Classification of local conformal nets.
Case $c < 1$

By Yasuyuki Kawahigashi and Roberto Longo*

Dedicated to Masamichi Takesaki on the occasion of his seventieth birthday

Abstract

We completely classify diffeomorphism covariant local nets of von Neumann algebras on the circle with central charge c less than 1. The irreducible ones are in bijective correspondence with the pairs of A-D_{2n}-$E_{6,8}$ Dynkin diagrams such that the difference of their Coxeter numbers is equal to 1.

We first identify the nets generated by irreducible representations of the Virasoro algebra for $c < 1$ with certain coset nets. Then, by using the classification of modular invariants for the minimal models by Cappelli-Itzykson-Zuber and the method of α-induction in subfactor theory, we classify all local irreducible extensions of the Virasoro nets for $c < 1$ and infer our main classification result. As an application, we identify in our classification list certain concrete coset nets studied in the literature.

1. Introduction

Conformal field theory on S^1 has been extensively studied in recent years by different methods with important motivation coming from various branches of theoretical physics (two-dimensional critical phenomena, holography, . . .) and mathematics (quantum groups, subfactors, topological invariants in three dimensions, . . .).

In various approaches to the subject, it is unclear whether different models are to be regarded as equivalent or to contain the same physical information. This becomes clearer by considering the operator algebra generated by smeared fields localized in a given interval I of S^1 and taking its closure $\mathcal{A}(I)$ in the weak operator topology. The relative positions of the various von Neumann

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algebras $\mathcal{A}(I)$, namely the net $I \to \mathcal{A}(I)$, essentially encode all the structural information, in particular the fields can be constructed out of a net [18].

One can describe local conformal nets by a natural set of axioms. The classification of such nets is certainly a well-posed problem and obviously one of the basic ones of the subject. Note that the isomorphism class of a given net corresponds to the Borchers' class for the generating field.

Our aim in this paper is to give a first general and complete classification of local conformal nets on S^1 when the central charge c is less than 1, where the central charge is the one associated with the representation of the Virasoro algebra (or, in physical terms, with the stress-energy tensor) canonically associated with the irreducible local conformal net, as we will explain.

Haag-Kastler nets of operator algebras have been studied in algebraic quantum field theory for a long time (see [29], for example). More recently, (irreducible, local) conformal nets of von Neumann algebras on S^1 have been studied; see [8], [12], [13], [18], [19], [21], [26], [27], [66], [67], [68], [69], [70]. Although a complete classification seems to be still out of reach, we will take a first step by classifying the discrete series.

In general, it is not clear what kinds of axioms we should impose on conformal nets, beside the general ones, in order to obtain an interesting mathematical structure or classification theory. A set of conditions studied by us in [40], called *complete rationality*, selects a basic class of nets. Complete rationality consists of the following three requirements:

1. Split property.
2. Strong additivity.
3. Finiteness of the Jones index for the 2-interval inclusion.

Properties 1 and 2 are quite general and well studied (see e.g. [16], [27]). The third condition means the following. Split the circle S^1 into four proper intervals and label their interiors by I_1, I_2, I_3, I_4 in clockwise order. Then, for a local net $\mathcal{A}$, we have an inclusion

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))',$$

the "2-interval inclusion" of the net; its index, called the *μ-index* of $\mathcal{A}$, is required to be finite.

Under the assumption of complete rationality, we have proved in [40] that the net has only finitely many inequivalent irreducible representations, all have finite statistical dimensions, and the associated braiding is nondegenerate. That is, irreducible Doplicher-Haag-Roberts (DHR) endomorphisms of the net (which basically corresponds to primary fields) produce a modular tensor category in the sense of [62]. Such finiteness of the set of irreducible representations ("rationality", cf. [2]) is often difficult to prove by other methods. Furthermore, the nondegeneracy of the braiding, also called *modularity*

or invertibility of the S-matrix, plays an important role in the theory of topological invariants [62], particularly of Reshetikhin-Turaev type, and is usually the hardest to prove among the axioms of modular tensor category. Thus our results in [40] show that complete rationality specifies a class of conformal nets with the right rational behavior.

The finiteness of the μ-index may be difficult to verify directly in concrete models as in [66], but once this is established for some net, then it passes to subnets or extensions with finite index. Strong additivity is also often difficult to check, but recently one of us has proved in [45] that complete rationality also passes to a subnet or extension with finite index. In this way, we now know that large classes of coset models [67] and orbifold models [70] are completely rational.

Now consider an irreducible local conformal net $\mathcal{A}$ on S^1. Because of diffeomorphism covariance, $\mathcal{A}$ canonically contains a subnet $\mathcal{A}_{\mathrm{Vir}}$ generated by a unitary projective representation of the diffeomorphism group of S^1; thus we have a representation of the Virasoro algebra. (In physical terms, this appears by the Lüscher-Mack theorem as Fourier modes of a chiral component of the stress-energy tensor T,

$$T(z) = \sum L_n z^{-n-2}, \qquad [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}.)$$

This representation decomposes into irreducible representations, all with the same central charge $c > 0$, that is clearly an invariant for $\mathcal{A}$. As is well known either $c \geq 1$ or c takes a discrete set of values [20].

Our first observation is that if c belongs to the discrete series, then $\mathcal{A}_{\mathrm{Vir}}$ is an irreducible subnet with finite index of $\mathcal{A}$. The classification problem for $c < 1$ thus becomes the classification of irreducible local finite-index extensions $\mathcal{A}$ of the Virasoro nets for $c < 1$. We shall show that the nets $\mathcal{A}_{\mathrm{Vir}}$ are completely rational if $c < 1$, and so must be the original nets $\mathcal{A}$.

Thus, while our main result concerns nets of single factors, our main tool is the theory of nets of *subfactors*. This is the key of our approach.

The outline of this paper is as follows. We first identify the Virasoro nets with central charge less than one and the coset net arising from the diagonal embedding $\mathrm{SU}(2)_{m-1} \subset \mathrm{SU}(2)_{m-2} \times \mathrm{SU}(2)_1$ studied in [67], as naturally expected from the coset construction of [23]. Then it follows from [45] that the Virasoro nets with central charge less than 1 are completely rational.

Next we study the extensions of the Virasoro nets with central charge less than 1. If we have an extension, we can apply the machinery of α-induction, which was introduced in [46] and further studied in [64], [65], [3], [4], [5], [6], [7]. This is a method producing endomorphisms of the extended net from DHR endomorphisms of the smaller net using a braiding, but the extended endomorphisms are not DHR endomorphisms in general. For two irreducible DHR endomorphisms λ, μ of the smaller net, we can make extensions $\alpha_\lambda^+, \alpha_\mu^-$

using positive and negative braidings, respectively. Then we have a nonnegative integer $Z_{\lambda\mu} = \dim \mathrm{Hom}(\alpha_\lambda^+, \alpha_\mu^-)$. Recall that a completely rational net produces a unitary representation of $\mathrm{SL}(2, \mathbb{Z})$ by [54] and [40] in general. Then [5, Cor. 5.8] says that this matrix Z with nonnegative integer entries and normalization $Z_{00} = 1$ is in the commutant of this unitary representation, regardless of whether the extension is local or not, and this gives a very strong constraint on possible extensions of the Virasoro net. Such a matrix Z is called a *modular invariant* in general and has been extensively studied in conformal field theory. (See [14, Ch. 10] for example.) For a given unitary representation of $\mathrm{SL}(2, \mathbb{Z})$, the number of modular invariants is always finite and often very small, such as 1, 2, or 3, in concrete examples. The complete classification of modular invariants for a given representation of $\mathrm{SL}(2, \mathbb{Z})$ was first given in [11] for the case of the $\mathrm{SU}(2)_k$ WZW-models and the minimal models, and several more classification results have been obtained by Gannon. (See [22] and references there.)

Our approach to the classification problem of local extensions of a given net makes use of the classification of the modular invariants. For any local extension, we have indeed a modular invariant coming from the theory of α-induction as explained above. For each modular invariant in the classification list, we check the existence and uniqueness of corresponding extensions. In complete generality, we expect neither existence nor uniqueness, but this approach is often powerful enough to get a complete classification in concrete examples. This is the case of $\mathrm{SU}(2)_k$. (Such a classification is implicit in [6], though not explicitly stated there in this way. See Theorem 2.4 below.) Also along this approach, we obtain a complete classification of the local extensions of the Virasoro nets with central charge less than 1 in Theorem 4.1. By the stated canonical appearance of the Virasoro nets as subnets, we derive our final classification in Theorem 5.1. That is, our labeling of a conformal net in terms of pairs of Dynkin diagrams is given as follows. For a given conformal net with central charge $c < 1$, we have a Virasoro subnet. Then the α-induction applied to this extension of the Virasoro net produces a modular invariant $Z_{\lambda\mu}$ as above and such a matrix is labeled with a pair of Dynkin diagrams as in [11]. This labeling gives a complete classification of such conformal nets.

Some extensions of the Virasoro nets in our list have been studied or conjectured by other authors [3], [69] (they are related to the notion of W-algebra in the physical literature). Since our classification is complete, it is not difficult to identify them in our list. This will be done in Section 6.

Before closing this introduction we indicate possible background references to aid the readers; some have been already mentioned. Expositions of the basic structure of conformal nets on S^1 and subnets are contained in [26] and [46], respectively. Jones index theory [34] is discussed in [43] in connection with quantum field theory. Concerning modular invariants and α-induction one can

look at references [3], [5], [6]. The books [14], [29], [17], [35] deal respectively with conformal field theory from the physical viewpoint, algebraic quantum field theory, subfactors and connections with mathematical physics and infinite dimensional Lie algebras.

2. Preliminaries

In this section, we recall and prepare necessary results on extensions of completely rational nets in connection with extensions of the Virasoro nets.

2.1. *Conformal nets on S^1.* We denote by $\mathcal{I}$ the family of proper intervals of S^1. A *net* $\mathcal{A}$ of von Neumann algebras on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

from $\mathcal{I}$ to von Neumann algebras on a fixed Hilbert space $\mathcal{H}$ that satisfies:

A. *Isotony.* If $I_1 \subset I_2$ belong to $\mathcal{I}$, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2).$$

The net $\mathcal{A}$ is called *local* if it satisfies:

B. *Locality.* If $I_1, I_2 \in \mathcal{I}$ and $I_1 \cap I_2 = \emptyset$ then

$$[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\},$$

where brackets denote the commutator.

