal{Q}}^y_k}{\times} \check{\mathcal{E}}^i_k \to \check{\mathcal{E}}^i_k$ and ev is the evaluation map $\mathcal{E}^i_k \underset{\overline{\mathcal{Q}}^y_k}{\times} \check{\mathcal{E}}^i_k \to \breve{\mathbb{A}}^1$.

However by looking at another Cartesian square:

$$\begin{CD}
\check{\mathcal{E}}^i_k @<{'\,\mathrm{pr}^i_k\times\mathrm{id}}<< \mathcal{E}^i_k \underset{\overline{\mathcal{Q}}^y_k}{\times} \check{\mathcal{E}}^i_k \\
@V{'\check{\mathrm{pr}}^i_k}VV @VV{\mathrm{id}\times{}'\check{\mathrm{pr}}^i_k}V \\
\overline{\mathcal{Q}}^y_k @<{'\,\mathrm{pr}^i_k}<< \mathcal{E}^i_k,
\end{CD}$$

we obtain (19) which implies

$$\mathrm{Four}^{-1}(\mathcal{F}') \simeq {}'\,\mathrm{act}^{i\,*}_k\left({}'\check{\mathrm{pr}}^i_{k!}(\mathcal{F}')\right),$$

which is what we had to show.

The last assertion that $\pi_{k!}(\mathcal{F}') \to \pi_{k*}(\mathcal{F}')$ also follows from the above diagram by the fact that the $!-$ and $*-$ Fourier transforms coincide. $\square$

5.11. Finally, we are ready to construct the functor

$$W_{k,k+1,ex} : \mathrm{D}^W(\overline{\mathcal{Q}}_k) \to \mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex}).$$

The construction of the corresponding functor on the level of abelian categories, i.e., $W_{k,k+1,ex} : \mathrm{P}^W(\overline{\mathcal{Q}}_k) \to \mathrm{P}^W(\overline{\mathcal{Q}}_{k+1,ex})$, follows immediately from Proposition 5.10, because an object $\mathcal{F} \in \mathrm{P}^W(\overline{\mathcal{Q}}_{k+1,ex})$ can be glued from its restrictions $\mathcal{F}|_{\overline{\mathcal{Q}}^y_{k+1,ex}}$.

Moreover, Proposition 5.10 implies that $\pi_{k+1,ex,k!}$ maps $\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})$ to $\mathrm{D}^W(\overline{\mathcal{Q}}_k)$ and the induced functor $\mathrm{P}^W(\overline{\mathcal{Q}}_{k+1,ex}) \to \mathrm{P}^W(\overline{\mathcal{Q}}_k)$ is an equivalence inverse to $W_{k,k+1,ex}$.

Thus, it remains to show that $\pi_{k+1,ex,k!} : \mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex}) \to \mathrm{D}^W(\overline{\mathcal{Q}}_k)$ is an equivalence.

First, let us notice that for any substack $U \subset \overline{\mathcal{Q}}_k$ of finite type, as in Proposition 5.7, one can find an integer $i''(U)$ such that for $i \geq i''(U)$ the image of each $U \cap \overline{\mathcal{Q}}^y_k$ under the action of the entire groupoid $\mathcal{H}_{\mathcal{N}^y_k}$ equals that of $\mathcal{H}^i_{\mathcal{N}^y_k}$. Hence, any substack U of finite type can be replaced by a bigger one, which is also of finite type, such that each $U \cap \overline{\mathcal{Q}}^y_k$ is $\mathcal{H}_{\mathcal{N}^y_k}$-stable. Let $U_{k+1,ex}$ denote its preimage in $\overline{\mathcal{Q}}_{k+1,ex}$.

Then the categories $\mathrm{D}^W(U)$ and $\mathrm{D}^W(U_{k+1,ex})$ make sense, and we have the functor $\pi_{k+1,ex,k!} : \mathrm{D}^W(U_{k+1,ex}) \to \mathrm{D}^W(U)$, and it suffices to show that it is an equivalence.

We claim that this functor is fully-faithful. Since U intersects only finitely many strata ${}^d\overline{\mathcal{Q}}_k$, it suffices to check that for two objects $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{D}^W(U_{k+1,ex})$

$$\mathrm{Hom}_{\mathrm{D}({}^d\overline{\mathcal{Q}}_{k+1,ex})}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Hom}_{\mathrm{D}({}^d\overline{\mathcal{Q}}_k)}\left(\pi_{k+1,ex,k!}(\mathcal{F}_1), \pi_{k+1,ex,k!}(\mathcal{F}_2)\right)$$

is an isomorphism. But we know that from Lemma 5.3.

To finish the proof, we must show that $\pi_{k+1,ex,k!} : \mathrm{D}^W(U_{k+1,ex}) \to \mathrm{D}^W(U)$ is surjective on objects. However, this we know, because every object of $\mathrm{D}^W(U_{k+1,ex})$ is obtained by gluing finitely many perverse sheaves, and we know already that $\pi_{k+1,ex,k!} : \mathrm{P}^W(\overline{\mathcal{Q}}_{k+1,ex}) \to \mathrm{P}^W(\overline{\mathcal{Q}}_k)$ is an equivalence.

5.12. Thus, Theorem 5.1 is proved.

We define the functor $W_{k,k+1} : \mathrm{D}^W(\overline{\mathcal{Q}}_k) \to \mathrm{D}^W(\overline{\mathcal{Q}}_{k+1})$ as the composition of $W_{k,k+1,ex}$ followed by the restriction $\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex}) \to \mathrm{D}^W(\overline{\mathcal{Q}}_{k+1})$. By construction, $W_{k,k+1}$ is exact.

For two integers $1 \leq k < k' \leq n$ we define $W_{k,k'} : \mathrm{D}^W(\overline{\mathcal{Q}}_k) \to \mathrm{D}^W(\overline{\mathcal{Q}}_{k'})$ as the composition $W_{k'-1,k'} \circ \cdots \circ W_{k,k+1}$. Finally, we set $W : \mathrm{D}^W(\mathrm{Bun}'_n) \to \mathrm{D}^W(\overline{\mathcal{Q}})$ to be $W_{1,n}$. All these functors are exact.

5.13. Recall that in Section 4.16 we said that the categories $\mathrm{D}^W(S \times \overline{\mathcal{Q}}_k)$, $\mathrm{D}^W(S \times \overline{\mathcal{Q}}_{k,ex})$ can be introduced for an arbitrary base scheme S. Similarly, one has the functors $W_{k,k+1,ex} : \mathrm{D}^W(S \times \overline{\mathcal{Q}}_k) \to \mathrm{D}^W(S \times \overline{\mathcal{Q}}_{k+1,ex})$, which are equivalences of categories, and the corresponding functors

$$W_{k,k'} : \mathrm{D}^W(S \times \overline{\mathcal{Q}}_k) \to \mathrm{D}^W(S \times \overline{\mathcal{Q}}_{k'}).$$

All these functors map perverse sheaves to perverse sheaves, and commute with the Verdier duality.

Moreover, for a morphism of schemes $f : S_1 \to S_2$, the !- and $*$- direct and inverse image functors $\mathrm{D}^W(S_1 \times \overline{\mathcal{Q}}_k) \rightleftarrows \mathrm{D}^W(S_2 \times \overline{\mathcal{Q}}_k)$ and $\mathrm{D}^W(S_1 \times \overline{\mathcal{Q}}_{k+1,ex}) \rightleftarrows \mathrm{D}^W(S_2 \times \overline{\mathcal{Q}}_{k+1,ex})$ commute in the natural sense with $W_{k,k'}$. This is evident for the $*$-inverse image and the !-direct image via the description of the quasi-inverse functor as $\pi_{k+1,ex,k!}$, and for the !-inverse image and the $*$-direct image as $\pi_{k+1,ex,k*}$.

To conclude, we note that the group $\mathbb{G}_m$ acts on all the stacks $\overline{\mathcal{Q}}_k$ by simultaneously scaling the maps κ_i. Thus, it makes sense to talk about the equivariant derived categories $\mathrm{D}^{\mathbb{G}_m}(\overline{\mathcal{Q}}_k)$.

We introduce the equivariant version of the Whittaker category $\mathrm{D}^{\mathbb{G}_m,W}(\overline{\mathcal{Q}}_k)$ as the full triangulated subcategory of $\mathrm{D}^{\mathbb{G}_m}(\overline{\mathcal{Q}}_k)$ consisting of objects whose perverse cohomologies belong to $\mathrm{D}^W(\overline{\mathcal{Q}}_k)$. Thus, we have an equivariant version of Theorem 5.1, and, in particular, the equivalences $W_{k,k+1,ex} : \mathrm{D}^{\mathbb{G}_m,W}(\overline{\mathcal{Q}}_k) \to \mathrm{D}^{\mathbb{G}_m,W}(\overline{\mathcal{Q}}_{k+1,ex})$, and the Whittaker functors $W_{k,k'} : \mathrm{D}^{\mathbb{G}_m,W}(\overline{\mathcal{Q}}_k) \to \mathrm{D}^{\mathbb{G}_m,W}(\overline{\mathcal{Q}}_{k'})$.

5.14. The rest of this section will not be used in the sequel. We would like to compare the Whittaker functor $W : \mathrm{D}(\mathrm{Bun}'_n) \to \mathrm{D}^W(\overline{\mathcal{Q}})$ defined above with another functor of related nature introduced by G. Laumon in [11].

For an integer k, let Coh_k denote the stack of coherent sheaves of generic rank k; let Coh'_k denote the stack of pairs: $(\mathcal{M}, \kappa)$, where $\mathcal{M} \in \mathrm{Coh}_k$, and κ is an injective map of sheaves $\Omega^{k-1} \to \mathcal{M}$. Let, in addition, $\mathrm{Coh}'_{k,ex} \supset \mathrm{Coh}'_k$ denote the stack of pairs $(\mathcal{M}, \kappa)$ as before, but where we omit the condition that κ be injective.

We have a functor $W^{\mathrm{Coh}}_{k,k-1} : \mathrm{D}(\mathrm{Coh}'_k) \to \mathrm{D}(\mathrm{Coh}'_{k-1})$. Namely, note that Coh'_k and $\mathrm{Coh}'_{k-1,ex}$ form a pair of mutually dual vector bundles over Coh_{k-1}. We set $W^{\mathrm{Coh}}_{k,k-1}$ to be the composition of the Fourier transform functor $\mathrm{D}(\mathrm{Coh}'_k)$ $\to \mathrm{D}(\mathrm{Coh}'_{k-1,ex})$ followed by the restriction $\mathrm{D}(\mathrm{Coh}'_{k-1,ex}) \to \mathrm{D}(\mathrm{Coh}'_{k-1})$.

By composing, for any n we obtain a functor $W^{\mathrm{Coh}}_n : \mathrm{D}(\mathrm{Coh}'_n) \to \mathrm{D}(\mathrm{Coh}'_1)$.

Recall now the stack $\widetilde{\mathcal{Q}}$ of [7]. We have a natural smooth projection $\phi^{\mathrm{Coh}} : \widetilde{\mathcal{Q}} \to \mathrm{D}(\mathrm{Coh}'_1)$, and a map $\mathrm{ev} : \widetilde{\mathcal{Q}} \to \mathbb{A}^1$.

We define the functor $\widetilde{W}^{\mathrm{Coh}}_n : \mathrm{D}(\mathrm{Coh}'_n) \to \mathrm{D}(\widetilde{\mathcal{Q}})$ by

$$\mathcal{F} \mapsto \phi^{\mathrm{Coh}*}\left(W^{\mathrm{Coh}}_n(\mathcal{F})\right)[d] \otimes \mathrm{ev}^*(\text{A-Sch}),$$

where $[d]$ is the shift by $\dim.\mathrm{rel}.(\widetilde{\mathcal{Q}}, \mathrm{Coh}'_1)$.

Recall also that we have a map $\nu : \widetilde{\mathcal{Q}} \to \overline{\mathcal{Q}} = \mathcal{Q}_n$. Finally note that Coh'_n contains $\mathrm{Bun}'_n = \mathcal{Q}_1$ as an open substack.

PROPOSITION 5.15. *Let* $\mathcal{F}$ *be an object of* $\mathrm{D}(\mathrm{Bun}'_n)$, *and let* $\mathcal{F}'$ *be its any extension to an object of* $\mathrm{D}(\mathrm{Coh}'_n)$. *Then*

$$\nu_! \left(\widetilde{W}_n^{\mathrm{Coh}}(\mathcal{F}')\right) \simeq \nu_* \left(\widetilde{W}_n^{\mathrm{Coh}}(\mathcal{F}')\right) \simeq W(\mathcal{F}).$$

Instead of giving the proof of this statement, we will sketch the argument when $n = 2$. The proof in the general case follows the same lines.

5.16. Consider the following set-up: Let $\mathcal{Y}$ be a base, and $\mathcal{E}_1$, $\mathcal{E}_2$ be two vector bundles viewed as group-schemes over $\mathcal{Y}$, and $\mathrm{p} : \mathcal{E}_1 \to \mathcal{E}_2$ a map. Suppose that both $\mathcal{E}_1$ and $\mathcal{E}_2$ act on a scheme $\mathcal{X}$ over $\mathcal{Y}$, i.e., we have the action maps

$$\mathrm{act}_i : \mathcal{E}_i \underset{\mathcal{Y}}{\times} \mathcal{X} \to \mathcal{X},$$

with $\mathrm{act}_1 = \mathrm{act}_2 \circ \mathrm{p}$.

Consider the functors $\mathrm{F}_i : \mathrm{D}(\mathcal{X}) \to \mathrm{D}(\check{\mathcal{E}}_i \underset{\mathcal{Y}}{\times} \mathcal{X})$, where $\check{\mathcal{E}}_i$ is the dual vector bundle, given by

$$\mathcal{F} \mapsto \mathrm{Four}\left(\mathrm{act}_i^*(\mathcal{F})[d_i]\right),$$

where $d_i = \mathrm{dim.rel.}(\mathcal{E}_i, \mathcal{Y})$. Then for $\mathcal{F} \in \mathrm{D}(\mathcal{X})$, we have:

$$\mathrm{F}_1(\mathcal{F}) \simeq (\check{\mathrm{p}} \times \mathrm{id})_!(\mathrm{F}_2(\mathcal{F})),$$

where $\check{\mathrm{p}} \times \mathrm{id}$ is the natural map $\check{\mathcal{E}}_2 \underset{\mathcal{Y}}{\times} \mathcal{X} \to \check{\mathcal{E}}_1 \underset{\mathcal{Y}}{\times} \mathcal{X}$.

We apply the above observation in the following circumstances: We set $\mathcal{Y} = \mathrm{Coh}_1$, $\mathcal{X} := \mathrm{Coh}'_2$. The vector bundle $\mathcal{E}_2$ is isomorphic to $\mathcal{X} := \mathrm{Coh}'_2$ itself; i.e., its fiber at $\mathcal{L} \in \mathrm{Coh}_1$ is the stack of extensions

$$0 \to \Omega \to \mathcal{M} \to \mathcal{L} \to 0.$$

The vector bundle $\mathcal{E}_1$ has its fiber over $\mathcal{L}$ as above the stack of extensions

$$0 \to \Omega \to \mathcal{M} \to \det(\mathcal{L}) \to 0,$$

where $\det(\mathcal{L})$ is the determinant of $\mathcal{L}$. The map $\mathrm{p} : \mathcal{E}_1 \to \mathcal{E}_2$ comes from the canonical map of sheaves $\mathcal{L} \to \det(\mathcal{L})$.

