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Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers

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Abstract

Inspired by Lorenz' remarkable chaotic flow, we describe in this paper the structure of all C^1 robust transitive sets with singularities for flows on closed 3-manifolds: they are partially hyperbolic with volume-expanding central direction, and are either attractors or repellers. In particular, any C^1 robust attractor with singularities for flows on closed 3-manifolds always has an invariant foliation whose leaves are forward contracted by the flow, and has positive Lyapunov exponent at every orbit, showing that *any* C^1 robust attractor resembles a geometric Lorenz attractor.

1. Introduction

A long-time goal in the theory of dynamical systems has been to describe and characterize systems exhibiting dynamical properties that are preserved under small perturbations. A cornerstone in this direction was the Stability Conjecture (Palis-Smale [30]), establishing that those systems that are identical, up to a continuous change of coordinates of phase space, to all nearby systems are characterized as the hyperbolic ones. Sufficient conditions for structural stability were proved by Robbin [36] (for $r \geq 2$), de Melo [6] and Robinson [38] (for $r = 1$). Their necessity was reduced to showing that structural stability implies hyperbolicity (Robinson [37]). And that was proved by Mañé [23] in the discrete case (for $r = 1$) and Hayashi [13] in the framework of flows (for $r = 1$).

This has important consequences because there is a rich theory of hyperbolic systems describing their geometric and ergodic properties. In particular, by Smale's spectral decomposition theorem [39], one has a description of the nonwandering set of a structural stable system as a finite number of disjoint *compact maximal invariant and transitive sets*, each of these pieces being well understood from both the deterministic and statistical points of view. Fur-

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thermore, such a decomposition persists under small C^1 perturbations. This naturally leads to the study of isolated transitive sets that remain transitive for all nearby systems (robustness).

What can one say about the dynamics of robust transitive sets? Is there a characterization of such sets that also gives dynamical information about them? In the case of 3-flows, a striking example is the Lorenz attractor [19], given by the solutions of the polynomial vector field in $\mathbb{R}^3$:

$$X(x,y,z) = \begin{cases} \dot{x} = -\alpha x + \alpha y \\ \dot{y} = \beta x - y - xz \\ \dot{z} = -\gamma z + xy\,, \end{cases} \tag{1}$$

where α, β, γ are real parameters. Numerical experiments performed by Lorenz (for $\alpha = 10, \beta = 28$ and $\gamma = 8/3$) suggested the existence, in a robust way, of a strange attractor toward which a full neighborhood of positive trajectories of the above system tends. That is, the strange attractor could not be destroyed by any perturbation of the parameters. Most important, the attractor contains an equilibrium point $(0,0,0)$, and periodic points accumulating on it, and hence can not be hyperbolic. Notably, only now, three and a half decades after this remarkable work, did Tucker prove [40] that the solutions of (1) satisfy such a property for values α, β, γ near the ones considered by Lorenz.

However, in the mid-seventies, the existence of robust nonhyperbolic attractors was proved for flows (introduced in [1] and [11]), which we now call geometric models for Lorenz attractors. In particular, they exhibit, in a robust way, an attracting transitive set with an equilibrium (singularity). For such models, the eigenvalues $\lambda_i, 1 \leq i \leq 3$, associated to the singularity are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$. In the definition of geometrical models, another key requirement was the existence of an invariant foliation whose leaves are forward contracted by the flow. Apart from some other technical assumptions, these features allow one to extract very complete topological, dynamical and ergodic information about these geometrical Lorenz models [12]. The question we address here is whether such features are present for any robust transitive set.

Indeed, the main aim of our paper is to describe the dynamical structure of compact transitive sets (there are dense orbits) of flows on 3-manifolds which are *robust* under small C^1 perturbations. We shall prove that *C^1 robust transitive sets with singularities on closed 3-manifolds are either proper attractors or proper repellers.* We shall also show that the singularities lying in a C^1 robust transitive set of a 3-flow are Lorenz-like: *the eigenvalues at the singularities satisfy the same inequalities as the corresponding ones at the singularity in a Lorenz geometrical model.* As already observed, the presence of a singularity prevents these attractors from being hyperbolic. On the other hand, we are going to prove that robustness does imply a weaker form of hyperbolicity:

C^1 robust attractors for 3-flows are partially hyperbolic with a volume-expanding central direction.

A first consequence from this is that every orbit in *any* robust attractor has a direction of exponential divergence from nearby orbits (positive Lyapunov exponent). Another consequence is that robust attractors always admit an invariant foliation whose leaves are forward contracted by the flow, showing that any robust attractor with singularities displays similar properties to those of the geometrical Lorenz model. In particular, in view of the result of Tucker [40], the Lorenz attractor generated by the Lorenz equations (1) much resembles a geometrical one.

To state our results in a precise way, let us fix some notation and recall some definitions and results proved elsewhere.

Throughout, M is a boundaryless compact manifold and $\mathcal{X}^r(M)$ denotes the space of C^r vector fields on M endowed with the C^r topology, $r \geq 1$. If $X \in \mathcal{X}^r(M)$, X_t, $t \in \mathbb{R}$, denotes the flow induced by X.

1.1. *Robust transitive sets are attractors or repellers.* A compact invariant set Λ of X is *isolated* if there exists an open set $U \supset \Lambda$, called an *isolating block*, such that

$$\Lambda = \bigcap_{t\in\mathbb{R}} X_t(U).$$

If U can be chosen such that $X_t(U) \subset U$ for $t > 0$, we say that the isolated set Λ is an *attracting set*.

A compact invariant set Λ of X is *transitive* if it coincides with the ω-limit set of an X-orbit. An *attractor* is a transitive attracting set. A *repeller* is an attractor for the reversed vector field $-X$. An attractor (or repeller) which is not the whole manifold is called *proper*. An invariant set of X is *nontrivial* if it is neither a periodic point nor a singularity.

Definition 1.1. An isolated set Λ of a C^1 vector field X is *robust transitive* if it has an isolating block U such that

$$\Lambda_Y(U) = \bigcap_{t\in\mathbb{R}} Y_t(U)$$

is both transitive and nontrivial for any Y C^1-close to X.

THEOREM A. *A robust transitive set containing singularities of a flow on a closed 3-manifold is either a proper attractor or a proper repeller.*

As a matter-of-fact, Theorem A will follow from a general result on n-manifolds, $n \geq 3$, settling sufficient conditions for an isolated set to be an attracting set: (a) all its periodic points and singularities are hyperbolic and (b) it robustly contains the unstable manifold of either a periodic point or a singularity (Theorem D). This will be established in Section 2.

Theorem A is false in dimension bigger than three; a counterexample can be obtained by multiplying the geometric Lorenz attractor by a hyperbolic system in such a way that the directions supporting the Lorenz flow are normally hyperbolic. It is false as well in the context of boundary-preserving vector fields on 3-manifolds with boundary [17]. The converse to Theorem A is also not true: proper attractors (or repellers) with singularities are not necessarily robust transitive, even if their periodic points and singularities are hyperbolic in a robust way.

Let us describe a global consequence of Theorem A, improving a result in [9]. To state it, we recall that a vector field X on a manifold M is *Anosov* if M is a hyperbolic set of X. We say that X is *Axiom* A if its nonwandering set $\Omega(X)$ decomposes into two disjoint invariant sets $\Omega_0 \bigcup \Omega_1$, where Ω_0 consists of finitely many hyperbolic singularities and Ω_1 is a hyperbolic set which is the closure of the (nontrivial) periodic orbits.

Corollary 1.2. *C^1 vector fields on a closed 3-manifold with robust transitive nonwandering sets are Anosov.*

Indeed, let X be a C^1 vector field satisfying the hypothesis of the corollary. If the nonwandering set $\Omega(X)$ has singularities, then $\Omega(X)$ is either a proper attractor or a proper repeller of X by Theorem A, which is impossible. Then $\Omega(X)$ is a robust transitive set without singularities. By [9], [41] we conclude that $\Omega(X)$ is hyperbolic. Consequently, X is Axiom A with a unique basic set in its spectral decomposition. Since Axiom A vector fields always exhibit at least one attractor and $\Omega(X)$ is the unique basic set of X, it follows that $\Omega(X)$ is an attractor. But clearly this is possible only if $\Omega(X)$ is the whole manifold. As $\Omega(X)$ is hyperbolic, we conclude that X is Anosov as desired. □

Here we observe that the conclusion of the last corollary holds, replacing in its statement nonwandering set by limit-set [31].

1.2. *The singularities of robust attractors are Lorenz-like.* To motivate the next theorem, recall that the geometric Lorenz attractor L is a proper robust transitive set with a hyperbolic singularity σ such that if $\lambda_i, 1 \leq i \leq 3$, are the eigenvalues of L at σ, then λ_i, $1 \leq i \leq 3$, are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$ [12]. Inspired by this property we introduce the following definition.

Definition 1.3. A singularity σ is *Lorenz-like* for X if the eigenvalues $\lambda_i, 1 \leq i \leq 3$, of $DX(\sigma)$ are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$.

If σ is a Lorenz-like singularity for X then the strong stable manifold $W^{ss}_X(\sigma)$ exists. Moreover, $\dim(W^{ss}_X(\sigma)) = 1$, and $W^{ss}_X(\sigma)$ is tangent to the eigenvector direction associated to λ_2. Given a vector field $X \in \mathcal{X}^r(M)$, we let $\mathrm{Sing}(X)$ be the set of singularities of X. If Λ is a compact invariant set of X we let $\mathrm{Sing}_X(\Lambda)$ be the set of singularities of X in Λ.

The next result shows that the singularities of robust transitive sets on closed 3-manifolds are Lorenz-like.

THEOREM B. *Let Λ be a robust singular transitive set of $X \in \mathcal{X}^1(M)$. Then, either for $Y = X$ or $Y = -X$, every $\sigma \in \mathrm{Sing}_Y(\Lambda)$ is Lorenz-like for Y and satisfies*

$$W_Y^{ss}(\sigma) \cap \Lambda = \{\sigma\}.$$

The following result is a direct consequence of Theorem B. A *robust attractor* of a C^1 vector field X is an attractor of X that is also a robust transitive set of X.

COROLLARY 1.4. *Every singularity of a robust attractor of X on a closed 3-manifold is Lorenz-like for X.*

In light of these results, a natural question arises: can one achieve a general description of the structure for robust attractors? In this direction we prove: if Λ is a robust attractor for X containing singularities then it is partially hyperbolic with volume-expanding central direction.

1.3. *Robust attractors are singular-hyperbolic.* To state this result in a precise way, let us introduce some definitions and notations.

Definition 1.5. Let Λ be a compact invariant transitive set of $X \in \mathcal{X}^r(M)$, $c > 0$, and $0 < \lambda < 1$. We say that Λ has a (c, λ)-*dominated splitting* if the bundle over Λ can be written as a continuous DX_t-invariant sum of sub-bundles

$$T_\Lambda = E^s \oplus E^{cu},$$

such that for every $T > 0$, and every $x \in \Lambda$,

(a) E^s is one-dimensional,

(b) The bundle E^{cu} contains the direction of X, and

$$\|DX_T/E_x^s\|.\|DX_{-T}/E_{X_T(x)}^{cu}\| < c\,\lambda^T.$$

E^{cu} is called the *central direction* of T_Λ.

A compact invariant transitive set Λ of X is *partially hyperbolic* if Λ has a (c, λ)-dominated splitting $T_\Lambda M = E^s \oplus E^{cu}$ such that the bundle E^s is uniformly contracting; that is, for every $T > 0$, and every $x \in \Lambda$,

$$\|DX_T/E_x^s\| < c\,\lambda^T.$$

For $x \in \Lambda$ and $t \in I\!R$ we let $J_t^c(x)$ be the absolute value of the determinant of the linear map $DX_t/E_x^{cu} : E_x^{cu} \to E_{X_t(x)}^{cu}$. We say that the subbundle E_Λ^{cu} of the partially hyperbolic set Λ is *volume-expanding* if

$$J_t^c(x) \geq c\, e^{\lambda t},$$

for every $x \in \Lambda$ and $t \geq 0$ (in this case we say that E_Λ^{cu} is (c, λ)-*volume-expanding* to indicate the dependence on c, λ).

