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The space of embedded minimal surfaces of fixed genus in a 3-manifold IV; Locally simply connected

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0. Introduction

This paper is the fourth in a series where we describe the space of all embedded minimal surfaces of fixed genus in a fixed (but arbitrary) closed 3-manifold. The key is to understand the structure of an embedded minimal disk in a ball in $\mathbf{R}^3$. This was undertaken in [CM3], [CM4] and the global version of it will be completed here; see the discussion around Figure 12 for the local case and [CM15] for some more details.

Our main results are Theorem 0.1 (the lamination theorem) and Theorem 0.2 (the one-sided curvature estimate). The lamination theorem is stated in the global case where the lamination is, in fact, a foliation. The first four papers of this series show that every embedded minimal disk is either a graph of a function or is a double spiral staircase where each staircase is a multi-valued graph. This is done by showing that if the curvature is large at some point (and hence the surface is not a graph), then it is a double spiral staircase like the helicoid. To prove that such a disk is a double spiral staircase, we showed in the first three papers of the series that it is built out of N-valued graphs where N is a fixed number. In this paper we will deal with how the multi-valued graphs fit together and, in particular, prove regularity of the set of points of large curvature – the axis of the double spiral staircase.

The first theorem is the global version of the statement that every embedded minimal disk is a double spiral staircase.

Theorem 0.1 (see Figure 1). *Let $\Sigma_i \subset B_{R_i} = B_{R_i}(0) \subset \mathbf{R}^3$ be a sequence of embedded minimal disks with $\partial \Sigma_i \subset \partial B_{R_i}$ where $R_i \to \infty$. If*

$$\sup_{B_1 \cap \Sigma_i} |A|^2 \to \infty \, ,$$

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then there exists a subsequence, Σ_j*, and a Lipschitz curve* $\mathcal{S} : \mathbf{R} \to \mathbf{R}^3$ *such that after a rotation of* $\mathbf{R}^3$:

(1) $x_3(\mathcal{S}(t)) = t$. (*That is,* $\mathcal{S}$ *is a graph over the* x_3*-axis.*)

(2) *Each* Σ_j *consists of exactly two multi-valued graphs away from* $\mathcal{S}$ (*which spiral together*).

(3) *For each* $1 > \alpha > 0$, $\Sigma_j \setminus \mathcal{S}$ *converges in the* C^α*-topology to the foliation,* $\mathcal{F} = \{x_3 = t\}_t$*, of* $\mathbf{R}^3$.

(4) $\sup_{B_r(\mathcal{S}(t))\cap\Sigma_j} |A|^2 \to \infty$ *as* $j \to \infty$ *for all* $r > 0$, $t \in \mathbf{R}$. (*The curvatures blow up along* $\mathcal{S}$.)

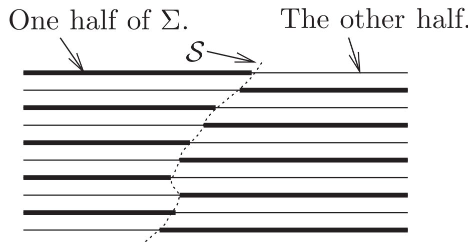

Figure 1: Theorem 0.1 — the singular set, $\mathcal{S}$, and the two multi-valued graphs.

In (2) and (3) that $\Sigma_j \setminus \mathcal{S}$ are multi-valued graphs and converge to $\mathcal{F}$ means that, for each compact subset $K \subset \mathbf{R}^3 \setminus \mathcal{S}$ and j sufficiently large, $K \cap \Sigma_j$ consists of multi-valued graphs over (part of) $\{x_3 = 0\}$ and $K \cap \Sigma_j \to K \cap \mathcal{F}$.

This theorem (like many of the results below) is modeled by the helicoid and its rescalings. The helicoid is the minimal surface Σ^2 in $\mathbf{R}^3$ parametrized by

$$(s \cos t, s \sin t, -t) \text{ where } s,\ t \in \mathbf{R}\,.$$

We have chosen to normalize so that the helicoid spirals down as we move counter-clockwise. Take a sequence $\Sigma_i = a_i\,\Sigma$ of rescaled helicoids where $a_i \to 0$. Since the helicoid has cubic volume growth, the density of the rescaled helicoids is unbounded as $i \to \infty$. The curvature is blowing up along the vertical axis. The sequence converges (away from the vertical axis) to a foliation by flat parallel planes. The singular set $\mathcal{S}$ (the vertical axis) then consists of removable singularities.

The second main theorem asserts that every embedded minimal disk lying above a plane, and coming close to the plane near the origin, is a graph. Precisely this is the following:

Theorem 0.2 (see Figure 2). *There exists* $\varepsilon > 0$, *so that if*

$$\Sigma \subset B_{2r_0} \cap \{x_3 > 0\} \subset \mathbf{R}^3$$

is an embedded minimal disk with $\partial\Sigma \subset \partial B_{2\,r_0}$, *then for all components* Σ' *of* $B_{r_0} \cap \Sigma$ *which intersect* $B_{\varepsilon r_0}$,

$$\sup_{\Sigma'} |A_\Sigma|^2 \leq r_0^{-2}\,. \tag{0.3}$$

By the minimal surface equation and the fact that Σ' has points close to a plane, it is not hard to see that, for $\varepsilon > 0$ sufficiently small, (0.3) is equivalent to the statement that Σ' is a graph over the plane $\{x_3 = 0\}$.

An embedded minimal surface Σ which is as in Theorem 0.2 is said to satisfy the (ε, r_0)-*effective one-sided Reifenberg condition*; cf. Appendix A. We will often refer to Theorem 0.2 as *the one-sided curvature estimate* since it gives a curvature estimate for disks on one side of a plane.

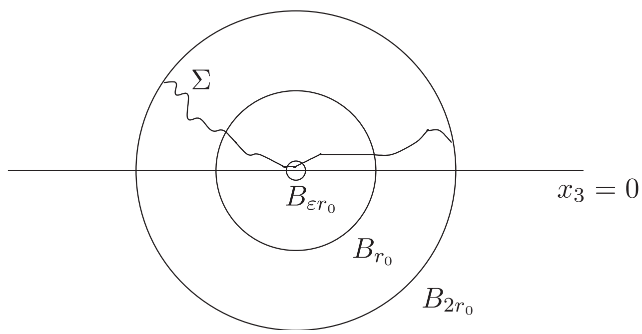

Figure 2: Theorem 0.2 — the one-sided curvature estimate for an embedded minimal disk Σ in a half-space with $\partial\Sigma \subset \partial B_{2r_0}$: The components of $B_{r_0} \cap \Sigma$ intersecting $B_{\varepsilon r_0}$ are graphs.

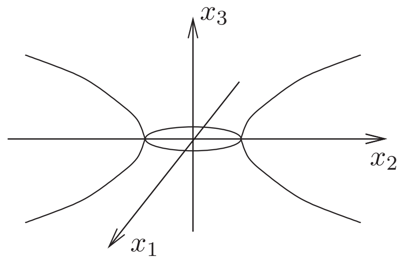

Figure 3: The catenoid given by revolving $x_1 = \cosh x_3$ around the x_3-axis.

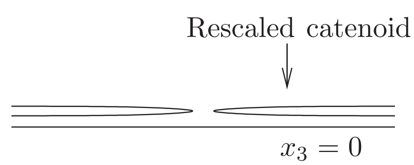

Figure 4: Rescaling the catenoid shows that simply connectedness is needed in the one-sided curvature estimate.

Note that the assumption in Theorem 0.2 that Σ is simply connected is crucial as can be seen from the example of a rescaled catenoid. The catenoid (see Figure 3) is the minimal surface in $\mathbf{R}^3$ parametrized by

$$(\cosh s\, \cos t, \cosh s\, \sin t, s) \text{ where } s,\, t \in \mathbf{R}\,.$$

Under rescalings the catenoid converges (with multiplicity two) to the flat plane; see Figure 4. Likewise, by considering the universal cover of the catenoid, one sees that being embedded, and not just immersed, is needed in Theorem 0.2. The following corollary is an almost immediate consequence of Theorem 0.2:

Corollary 0.4 (see Figure 5). *There exist $c > 1$, $\varepsilon > 0$ so that the following holds*:

Let Σ_1 and $\Sigma_2 \subset B_{cr_0} \subset \mathbf{R}^3$ be disjoint embedded minimal surfaces with $\partial \Sigma_i \subset \partial B_{cr_0}$ and $B_{\varepsilon r_0} \cap \Sigma_i \neq \emptyset$. If Σ_1 is a disk, then for all components Σ_1' of $B_{r_0} \cap \Sigma_1$ which intersect $B_{\varepsilon r_0}$

$$\sup_{\Sigma_1'} |A|^2 \leq r_0^{-2}\,. \tag{0.5}$$

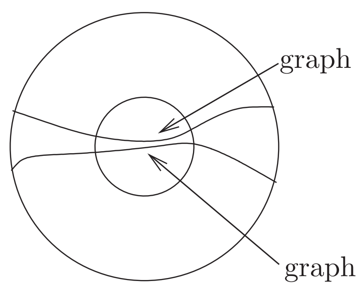

Figure 5: Corollary 0.4: Two sufficiently close components of an embedded minimal disk must each be a graph.

To explain how these theorems are proved by the results of [CM3]–[CM5] and [CM7], we will need some notation for multi-valued graphs. Let D_r be the disk in the plane centered at the origin and of radius r and let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbf{C} \setminus \{0\}$ with global polar coordinates (ρ, θ) so $\rho > 0$ and $\theta \in \mathbf{R}$. Given $0 \leq r \leq s$ and $\theta_1 \leq \theta_2$, define the "rectangle" $S_{r,s}^{\theta_1,\theta_2} \subset \mathcal{P}$ by

$$S_{r,s}^{\theta_1,\theta_2} = \{(\rho, \theta) \,|\, r \leq \rho \leq s\,,\, \theta_1 \leq \theta \leq \theta_2\}\,.$$

An N-valued graph of a function u on the annulus $D_s \setminus D_r$ is a single-valued graph over

$$S_{r,s}^{-N\pi,N\pi} = \{(\rho, \theta) \,|\, r \leq \rho \leq s\,,\, |\theta| \leq N\,\pi\}\,.$$

($\Sigma_{r,s}^{\theta_1,\theta_2}$ will denote the subgraph of Σ over the rectangle $S_{r,s}^{\theta_1,\theta_2}$). The multi-valued graphs to be considered will never close up; in fact they will all be embedded. Note that embedded means that the separation never vanishes. Here the separation (see Figure 6) is the function given by

$$w(\rho,\theta) = u(\rho,\theta+2\pi) - u(\rho,\theta)\,.$$

If Σ is the helicoid (see Figure 7), then

$$\Sigma \setminus x_3 - \text{axis} = \Sigma_1 \cup \Sigma_2\,,$$

where Σ_1, Σ_2 are ∞-valued graphs and Σ_1 is the graph of the function $u_1(\rho,\theta) = -\theta$ and Σ_2 is the graph of the function $u_2(\rho,\theta) = -\theta+\pi$. In either case the separation $w = -2\,\pi$. A multi-valued minimal graph is a multi-valued graph of a function u satisfying the minimal surface equation.

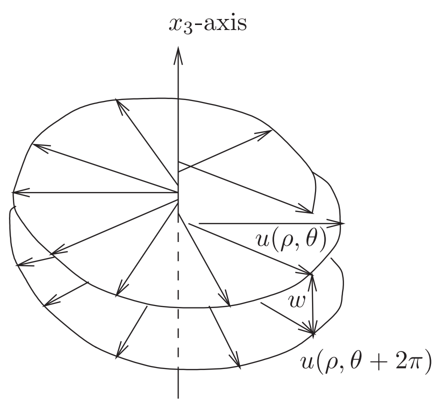

Figure 6: The separation of a multi-valued graph. (Here the multi-valued graph is shown with negative separation.)

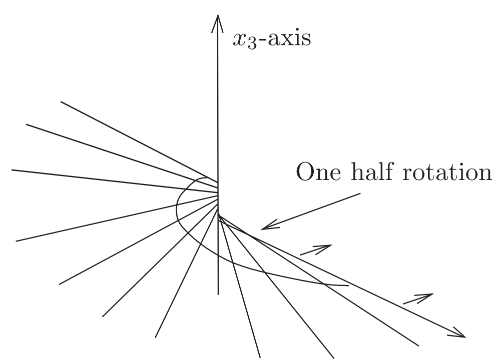

Figure 7: The helicoid is obtained by gluing together two ∞-valued graphs along a line. The two multi-valued graphs are given in polar coordinates by $u_1(\rho,\theta) = -\theta$ and $u_2(\rho,\theta) = -\theta+\pi$. In either case $w(\rho,\theta) = -2\,\pi$.

In this paper, as in [CM7], we have normalized so that the embedded multi-valued graphs have negative separation. We can achieve this after possibly reflecting in a plane.

The proof of Theorem 0.1 has the following three main steps; see Figure 8:

A. Fix an integer N (the "large curvature" in what follows will depend on N). If an embedded minimal disk Σ is not a graph (or equivalently if the curvature is large at some point), then it contains an N-valued minimal graph which initially is shown to exist on the scale of $1/\max|A|$. That is, the N-valued graph is initially shown to be defined on an annulus with both inner and outer radii inversely proportional to $\max|A|$.

B. Such a potentially small N-valued graph sitting inside Σ can then be seen to extend as an N-valued graph inside Σ almost all the way to the boundary. That is, the small N-valued graph can be extended to an N-valued graph defined on an annulus where the outer radius of the annulus is proportional to R. Here R is the radius of the ball in $\mathbf{R}^3$ containing the boundary of Σ.

C. The N-valued graph not only extends horizontally (i.e., tangent to the initial sheets) but also vertically (i.e., transversally to the sheets). That is, once there are N sheets there are many more and, in fact, the disk Σ consists of two multi-valued graphs glued together along an axis.

A was proved in [CM4], B was proved in [CM3], and C will be proved in this paper together with the regularity of the set of points with large curvature — the axis of the double spiral staircase.

Using [CM3], we showed in [CM4] that an embedded minimal disk in a ball in $\mathbf{R}^3$ with large curvature at a point contains an almost flat multi-valued graph nearby. Namely, we showed:

THEOREM 0.6 (Theorem 0.2 of [CM4]. See A and B in Figure 8). *Given $N \in \mathbf{Z}_+$ and $\varepsilon > 0$, there exist C_1 and $C_2 > 0$ so that the following holds*:

Let $0 \in \Sigma^2 \subset B_R \subset \mathbf{R}^3$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_R$. If for some $R > r_0 > 0$,

$$\max_{B_{r_0}\cap\Sigma} |A|^2 \geq 4\, C_1^2\, r_0^{-2}\,,$$

then there exists (after a rotation of $\mathbf{R}^3$) an N-valued graph Σ_g over $D_{R/C_2} \setminus D_{2r_0}$ with gradient $\leq \varepsilon$ and contained in $\Sigma \cap \{x_3^2 \leq \varepsilon^2\,(x_1^2 + x_2^2)\}$.

An important consequence of Theorem 0.6 is (see Theorem 5.8 of [CM4]):

Let $\Sigma_i \subset B_{2R}$ be a sequence of embedded minimal disks with $\partial\Sigma_i \subset \partial B_{2R}$. Clearly (after possibly going to a subsequence) either (1) or (2) occurs:

(1) $\sup_{B_R\cap\Sigma_i} |A|^2 \leq C < \infty$ for some constant C.

(2) $\sup_{B_R\cap\Sigma_i} |A|^2 \to \infty$.