Note that the action of $\mathcal{E}_1$ preserves the open substack $\mathrm{Bun}'_2 \subset \mathrm{Coh}'_2$, and $\check{\mathcal{E}}_1 \underset{\mathrm{Coh}_1}{\times} \mathrm{Bun}'_2 \simeq \overline{\mathcal{Q}}$. Moreover, the functor

$$\mathrm{F}_1|_{\mathrm{Bun}'_2} : \mathrm{D}(\mathrm{Bun}'_2) \to \check{\mathcal{E}}_1 \underset{\mathrm{Coh}_1}{\times} \mathrm{Bun}'_2$$

identifies with $W : \mathrm{D}(\mathrm{Bun}'_2) \to \mathrm{D}^W(\overline{\mathcal{Q}})$.

To prove the assertion of the proposition, it suffices to notice that we have a Cartesian square:

$$\begin{array}{ccccc}
\check{\mathcal{E}}_2 \underset{\mathrm{Coh}_1}{\times} \mathrm{Coh}'_2 & \longleftarrow & \check{\mathcal{E}}_2 \underset{\mathrm{Coh}_1}{\times} \mathrm{Bun}'_2 & \xrightarrow{\sim} & \widetilde{\mathcal{Q}} \\
{\scriptstyle \check{\mathfrak{p}}\times\mathrm{id}}\downarrow & & {\scriptstyle \check{\mathfrak{p}}\times\mathrm{id}}\downarrow & & {\scriptstyle \nu}\downarrow \\
\check{\mathcal{E}}_1 \underset{\mathrm{Coh}_1}{\times} \mathrm{Coh}'_2 & \longleftarrow & \check{\mathcal{E}}_1 \underset{\mathrm{Coh}_1}{\times} \mathrm{Bun}'_2 & \xrightarrow{\sim} & \overline{\mathcal{Q}}.
\end{array}$$

6. Cuspidality

6.1. Let us first recall the notion of cuspidality on Bun_n. For $n = n_1 + n_2$, let $\mathrm{Fl}^n_{n_1,n_2}$ denote the stack of extensions

$$0 \to \mathcal{M}^1 \to \mathcal{M} \to \mathcal{M}^2 \to 0,$$

where $\mathcal{M}^i \in \mathrm{Bun}_{n_i}$.

We have the natural projection $\mathfrak{p}_{n_1,n_2} : \mathrm{Fl}^n_{n_1,n_2} \to \mathrm{Bun}_n$, which remembers the middle term of the above short exact sequence, and the projection $\mathfrak{q}_{n_1,n_2} : \mathrm{Fl}^n_{n_1,n_2} \to \mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2}$, which remembers $(\mathcal{M}^1, \mathcal{M}^2)$.

The projection $\mathfrak{q}_{n_1,n_2}$ is in general non-representable, but is a generalized vector bundle with the fiber over $(\mathcal{M}^1, \mathcal{M}^2) \in \mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2}$ being the stack of extensions of $\mathcal{M}^2$ by means of $\mathcal{M}^1$. Therefore, the direct image functors $\mathfrak{q}_{n_1,n_2}! : \mathrm{D}(\mathrm{Fl}^n_{n_1,n_2}) \to \mathrm{D}(\mathrm{Bun}_n)$ are well-defined.

The constant term functors $\mathrm{CT}^n_{n_1,n_2} : \mathrm{D}(\mathrm{Bun}_n) \to \mathrm{D}(\mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2})$ are defined by

$$\mathcal{F} \mapsto \mathfrak{q}_{n_1,n_2!}\left(\mathfrak{p}^*_{n_1,n_2}(\mathcal{F})\right).$$

Recall that an object $\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n)$ is called cuspidal if $\mathrm{CT}^n_{n_1,n_2}(\mathcal{F}) = 0$ for all $1 \leq n_1, n_2 < n$.

Since the projection $\mathfrak{q}$ is not proper, the functor $\mathrm{CT}^n_{n_1,n_2}$ does not commute with the Verdier duality. Therefore, if $\mathcal{F}$ is cuspidal, it will not in general be true that $\mathbb{D}(\mathcal{F})$ is cuspidal.

6.2. We will now introduce the notion of cuspidality on the stacks $\overline{\mathcal{Q}}_k$. For n_1 as above and $k \leq n_1$, let $\overline{\mathcal{Q}}_{n_1,k}$ denote the stack classifying the data of $\mathcal{M}^1 \in \mathrm{Bun}_{n_1}$, and a collection of nonzero maps $\kappa_{n_1,i} : \Omega^{n-1+\cdots+n-i} \to \Lambda^i(\mathcal{M}^1)$ for $1 \leq i \leq k$, satisfying the Plücker relations.

For $k \leq n_1$, let $\mathrm{Fl}^{\overline{\mathcal{Q}}_k}_{n_1,n_2}$ be the stack classifying the data of a short exact sequence

$$0 \to \mathcal{M}^1 \to \mathcal{M} \to \mathcal{M}^2 \to 0,$$

as in the definition of $\mathrm{Fl}^n_{n_1,n_2}$, and a collection of nonzero maps $\kappa_i : \Omega^{n-1+\cdots+n-i} \to \Lambda^i(\mathcal{M}^1)$ for $1 \leq i \leq k$, which satisfy the Plücker relations.

We have a natural map $\mathfrak{q}_{n_1,n_2,k} : \mathrm{Fl}^{\overline{\mathcal{Q}}_k}_{n_1,n_2} \to \overline{\mathcal{Q}}_{n_1,k} \times \mathrm{Bun}_{n_2}$, which makes the following square Cartesian:

$$\begin{tikzcd} \mathrm{Fl}^{\overline{\mathcal{Q}}_k}_{n_1,n_2} \arrow[r, "\mathfrak{q}_{n_1,n_2,k}"] \arrow[d] & \overline{\mathcal{Q}}_{n_1,k} \times \mathrm{Bun}_{n_2} \arrow[d] \\ \mathrm{Fl}^{n}_{n_1,n_2} \arrow[r, "\mathfrak{q}_{n_1,n_2}"] & \mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2} . \end{tikzcd}$$

In addition, we have a map $\mathfrak{p}_{n_1,n_2,k} : \mathrm{Fl}^{\overline{\mathcal{Q}}_k}_{n_1,n_2} \to \overline{\mathcal{Q}}_k$.

For $k \leq n_1$ we define the constant term functors

$$\mathrm{CT}^{\overline{\mathcal{Q}}_k}_{n_1,n_2} : \mathrm{D}^W(\overline{\mathcal{Q}}_k) \to \mathrm{D}(\overline{\mathcal{Q}}_{n_1,k} \times \mathrm{Bun}_{n_2})$$

by $\mathcal{F} \mapsto \mathfrak{q}_{n_1,n_2,k!}\left(\mathfrak{p}^*_{n_1,n_2,k}(\mathcal{F})\right)$.

We call an object $\mathcal{F} \in \mathrm{D}^W(\overline{\mathcal{Q}}_k)$ cuspidal if $\mathrm{CT}^{\overline{\mathcal{Q}}_k}_{n_1,n_2}(\mathcal{F}) = 0$ for all $k \leq n_1 < n$.

In principle, one can introduce the constant term functors also for $k > n_1$, and properly speaking, a complex $\mathcal{F} \in \mathrm{D}(\overline{\mathcal{Q}}_k)$ should be called cuspidal if *all* the constant term functors vanish when applied to it, including those with $k > n_1$. However, for objects of the Whittaker category these other functors vanish automatically, so the two notions coincide.

Let π denote the natural projection $\overline{\mathcal{Q}}_1 \simeq \mathrm{Bun}'_n \to \mathrm{Bun}_n$. It is easy to see that for $\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n)$,

$$\mathrm{CT}^{\overline{\mathcal{Q}}_1}_{n_1,n_2}(\pi^*(\mathcal{F})) \simeq (\pi_{n_1} \times \mathrm{id})^* \left(\mathrm{CT}^{n}_{n_1,n_2}(\mathcal{F})\right),$$

where $\pi_{n_1} \times \mathrm{id}$ denotes the natural map $\overline{\mathcal{Q}}_{n_1,1} \times \mathrm{Bun}_{n_2} \to \mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2}$. Therefore, if an object $\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n)$ is cuspidal, then so is $\pi^*(\mathcal{F})$.

6.3. The main result of this section is the following theorem:

THEOREM 6.4. *Let $\mathcal{F}_1 \in \mathrm{D}(\mathrm{Bun}'_n)$ be cuspidal and $\mathcal{F}_2 \in \mathrm{D}(\mathrm{Bun}'_n)$ be any object. Then the map* $\mathrm{Hom}_{\mathrm{D}(\mathrm{Bun}'_n)}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Hom}_{\mathrm{D}^W(\overline{\mathcal{Q}})}(W(\mathcal{F}_1), W(\mathcal{F}_2))$ *is an isomorphism.*

Of course, along with Theorem 6.4 as it is stated, we have its $\mathbb{G}_m$-equivariant version, and a version involving a base S; cf. Section 5.13.

Theorem 6.4 follows by induction from the following assertion:

PROPOSITION 6.5. (1) *The functor $W_{k,k+1} : \mathrm{D}^W(\overline{\mathcal{Q}}_k) \to \mathrm{D}^W(\overline{\mathcal{Q}}_{k+1})$ maps cuspidal objects to cuspidal.*

(2) *If $\mathcal{F} \in \mathrm{D}^W(\overline{\mathcal{Q}}_k)$ is cuspidal, then the $*$-restriction of $W_{k,k+1,ex}(\mathcal{F})$ to $\overline{\mathcal{Q}}_{k+1} - \overline{\mathcal{Q}}_{k+1,ex}$ is zero.*

Indeed, to prove Theorem 6.4, it suffices to show that if we have two objects $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{D}^W(\overline{\mathcal{Q}}_k)$ with $\mathcal{F}_1$ cuspidal, then

$$\mathrm{Hom}_{\mathrm{D}^W(\overline{\mathcal{Q}}_k)}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Hom}_{\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1})}(W_{k,k+1}(\mathcal{F}_1), W_{k,k+1}(\mathcal{F}_2))$$

is an isomorphism. However, by Theorem 5.1, we know that

$$\mathrm{Hom}_{\mathrm{D}^W(\overline{\mathcal{Q}}_k)}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Hom}_{\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})}(W_{k,k+1,ex}(\mathcal{F}_1), W_{k,k+1,ex}(\mathcal{F}_2))$$

is an isomorphism. And now, the condition that

$$W_{k,k+1,ex}(\mathcal{F}_1)|_{\overline{\mathcal{Q}}_{k+1} - \overline{\mathcal{Q}}_{k+1,ex}} = 0$$

means that

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})}(W_{k,k+1,ex}(\mathcal{F}_1), W_{k,k+1,ex}(\mathcal{F}_2)) \\
&\qquad := \mathrm{Hom}_{\mathrm{D}(\overline{\mathcal{Q}}_{k+1,ex})}(W_{k,k+1,ex}(\mathcal{F}_1), W_{k,k+1,ex}(\mathcal{F}_2)) \\
&\qquad \simeq \mathrm{Hom}_{\mathrm{D}(\overline{\mathcal{Q}}_{k+1})}(W_{k,k+1}(\mathcal{F}_1), W_{k,k+1}(\mathcal{F}_2)) \\
&\qquad =: \mathrm{Hom}_{\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1})}(W_{k,k+1}(\mathcal{F}_1), W_{k,k+1}(\mathcal{F}_2)).
\end{aligned}$$

6.6. *Proof of Proposition* 6.5(1). Let $n_1 \geq k+1$. Note that in addition to the stack $\overline{\mathcal{Q}}_{n_1,k}$, one can introduces its "*ex*" version $\overline{\mathcal{Q}}_{n_1,k+1,ex}$. Moreover, proceeding just as in Sections 4 and 5, we introduce the categories $\mathrm{D}^W(\overline{\mathcal{Q}}_{n_1,k})$, $\mathrm{D}^W(\overline{\mathcal{Q}}_{n_1,k+1,ex})$, and the functors $W_{n_1,k,k+1,ex} : \mathrm{D}^W(\overline{\mathcal{Q}}_{n_1,k}) \to \mathrm{D}^W(\overline{\mathcal{Q}}_{n_1,k+1,ex})$ and $W_{n_1,k,k+1} : \mathrm{D}^W(\overline{\mathcal{Q}}_{n_1,k}) \to \mathrm{D}^W(\overline{\mathcal{Q}}_{n_1,k+1})$.

In addition, we can introduce a stack $\mathrm{Fl}_{n_1,n_2}^{\overline{\mathcal{Q}}_{k+1,ex}}$, which fits into the diagram:

$$\begin{tikzcd}
\overline{\mathcal{Q}}_{k+1,ex} \arrow[d, "\pi_{k+1,ex,k}"'] & \mathrm{Fl}_{n_1,n_2}^{\overline{\mathcal{Q}}_{k+1,ex}} \arrow[l, "\mathfrak{p}_{n_1,n_2,k+1,ex}"'] \arrow[r, "\mathfrak{q}_{n_1,n_2,k+1,ex}"] \arrow[d] & \overline{\mathcal{Q}}_{n_1,k+1,ex} \times \mathrm{Bun}_{n_2} \arrow[d, "\pi_{n_1,k+1,ex,k} \times \mathrm{id}"] \\
\overline{\mathcal{Q}}_k \arrow[d] & \mathrm{Fl}_{n_1,n_2}^{\overline{\mathcal{Q}}_k} \arrow[l, "\mathfrak{p}_{n_1,n_2,k}"'] \arrow[r, "\mathfrak{q}_{n_1,n_2,k}"] \arrow[d] & \overline{\mathcal{Q}}_{n_1,k} \times \mathrm{Bun}_{n_2} \arrow[d] \\
\mathrm{Bun}_n & \mathrm{Fl}_{n_1,n_2}^n \arrow[l, "\mathfrak{p}_{n_1,n_2}"'] \arrow[r, "\mathfrak{q}_{n_1,n_2}"] & \mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2} .
\end{tikzcd}$$

In this diagram the right portion consists of Cartesian squares.

Using the stack $\mathrm{Fl}_{n_1,n_2}^{\overline{\mathcal{Q}}_{k+1,ex}}$ we introduce the functor

$$\mathrm{CT}_{n_1,n_2}^{\overline{\mathcal{Q}}_{k+1,ex}} : \mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex}) \to \mathrm{D}(\overline{\mathcal{Q}}_{n_1,k+1,ex} \times \mathrm{Bun}_{n_2}).$$

LEMMA 6.7. *The functor* $\mathrm{CT}_{n_1,n_2}^{\overline{\mathcal{Q}}_{k+1,ex}}$ *maps* $\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})$ *to* $\mathrm{D}^W(\overline{\mathcal{Q}}_{n_1,k+1,ex} \times \mathrm{Bun}_{n_2})$.