Definition 1.6. Let Λ be a compact invariant transitive set of $X \in \mathcal{X}^r(M)$ with singularities. We say that Λ is a *singular-hyperbolic set* for X if all the singularities of Λ are hyperbolic, and Λ is partially hyperbolic with volume-expanding central direction.

We shall prove the following result.

THEOREM C. *Robust attractors of $X \in \mathcal{X}^1(M)$ containing singularities are singular-hyperbolic sets for X.*

We note that robust attractors cannot be C^1 approximated by vector fields presenting either attracting or repelling periodic points. This implies that, on closed 3-manifolds, any periodic point lying in a robust attractor is hyperbolic of saddle-type. Thus, as in [18, Th. A], we conclude that robust attractors *without singularities* on closed 3-manifolds are hyperbolic. Therefore we have the following dichotomy:

COROLLARY 1.7. *Let Λ be a robust attractor of $X \in \mathcal{X}^1(M)$. Then Λ is either hyperbolic or singular-hyperbolic.*

1.4. *Dynamical consequences of singular-hyperbolicity.* In the theory of differentiable dynamics for flows, i.e., in the study of the asymptotic behavior of orbits $\{X_t(x)\}_{t\in\mathbb{R}}$ for $X \in \mathcal{X}^r(M)$, $r \geq 1$, a fundamental problem is to understand how the behavior of the tangent map DX controls or determines the dynamics of the flow X_t.

So far, this program has been solved for hyperbolic dynamics: there is a complete description of the dynamics of a system under the assumption that the tangent map has a hyperbolic structure.

Under the sole assumption of singular-hyperbolicity one can show that at each point there exists a strong stable manifold; more precisely, the set is a subset of a lamination by strong stable manifolds. It is also possible to show the existence of local central manifolds tangent to the central unstable direction [15]. Although these central manifolds do not behave as unstable ones, in the sense that points in it are not necessarily asymptotic in the past, using the fact that the flow along the central unstable direction expands volume, we can obtain some remarkable properties.

We shall list some of these properties that give us a nice description of the dynamics of robust transitive sets with singularities, and in particular, for robust attractors. The proofs of the results below are in Section 5.

The first two properties do not depend either on the fact that the set is robust transitive or an attractor, but only on the fact that the flow expands volume in the central direction.

PROPOSITION 1.8. *Let Λ be a singular-hyperbolic compact set of $X \in \mathcal{X}^1(M)$. Then any invariant compact set $\Gamma \subset \Lambda$ without singularities is a hyperbolic set.*

Recall that, given $x \in M$, and $v \in T_xM$, the Lyapunov exponent of x in the direction of v is

$$\gamma(x,v) = \liminf_{t\to\infty} \frac{1}{t} \log \|DX_t(x)v\| .$$

We say that x has positive Lyapunov exponent if there is $v \in T_xM$ such that $\gamma(x,v) > 0$.

The next two results show that important features of hyperbolic attractors and of the geometric Lorenz attractor are present for singular-hyperbolic attractors, and so, for robust attractors with singularities:

PROPOSITION 1.9. *A singular-hyperbolic attractor Λ of $X \in \mathcal{X}^1(M)$ has uniform positive Lyapunov exponent at every orbit.*

The last property proved in this paper is the following.

PROPOSITION 1.10. *For X in a residual (set containing a dense G_δ) subset of $\mathcal{X}^1(M)$, each robust transitive set with singularities is the closure of the stable or unstable manifold of one of its hyperbolic periodic points.*

We note that in [29] it was proved that a singular-hyperbolic set Λ of a 3-flow is expansive with respect to initial data; i.e., there is $\delta > 0$ such that for any pair of distinct points $x, y \in \Lambda$, if $\operatorname{dist}(X_t(x), X_t(y)) < \delta$ for all $t \in \mathbb{R}$ then x is in the orbit of y.

Finally, it was proved in [4] that if Λ is a singular-hyperbolic attractor of a 3-flow X then the central direction $E^{cu}_{\widetilde{\Lambda}}$ can be continuously decomposed into $E^u \oplus [X]$, with the E^u direction being nonuniformly hyperbolic ([28], [32]). Here $\widetilde{\Lambda} = \Lambda \setminus \cup_{\sigma\in \mathrm{Sing}_X(\Lambda)} W^u(\sigma)$.

1.5. *Related results and comments.* We note that for diffeomorphisms in dimension two, any robust transitive set is a hyperbolic set [22]. The corresponding result for 3-flows without singularities can be easily obtained from [18, Th. A]. However, in the presence of singularities, this result cannot be applied: a singularity is an obstruction to consider the flow as the suspension of a 2-diffeomorphism. On the other hand, for diffeomorphisms on 3-manifolds it has recently been proved that any robust transitive set is partially hyperbolic [8]. Again, this result cannot be applied to the time-one diffeomorphism X_1 to prove Theorem C: if Λ is a saddle-type periodic point of X then Λ is a robust transitive set for X, but not necessarily a robust transitive set for X_1. Moreover, such a Λ cannot be approximated by robust transitive sets for diffeomorphisms C^1-close to X_1. Indeed, since Λ is normally hyperbolic, it is persistent, [20]. So, for any g nearby X_1, the maximal invariant set Λ_g of g in a neighborhood U of Λ is diffeomorphic to S^1. Since the set of diffeomorphisms g C^1 close to X_1 such that the restriction of g to Λ_g has an attracting periodic point is open, our statement follows.

We also point out that a transitive singular-hyperbolic set is not necessarily a robust transitive set, even in the case that the set is an attractor; see [17] and [27]. So, the converse of our results requires extra conditions that are yet unknown. Anyway, we conjecture that generically, transitive singular-hyperbolic attractors or repellers are robust transitive in the C^∞ topology.

1.6. *Brief sketches of the main results.* This paper is organized as follows. Theorems A and B are proved in Section 2. This section is independent of the remainder of the paper.

To prove Theorem A we first obtain a sufficient condition for a transitive isolated set with hyperbolic critical elements of a C^1 vector field on a n-manifold, $n \geq 3$, to be an attractor (Theorem D). We use this to prove that a robust transitive set whose critical elements are hyperbolic is an attractor if it contains a singularity whose unstable manifold has dimension one (Theorem E). This implies that C^1 robust transitive sets with singularities on closed 3-manifolds are either proper attractors or proper repellers, leading to Theorem A.

To obtain the characterization of singularities in a robust transitive set as Lorenz-like ones (Theorem B), we reason by contradiction. Using the Connecting Lemma [13], we can produce special types of cycles (inclination-flip) associated to a singularity leading to nearby vector fields which exhibit attracting or repelling periodic points. This contradicts the robustness of the transitivity condition.

Theorem C is proved in Section 3. We start by proposing an invariant splitting over the periodic points lying in Λ and prove two basic facts, Theorems 3.6 and 3.7, establishing uniform estimates on angles between stable, unstable, and central unstable bundles for periodic points. Roughly speaking, if such angles are not uniformly bounded away from zero, we construct a new vector field near the original one exhibiting either a sink or a repeller, yielding a contradiction. Such a perturbation is obtained using Lemma 3.1, which is a version for flows of a result in [10]. This allows us to prove that the splitting proposed for the periodic points is partially hyperbolic with volume-expanding central direction. Afterwards, we extend this splitting to the closure of the periodic points. The main objective is to prove that the splitting proposed for the periodic points is compatible with the local partial hyperbolic splitting at the singularities. This is expressed by Proposition 4.1. For this, we use two facts: (a) the linear Poincaré flow has a dominated splitting outside the singularities ([41, Th. 3.8]) and (b) the nonwandering set outside a neighborhood of the singularities is hyperbolic (Lemma 4.3). We next extend this splitting to all of Λ, obtaining Theorem C. Theorems 3.6 and 3.7 are proved in Section 4.

The results in this paper were announced in [26].

2. Attractors and isolated sets for C^1 flows

In this section we shall prove Theorems A, and B.

Our approach to understand, from the dynamical point of view, robust transitive sets for 3-flows is the following. We start by focusing on isolated sets, obtaining sufficient conditions for an isolated set of a C^1 flow on a n-manifold, $n \geq 3$, to be an attractor: (a) all its periodic points and singularities are hyperbolic and (b) it contains, in a robust way, the unstable manifold of either a periodic point or a singularity . Using this we prove that isolated sets whose periodic points and singularities are hyperbolic and which are either robustly nontrivial and transitive (robust transitive) or robustly the closure of their periodic points (C^1 robust periodic) are attractors if they contain a singularity with one-dimensional unstable manifold. In particular, robust transitive sets with singularities on closed 3-manifolds are either proper attractors or proper repellers, proving Theorem A. Afterward we characterize the singularities on a robust transitive set on 3-manifolds as Lorenz-like, obtaining Theorem B.

In order to state the results in a precise way, let us recall some definitions and fix the notation.

A point $p \in M$ is a singularity of X if $X(p) = 0$ and p is a periodic point of X if $X(p) \neq 0$ and there is $t > 0$ such that $X_t(p) = p$. The minimal $t \in \mathbb{R}^+$ satisfying $X_t(p) = p$ is called the period of p and is denoted by t_p.

A point $p \in M$ is a *critical element* of X if p is either a singularity or a periodic point of X. The set of critical elements of X is denoted by $\mathrm{Crit}(X)$. If $A \subset M$, the set of critical elements of X lying in A is denoted by $\mathrm{Crit}_X(A)$.

We say that $p \in \mathrm{Crit}(X)$ is hyperbolic if its orbit is hyperbolic. When p is a periodic point (respectively a singularity) this is equivalent to saying that its Poincaré map has no eigenvalues with modulus one (respectively $DX(p)$ has no eigenvalues with zero real parts). If $p \in \mathrm{Crit}(X)$ is hyperbolic then there are well defined invariant manifolds $W^s_X(p)$ (stable manifold) and $W^u_X(p)$ (unstable manifold) [15]. Moreover, there is a continuation $p(Y) \in \mathrm{Crit}(Y)$ for Y C^r-close to X.

Note that elementary topological dynamics imply that an attractor containing a hyperbolic critical element is a transitive isolated set containing the unstable manifold of this hyperbolic critical element. The converse, although false in general, is true for a residual subset of C^1 vector fields [3]. We derive a sufficient condition for the validity of the converse to this result inspired by the following well known property of hyperbolic attractors [31]: If Λ is a hyperbolic attractor of a vector field X, then there is an isolating block U of Λ and $x_0 \in \mathrm{Crit}_X(\Lambda)$ such that $W^u_Y(x_0(Y)) \subset U$ for every Y close to X. This property motivates the following definition.

Definition 2.1. Let Λ be an isolated set of a C^r vector field X, $r \geq 1$. We say that Λ *robustly contains the unstable manifold of a critical element* if there are $x_0 \in \mathrm{Crit}_X(\Lambda)$ hyperbolic, an isolating block U of Λ and a neighborhood $\mathcal{U}$ of X in the space of C^r vector fields such that

$$W^u_Y(x_0(Y)) \subset U, \quad \text{for all} \quad Y \in \mathcal{U}.$$

With this definition in mind we have the following result.

THEOREM D. *Let Λ be a transitive isolated set of $X \in \mathcal{X}^1(M^n)$, $n \geq 3$, and suppose that every $x \in \mathrm{Crit}_X(\Lambda)$ is hyperbolic. If Λ robustly contains the unstable manifold of a critical element then Λ is an attractor.*

Next we derive an application of Theorem D. For this let us introduce the following notation and definitions. If $A \subset M$, then $\mathrm{Cl}(A)$ denotes the closure of A, and $\mathrm{int}(A)$ denotes the interior of A. The set of periodic points of $X \in \mathcal{X}^r(M)$ is denoted by $\mathrm{Per}(X)$, and the set of periodic points of X in A is denoted by $\mathrm{Per}_X(A)$.

Definition 2.2. Let Λ be an isolated set of a C^r vector field X, $r \geq 1$. We say that Λ is C^r *robust periodic* if there are an isolating block U of Λ and a neighborhood $\mathcal{U}$ of X in the space of all C^r vector fields such that

$$\Lambda_Y(U) = \mathrm{Cl}(\mathrm{Per}_Y(\Lambda_Y(U)), \quad \forall \quad Y \in \mathcal{U}.$$

Examples of C^1 robust periodic sets are the hyperbolic attractors and the geometric Lorenz attractor [12]. These examples are also C^1 robust transitive. On the other hand, the singular horseshoe [17] and the example in [27] are neither C^1 robust transitive nor C^1 robust periodic. These examples motivate the question whether *all* C^1 robust transitive sets for vector fields are C^1 robust periodic.