The net $\mathcal{A}$ is called *Möbius covariant* if in addition satisfies the following properties C, D, E:

C. *Möbius covariance.*[1] There exists a strongly continuous unitary representation U of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathcal{H}$ such that

$$\mathrm{U}(g)\mathcal{A}(I)\mathrm{U}(g)^* = \mathcal{A}(gI), \quad g \in \mathrm{PSL}(2, \mathbb{R}),\ I \in \mathcal{I}.$$

Here $\mathrm{PSL}(2, \mathbb{R})$ acts on S^1 by Möbius transformations.

D. *Positivity of the energy.* The generator of the one-parameter rotation subgroup of U (conformal Hamiltonian) is positive.

E. *Existence of the vacuum.* There exists a unit U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for the von Neumann algebra $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

(Here the lattice symbol $\bigvee$ denotes the von Neumann algebra generated.)

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[1]Möbius covariant nets are often called *conformal* nets. In this paper however we shall reserve the term 'conformal' to indicate diffeomorphism covariant nets.

Let $\mathcal{A}$ be an irreducible Möbius covariant net. By the Reeh-Schlieder theorem the vacuum vector Ω is cyclic and separating for each $\mathcal{A}(I)$. The Bisognano-Wichmann property then holds [8], [21]: the Tomita-Takesaki modular operator Δ_I and conjugation J_I associated with $(\mathcal{A}(I), \Omega)$, $I \in \mathcal{I}$, are given by

(1) $$\mathrm{U}(\Lambda_I(2\pi t)) = \Delta_I^{it}, \ t \in \mathbb{R}, \qquad \mathrm{U}(r_I) = J_I,$$

where Λ_I is the one-parameter subgroup of $\mathrm{PSL}(2, \mathbb{R})$ of special conformal transformations preserving I and $\mathrm{U}(r_I)$ implements a geometric action on $\mathcal{A}$ corresponding to the Möbius reflection r_I on S^1 mapping I onto I', i.e. fixing the boundary points of I, see [8].

This immediately implies Haag duality (see [28], [10]):

$$\mathcal{A}(I)' = \mathcal{A}(I'), \quad I \in \mathcal{I} \ ,$$

where $I' \equiv S^1 \smallsetminus I$.

We shall say that a Möbius covariant net $\mathcal{A}$ is *irreducible* if $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$. Indeed $\mathcal{A}$ is irreducible if and only if Ω is the unique U-invariant vector (up to scalar multiples), and if and only if the local von Neumann algebras $\mathcal{A}(I)$ are factors. In this case they are III_1-factors (unless $\mathcal{A}(I) = \mathbb{C}$ identically); see [26].

Because of Lemma 2.1 below, we may always consider irreducible nets. Hence, from now on, we shall make the assumption:

F. *Irreducibility.* The net $\mathcal{A}$ is irreducible.

Let $\mathrm{Diff}(S^1)$ be the group of orientation-preserving smooth diffeomorphisms of S^1. As is well known $\mathrm{Diff}(S^1)$ is an infinite dimensional Lie group whose Lie algebra is the Virasoro algebra (see [53], [35]).

By a *conformal net* (or diffeomorphism covariant net) $\mathcal{A}$ we shall mean a Möbius covariant net such that the following holds:

G. *Conformal covariance.* There exists a projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of $\mathrm{PSL}(2, \mathbb{R})$ such that for all $I \in \mathcal{I}$ we have

$$\begin{aligned} \mathrm{U}(g)\mathcal{A}(I)\mathrm{U}(g)^* &= \mathcal{A}(gI), & g \in \mathrm{Diff}(S^1), \\ \mathrm{U}(g)A\mathrm{U}(g)^* &= A, & A \in \mathcal{A}(I), \ g \in \mathrm{Diff}(I'), \end{aligned}$$

where $\mathrm{Diff}(I)$ denotes the group of smooth diffeomorphisms g of S^1 such that $g(t) = t$ for all $t \in I'$.

If $\mathcal{A}$ is a local conformal net on S^1 then, by Haag duality,

$$\mathrm{U}(\mathrm{Diff}(I)) \subset \mathcal{A}(I).$$

Notice that, in general, $\mathrm{U}(g)\Omega \neq \Omega$, $g \in \mathrm{Diff}(S^1)$. Otherwise the Reeh-Schlieder theorem would be violated.

LEMMA 2.1. *Let $\mathcal{A}$ be a local Möbius (resp. diffeomorphism) covariant net. The center Z of $\mathcal{A}(I)$ does not depend on the interval I and $\mathcal{A}$ has a decomposition*

$$\mathcal{A}(I) = \int_X^\oplus \mathcal{A}_\lambda(I)\,d\mu(\lambda)$$

where the nets $\mathcal{A}_\lambda$ are Möbius (resp. diffeomorphism) covariant and irreducible. The decomposition is unique (up to a set of measure 0). Here $Z = L^\infty(X,\mu)$.[2]

Proof. Assume $\mathcal{A}$ to be Möbius covariant. Given a vector $\xi \in \mathcal{H}$, $U(\Lambda_I(t))\xi = \xi$, $\forall t \in \mathbb{R}$, if and only if $U(g)\xi = \xi$, $\forall g \in \mathrm{PSL}(2,\mathbb{R})$; see [26]. Hence if $I \subset \tilde{I}$ are intervals and $A \in \mathcal{A}(\tilde{I})$, the vector $A\Omega$ is fixed by $U(\Lambda_I(\cdot))$ if and only if it is fixed by $U(\Lambda_{\tilde{I}}(\cdot))$. Thus A is fixed by the modular group of $(\mathcal{A}(I),\Omega)$ if and only if it is fixed by the modular group of $(\mathcal{A}(\tilde{I}),\Omega)$. In other words the centralizer Z_ω of $\mathcal{A}(I)$ is independent of I; hence, by locality, it is contained in the center of any $\mathcal{A}(I)$. Since the center is always contained in the centralizer, it follows that Z_ω must be the common center of all the $\mathcal{A}(I)$'s. The statement is now an immediate consequence of the uniqueness of the direct integral decomposition of a von Neumann algebra into factors.

Furthermore, if $\mathcal{A}$ is diffeomorphism covariant, then the fiber $\mathcal{A}_\lambda$ in the decomposition is diffeomorphism covariant too. Indeed $\mathrm{Diff}(I) \subset \mathcal{A}(I)$ decomposes through the space X and so does $\mathrm{Diff}(S^1)$, which is generated by $\{\mathrm{Diff}(I), I \in \mathcal{I}\}$ (cf. e.g. [42]).   □

Before concluding this subsection, we explicitly say that two conformal nets $\mathcal{A}_1$ and $\mathcal{A}_2$ are *isomorphic* if there is a unitary V from the Hilbert space of $\mathcal{A}_1$ to the Hilbert space of $\mathcal{A}_2$, mapping the vacuum vector of $\mathcal{A}_1$ to the vacuum vector of $\mathcal{A}_2$, such that $V\mathcal{A}_1(I)V^* = \mathcal{A}_2(I)$ for all $I \in \mathcal{I}$. Then V also intertwines the Möbius covariance representations of $\mathcal{A}_1$ and $\mathcal{A}_2$ [8], because of the uniqueness of these representations due to eq. (1). Our classification will be up to isomorphism. Yet, as a consequence of these results, our classification will indeed be up to the *a priori* weaker notion of isomorphism where V is not assumed to preserve the vacuum vector.

Note also that, by Haag duality, two fields generate isomorphic nets if and only if they are relatively local, that is, belong to the same Borchers class (see [29]).

2.1.1. *Representations.*   Let $\mathcal{A}$ be an irreducible local Möbius covariant (resp. conformal) net. A *representation* π of $\mathcal{A}$ is a map

$$I \in \mathcal{I} \to \pi_I \ ,$$

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[2]If $\mathcal{H}$ is nonseparable the decomposition should be stated in a more general form.

where π_I is a representation of $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}_\pi$ such that

$$\pi_{\tilde{I}}\!\upharpoonright_{\mathcal{A}(I)} = \pi_I, \quad I \subset \tilde{I} \ .$$

We shall always implicitly assume that π is locally normal, namely π_I is normal for all $I \in \mathcal{I}$, which is automatic if $\mathcal{H}_\pi$ is separable [60].

We shall say that π is Möbius (resp. conformal) covariant if there exists a positive energy representation U_π of $\mathrm{PSL}(2,\mathbb{R})\tilde{}$ (resp. of $\mathrm{Diff}(S^1)\tilde{}$) such that

$$U_\pi(g)\mathcal{A}(I)U_\pi(g)^{-1} = \mathcal{A}(gI), \ g \in \mathrm{PSL}(2,\mathbb{R})\tilde{} \quad (\text{resp. } g \in \mathrm{Diff}(S^1)\tilde{}).$$

(Here $\mathrm{PSL}(2,\mathbb{R})\tilde{}$ denotes the universal central cover of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{Diff}(S^1)\tilde{}$ the corresponding central extension of $\mathrm{Diff}(S^1)$.) The identity representation of $\mathcal{A}$ is called the *vacuum* representation; if convenient, it will be denoted by π_0.

A representation ρ is *localized* in an interval I_0 if $\mathcal{H}_\rho = \mathcal{H}$ and $\rho_{I_0'} = \mathrm{id}$. Given an interval I_0 and a representation π on a separable Hilbert space, there is a representation ρ unitarily equivalent to π and localized in I_0. This is due to the type III factor property. If ρ is a representation localized in I_0, then by Haag duality ρ_I is an endomorphism of $\mathcal{A}(I)$ if $I \supset I_0$. The endomorphism ρ is called a DHR endomorphism [15] localized in I_0. The *index* of a representation ρ is the Jones index $[\rho_{I'}(\mathcal{A}(I'))' : \rho_I(\mathcal{A}(I))]$ for any interval I or, equivalently, the Jones index $[\mathcal{A}(I) : \rho_I(\mathcal{A}(I))]$ of ρ_I, if $I \supset I_0$. The (statistical) *dimension* $d(\rho)$ of ρ is the square root of the index.

The unitary equivalence $[\rho]$ class of a representation ρ of $\mathcal{A}$ is called a *sector* of $\mathcal{A}$.

2.1.2. *Subnets.* Let $\mathcal{A}$ be a Möbius covariant (resp. conformal) net on S^1 and U the unitary covariance representation of the Möbius group (resp. of $\mathrm{Diff}(S^1)$).

A Möbius covariant (resp. conformal) *subnet* $\mathcal{B}$ of $\mathcal{A}$ is an isotonic map $I \in \mathcal{I} \to \mathcal{B}(I)$ that associates to each interval I a von Neumann subalgebra $\mathcal{B}(I)$ of $\mathcal{A}(I)$ with $U(g)\mathcal{B}(I)U(g)^* = \mathcal{B}(gI)$ for all g in the Möbius group (resp. in $\mathrm{Diff}(S^1)$).