Proof. We will use the description of $\mathrm{D}^W(\overline{\mathcal{Q}}_{n_1,k+1,ex})$ similar to that of Proposition 4.13. For a string of integers $\overline{d} = d_1, \dots, d_k$, let ${}^{\overline{d}}\overline{\mathcal{Q}}_{n_1,k+1,ex}{}' \subset$

${}^{\overline{d}}\overline{\mathcal{Q}}_{n_1,k+1,ex}$ be the corresponding locally closed substacks of $\overline{\mathcal{Q}}_{n_1,k+1,ex}$. Let also ${}^{\overline{d}}\mathcal{P}_{n_1,k}$ be the stack classifying the data of $(D_1,\dots,D_k,\mathcal{M}^{1\prime},\Omega^{n-k-1}\to\mathcal{M}^{1\prime})$, as in the definition of ${}^{\overline{d}}\mathcal{P}_k$, with the difference that now $\mathcal{M}^{1\prime}$ is a vector bundle of rank n_1-k. We have a smooth map $\phi_{n_1,k}:{}^{\overline{d}}\overline{\mathcal{Q}}_{n_1,k+1,ex}{}'\to{}^{\overline{d}}\mathcal{P}_{n_1,k}$.

To prove the lemma it is sufficient to show that for $\mathcal{F}\in \mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})$, the restriction of $\mathrm{CT}^{\overline{\mathcal{Q}}_{k+1,ex}}_{n_1,n_2}(\mathcal{F})$ to each ${}^{\overline{d}}\overline{\mathcal{Q}}_{n_1,k+1,ex}{}'$ is isomorphic to the pull-back of a complex on ${}^{\overline{d}}\mathcal{P}_{n_1,k}$, tensored by an appropriate Artin-Schreier sheaf.

Consider the fiber product

$$Z := \mathrm{Fl}^{n-k}_{n_1-k,n_2}\underset{\mathrm{Bun}_{n_1-k}}{\times}{}^{\overline{d}}\mathcal{P}_{n_1,k}.$$

Let ${}^{\overline{d}}\,\mathrm{Fl}^{\overline{\mathcal{Q}}_{k+1,ex}}_{n_1,n_2}$ be the preimage in $\mathrm{Fl}^{\overline{\mathcal{Q}}_{k+1,ex}}_{n_1,n_2}$ of the substack ${}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}\subset\overline{\mathcal{Q}}_{k+1,ex}$ under $\mathfrak{p}_{n_1,n_2}$. We have a commutative diagram

$$\begin{tikzcd}
{}^{\overline{d}}\overline{\mathcal{Q}}_{k+1,ex}{}' \arrow[d, "\phi_k"'] & {}^{\overline{d}}\,\mathrm{Fl}^{\overline{\mathcal{Q}}_{k+1,ex}}_{n_1,n_2} \arrow[l, "\mathfrak{p}_{n_1,n_2}"'] \arrow[r, "\mathfrak{q}_{n_1,n_2}"] \arrow[d] & {}^{\overline{d}}\overline{\mathcal{Q}}_{n_1,k+1,ex}{}'\times\mathrm{Bun}_{n_2} \arrow[d, "\phi_{n_1,k}\times\mathrm{id}"'] \\
{}^{\overline{d}}\mathcal{P}_k & Z \arrow[l] \arrow[r] & {}^{\overline{d}}\mathcal{P}_{n_1,k}\times\mathrm{Bun}_{n_2}\,.
\end{tikzcd}$$

The right portion of this diagram is not Cartesian. However, the map

$${}^{\overline{d}}\,\mathrm{Fl}^{\overline{\mathcal{Q}}_{k+1,ex}}_{n_1,n_2}\to Z\underset{{}^{\overline{d}}\mathcal{P}_{n_1,k}\times\mathrm{Bun}_{n_2}}{\times}({}^{\overline{d}}\overline{\mathcal{Q}}_{n_1,k+1,ex}{}'\times\mathrm{Bun}_{n_2})$$

is smooth with contractible fibers. Hence, the assertion of the lemma follows from the projection formula. $\square$

By the lemma above, part (1) of Proposition 6.5 would follow once we are able to establish an isomorphism of functors:

$$\mathrm{CT}^{\overline{\mathcal{Q}}_{k+1,ex}}_{n_1,n_2}\circ W_{k,k+1,ex}\simeq(W_{n_1,k,k+1,ex}\times\mathrm{id})\circ\mathrm{CT}^{\overline{\mathcal{Q}}_k}_{n_1,n_2},\tag{20}$$

both of which map from $\mathrm{D}^W(\overline{\mathcal{Q}}_k)$ to $\mathrm{D}^W(\overline{\mathcal{Q}}_{n_1,k+1,ex}\times\mathrm{Bun}_{n_2})$.

Observe that the functor $\mathrm{CT}^{\overline{\mathcal{Q}}_k}_{n_1,n_2}$, has a natural right adjoint, which we will denote by $\mathrm{Eis}^{\overline{\mathcal{Q}}_k}_{n_1,n_2}$, that maps $\mathcal{F}\in\mathrm{D}(\overline{\mathcal{Q}}_{n_1,k+1}\times\mathrm{Bun}_{n_2})$ to $\mathfrak{p}_{n_1,n_2,k*}\Big(\mathfrak{q}^!_{n_1,n_2,k}(\mathcal{F})\Big)$. This functor also maps $\mathrm{D}^W(\overline{\mathcal{Q}}_{n_1,k}\times\mathrm{Bun}_{n_2})$ to $\mathrm{D}^W(\overline{\mathcal{Q}}_k)$.

Similarly, we have a right adjoint of $\mathrm{CT}^{\overline{\mathcal{Q}}_{k+1,ex}}_{n_1,n_2}$

$$\mathrm{Eis}^{\overline{\mathcal{Q}}_{k+1,ex}}_{n_1,n_2}:\mathrm{D}^W(\overline{\mathcal{Q}}_{n_1,k+1,ex}\times\mathrm{Bun}_{n_2})\to\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex}).$$

To prove (20) it suffices to verify the isomorphism on the level of the corresponding adjoint functors. In other words, we must show that

$$\pi_{k+1,ex,k*}\circ\mathrm{Eis}^{\overline{\mathcal{Q}}_{k+1,ex}}_{n_1,n_2}\simeq\mathrm{Eis}^{\overline{\mathcal{Q}}_k}_{n_1,n_2}\circ(\pi_{n_1,k+1,ex,k}\times\mathrm{id})_*.$$

However, the latter isomorphism follows from base change.

6.8. *Proof of Proposition* 6.5(2). Note that $\overline{\mathcal{Q}}_{k+1,ex} - \overline{\mathcal{Q}}_{k+1} \subset \overline{\mathcal{Q}}_{k+1}$ is naturally isomorphic to $\overline{\mathcal{Q}}_k$. We would like to calculate $W_{k,k+1,ex}(\mathcal{F})|_{\overline{\mathcal{Q}}_k}$ in terms of $\mathrm{CT}^{\overline{\mathcal{Q}}_k}_{n_1,n_2}$ for $n_1 = k$.

Recall that to a string of integers $\overline{d} = d_1, \ldots, d_k$ we associated a locally closed substack ${}^{\overline{d}}\overline{\mathcal{Q}}_k \subset \overline{\mathcal{Q}}_k$.

Note now that we have a natural map $\psi_k : {}^{\overline{d}}\overline{\mathcal{Q}}_k \to \overline{\mathcal{Q}}_{k,k} \times \mathrm{Bun}_{n-k}$. Namely, we can think of a point of ${}^{\overline{d}}\overline{\mathcal{Q}}_k$ as a data of

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_k \subset \mathcal{M},$$

and identifications $\mathcal{M}_i/\mathcal{M}_{i-1} \simeq \Omega^{n-i}(D_i - D_{i-1})$ for $(D_1, \ldots, D_k) \in X^{(\overline{d})}$.

The corresponding point of $\overline{\mathcal{Q}}_{k,k} \times \mathrm{Bun}_{n-k}$ is $\mathcal{M}^1 = \mathcal{M}_k$, with the data of $\kappa_{n_1,i}$ being given by the old κ_i's, and $\mathcal{M}^2 := \mathcal{M}/\mathcal{M}_k$.

We claim that up to a cohomological shift, for $\mathcal{F} \in \mathrm{D}^W(\overline{\mathcal{Q}}_k)$,

$$W_{k,k+1,ex}(\mathcal{F})|_{{}^{\overline{d}}\overline{\mathcal{Q}}_k} \simeq \psi_k^*(\mathrm{CT}^{\overline{\mathcal{Q}}_k}_{k,n-k}(\mathcal{F})). \tag{21}$$

This follows immediately from the description of the functor ${}^{\overline{d}}W_{k,k+1,ex}$ in Section 5.2. Thus, part (2) of Proposition 6.5 follows, because to show that $W_{k,k+1,ex}(\mathcal{F})|_{\overline{\mathcal{Q}}_k} = 0$ for $\mathcal{F}$ cuspidal, it is enough to show that for all $\overline{d}$ as above $W_{k,k+1,ex}(\mathcal{F})|_{{}^{\overline{d}}\overline{\mathcal{Q}}_k} = 0$, and the latter is given by (21).

Note that in the course of the proof we have shown that $W_{k,k+1,ex}(\mathcal{F})|_{\overline{\mathcal{Q}}_k} = 0$ *if and only if* $\mathrm{CT}^{\overline{\mathcal{Q}}_k}_{k,n-k}(\mathcal{F}) = 0$. This is because the stack $\overline{\mathcal{Q}}_{k,k}$ is also stratified by means of ${}^{\overline{d}}\overline{\mathcal{Q}}_{k,k}$, and for every $\overline{d}$ the map

$$\psi : {}^{\overline{d}}\overline{\mathcal{Q}}_k \to {}^{\overline{d}}\overline{\mathcal{Q}}_{k,k} \times \mathrm{Bun}_{n-k}$$

is surjective.

6.9. Thus, Theorem 6.4 is proved. We will now give another categorical interpretation of it. Let $\mathrm{D}^W_{\mathrm{cusp}}(\overline{\mathcal{Q}}_k)$ denote the full subcategory consisting of cuspidal objects in $\mathrm{D}^W(\overline{\mathcal{Q}}_k)$. This is evidently a triangulated subcategory in $\mathrm{D}^W(\overline{\mathcal{Q}}_k)$.

Now, let $\mathrm{D}^W_{\mathrm{degen}}(\overline{\mathcal{Q}}_k) \subset \mathrm{D}^W(\overline{\mathcal{Q}}_k)$ denote the (full triangulated) subcategory of those objects $\mathcal{F}$ for which $W_{k,n}(\mathcal{F}) = 0$. Let $\widetilde{\mathrm{D}}^W(\overline{\mathcal{Q}}_k)$ denote the quotient triangulated category $\mathrm{D}^W(\overline{\mathcal{Q}}_k)/\,\mathrm{D}^W_{\mathrm{degen}}(\overline{\mathcal{Q}}_k)$.

Consider the composition

$$\mathrm{D}^W_{\mathrm{cusp}}(\overline{\mathcal{Q}}_k) \to \mathrm{D}^W(\overline{\mathcal{Q}}_k) \to \widetilde{\mathrm{D}}^W(\overline{\mathcal{Q}}_k).$$

THEOREM 6.10. (1) *The above functor* $\mathrm{D}^W_{\mathrm{cusp}}(\overline{\mathcal{Q}}_k) \to \widetilde{\mathrm{D}}^W(\overline{\mathcal{Q}}_k)$ *is an equivalence of categories.*

(2) *The functor* $\mathrm{D}^W_{\mathrm{cusp}}(\overline{\mathcal{Q}}_k) \xrightarrow{W_{k,n}} \mathrm{D}^W(\overline{\mathcal{Q}})$ *is an equivalence as well.*

6.11. *Proof of Theorem* 6.10. Let $W_{k,k+1}^{-1} : \mathrm{D}^W(\overline{\mathcal{Q}}_{k+1}) \to \mathrm{D}^W(\overline{\mathcal{Q}}_k)$ be defined by sending $\mathcal{F}' \in \mathrm{D}^W(\overline{\mathcal{Q}}_{k+1})$ to $\pi_{k+1,ex,k!}(\mathcal{F}'')$, where $\mathcal{F}''$ is the !-extension from $\overline{\mathcal{Q}}_{k+1}$ to $\overline{\mathcal{Q}}_{k+1,ex}$. Since $W_{k,k+1,ex}$ is an equivalence, we have an isomorphism of functors

$$W_{k,k+1} \circ W_{k,k+1}^{-1} \simeq \mathrm{id}_{\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1})},$$

and an adjunction map $W_{k,k+1}^{-1} \circ W_{k,k+1} \to \mathrm{id}_{\mathrm{D}^W(\overline{\mathcal{Q}}_k)}$.

By construction, $W_{k,k+1}$ induces a functor $\widetilde{\mathrm{D}}^W(\overline{\mathcal{Q}}_k) \to \widetilde{\mathrm{D}}^W(\overline{\mathcal{Q}}_{k+1})$. We claim this functor is an equivalence for every k.

Indeed, it is easy to see that the functor

$$\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1}) \xrightarrow{W_{k,k+1}^{-1}} \mathrm{D}^W(\overline{\mathcal{Q}}_k) \to \widetilde{\mathrm{D}}^W(\overline{\mathcal{Q}}_k)$$

factors through $\widetilde{\mathrm{D}}^W(\overline{\mathcal{Q}}_{k+1})$ and defines a quasi-inverse for $W_{k,k+1}$. Hence, $W_{k,n} : \widetilde{\mathrm{D}}^W(\overline{\mathcal{Q}}_k) \to \widetilde{\mathrm{D}}^W(\overline{\mathcal{Q}}_n) = \mathrm{D}^W(\overline{\mathcal{Q}})$ is an equivalence as well.

Thus, it remains to prove the first assertion of the theorem. For that, it is enough to show that $W_{k,k+1}$ induces an equivalence $\mathrm{D}^W_{\mathrm{cusp}}(\overline{\mathcal{Q}}_k) \to \mathrm{D}^W_{\mathrm{cusp}}(\overline{\mathcal{Q}}_{k+1})$ for every k. The fact that the image of $\mathrm{D}^W_{\mathrm{cusp}}(\overline{\mathcal{Q}}_k)$ under $W_{k,k+1}$ belongs to $\mathrm{D}^W_{\mathrm{cusp}}(\overline{\mathcal{Q}}_k)$ was proved in Proposition 6.5.

We claim that $W_{k,k+1}^{-1}$ defines a quasi-inverse. Indeed, for $\mathcal{F} \in \mathrm{D}^W_{\mathrm{cusp}}(\overline{\mathcal{Q}}_{k+1})$ to show that $W_{k,k+1}^{-1}(\mathcal{F}) \in \mathrm{D}^W_{\mathrm{cusp}}(\overline{\mathcal{Q}}_k)$ we must verify that $\mathrm{CT}^{\overline{\mathcal{Q}}_k}_{n_1,n_2}(W_{k,k+1}^{-1}(\mathcal{F})) = 0$ for $n_1 \geq k$.

Suppose first that $n_1 \geq k+1$. Then, since $W_{n_1,k,k+1,ex}$ is an equivalence, what we need follows immediately from (20). For $n_1 = k$, the needed assertion follows from the last remark of Section 6.8.