The geometric Lorenz attractor [12] is a robust transitive (periodic) set, and it is an attractor satisfying: (a) all its periodic points are hyperbolic and (b) it contains a singularity whose unstable manifold has dimension one. The result below shows that such conditions suffice for a robust transitive (periodic) set to be an attractor.

THEOREM E. *Let Λ be either a robust transitive or a transitive C^1 robust periodic set of $X \in \mathcal{X}^1(M^n)$, $n \geq 3$. If*

1. *every $x \in \mathrm{Crit}_X(\Lambda)$ is hyperbolic and*

2. *Λ has a singularity whose unstable manifold is one-dimensional,*

then Λ is an attractor of X.

This theorem follows from Theorem D by proving that Λ robustly contains the unstable manifold of the singularity in the hypothesis (2) above.

To prove these results, let us establish in a precise way some notation and results that will be used to obtain the proofs. Throughout, M denotes a compact boundaryless manifold with dimension $n \geq 3$. First we shall obtain a sufficient condition for an isolated invariant set of $X \in \mathcal{X}^1(M)$ to be an attractor. For this we proceed as follows.

Given $p \in M$, $\mathcal{O}_X(p)$ denotes the orbit of p by X. If $\mathcal{O}_X(p)$, $p \in \mathrm{Crit}(X)$, is hyperbolic and $x \in \mathcal{O}_X(p)$ then there are well-defined invariant manifolds $W^s_X(x)$, the stable manifold at x, and $W^u_X(x)$, the unstable manifold at x. Given a hyperbolic $x \in \mathrm{Crit}(X)$, and Y C^r-close to X, we denote by $x(Y) \in \mathrm{Crit}(Y)$ the continuation of x.

The following two results are used to connect unstable manifolds to suitable points in M. For the proofs of these results see [2], [13], [14], [42].

THEOREM 2.3 (The connecting lemma). *Let $X \in \mathcal{X}^1(M)$ and $\sigma \in \mathrm{Sing}(X)$ be hyperbolic. Suppose that there are $p \in W^u_X(\sigma) \setminus \{\sigma\}$ and $q \in M \setminus \mathrm{Crit}(X)$ such that:*

(H1) *For all neighborhoods U, V of p, q (respectively) there is $x \in U$ such that $X_t(x) \in V$ for some $t \geq 0$.*

Then there are Y arbitrarily C^1-close to X and $T > 0$ such that $p \in W^u_Y(\sigma(Y))$ and $Y_T(p) = q$. If in addition $q \in W^s_X(x) \setminus \mathcal{O}_X(x)$ for some $x \in \mathrm{Crit}(X)$ hyperbolic, then Y can be chosen so that $q \in W^s_Y(x(Y)) \setminus \mathcal{O}_Y(x(Y))$.

THEOREM 2.4. *Let $X \in \mathcal{X}^1(M)$ and $\sigma \in \mathrm{Sing}(X)$ be hyperbolic. Suppose that there are $p \in W^u_X(\sigma) \setminus \{\sigma\}$ and $q, x \in M \setminus \mathrm{Crit}(X)$ such that:*

(H2) *For all neighborhoods U, V, W of p, q, x (respectively) there are $x_p \in U$ and $x_q \in V$ such that $X_{t_p}(x_p) \in W$ and $X_{t_q}(x_q) \in W$ for some $t_p > 0$, $t_q < 0$.*

Then there are Y arbitrarily C^1-close to X and $T > 0$ such that $p \in W^u_Y(\sigma(Y))$ and $Y_T(p) = q$.

The following lemma is well-known; see for instance [5, p. 3]. Recall that given an isolated set Λ of $X \in \mathcal{X}^r(M)$ with isolating block U, we denote by $\Lambda_Y(U) = \cap_{t\in\mathbb{R}} Y_t(U)$ the maximal invariant set of Y in U for every $Y \in \mathcal{X}^r(M)$.

LEMMA 2.5. *Let Λ be an isolated set of $X \in \mathcal{X}^r(M)$, $r \geq 0$. Then, for every isolating block U of Λ and every open set V containing Λ, there is a neighborhood $\mathcal{U}_0$ of X in $\mathcal{X}^r(M)$ such that*

$$\Lambda_Y(U) \subset V, \quad \forall \quad Y \in \mathcal{U}_0.$$

LEMMA 2.6. *If Λ is an attracting set and a repelling set of $X \in \mathcal{X}^1(M)$, then $\Lambda = M$.*

Proof. Suppose that Λ is an attracting set and a repelling set of X. Then there are neighborhoods V_1 and V_2 of Λ satisfying $X_t(V_1) \subset V_1$, $X_{-t}(V_2) \subset V_2$ (for all $t \geq 0$),

$$\Lambda = \bigcap_{t\geq 0} X_t(V_1) \quad \text{and} \quad \Lambda = \bigcap_{t\geq 0} X_{-t}(V_2).$$

Define $U_1 = \mathrm{int}(V_1)$ and $U_2 = \mathrm{int}(V_2)$. Clearly $X_t(U_1) \subset U_1$ and $X_{-t}(U_2) \subset U_2$ (for all $t \geq 0$) since X_t is a diffeomorphism. As U_2 is open and contains Λ, the first equality implies that there is $t_2 > 0$ such that $X_{t_2}(V_1) \subset U_2$ (see for instance [16, Lemma 1.6]). As $X_{t_2}(U_1) \subset X_{t_2}(V_1)$ it follows that $U_1 \subset X_{-t_2}(U_2) \subset U_2$ proving

$$U_1 \subset U_2.$$

Similarly, as U_2 is open and contains Λ, the second equality implies that there is $t_1 > 0$ such that $X_{-t_1}(V_2) \subset U_1$. As $X_{-t_1}(U_2) \subset X_{-t_1}(V_2)$ it follows that $U_2 \subset X_{t_1}(U_1) \subset U_1$ proving

$$U_2 \subset U_1.$$

Thus, $U_1 = U_2$. From this we obtain

$$X_t(U_1) = U_1, \quad \forall t \geq 0$$

proving $\Lambda = U_1$. As Λ is compact, by assumption we conclude that Λ is open and closed. As M is connected and Λ is not empty we obtain that $\Lambda = M$ as desired. ◻

The lemma below gives a sufficient condition for an isolated set to be attracting.

LEMMA 2.7. *Let Λ be an isolated set of $X \in \mathcal{X}^1(M)$. If there are an isolating block U of Λ and an open set W containing Λ such that $X_t(W) \subset U$ for every $t \geq 0$, then Λ is an attracting set of X.*

Proof. Let Λ and X be as in the statement. To prove that Λ is attracting we have to find a neighborhood V of Λ such that $X_t(V) \subset V$ for all $t > 0$ and

$$\Lambda = \cap_{t \leq 0} X_t(V). \tag{2}$$

To construct V we let W be the open set in the statement of the lemma and define

$$V = \cup_{t>0} X_t(W).$$

Clearly V is a neighborhood of Λ satisfying $X_t(V) \subset V$, for all $t > 0$.

We claim that V satisfies (2). Indeed, as $X_t(W) \subset U$ for every $t > 0$ we have that $V \subset U$ and so

$$\cap_{t \in \mathbb{R}} X_t(V) \subset \Lambda$$

because U is an isolating block of Λ. But $V \subset X_t(V)$ for every $t \leq 0$ since V is forward invariant. So, $V \subset \cap_{t \leq 0} X_t(V)$. From this we have

$$\begin{aligned} \cap_{t \geq 0} X_t(V) &\subset V \cap (\cap_{t>0} X_t(V)) \\ &\subset (\cap_{t \leq 0} X_t(V)) \cap (\cap_{t>0} X_t(V)) = \cap_{t \in \mathbb{R}} X_t(V). \end{aligned}$$

Thus,

$$\cap_{t \geq 0} X_t(V) \subset \Lambda.$$

Now, as $\Lambda \subset V$ and Λ is invariant, we have $\Lambda \subset X_t(V)$ for every $t \geq 0$. Then

$$\Lambda \subset \cap_{t \geq 0} X_t(V),$$

proving (2). □

2.1. *Proof of Theorems* D *and* E. The proof of Theorem D is based on the following lemma.

LEMMA 2.8. *Let Λ be a transitive isolated set of $X \in \mathcal{X}^1(M)$ such that every $x \in \mathrm{Crit}_X(\Lambda)$ is hyperbolic. Suppose that the following condition holds*:

(H3) *There are $x_0 \in \mathrm{Crit}_X(\Lambda)$, an isolating block U of Λ and a neighborhood $\mathcal{U}$ of X in $\mathcal{X}^1(M)$ such that*

$$W^u_Y(x_0(Y)) \subset U, \quad \forall \quad Y \in \mathcal{U}.$$

Then $W^u_X(x) \subset \Lambda$ for every $x \in \mathrm{Crit}_X(\Lambda)$.

Proof. Let x_0, U and $\mathcal{U}$ be as in (H3). By assumption $\mathcal{O}_X(x_0)$ is hyperbolic. If $\mathcal{O}_X(x_0)$ is attracting then $\Lambda = \mathcal{O}_X(x_0)$ since Λ is transitive and we are done. We can then assume that $\mathcal{O}_X(x_0)$ is not attracting. Thus, $W^u_X(x_0) \setminus \mathcal{O}_X(x_0) \neq \emptyset$.

By contradiction, suppose that there is $x \in \mathrm{Crit}_X(\Lambda)$ such that $W^u_X(x)$ is not contained in Λ. Then $W^u_X(x)$ is not contained in $\mathrm{Cl}(U)$. As $M \setminus \mathrm{Cl}(U)$ is open there is a cross-section $\Sigma \subset M \setminus \mathrm{Cl}(U)$ of X such that $W^u_X(x) \cap \Sigma \neq \emptyset$ is transversal. Shrinking $\mathcal{U}$ if necessary we may assume that $W^u_Z(x(Z)) \cap \Sigma \neq \emptyset$ is transversal for all $Z \in \mathcal{U}$.

Now, $W^u_X(x_0) \subset \Lambda$ by (H3) applied to $Y = X$. Choose $p \in W^u_X(x_0) \setminus \mathcal{O}_X(x_0)$. As Λ is transitive and $p, x \in \Lambda$, there is $q \in W^s_X(x) \setminus \mathcal{O}_X(x)$ such that p, q satisfy (H1) in Theorem 2.3. Indeed, the dense orbit of Λ accumulates both p and x. Then, by Theorem 2.3, there are $Z \in \mathcal{U}$ and $T > 0$ such that $p \in W^u_Z(x(Z))$, $q \in W^s_Z(x(Z))$ and $Z_T(p) = q$. In other words, $\mathcal{O}_Z(q)$ is a saddle connection between $x_0(Z)$ and $x(Z)$. On the other hand, as $Z \in \mathcal{U}$, we have that $W^u_Z(x(Z)) \cap \Sigma \neq \emptyset$ is transversal. It follows from the Inclination Lemma [7] that $Z_t(\Sigma)$ accumulates on q as $t \to \infty$. This allows us to break the saddle-connection $\mathcal{O}_Z(q)$ in the standard way in order to find $Z' \in \mathcal{U}$ such that $W^u_{Z'}(x_0(Z')) \cap \Sigma \neq \emptyset$ (see the proof of [7, Lemma 2.4, p. 101]). In particular, $W^u_{Z'}(x_0(Z'))$ is not contained in U. This contradicts (H3) and the lemma follows. □

Proof of Theorem D. Let Λ and X be as in the statement of Theorem D. It follows that there are $x_0 \in \mathrm{Crit}_X(\Lambda)$, U and $\mathcal{U}$ such that (H3) holds.

Next we prove that Λ satisfies the hypothesis of Lemma 2.7, that is, there is an open set W containing Λ such that $X_t(W) \subset U$ for every $t \geq 0$.

Suppose that such a W does not exist. Then, there are sequences $x_n \to x \in \Lambda$ and $t_n > 0$ such that $X_{t_n}(x_n) \in M \setminus U$. By compactness we can assume that $X_{t_n}(x_n) \to q$ for some $q \in \mathrm{Cl}(M \setminus U)$.