If $\mathcal{A}$ is local and irreducible, then the modular group of $(\mathcal{A}(I), \Omega)$ is ergodic and so is its restriction to $\mathcal{B}(I)$; thus each $\mathcal{B}(I)$ is a factor. By the Reeh-Schlieder theorem the Hilbert space $\mathcal{H}_0 \equiv \overline{\mathcal{B}(I)\Omega}$ is independent of I. The restriction of $\mathcal{B}$ to $\mathcal{H}_0$ is then an irreducible local Möbius covariant net on $\mathcal{H}_0$ and we denote it here by $\mathcal{B}_0$. The vector Ω is separating for $\mathcal{B}(I)$; therefore the map $B \in \mathcal{B}(I) \to B|_{\mathcal{H}_0} \in \mathcal{B}_0(I)$ is an isomorphism. Its inverse thus defines a representation of $\mathcal{B}_0$ that we shall call the restriction to $\mathcal{B}$ of the vacuum representation of $\mathcal{A}$ (as a sector this is given by the dual canonical endomorphism of $\mathcal{A}$ in $\mathcal{B}$). Indeed we shall sometimes identify $\mathcal{B}(I)$ and $\mathcal{B}_0(I)$ although, properly speaking, $\mathcal{B}$ is not a Möbius covariant net because Ω is not cyclic. Note that if $\mathcal{A}$ is conformal and $U(\mathrm{Diff}(I)) \subset \mathcal{B}(I)$ then $\mathcal{B}_0$ is a conformal net (compare with Prop. 6.2).

If $\mathcal{B}$ is a subnet of $\mathcal{A}$ we shall denote here $\mathcal{B}''$ the von Neumann algebra generated by all the algebras $\mathcal{B}(I)$ as I varies in the intervals $\mathcal{I}$. The subnet $\mathcal{B}$ of $\mathcal{A}$ is said to be *irreducible* if $\mathcal{B}' \cap \mathcal{A}(I) = \mathbb{C}$ (if $\mathcal{B}$ is strongly additive this is equivalent to $\mathcal{B}(I)' \cap \mathcal{A}(I) = \mathbb{C}$). If $[\mathcal{A} : \mathcal{B}] < \infty$ then $\mathcal{B}$ is automatically irreducible.

The following lemma will be used in the paper.

LEMMA 2.2. *Let $\mathcal{A}$ be a Möbius covariant net on S^1 and $\mathcal{B}$ a Möbius covariant subnet. Then $\mathcal{B}'' \cap \mathcal{A}(I) = \mathcal{B}(I)$ for any given $I \in \mathcal{I}$.*

Proof . By equation (1), $\mathcal{B}(I)$ is globally invariant under the modular group of $(\mathcal{A}(I), \Omega)$; thus by Takesaki's theorem there exists a vacuum-preserving conditional expectation from $\mathcal{A}(I)$ to $\mathcal{B}(I)$ and an operator $A \in \mathcal{A}(I)$ belongs to $\mathcal{B}(I)$ if and only if $A\Omega \in \overline{\mathcal{B}(I)\Omega}$. By the Reeh-Schlieder theorem $\overline{\mathcal{B}''\Omega} = \overline{\mathcal{B}(I)\Omega}$ and this immediately entails the statement. $\square$

2.2. *Virasoro algebra and Virasoro nets.* The Virasoro algebra is the infinite dimensional Lie algebra generated by elements $\{L_n \mid n \in \mathbb{Z}\}$ and c with relations

(2) $$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

and $[L_n, c] = 0$. It is the (complexification of) the unique, nontrivial one-dimensional central extension of the Lie algebra of $\mathrm{Diff}(S^1)$.

We shall only consider unitary representations of the Virasoro algebra (i.e. $L_n^* = L_{-n}$ in the representation space) with positive energy (i.e. $L_0 > 0$ in the representation space), indeed the ones associated with a projective unitary representation of $\mathrm{Diff}(S^1)$.

In any irreducible representation the *central charge c* is a scalar, indeed $c = 1 - 6/m(m + 1)$, $(m = 2, 3, 4, \dots)$ or $c \geq 1$ [20] and all these values are allowed [23].

For every admissible value of c there is exactly one irreducible (unitary, positive energy) representation U of the Virasoro algebra (i.e. projective unitary representation of $\mathrm{Diff}(S^1)$) such that the lowest eigenvalue of the conformal Hamiltonian L_0 (i.e. the *spin*) is 0; this is the vacuum representation with central charge c. One can then define the Virasoro net

$$\mathrm{Vir}_c(I) \equiv \mathrm{U}(\mathrm{Diff}(I))''.$$

Any other projective unitary irreducible representation of $\mathrm{Diff}(S^1)$ with a given central charge c is uniquely determined by its spin. Indeed, as we shall see, these representations with central charge c correspond bijectively to the irreducible representations (in the sense of Subsection 2.1.1) of the Vir_c net; namely, their equivalence classes correspond to the irreducible sectors of the Vir_c net.

In conformal field theory, the Vir_c net for $c < 1$ are studied under the name of *minimal models* (see [14, Ch. 7, 8], for example). Notice that they are indeed minimal in the sense they contain no nontrivial subnet [12].

For the central charge $c = 1 - 6/m(m + 1)$, $(m = 2, 3, 4, \dots)$, we have $m(m - 1)/2$ characters $\chi_{(p,q)}$ of the minimal model labeled with (p, q), $1 \le p \le m - 1$, $1 \le q \le m$ with the identification $\chi_{(p,q)} = \chi_{(m-p,m+1-q)}$, as in [14, Subsec. 7.3.4]. They have fusion rules as in [14, Subsec. 7.3.3] and they are given as follows.

$$(3) \qquad \chi_{(p,q)}\chi_{(p',q')} = \bigoplus_{\substack{r=|p-p'|+1,\, r+p+p':\text{odd}}}^{\min(p+p'-1,2m-p-p'-1)} \bigoplus_{\substack{s=|q-q'|+1,\, s+q+q':\text{odd}}}^{\min(q+q'-1,2(m+1)-q-q'-1)} \chi_{(r,s)}.$$

Note that here the product $\chi_{(p,q)}\chi_{(p',q')}$ denotes the fusion of characters and not their pointwise product as functions.

For the character $\chi_{(p,q)}$, we have a spin

$$(4) \qquad\qquad\qquad h_{p,q} = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}$$

by [23]. (Also see [14, Subsec. 7.3.3].) The characters $\{\chi_{(p,q)}\}_{p,q}$ have the S, T-matrices of Kac-Petersen as in [14, Sec. 10.6].

2.3. *Virasoro nets and classification of the modular invariants.* Cappelli-Itzykson-Zuber [11] and Kato [36] have made an *A-D-E* classification of the modular invariant matrices for $\mathrm{SU}(2)_k$. That is, for the unitary representation of the group $\mathrm{SL}(2, \mathbb{Z})$ arising from $\mathrm{SU}(2)_k$ as in [14, Subsec. 17.1.1], they classified matrices Z with nonnegative integer entries in the commutant of this unitary representation, up to the normalization $Z_{00} = 1$. Such matrices are called *modular invariants* of $\mathrm{SU}(2)_k$ and labeled with Dynkin diagrams A_n, D_n, $E_{6,7,8}$ by looking at the diagonal entries of the matrices as in the table (17.114) in [14]. Based on this classification, Cappelli-Itzykson-Zuber [11] also gave a classification of the modular invariant matrices for the above minimal models and the unitary representations of $\mathrm{SL}(2, \mathbb{Z})$ arising from the S, T-matrices mentioned at the end of the previous subsection. From our viewpoint, we will regard this as a classification of matrices with nonnegative integer entries in the commutant of the unitary representations of $\mathrm{SL}(2, \mathbb{Z})$ arising from the Virasoro net Vir_c with $c < 1$. Such modular invariants of the minimal models are labeled with pairs of Dynkin diagrams of *A-D-E* type such that the difference of their Coxeter numbers is 1. The classification tables are given in Table 1 for so-called type I (block-diagonal) modular invariants, where each modular invariant $(Z_{(p,q),(p',q')})_{(p,q),(p',q')}$ is listed in the form $\sum Z_{(p,q),(p',q')}\chi_{(p,q)}\overline{\chi_{(p',q')}}$, and we refer to [14, Table 10.4] for the type II modular invariants, since we are mainly concerned with type I modular invariants in this paper. (Note that the coefficient $1/2$ in the table arises from a double counting due to the

Label	$\sum Z_{(p,q),(p',q')}\chi_{(p,q)}\overline{\chi_{(p',q')}}$
(A_{n-1}, A_n)	$\displaystyle\sum_{p,q} \left\|\chi_{(p,q)}\right\|^2/2$
(A_{4n}, D_{2n+2})	$\displaystyle\sum_{q:\text{ odd}} \left\|\chi_{(p,q)} + \chi_{(p,4n+2-q)}\right\|^2/2$
(D_{2n+2}, A_{4n+2})	$\displaystyle\sum_{p:\text{ odd}} \left\|\chi_{(p,q)} + \chi_{(4n+2-p,q)}\right\|^2/2$
(A_{10}, E_6)	$\displaystyle\sum_{p=1}^{10} \left\{\left\|\chi_{(p,1)} + \chi_{(p,7)}\right\|^2 + \left\|\chi_{(p,4)} + \chi_{(p,8)}\right\|^2 + \left\|\chi_{(p,5)} + \chi_{(p,11)}\right\|^2\right\}/2$
(E_6, A_{12})	$\displaystyle\sum_{q=1}^{12} \left\{\left\|\chi_{(1,q)} + \chi_{(7,q)}\right\|^2 + \left\|\chi_{(4,q)} + \chi_{(8,q)}\right\|^2 + \left\|\chi_{(5,q)} + \chi_{(11,q)}\right\|^2\right\}/2$
(A_{28}, E_8)	$\displaystyle\sum_{p=1}^{28} \left\{\left\|\chi_{(p,1)} + \chi_{(p,11)} + \chi_{(p,19)} + \chi_{(p,29)}\right\|^2 + \left\|\chi_{(p,7)} + \chi_{(p,13)} + \chi_{(p,17)} + \chi_{(p,23)}\right\|^2\right\}/2$
(E_8, A_{30})	$\displaystyle\sum_{q=1}^{30} \left\{\left\|\chi_{(1,q)} + \chi_{(11,q)} + \chi_{(19,q)} + \chi_{(29,q)}\right\|^2 + \left\|\chi_{(7,q)} + \chi_{(13,q)} + \chi_{(17,q)} + \chi_{(23,q)}\right\|^2\right\}/2$

Table 1: Type I modular invariants of the minimal models

identification $\chi_{(p,q)} = \chi_{(m-p,m+1-q)}$.) Here the labels come from the diagonal entries of the matrices again, but we will give our subfactor interpretation of this labeling later.