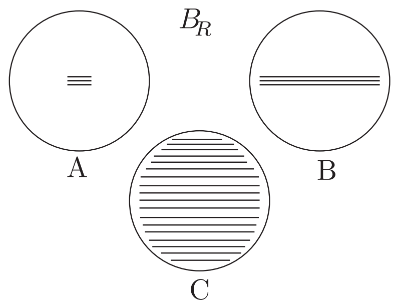

Figure 8: Proving Theorem 0.1. A. Finding a small N-valued graph in Σ. B. Extending it in Σ to a large N-valued graph. C. Extending the number of sheets. (A follows from [CM4] and B follows from [CM3].)

In (1) (by a standard argument), $\mathcal{B}_s(y_i)$ is a graph for all $y_i \in B_R \cap \Sigma_i$, where s depends only on C. In (2) (by Theorem 5.8 of [CM4]) if $y_i \in B_R \cap \Sigma_i$ with

$$|A|^2(y_i) \to \infty\,,$$

then we can (after passing to a subsequence) assume that $y_i \to y$, each Σ_i contains a 2-valued graph $\Sigma_{d,i}$ over $D_{R/C_2}(y) \setminus D_{\varepsilon_i}(y)$ with $\varepsilon_i \to 0$, and $\Sigma_{d,i}$ converges to a graph $y \in \Sigma_\infty$ over $D_{R/C_2}(y)$. In either case in the limit there is a smooth minimal graph through each point in the support.

The multi-valued graphs given by Theorem 0.6 should be thought of as the basic building blocks for an embedded minimal disk. In fact, using a standard blow up argument, we showed in [CM4] (Corollary 4.14 combined with Proposition 4.15 there) that Theorem 0.6 was a consequence of the next theorem. This next theorem will be used to construct the actual building blocks starting off on the smallest possible scale:

THEOREM 0.7 ([CM4]). *Given $N \in \mathbf{Z}_+$ and $\varepsilon > 0$, there exist C_1, C_2, $C_3 > 0$ so that the following holds*:

Let $0 \in \Sigma^2 \subset B_R \subset \mathbf{R}^3$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_R$. If for some r_0 with $R > r_0 > 0$,

$$\sup_{B_{r_0}\cap\Sigma} |A|^2 \leq 4\,|A|^2(0) = 4\,C_1^2\,r_0^{-2}\,,$$

then there exists (*after a rotation*) *an N-valued graph*

$$\Sigma_g \subset \Sigma \cap \{x_3^2 \leq \varepsilon^2\,(x_1^2 + x_2^2)\}$$

over $D_{R/C_2} \setminus D_{r_0}$ with gradient $\leq \varepsilon$ and separation $\geq C_3\,r_0$ over ∂D_{r_0}.

It will be important for the application of Theorem 0.7 here that the initial separation of the sheets is proportional to the initial scale that the graph starts off on.

Theorems 0.1 and 0.2 deal with how the building blocks fit together. A consequence of Theorem 0.1 is that if an embedded minimal disk starts to spiral very tightly, then it can unwind only very slowly. That is, in a whole extrinsic tubular neighborhood it continues to spiral tightly and fills up almost the entire space.

Let us also briefly outline the proof of the one-sided curvature estimate, i.e., Theorem 0.2. Suppose that Σ is an embedded minimal disk in the half-space $\{x_3 > 0\}$. We prove the curvature estimate by contradiction; so suppose that Σ has low points with large curvature. Starting at such a point, we decompose (see Corollary III.1.3) Σ into disjoint multi-valued graphs using the existence of nearby points with large curvature (see Proposition I.0.11), a blow up argument, and [CM3], [CM4]. The key point is then to show (see Proposition III.2.2 and Figure 9) that we can in fact find such a decomposition where the "next" multi-valued graph starts off a definite amount below where the previous multi-valued graph started off. In fact, what we show is that this definite amount is a fixed fraction of the distance between where the two graphs started off. Iterating this eventually forces Σ to have points where $x_3 < 0$, which is the desired contradiction.

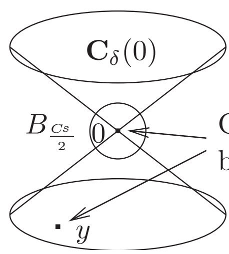

Figure 9: Two consecutive blow up points satisfying (III.2.1).

To prove this key proposition (Proposition III.2.2) about where the next multi-valued graph starts off, we use two decompositions and two kinds of blow up points. The first decomposition which is Corollary III.1.3 uses the more standard blow up points given by (III.1.1). These are pairs (y, s) of a point $y \in \Sigma$ and a radius $s > 0$ such that

$$\sup_{\mathcal{B}_{8s}(y)} |A|^2 \leq 4|A|^2(y) = 4C_1^2 s^{-2} .$$

The point about such a pair (y, s) is that by [CM3], [CM4] (and an argument in Part II which allows us replace extrinsic balls by intrinsic ones), Σ contains a multi-valued graph near y starting off on the scale s. (This is assuming that C_1 is a sufficiently large constant given by [CM3], [CM4].) The second kind of blow up points are the ones satisfying (III.2.1). Basically (III.2.1) is (III.1.1) (except for a technical issue) where 8 is replaced by some really large constant C. The point will then be that we can find blow up points satisfying

(III.2.1) so that the distance between them is proportional to the sum of the scales. Moreover, between consecutive blow up points satisfying (III.2.1), we can find a bunch of blow up points satisfying (III.1.1); see Figure 10. The advantage is now that if we look between blow up points satisfying (III.2.1), then the height of the multi-valued graph given by such a pair grows like a small power of the distance whereas the separation between the sheets in a multi-valued graph given by (III.1.1) decays like a small power of the distance; see Figure 11. Now since the number of blow up points satisfying (III.1.1) (between two consecutive blow up points satisfying (III.2.1)) grows almost linearly, even though the height of the graph coming from the blow up point satisfying (III.2.1) could move up (and thus work against us), the sum of the separations of the graphs coming from the points satisfying (III.1.1) dominates the other term. Thus the next blow up point satisfying (III.2.1) (which lies below all the other graphs) is forced to be a definite amount lower than the previous blow up point satisfying (III.2.1).

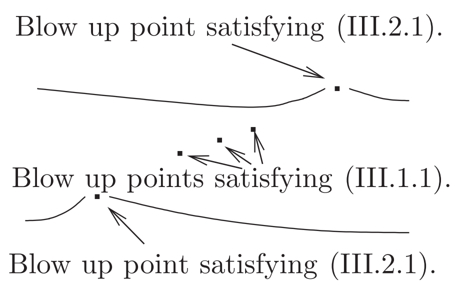

Figure 10: Between two consecutive blow up points satisfying (III.2.1) there are a bunch of blow up points satisfying (III.1.1).

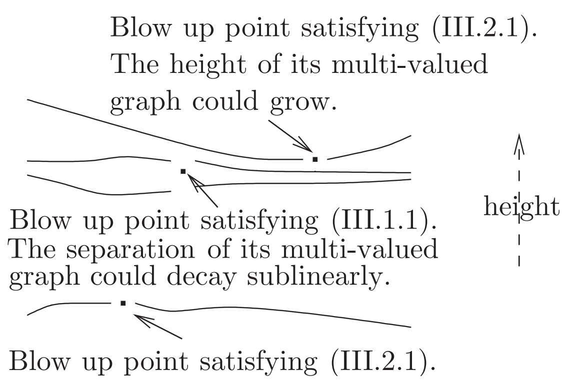

Figure 11: Measuring height. Blow up points and corresponding multi-valued graphs.

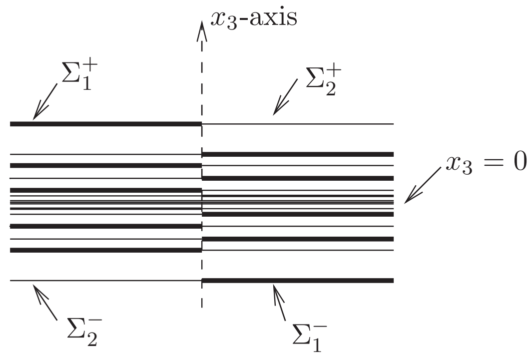

Figure 12: A schematic picture of the limit in [CM15] which is not smooth and not proper (the dotted x_3-axis is part of the limit). The limit contains four multi-valued graphs joined at the x_3-axis; Σ_1^+, Σ_2^+ above the plane $x_3 = 0$ and Σ_1^-, Σ_2^- below the plane. Each of the four spirals into the plane.

Finally, we discuss the differences between the so-called local and global cases. The local case is where we have a sequence of embedded minimal disks in a ball of fixed radius in $\mathbf{R}^3$ - the global case (Theorem 0.1) is where the disks are in a sequence of expanding balls with radii tending to infinity. The main difference between these cases is that in the local case we can get limits with singularities. In the global case this does not happen because any limit is a foliation by flat parallel planes (if the curvatures of the sequence are blowing up). However, in both the local and global cases, we always get a double spiral staircase.

Recall that a surface $\Sigma \subset \mathbf{R}^3$ is said to be properly embedded if it is embedded and the intersection of Σ with any compact subset of $\mathbf{R}^3$ is compact. We say that a lamination or foliation is proper if each leaf is proper.

To illustrate the key issue for the failure of properness, suppose that $\Sigma_i \subset B_{R_i}$ is a sequence of minimal disks with $\partial \Sigma_i \subset \partial B_{R_i}$ and $|A|^2(0) \to \infty$ as $i \to \infty$. In the global case, where $R_i \to \infty$, Theorem 0.1 gives a subsequence of the Σ_i converging off of a Lipschitz curve to a foliation by parallel planes. In particular, the limit is a (smooth) foliation which is proper. However, we showed in [CM15] that smoothness and properness of the limit can fail in the local case; see Figure 12.

In either the local or global case, we get a sequence of 2-valued graphs which converges to a minimal graph Σ_0 through 0 (this graph is a plane in the global case). Furthermore, by the one-sided curvature estimate (see Corollary I.1.9), the intersection of Σ_i with a low cone about Σ_0 consists of multi-valued graphs for i large. There are now two possibilities:

- The multi-valued graphs in this low cone close up in the limit.
- The limits of these multi-valued graphs spiral infinitely into Σ_0.

In the first case, where properness holds, the sequence converges to a foliation in a neighborhood of 0. In the second case, where properness fails, the sequence converges to a lamination away from 0 but cannot be extended smoothly to any neighborhood of 0. The proof of properness in the global case is given in Lemma I.1.10 below.

Let x_1, x_2, x_3 be the standard coordinates on $\mathbf{R}^3$ and $\Pi : \mathbf{R}^3 \to \mathbf{R}^2$ orthogonal projection to $\{x_3 = 0\}$. For $y \in S \subset \Sigma \subset \mathbf{R}^3$ and $s > 0$, the extrinsic and intrinsic balls are $B_s(y)$ and $\mathcal{B}_s(y)$, respectively, and $\Sigma_{y,s}$ is the component of $B_s(y) \cap \Sigma$ containing y. D_s denotes the disk $B_s(0) \cap \{x_3 = 0\}$. Also, K_Σ is the sectional curvature of a smooth compact surface Σ and when Σ is immersed A_Σ will be its second fundamental form. When Σ is oriented, $\mathbf{n}_\Sigma$ is the unit normal.

The reader may find it useful also to look at the survey [CM13] and the expository article [CM14] for an outline of our results, and their proofs, about embedded minimal disks and how these results fit together. The article [CM14] is the best to start with.

This paper completes the results announced in [CM11] and [CM12].

Since the announcements of our results, a number of interesting theorems have been proved using our Theorems 0.1 and 0.2. For instance, in [CM10], using Theorem 0.2, we gave an alternative proof of the so-called generalized Nitsche conjecture originally proved by P. Collin by very different arguments; cf. also [Ro]. In [CM8], using Theorem 0.2 and [CM5], we proved that any embedded minimal annulus in a ball (with boundary in the boundary of the ball and) with a small neck can be decomposed by a simple closed geodesic into two graphical sub–annuli. Moreover, we gave a sharp bound for the length of this closed geodesic in terms of the separation (or height) between the graphical sub–annuli. This serves to illustrate our "pair of pants" decomposition from [CM6] in the special case where the embedded minimal planar domain is an annulus.

Using Theorems 0.1, 0.2, W. Meeks and H. Rosenberg proved that the plane and helicoid are the only complete properly embedded simply-connected minimal surfaces in $\mathbf{R}^3$, [MeRo]. Recall that if we take a sequence of rescalings of the helicoid, then the singular set $\mathcal{S}$ for the convergence is the vertical axis perpendicular to the leaves of the foliation. In [Me1], W. Meeks used this fact together with the uniqueness of the helicoid to prove that the singular set $\mathcal{S}$ in Theorem 0.1 is always a straight line perpendicular to the foliation, cf. also [Me2] for finer metric space structure. Very recently, W. Meeks and M. Weber have constructed a *local* example (i.e., a sequence of embedded minimal surfaces in a tubular neighborhood of a circle whose intersections with every sufficiently small ball are disks) where $\mathcal{S}$ is a circle, [MeWe].

Recently, our results here played a key role in our proof of the Calabi-Yau conjectures for embedded surfaces in [CM17]; cf. [JXa] and [Na]. The main result in [CM17] was a chord-arc bound for *possibly non-compact* embedded minimal disks, relating the extrinsic and intrinsic distances. This chord-arc bound implies that a complete embedded minimal disk in $\mathbf{R}^3$ is *properly* embedded. As an immediate consequence, we get intrinsic versions of all of the results of this paper. See [CM17] for more details and more general results.

In the case where Σ has infinite topology (e.g., when Σ is one of the Riemann examples), a number of interesting results have been obtained relying on our results. This is an area of much current research, see [CM6], the work of Meeks, J. Perez and A. Ros, [MePRs1], [MePRs2], and [MePRs3], the survey [MeP] and references therein.

Part I. The proof of Theorem 0.1 assuming Theorem 0.2 and short curves

In this part we will show how Theorem 0.1 follows from Theorem 0.2, the results about existence of multi-valued graphs from [CM3], [CM4] which were recalled in the introduction, Corollary III.3.5 of [CM5], and the results about properness of embedded disks from [CM7] (once we see that the conditions in corollary 0.7 of [CM7] are satisfied). The remaining parts of this paper are devoted to showing Theorem 0.2 (Part II.2) and that Corollary 0.7 of [CM7] applies (Part IV; see, in particular, Theorem I.0.10 below).

We will use several times that given $\alpha > 0$, Proposition II.2.12 of [CM3] gives a constant N_g so that if u satisfies the minimal surface equation on

$$S^{-N_g, 2\pi+N_g}_{\mathrm{e}^{-N_g},\ \mathrm{e}^{N_g} R}$$

with $|\nabla u| \leq 1$, and $w < 0$, then

$$\rho\,|\operatorname{Hess}_u| + \rho\,|\nabla w|/|w| \leq \alpha \text{ on } S^{0,2\pi}_{1,R}\,.$$

Theorem 3.36 of [CM9] then yields

$$|\nabla u - \nabla u(1,0)| \leq C\alpha\,.$$

We can therefore assume (after rotating $\mathbf{R}^3$ so that $\nabla u(1,0) = 0$) that

$$|\nabla u| + \rho\,|\operatorname{Hess}_u| + 4\,\rho\,|\nabla w|/|w| + \rho^2\,|\operatorname{Hess}_w|/|w| \leq \varepsilon < 1/(2\pi)\,. \tag{I.0.8}$$

The bound on $|\operatorname{Hess}_w|$ follows from the other bounds and standard elliptic theory. In what follows, we will assume that $w < 0$. (This normalizes the graph of u to spiral downward; this can be achieved after possibly reflecting in a plane.)