Fix an open set $V \subset \mathrm{Cl}(V) \subset U$ containing Λ. As $q \in \mathrm{Cl}(M \setminus U)$,

$$\mathrm{Cl}(M \setminus U) \subset M \setminus \mathrm{int}(U), \quad \text{and} \quad M \setminus \mathrm{int}(U) \subset M \setminus \mathrm{Cl}(V)$$

we have that

$$q \notin \mathrm{Cl}(V).$$

By Lemma 2.5 there is a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of X such that

$$\Lambda_Y(U) \subset V, \quad \forall\, Y \in \mathcal{U}_0. \tag{3}$$

Then the hypothesis (H3), the invariance of $W^u_Y(x_0(Y))$ and the relation (3) imply

$$W^u_Y(x_0(Y)) \subset V \subset \mathrm{Cl}(V), \quad \forall\, Y \in \mathcal{U}_0. \tag{4}$$

Now we have two cases:

(1) $x \notin \mathrm{Crit}(X)$.

(2) $x \in \mathrm{Crit}(X)$.

In Case (1) we obtain a contradiction as follows. Let $\mathcal{O}_X(z)$ be the dense orbit of Λ, i.e. $\Lambda = \omega_X(z)$. Fix $p \in W^u_X(x_0) \setminus \mathcal{O}_X(x_0)$. Then $p \in \Lambda$ by (H3) applied to $Y = X$. As $x \in \Lambda$ we can choose sequences $z_n \in \mathcal{O}_X(z)$ and $t'_n > 0$ such that

$$z_n \to p \quad \text{and} \quad X_{t'_n}(z_n) \to x.$$

It follows that p, q, x satisfy (H2) of Theorem 2.4 for $Y = X$. Then, by Theorem 2.4, there is $Z \in \mathcal{U}_0$ such that $q \in W^u_Z(x_0(Z))$. As $q \notin \mathrm{Cl}(V)$ we have that $W^u_Z(x_0(Z))$ is not contained in U. And this is a contradiction by (4) since $Z \in \mathcal{U}_0$.

In Case (2) we use (H3) to obtain a contradiction as follows. By assumption $\mathcal{O}_X(x)$ is a hyperbolic closed orbit. Clearly $\mathcal{O}_X(x)$ is neither attracting nor repelling. In particular, $W^u_X(x) \setminus \mathcal{O}_X(x) \neq \emptyset$. But $x_n \notin W^s_X(x)$ since $x_n \to x$ and $X_{t_n}(x_n) \notin U$. Then, using a linearizing coordinate given by the Grobman-Hartman Theorem around $\mathcal{O}_X(x)$ (see references in [31]), we can find x'_n in the positive orbit of x_n such that $x'_n \to r \in W^u_X(x) \setminus \mathcal{O}_X(x)$. Note that $r \notin \mathrm{Crit}(X)$ and that there are $t'_n > 0$ such that $X_{t'_n}(x'_n) \to q$.

Since (H3) holds, by Lemma 2.8 we have $W^u_X(x) \subset \Lambda$. This implies that $r \in \Lambda$. Then we have Case (1) replacing x by r, t_n by t'_n and x_n by x'_n. As Case (1) results in a contradiction, we conclude that Case (2) also does.

Hence Λ satisfies the hypothesis of Lemma 2.7, and Theorem D follows. □

Proof of Theorem E. Let Λ be either a robust transitive set or a transitive C^1 robust periodic set of $X \in \mathcal{X}^1(M)$ satisfying the following hypothesis:

(1) Every critical element of X in Λ is hyperbolic.

(2) Λ contains a singularity σ with $\dim(W^u_X(\sigma)) = 1$.

On one hand, if Λ is robust transitive, we can fix by Definition 1.1 a neighborhood $\mathcal{U}$ of X and an isolating block U of Λ such that $\Lambda_Y(U)$ is a nontrivial transitive set of Y, for all $Y \in \mathcal{U}$. Clearly, we can assume that the continuation $\sigma(Y)$ is well defined for all $Y \in \mathcal{U}$. As transitive sets are connected sets, we have the following additional property:

(C) $\Lambda_Y(U)$ is connected, for all $Y \in \mathcal{U}$.

On the other hand, if Λ is C^1 robust periodic, we can fix by Definition 2.2 a neighborhood $\mathcal{U}$ of X and an isolating block U of Λ such that $\Lambda_Y(U) = \mathrm{Cl}(\mathrm{Per}(\Lambda_Y(U)))$, for all $Y \in \mathcal{U}$. As before we can assume that $\sigma(Y)$ is well defined for all $Y \in \mathcal{U}$. In this case we have the following additional property.

(C$'$) $\sigma(Y) \in \mathrm{Cl}(\mathrm{Per}_Y(\Lambda_Y(U)))$, for all $Y \in \mathcal{U}$.

Now we have the following claim.

CLAIM 2.9. Λ *robustly contains the unstable manifold of a critical element.*

By Definition 2.1 it suffices to prove

$$W^u_Y(\sigma(Y)) \subset \mathrm{Cl}(U), \quad \forall\, Y \in \mathcal{U},$$

where $\mathcal{U}$ is the neighborhood of X described in either Property (C) or (C′).

By contradiction suppose that $\exists Y \in \mathcal{U}$ such that $W^u_Y(\sigma(Y))$ *is not contained in* U. By hypothesis (2) above it follows that $W^u_X(\sigma) \setminus \{\sigma\}$ has two branches which we denote by w^+ and w^- respectively. Fix $q^+ \in w^+$ and $q^- \in w^-$. Denote by $q^{\pm}(Y)$ the continuation of $q^{\pm}$ for Y close to X. We can assume that the $q^{\pm}(Y)$ are well defined for all $Y \in \mathcal{U}$.

As $q^{\pm}(Y) \in W^u_Y(\sigma(Y))$, the negative orbit of $q^{\pm}(Y)$ converges to $\sigma(Y) \in \mathrm{int}(U) \subset U$. If the positive orbit of $q^{\pm}(Y)$ is in U, then $W^u_Y(\sigma(Y)) \subset U$, which is a contradiction. Consequently the positive orbit of either $q^+(Y)$ or $q^-(Y)$ leaves U. It follows that there is $t > 0$ such that either $Y_t(q^+(Y))$ or $Y_t(q^-(Y)) \notin U$. Assume the first case. The other case is analogous. As $M \setminus U$ is open, the continuous dependence of the unstable manifolds implies that there is a neighborhood $\mathcal{U}' \subset \mathcal{U}$ of Y such that

$$Z_t(q^+(Z)) \notin U, \quad \forall\, Z \in \mathcal{U}'. \tag{5}$$

Now we split the proof into two cases.

Case I: Λ *is robust transitive.* In this case $\Lambda_Y(U)$ is a nontrivial transitive set of Y. Fix $z \in \Lambda_Y(U)$ such that $\omega_Y(z) = \Lambda_Y(U)$. As $\sigma(Y) \in \Lambda_Y(U)$ it follows that either $q^+(Y)$ or $q^-(Y) \in \omega_Y(z)$. As $Y \in \mathcal{U}'$, the relation (5) implies $q^-(Y) \in \omega_Y(z)$. Thus, there is a sequence $z_n \in \mathcal{O}_Y(z)$ converging to $q^-(Y)$. Similarly there is a sequence $t_n > 0$ such that $Y_{t_n}(z_n) \to q$ for some $q \in W^s_Y(\sigma(Y) \setminus \{\sigma(Y)\}$. Define $p = q^-(Y)$.

It follows that p, q, Y satisfy (H1) in Theorem 2.3, and so, there is $Z \in \mathcal{U}'$ such that $q^-(Z) \in W^s_Z(\sigma(Z))$. This gives a homoclinic connection associated to $\sigma(Z)$. Breaking this connection as in the proof of Lemma 2.8, we can find $Z' \in \mathcal{U}'$ close to Z and $t' > 0$ such that

$$Z'_{t'}(q^-(Z')) \notin U. \tag{6}$$

Now, (5), (6) together with [7, Grobman-Harman Theorem] imply that the set $\{\sigma(Z')\}$ is isolated in $\Lambda_Z(U)$. But $\Lambda_{Z'}(U)$ is connected by Property (C) since $Z' \in \mathcal{U}' \subset \mathcal{U}$. Then $\Lambda_{Z'}(U) = \{\sigma(Z')\}$, a contradiction since $\Lambda_{Z'}(U)$ is nontrivial. This proves Claim 2.9 in the present case.

Case II: Λ *is C^1 robust periodic.* The proof is similar to the previous one. In this case $\Lambda_Y(U)$ is the closure of its periodic orbits and $\dim(W^u_Y(\sigma(Y)) = 1$.

As the periodic points of $\Lambda_Y(U)$ do accumulate either $q^+(Y)$ or $q^-(Y)$, relation (5) implies that there is a sequence $p_n \in \mathrm{Per}_Y(\Lambda_Y(U))$ such that $p_n \to q^-(Y)$. Clearly there is another sequence $p'_n \in \mathcal{O}_Y(p_n)$ now converging to some $q \in W^s_Y(\sigma(Y) \setminus \{\sigma(Y)\}$. Set $p = q^-(Y)$.

Again p, q, Y satisfy (H1) in Theorem 2.3, and so, there is $Z \in \mathcal{U}'$ such that $q^-(Z) \in W^s_Z(\sigma(Z))$. As before we have a homoclinic connection associated to $\sigma(Z)$. Breaking this connection we can find $Z' \in \mathcal{U}'$ close to Z and $t' > 0$ such that

$$Z'_{t'}(q^-(Z')) \notin U.$$

Again this relation together with [7, Grobman-Harman Theorem] and the relation (5) would imply that every periodic point of Z' passing close to $\sigma(Z')$ is not contained in $\Lambda_{Z'}(U)$. But this contradicts Property (C′) since $Z' \in \mathcal{U}' \subset \mathcal{U}$. This completes the proof of Claim 2.9 in the final case.

It follows that Λ is an attractor by hypothesis (1) above, Theorem D and Claim 2.9. This completes the proof of Theorem E. ☐

2.2. *Proof of Theorems* A *and* B. In this section M is a closed 3-manifold and Λ is a robust transitive set of $X \in \mathcal{X}^1(M)$. Recall that the set of periodic points of X in Λ is denoted by $\mathrm{Per}_X(\Lambda)$, the set of singularities of X in Λ is denoted by $\mathrm{Sing}_X(\Lambda)$, and the set of critical elements of X in Λ is denoted by $\mathrm{Crit}_X(\Lambda)$.

By Definition 1.1 we can fix an isolating block U of Λ and a neighborhood $\mathcal{U}_U$ of X such that $\Lambda_Y(U) = \cap_{t\in\mathbb{R}} Y_t(U)$ is a nontrivial transitive set of Y, for all $Y \in \mathcal{U}_U$.

A sink (respectively source) of a vector field is a hyperbolic attracting (respectively repelling) critical element. Since $\dim(M) = 3$, robustness of transitivity implies that $X \in \mathcal{U}_U$ cannot be C^1-approximated by vector fields exhibiting either sinks or sources in U. And this easily implies the following result:

LEMMA 2.10. *Let $X \in \mathcal{U}_U$. Then X has neither sinks nor sources in U, and any $p \in \mathrm{Per}(\Lambda_X(U))$ is hyperbolic.*

LEMMA 2.11. *Let $Y \in \mathcal{U}_U$ and $\sigma \in \mathrm{Sing}(\Lambda_Y(U))$. Then,*

1. *The eigenvalues of σ are real.*
2. *If $\lambda_2(\sigma) \le \lambda_3(\sigma) \le \lambda_1(\sigma)$ are the eigenvalues of σ, then*
$$\lambda_2(\sigma) < 0 < \lambda_1(\sigma).$$
3. *If $\lambda_i(\sigma)$ are as above, then*
(a)$\lambda_3(\sigma) < 0 \Longrightarrow -\lambda_3(\sigma) < \lambda_1(\sigma)$;
(b) $\lambda_3(\sigma) > 0 \Longrightarrow -\lambda_3(\sigma) > \lambda_2(\sigma)$.