2.4. *Q-systems and classification.* Let M be an infinite factor. A Q-system (ρ, V, W) in [44] is a triple of an endomorphism of M and isometries $V \in \text{Hom}(\text{id}, \rho)$, $W \in \text{Hom}(\rho, \rho^2)$ satisfying the following identities:

$$V^*W = \rho(V^*)W \in \mathbb{R}_+,$$
$$\rho(W)W = W^2.$$

The abstract notion of a Q-system for tensor categories is contained in [47]. (We had another identity in addition to the above in [44] as the definition of a Q-system, but it was proved to be redundant in [47].)

If $N \subset M$ is a finite-index subfactor, the associated canonical endomorphism gives rise to a Q-system. Conversely any Q-system determines a subfactor N of M such that ρ is the canonical endomorphism for $N \subset M$: N is given by

$$N = \{x \in M \mid Wx = \rho(x)W\}.$$

We say (ρ, V, W) is irreducible when $\dim \text{Hom}(\text{id}, \rho) = 1$. We say that two Q-systems (ρ, V_1, W_1) and (ρ, V_2, W_2) are *equivalent* if we have a unitary $u \in \text{Hom}(\rho, \rho)$ satisfying

$$V_2 = uV_1, \qquad W_2 = u\rho(u)W_1u^*.$$

This equivalence of Q-systems is equivalent to inner conjugacy of the corresponding subfactors.

Subfactors $N \subset M$ and extensions $\tilde{M} \supset M$ of M are naturally related by Jones basic construction (or by the canonical endomorphism). The problem we are interested in is a classification of Q-systems up to equivalence when a system of endomorphisms is given and ρ is a direct sum of endomorphisms in the system.

2.5. *Classification of local extensions of the* $\mathrm{SU}(2)_k$ *net.* As a preliminary to our main classification theorem, we first deal with local extensions of the $\mathrm{SU}(2)_k$ net. The $\mathrm{SU}(n)_k$ net was constructed in [63] using a representation of the loop group [53]. By the results on the fusion rules in [63] and the spin-statistics theorem [26], we know that the usual S- and T-matrices of $\mathrm{SU}(n)_k$ as in [14, Sec. 17.1.1] and those arising from the braiding on the $\mathrm{SU}(n)_k$ net as in [54] coincide.

We start with the following result.

PROPOSITION 2.3. *Let* $\mathcal{A}$ *be a Möbius covariant net on the circle. Suppose that* $\mathcal{A}$ *admits only finitely many irreducible* DHR *sectors and each sector is sum of sectors with finite statistical dimension. If* $\mathcal{B}$ *is an irreducible local extension of* $\mathcal{A}$, *then the index* $[\mathcal{B} : \mathcal{A}]$ *is finite.*

Proof. As in [45, Lemma 13], we have a vacuum-preserving conditional expectation $\mathcal{B}(I) \to \mathcal{A}(I)$. The dual canonical endomorphism θ for $\mathcal{A}(I) \subset \mathcal{B}(I)$ decomposes into DHR endomorphisms of the net $\mathcal{A}$, but we have only finitely many such endomorphisms of finite statistical dimensions by assumption. Then the result in [33, p. 39] shows that multiplicity of each such DHR endomorphism in θ is finite; thus the index $(= d(\theta))$ is also finite. $\square$

We are interested in the classification problem of irreducible local extensions $\mathcal{B}$ when $\mathcal{A}$ is given. (Note that if we have finite index $[\mathcal{B} : \mathcal{A}]$, then the irreducibility holds automatically by [3, I, Corollary 3.6], [13].) The basic case of this problem is the one where $\mathcal{A}(I)$ is given from $\mathrm{SU}(2)_k$ as in [63]. In this case, the following classification result is implicit in [6], but for the sake of completeness, we state and give a proof to it here as follows. Note that G_2 in Table 2 means the exceptional Lie group G_2.

THEOREM 2.4. *The irreducible local extensions of the* $\mathrm{SU}(2)_k$ *net are in a bijective correspondence to the Dynkin diagrams of type* A_n, D_{2n}, E_6, E_8 *as in Table* 2.

Proof. The $\mathrm{SU}(2)_k$ net $\mathcal{A}$ is completely rational by [66]; thus any local extension $\mathcal{B}$ is of finite index by [40, Cor. 39] and Proposition 2.3. For a fixed interval I, we have a subfactor $\mathcal{A}(I) \subset \mathcal{B}(I)$ and can apply the α-induction for the system Δ of DHR endomorphisms of $\mathcal{A}$. Then the matrix Z given by $Z_{\lambda\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$ is a modular invariant for $\mathrm{SU}(2)_k$ by

level k	Dynkin diagram	Description
$n-1, (n \geq 1)$	A_n	$\mathrm{SU}(2)_k$ itself
$4n-4, (n \geq 2)$	D_{2n}	Simple current extension of index 2
10	E_6	Conformal inclusion $\mathrm{SU}(2)_{10} \subset \mathrm{SO}(5)_1$
28	E_8	Conformal inclusion $\mathrm{SU}(2)_{28} \subset (G_2)_1$

Table 2: Local extensions of the $\mathrm{SU}(2)_k$ net

[5, Cor. 5.8] and thus one of the matrices listed in [11]. Now we have the locality of $\mathcal{B}$, so that $Z_{\lambda,0} = \langle \alpha_\lambda^+, \mathrm{id} \rangle = \langle \lambda, \theta \rangle$, where θ is the dual canonical endomorphism for $\mathcal{A}(I) \subset \mathcal{B}(I)$ by [64], and the modular invariant matrix Z must be block-diagonal, which is said to be of type I as in Table 1. Considering the classification of [11], we have only the following possibilities for θ.

$\theta = \mathrm{id}$,   for the type A_{k+1} modular invariant at level k,

$\theta = \lambda_0 \oplus \lambda_{4n-4}$,   for the type D_{2n} modular invariant at level $k = 4n - 4$,

$\theta = \lambda_0 \oplus \lambda_6$,   for the type E_6 modular invariant at level $k = 12$,

$\theta = \lambda_0 \oplus \lambda_{10} \oplus \lambda_{18} \oplus \lambda_{28}$,   for the type E_8 modular invariant at level $k = 28$.

By [64], [3, II, Sec. 3], we know that all these cases indeed occur, and we have the unique Q-system for each case by [41, Sec. 6]. (In [41, Def. 1.1], Conditions 1 and 3 correspond to the axioms of the Q-system in Subsection 2.4, Condition 4 corresponds to irreducibility, and Condition 3 corresponds to chiral locality in [46, Th. 4.9] in the sense of [5, p. 454].) By [46, Th. 4.9], we conclude that the local extensions are classified as desired. $\square$

Remark 2.5. The proof of uniqueness for the E_8 case in [41, Sec. 6] uses vertex operator algebras. Izumi has recently given a direct proof of uniquenss of the Q-system using an intermediate extension. We later obtained another proof based on 2-cohomology vanishing for the tensor category $\mathrm{SU}(2)_k$ in [39]. An outline of the arguments is as follows.

Suppose there are two Q-systems for this dual canonical endomorphism of an injective type III_1 factor M. We need to prove that the two corresponding subfactors $N_1 \subset M$ and $N_2 \subset M$ are inner conjugate. First, it is easy to prove that the paragroups of these two subfactors are isomorphic to that of the Goodman-de la Harpe-Jones subfactor [24, Sec. 4.5] arising from E_8. Thus we may assume that these two subfactors are conjugate. From this, one shows that the two Q-systems differ only by a "2-cocycle" of the even part of the tensor category $\mathrm{SU}(2)_{28}$. Using the facts that the fusion rules of $\mathrm{SU}(2)_k$ have no multiplicities and that all the $6j$-symbols are nonzero, one proves that any such 2-cocycle is trivial. This implies that the two Q-systems are equivalent.

3. The Virasoro nets as cosets

Based on the coset construction of projective unitary representations of the Virasoro algebras with central charge less than 1 by Goddard-Kent-Olive [23], it is natural to expect that the Virasoro net on the circle with central charge $c = 1 - 6/m(m + 1)$ and the coset model arising from the diagonal embedding $\mathrm{SU}(2)_{m-1} \subset \mathrm{SU}(2)_{m-2} \times \mathrm{SU}(2)_1$ as in [67] are isomorphic. We prove the isomorphism here. This, in particular, implies that the Virasoro nets with central charge less than 1 are completely rational in the sense of [40].

LEMMA 3.1. *If $\mathcal{A}$ is a* Vir *net, then every Möbius covariant representation π of $\mathcal{A}$ is* $\mathrm{Diff}(S^1)$ *covariant.*

Proof. Indeed $\mathcal{A}(I)$ is generated by $\mathrm{U}(\mathrm{Diff}(I))$, where U is an irreducible projective unitary representation of $\mathrm{Diff}(S^1)$, and $\mathrm{U}(g)$ clearly implements the covariance action of g on $\mathcal{A}$ if g belongs to $\mathrm{Diff}(I)$. Thus $\pi_I(\mathrm{U}(g))$ implements the covariance action of g in the representation π. As $\mathrm{Diff}(S^1)$ is generated by $\mathrm{Diff}(I)$ as I varies in the intervals, the full $\mathrm{Diff}(S^1)$ acts covariantly. The positivity of the energy holds by the Möbius covariance assumption.   □

LEMMA 3.2. *Let $\mathcal{A}$ be an irreducible Möbius covariant local net, $\mathcal{B}$ and $\mathcal{C}$ mutually commuting subnets of $\mathcal{A}$. Suppose the restriction to $\mathcal{B} \vee \mathcal{C} \simeq \mathcal{B} \otimes \mathcal{C}$ of the vacuum representation π_0 of $\mathcal{A}$ has the (finite or infinite) expansion*

$$(5) \qquad \pi_0|_{\mathcal{B} \vee \mathcal{C}} = \bigoplus_{i=0}^{n} \rho_i \otimes \sigma_i,$$

where ρ_0 is the vacuum representation of $\mathcal{B}$, σ_0 is the vacuum representation of $\mathcal{C}$, and ρ_0 is disjoint from ρ_i if $i \neq 0$. Then $\mathcal{C}(I) = \mathcal{B}' \cap \mathcal{A}(I)$.