If Σ is an embedded graph of u over $S^{-3\pi,N+3\pi}_{1/2,2R}$, then E is the region over $D_R \setminus D_1$ between the top and bottom sheets of the concentric subgraph over $S^{-2\pi,N+2\pi}_{1,R}$ (recall that, possibly after reflection, we can assume $w < 0$). That is, when N is even, E is the set (see Figure 13) of all

$$\{(r\cos\theta, r\sin\theta, t)\,|\,1 \leq r \leq R \text{ and } -2\pi \leq \theta < 0\}$$

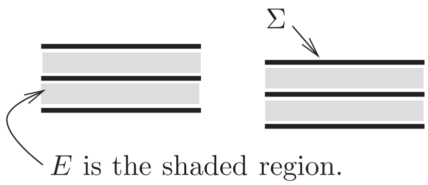

Figure 13: The set E in (I.0.9).

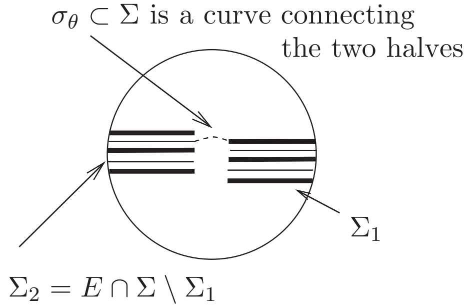

Figure 14: Theorem I.0.10 — the existence of the other half and the short curves, σ_θ, connecting the two halves.

which satisfy

$$u(r, \theta + (N+2)\pi) < t < u(r, \theta)\,. \tag{I.0.9}$$

To apply Corollary 0.7 of [CM7] we need the following result (which will be proved in Part IV) on existence of "the other half" of an embedded minimal disk and short curves, σ_θ, connecting the two halves:

Theorem I.0.10 (See Figure 14). *There exist C, R_0, N_0, $\varepsilon > 0$ so that for $N \geq N_0$ the following holds*:

Let $\Sigma \subset B_{4R}$ be an embedded minimal disk with $\partial \Sigma \subset \partial B_{4R}$. If $R \geq R_0$ and $\Sigma_1 \subset \Sigma$ is a (multi-valued) graph of a function u_1 with $|\nabla u_1| \leq \varepsilon$ over

$$S_{1/2,2R}^{-3\pi, N+3\pi}\,,$$

then $E \cap \Sigma \setminus \Sigma_1$ is a graph of a function u_2 over $S_{1,R}^{0,N+2\pi}$ with

$$u_1(1, 2\pi) < u_2(1, 0) < u_1(1, 0)\,.$$

Moreover, for all $0 \leq \theta \leq N + 2\pi$, a curve

$$\sigma_\theta \subset \{x_1^2 + x_2^2 \leq 1\} \cap \Sigma$$

with length $\leq C$ connects the image of u_1 over $(1, \theta)$ with the image of u_2 over $(1, \theta)$.

The main example of the "two halves" of an embedded minimal disk and short curves connecting them comes from the helicoid. Namely, let Σ be the helicoid, i.e.,

$$\Sigma = \{(\rho \cos\theta, \rho \sin\theta, -\theta) \,|\, \rho,\, \theta \in \mathbf{R}\}\,,$$

then $\Sigma \setminus \{\rho = 0\}$ consists of two ∞-valued graphs, Σ_1 and Σ_2, and the curves σ_θ given by

$$\Sigma \cap \{x_3 = -\theta\} \cup \{(-\cos\tau, -\sin\tau, -\tau) \,|\, \theta \leq \tau \leq \theta + \pi\}$$

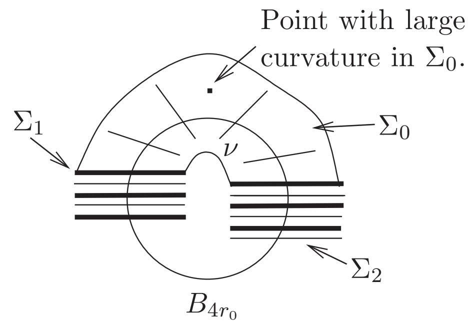

Figure 15: Proposition I.0.11 — existence of nearby points with large curvature.

are the short curves connecting the two halves. Theorem I.0.10 asserts that this is the general picture.

We will use the following result from [CM5] to get nearby points with large curvature (here, as before, $\Sigma_{y,s}$ is the component of $B_s(y) \cap \Sigma$ containing y):

Proposition I.0.11 (Corollary III.3.5 of [CM5]. See Figure 15). *Given C_1, there exists C_2 so that the following holds*:

Let $0 \in \Sigma \subset B_{2C_2 r_0}$ be an embedded minimal disk. Suppose that

$$\Sigma_1 \text{ and } \Sigma_2 \subset \Sigma \cap \{x_3^2 \leq (x_1^2 + x_2^2)\}$$

are graphs of functions u_i satisfying (I.0.8) *on $S^{-2\pi,2\pi}_{r_0,C_2 r_0}$ with*

$$u_1(r_0, 2\pi) < u_2(r_0, 0) < u_1(r_0, 0)\,,$$

and $\nu \subset \partial \Sigma_{0,2r_0}$ is a curve from Σ_1 to Σ_2. Let Σ_0 be the component of

$$\Sigma_{0,C_2 r_0} \setminus (\Sigma_1 \cup \Sigma_2 \cup \nu)$$

which does not contain Σ_{0,r_0}.

If either:

- $\partial \Sigma \subset \partial B_{2C_2 r_0}$, *or*
- Σ *is stable and* Σ_0 *does not intersect* $\partial \Sigma$,

then

$$\sup_{x \in \Sigma_0 \setminus B_{4r_0}} |x|^2\,|A|^2(x) \geq 4\,C_1^2\,. \tag{I.0.12}$$

Note that by the curvature estimate for stable surfaces, [Sc], [CM2], when Σ is stable then the conclusion of Proposition I.0.11 is that no such surface exists for C_1, C_2 sufficiently large.

I.1. Regularity of the singular set

If $\delta > 0$ and $z \in \mathbf{R}^3$, then we denote by $\mathbf{C}_\delta(z)$ the (convex) cone with vertex z, cone angle $(\pi/2 - \arctan\delta)$, and axis parallel to the x_3-axis. That is (see Figure 16),

$$\mathbf{C}_\delta(z) = \{x \in \mathbf{R}^3 \,|\, (x_3 - z_3)^2 \geq \delta^2\left((x_1 - z_1)^2 + (x_2 - z_2)^2\right)\}. \tag{I.1.1}$$

Lemma I.1.2 (see Figure 16). *Let $0 \in \mathcal{S} \subset \mathbf{R}^3$ be a closed set such that for some $\delta > 0$ and each $z \in \mathcal{S}$, such an $\mathcal{S} \subset \mathbf{C}_\delta(z)$. If for all $t \in x_3(\mathcal{S})$ and all $\varepsilon > 0$,*

$$\mathcal{S} \cap \{t < x_3 < t + \varepsilon\} \neq \emptyset,$$
$$\mathcal{S} \cap \{t - \varepsilon < x_3 < t\} \neq \emptyset,$$

then $\mathcal{S} \cap \{x_3 = t\}$ consists of exactly one point $\mathcal{S}_t$ for all $t \in \mathbf{R}$, and $t \to \mathcal{S}_t$ is a Lipschitz parametrization of $\mathcal{S}$. In fact,

$$|t_2 - t_1| \leq |\mathcal{S}_{t_2} - \mathcal{S}_{t_1}| \leq \sqrt{1 + \delta^{-2}}\,|t_2 - t_1|. \tag{I.1.3}$$

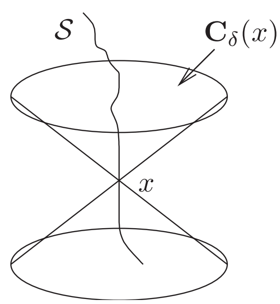

Figure 16: It follows from the one-sided curvature estimate that the singular set has the cone property and hence is a Lipschitz curve; see Lemma I.1.2.

Proof. First, by the cone property, it follows that $\mathcal{S} \cap \{x_3 = t\}$ consists of at most one point for each $t \in \mathbf{R}$. Assume that $\mathcal{S} \cap \{x_3 = t_0\} = \emptyset$ for some t_0. Since $\mathcal{S} \subset \mathbf{R}^3$ is a nonempty closed set and

$$x_3 : \mathcal{S} \subset \mathbf{C}_\delta(0) \to \mathbf{R}$$

is proper, $x_3(\mathcal{S}) \subset \mathbf{R}$ is also closed (and nonempty). Let $t_s \in x_3(\mathcal{S})$ be the closest point in $x_3(\mathcal{S})$ to t_0. The desired contradiction now easily follows since either $\mathcal{S} \cap \{t_s < x_3 < t_0\}$ or $\mathcal{S} \cap \{t_0 < x_3 < t_s\}$ is nonempty by assumption.

It follows that $t \to \mathcal{S}_t$ is a well-defined curve (from $\mathbf{R}$ to $\mathcal{S}$). Moreover, since

$$\mathcal{S}_{t_2} \subset \{x_3 = t_1 + (t_2 - t_1)\} \cap \mathbf{C}_\delta(\mathcal{S}_{t_1}) \subset B_{\sqrt{1+\delta^{-2}}|t_2 - t_1|}(\mathcal{S}_{t_1}),$$

(I.1.3) follows. □

We will refer loosely to a set $\mathcal{S}$ as in Lemma I.1.2 as having the cone property. Next we will see, by a very general compactness argument, that for any sequence of surfaces in $\mathbf{R}^3$, after possibly going to a subsequence, then there is a well defined notion of points where the second fundamental form of the sequence blows up. The set of such points will be the $\mathcal{S}$ in Lemma I.1.2 below; we observe in Corollary I.1.9 below that $\mathcal{S}$ has the cone property.

Lemma I.1.4. *Let $\Sigma_i \subset B_{R_i}$ with $\partial \Sigma_i \subset \partial B_{R_i}$ and $R_i \to \infty$ be a sequence of (smooth) compact surfaces. After passing to a subsequence, Σ_j, we may assume that for each $x \in \mathbf{R}^3$:*

- *Either $\sup_{B_r(x)\cap\Sigma_j} |A|^2 \to \infty$ for all $r > 0$.*
- *Or $\sup_j \sup_{B_r(x)\cap\Sigma_j} |A|^2 < \infty$ for some $r > 0$.*

Proof. For $r > 0$ and an integer n, define a sequence of functions on $\mathbf{R}^3$ by

$$\mathcal{A}_{i,r,n}(x) = \min\{n, \sup_{B_r(x)\cap\Sigma_i} |A|^2\}\,, \tag{I.1.5}$$

where $\sup_{B_r(x)\cap\Sigma_i} |A|^2 = 0$ if $B_r(x) \cap \Sigma_i = \emptyset$. Set

$$\mathcal{D}_{i,r,n} = \lim_{k\to\infty} 2^{-k} \sum_{m=0}^{2^k-1} \mathcal{A}_{i,(1+m2^{-k})r,n}\,, \tag{I.1.6}$$

with $\mathcal{D}_{i,r,n}$ continuous and $\mathcal{A}_{i,2r,n} \geq \mathcal{D}_{i,r,n} \geq \mathcal{A}_{i,r,n}$. Let $\nu_{i,r,n}$ be the (bounded) functionals,

$$\nu_{i,r,n}(\phi) = \int_{B_n} \phi\, \mathcal{D}_{i,r,n} \text{ for } \phi \in L^2(\mathbf{R}^3)\,. \tag{I.1.7}$$

By standard compactness for fixed r, n, after passing to a subsequence, we see that $\nu_{j,r,n} \to \nu_{r,n}$ weakly. In fact (since the unit ball in $L^2(\mathbf{R}^3)$ has a countable basis), by an easy diagonal argument after passing to a subsequence we may assume that for all $n, m \geq 1$ fixed

$$\nu_{j,2^{-m},n} \to \nu_{2^{-m},n} \text{ weakly}\,.$$

Note that if $x \in \mathbf{R}^3$ and for all m, n with $n \geq |x| + 1$ (identify $B_{2^{-m}}(x)$ with its characteristic function),

$$\nu_{2^{-m},n}(B_{2^{-m}}(x)) \geq n \operatorname{Vol}(B_{2^{-m}})\,, \tag{I.1.8}$$

then for each fixed $r > 0$, we have

$$\sup_{B_r(x)\cap\Sigma_j} |A|^2 \to \infty\,.$$

On the other hand, if for some $n \geq |x| + 1$, m, (I.1.8) fails at x, then

$$\sup_j \sup_{B_r(x)\cap\Sigma_j} |A|^2 < \infty \text{ for } r = 2^{-m-1}\,. \qquad \square$$

To implement Lemma I.1.2 in the proof of Theorem 0.1, we will need the following (direct) consequence of Theorem 0.2 with Σ_d playing the role of the plane (and the maximum principle as in Appendix C):

COROLLARY I.1.9 (see Figure 17). *There exists $\delta_0 > 0$ so that the following holds*:

Let $\Sigma \subset B_{2R}$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_{2R}$. If Σ contains a 2-valued graph $\Sigma_d \subset \{x_3^2 \leq \delta_0^2\,(x_1^2 + x_2^2)\}$ over $D_R \setminus D_{r_0}$ with gradient $\leq \delta_0$, then each component of

$$B_{R/2} \cap \Sigma \setminus (\mathbf{C}_{\delta_0}(0) \cup B_{2r_0})$$

is a multi-valued graph with gradient ≤ 1.

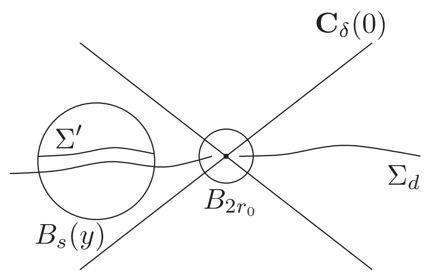

Figure 17: Corollary I.1.9: With Σ_d playing the role of $x_3 = 0$, by the one-sided estimate, Σ consists of multi-valued graphs away from a cone.

Note that since Σ is compact and embedded, the multi-valued graphs given by Corollary I.1.9 spiral through the cone. That is, if a graph did close up, then the graph containing Σ_d would be forced to accumulate into it, contradicting compactness.

Another result needed to apply Lemma I.1.2 is:

LEMMA I.1.10 (see Figure 18). *There exists $c_0 > 0$ so that the following holds*:

Let $\Sigma_i \subset B_{R_i}$ be a sequence of embedded minimal disks with $\partial\Sigma_i \subset \partial B_{R_i}$ and $R_i \to \infty$. If $\Sigma_{d,i} \subset \Sigma_i$ is a sequence of 2-valued graphs over $D_{R_i/C} \setminus D_{\varepsilon_i}$ with $\varepsilon_i \to 0$ and

$$\Sigma_{d,i} \to \{x_3 = 0\} \setminus \{0\}\,,$$

then

$$\sup_{B_1 \cap \Sigma_i \cap \{x_3 > c_0\}} |A|^2 \to \infty\,. \tag{I.1.11}$$

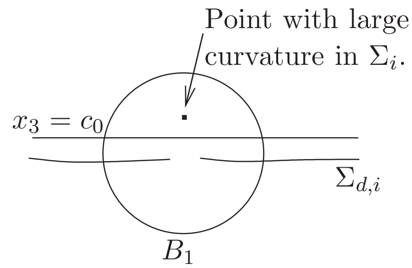

Figure 18: Lemma I.1.10 — point with large curvature in Σ_i above the plane $x_3 = c_0$ but near the center of the 2-valued graph $\Sigma_{d,i}$.