The fact that $W_{k,k+1} \circ W_{k,k+1}^{-1} \simeq \mathrm{id}$ we know already. It remains, therefore to show that for $\mathcal{F} \in \mathrm{D}^W_{\mathrm{cusp}}(\overline{\mathcal{Q}}_k)$,

$$W_{k,k+1}^{-1}(W_{k,k+1}(\mathcal{F})) \to \mathcal{F}$$

is an isomorphism. Let $\mathcal{F}'$ be the cone of the above map. We know that $\mathcal{F}' \in \mathrm{D}^W_{\mathrm{cusp}}(\overline{\mathcal{Q}}_k)$, and $W_{k,k+1}(\mathcal{F}') \simeq 0$. Hence, $\mathcal{F}' \simeq 0$ by Theorem 6.4.

6.12. As a corollary of Theorem 6.10 we obtain that the category $\mathrm{D}^W_{\mathrm{cusp}}(\overline{\mathcal{Q}}_k)$, and hence, in particular $\mathrm{D}_{\mathrm{cusp}}(\mathrm{Bun}_1)$, possesses a t-structure. Indeed, it is equivalent to the category $\mathrm{D}^W(\overline{\mathcal{Q}})$, for which the t-structure is manifest. Note that this t-structure does not coincide with the t-structure on the ambient category $\mathrm{D}^W(\overline{\mathcal{Q}}_k)$.

7. The Hecke functors

7.1. Recall the Hecke functor $\mathrm{H} : \mathrm{D}(\mathrm{Bun}_n) \to \mathrm{D}(X \times \mathrm{Bun}_n)$, which was defined using the stack $\mathcal{H} = \mathrm{Mod}_n^1$. In this section we will introduce Hecke functors that map from $\mathrm{D}(\overline{\mathcal{Q}}_k)$ to $\mathrm{D}(X \times \overline{\mathcal{Q}}_k)$. First we will consider the case of $\overline{\mathcal{Q}}_1 = \mathrm{Bun}_n'$.

Set $\mathcal{H}^{\mathrm{Bun}_n'} := \mathrm{Bun}_n' \underset{\mathrm{Bun}_n}{\times} \mathcal{H}$, where the map $\overleftarrow{h} : \mathcal{H} \to \mathrm{Bun}_n$ is used to define the fiber product.

We have a commutative diagram

$$\begin{tikzcd}
X \times \mathrm{Bun}_n' \arrow[d, "\mathrm{id} \times \pi"'] & \mathcal{H}^{\mathrm{Bun}_n'} \arrow[l, "s^{\mathrm{Bun}_n'} \times \overleftarrow{h}^{\mathrm{Bun}_n'}"'] \arrow[r, "\overrightarrow{h}^{\mathrm{Bun}_n'}"] \arrow[d] & \mathrm{Bun}_n' \arrow[d, "\pi"'] \\
X \times \mathrm{Bun}_n & \mathcal{H} \arrow[l, "s \times \overleftarrow{h}"'] \arrow[r, "\overrightarrow{h}"] & \mathrm{Bun}_n,
\end{tikzcd}$$

in which the left square is Cartesian. Indeed, the map $\overrightarrow{h}^{\mathrm{Bun}_n'}$ attaches to a point $(x, \mathcal{M} \hookrightarrow \mathcal{M}', \kappa : \Omega^{n-1} \to \mathcal{M}) \in \mathrm{Bun}_n' \underset{\mathrm{Bun}_n}{\times} \mathcal{H}$ the point $(\mathcal{M}', \kappa' : \Omega^{n-1} \to \mathcal{M}')$, where κ' is the composition $\Omega^{n-1} \xrightarrow{\kappa} \mathcal{M} \to \mathcal{M}'$.

We define the functor $\mathrm{H}^{\mathrm{Bun}_n'} : \mathrm{D}(\mathrm{Bun}_n') \to \mathrm{D}(X \times \mathrm{Bun}_n')$ by

$$\mathcal{F} \mapsto (s^{\mathrm{Bun}_n'} \times \overleftarrow{h}^{\mathrm{Bun}_n'})_! \left(\overrightarrow{h}^{\mathrm{Bun}_n' *}(\mathcal{F}) \right) [n-1].$$

Note that the functors $\mathrm{H}^{\mathrm{Bun}_n'}$ and H are compatible in the following way: for $\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n)$,

$$(\mathrm{id} \times \pi)^* (\mathrm{H}(\mathcal{F})) \simeq \mathrm{H}^{\mathrm{Bun}_n'} (\pi^*(\mathcal{F})) [1]. \tag{22}$$

Note also that since the map $\overrightarrow{h}^{\mathrm{Bun}_n'}$ is not smooth, the functor $\mathrm{H}^{\mathrm{Bun}_n'}$ does not commute with the Verdier duality. In particular, one could define its Verdier twin by $\mathcal{F} \mapsto (s^{\mathrm{Bun}_n'} \times \overleftarrow{h}^{\mathrm{Bun}_n'})_* \left(\overrightarrow{h}^{\mathrm{Bun}_n' !}(\mathcal{F}) \right) [1-n]$.

7.2. For $1 \leq k \leq n$ we introduce the appropriate Hecke functors in a similar fashion. Namely, we set $\mathcal{H}^{\overline{\mathcal{Q}}_k} := \overline{\mathcal{Q}}_k \underset{\mathrm{Bun}_n}{\times} \mathcal{H}$, which fits into a commutative diagram

$$\begin{tikzcd}
X \times \overline{\mathcal{Q}}_k \arrow[d, "\mathrm{id} \times \pi"'] & \mathcal{H}^{\overline{\mathcal{Q}}_k} \arrow[l, "s^{\overline{\mathcal{Q}}_k} \times \overleftarrow{h}^{\overline{\mathcal{Q}}_k}"'] \arrow[r, "\overrightarrow{h}^{\overline{\mathcal{Q}}_k}"] \arrow[d] & \mathrm{Bun}_n' \arrow[d, "\pi"'] \\
X \times \mathrm{Bun}_n & \mathcal{H} \arrow[l, "s \times \overleftarrow{h}"'] \arrow[r, "\overrightarrow{h}"] & \mathrm{Bun}_n,
\end{tikzcd}$$

in which the left square is Cartesian.

The functor $\mathrm{H}^{\overline{\mathcal{Q}}_k} : \mathrm{D}(\overline{\mathcal{Q}}_k) \to \mathrm{D}(X \times \overline{\mathcal{Q}}_k)$ is defined by means of

$$\mathcal{F} \mapsto (s^{\overline{\mathcal{Q}}_k} \times \overleftarrow{h}^{\overline{\mathcal{Q}}_k})_! \left(\overrightarrow{h}^{\overline{\mathcal{Q}}_k *}(\mathcal{F}) \right) [n-1].$$

Set ${}_x\mathcal{H}^{\overline{\mathcal{Q}}_k}$ to be the preimage of $x \in X$ in $\mathcal{H}^{\overline{\mathcal{Q}}_k}$. For a point $x \in X$ we will denote by ${}_x\,\mathrm{H}^{\overline{\mathcal{Q}}_k}$ the functor $\mathrm{D}(\overline{\mathcal{Q}}_k) \to \mathrm{D}(\overline{\mathcal{Q}}_k)$ obtained as a composition of $\mathrm{H}^{\overline{\mathcal{Q}}_k}$ followed by the $*$-restriction to $x \times \overline{\mathcal{Q}}_k \subset X \times \overline{\mathcal{Q}}_k$.

In other words, ${}_x\,\mathrm{H}^{\overline{\mathcal{Q}}_k}$ can be defined using the substack ${}_x\mathcal{H}^{\overline{\mathcal{Q}}_k}$ as a correspondence.

In a similar way we define the stack $\mathcal{H}^{\overline{\mathcal{Q}}_{k+1,ex}}$ and the corresponding functor $\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex}} : \mathrm{D}(\overline{\mathcal{Q}}_{k+1,ex}) \to \mathrm{D}(X \times \overline{\mathcal{Q}}_{k+1,ex})$.

Proposition 7.3. *The functor* $\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex}}$ *maps* $\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})$ *to* $\mathrm{D}^W(X \times \overline{\mathcal{Q}}_{k+1,ex})$.

Of course, as a corollary of this proposition we obtain that $\mathrm{H}^{\overline{\mathcal{Q}}_k}$ maps $\mathrm{D}^W(\overline{\mathcal{Q}}_k)$ to $\mathrm{D}^W(X \times \overline{\mathcal{Q}}_k)$.

Proof. To simplify the notation we will show that for any $x \in X$, the functor ${}_x\,\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex}} : \mathrm{D}(\overline{\mathcal{Q}}_{k+1,ex}) \to \mathrm{D}(\overline{\mathcal{Q}}_{k+1,ex})$ preserves the subcategory $\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})$.

Let $y \in X$ be a point different from x. It is easy to see that we have a well-defined functor ${}_x\,\mathrm{H}^{\overline{\mathcal{Q}}^y_{k+1,ex}} : \mathrm{D}(\overline{\mathcal{Q}}^y_{k+1,ex}) \to \mathrm{D}(\overline{\mathcal{Q}}^y_{k+1,ex})$, constructed using the stack that we will denote by ${}_x\mathcal{H}^{\overline{\mathcal{Q}}^y_{k+1,ex}}$. We will first show that this functor preserves $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$.

However, this is almost obvious from the definitions:

Recall the groupoid $\mathcal{H}_{N^y_k} \underset{\overline{\mathcal{Q}}^y_k}{\times} \overline{\mathcal{Q}}^y_{k+1,ex}$ acting on $\overline{\mathcal{Q}}^y_{k+1,ex}$. We claim that it lifts to the stack ${}_x\mathcal{H}^{\overline{\mathcal{Q}}^y_{k+1,ex}}$; i.e., we have a groupoid ${}_x\mathcal{H}^{\overline{\mathcal{Q}}^y_{k+1,ex}}_{N^y_k}$ which fits into two commutative diagrams

$$\begin{array}{ccccc}
{}_x\mathcal{H}^{\overline{\mathcal{Q}}^y_{k+1,ex}} & \xleftarrow{\mathrm{act}} & {}_x\mathcal{H}^{\overline{\mathcal{Q}}^y_{k+1,ex}}_{N^y_k} & \xrightarrow{\mathrm{pr}} & {}_x\mathcal{H}^{\overline{\mathcal{Q}}^y_{k+1,ex}} \\
\downarrow{\scriptstyle \overleftarrow{h}^{\overline{\mathcal{Q}}^y_{k+1,ex}}} & & \downarrow{\scriptstyle \overleftarrow{h}} & & \downarrow{\scriptstyle \overleftarrow{h}^{\overline{\mathcal{Q}}^y_{k+1,ex}}} \\
\overline{\mathcal{Q}}^y_{k+1,ex} & \xleftarrow{\mathrm{act}_{k,ex}} & \mathcal{H}_{N^y_k} \underset{\overline{\mathcal{Q}}^y_k}{\times} \overline{\mathcal{Q}}^y_{k+1,ex} & \xrightarrow{\mathrm{pr}_{k,ex}} & \overline{\mathcal{Q}}^y_{k+1,ex},
\end{array}$$

and

$$\begin{array}{ccccc}
{}_x\mathcal{H}^{\overline{\mathcal{Q}}^y_{k+1,ex}} & \xleftarrow{\mathrm{act}} & {}_x\mathcal{H}^{\overline{\mathcal{Q}}^y_{k+1,ex}}_{N^y_k} & \xrightarrow{\mathrm{pr}} & {}_x\mathcal{H}^{\overline{\mathcal{Q}}^y_{k+1,ex}} \\
\downarrow{\scriptstyle \overrightarrow{h}^{\overline{\mathcal{Q}}^y_{k+1,ex}}} & & \downarrow{\scriptstyle \overrightarrow{h}} & & \downarrow{\scriptstyle \overrightarrow{h}^{\overline{\mathcal{Q}}^y_{k+1,ex}}} \\
\overline{\mathcal{Q}}^y_{k+1,ex} & \xleftarrow{\mathrm{act}_{k,ex}} & \mathcal{H}_{N^y_k} \underset{\overline{\mathcal{Q}}^y_k}{\times} \overline{\mathcal{Q}}^y_{k+1,ex} & \xrightarrow{\mathrm{pr}_{k,ex}} & \overline{\mathcal{Q}}^y_{k+1,ex},
\end{array}$$

in both of which both squares are Cartesian. Moreover, the compositions

$$_x\mathcal{H}^{\overline{\mathcal{Q}}^y_{k+1,ex}}_{N^y_k} \xrightarrow{\overleftarrow{h}} \mathcal{H}_{N^y_k} \underset{\overline{\mathcal{Q}}^y_k}{\times} \overline{\mathcal{Q}}^y_{k+1,ex} \xrightarrow{\chi_k} \mathbb{A}^1$$

and

$$_x\mathcal{H}^{\overline{\mathcal{Q}}^y_{k+1,ex}}_{N^y_k} \xrightarrow{\overrightarrow{h}} \mathcal{H}_{N^y_k} \underset{\overline{\mathcal{Q}}^y_k}{\times} \overline{\mathcal{Q}}^y_{k+1,ex} \xrightarrow{\chi_k} \mathbb{A}^1$$

coincide. Therefore, if an object $\mathcal{F} \in \mathrm{D}(\overline{\mathcal{Q}}^y_{k+1,ex})$ satisfies the equivariance condition (14), then so does $\overleftarrow{h}^{\overline{\mathcal{Q}}^y_k}_! \left(\overrightarrow{h}^{\overline{\mathcal{Q}}^y_k *}(\mathcal{F})\right)[n-1]$.

Now let $\mathcal{F}$ be an arbitrary object of $\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})$. To show that $_x\,\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex}}(\mathcal{F})$ also belongs to $\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})$, from Lemma 4.8 it follows that it is sufficient to show that any irreducible sub-quotient of any perverse cohomology sheaf of $_x\,\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex}}(\mathcal{F})$ belongs to $\mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})$.

Let $\mathcal{K}$ be such a sub-quotient. Then there exists $y \in X$, such that the restriction of $\mathcal{K}$ to $\overline{\mathcal{Q}}^y_{k+1,ex}$ is nonzero. Hence, again by Lemma 4.8 and Corollary 4.14, it suffices to show that $\mathcal{K}|_{\overline{\mathcal{Q}}^y_{k+1,ex}}$ belongs to $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$. But above we have shown that the entire $_x\,\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex}}(\mathcal{F})|_{\overline{\mathcal{Q}}^y_{k+1,ex}}$ belongs to $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$, and hence also $\mathcal{K}|_{\overline{\mathcal{Q}}^y_{k+1,ex}}$, which is its sub-quotient. □

7.4. Our next goal is to show that the Hecke functors and Whittaker functors commute with each other.

PROPOSITION 7.5. *We have a natural isomorphism of functors*

$$\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex}} \circ W_{k,k+1,ex} \simeq (\mathrm{id} \times W_{k,k+1,ex}) \circ \mathrm{H}^{\overline{\mathcal{Q}}_k} : \mathrm{D}^W(\overline{\mathcal{Q}}_k) \to \mathrm{D}^W(X \times \overline{\mathcal{Q}}_{k+1,ex}).$$

Of course, the proposition implies that the functors $\mathrm{H}^{\overline{\mathcal{Q}}_{k+1}} \circ W_{k,k+1}$ and $(\mathrm{id} \times W_{k,k+1}) \circ \mathrm{H}^{\overline{\mathcal{Q}}_k}$ from $\mathrm{D}^W(\overline{\mathcal{Q}}_k)$ to $\mathrm{D}^W(X \times \overline{\mathcal{Q}}_{k+1})$ are isomorphic.