Proof. The Hilbert space $\mathcal{H}$ of $\mathcal{A}$ decomposes according to the expansion (5) as

$$\mathcal{H} = \bigoplus_{i=0}^{n} \mathcal{H}_i \otimes \mathcal{K}_i.$$

The vacuum vector Ω of $\mathcal{A}$ corresponds to $\Omega_{\mathcal{B}} \otimes \Omega_{\mathcal{C}} \in \mathcal{H}_0 \otimes \mathcal{K}_0$, where $\Omega_{\mathcal{B}}$ and $\Omega_{\mathcal{C}}$ are the vacuum vector of $\mathcal{B}$ and $\mathcal{C}$, because $\mathcal{H}_0 \otimes \mathcal{K}_0$ is, by assumption, the support of the representation $\rho_0 \otimes \sigma_0$. We then have

$$\pi_0(B) = \sum_{i=0}^{n} \rho_i(B) \otimes 1|_{\mathcal{K}_i}, \quad B \in \mathcal{B}(I).$$

and, as ρ_0 is disjoint from ρ_i if $i \neq 0$,

$$\pi_0(\mathcal{B})' = (1_{\mathcal{H}_0} \otimes B(\mathcal{K}_0)) \oplus \cdots$$

where we have set $\pi_0(\mathcal{B})' \equiv (\bigvee_{I \in \mathcal{I}} \mathcal{B}(I))'$ and the dots stand for operators on the orthogonal complement of $\mathcal{H}_0 \otimes \mathcal{K}_0$. It follows that if $X \in \pi_0(\mathcal{B})'$, then $X\Omega \in \mathcal{H}_0 \otimes \mathcal{K}_0$.

With $\mathcal{L}$ the subnet of $\mathcal{A}$ given by $\mathcal{L}(I) \equiv \mathcal{B}(I) \vee \mathcal{C}(I)$, we then have by the Reeh-Schlieder theorem

$$X \in \pi_0(\mathcal{B})' \cap \mathcal{A}(I) \implies X\Omega \in \overline{\mathcal{L}(I)\Omega} \implies X \in \mathcal{L}(I),$$

where the last implication follows by Lemma 2.2. As $\mathcal{L}(I) \simeq \mathcal{B}(I) \otimes \mathcal{C}(I)$ and X commutes with $\mathcal{B}(I)$, we have $X \in \mathcal{C}(I)$ as desired. $\square$

The proof of the following corollary was shown to the authors (independently) by F. Xu and S. Carpi. Concerning our original proof, see Remark 3.7 at the end of this section.

COROLLARY 3.3. *The Virasoro net on the circle with central charge $c = 1 - 6/m(m+1)$ and the coset net arising from the diagonal embedding $\mathrm{SU}(2)_{m-1} \subset \mathrm{SU}(2)_{m-2} \times \mathrm{SU}(2)_1$ are isomorphic.*

Proof. As shown in [23], Vir_c is a subnet of the above coset net for $c = 1 - 6/m(m+1)$. Moreover, the formula in [23, (2.20)], obtained by comparison of characters, shows in particular that the hypothesis in Lemma 3.2 hold true with $\mathcal{A}$ the $\mathrm{SU}(2)_{m-2} \times \mathrm{SU}(2)_1$ net, $\mathcal{B}$ the $\mathrm{SU}(2)_{m-1}$ subnet (coming from diagonal embedding) and $\mathcal{C}$ the Vir_c subnet. Thus the corollary follows. $\square$

COROLLARY 3.4. *The Virasoro net on the circle Vir_c with central charge $c < 1$ is completely rational.*

Proof. The Virasoro net on the circle Vir_c with central charge $c = 1 - 6/m(m+1)$ coincides with the coset net arising from the diagonal embedding $\mathrm{SU}(2)_{m-1} \subset \mathrm{SU}(2)_{m-2} \times \mathrm{SU}(2)_1$ by Corollary 3.3; thus it is completely rational by [45, Sec. 3.5.1]. $\square$

The next proposition shows in particular that the central charge is defined for any local irreducible conformal net.

PROPOSITION 3.5. *Let $\mathcal{B}$ be a local irreducible conformal net on the circle. Then it contains canonically a Virasoro net as a subnet. If its central charge c satisfies $c < 1$, then the Virasoro subnet is an irreducible subnet with finite index.*

Proof. Let U be the projective unitary representation of $\mathrm{Diff}(S^1)$ implementing the diffeomorphism covariance on $\mathcal{B}$ and set

$$\mathcal{B}_{\mathrm{Vir}}(I) = \mathrm{U}(\mathrm{Diff}(I))''.$$

Then U is the direct sum of the vacuum representations of Vir_c and another representation of Vir_c. Indeed, as $\mathcal{B}_{\mathrm{Vir}}$ is a subnet of $\mathcal{B}$, all the subrepresentations of $\mathcal{B}_{\mathrm{Vir}}$ are mutually locally normal, and so they have the same central charge c. Note that the central charge is well defined because U is a projective unirary representation.

Suppose now that $c < 1$. For an interval I we must show that $\mathcal{B}_{\mathrm{Vir}}(I)' \cap \mathcal{B}(I) = \mathbb{C}$. By its locality it is enough to show that

$$(\mathcal{B}_{\mathrm{Vir}}(I') \vee \mathcal{B}_{\mathrm{Vir}}(I))' \cap \mathcal{B}(I) = \mathbb{C}.$$

Because the net Vir is completely rational by Corollary 3.4, it is strongly additive in particular, and thus we have $\mathcal{B}_{\mathrm{Vir}}(I') \vee \mathcal{B}_{\mathrm{Vir}}(I)$ is equal to the weak closure of all the nets $\mathcal{B}_{\mathrm{Vir}}$. Then any X in $\mathcal{B}(I)$ that commutes with $\mathcal{B}_{\mathrm{Vir}}(I') \vee \mathcal{B}_{\mathrm{Vir}}(I)$ would commute with $\mathrm{U}(g)$ for any g in $\mathrm{Diff}(I)$ for every interval I. Now the group $\mathrm{Diff}(S^1)$ is generated by the subgroups $\mathrm{Diff}(I)$, so that X would commute with all $\mathrm{U}(\mathrm{Diff}(S^1))$; in particular it would be fixed by the modular group of $(\mathcal{B}(I), \Omega)$, which is ergodic. Thus X is to be a scalar.

Then $[\mathcal{B} : \mathcal{B}_{\mathrm{Vir}}] < \infty$ by Proposition 2.3 and Corollary 3.4. □

We remark that we can also prove that $\mathcal{B}_{\mathrm{Vir}}(I') \vee \mathcal{B}_{\mathrm{Vir}}(I)$ and the range of full net $\mathcal{B}_{\mathrm{Vir}}$ have the same weak closure as follows. Since $\mathcal{B}_{\mathrm{Vir}}$ is obtained as a direct sum of irreducible sectors ρ_i of $\mathcal{B}_{\mathrm{Vir}}$ localizable in I, it is enough to show that the intertwiners between ρ_i and ρ_j as endomorphisms of the factor $\mathrm{Vir}_c(I)$ are the same as the intertwiners between ρ_i and ρ_j as representations of Vir_c. Since each ρ_i has a finite index by complete rationality as in [40, Cor. 39], the result follows by the theorem of equivalence of local and global intertwiners in [26].

Given a local irreducible conformal net $\mathcal{B}$, the subnet $\mathcal{B}_{\mathrm{Vir}}$ constructed in Proposition 3.5 is the *Virasoro subnet* of $\mathcal{B}$. It is isomorphic to Vir_c for some c, except that the vacuum vector is not cyclic. Of course, if $\mathcal{B}$ is a Virasoro net, then $\mathcal{B}_{\mathrm{Vir}} = \mathcal{B}$ by construction.

Xu has constructed irreducible DHR endomorphisms of the coset net arising from the diagonal embedding $\mathrm{SU}(n) \subset \mathrm{SU}(n)_k \otimes \mathrm{SU}(n)_l$ and computed their fusion rules in [67, Th. 4.6]. In the case of the Virasoro net with central charge $c = 1 - 6/m(m + 1)$, this gives the following result. For $\mathrm{SU}(2)_{m-1} \subset \mathrm{SU}(2)_{m-2} \times \mathrm{SU}(2)_1$, we use a label $j = 0, 1, \ldots, m - 2$ for the irreducible DHR endomorphisms of $\mathrm{SU}(2)_{m-2}$. Similarly, we use $k = 0, 1, \ldots, m-1$ and $l = 0, 1$ for the irreducible DHR endomorphisms of $\mathrm{SU}(2)_{m-1}$ and $\mathrm{SU}(2)_1$, respectively. (The label "0" always denote the identity endomorphism.) Then the irreducible DHR endomorphisms of the Virasoro net are labeled with triples (j, k, l) with $j - k + l$ being even under identification $(j, k, l) = (m-2-j, m-1-k, 1-l)$. Since $l \in \{0, 1\}$ is uniquely determined by (j, k) under this parity condition, we may and do label them with pairs (j, k) under identification $(j, k) = (m-2-j, m-1-k)$. In order to identify these DHR

endomorphisms with characters of the minimal models, we use variables p, q with $p = j+1, q = k+1$. Then we have $p \in \{1, 2, \ldots, m-1\}$, $q \in \{1, 2, \ldots, m\}$. We denote the DHR endomorphism of the Virasoro net labeled with the pair (p, q) by $\lambda_{(p,q)}$. That is, we have $m(m-1)/2$ irreducible DHR sectors $[\lambda_{(p,q)}]$, $1 \le p \le m-1$, $1 \le q \le m$ with the identification $[\lambda_{(p,q)}] = [\lambda_{(m-p,m+1-q)}]$, and then their fusion rules are identical to the one in (3). Although the indices of these DHR sectors are not explicitly computed in [67], these fusion rules uniquely determine the indices by the Perron-Frobenius theorem. All the irreducible DHR sectors of the Virasoro net on the circle with central charge $c = 1 - 6/m(m+1)$ are given as $[\lambda_{(p,q)}]$ as above by [68, Prop. 3.7]. Note that the μ-index of the Virasoro net with central charge $c = 1 - 6/m(m+1)$ is

$$\frac{m(m+1)}{8 \sin^2 \frac{\pi}{m} \sin^2 \frac{\pi}{m+1}}$$

by [68, Lemma 3.6].