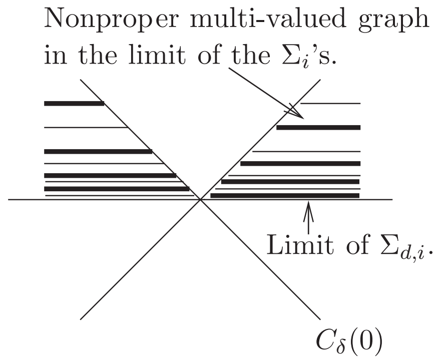

Figure 19: If Lemma I.1.10 failed, then by Corollary I.1.9 the limit of the Σ_i's would contain a nonproper multi-valued graph contradicting Corollary 0.7 of [CM7].

Proof. Suppose not (see Figure 19); assume that for each $c_0 > 0$, there is a sequence of embedded minimal disks Σ_i (and C_1 depending on both c_0 and the sequence) with

$$\sup_{B_1 \cap \Sigma_i \cap \{x_3 > c_0\}} |A|^2 \leq C_1 < \infty \tag{I.1.12}$$

and 2-valued graphs $\Sigma_{d,i} \subset \Sigma_i$ over $D_{R_i/C} \setminus D_{\varepsilon_i}$ with $\varepsilon_i \to 0$ and

$$\Sigma_{d,i} \to \{x_3 = 0\} \setminus \{0\}\,.$$

Increasing ε_i (yet still $\varepsilon_i \to 0$) and replacing R_i by $S_i \to \infty$, we can assume

$$\Sigma_{d,i} \subset \{x_3^2 \leq \varepsilon_i^2\,(x_1^2 + x_2^2)\}$$

is a 2-valued graph over $D_{4\,\mathrm{e}^{N_g} S_i} \setminus D_{\mathrm{e}^{-N_g}\varepsilon_i/2}$ with gradient $\leq \varepsilon_i$ (the constant N_g is given before (I.0.8)).

By Corollary I.1.9, each component of

$$B_{2\,\mathrm{e}^{N_g} S_i} \cap \Sigma_i \setminus (\mathbf{C}_{\delta_0}(0) \cup B_{\mathrm{e}^{-N_g}\varepsilon_i})$$

is a graph. Hence, by the Harnack inequality, if $\alpha > 0$ is sufficiently small and

$$q_i \in B_{S_i} \cap \Sigma_i \setminus (\mathbf{C}_\alpha(0) \cup B_{2\varepsilon_i}) ,$$

then for i large Σ_i contains an (N_g+1)-valued graph over $D_{\,\mathrm{e}^{N_g}|q_i|} \setminus D_{\,\mathrm{e}^{-N_g}|q_i|/2}$ with

$$\text{gradient } \leq \varepsilon < 1/(4\pi)$$

so that q_i is in the image of $\{|\theta| \leq \pi\}$ for this graph. Consequently, each component of

$$B_{S_i} \cap \Sigma_i \setminus (\mathbf{C}_\alpha(0) \cup B_{2\varepsilon_i})$$

is a multi-valued graph satisfying (I.0.8).

Fix h and ℓ with $0 < h < \alpha\, \ell$. We get points

$$z_i \in \{x_3 = h,\, x_1^2 + y_1^2 = \ell^2\} \cap \Sigma_i$$

and multi-valued graphs $\Sigma_{1,i}$ with

$$z_i \in \Sigma_{1,i} \subset \{x_3 > 0\} \cap \Sigma_i$$

defined over $S_{\ell/2,S_i/2}^{-3\pi,3\pi+N_i}$, with $N_i \to \infty$, so that z_i is in the image of $S_{\ell,\ell}^{-\pi,\pi}$, and so that $\Sigma_{1,i}$ spirals into $\{x_3 = 0\}$ (note that we have assumed that it spirals down; we can argue similarly in the other case). In particular, Theorem I.0.10 applies, giving the other multi-valued graphs $\Sigma_{2,i}$ so that:

- $\Sigma_{1,i}$ and $\Sigma_{2,i}$ spiral together, and
- $\Sigma_{2,i}$ is the only part of Σ_i between the sheets of $\Sigma_{1,i}$.

Moreover, Theorem I.0.10 also gives the short curves $\sigma_{\theta,i}$ connecting these. It now follows from Corollary 0.7 of [CM7] that the separations of the graph $\Sigma_{1,i}$ at z_i go to 0. Since this holds for all such h and ℓ, it follows that

$$\Sigma_i \setminus \mathbf{C}_\alpha(0) \to \mathcal{F} ,$$

where $\mathcal{F}$ is a foliation of $\mathbf{R}^3 \setminus \mathbf{C}_\alpha(0)$ by minimal annuli (all graphs over part of $\{x_3 = 0\}$).

Theorem 0.7 gives $0 < C_2 < \infty$ so that, given $r_0 > 0$, if $y_i \in \Sigma_i \setminus B_{3r_0}$, i is large, and

$$|y_i|^2\, |A|^2(y_i) > C_2 \tag{I.1.13}$$

then there is a 2-valued graph $\Sigma_{d,i}^{y_i} \subset \Sigma_i \setminus B_{C_3|y_i|}$ starting in

$$B_{C_4|y_i|}(y_i) \subset \{x_3 > C_3\, r_0\}$$

(by Theorem 0.7, $\Sigma_{d,i}^{y_i}$ starts in $B_{C_4|y_i|}(y_i)$ where $C_4 = C_4(C_2)$ and, by Corollary I.1.9, $y_i \in \mathbf{C}_{\delta_0/2}(0)$). Let $C_2' = C_2'(C_2) > 1$ be given by Proposition I.0.11 and set $r_0 = 1/(4C_2')$.

Choose $h_i, \ell_i \to 0$ with

$$\varepsilon_i < \ell_i < r_0/4\,,$$
$$0 < h_i < \alpha\, \ell_i\,,$$

and let $z_i, \Sigma_{1,i}, \Sigma_{2,i}$ be as above. Since $\partial \Sigma_{i,z_i,2r_0}$ is a simple closed curve, it must pass between the sheets of $\Sigma_{1,i}$. Since $\Sigma_{2,i}$ is the only part of Σ_i between the sheets of $\Sigma_{1,i}$, we can connect $\Sigma_{1,i}$ and $\Sigma_{2,i}$ by curves $\nu_i \subset \partial \Sigma_{i,z_i,2r_0}$ which are above $\Sigma_{1,i}$. We can now apply Proposition I.0.11 to get the points

$$y_i \in B_{1/2}(z_i) \cap \Sigma_i \setminus B_{2r_0}(z_i) \subset B_{1/2+4\ell_i} \setminus B_{3r_0}$$

as in (I.1.13).

To get the desired contradiction, observe that if $c_0 < C_3 r_0$, then the 2-valued graphs $\Sigma_{d,i}^{y_i}$ given by (I.1.13) and Theorem 0.7, have

$$\text{separation } \geq C_5 = C_5(C_1) > 0$$

(since $B_{C_4|y_i|}(y_i) \subset \{x_3 > C_3\, r_0\}$). Namely, this separation is on a fixed scale bounded away from zero even as $\Sigma_{d,i}^{y_i}$ extends out of $\mathbf{C}_\alpha(0)$, contradicting $\Sigma_i \setminus \mathbf{C}_\alpha(0) \to \mathcal{F}$. The lemma follows. □

I.2. Proof of Theorem 0.1

Proof of Theorem 0.1. By Lemma I.1.4, after passing to a subsequence (also denoted by Σ_i) we can assume that for each $x \in \mathbf{R}^3$ either

$$\sup_{B_r(x)\cap\Sigma_i} |A|^2 \to \infty \text{ for all } r > 0\,, \tag{I.2.1}$$

or $\sup_i \sup_{B_r(x)\cap\Sigma_i} |A|^2 < \infty$ for some $r > 0$. Let $\mathcal{S} \subset \mathbf{R}^3$ be the points where (I.2.1) holds. By assumption $B_1 \cap \mathcal{S} \neq \emptyset$. Thus, after a possible translation we may assume that $0 \in \mathcal{S}$ and it follows easily from the definition that $\mathcal{S}$ is closed. By Theorem 5.8 of [CM4] (and Bernstein's theorem; see for instance Theorem 1.16 of [CM1]), there exists a subsequence Σ_j and 2-valued graphs $\Sigma_{d,j} \subset \Sigma_j$ over $D_{R_j/C} \setminus D_{\varepsilon_j}$ with $\varepsilon_j \to 0$ such that

$$\Sigma_{d,j} \to \{x_3 = 0\} \setminus \{0\}$$

(after possibly rotating $\mathbf{R}^3$). (This fixes the subsequence and the coordinate system of $\mathbf{R}^3$.) Again by theorem 5.8 of [CM4] (and Bernstein's theorem) for each $\mathcal{S}_t \in \mathcal{S}$ there are 2-valued graphs $\Sigma_{d,j}^t \subset \Sigma_j$ over $D_{R_j/C}(\mathcal{S}_t) \setminus D_{\varepsilon_j}(\mathcal{S}_t)$ with $\varepsilon_j \to 0$ such that

$$\Sigma_{d,j}^t \to \{x_3 = t\} \setminus \{\mathcal{S}_t\}\,.$$

Hence, by Corollary I.1.9, $\mathcal{S} \subset \mathbf{C}_\delta(\mathcal{S}_t)$. By Lemma I.1.10 (and scaling), for all $t \in x_3(\mathcal{S})$ and all $\varepsilon > 0$, we have

$$\mathcal{S} \cap \{t < x_3 < t + \varepsilon\} \neq \emptyset\,,$$
$$\mathcal{S} \cap \{t - \varepsilon < x_3 < t\} \neq \emptyset\,.$$

It follows from Lemma I.1.2 that $t \to \mathcal{S}_t$ is a Lipschitz curve and $\Sigma_j \setminus \mathcal{S} \to \mathcal{F} \setminus \mathcal{S}$ in the C^α-topology for all $\alpha < 1$ (and with uniformly bounded curvatures on compact subsets of $\mathbf{R}^3 \setminus \mathcal{S}$; see also Appendix B). □

Part II. "The other half"

Theorem I.0.10 will follow by first showing that if an embedded minimal disk contains a multi-valued graph, then "between the sheets" of the graph the surface is another multi-valued graph - "the other half". Second, we show an intrinsic version of Theorem 0.7 and, third, using this intrinsic version, we construct in Part IV the short curves connecting the two halves.

II.1. "The other half" of an embedded minimal disk

We show first that any point between the sheets of a multi-valued graph must connect to it within a fixed extrinsic ball:

LEMMA II.1.1. *There exist $\varepsilon_s > 0$ and $C_s > 2$ so that the following holds*: *Let $0 \in \Sigma \subset B_R$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_R$. Suppose that*

$$\Sigma_d \subset \{x_3^2 \leq x_1^2 + x_2^2\} \cap \Sigma$$

is a 2-valued graph over $D_{3r_0} \setminus D_{r_0}$ with gradient $\leq \varepsilon_s$. If E_0 is the region over $D_{2r_0} \setminus D_{r_0}$ between the sheets of Σ_d, then

$$E_0 \cap \Sigma \subset \Sigma_{0,C_s\, r_0}\,.$$

Proof. Fix $\varepsilon_s > 0$ small and C_s large to be chosen. If the lemma fails, then there are disjoint components Σ_a and Σ_b of $B_{C_s\, r_0} \cap \Sigma$ with

$$\Sigma_d \subset \Sigma_a \text{ and } y \in E_0 \cap \Sigma_b\,.$$

By the maximum principle, Σ_a and Σ_b are disks. Let $\tilde{\eta}_y$ be the vertical segment (i.e., parallel to the x_3-axis) through y connecting the sheets of Σ_d. Fix a component η_y of $\tilde{\eta}_y \setminus \Sigma$ connecting Σ_b to $\Sigma \setminus \Sigma_b$. Let Ω be the component of $B_{C_s\, r_0} \setminus \Sigma$ containing η_y (so that $\partial\Sigma_b$ and η_y are linked in Ω). [MeYa] gives a stable disk $\Gamma \subset \Omega$ with $\partial\Gamma = \partial\Sigma_b$. By means of the linking, Γ intersects η_y. The curvature estimates of [Sc], [CM2] (cf. Lemma I.0.9 of [CM3]) give a constant C_s so that any component Γ_y of $B_{10r_0} \cap \Gamma$ intersecting η_y is a graph with bounded gradient over some plane; for ε_s small, this plane must be almost horizontal. Hence, Γ_y is forced to "cut the axis" (i.e., intersect the curve in Σ_d over ∂D_{r_0} connecting the top and bottom sheets), giving the desired contradiction. □

In the next proposition $\Sigma \subset B_{4R}$ with $\partial \Sigma \subset \partial B_{4R}$ is an embedded minimal surface and

$$\Sigma_1 \subset \{x_3^2 \leq x_1^2 + x_2^2\} \cap \Sigma$$

is an $(N+2)$-valued graph of u_1 over $D_{2R} \setminus D_{r_1}$ with $|\nabla u_1| \leq \varepsilon$ and $N \geq 6$. Let E_1 be the region over $D_R \setminus D_{2\,r_1}$ between the top and bottom sheets of the concentric $(N+1)$-valued subgraph in Σ_1. To be precise, E_1 is the set of all

$$\{(r\cos\theta, r\sin\theta, t) \,|\, 2r_1 \leq r \leq R\,,\, (N-1)\pi \leq \theta < (N+1)\pi\}$$

where r and θ satisfy

$$u_1(r,\theta) < t < u_1(r, \theta - 2N\pi)\,. \tag{II.1.2}$$

PROPOSITION II.1.3. *There exist $C_0 > C_s$ and $\varepsilon_0 > 0$ so that if Σ is a disk as above with*

$$R \geq C_0\, r_1 \text{ and } \varepsilon_0 \geq \varepsilon\,,$$

then $E_1 \cap \Sigma \setminus \Sigma_1$ is an (oppositely oriented) N-valued graph Σ_2.

Proof. Fix $z \in \Sigma_1$ over ∂D_{r_1}. Since $\partial \Sigma_{z,2r_1}$ is a simple closed curve, it must pass between the sheets of Σ_1 and hence through some other component Σ_2 of $E_1 \cap \Sigma$.

The version of the "estimate between the sheets" given in Theorem III.2.4 of [CM3] gives $\varepsilon_0 > 0$ so that $E_1 \cap \Sigma$ is locally graphical (i.e., if $z \in E_1 \cap \Sigma$, then $\langle \mathbf{n}_\Sigma(z), (0,0,1)\rangle \neq 0$). It follows that each component of $E_1 \cap \Sigma$ is an N-valued graph.