Proof. Let us prove (1). By contradiction, suppose that there is $Y \in \mathcal{U}_U$ and $\sigma \in \mathrm{Sing}(\Lambda_Y(U))$ with a complex eigenvalue ω. We can assume that σ is hyperbolic. As $\dim(M) = 3$, the remaining eigenvalue λ of σ is real. We have either $\mathrm{Re}(\omega) < 0 < \lambda$ or $\lambda < 0 < \mathrm{Re}(\omega)$. Reversing the flow direction if necessary we can assume the first case. We can further assume that Y is C^∞ and

$$\frac{\lambda}{-\mathrm{Re}(\omega)} \neq 1. \tag{7}$$

By Theorem 2.3, we can assume that there is a homoclinic loop $\Gamma \subset \Lambda_Y(U)$ associated to σ. Then Γ is a *Shilnikov bifurcation* [43]. By well-known results [43, p. 227] (see also [35, p. 13]), and by (7), there is a C^1 vector field Z arbitrarily C^1 close to Y exhibiting a sink or a source in $\Lambda_Z(U)$. This yields a contradiction by Lemma 2.10 and proves (1).

Thus, we can arrange the eigenvalues $\lambda_1(\sigma)$, $\lambda_2(\sigma)$, $\lambda_3(\sigma)$ of σ in such a way that

$$\lambda_2(\sigma) \le \lambda_3(\sigma) \le \lambda_1(\sigma).$$

By Lemma 2.10 we have that $\lambda_2(\sigma) < 0$ and $\lambda_1(\sigma) > 0$. This proves (2).

Let us prove (3). For this we can apply [43, Th. 3.2.12, p. 219] in order to prove that there is Z arbitrarily C^1 close to Y exhibiting a sink in $\Lambda_Z(U)$ (if (a) fails) or a source in $\Lambda_Z(U)$ (if (b) fails). This is a contradiction as before, proving (3). □

LEMMA 2.12. *There is no $Y \in \mathcal{U}_U$ exhibiting two hyperbolic singularities in $\Lambda_Y(U)$ with different unstable manifold dimensions.*

Proof. Suppose by contradiction that there is $Y \in \mathcal{U}_U$ exhibiting two hyperbolic singularities with different unstable manifold dimensions in $\Lambda_Y(U)$. Note that $\Lambda' = \Lambda_Y(U)$ is a robust transitive set of Y and $-Y$ respectively. By [7, Kupka-Smale Theorem] we can assume that all the critical elements of Y in Λ' are hyperbolic. As $\dim(M) = 3$ and Y has two hyperbolic singularities with different unstable manifold dimensions, it follows that both Y and $-Y$ have a singularity in Λ' whose unstable manifold has dimension one. Then, by Theorem E applied to Y and $-Y$ respectively, Λ' is a proper attractor and a proper repeller of Y. In particular, Λ' is an attracting set and a repelling set of Y. It would follow from Lemma 2.6 that $\Lambda' = M$. But this is a contradiction since Λ' is proper. □

COROLLARY 2.13. *If $Y \in \mathcal{U}_U$, then every critical element of Y in $\Lambda_Y(U)$ is hyperbolic.*

Proof. By Lemma 2.10 every periodic point of Y in $\Lambda_Y(U)$ is hyperbolic, for all $Y \in \mathcal{U}$. It remains to prove that every $\sigma \in \mathrm{Sing}_Y(\Lambda_Y(U))$ is hyperbolic, for all $Y \in \mathcal{U}_U$. By Lemma 2.11 the eigenvalues $\lambda_1(\sigma), \lambda_2(\sigma), \lambda_3(\sigma)$ of σ are real and satisfy $\lambda_2(\sigma) < 0 < \lambda_1(\sigma)$. Then, to prove that σ is hyperbolic, we only have to prove that $\lambda_3(\sigma) \neq 0$. If $\lambda_3(\sigma) = 0$, then σ is a generic saddle-node singularity (after a small perturbation if necessary). Unfolding this saddle-node we obtain $Y' \in \mathcal{U}_U$ close to Y having two hyperbolic singularities with different unstable manifold dimensions in $\Lambda_{Y'}(U)$. This contradicts Lemma 2.12 and the proof follows. $\square$

Proof of Theorem A. Let Λ be a robust transitive set with singularities of $X \in \mathcal{X}^1(M)$ with $\dim(M) = 3$. By Corollary 2.13 applied to $Y = X$ we have that every critical element of X in Λ is hyperbolic. So, Λ satisfies the hypothesis (1) of Theorem E. As $\dim(M) = 3$ and Λ is nontrivial, if Λ has a singularity, then this singularity has unstable manifold dimension equal to one either for X or $-X$. So Λ also satisfies hypothesis (2) of Theorem E either for X or $-X$. Applying Theorem E we have that Λ is an attractor (in the first case) or a repeller (in the second case).

We shall prove that Λ is proper in the first case. The proof is similar in the second case. If $\Lambda = M$ then we would have $U = M$. From this it would follow that $\Omega(X) = M$ and, moreover, that X cannot be C^1 approximated by vector fields exhibiting attracting or repelling critical elements. It would follow from the Theorem in [9, p. 60] that X is Anosov. But this is a contradiction since Λ (and so X) has a singularity and Anosov vector fields do not. This finishes the proof of Theorem A. $\square$

Now we prove Theorem B, starting with the following corollary.

COROLLARY 2.14. *If $Y \in \mathcal{U}_U$ then, either for $Z = Y$ or $Z = -Y$, every singularity of Z in $\Lambda_Z(U)$ is Lorenz-like.*

Proof. Apply Lemmas 2.11, 2.12 and Corollary 2.13. $\square$

Before we continue with the proof, let us recall the concept of linear Poincaré flow and deduce a result for $X \in \mathcal{U}_U$ that will be used in the sequel.

Linear Poincaré flow. Let Λ be an invariant set without singularities of a vector field X. Denote by N_Λ the sub-bundle over Λ such that the fiber N_q at $q \in \Lambda$ is the orthogonal complement of the direction generated by $X(q)$ in T_qM.

For any $t \in \mathbb{R}$ and $v \in N_q$ define $P_t(v)$ as the orthogonal projection of $DX_t(v)$ onto $N_{X_t(q)}$. The flow P_t is called the *linear Poincaré flow* of X over Λ.

Given $X \in \mathcal{U}_U$ set

$$\Lambda_X^*(U) = \Lambda_X(U) \setminus \mathrm{Sing}_X(\Lambda_X(U)).$$

By Theorem A, we can assume that $\Lambda_X(U)$ is a proper attractor of X. Thus, there is a neighborhood $\mathcal{U}' \subset \mathcal{U}_U$ of X such that if $Y \in \mathcal{U}'$, $x \in \mathrm{Per}(Y)$ and $\mathcal{O}_Y(x) \cap U \neq \emptyset$, then

$$\mathcal{O}_Y(x) \subset \Lambda_Y(U). \tag{8}$$

In what follows, $[X]$ stands for the bundle spanned by the flow direction, and P_t^X stands for the linear Poincaré flow of X over $\Lambda_X^*(U)$. By Lemma 2.10, the fact that $\Lambda_X^*(U) \subset \Omega(X)$, (8), and the same arguments as in [9, Th. 3.2] (see also [41, Th. 3.8]) we obtain

Theorem 2.15 (Dominated splitting for the LPF). *Let $X \in \mathcal{U}' \subset \mathcal{U}_U$. Then there exists an invariant, continuous and dominated splitting*

$$N_{\Lambda_X^*(U)} = N^{s,X} \oplus N^{u,X}$$

for the linear Poincaré flow P_t of X. Moreover, the following hold:

1. *For all hyperbolic sets $\Gamma \subset \Lambda_X^*(U)$ with splitting $E_\Gamma^{s,X} \oplus [X] \oplus E_\Gamma^{u,X}$, if $x \in \Gamma$ then*

$$E_x^{s,X} \subset N_x^{s,X} \oplus [X(x)], \quad E_x^{u,X} \subset N_p^{u,X} \oplus [X(x)].$$

2. *If $Y_n \to X$ and $x_n \to x$ with $x_n \in \Lambda_{Y_n}^*(U), x \in \Lambda_X^*(U)$ then $N_{x_n}^{s,Y_n} \to N_x^{s,X}$ and $N_{x_n}^{u,Y_n} \to N_x^{u,X}$.*

Lemma 2.16. *If $\sigma \in \mathrm{Sing}_X(\Lambda)$ then the following properties hold:*

(1) *If $\lambda_2(\sigma) < \lambda_3(\sigma) < 0$, then σ is Lorenz-like for X and*

$$W_X^{ss}(\sigma) \cap \Lambda = \{\sigma\}.$$

(2) *If $0 < \lambda_3(\sigma) < \lambda_1(\sigma)$, then σ is Lorenz-like for $-X$ and*

$$W_X^{uu}(\sigma) \cap \Lambda = \{\sigma\}.$$

Proof. To prove (1) we assume that $\lambda_2(\sigma) < \lambda_3(\sigma) < 0$. Then, σ is Lorenz-like for X by Corollary 2.14.

We assume by contradiction that

$$W_X^{ss}(\sigma) \cap \Lambda \neq \{\sigma\}.$$

By Theorem 2.3, as Λ is transitive, there is $Z \in \mathcal{U}_U$ exhibiting a homoclinic connection

$$\Gamma \subset W_Z^u(\sigma(Z)) \cap W_Z^{ss}(\sigma(Z)).$$

This connection is called *orbit-flip*. By using [25, Claim 7.3] and the results in [24] we can approximate Z by $Y \in \mathcal{U}_U$ exhibiting a homoclinic connection

$$\Gamma' \subset W^u_Y(\sigma(Y)) \cap (W^s_Y(\sigma(Y)) \setminus W^{ss}_Y(\sigma(Y)))$$

so that there is a center-unstable manifold $W^{cu}_Y(\sigma(Y))$ containing Γ' and tangent to $W^s_Y(\sigma(Y))$ along Γ'. This connection is called *inclination-flip*. The existence of inclination-flip connections contradicts the existence of the dominated splitting in Theorem 2.15 as in [25, Th. 7.1, p. 374]. This contradiction proves (1).

The proof of (2) follows from the above argument applied to $-X$. We leave the details to the reader. □

Proof of Theorem B. Let Λ be a robust transitive set of $X \in \mathcal{X}^1(M)$ with $\dim(M) = 3$. By Corollary 2.14, if $\sigma \in \mathrm{Sing}_X(\Lambda)$, then σ is Lorenz-like either for X or $-X$. If σ is Lorenz-like for X we have that $W^{ss}_X(\sigma) \cap \Lambda = \{\sigma\}$ by Lemma 2.16-(1) applied to $Y = X$. If σ is Lorenz-like for $-X$ we have that $W^{uu}_X(\sigma) \cap \Lambda = \{\sigma\}$ by Lemma 2.16-(2) again applied to $Y = X$. As $W^{ss}_{-X}(\sigma) = W^{uu}_X(\sigma)$ the result follows. □

3. Attractors and singular-hyperbolicity

Throughout this section M is a boundaryless compact 3-manifold. The main goal here is the proof of Theorem C.

Let Λ be a robust attractor of $X \in \mathcal{X}^1(M)$, U an isolating block of Λ, and $\mathcal{U}_U$ a neighborhood of X such that for all $Y \in \mathcal{U}_U$, $\Lambda_Y(U) = \cap_{t\in\mathbb{R}} Y_t(U)$ is transitive. By definition, $\Lambda = \Lambda_X(U)$. As we pointed out before (Lemma 2.10 and Corollary 2.13), for all $Y \in \mathcal{U}_U$, all the singularities of $\Lambda_Y(U)$ are Lorenz-like and all the critical elements in $\Lambda_Y(U)$ are hyperbolic of saddle type.