Next we need statistical phases of the DHR sectors $[\lambda_{(p,q)}]$. Recall that an irreducible DHR endomorphism $r \in \{0, 1, \ldots, n\}$ of $\mathrm{SU}(2)_n$ has the statistical phase $\exp(2\pi r(r+2)i/4(n+2))$. This shows that for the triple (j, k, l), the statistical phase of the DHR endomorphism l of $\mathrm{SU}(2)_1$ is given by $\exp(2\pi(j-k)^2 i/4)$, because of the condition $j - k + l \in 2\mathbb{Z}$. Then by [69, Th. 4.6.(i)] and [4, Lemma 6.1], we obtain that the statistical phase of the DHR endomorphism $[\lambda_{(p,q)}]$ is

$$\exp 2\pi i \left(\frac{(m+1)p^2 - mq^2 - 1 + m(m+1)(p-q)^2}{4m(m+1)} \right),$$

which is equal to $\exp(2\pi i h_{p,q})$ with $h_{p,q}$ as in (4). Thus the S, T-matrices of Kac-Petersen in [14, Sec. 10.6] and the S, T-matrices for the DHR sectors $[\lambda_{(p,q)}]$ defined from the braiding as in [54] coincide. This shows that the unitary representations of $\mathrm{SL}(2, \mathbb{Z})$ studied in [11] for the minimal models and those arising from the braidings on the Virasoro nets are identical. So when we say the modular invariants for the Virasoro nets, we mean those in [11].

COROLLARY 3.6. *There is a natural bijection between representations of the* Vir_c *net and projective unitary (positive energy) representations of the group* $\mathrm{Diff}(S^1)$ *with central charge* $c < 1$.

Proof. If π is a representation of Vir_c, then the irreducible sectors are automatically Möbius covariant with positivity of the energy [25] because they have finite index and Vir_c is strongly additive by Corollary 3.4. Thus all sectors are diffeomorphism covariant by Lemma 3.1 and the associated covariance representation U_π is a projective unitary representation of $\mathrm{Diff}(S^1)$. The converse follows from the above description of the DHR sectors. $\square$

Remark 3.7. We give a remark about the thesis [42] of Loke. He constructed irreducible DHR endomorphisms of the Virasoro net with $c < 1$ using the discrete series of projective unitary representations of $\mathrm{Diff}(S^1)$ and computed their fusion rules, which coincides with the one given above. However, his proof of strong additivity contains a serious gap and this affects the entire results in [42]. So we have avoided using his results here. (The proof of strong additivity in [63, Th. E] also has a similar trouble, but the arguments in [61] gives a correct proof of the strong additivity of the $\mathrm{SU}(n)_k$-net and the results in [63] are not affected.) A. Wassermann informed us that he can fix this error and recover the results in [42]. (Note that the strong additivity for Vir_c with $c < 1$ follows from our Corollary 3.4.) If we can use the results in [42] directly, we can give an alternate proof of the results in this section as follows. First, Loke's results imply that the Virasoro nets are rational in the sense that we have only finitely many irreducible DHR endomorphisms and that all of them have finite indices. This is enough for showing that the Virasoro net with $c < 1$ is contained in the corresponding coset net irreducibly as in the remark after the proof of Proposition 3.5. Then Proposition 2.3 implies that the index is finite and this already shows that the Virasoro net is completely rational by [45]. Then by comparing the μ-indices of the Virasoro net and the coset net, we conclude that the two nets are equal.

4. Classification of local extensions of the Virasoro nets

By [11], we have a complete classification of the modular invariants for the Virasoro nets with central charge $c = 1 - 6/m(m + 1) < 1$, $m = 2, 3, 4, \ldots$. If each modular invariant is realized with α-induction for an extension $\mathrm{Vir}_c \subset \mathcal{B}$ as in [5, Cor. 5.8], then we have the numbers of irreducible morphisms as in Tables 3, 4 by a similar method to the one used in [6, Table 1, p. 774], where $|_{\mathcal{A}}\Delta_{\mathcal{B}}|$, $|_{\mathcal{B}}\Delta_{\mathcal{B}}|$, $|_{\mathcal{B}}\Delta_{\mathcal{B}}^+|$, and $|_{\mathcal{B}}\Delta_{\mathcal{B}}^0|$ denote the numbers of irreducible $\mathcal{A}$-$\mathcal{B}$ sectors, $\mathcal{B}$-$\mathcal{B}$ sectors, $\mathcal{B}$-$\mathcal{B}$ sectors arising from $\alpha^\pm$-induction, and the ambichiral $\mathcal{B}$-$\mathcal{B}$ sectors, respectively. (The ambichiral sectors are those arising from both α^+- and α^--induction, as in [6, p. 741].) We will prove that the entries in Table 3 correspond bijectively to local extensions of the Virasoro nets and that each entry in Table 4 is realized with a nonlocal extension of the Virasoro net. (For the labels for Z in Table 3, see Table 1.)

THEOREM 4.1. *The local irreducible extensions of the Virasoro nets on the circle with central charge less than* 1 *correspond bijectively to the entries in Table* 3.

Note that the index $[\mathcal{B} : \mathcal{A}]$ in the seven cases in Table 3 are 1, 2, 2, $3 + \sqrt{3}$, $3 + \sqrt{3}$, $\sqrt{30 - 6\sqrt{5}}/2\sin(\pi/30) = 19.479\cdots$, $\sqrt{30 - 6\sqrt{5}}/2\sin(\pi/30) = 19.479\cdots$, respectively.

m	Labels for Z	$|_\mathcal{A}\Delta_\mathcal{B}|$	$|_\mathcal{B}\Delta_\mathcal{B}|$	$|_\mathcal{B}\Delta_\mathcal{B}^+|$	$|_\mathcal{B}\Delta_\mathcal{B}^0|$
n	(A_{n-1}, A_n)	$n(n-1)/2$	$n(n-1)/2$	$n(n-1)/2$	$n(n-1)/2$
$4n+1$	(A_{4n}, D_{2n+2})	$2n(2n+2)$	$2n(4n+4)$	$2n(2n+2)$	$2n(n+2)$
$4n+2$	(D_{2n+2}, A_{4n+2})	$(2n+1)(2n+2)$	$(2n+1)(4n+4)$	$(2n+1)(2n+2)$	$(2n+1)(n+2)$
11	(A_{10}, E_6)	30	60	30	15
12	(E_6, A_{12})	36	72	36	18
29	(A_{28}, E_8)	112	448	112	28
30	(E_8, A_{30})	120	480	120	30

Table 3: Type I modular invariants for the Virasoro nets

m	Labels for Z	$|_\mathcal{A}\Delta_\mathcal{B}|$	$|_\mathcal{B}\Delta_\mathcal{B}|$	$|_\mathcal{B}\Delta_\mathcal{B}^+|$	$|_\mathcal{B}\Delta_\mathcal{B}^0|$
$4n$	(D_{2n+1}, A_{4n})	$2n(2n+1)$	$2n(4n-1)$	$2n(4n-1)$	$2n(4n-1)$
$4n+3$	(A_{4n+2}, D_{2n+3})	$(2n+1)(2n+3)$	$(2n+1)(4n+3)$	$(2n+1)(4n+3)$	$(2n+1)(4n+3)$
17	(A_{16}, E_7)	56	136	80	48
18	(E_7, A_{18})	63	153	90	54

Table 4: Type II modular invariants for the Virasoro nets

THEOREM 4.2. *Each entry in Table 4 is realized by α-induction for a nonlocal (but relatively local) extension of the Virasoro net with central charge $c = 1 - 6/m(m+1)$.*

Proofs of these theorems are given in the following subsections.

Remark 4.3. Here we make explicit that every irreducible net extension $\mathcal{A}$ of Vir_c, $c < 1$, is diffeomorphism covariant.

First note that every representation ρ of Vir_c is diffeomorphism covariant; indeed we can assume that $d(\rho) < \infty$ (by decomposition into irreducibles); thus ρ is Möbius covariant with positive energy by [25] because Vir_c is strongly additive. Then ρ is diffeomorphism covariant by Lemma 3.1.

Now fix an interval $I \subset S^1$ and consider a canonical endomorphism γ_I of $\mathcal{A}(I)$ into $\mathrm{Vir}_c(I)$ so that $\theta_I \equiv \gamma_I \upharpoonright_{\mathrm{Vir}_c(I)}$ is the restriction of a DHR endomorphism θ localized in I. With z_θ the covariance cocycle of θ, the covariant action of $\mathrm{Diff}(S^1)$ on $\mathcal{A}$ is given by

$$\tilde{\alpha}_g(X) = \alpha_g(X), \qquad \tilde{\alpha}_g(T) = z_\theta(g)^*T, \qquad g \in \mathrm{Diff}(S^1)$$

where X is a local operator of Vir_c, $T \in \mathcal{A}(I)$ is isometry intertwining the identity and γ_I and α is the covariant action of $\mathrm{Diff}(S^1)$ on Vir_c (cf. [45]).

4.1. *Simple current extensions.* First we handle the easier case, the simple current extensions of index 2 in Theorem 4.2.

Let $\mathcal{A}$ be the Virasoro net with central charge $c = 1 - 6/m(m+1)$. We have irreducible DHR endomorphisms $\lambda_{(p,q)}$ as in Subsection 2.2. The statistics phase of the sector $\lambda_{(m-1,1)}$ is $\exp(\pi i(m-1)(m-2)/2)$ by (4). This is equal

to 1 if $m \equiv 1, 2 \mod 4$, and -1 if $m \equiv 0, 3 \mod 4$. In both cases, we can take an automorphism σ with $\sigma^2 = 1$ within the unitary equivalence class of the sector $[\lambda_{(m-1,1)}]$ by [55, Lemma 4.4]. It is clear that $\rho = \mathrm{id} \oplus \sigma$ is an endomorphism of a Q-system, so we can make an irreducible extension $\mathcal{B}$ with index 2 by [46, Th. 4.9]. By [3, II, Cor. 3.7], the extension is local if and only if $m \equiv 1, 2 \mod 4$. The extensions are unique for each m, because of triviality of $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{T})$ and [32], and we get the modular invariants as in Tables 3, 4. (See [3, II, Sec. 3] for similar computations.)

4.2. *The four exceptional cases.* We next handle the remaining four exceptional cases in Theorem 4.2, and first deal with the case $m = 11$ for the modular invariants (A_{10}, E_6). The other three cases can be handled in very similar ways.