Fix a component Ω of $B_{4R} \setminus \Sigma$. We show next that Σ_2 is the only other component of $E_1 \cap \Sigma$ (i.e., $E_1 \cap \Sigma \subset \Sigma_1 \cup \Sigma_2$). If not, then there is a third component Σ_3 which is also an N-valued graph. An easy argument (using orientations) shows that there must then be a fourth component Σ_4 of $E_1 \cap \Sigma$. By the fact that each Σ_i is a multi-valued graph, it follows easily that we can choose two of these four which cannot be connected in $\Omega \cap E_1$; call these Σ_{i_1} and Σ_{i_2}. The rest of this argument uses these components to find a stable $\Gamma \subset \Omega$ which has points of large curvature by Proposition I.0.11, contradicting the curvature estimates from stability. First, we construct $\partial \Gamma$. Let $\sigma_j \subset \Sigma_{i_j}$ be the images of $\{\theta = 0\}$ from $\{x_1^2 + x_2^2 = 4r_1^2\}$ to ∂B_R and set

$$y_j = \{x_1^2 + x_2^2 = 4r_1^2\} \cap \partial \sigma_j\,.$$

By Lemma II.1.1, we can connect the points y_1 and y_2 by a curve $\sigma_0 \subset B_{C_s r_1} \cap \Sigma$. By the maximum principle, each component of $B_R \cap \Sigma$ is a disk. Therefore, we can add a segment in $\partial B_R \cap \Sigma$ to $\sigma_0 \cup \sigma_1 \cup \sigma_2$ to get a closed curve $\sigma \subset \Sigma$. A result of [MeYa] then gives a stable embedded minimal disk $\Gamma \subset \Omega$ with $\partial \Gamma = \sigma$.

Now that we have Γ, we show that Proposition I.0.11 applies. Namely, let (the disk) $\Gamma_{2C_s\, r_1}(\sigma_0)$ be the component of $B_{2C_s\, r_1} \cap \Gamma$ containing σ_0, so that

$\partial \Gamma_{2C_s r_1}(\sigma_0)$ contains a curve $\nu \subset \partial B_{2C_s r_1}$ connecting σ_1 to σ_2. Since σ_1 and σ_2 are in the middle sheets of Σ_{i_1} and Σ_{i_2} (and Γ is stable), we get that Γ contains two disjoint $(N/2-1)$-valued graphs Γ_1 and Γ_2 in E_1 which spiral together and ν connects these (note that $E_1 \cap \Gamma$ may contain many components; at least two of these, say Γ_1, Γ_2, spiral together). Let Γ_0 be the component of

$$\Gamma_{R/2}(\sigma_0) \setminus (\nu \cup \Gamma_1 \cup \Gamma_2)$$

which does not contain $\Gamma_{2C_s r_1}(\sigma_0)$. It is easy to see that $\Gamma_0 \cap \partial \Gamma = \emptyset$; in fact, if $x \in \Gamma_0$, then

$$\mathrm{dist}_\Gamma(x, \partial\Gamma) \geq |x|/2\,.$$

Therefore, for R/r_1 sufficiently large, Proposition I.0.11 gives an interior point of large curvature, contradicting the curvature estimate for stable surfaces. We conclude that

$$E_1 \cap \Sigma \subset \Sigma_1 \cup \Sigma_2\,.$$

Finally, it follows easily that Σ_2 is oppositely oriented. □

The proof of Proposition II.1.3 can be simplified when Σ is in a slab. In this case, [Sc], [CM2] and the gradient estimate (cf. Lemma I.0.9 of [CM3]) force Γ to spiral indefinitely if it leaves E_1.

II.2. An intrinsic version of Theorem 0.7

We will first show a "chord-arc" type result (relating extrinsic and intrinsic distances) assuming a curvature bound on an intrinsic ball.

Lemma II.2.1 (cf. Lemma III.1.3 in [CM5]). *Given R_0, there exists R_1 so that the following holds*:

If $0 \in \Sigma \subset B_{R_1}$ is an embedded minimal surface with $\partial\Sigma \subset \partial B_{R_1}$ and

$$\sup_{\mathcal{B}_{R_1}} |A|^2 \leq 4\,,$$

then

$$\Sigma_{0,R_0} \subset \mathcal{B}_{R_1}\,.$$

Proof. Let $\tilde\Sigma$ be the universal cover of Σ and $\tilde\Pi : \tilde\Sigma \to \Sigma$ the covering map. With the definition of δ-stable as in section 2 of [CM4], the argument of [CM2] (i.e., curvature estimates for 1/2-stable surfaces) gives $C > 10$ so that if $\mathcal{B}_{CR_0/2}(\tilde z) \subset \tilde\Sigma$ is 1/2-stable and $\tilde\Pi(\tilde z) = z$, then

$$\tilde\Pi : \mathcal{B}_{5R_0}(\tilde z) \to \mathcal{B}_{5R_0}(z)$$

is one-to-one and $\mathcal{B}_{5R_0}(z)$ is a graph with

$$B_{4R_0}(z) \cap \partial \mathcal{B}_{5R_0}(z) = \emptyset\,.$$

Corollary 2.13 in [CM4] gives $\varepsilon = \varepsilon(CR_0) > 0$ so that if $|z_1 - z_2| < \varepsilon$ and $|A|^2 \leq 4$ on (the disjoint balls) $\mathcal{B}_{CR_0}(z_i)$, then each

$$\mathcal{B}_{CR_0/2}(\tilde{z}_i) \subset \tilde{\Sigma}$$

is 1/2-stable where $\tilde{\Pi}(\tilde{z}_i) = z_i$.

We claim that there exists n so that

$$\Sigma_{0,R_0} \subset \mathcal{B}_{(2n+1)\,CR_0}\,.$$

Suppose not; we get a curve $\sigma \subset \Sigma_{0,R_0} \subset B_{R_0}$ from 0 to $\partial \mathcal{B}_{(2n+1)\,CR_0}$. For $i = 1, \dots, n$, fix points $z_i \in \partial \mathcal{B}_{2i\,CR_0} \cap \sigma$. It follows that the intrinsic balls $\mathcal{B}_{CR_0}(z_i)$:

- are disjoint;
- have centers in $B_{R_0} \subset \mathbf{R}^3$;
- have $|A|^2 \leq 4$.

In particular, there exist i_1 and i_2 with

$$0 < |z_{i_1} - z_{i_2}| < C'\, R_0\, n^{-1/3} < \varepsilon\,,$$

and, by Corollary 2.13 in [CM4], each $\mathcal{B}_{CR_0/2}(\tilde{z}_{i_j}) \subset \tilde{\Sigma}$ is 1/2-stable where $\tilde{\Pi}(\tilde{z}_{i_j}) = z_{i_j}$. By [CM2], each $\mathcal{B}_{5R_0}(z_{i_j})$ is a graph with $B_{4R_0}(z_{i_j}) \cap \partial \mathcal{B}_{5R_0}(z_{i_j}) = \emptyset$. In particular,

$$B_{R_0} \cap \partial \mathcal{B}_{5R_0}(z_{i_j}) = \emptyset\,.$$

This contradicts the fact that $\sigma \subset B_{R_0}$ connects z_{i_j} to $\partial \mathcal{B}_{CR_0}(z_{i_j})$. □

An immediate consequence of Lemma II.2.1, is that we can improve Theorem 0.7 (and hence also, by an intrinsic blow-up argument, Theorem 0.6) by observing that the multi-valued graph can actually be chosen to be intrinsically nearby where the curvature is large (as opposed to extrinsically nearby):

THEOREM II.2.2. *Given $N \in \mathbf{Z}_+$ and $\varepsilon > 0$, there exist $C_1, C_2, C_3 > 0$ so that the following holds*:

Let $0 \in \Sigma^2 \subset B_R \subset \mathbf{R}^3$ be an embedded minimal disk with $\partial \Sigma \subset \partial B_R$. If for some r_0 with $0 < r_0 < R$,

$$\sup_{\mathcal{B}_{r_0}} |A|^2 \leq 4\,|A|^2(0) = 4\,C_1^2\,r_0^{-2}\,,$$

then there exists (after a rotation of $\mathbf{R}^3$) an N-valued graph

$$\Sigma_g \subset \Sigma \cap \{x_3^2 \leq \varepsilon^2\,(x_1^2 + x_2^2)\}$$

over $D_{R/C_2} \backslash D_{r_0}$ with gradient $\leq \varepsilon$, separation $\geq C_3\, r_0$ over ∂D_{r_0}, and $\text{dist}_\Sigma(0, \Sigma_g) \leq 2\, r_0$.

Proof. By combining Theorems 0.4 and 0.6 of [CM4], we get C_0, C_2, C_3 so that if

$$\sup_{\Sigma_{0,r_0}} |A|^2 \leq 4\,|A|^2(0) = 4\,C_0^2\,r_0^{-2}\,,$$

then we get (after a rotation) an N-valued graph Σ_g over $D_{R/C_2} \setminus D_{r_0}$ with gradient $\leq \varepsilon$,

$$\Sigma_g \subset \Sigma \cap \{x_3^2 \leq \varepsilon^2\,(x_1^2 + x_2^2)\}\,,$$

separation $\geq C_3\,r_0$ over ∂D_{r_0}, where Σ_y intersects Σ_{0,r_0}. Namely, Theorem 0.4 of [CM4] gives an initial N-valued graph contained in Σ_{0,r_0} and then Theorem 0.6 of [CM4] extends this out to $\partial D_{R/C_2}$. Let C_1 be the R_1 from Lemma II.2.1 with $R_0 = C_0$. By rescaling, we can assume that

$$\sup_{\mathcal{B}_{C_1}} |A|^2 \leq 4\,|A|^2(0) = 4\,.$$

By Lemma II.2.1, we have that

$$\Sigma_{0,C_0} \subset \mathcal{B}_{C_1}\,,$$

and so we conclude that

$$\sup_{\Sigma_{0,C_0}} |A|^2 \leq 4\,.$$

Theorems 0.4 and 0.6 of [CM4] now give the desired Σ_g. □

A standard blowup argument gives points as in Theorem II.2.2 (with $C_4 = 1$ and $s = r_0$):

LEMMA II.2.3 (Lemma 5.1 of [CM4]). *Given C_1 and C_4, if $\mathcal{B}_{C_1C_4}(0) \subset \Sigma$ is an immersed surface and*

$$|A|^2(0) \geq 4\,,$$

then there exists $\mathcal{B}_{C_4 s}(z) \subset \mathcal{B}_{C_1C_4}(0)$ with

$$\sup_{\mathcal{B}_{C_4 s}(z)} |A|^2 \leq 4\,|A|^2(z) = 4\,C_1^2\,s^{-2}\,. \tag{II.2.4}$$

Proof. This follows as in Lemma 5.1 of [CM4], except we define F intrinsically on $\mathcal{B}_{C_1C_4}(0)$ by

$$F = d^2\,|A|^2$$

where $d(x) = C_1C_4 - \operatorname{dist}_\Sigma(x,0)$ (so that $F = 0$ on $\partial\mathcal{B}_{C_1C_4}(0)$ and $F(0) \geq 4(C_1C_4)^2$). Let $F(z)$ be the maximum of F and set $s = C_1/|A|(z)$. It follows that

$$\sup_{\mathcal{B}_{d(z)/2}(z)} |A|^2 \leq 4\,|A|^2(z)$$

and (using $F(z) \geq (C_1C_4)^2$) we get $2C_4 s \leq d(z)$, giving (II.2.4). □

Part III. The stacking and the proof of Theorem 0.2

This part deals with how the multi-valued graphs given by [CM4] fit together. As mentioned in the introduction, a general embedded minimal disk with large curvature at some point should be thought of as obtained by stacking such graphs on top of each other.

III.1. Decomposing disks into multi-valued graphs

Fix $N > 6$ large and $1/10 > \varepsilon > 0$ small. We will choose $\varepsilon_g > 0$ small depending on ε and then let $N_g = N_g(\varepsilon_g)$ be given by Proposition II.2.12 of [CM3]. Below Σ will be an embedded minimal disk. Theorem II.2.2 gives C_1, C_2, C_3 (depending on ε_g, N, and N_g) so that if $B_R(y) \cap \partial\Sigma = \emptyset$ and the pair (y, s) satisfies

$$\sup_{\mathcal{B}_{8s}(y)} |A|^2 \leq 4\,|A|^2(y) = 4\,C_1^2\,s^{-2}\,, \tag{III.1.1}$$

then (after a rotation) we get an $(N{+}N_g{+}4)$-valued graph $\tilde{\Sigma}_1$ over $D_{2\,\mathrm{e}^{N_g} R/C_2}(p) \setminus D_{\mathrm{e}^{-N_g} s/2}(p)$ (where $p = (y_1, y_2, 0)$) satisfying:

- gradient $\leq \varepsilon_g$;
- separation $\geq C_3\, s$ over $\partial D_s(p)$;
- $\mathrm{dist}_\Sigma(y, \tilde{\Sigma}_1) \leq 2s$.

In particular, by Proposition II.2.12 of [CM3] and the version of the "estimate between the sheets" given in Theorem III.2.4 of [CM3], we can choose $\varepsilon_g = \varepsilon_g(\varepsilon) > 0$ so that

(1) The concentric $(N{+}3)$-valued subgraph $\hat{\Sigma}_1$ over $D_{R/C_2}(p) \setminus D_s(p)$ satisfies (I.0.8).

(2) Each component of Σ between the sheets of $\hat{\Sigma}_1$ (as in (II.1.2)) is an $(N+2)$-valued graph also satisfying (I.0.8).

In the remainder of this section, C_1, C_2, C_3 will be fixed.

Let ε_0, C_0 be from Proposition II.1.3 and suppose that $\varepsilon < \varepsilon_0$. If $s < R/(8C_2C_0)$ for such a pair (y, s), then Proposition II.1.3 applies. Let $\hat{E}$ and E be the regions between the sheets of the concentric $(N+2)$-valued and $(N+1)$-valued, respectively, subgraphs of $\hat{\Sigma}_1$; these are defined over

$$D_{R/C_2}(p) \setminus D_s(p)\,.$$

By Proposition II.1.3 (and (2) above), we have that

$$\hat{E} \cap \Sigma \setminus \hat{\Sigma}_1$$

is an $(N+1)$-valued graph $\hat{\Sigma}_2$; similarly, $E \cap \Sigma \setminus \hat{\Sigma}_1$ is an N-valued graph $\Sigma_2 \subset \hat{\Sigma}_2$. Let $\Sigma_1 \subset \hat{\Sigma}_1$ be the concentric N-valued subgraph. Since $\partial \Sigma_{y,4s}$ is a simple closed curve, it must pass through $E \setminus \Sigma_1$. Therefore, since Σ_2 is the only other part of Σ in E, we can connect Σ_1 and Σ_2 by curves ν_+ and ν_- with

$$\nu_\pm \subset \partial B_{4s}(y) \cap \Sigma$$

which are above and below E, respectively. This gives components $\Sigma_\pm$ of

$$\Sigma_{y,R/(2C_2)} \setminus (\Sigma_1 \cup \Sigma_2 \cup \nu_\pm)$$

which do not contain $\Sigma_{y,s}$ and which are above and below E, respectively (these will be the Σ_0's for Proposition I.0.11).

Given a pair satisfying (III.1.1), Proposition I.0.11 and Lemma II.2.3 easily give two nearby pairs (one above and one below):

LEMMA III.1.2. *There exists $C_4 > 1$ so that the following holds*:

Let $0 \in \Sigma \subset B_{3R}$ be an embedded minimal disk with $\partial \Sigma \subset \partial B_{3R}$. If the pair $(0,s)$ satisfies (III.1.1) *and*

$$s < \min\{R/(2C_4), R/(8C_2C_0)\}\,,$$

then we get a pair (y_-, s_-) also satisfying (III.1.1) *with $y_- \in \Sigma_-$ and*

$$\Sigma_{y_-,4s_-} \subset \Sigma_{0,C_4 s} \setminus B_{4s}\,.$$

Moreover, the N-valued graphs corresponding to $(0,s)$ and (y_-,s_-) are disjoint.