The strategy to prove Theorem C is the following: given $X \in \mathcal{U}_U$ we show that there exist a neighborhood $\mathcal{V}$ of X, $c > 0$, $0 < \lambda < 1$ and $T_0 > 0$ such that for all $Y \in \mathcal{V}$, the set $\mathrm{Per}^{T_0}_Y(\Lambda_Y(U)) = \{y \in \mathrm{Per}_Y(\Lambda_Y(U)) : t_y > T_0\}$ has a continuous invariant (c, λ)-dominated splitting $E^s \oplus E^{cu}$, with $\dim(E^s) = 1$. Here t_y is the period of y. Then, using the Closing Lemma [33] and the robust transitivity, we induce a dominated splitting over $\Lambda_X(U)$. To do so, the natural question that arises regards splitting around the singularities. By Theorem B they are Lorenz-like, and in particular, they also have the local hyperbolic bundle $\hat{E}^{ss}$ associated to the strongest contracting eigenvalue of $DX(\sigma)$, and the central bundle $\hat{E}^{cu}$ associated to the remaining eigenvalues of $DX(\sigma)$. Thus, these bundles induce a local partial hyperbolic splitting around the singularities, $\hat{E}^{ss} \oplus \hat{E}^{cu}$. The main idea is to prove that the splitting proposed for the periodic points is compatible with the local partial hyperbolic splitting at the singularities. Proposition 4.1 expresses this fact. Finally we prove that E^s is contracting and that the central direction E^{cu} is volume expanding, concluding the proof of Theorem C.

We point out that the splitting for the Linear Poincaré Flow obtained in Theorem 2.15 is not invariant by DX_t. When $\Lambda_X^*(U) = \Lambda_X(U) \setminus \mathrm{Sing}_X(\Lambda_X(U))$ is closed, this splitting induces a hyperbolic one for X, see [9, Prop. 1.1] and [18, Th. A]. The arguments used there do not apply here, since $\Lambda_X^*(U)$ is not closed. We also note that a hyperbolic splitting for X over $\Lambda_X^*(U)$ cannot be extended to a hyperbolic one over $\mathrm{Cl}(\Lambda_X^*(U))$: the presence of a singularity is an obstruction to it. On the other hand, Theorem C shows that this fact is not an obstruction to the existence of a partially hyperbolic structure for X over $\mathrm{Cl}(\Lambda_X^*(U))$.

3.1. *Preliminary results.* We start by establishing some notation, definitions and preliminary results.

Recall that given a vector field X we denote with DX the derivative of the vector field. With $X_t(q)$ we set the flow induced by X at $(t, q) \in \mathbb{R} \times M$ and $DX_t(q)$ the derivative of X at (t, q). Observe that $X_0(q) = q$ for every $q \in M$ and that $\partial_t X_t(q) = X(X_t(q))$. Moreover, for each $t \in \mathbb{R}$ fixed, $X_t : M \to M$ is a diffeomorphism on M. Then $X_0 = \mathrm{Id}$, the identity map of M, and $X_{t+s} = X_t \circ X_s$ for every $t, s \in \mathbb{R}$ and $\partial_s DX_s(X_t(q))|_{s=0} = DX(X_t(q))$. We set $\|.\|$ for the C^1 norm in $\mathcal{X}^1(M)$. Given any $\delta > 0$, set $B_\delta(X_{[a,b]}(q))$ the δ-neighborhood of the orbit segment $X_{[a,b]}(q) = \{X_t(q), a \leq t \leq b\}$.

To simplify notation, given $x \in M$, a subspace $L_x \subset T_x M$, and $t \in \mathbb{R}$, DX_t/L_x stands for the restriction of $DX_t(x)$ to L_x. Also, $[X(x)]$ stands for the bundle spanned by $X(x)$.

We shall use an extension for flows of a result in [10] stated below. This result allows us to locally change the derivative of the flow along a compact trajectory. To simplify notation, since this result is a local one, we shall state it for flows on compact sets of $\mathbb{R}^n$. Taking local charts we obtain the same result for flows on compact boundaryless 3-manifolds. Then, only in the lemma below, M is a compact set of $\mathbb{R}^n$.

LEMMA 3.1. *Given $\varepsilon_0 > 0$, $Y \in \mathcal{X}^2(M)$, an orbit segment $Y_{[a,b]}(p)$, a neighborhood U of $Y_{[a,b]}(p)$ and a parametrized family of invertible linear maps $A_t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $t \in [a, b]$, C^2 with respect to the parameter t, such that*

a) $A_0 = \mathrm{Id}$ *and* $A_t(Y(Y_s(q))) = Y(Y_{t+s}(q))$,

b) $\|\partial_s A_{t+s} A_t^{-1}|_{s=0} - DY(Y_t(p))\| < \varepsilon$, *with* $\varepsilon < \varepsilon_0$,

then there is $Z \in \mathcal{U}$, $Z \in \mathcal{X}^1(M)$ such that $\|Y - Z\| \leq \varepsilon$, Z coincides with Y in $M \setminus U$, $Z_s(p) = Y_s(p)$ for every $s \in [a, b]$, and $DZ_t(p) = A_t$ for every $t \in [a, b]$.

Remark 3.2. Note that if there is Z such that $DZ_t(p) = A_t$ and $Z_t(p) = Y_t(p)$, $0 \le t \le T$, then, necessarily, A_t has to preserve the flow direction. Condition a) above requires this. Moreover,

$$\partial_s A_{t+s} A_t^{-1}|_{s=0} = \frac{\partial}{\partial s} D_p Z_{t+s} D_{Z_t(p)} Z_{-t}|_{s=0} = \frac{\partial}{\partial s} D_{Z_t(p)} Z_s|_{s=0} = DZ(Z_t(p)),$$

so, condition b) simply requires that DZ be near DY along the given orbit segment $Y_{[a,b]}(p)$.

We also point out that although we start with a C^2 vector field Y we obtain Z only of class C^1 and C^1 near Y. Increasing the class of differentiability of the initial vector field Y and of the family A_t with respect to the parameter t we increase the class of differentiability of Z. But even in this setting the best we can get about closeness is C^1 [34].

Using this lemma we can perturb a C^2 vector field Y to obtain Z of class C^1 that coincides with Y on $M \setminus U$ and on the orbit segment $Y_{[a,b]}$, but such that the derivative of Z_t along this orbit segment is the given parametrized family of linear maps A_t.

To prove our results we shall also use the Ergodic Closing Lemma for flows [22], [41], which shows that any invariant measure can be approximated by one supported on critical elements. To announce it, let us introduce the set of points in M which are strongly closed:

Definition 3.3. A point $x \in M \setminus \mathrm{Sing}(X)$ is *δ-strongly closed* if for any neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X, there are $Z \in \mathcal{U}$, $z \in M$, and $T > 0$ such that $Z_T(z) = z$, $X = Z$ on $M \setminus B_\delta(X_{[0,T]}(x))$ and $\mathrm{dist}(Z_t(z), X_t(x)) < \delta$, for all $0 \le t \le T$.

Denote by $\Sigma(X)$ the set of points of M which are δ- strongly closed for any δ sufficiently small.

THEOREM 3.4 (Ergodic Closing Lemma for flows, [22], [41]). *Let μ be any X-invariant Borel probability measure. Then $\mu(\mathrm{Sing}(X) \cup \Sigma(X)) = 1$.*

3.2. *Uniformly dominated splitting over $T_{\mathrm{Per}_Y^{T_0}(\Lambda_Y(U))}M$.* Let $\Lambda_Y(U)$ be a robust attractor of $Y \in \mathcal{U}_U$, where U and $\mathcal{U}_U$ are as in the previous section. Since any $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ is hyperbolic of saddle type, the tangent bundle of M over p can be written as

$$T_pM = E^s_p \oplus [Y(p)] \oplus E^u_p\,,$$

where E^s_p is the eigenspace associated to the contracting eigenvalue of $DY_{t_p}(p)$, E^u_p is the eigenspace associated to the expanding eigenvalue of $DY_{t_p}(p)$. Here t_p is the period of p.

Note that $E^s_p \subset N^s_p \oplus [Y(p)]$ and $E^u_p \subset N^u_p \oplus [Y(y)]$, where $N^s \oplus N^u$ is the splitting for the linear Poincaré flow.

Observe that, if we consider the previous splitting over all $\mathrm{Per}_Y(\Lambda_Y(U))$, the presence of a singularity in $\mathrm{Cl}(\mathrm{Per}_Y(\Lambda_Y(U)))$ is an obstruction to the extension of the stable and unstable bundles E^s and E^u to $\mathrm{Cl}(\mathrm{Per}_Y(\Lambda_Y(U)))$. Indeed, near a singularity, the angle between either E^u and the direction of the flow or E^s and the direction of the flow goes to zero. To bypass this difficulty, we introduce the following definition:

Definition 3.5. Given $Y \in \mathcal{U}_U$, we set, for any $p \in \mathrm{Per}_Y(\Lambda_Y(U))$, the following splitting:

$$T_pM = E_p^{s,Y} \oplus E_p^{cu,Y},$$

where $E_p^{cu,Y} = [Y(p)] \oplus E_p^u$. And we set over $\mathrm{Per}_Y(\Lambda_Y(U))$ the splitting

$$T_{\mathrm{Per}_Y(\Lambda_Y(U))}M = \cup_{p \in \mathrm{Per}_Y(\Lambda_Y(U))}(E_p^{s,Y} \oplus E_p^{cu,Y}).$$

When no confusion is possible, we drop the Y-dependence on the bundle defined above. To simplify notation, we denote the restriction of $DY_T(p)$, $T \in \mathbb{R}$, $p \in \mathrm{Per}(\Lambda_Y(U))$, to $E_p^{s,Y}$ (respectively $E_p^{cu,Y}$) simply by DY_T/E_p^s (respectively DY_T/E_p^{cu}).

We shall prove that the splitting over $\mathrm{Per}_Y(\Lambda_Y(U))$ given by Definition 3.5 is a DY_t-invariant uniformly dominated splitting along periodic points with large period. That is, we shall prove

THEOREM F. *Given $X \in \mathcal{U}_U$ there are a neighborhood $\mathcal{V} \subset \mathcal{U}_U$, $0 < \lambda < 1$, $c > 0$, and $T_0 > 0$ such that for every $Y \in \mathcal{V}$, if $p \in \mathrm{Per}_Y^{T_0}(\Lambda_Y(U))$ and $T > 0$ then*

$$\|DY_T/E_p^s\| \, \|DY_{-T}/E_{Y_T(p)}^{cu}\| < c\,\lambda^T.$$

Theorem F will be proved in Section 3.6, with the help of Theorems 3.6 and 3.7 below. The proofs of these theorems are in Section 4.

Theorem 3.6 establishes, first, that the periodic points are uniformly hyperbolic, i.e., the periodic points are of saddle-type and the Lyapunov exponents are uniformly bounded away from zero. Second, that the angle between the stable and the unstable eigenspace at periodic points are uniformly bounded away from zero.

THEOREM 3.6. *Given $X \in \mathcal{U}_U$, there are a neighborhood $\mathcal{V} \subset \mathcal{U}_{\mathcal{U}}$ of X and constants $0 < \lambda < 1$ and $c > 0$, such that for every $Y \in \mathcal{V}$, if $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ and t_p is the period of p then*

a) 1. $\|DY_{t_p}/E^s_p\| < \lambda^{t_p}$ (*uniform contraction on the period*)

2. $\|DY_{-t_p}/E^u_p\| < \lambda^{t_p}$ (*uniform expansion on the period*).

b) $$\alpha(E^s_p, E^u_p) > c\,.$$

Theorem 3.7 is a strong version of Theorem 3.6-b). It establishes that at periodic points, the angle between the stable and the central unstable bundles is uniformly bounded away from zero. To announce it, let us introduce some notation.

If $x \in M$, the angle between $v_x, w_x \in T_xM$ is denoted by $\alpha(v_x, w_x)$. Given two subspaces $A \subset T_xM$ and $B \subset T_xM$ the angle $\alpha(A, B)$ between A and B is defined as $\alpha(A, B) = \inf\{\alpha(v_x, w_x),\, v_x, w_x \in T_xM\}$.

THEOREM 3.7. *Given $X \in \mathcal{U}_U$, there are a neighborhood $\mathcal{V} \subset \mathcal{U}_U$ of X and a positive constant C such that for every $Y \in \mathcal{V}$ and $p \in \mathrm{Per}_Y(\Lambda_Y(U))$,*

$$\alpha(E^s_p, E^{cu}_p) > C \quad (\textit{angle uniformly bounded away from zero})\,.$$

We shall prove that if Theorem F fails then we can create a periodic point for a nearby flow with the angle between the stable and the central unstable bundles arbitrarily small, which yields a contradiction to Theorem 3.7. In proving the existence of such a periodic point for a nearby flow we use Theorem 3.6.