Let $\mathcal{A}$ be the Virasoro net with central charge $c = 21/22$. Fix an interval I on the circle and consider the set of DHR endomorphisms of the net $\mathcal{A}$ localized in I as in Subsection 2.2. Then consider the subset $\{\lambda_{(1,1)}, \lambda_{(1,2)}, \ldots, \lambda_{(1,11)}\}$ of the DHR endomorphisms. By the fusion rules (3), this system is closed under composition and conjugation, and the fusion rules are the same as for $\mathrm{SU}(2)_{10}$. So the subfactor $\lambda_{(1,2)}(\mathcal{A}(I)) \subset \mathcal{A}(I)$ has the principal graph A_{11} and the fusion rules and the quantum $6j$-symbols for the subsystem $\{\lambda_{(1,1)}, \lambda_{(1,3)}, \lambda_{(1,5)}, \ldots, \lambda_{(1,11)}\}$ of the DHR endomorphisms are the same as those for the usual Jones subfactor with principal graph A_{11} and are uniquely determined. (See [48], [37], [17, Ch. 9–12].) Since we already know by Theorem 2.4 that the endomorphism $\lambda_0 \oplus \lambda_6$ gives a Q-system uniquely for the system of irreducible DHR sectors $\{\lambda_0, \lambda_1, \ldots, \lambda_{10}\}$ for the $\mathrm{SU}(2)_{10}$ net, we also know that the endomorphism $\lambda_{(1,1)} \oplus \lambda_{(1,7)}$ gives a Q-system uniquely, by the above identification of the fusion rules and quantum $6j$-symbols. By [46, Th. 4.9], we can make an irreducible extension $\mathcal{B}$ of $\mathcal{A}$ using this Q-system, but the locality criterion in [46, Th. 4.9] depends on the braiding structure of the system, and the standard braiding on the $\mathrm{SU}(2)_{10}$ net and the braiding we know have on $\{\lambda_{(1,1)}, \lambda_{(1,2)}, \ldots, \lambda_{(1,11)}\}$ from the Virasoro net are not the same, since their spins are different. So we need an extra argument for showing the locality of the extension.

Even when the extension is not local, we can apply the α-induction to the subfactor $\mathcal{A}(I) \subset \mathcal{B}(I)$ and then the matrix Z given by $Z_{\lambda\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$ is a modular invariant for the S and T matrices arising from the minimal model by [5, Cor. 5.8]. (Recall that the braiding is now nondegenerate.) By the Cappelli-Itzykson-Zuber classification [11], we have only three possibilities for this matrix at $m = 11$. It is now easy to count the number of $\mathcal{A}(I)$-$\mathcal{B}(I)$ sectors arising from all the DHR sectors of $\mathcal{A}$ and the embedding $\iota : \mathcal{A}(I) \subset \mathcal{B}(I)$ as in [5], [6], and the number is 30. Then by [5] and the Tables 3, 4, we conclude that the matrix Z is of type (A_{10}, E_6). Then by a criterion of locality due to Böckenhauer-Evans [4, Prop. 3.2], we conclude from this modular invariant matrix that the extension $\mathcal{B}$ is local. The uniqueness of $\mathcal{B}$ also follows from

the above argument. (Uniqueness in Theorem 2.4 is under an assumption of locality, but the above argument based on [4] shows that an extension is automatically local in this setting.)

In the case of $m = 12$ for the modular invariant (E_6, A_{12}), we now use the system $\{\lambda_{(1,1)}, \lambda_{(2,1)}, \ldots, \lambda_{(11,1)}\}$. Then the rest of the arguments are the same as above. The cases $m = 29$ for the modular invariant (A_{28}, E_8) and $m = 30$ for the modular invariant (E_8, A_{30}) are handled in similar ways.

Remark 4.4. In the above cases, we can determine the isomorphism class of the subfactors $\mathcal{A}(I) \subset \mathcal{B}(I)$ for a fixed interval I as follows. Let $m = 11$. By the same arguments as in [6, App.], we conclude that the subfactor $\mathcal{A}(I) \subset \mathcal{B}(I)$ is the Goodman-de la Harpe-Jones subfactor [24, Sec. 4.5] of index $3 + \sqrt{3}$ arising from the Dynkin diagram E_6. We get the isomorphic subfactor also for $m = 12$. The cases $m = 29, 30$ give the Goodman-de la Harpe-Jones subfactor arising from E_8.

4.3. *Nonlocal extensions.* We now explain how to prove Theorem 4.2. We have already seen the case of D_{odd} above. In the case of $m = 17, 18$ for the modular invariants of type (A_{16}, E_7), (E_7, A_{18}), respectively, we can make Q-systems in very similar ways to the above cases. Then we can make the extensions $\mathcal{B}(I)$, but the criterion in [4, Prop. 3.2] shows that they are not local. The extensions are relatively local by [46, Th. 4.9].

4.4. *The case $c = 1$.* By [56], we know that the Virasoro net for $c = 1$ is the fixed-point net of the $\mathrm{SU}(2)_1$ net with the action of $\mathrm{SU}(2)$. That is, for each closed subgroup of $\mathrm{SU}(2)$, we have a fixed point net, which is an irreducible local extension of the Virasoro net with $c = 1$. Such subgroups are labeled with affine *A-D-E* diagrams and we have infinitely many such subgroups. (See [24, Sec. 4.7.d], for example.) Thus finiteness of local extensions fails for the case $c = 1$.

Note also that, if $c > 1$, Vir_c is not strongly additive [10] and all sectors except the identity are expected to be infinite-dimensional [56].

5. Classification of conformal nets

We now give our main result.

THEOREM 5.1. *The local (irreducible) conformal nets on the circle with central charge less than 1 correspond bijectively to the entries in Table 3.*

Proof. By Proposition 3.5, a conformal net $\mathcal{B}$ on the circle with central charge less than 1 contains a Virasoro net as an irreducible subnet. Thus Theorem 4.1 gives the desired conclusion. $\qquad\square$

In this theorem, the correspondence between such conformal nets and pairs of Dynkin diagrams is given explicitly as follows. Let $\mathcal{B}$ be such a net with central charge $c < 1$ and Vir_c its canonical Virasoro subnet as above. Fix an interval $I \subset S^1$. For a DHR endomorphism $\lambda(p, q)$ of Vir_c localized in I, we have $\alpha^{\pm}$-induced endomorphism $\alpha^{\pm}_{\lambda(p,q)}$ of $\mathcal{B}(I)$. We denote this endomorphism simply by $\alpha^{\pm}_{(p,q)}$. Then we have two subfactors $\alpha^+_{(2,1)}(\mathcal{B}(I)) \subset \mathcal{B}(I)$ and $\alpha^+_{(1,2)}(\mathcal{B}(I)) \subset \mathcal{B}(I)$ and the index values are both below 4. Let (G, G') be the pair of the corresponding principal graphs of these two subfactors. The above main theorem says that the map from $\mathcal{B}$ to (G, G') gives a bijection from the set of isomorphism classes of such nets to the set of pairs (G, G') of A_n-D_{2n}-$E_{6,8}$ Dynkin diagrams such that the Coxeter number of G is smaller than that of G' by 1.

6. Applications and remarks

In this section, we identify some coset nets studied in [3], [69] in our classification list, as applications of our main results.

6.1. *Certain coset nets and extensions of the Virasoro nets.*   In [69, Sec. 3.7], Xu considered the three coset nets arising from $\mathrm{SU}(2)_8 \subset \mathrm{SU}(3)_2$, $\mathrm{SU}(3)_2 \subset \mathrm{SU}(3)_1 \times \mathrm{SU}(3)_1$, $\mathrm{U}(1)_6 \subset \mathrm{SU}(2)_3$, all at central charge $4/5$. He found that all have six simple objects in the tensor categories of the DHR endomorphisms and give the same invariants for 3-manifolds. Our classification Theorem 5.1 shows that these three nets are indeed isomorphic as follows.

Theorem 5.1 shows that we have only two conformal nets at central charge $4/5$. One is the Virasoro net itself with $m = 5$ that has 10 irreducible DHR endomorphisms, and the other is its simple current extension of index 2 that has 6 irreducible DHR endomorphisms. This implies that all the three cosets above are isomorphic to the latter.

6.2. *More coset nets and extensions of the Virasoro nets.*   For the local extensions of the Virasoro nets corresponding to the modular invariants (E_6, A_{12}), (E_8, A_{30}), Böckenhauer-Evans [3, II, Subsec. 5.2] say that "the natural candidates" are the cosets arising from $\mathrm{SU}(2)_{11} \subset \mathrm{SO}(5)_1 \times \mathrm{SU}(2)_1$ and $\mathrm{SU}(2)_{29} \subset (G_2)_1 \times \mathrm{SU}(2)_1$, respectively, but they were unable to prove that these cosets indeed produce the desired local extensions. (For the modular invariants (A_{10}, E_6), (A_{28}, E_8), they also say that "there is no such natural candidate" in [3, II, Subsec. 5.2].) It is obvious that the above two cosets give local irreducible extensions of the Virasoro nets, but the problem is that the index might be 1. Here we already have a complete classification of local irreducible extensions of the Virasoro nets, and using it, we can prove that the above two cosets indeed coincide with the extension we have constructed above.

First we consider the case of the modular invariant (E_6, A_{12}). Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be the nets corresponding to $\mathrm{SU}(2)_{11}$, $\mathrm{SU}(2)_{10} \times \mathrm{SU}(2)_1$, $\mathrm{SO}(5)_1 \times \mathrm{SU}(2)_1$, respectively. We have natural inclusions $\mathcal{A}(I) \subset \mathcal{B}(I) \subset \mathcal{C}(I)$, and define the coset nets by $\mathcal{D}(I) = \mathcal{A}(I)' \cap \mathcal{B}(I)$, $\mathcal{E}(I) = \mathcal{A}(I)' \cap \mathcal{C}(I)$. We know that the net $\mathcal{D}(I)$ is the Virasoro net with central charge $25/26$ and will prove that the extension $\mathcal{E}$ is the one corresponding to the entry (E_6, A_{12}) in Table 3 in Theorem 4.1.