Proof. Proposition I.0.11 gives $C_4 = C_4(C_1)$ and a point $z \in \Sigma_{0,C_4 s/2} \cap \Sigma_- \setminus B_{8s}$ with

$$|z|^2\,|A|^2(z) \geq 4(8C_1)^2\,.$$

Since $\hat{E} \cap \Sigma$ consists of the multi-valued graphs $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$,

$$|x|^2\,|A|^2(x) \leq C \text{ on } \hat{E} \cap B_{C_4 s} \cap \Sigma_- \setminus B_{2s}$$

for C small (C can be made arbitrarily small by choosing ε even smaller). Hence, $z \notin \hat{E}$ and so

$$\mathcal{B}_{|z|/2}(z) \cap E = \emptyset\,.$$

Applying Lemma II.2.3 on $\mathcal{B}_{|z|/2}(z)$, we get a pair (y_-,s_-) satisfying (III.1.1) with

$$\mathcal{B}_{8s_-}(y_-) \subset \mathcal{B}_{|z|/2}(z)$$

($\subset \Sigma_- \setminus E$). It follows that

$$\Sigma_{y_-,4s_-} \subset \Sigma_{0,C_4 s} \setminus B_{4s}$$

and the corresponding N-valued graphs are disjoint. $\square$

Let C_4 be given by Lemma III.1.2. Iterating the construction of Lemma III.1.2, we can decompose an embedded minimal disk into basic building blocks ordered by heights (the points p_i in Corollary III.1.3 are the projections to $\{x_3 = 0\}$ of the blowup points y_i):

Corollary III.1.3. *There exist $C_5 > 1$, $\tilde{C}_3 > 0$ so that the following holds*:

Let $\Sigma \subset B_{C_5 R}$ be an embedded minimal disk with $\partial \Sigma \subset \partial B_{C_5 R}$. If the pair (y_0, s_0) satisfies (III.1.1) *with*

$$B_{C_4}(y_0) \subset B_R \,,$$

then there exist pairs $\{(y_i, s_i)\}$ (for $i > 0$) satisfying (III.1.1) *with $y_i \in \Sigma_-$ and corresponding (disjoint) N-valued graphs $\Sigma_i \subset \Sigma$ of u_i over $D_{2R}(0) \setminus D_{2s_i}(p_i)$ with gradient $\leq 2\varepsilon$, separation $\geq \tilde{C}_3\, s_i$ over $\partial D_{2s_i}(p_i)$, and satisfying*:

$$\text{(III.1.4)} \qquad \textit{If } i < j \textit{ and both } u_i \textit{ and } u_j \textit{ are defined at } p, \textit{ then } u_j(p) < u_i(p)\,;$$

$$\text{(III.1.5)} \qquad \Sigma_{y_{i+1}, 4s_{i+1}} \subset \Sigma_{y_i, C_4 s_i} \setminus B_{4s_i}(y_i) \;\; \textit{and} \;\; \cup_i B_{C_4 s_i}(y_i) \setminus B_R \neq \emptyset \,.$$

Proof. Starting with (y_0, s_0), we can apply Lemma III.1.2 repeatedly, until the second part of (III.1.5) holds, to find bottom N-valued graphs giving (III.1.4) and the first part of (III.1.5). Each N-valued graph is a graph over some plane with gradient $\leq \varepsilon$. Since Σ is embedded, we can take these to be graphs over a fixed plane with gradient $\leq 2\varepsilon$ (after possibly taking $C_5 > 3C_2+1$ larger). Now $\tilde{C}_3 > 0$ is just a fixed fraction of C_3. □

In the next lemma and corollary, $\Sigma \subset B_{C_5 R}$ is an embedded minimal disk with $\partial \Sigma \subset \partial B_{C_5 R}$.

Lemma III.1.6. *If (y, s) satisfies* (III.1.1) *and $B_s(y) \subset B_{R/2}$, then the corresponding 2-valued graph over $D_R(0) \setminus D_s(p)$ (after a rotation) has*

$$\textit{separation } \geq C_3\, (s/R)^{\varepsilon}\, s/2 \textit{ over } \partial D_R(0)\,.$$

Proof. By the discussion around (III.1.1), we see that the separation $|w|$ is $\geq C_3\, s$ at $\partial D_s(p)$ and

$$|\nabla \log |w|| \leq \varepsilon / \rho_p \text{ on } D_{2R}(p) \setminus D_s(p)\,.$$

Since $D_s(p) \subset D_{R/2}(0)$, integrating gives

$$\text{(III.1.7)} \qquad \min_{\partial D_R(0)} |w| \geq \min_{D_{2R}(p) \setminus D_{R/2}(p)} |w| \geq C_3\, (s/(2R))^{\varepsilon}\, s\,. \qquad \square$$

Corollary III.1.8. *There exists $C_6 > 0$ so that if the pair $(0, s)$ satisfies* (III.1.1) *and for some $4C_4^2\, s < \ell < R$,*

$$\sup_{B_\ell(0) \cap \Sigma_-} |A|^2 \leq 5 C_1^2 s^{-2}\,,$$

then there exists a pair (z,r) *satisfying* (III.1.1) *with* $\Sigma_{z,r} \subset \Sigma_{0,\ell/2}$, *so that the separation at* $\partial D_\ell(0)$ *between the* 2*-valued graphs* Σ_0, Σ_z, *corresponding to* $(0,s)$, (z,r), *is* $\geq C_6\,(s/\ell)^\varepsilon\,\ell$, *and* $\Sigma_z \subset \Sigma_-$.

Proof. Set $(y_0,s_0) = (0,s)$ and let (y_i,s_i), Σ_i, u_i, and p_i be given by Corollary III.1.3. Let i_0 be the first i with

$$B_{C_4 s_{i_0}}(y_{i_0}) \setminus B_{\ell/2}(0) \neq \emptyset\,.$$

Set $(z,r) = (y_{i_0-1}, s_{i_0-1})$. It follows for $i < i_0$ that $B_{s_i}(y_i) \subset B_{\ell/2}(0)$ and $s_i \geq s/2$ since

$$\sup_{B_\ell(0)\cap\Sigma_-} |A|^2 \leq 5C_1^2 s^{-2}\,.$$

Hence, by Lemma III.1.6 (as in Corollary III.1.3), we get that Σ_i has separation

$$\geq \tilde{C}_3\,(s/\ell)^\varepsilon\, s_i/4$$

at $\partial D_\ell(0)$ for $i < i_0$. By (III.1.5),

$$\ell/4 \leq \sum_{i\leq i_0} C_4 s_i \leq (1+C_4)\sum_{i<i_0} C_4 s_i\,.$$

Since the Σ_i's are ordered by height, the separation at $\partial D_\ell(0)$ between Σ_0 and $\Sigma_z = \Sigma_{i_0-1}$ is

$$\geq \sum_{i<i_0} \tilde{C}_3\,(s/\ell)^\varepsilon\, s_i/4 \geq C_6\,(s/\ell)^\varepsilon\,\ell\,. \qquad \square$$

III.2. Stacking multi-valued graphs and Theorem 0.2

If (y,s) satisfies (III.1.1), then Σ_y is the corresponding 2-valued graph and $\Sigma_{y,-}$ the portion of Σ below Σ_y. Given $C > 8$, we will consider such pairs which in addition satisfy

$$\text{(III.2.1)} \qquad \sup_{B_{Cs}(y)\cap\Sigma_{y,-}} |A|^2 \leq 4\,|A|^2(y) = 4C_1^2 s^{-2}\,.$$

Using Section III.1, we show next that a pair $(0,s)$ satisfying (III.2.1) has a nearby pair with a definite height below Σ_0. In this section, $\Sigma \subset B_{C_5 R}$ is an embedded minimal disk with $\partial\Sigma \subset \partial B_{C_5 R}$.

PROPOSITION III.2.2 (see Figure 9). *There exist* $C, \bar{C} > 10\,C_4^2$ *and* $\delta > 0$ *so that if the pair* $(0,s)$ *satisfies* (III.2.1) *with* $s < R/\bar{C}$, $\Sigma_0 \subset \Sigma$ *is over* $D_R \setminus D_s$ (*without a rotation*), *and*

$$\nabla u((Rs)^{1/2}, 0) = 0\,,$$

then there is a pair (y,t) *satisfying* (III.2.1) *with*

$$y \in \mathbf{C}_\delta(0) \cap \Sigma_- \setminus B_{Cs/2}\,.$$

Proof. We will choose C large below and then set $\delta = \delta(C) > 0$ and $\bar{C} = \bar{C}(C)$. Note first that (since $\nabla u((Rs)^{1/2}, 0) = 0$) Corollary 1.14 of [CM7] gives for $s \leq \rho \leq (Rs)^{1/2}$ that

$$|\nabla u(\rho, \theta)| \leq C_a \, (\rho/s)^{-5/12} \, .$$

Integrating this, we get for $s \leq \rho \leq (Rs)^{1/2}$ that

$$\text{(III.2.3)} \qquad |u(\rho, \theta)| \leq s + C_a \int_s^\rho (\tau/s)^{-5/12} \, d\tau \leq (1 + 2C_a) \, (s/\rho)^{5/12} \, \rho \, .$$

Proposition I.0.11 gives $C_b(C_1, C)$ and a point $z_0 \in B_{C_b s} \cap \Sigma_- \setminus B_{4s}$ with

$$|A|^2(z_0) \geq 5 \, C^2 \, C_1^2 \, |z_0|^{-2} \, .$$

Define the set $\mathcal{A}$ by

$$\text{(III.2.4)} \qquad \mathcal{A} = \{ x \in B_{C_b s} \cap \Sigma_- \, | \, |A|^2(x) \geq 5 \, C^2 \, C_1^2 |x|^{-2} \} \, ,$$

(so $z_0 \in \mathcal{A}$) and let $x_0 \in \mathcal{A}$ satisfy $|x_0| = \inf_{x \in \mathcal{A}} |x|$. Consequently, by (III.2.1),

$$|A|^2 \leq 5 \, C_1^2 \, s^{-2} \text{ on } B_{|x_0|} \cap \Sigma_- \, ,$$
$$C s \leq |x_0| \leq C_b \, s \, .$$

An obvious extrinsic version of Lemma II.2.3 (cf. Theorem A.9) gives a pair (y, t) satisfying (III.2.1) with $B_{Ct}(y) \subset B_{|x_0|/2}(x_0)$. We can assume $|p| \geq 4|y|/5$.

Since $|A|^2 \leq 5 \, C_1^2 \, s^{-2}$ on $B_{|x_0|/2} \cap \Sigma_-$ and $(0, s)$ satisfies (III.2.1) and hence (III.1.1), it follows that Corollary III.1.8 (with $\ell = |p|$) gives a pair (z, r) also satisfying (III.1.1) with

$$\Sigma_z \subset \Sigma_- \text{ and } \Sigma_{z,r} \subset \Sigma_{0, \frac{|p|}{2}} \, .$$

Thus the separation between Σ_0 and Σ_z is at least $C_c \, (s/|y|)^\varepsilon \, |y|$ at p. However, since Σ is embedded, then Σ_y must be below both Σ_0 and Σ_z. Combining this with (III.2.3) gives

$$\text{(III.2.5)} \qquad \frac{|x_3|(y)}{|y|} \geq C_c \left(\frac{s}{|y|} \right)^\varepsilon - (1 + 2 \, C_a) \left(\frac{s}{|y|} \right)^{5/12} \, .$$

Since $C \leq 2|y|/s \leq 3C_b$, the proposition follows from (III.2.5) by choosing C sufficiently large and then setting $\bar{C} = \bar{C}(C, C_5)$ and $C_b = C_b(C)$ (where $\bar{C}$ is chosen so that $C_b \, s \leq (Rs)^{1/2}$). □

We next prove Theorem 0.2. Namely, iterating Proposition III.2.2, we show that if the curvature of an embedded minimal disk were large at a point, then it would be forced to grow out of the half-space it was assumed to lie in. First we need:

LEMMA III.2.6. *Given C and $\delta > 0$, there exists $\varepsilon_1 > 0$ so that the following holds*:

Let $\Sigma \subset B_{2r_0} \cap \{x_3 > 0\}$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_{2r_0}$. If

$$\sup_{\Sigma \cap \{x_3 \leq \delta\, r_0\}} |A|^2 \leq C\, r_0^{-2}\,,$$

then for any component Σ' of $B_{r_0} \cap \Sigma$ which intersects $B_{\varepsilon_1 r_0}$,

$$\sup_{\Sigma'} |A|^2 \leq r_0^{-2}\,.$$

Proof. If $y \in B_{r_0} \cap \Sigma \cap \{x_3 \leq \delta\, r_0/4\}$, then

$$\sup_{\Sigma_{y,\delta r_0/2}} |\nabla x_3|^2 \leq C\, x_3^2(y)\, \delta^{-2}\, r_0^{-2}$$

(by the gradient estimate) and hence $\Sigma_{y,\delta r_0/2}$ is a graph for $x_3(y)/(\delta\, r_0)$ small; cf. Lemma A.3. The lemma follows by applying this to a chain of balls as in the proof of Lemma 2.10 in [CM8] or the theorem in [CM10]. □

Let $C_1, \ldots, C_6$ be as above and $\delta, C, \bar{C}$ be from Proposition III.2.2.

Proof of Theorem 0.2. By Lemma III.2.6 (and scaling), it suffices to find $d > 0$ and $\hat{C} > 1$ so that if $\Sigma \subset B_{4C_5\hat{C}R} \cap \{x_3 > 0\}$ and $\partial\Sigma \subset \partial B_{4C_5\hat{C}R}$, then

$$\text{(III.2.7)} \qquad \sup_{B_{dR} \cap \Sigma} |A|^2 \leq 4\, C^2\, C_1^2\, (dR)^{-2}\,.$$

Suppose that (III.2.7) fails; we will get a contradiction. An obvious extrinsic version of Lemma II.2.3 gives a pair (y_0, s_0) satisfying (III.2.1) with $B_{Cs_0}(y_0) \subset B_{2dR}$. Let Σ_0 be the corresponding N-valued graph of u_0 over $D_{\hat{C}R} \setminus D_{s_0}(p_0)$ and Σ_- the portion of Σ below Σ_0.

To apply Proposition III.2.2, we will need that if (y, s) satisfies (III.2.1) with $y \in B_{2R} \cap \Sigma_-$ (where Σ_y is a graph of u over $D_{\hat{C}R} \setminus D_s(p)$), then

$$\text{(III.2.8)} \qquad s \leq R/\bar{C} \text{ and } |\nabla u((\hat{C}Rs)^{1/2}, 0)| < \delta/4\,.$$

To see (III.2.8), observe first that the sublinear growth proven in Proposition II.2.12 of [CM3] applies to the positive function u_0 so we get that

$$\sup_{D_{6R}} u_0 \leq 2dR(6/d)^{\varepsilon} \leq 12\, R\, d^{1-\varepsilon}\,.$$

It follows that

$$B_{6R} \cap \Sigma_- \subset \{0 < x_3 < 12\, R\, d^{1-\varepsilon}\}\,.$$

This bound on the height implies a bound on the radius s of the blow up pair

$$s \leq C_a\, R\, d^{1-\varepsilon}\,.$$

Likewise, this height bound and the gradient estimate (since the height function is harmonic on a minimal surface) give

$$\sup_{\partial D_{4R}} |\nabla u| \leq C_b \, d^{1-\varepsilon} \, .$$

Lemma 1.8 of [CM7] and the mean value inequality (as in corollary 1.14 of [CM7]) give

$$| \operatorname{Hess}_u | \leq C_c (\hat{C} R)^{-5/12} \, \rho^{-7/12}$$

for $(\hat{C} R s)^{1/2} \leq \rho \leq \hat{C} R$. Combining these at $\rho = (\hat{C} R s)^{1/2}$ and $\theta = 0$, we get that

$$\text{(III.2.9)} \qquad \begin{aligned} |\nabla u| &\leq C_b \, d^{1-\varepsilon} + C_c (\hat{C} R)^{-5/12} \int_0^{8R} t^{-7/12} \, dt \\ &= C_b \, d^{1-\varepsilon} + C_c' \hat{C}^{-5/12} \, . \end{aligned}$$

In particular, for $d > 0$ small and $\hat{C}$ large, (III.2.9) gives (III.2.8).