Assuming Theorem F, we establish in the next section the extension of the splitting given in Definition 3.5 to all of $\Lambda_X(U)$. Afterward, with the help of Theorem 3.6, we prove that the bundle E^s is uniformly contracting and that the bundle E^{cu} is volume expanding. The role of Theorem 3.6 in the proof that E^s is uniformly contracted (respectively E^{cu} is volume expanding) is that the opposite assumption leads to the creation of periodic points for flows nearby the original one with contraction (respectively expansion) along the stable (respectively unstable) bundle arbitrarily small, contradicting the first part of Theorem 3.6.

All of these facts together prove Theorem C.

3.3. *Dominated splitting over $\Lambda_X(U)$.* In order to induce a dominated splitting over $\Lambda_X(U)$ using the dominated splitting over $\mathrm{Per}^{T_0}_Y(\Lambda_Y(U))$ for flows near X given by Definition 3.5, we proceed as follows. First observe that since $\Lambda_Y(U)$ is a proper attractor for every Y C^1-close to X, we can assume, without loss of generality, that for all $Y \in \mathcal{V}$, and $x \in \mathrm{Per}(Y)$ with $\mathcal{O}_Y(x) \cap U \neq \emptyset$,

$$\mathcal{O}_Y(x) \subset \Lambda_Y(U). \tag{9}$$

On the other hand, since $\Lambda_X(U)$ is a nontrivial transitive set, we get that $\Lambda_X(U) \setminus \{p \in \mathrm{Per}_X(\Lambda_X(U)) : t_p < T_0\}$ is dense in $\Lambda_X(U)$. So, to induce an invariant splitting over $\Lambda_X(U)$ it is enough to do so over

$$\Lambda_X(U) \setminus \{p \in \mathrm{Per}_X(\Lambda_X(U)) : t_p < T_0\}$$

(see [21] and references therein). For this we proceed as follows.

Given $X \in \mathcal{U}_U$, let $K(X) \subset \Lambda_X(U) \setminus \{p \in \mathrm{Per}_X(\Lambda_X(U)) : t_p < T_0\}$ be such that $\forall x \in K(X)$, $X_t(x) \notin K(X)$ if $t \neq 0$. In other words, $K(X)$ is the quotient $\Lambda_X(U) \setminus \{p \in \mathrm{Per}_X(\Lambda_X(U)) : t_p < T_0\}/\sim$, where $\sim$ is the equivalence relation given by $x \sim y$ if and only if $x \in \mathcal{O}_X(y)$. Since $\Lambda_X(U) = \omega(z)$ for some $z \in M$, we have that for any $x \in K(X)$ there exists $t_n > 0$ such that $X_{t_n}(z) \to x$. Then, by the C^1 Closing Lemma [33], there exist $Y^n \to X$, $y_n \to x$ such that $y_n \in \mathrm{Per}(Y^n)$. We can assume that $Y^n \in \mathcal{U}_U$ for all n. In particular, inclusion (9) holds for all n, and so $\mathcal{O}_{Y^n}(y_n) \subset \Lambda_{Y^n}(U)$. Moreover, since the period for the periodic points in $K(X)$ are larger than T_0, we can also assume that $t_{y_n} > T_0$ for all n. Thus, the (c, λ)-dominated splitting over $\mathrm{Per}^{T_0}_{Y^n}(\Lambda_{Y^n}(U))$ given by Theorem F, $E^{s,Y^n} \oplus E^{cu,Y^n}$, is well-defined. Take a converging subsequence $E^{s,Y^{n_k}}_{y_{n_k}} \oplus E^{cu,Y^{n_k}}_{y_{n_k}}$ and set

$$E^{s,X}_x = \lim_{k\to\infty} E^{s,Y^{n_k}}_{y_{n_k}}, \qquad E^{cu,X}_x = \lim_{k\to\infty} E^{cu,Y^{n_k}}_{y_{n_k}}.$$

Since $E^{s,Y^n} \oplus E^{cu,Y^n}$ is a (c, λ)-dominated splitting for all n then so is $E^{s,X}_x \oplus E^{cu,X}_x$. Moreover, $\dim(E^{s,X}_x) = 1$ and $\dim(E^{cu,X}_x) = 2$, for all $x \in K(X)$.

Set, along $X_t(x)$, $t \in \mathbb{R}$, the eigenspaces

$$E^{s,X}_{X_t(x)} = DX_t(E^{s,X}_x), \quad \text{and} \quad E^{cu,X}_{X_t(x)} = DX_t(E^{cu,X}_x).$$

Since for every n the splitting over $\mathrm{Per}^{T_0}_{Y^n}(\Lambda_{Y^n}(U))$ is (c, λ)-dominated, it follows that the splitting above defined for X along the orbits of points in $K(X)$ is also (c, λ)-dominated. Furthermore, $\dim(E^{s,X}_{X_t(x)}) = 1$ and $\dim(E^{cu,X}_{X_t(x)}) = 2$ for all $t \in \mathbb{R}$. This provides the desired extension to $\Lambda_X(U)$.

We denote by $E^s \oplus E^{cu}$ the splitting over $\Lambda_X(U)$ obtained in this way, and since this splitting is uniformly dominated we also obtain that $E^s \oplus E^{cu}$ varies continuously with X [15].

When necessary we denote by $E^{s,Y} \oplus E^{cu,Y}$ the above splitting for Y near X.

Remark 3.8. If $\sigma \in \mathrm{Sing}_X(\Lambda_X(U))$ then E^s_σ is the eigenspace $\hat{E}^{\mathrm{ss}}_\sigma$ associated to the strongest contracting eigenvalue of $DX(\sigma)$, and E^{cu}_σ is the two dimensional eigenspace associated to the remaining eigenvalues of $DX(\sigma)$. This follows from the uniqueness of dominated splittings [9], [23].

3.4. *E^s is uniformly contracting.* We start by proving the following two elementary lemmas.

LEMMA 3.9. *If* $\lim_{t\to\infty}\inf\|DX_t/E^s_x\| = 0$ *for all* $x\in\Lambda_X(U)$ *then there is* $T_0>0$ *such that for any* $x\in\Lambda_X(U)$,

$$\|DX_{T_0}/E^s_x\| < \frac{1}{2}\,.$$

Proof. For each $x\in\Lambda_X(U)$ there is t_x such that $\|DX_{t_x}/E^s_x\| < 1/3$. Hence, for each x there is a neighborhood $B(x)$ such that for all $y\in B(x)$ we have $\|DX_{t_x}/E^s_y\| < 1/2$.

As $\Lambda_X(U)$ is compact, there are $B(x_i)$, $1\le i\le n$, such that $\Lambda_X(U)\subset \cup_{1\le i\le n}B(x_i)$.

Let $K_0=\sup\{\|DX_t/E^s_y\|, y\in B(x_i), 0\le t\le t_{x_i}, 0\le i\le n\}$, and j_0 be such that

$$K_0\,\frac{1}{2^{j_0}} < \frac{1}{2}\,.$$

Fix $T_0 > j_0\sup\{t_{x_i}, 1\le i\le n\}$. We claim that T_0 satisfies the lemma. Indeed, given $y\in\Lambda_X(U)$, we have that $y\in B(x_{i_1})$ for some $1\le i_1\le n$. Let $t_{i_1},\dots,t_{i_k},t_{i_{k+1}}$ satisfy

(a) $X_{t_{i_1}+\cdots+t_{i_j}}(y)\in B(x_{i_{j+1}})$, $1\le j\le k$,

(b) $t_{i_1}+\cdots+t_{i_k}\le T_0\le t_{i_1}+\cdots+t_{i_{k+1}}$.

Observe that $k\ge j_0$. Then, if $\ell_j = t_{i_1}+\cdots+t_{i_j}$, $1\le j\le k+1$,

$$\|DX_{T_0}/E^s_y\| \le \|DX_{T_0-\ell_k}/E^s_{X_{\ell_k}}\|\,.\,\prod_{j=1}^{k}\|DX_{t_{i_j}}/E^s_{X_{\ell_{j-1}}(y)}\| < K_0\,\frac{1}{2^k} < \frac{1}{2}\,.$$

The proof is complete. □

LEMMA 3.10. *If there is* $T_0>0$ *such that* $\|DX_{T_0}/E^s_x\|<1/2$ *for all* $x\in\Lambda_X(U)$ *then there are* $c>0$, $0<\lambda<1$ *such that for all* $x\in\Lambda_X(U)$,

$$\|DX_T/E^s_x\| < c\,\lambda^T, \quad \forall\ T>0.$$

Proof. Let $K_1=\sup\{\|DX_r\|, 0\le r\le T_0\}$. Choose $\lambda<1$ such that $1/2<\lambda^{T_0}$, and $c>0$ such that $K_1<c\,\lambda^r$ for $0\le r\le T_0$.

For any $x\in\Lambda_X(U)$, and all $T>0$, we have $T=nT_0+r$, $r<T_0$. Then,

$$\begin{aligned}\|DX_T/E^s_x\| &= \|DX_r/E^s_{X_r(x)}\|\,.\,\prod_{j=0}^{n-1}\|DX_{jT_0}/E^s_{X_{jT_0}(x)}\|\\ &< K_1(1/2)^n < c\,\lambda^r\,(\lambda^{T_0})^n < c\,\lambda^T,\end{aligned}$$

proving the lemma. □

Lemmas 3.9 and 3.10 imply that to prove the bundle E^s is uniformly contracting it is enough to prove that

$$\lim_{t\to\infty} \inf \|DX_t/E^s_x\| = 0,$$

for every $x \in \Lambda_X(U)$. Suppose, by contradiction, that there is $x \in \Lambda_X(U)$ such that

$$\lim_{t\to\infty} \inf \|DX_t/E^s_x\| > 0.$$

Then, there is $s_n \to \infty$ as $n \to \infty$ such that

$$\lim_{s_n\to\infty} \frac{1}{s_n} \log \|DX_{s_n}/E^s_x\| \geq 0. \tag{10}$$

Let $C^0(\Lambda_X(U))$ be the set of real continuous functions defined on $\Lambda_X(U)$ with the C^0 topology, and define the sequence of continuous operators

$$\begin{aligned}\Psi_n : C^0(\Lambda_X(U)) &\to \mathbb{R},\\ \varphi &\longmapsto \frac{1}{s_n}\int_0^{s_n} \varphi(X_s(x))ds.\end{aligned}$$

There exists a convergent subsequence of Ψ_n, which we still denote by Ψ_n, converging to a continuous map $\Psi : C^0(\Lambda_X(U)) \to \mathbb{R}$. Let $\mathcal{M}(\Lambda_X(U))$ be the space of measures with support on $\Lambda_X(U)$. By the Theorem of Riez, there exists $\mu \in \mathcal{M}(\Lambda_X(U))$ such that

$$\int_{\Lambda_X(U)} \varphi d\mu = \lim_{s_n\to\infty} \frac{1}{s_n}\int_0^{s_n} \varphi(X_s(x))ds = \Psi(\varphi), \tag{11}$$

for every continuous map φ defined on $\Lambda_X(U)$. It is clear that such μ is invariant by the flow X_t.

Define $\varphi_X : C^0(\Lambda_X(U)) \longrightarrow \mathbb{R}$ by

$$\varphi_X(p) = \partial_h(\log\|DX_h/E^s_p\|)_{h=0} = \lim_{h\to 0}\frac{1}{h}\log\|DX_h/E^s_p\|.$$

This map is continuous, and so it satisfies (11).