Proof. As in the proof of the previous proposition, in order to simplify the notation, we will consider the functors $_x\,\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex}}$ and $_x\,\mathrm{H}^{\overline{\mathcal{Q}}_k}$ instead of $\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex}}$ and $\mathrm{H}^{\overline{\mathcal{Q}}_k}$.

In fact, from the proof of Proposition 7.3 given above one can directly deduce that for $y \neq x$, $_x\,\mathrm{H}^{\overline{\mathcal{Q}}^y_{k+1,ex}} \circ W^y_{k,k+1,ex} \simeq W^y_{k,k+1,ex} \circ {_x\,\mathrm{H}^{\overline{\mathcal{Q}}^y_k}}$, using the definition of $W^y_{k,k+1,ex}$ via the Fourier transform functor as in Section 5.9. We will proceed differently. Namely, we will prove that for $\mathcal{F} \in \mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})$,

$$\pi_{k+1,ex,k!}\left({_x\,\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex}}(\mathcal{F})}\right) \simeq {_x\,\mathrm{H}^{\overline{\mathcal{Q}}_k}}\left(\pi_{k+1,ex,k!}(\mathcal{F})\right), \tag{23}$$

which is equivalent to the statement of Proposition 7.5, since $\pi_{k+1,ex,k!}$ induces an equivalence of categories.

For a point $x \in X$, let $\overline{\mathcal{Q}}_{k+1,ex,x}$ denote the stack that classifies the data $(\mathcal{M}, \kappa_1, \ldots, \kappa_k, \kappa_{k+1})$ as before, with the difference that now the last map $\kappa_{k+1} : \Omega^{n-1+\cdots+n-(k+1)} \to \Lambda^{k+1}(\mathcal{M})$ is allowed to have a simple pole at x. We have a natural closed embedding $\overline{\mathcal{Q}}_{k+1,ex} \hookrightarrow \overline{\mathcal{Q}}_{k+1,ex,x}$.

Let ${}_x\,\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex,x}}$ denote the Cartesian product

$$ {}_x\,\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex,x}} := {}_x\,\mathrm{H}^{\overline{\mathcal{Q}}_k} \underset{\overline{\mathcal{Q}}_k}{\times} \overline{\mathcal{Q}}_{k+1,ex}, $$

where we have used the map $\overrightarrow{h}^{\overline{\mathcal{Q}}_k} : {}_x\,\mathrm{H}^{\overline{\mathcal{Q}}_k} \to \overline{\mathcal{Q}}_k$ to define the product.

We have a commutative diagram:

$$\begin{tikzcd}
\overline{\mathcal{Q}}_{k+1,ex,x} \arrow[d, "\pi_{k+1,ex,k,x}"'] & {}_x\,\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex,x}} \arrow[l, "\overleftarrow{h}^{\overline{\mathcal{Q}}_{k+1,ex,x}}"'] \arrow[r, "\overrightarrow{h}^{\overline{\mathcal{Q}}_{k+1,ex,x}}"] \arrow[d] & \overline{\mathcal{Q}}_{k+1,ex} \arrow[d, "\pi_{k+1,ex,k}"'] \\
\overline{\mathcal{Q}}_k & {}_x\,\mathrm{H}^{\overline{\mathcal{Q}}_k} \arrow[l, "\overleftarrow{h}^{\overline{\mathcal{Q}}_k}"'] \arrow[r, "\overrightarrow{h}^{\overline{\mathcal{Q}}_k}"] & \overline{\mathcal{Q}}_k,
\end{tikzcd}$$

in which the right square is Cartesian.

By base change, for $\mathcal{F} \in \mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})$, the right-hand side of (23) equals

$$ (\pi_{k+1,ex,k,x})_! \left(\overleftarrow{h}_!^{\overline{\mathcal{Q}}_{k+1,ex,x}} \left(\overrightarrow{h}^{\overline{\mathcal{Q}}_{k+1,ex,x}\,*}(\mathcal{F}) \right) \right). \tag{24} $$

LEMMA 7.6. *For* $\mathcal{F} \in \mathrm{D}^W(\overline{\mathcal{Q}}_{k+1,ex})$, *the object*

$$ \overleftarrow{h}_!^{\overline{\mathcal{Q}}_{k+1,ex,x}} \left(\overrightarrow{h}^{\overline{\mathcal{Q}}_{k+1,ex,x}\,*}(\mathcal{F}) \right) \in \mathrm{D}(\overline{\mathcal{Q}}_{k+1,ex,x}) $$

is supported on $\overline{\mathcal{Q}}_{k+1,ex}$.

Proof. For $y \neq x$ let $\overline{\mathcal{Q}}^y_{k+1,ex,x}$ denote the open substack of $\overline{\mathcal{Q}}_{k+1,ex,x}$ equal to the preimage of $\overline{\mathcal{Q}}^y_k$ under $\pi_{k+1,ex,k,x}$. It would be sufficient to show that for any such y, the restriction of $\overleftarrow{h}_!^{\overline{\mathcal{Q}}_{k+1,ex,x}} \left(\overrightarrow{h}^{\overline{\mathcal{Q}}_{k+1,ex,x}\,*}(\mathcal{F}) \right)$ (as in the lemma) to $\overline{\mathcal{Q}}^y_{k+1,ex,x}$ is supported on $\overline{\mathcal{Q}}^y_{k+1,ex}$.

As in Section 4.7 we can introduce the category $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex,x})$, and, as in Proposition 7.3, we show that the Hecke functor $\mathcal{F} \mapsto \overleftarrow{h}_!^{\overline{\mathcal{Q}}_{k+1,ex,x}} \left(\overrightarrow{h}^{\overline{\mathcal{Q}}_{k+1,ex,x}\,*}(\mathcal{F}) \right)$ maps $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex})$ to $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex,x})$.

However, we claim that every object of the category $\mathrm{D}^W(\overline{\mathcal{Q}}^y_{k+1,ex,x})$ is supported on $\overline{\mathcal{Q}}^y_{k+1,ex}$. We show this by introducing a stratification on $\overline{\mathcal{Q}}^y_{k+1,ex,x}$ analogous to the stratification by ${}^d\overline{\mathcal{Q}}^y_{k+1,ex}$ on $\overline{\mathcal{Q}}^y_{k+1,ex}$ and using an analog of Proposition 4.13(1). $\square$

To finish the proof of Proposition 7.5, we observe that there is another diagram:

$$\begin{tikzcd}
\overline{\mathcal{Q}}_k \arrow[d, "\mathrm{id}"'] & \overline{\mathcal{Q}}_{k+1,ex} \arrow[l, "\pi_{k+1,ex,k}"'] \arrow[d] & {}_x\,\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex}} \arrow[l, "\overleftarrow{h}^{\overline{\mathcal{Q}}_{k+1,ex}}"'] \arrow[r, "\overrightarrow{h}^{\overline{\mathcal{Q}}_{k+1,ex}}"] \arrow[d] & \overline{\mathcal{Q}}_{k+1,ex} \arrow[d, "\mathrm{id}"'] \\
\overline{\mathcal{Q}}_k & \overline{\mathcal{Q}}_{k+1,ex,x} \arrow[l, "\pi_{k+1,ex,k,x}"'] & {}_x\,\mathrm{H}^{\overline{\mathcal{Q}}_{k+1,ex,x}} \arrow[l, "\overleftarrow{h}^{\overline{\mathcal{Q}}_{k+1,ex,x}}"'] \arrow[r, "\overrightarrow{h}^{\overline{\mathcal{Q}}_{k+1,ex,x}}"] & \overline{\mathcal{Q}}_{k+1,ex},
\end{tikzcd}$$

in which the middle square is Cartesian.

Therefore, by Lemma 7.6, the expression in (24) can be rewritten as

$$(\pi_{k+1,ex,k})_!\left(\overleftarrow{h}_!^{\overline{\mathcal{Q}}_{k+1,ex}}\left(\overrightarrow{h}^{\overline{\mathcal{Q}}_{k+1,ex}*}(\mathcal{F})\right)\right),$$

which equals the expression on the left-hand side of (23). $\square$

7.7. The following theorem is one of the main technical results of this paper:

THEOREM 7.8. *The functor* $\mathrm{H}^{\overline{\mathcal{Q}}_n} : \mathrm{D}(\overline{\mathcal{Q}}_n) \to \mathrm{D}(X \times \overline{\mathcal{Q}}_n)$ *is right-exact.*

The rest of this section is devoted to the proof of this theorem. Let us restrict our attention to the connected component of $\overline{\mathcal{Q}}_n$ corresponding to vector bundles $\mathcal{M}$ of a fixed degree. We set $d = \deg(\Lambda^n(\mathcal{M})) - \deg(\Omega^{n-1+n-2+\cdots+1+0})$. According to the conventions of [7], the corresponding connected component of Bun_n is denoted by Bun_n^d, and we keep similar notation for $\overline{\mathcal{Q}}_n^d$.

The data of κ_n in the definition of $\overline{\mathcal{Q}}_n$ define a map $\tau_d : \overline{\mathcal{Q}}_n^d \to X^{(d)}$. Observe that we have a commutative diagram:

$$\begin{tikzcd}
\mathcal{H}^{\overline{\mathcal{Q}}_n} \arrow[r, "\overrightarrow{h}^{\overline{\mathcal{Q}}_n}"] \arrow[d, "s^{\overline{\mathcal{Q}}_n} \times \overleftarrow{h}^{\overline{\mathcal{Q}}_n}"'] & \overline{\mathcal{Q}}_n^{d+1} \arrow[d, "\tau_{d+1}"'] \\
X \times \overline{\mathcal{Q}}_n^d \arrow[r] & X^{(d+1)},
\end{tikzcd}$$

where the bottom horizontal arrow is the composition

$$X \times \overline{\mathcal{Q}}_n^d \xrightarrow{\mathrm{id} \times \tau_d} X \times X^{(d)} \to X^{(d+1)}.$$

From the above diagram we obtain the following:

LEMMA 7.9. *For a given point* $(\mathcal{M}', \kappa_1', \ldots, \kappa_n') \in \overline{\mathcal{Q}}_n^{d+1}$, *its preimage in* $\mathcal{H}^{\overline{\mathcal{Q}}_n}$ *is contained in*

$$\bigcup_{x \in \mathrm{supp}(D_n')} {}_x\mathcal{H}^{\overline{\mathcal{Q}}_n},$$

where $D_n' \in X^{(d+1)}$ *is the image of the above point under* τ_{d+1}.

The proof of Theorem 7.8 will be obtained from the following general result: Let

$$\mathcal{Y} \xleftarrow{f} Z \xrightarrow{f'} \mathcal{Y}'$$

be a diagram of stacks with the morphism f representable. Suppose that Z can be decomposed into locally closed substacks $Z = \cup Z_\alpha$ (the decomposition being locally finite) such that if we denote by m_α (resp., m'_α) the maximum of the dimensions of fibers of $f : Z_\alpha \to \mathcal{Y}$ (resp., $f' : Z_\alpha \to \mathcal{Y}'$), we have:

$$m_\alpha + m'_\alpha \leq m$$

for some integer m.

LEMMA 7.10. *Under the above circumstances, the functor* $\mathrm{D}(\mathcal{Y}') \to \mathrm{D}(\mathcal{Y})$ *given by*

$$\mathcal{F} \mapsto f_! \left(f'^*(\mathcal{F})\right)$$

sends objects of $\mathrm{D}(\mathcal{Y}')^{\leq 0}$ *to* $\mathrm{D}(\mathcal{Y})^{\leq m}$.

The proof of the lemma follows from the definition of the perverse t-structure.

We apply this lemma for $\mathcal{Y} = X \times \overline{\mathcal{Q}}_n$, $\mathcal{Y}' = \overline{\mathcal{Q}}_n$, $Z = \mathcal{H}^{\overline{\mathcal{Q}}_n}$, $f = s^{\overline{\mathcal{Q}}_n} \times \overleftarrow{h}^{\overline{\mathcal{Q}}_n}$, $f' = \overrightarrow{h}^{\overline{\mathcal{Q}}_n}$, and $m = n-1$. Thus, our task is to find a suitable stratification of $\mathcal{H}^{\overline{\mathcal{Q}}_n}$.

7.11. For two strings of nonnegative integers $\overline{d}^1 = d^1_1, \ldots, d^1_n$, $\overline{d}^2 = d^2_1, \ldots, d^2_n$ with $d^2_n = d^1_n + 1$, and $d^1_i \leq d^2_i \leq d^1_i + 1$, let ${}^{\overline{d}^1, \overline{d}^2}\mathcal{H}^{\overline{\mathcal{Q}}_n}$ denote the following locally closed substack of $\mathcal{H}^{\overline{\mathcal{Q}}_n}$:

Recall that $\mathcal{H}^{\overline{\mathcal{Q}}_n}$ classifies the data of

$$(x \in X,\ \mathcal{M} \in \mathrm{Bun}_n,\ \kappa_i : \Omega^{n-1+\cdots+n-i} \to \Lambda^i(\mathcal{M}),\ \mathcal{M}' \in \mathrm{Bun}_n,\ \beta : \mathcal{M} \hookrightarrow \mathcal{M}'),$$

where $\mathcal{M}'/\mathcal{M}$ is a skyscraper at x. We say that such a point as x belongs to ${}^{\overline{d}^1, \overline{d}^2}\mathcal{H}^{\overline{\mathcal{Q}}_n}$ if

(a) Each map $\kappa_i : \Omega^{n-1+\cdots+n-i} \to \Lambda^i(\mathcal{M})$ has a zero of order d^1_i at x.

(b) Each composed map $\kappa'_i : \Omega^{n-1+\cdots+n-i} \to \Lambda^i(\mathcal{M}')$ has a zero of order d^2_i at x.

As in the case of $\overline{\mathcal{Q}}_k = \cup\, {}^{\overline{d}}\overline{\mathcal{Q}}_k$, it is easy to show that the substacks ${}^{\overline{d}^1, \overline{d}^2}\mathcal{H}^{\overline{\mathcal{Q}}_n}$ define a locally finite decomposition of $\mathcal{H}^{\overline{\mathcal{Q}}_n}$ into locally closed substacks.

We now need to verify the estimate on the dimensions of fibers ${}^{\overline{d}^1, \overline{d}^2}\mathcal{H}^{\overline{\mathcal{Q}}_n}$ under the maps $(s^{\overline{\mathcal{Q}}_n} \times \overleftarrow{h}^{\overline{\mathcal{Q}}_n})$ and $\overrightarrow{h}^{\overline{\mathcal{Q}}_n}$.