Repeatedly applying Proposition III.2.2 (using (III.2.8)) gives (y_{i+1}, s_{i+1}) satisfying (III.2.1) with

$$y_{i+1} \in \mathbf{C}_{\delta/2}(y_i) \cap \Sigma_- \setminus B_{C s_i / 2}(y_i) \, .$$

After choosing $d > 0$ even smaller, it follows that the y_i's must leave the half-space before they leave B_R. □

Proof of Corollary 0.4. Using $\Sigma_1 \cup \Sigma_2$ as a barrier, the existence theory of [MeYa] and a linking argument (cf. Lemma II.1.1) give a stable surface

$$\Gamma \subset B_{c r_0} \setminus (\Sigma_1 \cup \Sigma_2)$$

with $\partial \Gamma \subset \partial B_{c r_0}$ and $B_{\varepsilon r_0} \cap \Gamma \neq \emptyset$. Estimates for stable surfaces give a graphical component of $B_{2 r_0} \cap \Gamma$ which intersects $B_{\varepsilon r_0}$. The corollary now follows from Theorem 0.2. □

Part IV. The short connecting curves and Theorem I.0.10

We first combine Lemmas II.1.1 and II.2.1 to see that any curve in Σ which intersects both above and below a multi-valued graph (with a curvature bound on an intrinsic ball) connects to it in a fixed intrinsic ball:

Corollary IV.0.10. *Given C_1, there exists $C_4 > 1$ so that the following holds*:

Let Σ, Σ_d, E_0, and r_0 be as in Lemma II.1.1. *If a curve $\eta \subset B_{2 r_0} \cap \Sigma$ connects points in $\partial B_{2 r_0}$ above and below E_0 and*

$$\sup_{\mathcal{B}_{C_4 r_0}} |A|^2 \leq 4 \, C_1^2 \, r_0^{-2} \, ,$$

then $\eta \subset \mathcal{B}_{C_4 r_0}$.

Proof. Let $\Sigma_{2r_0}(\eta)$ be the component of $B_{2r_0} \cap \Sigma$ containing η. By the maximum principle, we have that $\Sigma_{2r_0}(\eta)$ is a disk and hence $\partial \Sigma_{2r_0}(\eta)$ must pass through E_0 (to connect the points on opposite sides of E_0). Hence, by Lemma II.1.1, we get

$$\Sigma_{2r_0}(\eta) \subset \Sigma_{0,C_s r_0}\,.$$

Finally, Lemma II.2.1 yields $\Sigma_{0,C_s r_0} \subset \mathcal{B}_{C_4 r_0}$, proving the corollary. □

Proof of Theorem I.0.10. Fix $\varepsilon > 0$ with

$$\varepsilon < \min\{\varepsilon_0, \varepsilon_s\}$$

(ε_0 is given by Proposition II.1.3 and ε_s is from Lemma II.1.1). Choose N_0 and R_0 large so that Proposition II.1.3 gives "the other half" Σ_2. If Σ_1 comes from an intrinsic blow up point, then it follows from Lemma II.1.1, that there are short curves connecting Σ_1 and Σ_2. While it is *a priori* not clear that every multi-valued graph arises this way, Theorem II.2.2 implies that every multi-valued graph is intrinsically near one of these. We use this below to produce the short curves σ_θ in general.

Suppose that no σ_θ with length $\leq C$ exists for some θ; we get points y_1 and y_2 with

$$\begin{aligned} y_i &\in \{x_1^2 + x_2^2 = 1\} \cap \Sigma_i\,,\\ C &< \operatorname{dist}_\Sigma(y_1, y_2)\,, \end{aligned}$$

and so that y_1 and y_2 are in consecutive sheets of Σ (i.e., y_1 and y_2 can be connected by a segment parallel to the x_3-axis which does not otherwise intersect Σ). See Figure 20. We will get a contradiction from this for C large.

Since $\partial \Sigma_{y_1,4}$ is a simple closed curve, it must pass through $E \setminus \Sigma_1$. See Figure 21. Therefore, since Σ_2 is the only other part of Σ in E, we can connect Σ_1 and Σ_2 by a curve in $\partial \Sigma_{y_1,4}$. Connecting the endpoints of this curve to y_1 and y_2 gives a curve $\eta \subset \Sigma_{y_1,4}$ from y_1 to y_2. Since $\mathcal{B}_4(y_i)$ is not a graph, we have

$$\sup_{\mathcal{B}_4(y_i)} |A|^2 \geq C_0 > 0\,.$$

Let C_1 and C_2 (depending on ε and some fixed $N > 6$) be given by Theorem II.2.2 and $C_4 = C_4(C_1)$ given by Corollary IV.0.10. Lemma II.2.3 gives pairs (z_i, s_i) satisfying (II.2.4) with

$$\mathcal{B}_{C_4 s_i}(z_i) \subset \mathcal{B}_{C'}(y_i)$$

where C' does not depend on C. Let $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$ be the multi-valued graphs given by Theorem II.2.2 and $\hat{E}_i$ the regions between the sheets. Since

$$\operatorname{dist}_\Sigma(z_i, \hat{\Sigma}_i) \leq 2\, s_i\,,$$

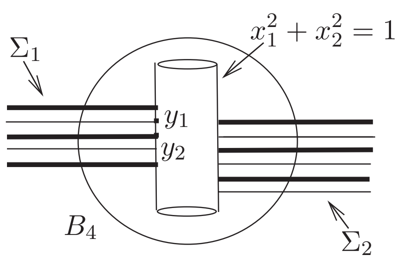

Figure 20: If Theorem I.0.10 fails, then there are points $y_1 \in \Sigma_1$ and $y_2 \in \Sigma_2$ in consecutive sheets which are intrinsically far apart.

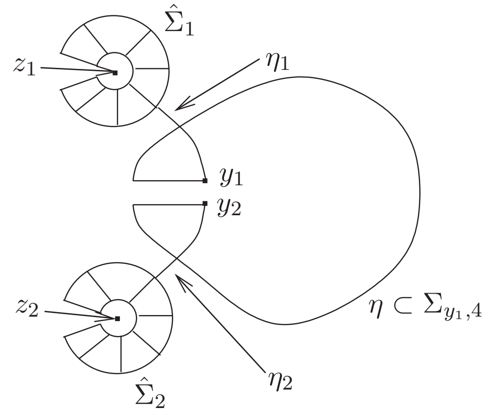

Figure 21: Proof of Theorem I.0.10. The blowup points z_1, z_2 and the corresponding multi-valued graphs $\hat{\Sigma}_1$, $\hat{\Sigma}_2$ and the curves η_i connecting y_i with $\hat{\Sigma}_i$.

we can choose curves η_i from y_i to $\mathcal{B}_{2s_i}(z_i) \cap \hat{\Sigma}_i$ with length $\leq C'$. Combining Corollary IV.0.10, $\text{Length}(\eta_i) \leq C'$, and $\text{dist}_\Sigma(y_1, y_2) > C$, we see easily that, for C large, η_1 intersects only one side of $\hat{E}_2 \cup B_{2s_2}(z_2)$; similarly, η_2 intersects only one side of $\hat{E}_1 \cup B_{2s_1}(z_1)$.

We will next find a third pair (z_3, s_3) satisfying (II.2.4) which is between $\hat{E}_1 \cup B_{2s_1}(z_1)$ and $\hat{E}_2 \cup B_{2s_2}(z_2)$ but which is intrinsically far from η_1 and η_2; Corollary IV.0.10 will then give a contradiction. By combination of

- $\text{dist}_\Sigma(\eta_1, \eta_2) > C - 2C'$,
- η_1 intersects only one side of $\hat{E}_2 \cup B_{2s_2}(z_2)$,
- η_2 intersects only one side of $\hat{E}_1 \cup B_{2s_1}(z_1)$, and
- $\eta \subset \Sigma_{y_1,4}$ connects y_1, y_2,

it is easy to see that there is a point $y_3 \in \Sigma_{y_1,4}$ with

$$\text{dist}_\Sigma(y_3, \{\eta_1, \eta_2\}) > (C - 2C')/2$$

and so that

$$y_3 \text{ is between } \hat{E}_1 \cup B_{2s_1}(z_1) \text{ and } \hat{E}_2 \cup B_{2s_2}(z_2)\,.$$

(This last condition means that there is a curve η_{y_3} from $B_{2s_1}(z_1)$ to $B_{2s_2}(z_2)$ so that $y_3 \in \eta_{y_3}$ and η_{y_3} intersects only one side of each of $\hat{E}_1 \cup B_{2s_1}(z_1)$ and $\hat{E}_2 \cup B_{2s_2}(z_2)$.) As above, Lemma II.2.3 gives a pair (z_3, s_3) satisfying (II.2.4) with $\mathcal{B}_{C_4 s_3}(z_3) \subset \mathcal{B}_{C'}(y_3)$ and then Theorem II.2.2 gives corresponding $\hat{\Sigma}_3$, $\hat{E}_3$. Since C' does not depend on C, we can assume that

$$\text{(IV.0.11)} \qquad \text{dist}_\Sigma(z_3, \{\eta_1, \eta_2\}) > C/4 .$$

It follows easily from Corollary IV.0.10 that $\hat{E}_3$ is between $\hat{E}_1$ and $\hat{E}_2$ (since $\hat{\Sigma}_3$ is close to y_3 and y_3 is far from $\hat{\Sigma}_1, \hat{\Sigma}_2$). Moreover, it is easy to see that at least one of η_1, η_2 must intersect both sides of $\hat{E}_3 \cup B_{2s_3}(z_3)$ and, therefore, Corollary IV.0.10 gives

$$\text{(IV.0.12)} \qquad \text{dist}_\Sigma(\hat{\Sigma}_3, \{\eta_1, \eta_2\}) \leq C''$$

(C'' independent of C). For C large, (IV.0.11) contradicts (IV.0.12), giving the theorem. □

Appendix A.
One-sided Reifenberg condition and curvature estimates

We will show here curvature estimates for minimal hypersurfaces, $\Sigma^{n-1} \subset M^n$, which on all sufficiently small scales lie on one side of, but come close to, a hypersurface with small curvature. Such a minimal hypersurface is said to satisfy the *one-sided Reifenberg condition.* Note that no assumption on the topology is made. Inspired by the classical Reifenberg condition (cf. [ChC] and references therein) we make the definition:

Definition A.1. A subset, Γ, of M^n satisfies the (δ, r_0)-*one-sided Reifenberg condition* at $x \in \Gamma$ if for every $0 < \sigma \leq r_0$ and every $y \in B_{r_0 - \sigma}(x) \cap \Gamma$, there is a connected hypersurface, $L^{n-1}_{y,\sigma}$, with $\partial L_{y,\sigma} \subset \partial B_\sigma(y)$,

$$\text{(A.2)} \qquad B_{\delta\sigma}(y) \cap L_{y,\sigma} \neq \emptyset, \ \sup_{B_\sigma(y) \cap L} |A_L|^2 \leq \delta^2 \, \sigma^{-2} ,$$

and the component of $B_\sigma(y) \cap \overline{\Gamma}$ through y lies on one side of $L_{y,\sigma}$.

LEMMA A.3. *There exist $r_1(i_0, k, n) > 0$, $0 < \varepsilon_0 < 1$, and $C = C(n)$ so that for $\varepsilon \leq \varepsilon_0$ and $r_0 \leq r_1$ the following holds*:

Let $z \in \Sigma^{n-1} \subset B_{r_0} = B_{r_0}(z) \subset M^n$ be an embedded minimal hypersurface with $\partial\Sigma \subset \partial B_{r_0}$. If there is a connected hypersurface, L^{n-1}, with $\partial L \subset \partial B_{r_0}$, $B_{\varepsilon r_0} \cap L \neq \emptyset$,

$$\text{(A.4)} \qquad \sup_{B_{r_0} \cap L} |A_L|^2 \leq \varepsilon^2 \, r_0^{-2} ,$$

$$\text{(A.5)} \qquad \sup_{B_{r_0} \cap \Sigma} |A|^2 \leq \varepsilon_0^2 \, r_0^{-2} ,$$

and $B_{r_0} \cap \Sigma$ lies on one side of L, then

$$|A(z)|^2 \leq C \, \varepsilon^2 \, r_0^{-2} .$$

Proof. We will prove this for $B_{r_0} = B_{r_0}(0) \subset \mathbf{R}^n$ ($z = 0$, $r_1 = \infty$); the general case is similar (cf. [CM1]). Choose $\varepsilon_0 > 0$ so that the following holds:

If $\mathcal{B}_{2s}(y) \subset \Sigma$, $s \sup_{\mathcal{B}_{2s}(y)} |A| \leq 4\,\varepsilon_0$, and $t \leq 9s/5$, then $\in \Sigma_{y,t}$ is a graph over $T_y\Sigma$ with gradient $\leq t/s$ and

$$\inf_{y' \in \mathcal{B}_{2s}(y)} |y' - y| / \operatorname{dist}_\Sigma(y, y') > 9/10\,. \tag{A.6}$$

Using $B_{\varepsilon r_0} \cap L \neq \emptyset$, let $L_{\frac{r_0}{2}}$ be a component of $B_{\frac{r_0}{2}} \cap L$ containing some $y_L \in B_{\varepsilon r_0} \cap L$. By (A.4) and (A.6), we have

$$L_{\frac{r_0}{2}} \subset \mathcal{B}_{\frac{3r_0}{4}}(y_L)\,.$$

Hence, by (A.4), we can rotate $\mathbf{R}^n$ so that $L_{\frac{r_0}{2}}$ is a graph over $\{x_n = 0\}$ with $|\nabla_L x_n| \leq \varepsilon$ and

$$|x_n(L)| \leq 4\,\varepsilon\, r_0\,.$$

Since $L \cap \Sigma = \emptyset$, the function $x_n + 4\,\varepsilon\, r_0 > 0$ is harmonic on $\mathcal{B}_{\frac{r_0}{4}} \subset \Sigma$. By (A.5), the Harnack inequality (and $0 \in \Sigma$) yields $C = C(n)$ so that

$$0 < \sup_{\mathcal{B}_{\frac{r_0}{6}}} (x_n + 4\varepsilon\, r_0) \leq C \inf_{\mathcal{B}_{\frac{r_0}{6}}} (x_n + 4\,\varepsilon\, r_0) \leq 4\, C\, \varepsilon\, r_0\,. \tag{A.7}$$

Since $\mathcal{B}_{\frac{r_0}{2}}$ is a graph with bounded gradient, elliptic estimates give

$$\int_{\mathcal{B}_{\frac{r_0}{8}}} |A|^2 \leq C'\, r_0^{-4} \int_{\mathcal{B}_{\frac{r_0}{6}}} |x_n|^2\,, \tag{A.8}$$

where $C' = C'(n)$. Combining (A.7) and (A.8), the lemma follows from the mean value inequality since Simons' inequality (cf. [CM1]) gives

$$\Delta |A|^2 \geq -2\, |A|^4\,. \qquad \square$$

THEOREM A.9 (Curvature estimate). *There exist $\varepsilon_1(i_0,k,n)$ and $r_1(i_0,k,n)$ > 0 so that the following holds*:

If $r_0 \leq r_1$, $\Sigma^{n-1} \subset B_{r_0} = B_{r_0}(x) \subset M^n$ is an embedded minimal hypersurface with $\partial\Sigma \subset \partial B_{r_0}$, and Σ satisfies the (ε_1, r_0)-one-sided Reifenberg condition at x, then for $0 < \sigma \leq r_0$,

$$\sup_{B_{r_0 - \sigma} \cap \Sigma} |A|^2 \leq \sigma^{-2}\,.$$

Proof. Take $r_1 > 0$ as in Lemma A.3, and set

$$F = (r_0 - r)^2\, |A|^2\,.$$

Since $F \geq 0$, $F|\partial B_{r_0} \cap \Sigma = 0$, and Σ is compact, F achieves its supremum at $y \in \partial B_{r_0-\sigma} \cap \Sigma$ with $0 < \sigma \leq r_0$. If $F \leq 1$, the theorem follows trivially. Hence, we may suppose

$$F(y) = \sup_{B_{r_0}\cap\Sigma} F > 1\,.$$

With $\varepsilon_0 \leq 1$ as in Lemma A.3, define $s > 0$ by

$$s^2\,|A(y)|^2 = \varepsilon_0^2/4\,.$$

Since $F(y) = \sigma^2\,|A(y)|^2 > 1$ and $\varepsilon_0 \leq 1$, we have $2s < \sigma$. Since $F(y) > 1$,

$$\text{(A.10)} \qquad \sup_{B_s(y)\cap\Sigma} \left(\frac{\sigma}{2}\right)^2 |A|^2 \leq \sup_{B_{\frac{\sigma}{2}}(y)\cap\Sigma} \left(\frac{\sigma}{2}\right)^2 |A|^2 \leq \sup_{B_{\frac{\sigma}{2}}(y)\cap\Sigma} F = \sigma^2\,|A(y)|^2\,.$$

Multiplying (A.10) by $4\,s^2/\sigma^2$ gives $\sup_{B_s(y)\cap\Sigma} s^2\,|A|^2 \leq \varepsilon_0^2$. Hence, we have the (ε_1, r_0)-one-sided Reifenberg assumption, Lemma A.3 contradicting the choice of s if $C\,\varepsilon_1^2 < \varepsilon_0^2/4$. Therefore, $F \leq 1$ for this ε_1, and the theorem follows. □

Letting $r_0 \to \infty$ in Theorem A.9 gives the following Bernstein-type result:

COROLLARY A.11. *There exists $\varepsilon(n) > 0$ so that any connected properly embedded minimal hypersurface satisfying the (ε, ∞)-one-sided Reifenberg condition is a hyperplane.*

We close by giving a condition which implies the one-sided Reifenberg condition. Its proof (left to the reader) relies on a simple barrier argument (as in the proof of Corollary 0.4).

LEMMA A.12. *There exist $\varepsilon_0(i_0, k)$, $r_1(i_0, k) > 0$, and $c(i_0, k) \geq 1$ so that the following holds*:

Let $\Sigma \subset B_{r_0} = B_{r_0}(x) \subset M^3$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_{r_0}$. If $r_0 \leq r_1$ and for some $\varepsilon < \varepsilon_0$, all $\sigma < r_0$ and all $y \in B_{r_0-\sigma} \cap \Sigma$ there is a minimal surface

$$\Sigma_{y,\sigma} \subset B_\sigma(y) \setminus \Sigma$$

with $\partial\Sigma_{y,\sigma} \subset \partial B_\sigma(y)$ and

$$\Sigma_{y,\sigma} \cap B_{\varepsilon\,\sigma}(y) \neq \emptyset\,,$$

then Σ satisfies the $(c\,\varepsilon, r_0)$-one-sided Reifenberg condition at x.

Appendix B. Laminations

A codimension one *lamination* on a 3-manifold M^3 is a collection $\mathcal{L}$ of smooth disjoint surfaces (called leaves) such that $\cup_{\Lambda\in\mathcal{L}}\Lambda$ is closed. Moreover, for each $x \in M$ there exists an open neighborhood U of x and a coordinate chart, (U, Φ), with $\Phi(U) \subset \mathbf{R}^3$ so that in these coordinates the leaves in $\mathcal{L}$ pass through $\Phi(U)$ in slices of the form

$$(\mathbf{R} \times \{t\}) \cap \Phi(U)\,.$$

A *foliation* is a lamination for which the union of the leaves is all of M and a *minimal lamination* is a lamination whose leaves are minimal. Finally, a sequence of laminations is said to converge if the corresponding coordinate maps converge. Note that any (compact) embedded surface (connected or not) is a lamination.

PROPOSITION B.1. *Let M^3 be a fixed 3-manifold. If $\mathcal{L}_i \subset B_{2R}(x) \subset M$ is a sequence of minimal laminations with uniformly bounded curvatures (where each leaf has boundary contained in $\partial B_{2R}(x)$), then a subsequence, $\mathcal{L}_j$, converges in the C^α topology for any $\alpha < 1$ to a (Lipschitz) lamination $\mathcal{L}$ in $B_R(x)$ with minimal leaves.*

Proof. For convenience, we will assume that each lamination $\mathcal{L}_i$ has only finitely many leaves where the number of leaves may depend on i. This is all that is needed in the application of this proposition anyway. Fix $x_0 \in B_R(x)$. The proposition will follow once we construct uniform coordinate charts Φ_i on a ball $B_{r_0} = B_{r_0}(x_0)$, where $4r_0 \leq R$ is to be chosen.

By assumption, there exists C so that for each i and every $\Lambda \in \mathcal{L}_i$,

$$\sup_{B_{4r_0} \cap \Lambda} |A|^2 \leq C\, r_0^{-2}\,.$$

Replacing $r_0 > 0$ with a smaller radius, we may assume that $C > 0$ and $r_0 \sqrt{k}$ are as small as we wish and $r_0 < \frac{i_0}{2}$ (i_0 being the injectivity radius and k a bound for the curvature of M in B_{4r_0}). In fact, if (x_1, x_2, x_3) are exponential normal coordinates centered at x_0 on B_{r_0}, then

$$\cup_{\Lambda \in \mathcal{L}_i} B_{r_0} \cap \Lambda$$

gives a sequence of disconnected small curvature surfaces in these coordinates. By standard estimates for normal coordinates, the curvature is also small with respect to the Euclidean metric. Going to a further subsequence (possibly with r_0 even smaller), for each i every sheet of

$$\cup_{\Lambda \in \mathcal{L}_i} B_{2r_0}(0) \cap \Lambda$$

is a graph with small gradient over a subset of the $\mathbf{R}^2 \times \{0\}$ plane containing a ball of radius r_0 centered at the origin.

We claim that, in this ball, the sequence of laminations converges in the C^α topology to a lamination for any $\alpha < 1$. The coordinate chart Φ required by the definition of a lamination will be given by the Arzela-Ascoli theorem as a limit of a sequence of bi-Lipschitz maps

$$\Phi_i : (x_j)_j \to \mathbf{R}^3$$

with bi-Lipschitz constants close to one and defined on a slightly smaller concentric ball B_{sr_0} for some $s > 0$ to be determined. Furthermore, we will show that for each i fixed

$$\Phi_i^{-1}(B_{sr_0} \cap \cup_{\Lambda \in \mathcal{L}_i} \Lambda)$$

is the union of planes which are each parallel to $\mathbf{R}^2 \times \{0\} \subset \mathbf{R}^3$; cf. [So].

Set the map Φ_i by letting

$$\Phi_i^{-1}(y_1, y_2, y_3) = (y_1, y_2, \phi_i(y_1, y_2, y_3))\,,$$

where ϕ_i is defined as follows: Order the sheets of $B_{2r_0}(0) \cap_{\Lambda \in \mathcal{L}_i} \Lambda$ as $\Lambda_{i,k}$ for $k = 1, \cdots$ by increasing values of x_3 and let $\Lambda_{i,k}$ be the graph of the function $f_{i,k}$ over (part of) the $\mathbf{R}^2 \times \{0\}$ plane. The domain of $f_{i,k}$ contains the ball of radius r_0 around the origin in the $\mathbf{R}^2 \times \{0\}$ plane. With slight abuse of notation, we will also denote balls in $\mathbf{R}^2 \times \{0\}$ with radius t and center 0 by B_t. Set

$$w_{i,k} = f_{i,k+1} - f_{i,k}\,.$$

In the next equation, Δ, ∇, and div will be with respect to the Euclidean metric on $\mathbf{R}^2 \times \{0\}$. By a standard computation (cf. [Si, (7) on p. 333] or Chapter 1 of [CM1]), we have

$$\Delta w_{i,k} = \operatorname{div}(a\, \nabla w_{i,k}) + b\, \nabla w_{i,k} + c\, w_{i,k}\,, \tag{B.2}$$

where

- a is a matrix valued function.
- b is a vector valued function.
- c is a function.

Although a, b, and c depend on i, their scale invariant norms are small when C and $\sqrt{k}\, r_0$ are. By (B.2), the Schauder estimates and Harnack inequality (e.g., 6.2 and 8.20 of [GiTr]) applied to the positive function $w_{i,k}$ give

$$s r_0 \sup_{B_{s\, r_0}} |\nabla w_{i,k}| \leq C \sup_{B_{2s\, r_0}} w_{i,k} \leq \exp(\varepsilon_a\, s^{\beta}) \inf_{B_{2s\, r_0}} w_{i,k}\,, \tag{B.3}$$

where ε_a and $\beta > 0$ depend on the scale invariant norms of a, b, and c. Set $\mathbf{M}_{i,k} = f_{i,k}(0)$. In the region

$$\{(y_1, y_2, y_3) \in B_{r_0} \times [\mathbf{M}_{i,k}, \mathbf{M}_{i,k+1}]\}\,,$$

define the function ϕ_i by

$$\phi_i(y_1, y_2, y_3) = f_{i,k}(y_1, y_2) + \frac{y_3 - \mathbf{M}_{i,k}}{\mathbf{M}_{i,k+1} - \mathbf{M}_{i,k}} w_{i,k}(y_1, y_2)\,. \tag{B.4}$$

Hence

$$\nabla \phi_i = \nabla f_{i,k} + \frac{y_3 - \mathbf{M}_{i,k}}{\mathbf{M}_{i,k+1} - \mathbf{M}_{i,k}} \nabla w_{i,k} + \frac{w_{i,k}}{\mathbf{M}_{i,k+1} - \mathbf{M}_{i,k}} \frac{\partial}{\partial y_3} . \tag{B.5}$$

It follows easily from (B.3) and (B.5) that for each i the map Φ_i restricted to $B_{sr_0}(0) \subset \mathbf{R}^3$ is bi-Lipschitz with bi-Lipschitz constant close to one if s is sufficiently small. By the Arzela-Ascoli theorem, a subsequence of Φ_i converges in the C^α topology for any $\alpha < 1$ to a Lipschitz coordinate chart Φ with the properties that are required. The leaves in B_{r_0} are $C^{1,\alpha}$ limits of minimal graphs with bounded gradient, and hence minimal by elliptic regularity. □

Trivial examples show that the Lipschitz regularity above is optimal.

Appendix C. A standard consequence of the maximum principle

Using the maximum principle and the convexity of small extrinsic balls, we can bound the topology of the intersection of a minimal surface with a ball:

LEMMA C.1. *Let $\Sigma^2 \subset M^n$ be an immersed minimal surface with $\partial\Sigma \subset \partial B_{r_0}(x)$. Suppose that $\mathrm{K}_{M^n} \leq k$ and the injectivity radius of $M \geq i_0$. If $r_0 < \min\{\frac{i_0}{4}, \frac{\pi}{4\sqrt{k}}\}$, $B_t(y) \subset B_{r_0}(x)$, and $\gamma \subset B_t(y) \cap \Sigma$ is a closed one-cycle homologous to zero in $B_{r_0}(x) \cap \Sigma$, then γ is homologous to zero in $B_t(y) \cap \Sigma$.*

Proof. Apply the maximum principle to the function

$$f = \operatorname{dist}^2_M(y, \cdot)$$

on the 2-current that γ bounds. □

By Lemma C.1, if $y \in B_t(x) \cap \Sigma$ is connected, then

$$\chi(B_s(y) \cap \Sigma) \geq \chi(B_t(x) \cap \Sigma)$$

whenever we have

$$s + \operatorname{dist}_M(x, y) \leq t < \min\{\frac{i_0}{4}, \frac{\pi}{4\sqrt{k}}\} .$$

(The Euler characteristic is monotone.)

D. A generalization of Proposition II.1.3

In [CM6], the next proposition is needed when we deal with the analog of the genus one helicoid (cf. [HoKrWe]) where Σ (as above (II.1.2)) is not a disk. The *genus* of a surface Σ ($\operatorname{gen}(\Sigma)$) is the genus of the closed surface obtained by adding a disk to each boundary circle.

PROPOSITION D.1. *There exist* C_0 *and* ε_0 *so that if* $0 \in \Sigma$, $\partial\Sigma$ *is connected,* $\text{gen}(\Sigma) = \text{gen}(\Sigma_{0,r_1})$, $R \geq C_0\, r_1$, *and*

$$\varepsilon_0 \geq \varepsilon\,,$$

then $E_1 \cap \Sigma \setminus \Sigma_1$ *is an* (*oppositely oriented*) N*-valued graph* Σ_2.

Proof. Note that, by the maximum principle and elementary topology (as in part I of [CM5]), we have that $\Sigma \setminus \Sigma_{0,t}$ is an annulus for $r_1 \leq t < 4R$. The proof now follows that of Proposition II.1.3.

First, (a slight extension of) the "estimate between the sheets" given in theorem III.2.4 of [CM3] gives ε_0 so that $E_1 \cap \Sigma$ is locally graphical (this extension uses the fact that $\Sigma \setminus \Sigma_{0,r_1}$ is an annulus instead of that Σ is a disk; the proof of this extension is outlined in appendix A of [CM8]). As before, we get the second (oppositely oriented) multi-valued graph $\Sigma_2 \subset \Sigma$.

Second, we argue by contradiction to show that there are no other components of $E_1 \cap \Sigma$. Fix σ_1 and σ_2 as before. The proof of Lemma II.1.1 applies virtually without change (since at least one of Σ_a and Σ_b must be a disk), so σ_1 and σ_2 connect in $\Sigma_{0,C_s r_1}$. Hence, a curve

$$\sigma_0 \subset \partial \Sigma_{0,C_s r_1}$$

connects σ_1 and σ_2. Replace σ_i with $\sigma_i \setminus B_{C_s r_1}$, so that

$$\sigma_0 \cup \sigma_1 \cup \sigma_2 \subset \Sigma \setminus \Sigma_{0,r_1}$$

is a simple curve and

$$\partial(\sigma_0 \cup \sigma_1 \cup \sigma_2) \subset \partial \Sigma_{0,R}\,.$$

Let $\hat{\Sigma}$ be the component of

$$\Sigma_{0,R} \setminus (\sigma_0 \cup \sigma_1 \cup \sigma_2)$$

which does not intersect Σ_{0,r_1}. It follows that $\hat{\Sigma}$ has genus zero and connected boundary; i.e., it is a disk. Solve as above for the stable disk Γ with $\partial\Gamma = \partial\hat{\Sigma}$ so that Γ contains two disjoint $(N/2-1)$-valued graphs in E_1 which spiral together. For R/r_1 large, Proposition I.0.11 gives the point of large curvature, contradicting the curvature estimate for stable surfaces. □

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