The following diagram

$$
\begin{array}{ccc}
\mathcal{A}(I) \vee \mathcal{D}(I) & \subset & \mathcal{B}(I) \\
\cap & & \cap \\
\mathcal{A}(I) \vee \mathcal{E}(I) & \subset & \mathcal{C}(I)
\end{array}
$$

is a commuting square [51], [24, Ch. 4], and we have

$$
(6) \qquad [\mathcal{B}(I) : \mathcal{A}(I) \vee \mathcal{D}(I)] \leq [\mathcal{C}(I) : \mathcal{A}(I) \vee \mathcal{E}(I)] < \infty.
$$

Next note that the new coset net $\{\mathcal{E}(I)' \cap \mathcal{C}(I)\}$ gives an irreducible local extension of the net $\mathcal{A}$, but Theorem 2.4 implies that we have no strict extension of $\mathcal{A}$. Thus we have $\mathcal{E}(I)' \cap \mathcal{C}(I) = \mathcal{A}(I)$, and $\mathcal{A}(I), \mathcal{E}(I)$ are the relative commutants of each other in $\mathcal{C}(I)$. So we can consider the inclusion $\mathcal{A}(I) \otimes \mathcal{E}(I) \subset \mathcal{C}(I)$ and this is a canonical tensor product subfactor in the sense of Rehren [57], [58]. (See [57, ll. 22–24, p. 701].) Thus the dual canonical endomorphism for this subfactor is of the form $\bigoplus_j \sigma_j \otimes \pi(\sigma_j)$, where $\{\sigma_j\}$ is a closed subsystem of DHR endomorphisms of the net $\mathcal{A}$ and the map π is a bijection from this subsystem to a closed subsystem of DHR endomorphisms of the net $\mathcal{E}$, by [57, Cor. 3.5, ll. 3–12, p. 706]. This implies that the index $[\mathcal{C}(I) : \mathcal{A}(I) \vee \mathcal{E}(I)]$ is a square sum of the statistical dimensions of the irreducible DHR endomorphisms over a subsystem of the $\mathrm{SU}(2)_{11}$-system. We have only three possibilities for such a closed subsystem as follows.

(1) $\{\lambda_0 = \mathrm{id}\}$,

(2) The even part $\{\lambda_0, \lambda_2, \ldots, \lambda_{10}\}$,

(3) The entire system $\{\lambda_0, \lambda_1, \ldots, \lambda_{11}\}$.

The first case would violate the inequality (6). Recall that we have only two possibilities for $\mu_{\mathcal{E}}$ by Theorem 4.1 and that we also have equality

$$
(7) \qquad \mu_{\mathcal{A}} \mu_{\mathcal{E}} = \mu_{\mathcal{C}} [\mathcal{C}(I) : \mathcal{A}(I) \vee \mathcal{E}(I)]
$$

by [40, Prop. 24]. Then the third case of the above three would be incompatible with the above equality (7), and thus we conclude that the second case occurs. Then the above equality (7) easily shows that the extension $\mathcal{E}(I)$ is the one corresponding to the entry (E_6, A_{12}) in Table 3 in Theorem 4.1.

The case (E_8, A_{30}) can be proved with a very similar argument to the above. We now have three possibilities for the μ-index by Theorem 4.1 instead of two possibilities above, but this causes no problem, and we get the desired isomorphism.

6.3. *Subnet structure.*   As a consequence of our results, the subnet structure of a local conformal net with $c < 1$ is very simple.

Let $\mathcal{A}$ be a local irreducible conformal net on S^1 with $c < 1$. The projective unitary representation U of $\mathrm{Diff}(S^1)$ is given so that the central charge and the Virasoro subnet are well-defined. By our classification, the Virasoro subnet (up to conjugacy), thus the central charge, do not depend on the choice of the covariance representation U if $c < 1$.

The following elementary lemma is implicit in the literature.

LEMMA 6.1. *Every projective unitary finite-dimensional representation of* $\mathrm{Diff}(S^1)$ *is trivial.*

Proof.   Otherwise, passing to the infinitesimal representation, we have operators L_n and c on a finite-dimensional Hilbert space satisfying the Virasoro relations (2) and the unitarity conditions $L_n^* = L_{-n}$. Then $\{L_1, L_{-1}, L_0\}$ gives a unitary finite-dimensional representation of the Lie algebra $s\ell(2, \mathbb{R})$; thus $L_1 = L_{-1} = L_0 = 0$. Then for $m \neq 0$ we have $L_m = m^{-1}[L_m, L_0] = 0$ and also $c = 0$ due to the relations (2).   □

PROPOSITION 6.2. *Let* $\mathcal{A}$ *be a local conformal net and* $\mathcal{B} \subset \mathcal{A}$ *a conformal subnet with finite index. Then* $\mathcal{B}$ *contains the Virasoro subnet:* $\mathcal{B}(I) \supset \mathcal{A}_{\mathrm{Vir}}(I)$, $I \in \mathcal{I}$.

Proof. Let π_0 denote the vacuum representation of $\mathcal{A}$. As $[\mathcal{A} : \mathcal{B}] < \infty$ we have an irreducible decomposition

$$(8) \qquad\qquad \pi_0|_{\mathcal{B}} = \bigoplus_{i=0}^n n_i \rho_i,$$

with $n_i < \infty$. Accordingly the vacuum Hilbert space $\mathcal{H}$ of $\mathcal{A}$ decomposes as $\mathcal{H} = \bigoplus_i \mathcal{H}_i \otimes \mathcal{K}_i$ where $\dim \mathcal{K}_i = n_i$.

By assumption, the projective unitary representation U implements automorphisms of $\pi_0(\mathcal{B})''$, hence of its commutant $\pi_0(\mathcal{B})' \simeq \bigoplus_i 1|_{\mathcal{H}_i} \otimes B(\mathcal{K}_i)$ which is finite-dimensional. As $\mathrm{Diff}(S^1)$ is connected, AdU acts trivially on the center of $\pi_0(\mathcal{B})'$, hence it implements automorphisms on each simple summand of $\pi_0(\mathcal{B})'$, isomorphic to $B(\mathcal{K}_i)$; hence it gives rise to a finite-dimensional representation of $\mathrm{Diff}(S^1)$ that is unitary with respect to the tracial scalar product, and so must be trivial because of Lemma 6.1. It follows that U decomposes

according to eq. (8) as

$$U = \bigoplus_{i=0}^{n} U_i \otimes 1|_{\mathcal{K}_i}$$

where U_i is a covariance representation for ρ_i. Thus $U(\mathrm{Diff}(I)) \subset \bigoplus_i B(\mathcal{H}_i) \otimes 1|_{\mathcal{K}_i} = \pi_0(\mathcal{B})''$, so $\mathcal{A}_{\mathrm{Vir}}(I) \subset \pi_0(\mathcal{B})'' \cap \mathcal{A}(I)$ which equals $\mathcal{B}(I)$ by Lemma 2.2. $\square$

THEOREM 6.3. *Let $\mathcal{A}$ be an irreducible local conformal net with central charge $c < 1$. Let s be the number of finite-index conformal subnets, up to conjugacy (including $\mathcal{A}$ itself). Then $s \in \{1, 2, 3\}$. $\mathcal{A}$ is completely classified by the pair (m, s) where $c = 1 - 6/m(m+1)$. For any $m \in \mathbb{N}$ the possible values of s are*:

1. *$s = 1$ for all $m \in \mathbb{N}$;*

2. *$s = 2$ if $m = 1, 2 \bmod 4$, and if $m = 11, 12$;*

3. *$s = 3$ if $m = 29, 30$.*

The corresponding structure follows from Table 3.

Proof. The proof is immediate by the classification Theorem 5.1 and Proposition 6.2. $\square$

6.4. *Remarks on subfactors and commuting squares.* It is interesting to point out that our framework of nets of subfactors as in [46] can be regarded as a net version of the usual classification problem of subfactors [34]. The difference here is that the smaller net is fixed and we wish to classify extensions, while in the usual subfactor setting a larger factor is fixed and we would like to classify factors contained in it. In the subfactor theory, classifying subfactors and classifying extensions are equivalent problems because of Jones's basic construction [34] (as long as we have finite index), but this is not true in the setting of nets of subfactors. Here, the basic construction does not work and considering an extension and considering a subnet are not symmetric procedures. (For a net of subfactors $\mathcal{A} \subset \mathcal{B}$, the dual canonical endomorphism for $\mathcal{A}(I) \subset \mathcal{B}(I)$ decomposes into DHR endomorphisms of the net $\mathcal{A}$, but the canonical endomorphism for $\mathcal{A}(I) \subset \mathcal{B}(I)$ does not decompose into DHR endomorphisms of the net $\mathcal{B}$.)

To illustrate this point, consider the example of a completely rational net $SU(2)_1$. This net has an action of $SU(2)$ by internal symmetries, and so a fixed point subnet with respect to any finite subgroup of $SU(2)$. We have infinitely many such finite subgroups; thus the completely rational net $SU(2)_1$ has infinitely many irreducible subnets with finite index. On the other hand,

the number of irreducible extensions of a given completely rational net is always finite, since the number of mutually inequivalent Q-systems (ρ, V, W) is finite for a given ρ by [32] and we have only finitely many choices of ρ for a given completely rational net, and this finite number is often very small, as shown in the main body of this paper. In general, considering extensions gives much stronger constraints than considering subnets, and this allows an interesting classification in concrete models.

Notice now that a net of factors on the circle produces a tensor category of DHR endomorphisms. On the other hand a subfactor $N \subset M$ with finite index produces tensor categories of endomorphisms of N and M arising from the powers of (dual) canonical endomorphisms. In this analogy, complete rationality corresponds to the finite depth condition for subfactors, and the 2-interval inclusion has similarity to the construction in [46], or the quantum double construction, as explained in [40]. A net of subfactors corresponds to "an inclusion of one subfactor into another subfactor", that is, a commuting square of factors [51], studied in [38]. For any subfactor $N \subset M$ with finite index, we have a Jones subfactor $P \subset Q$ made of the Jones projections with the same index [34] such that we have a commuting square

$$
\begin{array}{ccc}
N & \subset & M \\
\cup & & \cup \\
P & \subset & Q.
\end{array}
$$

In this sense, the Jones subfactors are "minimal" among general subfactors. The Virasoro nets have a similar minimality among nets of factors with diffeomorphism covariance, they are contained in every local conformal net (but they do not admit any nontrivial subnet [12]). This similarity is a guide to understanding our work.

In the above example of a commuting square, we have no control over an inclusion $P \subset N$ in general, but in the case of Virasoro net, we do have control over the inclusion if the central charge is less than 1. This has enabled us to obtain our results. As often pointed out, the condition that the Jones index is less than 4 has some formal similarity to the condition that the central charge is less than 1. The results in this paper give further evidence for this similarity.

UNIVERSITY OF TOKYO, KOMABA, TOKYO, JAPAN
E-mail address: yasuyuki@ms.u-tokyo.ac.jp

UNIVERSITY OF ROME, "TOR VERGATA", ROME, ITALY
E-mail address: longo@mat.uniroma2.it

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