Let ${}^{\overline{d}^1, \overline{d}^2}_{x}\mathcal{H}^{\overline{\mathcal{Q}}_n}$ denote the intersection ${}^{\overline{d}^1, \overline{d}^2}\mathcal{H}^{\overline{\mathcal{Q}}_n} \cap {}_x\mathcal{H}^{\overline{\mathcal{Q}}_n}$. In view of Lemma 7.9, it suffices to check that for any fixed $x \in X$, the sum of the dimensions of fibers of

$$\overleftarrow{h}^{\overline{\mathcal{Q}}_n} : {}_x^{\overline{d}^1,\overline{d}^2}\,\mathcal{H}^{\overline{\mathcal{Q}}_n} \to \overline{\mathcal{Q}}_n \text{ and } \overrightarrow{h}^{\overline{\mathcal{Q}}_n} : {}_x^{\overline{d}^1,\overline{d}^2}\,\mathcal{H}^{\overline{\mathcal{Q}}_n} \to \overline{\mathcal{Q}}_n$$

does not exceed $n-1$.

For fixed $\overline{d}^1, \overline{d}^2$, let k be the first integer for which $d_k^2 = d_k^1 + 1$. We claim that the dimensions of the fibers of ${}_x^{\overline{d}^1,\overline{d}^2}\mathcal{H}^{\overline{\mathcal{Q}}_n}$ under $\overleftarrow{h}^{\overline{\mathcal{Q}}_n}$ are exactly $k-1$, and those for $\overrightarrow{h}^{\overline{\mathcal{Q}}_n}$ are $n-k$.

Indeed, let first $(\mathcal{M}, \kappa_1, \ldots, \kappa_n)$ be a point of $\overline{\mathcal{Q}}_n^d$ such that each κ_i has a zero of order d_i^1 at x. Then on the formal disk around x we have a filtration

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_n = \mathcal{M}$$

with $\mathcal{M}_i/\mathcal{M}_{i-1} \simeq \Omega^{n-1+\ldots n-i}\left((d_i^1 - d_{i-1}^1)(x)\right)$.

The variety of all possible upper modifications $\mathcal{M}'$ of $\mathcal{M}$ at the given x is the projective space $\mathbb{P}(\mathcal{M}_x)$. Now, the condition that the point that $\mathcal{M}'$ defines in ${}_x\mathcal{H}^{\overline{\mathcal{Q}}_n}$ belongs to ${}_x^{\overline{d}^1,\overline{d}^2}\mathcal{H}^{\overline{\mathcal{Q}}_n}$ with the above condition on $(\overline{d}^1, \overline{d}^2)$ means that the corresponding line $\ell \subset \mathcal{M}_x$ belongs to $(\mathcal{M}_k)_x \subset \mathcal{M}_x$, and does not belong to $(\mathcal{M}_{k-1})_x$.

The dimension of the variety of these lines is exactly $k-1$.

Similarly, if we start with a point $(\mathcal{M}', \kappa_1', \ldots, \kappa_n') \in \overline{\mathcal{Q}}_n^{d+1}$ with each κ_i' having a zero of order d_i^2 at x, we obtain a flag

$$0 = \mathcal{M}_0' \subset \mathcal{M}_1' \subset \cdots \subset \mathcal{M}_n' = \mathcal{M}'$$

defined on the formal disk around x, and

$$\mathcal{M}_i'/\mathcal{M}_{i-1}' \simeq \Omega^{n-1+\ldots n-i}\left((d_i^2 - d_{i-1}^2)(x)\right).$$

The variety of all possible lower modifications $\mathcal{M}$ of $\mathcal{M}'$ constitutes the projective space of hyperplanes in $\mathcal{M}_x'$. The condition that $\mathcal{M}$ defines a point of ${}_x^{\overline{d}^1,\overline{d}^2}\mathcal{H}^{\overline{\mathcal{Q}}_n}$ means that the corresponding hyperplane contains $(\mathcal{M}_{k-1}')_x$, and does not contain $(\mathcal{M}_k')_x$, and the variety of these hyperplanes has dimension $n-k$.

7.12. As usual, everything said in this section carries over to the relative situation; i.e., for a base S we have the Hecke functors $\mathrm{H}^{\overline{\mathcal{Q}}_k} : \mathrm{D}^W(S \times \overline{\mathcal{Q}}_k) \to \mathrm{D}^W(S \times X \times \overline{\mathcal{Q}}_k)$. Moreover, for $k = n$ this functor is right-exact.

Note, however, that for a map $g : S_1 \to S_2$, the functors $\mathrm{H}^{\overline{\mathcal{Q}}_k}$ commute only with the !-push forward and the $*$-pull back.

8. Construction of quotients

In this section we will complete the construction of the quotient categories. Recall the category $\widetilde{\mathrm{D}}(\mathrm{Bun}'_n)$ introduced in Section 6.9. A naive idea would be to define $\widetilde{\mathrm{D}}(\mathrm{Bun}_n)$ as a quotient of $\mathrm{D}(\mathrm{Bun}_n)$ by the kernel of the composition

$$\mathrm{D}(\mathrm{Bun}_n) \xrightarrow{\pi^*} \mathrm{D}(\mathrm{Bun}'_n) \to \widetilde{\mathrm{D}}(\mathrm{Bun}_n),$$

i.e., to "kill" those sheaves $\mathcal{F}$ on Bun_n, for which $\pi^*(\mathcal{F}) \in \mathrm{D}(\mathrm{Bun}'_n)$ is degenerate. However, this definition does not work, because, since the map $\pi : \mathrm{Bun}'_n \to \mathrm{Bun}_n$ is not smooth, the functor π^* is not exact, and the resulting kernel would not in general be compatible with the t-structure. To remedy this, we will "kill" even more objects in $\mathrm{D}(\mathrm{Bun}_n)$.

8.1. Let $\mathcal{U} \subset \mathrm{Bun}_n$ be the open substack corresponding to $\mathcal{M} \in \mathrm{Bun}_n$ for which $\mathrm{Ext}^1(\Omega^{n-1}, \mathcal{M}) = 0$. It is well-known that each $\mathcal{U} \cap \mathrm{Bun}_n^d$ is of finite type. Obviously, the map $\pi : \mathrm{Bun}'_n \to \mathrm{Bun}_n$ is smooth over $\mathcal{U}$. Set $\mathcal{V} = \mathrm{Bun}_n - \mathcal{U}$, $\mathcal{U}^d = \mathcal{U} \cap \mathrm{Bun}_n^d$, and $\mathcal{V}^d = \mathrm{Bun}_n^d - \mathcal{U}^d$.

Recall (cf. [7, §3.2]) that a vector bundle $\mathcal{M}$ is called very unstable if $\mathcal{M}$ can represented as a direct sum $\mathcal{M} = \mathcal{M}^1 \oplus \mathcal{M}^2$, with $\mathcal{M}^i \neq 0$, and $\mathrm{Ext}^1(\mathcal{M}_1, \mathcal{M}_2) = 0$.

It is well-known (cf. [7, Lemma 6.11]) that if $\mathcal{F}$ is a cuspidal object of $\mathrm{D}(\mathrm{Bun}_n)$, then its $*$-stalk at every very unstable point $\mathcal{M} \in \mathrm{Bun}_n$ vanishes. The following is also well-known (cf. [7, Lemma 3.3]):

Lemma 8.2. *There exists an integer d_0, depending only on the genus of X, such that for $d \geq d_0$ every point of $\mathcal{M} \in \mathcal{V}^d$ is very unstable.*

8.3. Let $\mathcal{V}' \subset \mathrm{Bun}'_n$, $\mathcal{U}' \subset \mathrm{Bun}'_n$ be the preimages of $\mathcal{V}$ and $\mathcal{U}$, respectively, in Bun'_n. We denote by $\jmath : \mathcal{U} \to \mathrm{Bun}_n$, $\jmath' : \mathcal{U}' \to \mathrm{Bun}'_n$ the corresponding open embeddings.

The category $\mathrm{D}(\mathcal{V}')$ is a full triangulated subcategory of $\mathrm{D}(\mathrm{Bun}'_n)$. It is compatible with the t-structure on $\mathrm{D}(\mathrm{Bun}'_n)$, cf. Section 2.8.

Recall now the subcategory $\mathrm{D}_{\mathrm{degen}}(\mathrm{Bun}'_n) \subset \mathrm{D}(\mathrm{Bun}'_n)$ of Section 6.9, which by definition consists of objects annihilated by the functor $W : \mathrm{D}(\mathrm{Bun}'_n) \to \mathrm{D}^W(\overline{\mathbb{Q}})$. Since the functor W is exact, $\mathrm{D}_{\mathrm{degen}}(\mathrm{Bun}'_n)$ is also compatible with the t-structure on $\mathrm{D}(\mathrm{Bun}'_n)$; cf. Lemma 2.10.

Let $\mathrm{D}(\mathcal{V}' + \mathrm{degen}) \subset \mathrm{D}(\mathrm{Bun}'_n)$ be the triangulated category generated by $\mathrm{D}(\mathcal{V}')$ and $\mathrm{D}_{\mathrm{degen}}(\mathrm{Bun}'_n)$, i.e., $\mathrm{D}(\mathcal{V}' + \mathrm{degen})$ is the minimal full triangulated subcategory of $\mathrm{D}(\mathrm{Bun}'_n)$, which contains both $\mathrm{D}(\mathcal{V}')$ and $\mathrm{D}_{\mathrm{degen}}(\mathrm{Bun}'_n)$.

We have:

Lemma 8.4. *Let $\mathcal{C}$ be a triangulated subcategory endowed with a t-structure and let $\mathcal{C}', \mathcal{C}'' \subset \mathcal{C}$ be two full triangulated subcategories, both compatible with the t-structure on $\mathcal{C}$. Let $\mathcal{C}' + \mathcal{C}'' \subset \mathcal{C}$ be the triangulated subcategory generated by $\mathcal{C}'$ and $\mathcal{C}''$. Then $\mathcal{C}' + \mathcal{C}''$ is also compatible with the t-structure on $\mathcal{C}$.*

Proof. By definition, $(\mathcal{C}' + \mathcal{C}'') \cap \mathrm{P}(\mathcal{C})$ is the full abelian subcategory of $\mathrm{P}(\mathcal{C})$, consisting of objects, which admit a finite filtration with successive quotients being objects of either $\mathrm{P}(\mathcal{C}')$ or $\mathrm{P}(\mathcal{C}'')$. Clearly, $(\mathcal{C}' + \mathcal{C}'') \cap \mathrm{P}(\mathcal{C})$ is a Serre subcategory of $\mathrm{P}(\mathcal{C})$.

Thus, we have to show that if $\mathcal{S}$ is an object of $\mathcal{C}' + \mathcal{C}''$, then so is $\tau^{\leq 0}(\mathcal{S})$. Suppose that $\mathcal{S}$ can be obtained by an iterated i-fold procedure of taking cones, starting from objects of either $\mathcal{C}'$ or $\mathcal{C}''$. By induction on i, we may assume that $\mathcal{S}$ fits into an exact triangle

$$\mathcal{S}_1 \to \mathcal{S} \to \mathcal{S}_2$$

with $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{C}' + \mathcal{C}''$ and $\tau^{\leq 0}(\mathcal{S}_1), \tau^{\leq 0}(\mathcal{S}_2)$ being also in $\mathcal{C}' + \mathcal{C}''$. Let $\mathcal{S}_3$ be the image of $h^0(\mathcal{S}_2)$ in $h^1(\mathcal{S}_1)$; it belongs to $(\mathcal{C}' + \mathcal{C}'') \cap \mathrm{P}(\mathcal{C})$, by the above. Let $\mathcal{S}_4$ be the cone of $\tau^{\leq 0}(\mathcal{S}_2) \to \mathcal{S}_3$. Then $\tau^{\leq 0}(\mathcal{S})$ fits into the exact triangle

$$\tau^{\leq 0}(\mathcal{S}_1) \to \tau^{\leq 0}(\mathcal{S}) \to \mathcal{S}_4. \qquad \square$$

By applying this lemma to $\mathrm{D}(\mathcal{V}' + \text{degen})$, we obtain from Proposition 2.11 that the quotient triangulated category

$$\tilde{\tilde{\mathrm{D}}}(\mathrm{Bun}'_n) := \mathrm{D}(\mathrm{Bun}'_n) / \mathrm{D}(\mathcal{V}' + \text{degen})$$

carries a t-structure.

For an arbitrary base scheme S, the category $\tilde{\tilde{\mathrm{D}}}(S \times \mathrm{Bun}'_n)$ is defined in a similar way, as a quotient of $\mathrm{D}(S \times \mathrm{Bun}'_n)$ by a subcategory denoted $\mathrm{D}(S, \mathcal{V}' + \text{degen})$. This quotient is stable under the standard functors; i.e., for a map $S_1 \to S_2$ the four functors $\mathrm{D}(S_1 \times \mathrm{Bun}'_n) \rightleftarrows \mathrm{D}(S_2 \times \mathrm{Bun}'_n)$ give rise to well-defined functors on the quotients $\tilde{\tilde{\mathrm{D}}}(S_1 \times \mathrm{Bun}'_n) \rightleftarrows \tilde{\tilde{\mathrm{D}}}(S_2 \times \mathrm{Bun}'_n)$.

Moreover, the Verdier duality functor on $\mathrm{D}(S \times \mathrm{Bun}'_n)$ descends to a well-defined self-functor on $\tilde{\tilde{\mathrm{D}}}(S \times \mathrm{Bun}'_n)$. Finally, the "tensor product along S" functor

$$\mathrm{D}(S) \times \mathrm{D}(S \times \mathrm{Bun}'_n) \to \mathrm{D}(S \times \mathrm{Bun}'_n)$$

is also well-defined on the quotient.

8.5. We define the functor $\overline{\pi_S^*} : \mathrm{D}(S \times \mathrm{Bun}_n) \to \tilde{\tilde{\mathrm{D}}}(S \times \mathrm{Bun}'_n)$ as follows. For $\mathcal{F} \in \mathrm{D}(S \times \mathrm{Bun}_n^d)$ we set $\overline{\pi_S^*}(\mathcal{F})$ to be the image of $(\mathrm{id} \times \pi)^*(\mathcal{F})[\dim(d)]$ under $\mathrm{D}(S \times \mathrm{Bun}'_n) \to \tilde{\tilde{\mathrm{D}}}(S \times \mathrm{Bun}'_n)$, where $\dim(d) = \dim.\mathrm{rel}.(\mathcal{U}', \mathcal{U}^d)$. Note that $\dim(d+1) = \dim(d) + 1$, by the Riemann-Roch theorem.

Proposition 8.6. *The functor $\overline{\pi_S^*}$ is exact. Moreover, it commutes with the Verdier duality, the tensor product along S, and for a map $S_1 \to S_2$ it is compatible with the four functors* $\mathrm{D}(S_1 \times \mathrm{Bun}_n) \rightleftarrows \mathrm{D}(S_2 \times \mathrm{Bun}_n)$ *and* $\tilde{\tilde{\mathrm{D}}}(S_1 \times \mathrm{Bun}'_n) \rightleftarrows \tilde{\tilde{\mathrm{D}}}(S_2 \times \mathrm{Bun}'_n)$.

Proof. The functor $\mathcal{F} \mapsto (\mathrm{id}\times\pi)^*(\mathcal{F})[\dim(d)]$ from $\mathrm{D}(S \times \mathrm{Bun}_n^d)$ to $\mathrm{D}(S \times \mathrm{Bun}'_n)$ is not exact, because the map π is not smooth. However, for a perverse sheaf $\mathcal{F} \in \mathrm{P}(S \times \mathrm{Bun}_n^d)$ all the nonzero cohomology sheaves of $(\mathrm{id}\times\pi)^*(\mathcal{F})[\dim(d)]$ are supported on $\mathcal{V}'$. Hence they vanish after the projection to $\widetilde{\widetilde{\mathrm{D}}}(S \times \mathrm{Bun}'_n)$. This establishes the exactness of $\overline{\pi_S^*}$.

The other assertions of the proposition follow in a similar way. For example, to show that $\overline{\pi_S^*}$ commutes with the Verdier duality functor, it suffices to observe that

$$\jmath'^* \circ (\mathrm{id}\times\pi)^*(\mathcal{F})[\dim(d)]) \simeq \jmath'^* \circ \mathbb{D} \circ (\mathrm{id}\times\pi)^*(\mathbb{D}\mathcal{F})[\dim(d)]),$$

and for any $\mathcal{F}' \in \mathrm{D}(S\times\mathrm{Bun}'_n)$ the map $\jmath'_!\circ\jmath'^*(\mathcal{F}') \to \mathcal{F}'$ becomes an isomorphism in $\widetilde{\widetilde{\mathrm{D}}}(S \times \mathrm{Bun}'_n)$. $\square$

8.7. Since the functor $\overline{\pi_S^*}$ is exact, the subcategory

$$\mathrm{D}_{\mathrm{degen}}(S \times \mathrm{Bun}_n) := \ker(\overline{\pi_S^*}) \subset \mathrm{D}(S \times \mathrm{Bun}_n)$$

is compatible with the t-structure.

We define the category $\widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n)$ as the quotient

$$\mathrm{D}(S \times \mathrm{Bun}_n)/\,\mathrm{D}_{\mathrm{degen}}(S \times \mathrm{Bun}_n).$$

By Proposition 2.11, $\widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n)$ inherits a t-structure from $\mathrm{D}(S \times \mathrm{Bun}_n)$. By Proposition 8.6, the standard six functors that act on $\mathrm{D}(S \times \mathrm{Bun}_n)$ are well-defined on the quotient $\widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n)$. Thus, it remains to show that $\widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n)$ satisfies Properties 1 and 2 of Section 2.12.

8.8. *Verification of Property* 1. We must show that the Hecke functor

$$\mathrm{H}_S : \mathrm{D}(S \times \mathrm{Bun}_n) \to \mathrm{D}(S \times X \times \mathrm{Bun}_n)$$

descends to the quotient $\widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n)$, and the corresponding functor $\widetilde{\mathrm{H}}_S$ is exact. To prove the fact that $\widetilde{\mathrm{H}}_S$ is well-defined, we must show that H_S maps $\ker(\overline{\pi_S^*})$ to $\ker(\overline{\pi_{S\times X}^*})$. By (22), cf. Section 7.1, this reduces to showing that the subcategory $\mathrm{D}(S, \mathcal{V}' + \mathrm{degen}) \subset \mathrm{D}(S \times \mathrm{Bun}'_n)$ is preserved by $\mathrm{H}_S^{\mathrm{Bun}'_n}$: $\mathrm{D}(S \times \mathrm{Bun}'_n) \to \mathrm{D}(S \times X \times \mathrm{Bun}'_n)$. For that, it suffices to show that $\mathrm{H}_S^{\mathrm{Bun}'_n}$ maps $\mathrm{D}_{\mathrm{degen}}(S\times\mathrm{Bun}'_n)$ to $\mathrm{D}_{\mathrm{degen}}(S\times X\times\mathrm{Bun}'_n)$ and $\mathrm{D}(S\times\mathcal{V}')$ to $\mathrm{D}(S\times X\times\mathcal{V}')$.

The former follows immediately from Proposition 7.5. To prove the latter, it suffices to observe that in the diagram

$$\mathrm{Bun}'_n \xleftarrow{\overleftarrow{h}^{\mathrm{Bun}'_n}} \mathcal{H}^{\mathrm{Bun}'_n} \xrightarrow{\overrightarrow{h}^{\mathrm{Bun}'_n}} \mathrm{Bun}'_n,$$

the subset $(\overrightarrow{h}^{\mathrm{Bun}'_n})^{-1}(\mathcal{V}')$ is contained in $(\overleftarrow{h}^{\mathrm{Bun}'_n})^{-1}(\mathcal{V}')$.

Now we will prove the exactness of $\widetilde{\mathrm{H}}_S$ on $\widetilde{\mathrm{D}}(S\times\mathrm{Bun}_n)$. Since the functor $\mathrm{H}_S : \mathrm{D}(S \times \mathrm{Bun}_n) \to \mathrm{D}(S \times X \times \mathrm{Bun}_n)$ commutes with the Verdier duality, it suffices to show that $\widetilde{\mathrm{H}}_S$ is right-exact on $\widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n)$.

We have the following general assertion: Let

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{F} & \mathcal{C}_2 \\ {\scriptstyle G_1}\downarrow & & {\scriptstyle G_2}\downarrow \\ \mathcal{C}'_1 & \xrightarrow{F'} & \mathcal{C}'_2 \end{array}$$

be a commutative diagram of functors between triangulated categories endowed with t-structures. Suppose that the functors F, and F' are exact, and the functor G_2 is right-exact (resp., left-exact, exact).

LEMMA 8.10. *Under the above circumstances*, G_1 *gives rise to a well-defined functor*

$$\widetilde{G_1} : \mathcal{C}_1/\ker(F) \to \mathcal{C}'_1/\ker(F'),$$

and the latter functor is right-exact (*resp*., *left-exact, exact*).

Proof. The fact that the functor $\widetilde{G_1} : \mathcal{C}_1/\ker(F) \to \mathcal{C}'_1/\ker(F')$ is well-defined is immediate. Let us assume that G_2 is right-exact. To prove that $\widetilde{G_1}$ is then also right-exact, we must show that for $\mathcal{S} \in \mathcal{C}_1^{\leq 0}$, the projection to $\mathcal{C}'_1/\ker(F')$ of $\tau^{>0}(G_1(\mathcal{S}))$ vanishes. This amounts to showing that $F'\left(\tau^{>0}(G_1(\mathcal{S}))\right) = 0$. Since F is exact,

$$F'\left(\tau^{>0}(G_1(\mathcal{S}))\right) \simeq \tau^{>0}\left(F' \circ G_1(\mathcal{S})\right),$$

which, in turn, is isomorphic to $\tau^{>0}\left(G_2 \circ F(\mathcal{S})\right)$. Since F is exact, $F(\mathcal{S}) \in \mathcal{C}_2^{\leq 0}$, and since G_2 is right-exact, $G_2 \circ F(\mathcal{S}) \in \mathcal{C}'^{\leq 0}_2$, which is what we had to show. $\square$

We apply this lemma first to $\mathcal{C}_1 = \mathrm{D}(S \times \mathrm{Bun}'_n)$, $\mathcal{C}'_1 = \mathrm{D}(S \times X \times \mathrm{Bun}'_n)$, $\mathcal{C}_2 = \mathrm{D}(S \times \overline{\mathcal{Q}})$, $\mathcal{C}'_2 = \mathrm{D}(S \times X \times \overline{\mathcal{Q}})$, $F, F' = W$, $G_1 = \mathrm{H}_S^{\mathrm{Bun}'_n}$, and $G_2 = \mathrm{H}_S^{\overline{\mathcal{Q}}}$.

From Theorem 7.8 we know that $\mathrm{H}_S^{\overline{\mathcal{Q}}}$ is right exact, which by Lemma 8.10 implies that $\widetilde{\mathrm{H}}_S^{\mathrm{Bun}'_n} : \widetilde{\mathrm{D}}(S \times \mathrm{Bun}'_n) \to \widetilde{\mathrm{D}}(S \times X \times \mathrm{Bun}'_n)$ is right exact. Hence, the corresponding functor $\widetilde{\widetilde{\mathrm{H}}}_S^{\mathrm{Bun}'_n} : \widetilde{\widetilde{\mathrm{D}}}(S \times \mathrm{Bun}'_n) \to \widetilde{\widetilde{\mathrm{D}}}(S \times X \times \mathrm{Bun}'_n)$ is also right-exact.

We apply Lemma 8.10 the second time to $\mathcal{C}_1 = \mathrm{D}(S \times \mathrm{Bun}_n)$, $\mathcal{C}'_1 = \mathrm{D}(S \times X \times \mathrm{Bun}_n)$, $\mathcal{C}_2 = \widetilde{\widetilde{\mathrm{D}}}(S \times \mathrm{Bun}'_n)$, $\mathcal{C}'_2 = \widetilde{\widetilde{\mathrm{D}}}(S \times X \times \mathrm{Bun}'_n)$, $F = \overline{\pi^*_S}$, $F' = \overline{\pi^*_{S \times X}}$, $G_1 = \mathrm{H}_S$, and $G_2 = \widetilde{\widetilde{\mathrm{H}}}_S^{\mathrm{Bun}'_n}$.

We conclude that $\widetilde{\mathrm{H}}_S$ is exact as a functor $\widetilde{\mathrm{D}}(S \times \mathrm{Bun}_n) \to \widetilde{\mathrm{D}}(S \times X \times \mathrm{Bun}_n)$.

8.10. *Verification of Property* 2. We must show that if $\mathcal{F}_1$ is a cuspidal object of $\mathrm{D}(\mathrm{Bun}_n^d)$ with $d \geq d_0$ (cf. Lemma 8.2) and $\mathcal{F}_2 \in \mathrm{D}_{\mathrm{degen}}(\mathrm{Bun}_n)$, then

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Bun}_n)}(\mathcal{F}_1, \mathcal{F}_2) = 0.$$

First, from Lemma 8.2, we obtain that $\mathcal{F}_1 \simeq \jmath_!(\jmath^*(\mathcal{F}_1))$. Therefore,

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Bun}_n)}(\mathcal{F}_1, \mathcal{F}_2) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{U})}(\mathcal{F}_1|_{\mathcal{U}}, \mathcal{F}_2|_{\mathcal{U}}).$$

Consider the natural map

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{U})}(\mathcal{F}_1|_{\mathcal{U}}, \mathcal{F}_2|_{\mathcal{U}}) \to \mathrm{Hom}_{\mathrm{D}^{\mathbb{G}_m}(\mathcal{U}')}\left(\pi^*(\mathcal{F}_1)|_{\mathcal{U}'}, \pi^*(\mathcal{F}_2)|_{\mathcal{U}'}\right). \tag{25}$$

We claim that this map is an injection. Indeed, the quotient stack $\mathcal{U}'/\mathbb{G}_m$ is fibration into projective spaces over $\mathcal{U}$, and the required injectivity follows from the fact the direct image of the constant sheaf from $\mathcal{U}'/\mathbb{G}_m$ to $\mathcal{U}$ contains the constant sheaf on $\mathcal{U}$ as a direct summand. Thus, it will be sufficient to show that the right-hand side of (25) vanishes. Note that since $\pi^*(\mathcal{F}_1)|_{\mathcal{V}'} = 0$, we can rewrite (25) as

$$\mathrm{Hom}_{\mathrm{D}^{\mathbb{G}_m}(\mathrm{Bun}'_n)}\left(\pi^*(\mathcal{F}_1), \pi^*(\mathcal{F}_2)\right).$$

We will show that for any $\mathcal{F}'_2 \in \mathrm{D}(\mathcal{V}' + \mathrm{degen})$,

$$\mathrm{Hom}_{\mathrm{D}^{\mathbb{G}_m}(\mathrm{Bun}'_n)}(\pi^*(\mathcal{F}_1), \mathcal{F}'_2) = 0.$$

By definition of $\mathrm{D}(\mathcal{V}' + \mathrm{degen})$, we must analyze two cases:

Case 1. $\mathcal{F}'_2 \in \mathrm{D}(\mathcal{V}')$. In this case the above Hom vanishes, because $\pi^*(\mathcal{F}_1)|_{\mathcal{V}'} = 0$, as was noticed before.

Case 2. $\mathcal{F}'_2 \in \mathrm{D}_{\mathrm{degen}}(\mathrm{Bun}'_n)$.

We know that $\mathcal{F}'_1 := \pi^*(\mathcal{F}_1)$ is cuspidal, and from Theorem 6.4 (or rather from its $\mathbb{G}_m$-equivariant version) we obtain that

$$\mathrm{Hom}_{\mathrm{D}^{\mathbb{G}_m}(\mathrm{Bun}'_n)}(\mathcal{F}'_1, \mathcal{F}'_2) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathbb{G}_m}(\overline{\mathcal{Q}}_n)}(W(\mathcal{F}'_1), W(\mathcal{F}'_2)) = 0,$$

since it was assumed that $W(\mathcal{F}'_2) = 0$.

Appendix

A.1. We will present now a different way of deducing Conjecture 1.3 from Theorem 2.2. This argument is due to A. Braverman.

By induction, we assume Conjecture 1.3 for all integers $n' < n$. It is enough to show that $\mathrm{Av}^d_E(\mathcal{F})$ vanishes for a perverse sheaf $\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n)$, where d is as in Conjecture 1.3. We know that $\mathrm{Av}^d_E(\mathcal{F})$ is a perverse sheaf (by Theorem 2.2) and that it is cuspidal, by Lemma 2.17.

Recall the functor $\mathrm{Av}^{-d}_{E^*}$, which is left and right adjoint to Av^d_E. Since

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Bun}_n)}(\mathrm{Av}^d_E(\mathcal{F}), \mathrm{Av}^d_E(\mathcal{F})) \simeq \mathrm{Hom}_{\mathrm{D}(\mathrm{Bun}_n)}(\mathrm{Av}^{-d}_{E^*}(\mathrm{Av}^d_E(\mathcal{F})), \mathcal{F}),$$

we obtain that it is enough to show that the functor $\mathrm{Av}^{-d}_{E^*}$ annihilates every cuspidal perverse sheaf.

The stack Bun_n admits a natural automorphism, which sends a bundle to its dual. This automorphism transforms the functor $\mathrm{Av}^{-d}_{E^*}$ to $\mathrm{Av}^d_{E^*}$. Since E^* is irreducible if and only if E is, we deduce that it is enough to show that $\mathrm{Av}^d_E(\mathcal{F}) = 0$, where $\mathcal{F}$ is both perverse and cuspidal.

By Lemma 2.3 and Theorem 2.2, the above vanishing is equivalent to a weaker statement. Namely, it is sufficient to show that for a cuspidal perverse sheaf $\mathcal{F}$, the Euler-Poincaré characteristic of the stalks of $\mathrm{Av}^d_E(\mathcal{F})$ is zero. Finally, by Lemma 2.4 we conclude that it is enough to show that the Euler-Poincaré characteristics of the stalks of $\mathrm{Av}^d_{E_0}(\mathcal{F})$ vanish, where E_0 is *the trivial local system* of rank equal to that of E, and $\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n)$ is cuspidal and perverse.

We will prove a stronger statement. Namely, we will show that the object $\mathrm{Av}^d_{E_0}(\mathcal{F})$ vanishes, where E_0 is a trivial local system of rank m, and $d > (2g-2)\cdot n \cdot m$ for *every cuspidal object* $\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n)$.

A.2. First, we express the functor $\mathrm{Av}^d_{E_0}$ in terms of the corresponding averaging functor for the trivial 1-dimensional local system.

PROPOSITION A.3. *Let a local system E be the direct sum $E = E_1 \oplus E_2$. Then, canonically*:

$$\mathrm{Av}^d_E(\mathcal{F}) \simeq \bigoplus \mathrm{Av}^{d_1}_{E_1} \circ \mathrm{Av}^{d_2}_{E_2}(\mathcal{F}),$$

where the direct sum is taken over all pairs (d_1, d_2) with $d_i \geq 0$, $d_1 + d_2 = 0$.

Proof. For two nonnegative integers d_1, d_2 consider the stack $\mathrm{Mod}^{d_1}_n \underset{\mathrm{Bun}_n}{\times} \mathrm{Mod}^{d_2}_n$, where the fiber product is formed using the maps $\overrightarrow{h} : \mathrm{Mod}^{d_1}_n \to \mathrm{Bun}_n$ and $\overleftarrow{h} : \mathrm{Mod}^{d_2}_n \to \mathrm{Bun}_n$. In other words, this stack classifies successive extensions $\mathcal{M} \subset \mathcal{M}' \subset \mathcal{M}''$, where $\mathcal{M}'/\mathcal{M}$ is of length d_1 and $\mathcal{M}''/\mathcal{M}'$ is of length d_2. There is a natural projection $\mathfrak{r}_{d_1,d_2} : \mathrm{Mod}^{d_1}_n \underset{\mathrm{Bun}_n}{\times} \mathrm{Mod}^{d_2}_n \to \mathrm{Mod}^d_n$, where $d = d_1 + d_2$.

We have:

$$\mathfrak{s}^*(\mathcal{L}^d_E) \simeq \underset{(d_1,d_2)}{\oplus} \mathfrak{r}_{d_1,d_2!}(\mathfrak{s}^*(\mathcal{L}^{d_1}_{E_1}) \boxtimes \mathfrak{s}^*(\mathcal{L}^{d_2}_{E_2})).$$

Indeed, the isomorphism is evident over the open substack $\overset{\circ}{\mathrm{Mod}}{}^d_n$, and it extends to the entire Mod^d_n, since the maps $\mathfrak{r}_{d_1,d_2}$ are small.

By definition, this implies the required property of the functor Av^d_E. □

The same proof shows that the functors $\mathrm{Av}^{d_1}_{E_1}$ and $\mathrm{Av}^{d_2}_{E_2}$ mutually commute.

Let Av^d, with the subscript omitted, denote the averaging functor with respect to the trivial 1-dimensional local system. Note that for the trivial

1-dimensional local system, Laumon's sheaf on Coh_0^d is the constant sheaf. Therefore, the functor Av^d is just

$$\mathcal{F} \mapsto \overleftarrow{h}_! \circ \overrightarrow{h}^!(\mathcal{F})[nd]. \tag{26}$$

From Proposition A.3 we obtain that

$$\mathrm{Av}_E^d(\mathcal{F}) \simeq \bigoplus_{\overline{d}} \mathrm{Av}^{d_1} \circ \cdots \circ \mathrm{Av}^{d_m}(\mathcal{F}),$$

where the direct sum is taken over the set of m-tuples of nonnegative integers $\overline{d} = (d_1, \ldots, d_m)$ with $d_1 + \cdots + d_m = d$. If $d > (2g-2) \cdot m \cdot n$, then for every such $\overline{d}$ at least one d_i satisfies $d_i > (2g-2) \cdot n$. Hence, we are reduced to showing the following:

THEOREM A.4. *If $\mathcal{F} \in \mathrm{D}(\mathrm{Bun}_n)$ is cuspidal, then $\mathrm{Av}^d(\mathcal{F}) = 0$ for $d > (2g-2) \cdot n$.*

This theorem is a geometric analog of the classical statement that the L-function of a cuspidal automorphic representation of GL_n over a function field is a polynomial. The proof will be a geometrization of the Jacquet-Godement proof of the above classical fact, in the spirit of how the functional equation is established for geometric Eisenstein series in [3, §7.3].

A.5. The starting point is the following observation, due to V. Drinfeld and proved in [3, §7.3]: Let $\mathcal{Y}$ be a stack and $\mathcal{E}_1, \mathcal{E}_2$ two vector bundles on it, and $p : \mathcal{E}_1 \to \mathcal{E}_2$ a map between them as coherent sheaves. Let K_p be the kernel of p, considered as a group-scheme over $\mathcal{Y}$ and φ be its projection onto $\mathcal{Y}$. Consider the object $\mathcal{K}_p$ of $\mathrm{D}(\mathcal{Y})$ equal to $\varphi_!(\overline{\mathbb{Q}_\ell}_{K_p})[\dim.\mathrm{rel}.(\mathcal{E}_1, \mathcal{Y})]$, where $\overline{\mathbb{Q}_\ell}_{K_p}$ denotes the constant sheaf on K_p. Let $\check{p} : \check{\mathcal{E}}_1 \to \check{\mathcal{E}}_2$ denote the dual map, and consider also the object $\mathcal{K}_{\check{p}} := \check{\varphi}_!(\overline{\mathbb{Q}_\ell}_{K_{\check{p}}})[\dim.\mathrm{rel}.(\check{\mathcal{E}}_2, \mathcal{Y})]$. We have:

LEMMA A.6. *There is a canonical isomorphism $\mathcal{K}_p \simeq \mathcal{K}_{\check{p}}$.*

We will apply this lemma in the following situation. Let $\mathcal{F}$ be a cuspidal object of $\mathrm{D}(\mathrm{Bun}_n)$ supported on a connected component $\mathrm{Bun}_n^{d'}$. As in Section 8.1, we can assume that $\mathcal{F}$ is the extension by zero from an open substack of finite type $U' \subset \mathrm{Bun}_n^{d'}$. Let U be a scheme of finite type, which maps smoothly to $\mathrm{Bun}_n^{d'-d}$; moreover, we can assume that U' was chosen large enough so that the image of $\overrightarrow{h} : U \underset{\mathrm{Bun}_n}{\times} \mathrm{Mod}_n^d \to \mathrm{Bun}_n^{d'}$ is contained in U'. We shall show that $\mathrm{Av}^d(\mathcal{F})|_U$ vanishes.

We set the base $\mathcal{Y}$ to be $U \times U'$. To define $\mathcal{E}_1$ and $\mathcal{E}_2$ we pick an arbitrary point $y \in X$ and let i be a large enough integer so that $\mathrm{Ext}^1(\mathcal{M}, \mathcal{M}'(i \cdot y)) = 0$, whenever $(\mathcal{M}, \mathcal{M}') \in \mathrm{Bun}_n \times \mathrm{Bun}_n$ is in the image of $U \times U'$. We set $\mathcal{E}_1$ (resp., $\mathcal{E}_2$) to be the vector bundle, whose fiber at a point of $U \times U'$ mapping to a point $(\mathcal{M}, \mathcal{M}')$ as above is $\mathrm{Hom}(\mathcal{M}, \mathcal{M}'(i \cdot y))$ (resp., $\mathrm{Hom}(\mathcal{M}, \mathcal{M}'(i \cdot y)/\mathcal{M}'))$.

The group-scheme K_p has as its fiber over $(\mathcal{M}, \mathcal{M}')$ the vector space $\mathrm{Hom}(\mathcal{M}, \mathcal{M}')$. By Serre's duality, the fiber of $K_{\check{p}}$ is $\mathrm{Hom}(\mathcal{M}', \mathcal{M} \otimes \Omega)$. Let $\overset{\circ}{K}_p$ (resp., $\overset{\circ}{K}_{\check{p}}$) be the open subscheme corresponding to the condition that the map of sheaves $\mathcal{M} \to \mathcal{M}'$ (resp., $\mathcal{M}' \to \mathcal{M} \otimes \Omega$) is injective. Note that if $d = \deg(\mathcal{M}') - \deg(\mathcal{M}) > (2g-2) \cdot n$, then $\overset{\circ}{K}_{\check{p}}$ is empty. Let $\overset{\circ}{\mathcal{K}}_p$ be $\varphi_!(\overline{\mathbb{Q}_\ell}_{\overset{\circ}{K}_p})[\dim.\mathrm{rel.}(\mathcal{E}_1, \mathcal{Y})]$ (resp., $\overset{\circ}{\mathcal{K}}_{\check{p}} = \varphi_!(\overline{\mathbb{Q}_\ell}_{\overset{\circ}{K}_{\check{p}}})[\dim.\mathrm{rel.}(\check{\mathcal{E}}_2, \mathcal{Y})]$). Finally, let $\mathcal{K}_p^c$ (resp., $\mathcal{K}_{\check{p}}^c$) denote the cone of the natural arrow $\overset{\circ}{\mathcal{K}}_p \to \mathcal{K}_p$ (resp., $\overset{\circ}{\mathcal{K}}_{\check{p}} \to \mathcal{K}_{\check{p}}$).

Let us denote by q, q' the projections from $U \times U'$ to U and U', respectively. Consider the two functors $\mathrm{D}(U') \to \mathrm{D}^-(U)$ defined by

$$\mathcal{F} \mapsto q_!(q'^*(\mathcal{F}) \otimes \mathcal{K}_p) \text{ and } q_!(q'^*(\mathcal{F}) \otimes \overset{\circ}{\mathcal{K}}_p).$$

Here $\mathrm{D}^-(U)$ denotes the derived category of sheaves, bounded from above, on U, which appears due to the fact that the map q is not representable. Note, however, that because of (26),

$$q_!(q'^*(\mathcal{F}) \otimes \overset{\circ}{\mathcal{K}}_p) \simeq \mathrm{Av}^d(\mathcal{F})|_U.$$

Taking into account Lemma A.6, we have reduced Theorem A.4 to the fact that the functors

$$\mathcal{F} \mapsto q_!(q'^*(\mathcal{F}) \otimes \mathcal{K}_p^c) \text{ and } \mathcal{F} \mapsto q_!(q'^*(\mathcal{F}) \otimes \mathcal{K}_{\check{p}}^c)$$

annihilate cuspidal objects. We will prove it in the case of $\mathcal{K}_p^c$, as the other assertion is completely analogous.

A.7. Let K_p^c denote the complement to $\overset{\circ}{K}_p$ in K_p. By definition, it can be decomposed into the union of n locally closed substacks, where the k-th substack, classifies the data of a pair of points $(u, u') \in U \times U'$ and a map between the corresponding sheaves $\mathcal{M} \to \mathcal{M}'$, which is of generic rank k, with k running from 0 to $n-1$. Each such substack admits a further decomposition into locally closed substacks according to the length of the torsion of the quotient $\mathcal{M}'/\mathcal{M}$.

It is enough to show that the correspondence $\mathrm{D}(U') \to \mathrm{D}^-(U)$ defined by the constant sheaf on each of these locally closed substacks annihilates $\mathcal{F} \in \mathrm{D}(U')$, provided that $\mathcal{F}$ is cuspidal.

Let us consider separately the cases when $k = 0$ and when $k > 0$. In the former case, the corresponding (closed) substack of K_p^c is the zero-section, i.e., the product $U \times U'$. Thus, we must show that $H_c(U', \mathcal{F}) = 0$, when $\mathcal{F}$ is cuspidal. In other words, we must show that $\mathrm{Hom}_{\mathrm{D(U)}}(\mathcal{F}, \overline{\mathbb{Q}_\ell}_U) = 0$. However, this follows from Section 8.10: with no restriction of generality we may assume that $d' \geq d_0$, and the object $\overline{\mathbb{Q}_\ell}_{\mathrm{Bun}_n^{d'}}$ clearly belongs to $\mathrm{D}_{\mathrm{degen}}(\mathrm{Bun}_n)$.

Now let us suppose that $k > 0$, and consider the stack

$$Z := \mathrm{Fl}^n_{n-k,k} \underset{\mathrm{Bun}_k}{\times} \mathrm{Mod}^a_k \underset{\mathrm{Bun}_k}{\times} \mathrm{Fl}^n_{k,n-k},$$

where we have used the map $(\overleftarrow{h} \times \overrightarrow{h}) : \mathrm{Mod}^a_k \to \mathrm{Bun}_k \times \mathrm{Bun}_k$ to define the fiber product. By definition, a point of Z contains the data of

$$0 \to \mathcal{M}_{n-k} \to \mathcal{M} \to \mathcal{M}_k \to 0;\ \mathcal{M}_k \hookrightarrow \mathcal{M}'_k;\ 0 \to \mathcal{M}'_k \to \mathcal{M}' \to \mathcal{M}'_{n-k},$$

where $\mathcal{M}_{n-k}, \mathcal{M}'_{n-k}$ are vector bundles of rank $n-k$, $\mathcal{M}_k, \mathcal{M}'_k$ are vector bundles of rank k, and the quotient $\mathcal{M}'_k/\mathcal{M}_k$ is of length a.

The stack Z maps to $\mathrm{Bun}_n \times \mathrm{Bun}_n$ when we remember the data of $(\mathcal{M}, \mathcal{M}')$ and note that the fiber product ${}_UZ_{U'} := U \underset{\mathrm{Bun}_n}{\times} Z \underset{\mathrm{Bun}_n}{\times} U'$ is the required locally closed substack of K^c_p. By taking the constant sheaf on ${}_UZ_{U'}$ we obtain a functor $\mathrm{D}(U') \to \mathrm{D}^-(U)$, and we have to show that this functor annihilates every cuspidal object $\mathcal{F} \in \mathrm{D}(U')$. However, this follows by base change from the following diagram:

$$\begin{tikzcd}
{}_UZ_{U'} \arrow[r] \arrow[d] & \mathrm{Fl}^n_{k,n-k} \underset{\mathrm{Bun}_n}{\times} U' \arrow[r] \arrow[d] & U' \\
U \underset{\mathrm{Bun}_n}{\times} \mathrm{Fl}^n_{n-k,k} \underset{\mathrm{Bun}_k}{\times} \mathrm{Mod}^a_k \arrow[r] \arrow[d] & \mathrm{Bun}_k & \\
U \underset{\mathrm{Bun}_n}{\times} \mathrm{Fl}^n_{n-k,k} \arrow[d] & & \\
U. & &
\end{tikzcd}$$

Indeed, the functor $\mathrm{D}(U') \to \mathrm{D}^-(\mathrm{Bun}_k)$ corresponding to the upper-right corner of the above diagram annihilates cuspidal objects, by the definition of the constant term functor $\mathrm{CT}^n_{k,n-k}$, because the vertical arrow $\mathrm{Fl}^n_{k,n-k} \underset{\mathrm{Bun}_n}{\times} U' \to \mathrm{Bun}_k$, appearing in the diagram, factors as

$$\mathrm{Fl}^n_{k,n-k} \underset{\mathrm{Bun}_n}{\times} U' \xrightarrow{\mathfrak{q}_{k,n-k}} \mathrm{Bun}_k \times \mathrm{Bun}_{n-k} \to \mathrm{Bun}_k\,.$$

University of Chicago, Chicago, IL
E-mail address: gaitsgde@math.uchicago.edu

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