On the other hand, for any $T \in \mathbb{R}$,

$$\begin{aligned}\frac{1}{T}\int_0^T \varphi_X(X_s(p))ds &= \frac{1}{T}\int_0^T \partial_h(\log\|DX_h/E^s_{X_s(p)}\|)_{h=0}ds\\ &= \frac{1}{T}\log\|DX_T/E^s_p\|.\end{aligned} \tag{12}$$

Combining (10), (11), and (12) we get

$$\int_{\Lambda_X(U)} \varphi_X d\mu \geq 0. \tag{13}$$

By the Ergodic Theorem of Birkhoff, we have that

$$\int_{\Lambda_X(U)} \varphi_X d\mu = \int_{\Lambda_X(U)} \lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi_X(X_s(y))dsd\mu(y). \tag{14}$$

Let Σ_X be the set of strongly closed points. Since μ is invariant and $\mathrm{Supp}(\mu) \subset \Lambda_X(U)$, Theorem 3.4 implies

$$\mu(\Lambda_X(U) \cap (\mathrm{Sing}_X(\Lambda_X(U) \cup \Sigma_X)) = 1\,.$$

We claim that $\mu(\Lambda_X(U) \cap \Sigma_X) > 0$. Indeed, otherwise $\mu(\mathrm{Sing}_X(\Lambda_X(U)) = 1$. Since $\mathrm{Sing}_X(\Lambda_X(U))$ is invariant, we get that $\mu = \delta_\sigma$, the Dirac measure at $\sigma \in \mathrm{Sing}_X(\Lambda_X(U))$. But $E^s_\sigma = \hat{E}^{\mathrm{ss}}_\sigma$ (recall Remark 3.8) and the fact that the eigenvalue λ_{ss} along $\hat{E}^{\mathrm{ss}}_\sigma$ satisfies $\lambda_{ss} < 0$ gives

$$\int_{\Lambda_X(U)} \varphi_X d\mu = \int_\sigma \varphi_X d\mu = \varphi_X(\sigma) < 0\,,$$

contradicting (13). So $\mu(\Lambda_X(U) \cap \Sigma_X) > 0$, as claimed.

By the ergodic decomposition for invariant measures, we can suppose that μ is ergodic. Hence $\mu(\Lambda_X(U) \cap \Sigma_X) = 1$.

Now, by (13) and (14) we obtain that there exists $y \in \Lambda_X(U) \cap \Sigma_X$ such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi_X(X_s(y))ds \geq 0. \tag{15}$$

Since $y \in \Sigma_X$, there are $\delta_n \to 0$ as $n \to \infty$, $Y^n \in \mathcal{U}_U$, $p_n \in \mathrm{Per}_{Y^n}(\Lambda_{Y^n}(U))$ with period t_n such that

$$\|Y^n - X\| < \delta_n, \quad \text{and} \quad \mathrm{dist}(Y^n_s(p_n), X_s(y)) < \delta_n,\ 0 \leq s \leq t_n,$$

where Y^n_s is the flow induced by Y^n. Observe that $t_n \to \infty$ as $n \to \infty$. Otherwise, $y \in \mathrm{Per}_X(\Lambda_X(U))$ and if t_y is the period of y, (12) and (15) imply that DX_{t_y}/E^s_y expands. Combining this fact with Theorem 3.6 (a2) gives that y is a repeller periodic point, contradicting Lemma 2.10.

Let $\gamma < 0$ be arbitrarily small. By (15) again, there is T_γ such that for $t \geq T_\gamma$

$$\frac{1}{t} \int_0^t \varphi_X(X_s(y))ds \geq \gamma\,. \tag{16}$$

Since $t_n \to \infty$ as $n \to \infty$, we can assume that $t_n > T_\gamma$ for every n. The continuity of the splitting $E^s \oplus E^{cu}$ over $T_{\Lambda_X(U)}M$ with the flow together with (16) give, for n big enough, that

$$\frac{1}{t_n} \log \|DY^n_{t_n}/E^{s,Y^n}_{p_n}\| \geq \gamma\ .$$

Thus

$$\|DY^n_{t_n}/E^{s,Y^n}_{p_n}\| \geq e^{\gamma t_n}\,.$$

Taking n sufficiently large and $\gamma < 0$ sufficiently small, this last inequality contradicts (a1) in Theorem 3.6.

This completes the proof that E^s is a uniformly contracting bundle. $\square$

3.5. *E^{cu} is uniformly volume expanding.* Using lemmas similar to Lemmas 3.9 and 3.10, one can see that to prove that E^{cu} is volume-expanding it is enough to prove

$$\lim_{t\to\infty} \inf |\det(DX_{-t}/E^{cu}_x)| = 0\,,$$

for every $x \in \Lambda_X(U)$.

Suppose, by contradiction, that there is $x \in \Lambda_X(U)$ such that

$$\lim_{t\to\infty} \inf |\det(DX_{-t}/E^{cu}_x)| > 0\,.$$

Then there is $s_n \to \infty$ as $n \to \infty$, such that

$$\lim_{n\to\infty} \frac{1}{s_n} \log|\det(DX_{-s_n}/E^{cu}_x)| \geq 0. \tag{17}$$

Let $C^0(\Lambda_X(U))$ be the set of real continuous functions defined on $\Lambda_X(U)$ with the C^0 topology, and define the sequence of continuous operators

$$\Psi_n : C^0(\Lambda_X(U)) \to \mathbb{R},$$
$$\varphi \longmapsto \frac{1}{s_n}\int_0^{s_n} \varphi(X_{-s}(x))ds.$$

Hence there exists a convergent subsequence of Ψ_n, which we still denote by Ψ_n, converging to a continuous map $\Psi : C^0(\Lambda_X(U)) \to \mathbb{R}$. Let $\mathcal{M}(\Lambda_X(U))$ be the space of measures with support on $\Lambda_X(U)$. By the Theorem of Riez, there exists $\mu \in \mathcal{M}(\Lambda_X(U))$ such that

$$\int_{\Lambda_X(U)} \varphi d\mu = \lim_{s_n\to\infty} \frac{1}{s_n}\int_0^{s_n} \varphi(X_{-s}(x))ds = \Psi(\varphi),$$

for every continuous map φ defined on $\Lambda_X(U)$. It is clear that such μ is invariant by the flow Y_t.

Now define $\varphi_X : C^0(\Lambda_X(U)) \longrightarrow \mathbb{R}$ by

$$\varphi_X(p) = \partial_h \left(\log|\det(DX_{-h}/E^{cu}_p)|\right)_{h=0} = \lim_{h\to 0}\frac{1}{h}\log|\det(DX_{-h}(p)/E^{cu}_p)|\,.$$

Thus,

$$\int_{\Lambda_X(U)} \varphi_X d\mu = \lim_{n\to\infty} \frac{1}{s_n}\int_0^{s_n} \varphi_X(X_{-s}(p))ds.$$

On the other hand, for any T,

$$\begin{aligned}\frac{1}{T}\int_0^T \partial_h &\left(\log|\det(DX_{-h}/E^{cu}_{X_{-s}(p)})|\right)_{h=0} ds \\ &= \frac{1}{T}\log|\det(DX_{-T}/E^{cu}_p)|\,,\end{aligned} \tag{18}$$

and using (17) we get

$$\int_{\Lambda_X(U)} \varphi_X d\mu \geq 0. \tag{19}$$

By the Ergodic Theorem of Birkhoff, we have that

$$\int_{\Lambda_X(U)} \varphi_X d\mu = \int_{\Lambda_X(U)} \lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi_X(X_{-s}(x))dsd\mu(x). \tag{20}$$

Let Σ_X be the set of strongly closed points. Arguing as in the previous section, we can assume that $\mu(\Lambda_X(U)\cap\Sigma_X)=1$ for any μ with $\mathrm{Supp}(\mu)\subset\Lambda_X(U)$.

By (19) and (20), there exists $y\in\Lambda_X(U)\cap\Sigma_X$ such that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi_X(X_{-s}(y))ds \geq 0\,. \tag{21}$$

Hence there are $\delta_n\to 0$ as $n\to\infty$, $Y^n\in\mathcal{U}_U$, $p_n\in \mathrm{Per}_{Y^n}(\Lambda_{Y_n}(U))$ with period t_n such that $\|Y^n-X\|<\delta_n$ and $\mathrm{dist}(Y^n_{-s}(p_n),X_{-s}(y))<\delta_n$, $0\le s\le t_n$, where Y^n_s is the flow induced by Y^n. Observe that $t_n\to\infty$ as $n\to\infty$. Otherwise, $y\in\mathrm{Per}_Y(\Lambda_Y(U))$, and if t_y is the period of y then (21) implies that DX_{t_y}/E^u_y contracts. This fact combined with Theorem 3.6 (a1) gives the fact that y is a contracting periodic point, contradicting Lemma 2.10.

Let $\gamma<0$ be arbitrarily small. By (21), there is $T_\gamma>0$ such that for $t\ge T_\gamma$,

$$\frac{1}{t}\int_0^t \varphi_X(X_{-s}(y))ds \geq \gamma\,. \tag{22}$$

Since $t_n\to\infty$ as $n\to\infty$, we can assume that $t_n>T_\gamma$ for every n.

The continuity of the splitting $E^s\oplus E^{cu}$ with the flow together with (18) and (22) imply that for n big enough

$$\frac{1}{t_n}\log|\det(DY^n_{-t_n}/E^{cu,Y^n}_{p_n})|\geq\gamma\ .$$

Hence,

$$|\det(DY^n_{-t_n}/E^{cu,Y^n}_{p_n})|\geq e^{\gamma t_n},$$

which implies that

$$|\det(DY^n_{t_n}/E^{cu,Y^n}_{p_n})|\leq e^{-\gamma t_n}=(e^{-\gamma})^{t_n}. \tag{23}$$

Since $\gamma<0$ is arbitrarily small, taking n sufficiently big we obtain that $e^{-\gamma}$ is arbitrarily close to one. Since for periodic orbits the equality

$$|\det(DY^n_{t_n}/E^{cu,Y^n}_{p_n})| = |DY^n_{t_n}/E^{u,Y^n}_{p_n})|$$

holds, and (23) contradicts (a2) in Theorem 3.6. This completes the proof that E^{cu} is volume-expanding. □

3.6. *Proof of Theorem* F. As already said, let us assume Theorems 3.6 and 3.7 and show how we obtain Theorem F. The central idea of the proof is to show that if Theorem F fails then we can obtain a flow near X exhibiting a periodic point with angle between the stable and the central bundles arbitrarily small, leading to a contradiction to Theorem 3.7.

We shall use the following notation: if $v_1, \dots, v_n$ are vectors, $[v_1, \dots, v_n]$ is the space spanned by $v_1, \dots, v_n$.

As in Lemma 3.10, one can easily see that to obtain Theorem F it is enough to prove that there exist a neighborhood $\mathcal{V} \subset \mathcal{U}_U$ of X, and $T_0 > 0$, such that for any $Y \in \mathcal{V}$, if $p \in \mathrm{Per}_Y^{T_0}(\Lambda_Y(U))$ then

$$\|DY_{T_0}/E_p^s\| \, \|DY_{-T_0}/E^{cu}_{Y_{T_0}(p)}\| \leq \frac{1}{2} \, . \tag{24}$$

So, we only have to prove (24). The proof goes by contradiction. If it does not hold then given $X \in \mathcal{U}_U$, we have that for any $T_0 > 0$ there is $Y \in \mathcal{U}_U \cap \mathcal{X}^\infty(M)$ C^1 arbitrarily close to X, and $y \in \mathrm{Per}_Y^{T_0}(\Lambda_Y(U))$, such that

$$\|DY_{T_0}/E_y^s\| \, \|DY_{-T_0}/E^{cu}_{Y_{T_0}(y)}\| > \frac{1}{2} \, . \tag{25}$$

Claim 3.11. *For any positive number T_0, there are $Y \in \mathcal{X}^\infty(M)$, C^1 arbitrarily close to X, $T > T_0$, $y \in \mathrm{Per}_Y(\Lambda_Y(U))$ with period t_y larger than T, and $v \in E^{cu}_{Y_T(y)}$ not collinear to $Y(Y_T(y))$, such that*

$$\|DY_T/E_y^s\| \, \|DY_{-T}(Y_T(y))(v)\| = 1 \, .$$

Proof. First we will prove that there are $Y \in \mathcal{X}^\infty(M)$ C^1-arbitrarily close to X, $T > T_0$, and $y \in \mathrm{Per}_Y^{T_0}(\Lambda_Y(U))$ with period t_y larger than T, and $v \in E^{cu}_{Y_T(y)}$, such that

$$\|DY_T/E_y^s\| \, \|DY_{-T}(Y_T(y))(v)\| = \frac{1}{2} \, . \tag{26}$$

Take Y of class C^∞ close to X, T_0 large enough and $y \in \mathrm{Per}_Y^{T_0}(\Lambda_Y(U))$ such that (25) holds. If for some $T_0 < T < t_y$

$$\|DY_T/E_y^s\| \, \|DY_{-T}/E^{cu}_{Y_T(y)}\| < \frac{1}{2} \, ,$$

then there is another intermediate time, that we continue to denote by T, $T_0 < T < t_y$, such that

$$\|DY_T/E_y^s\| \, \|DY_{-T}/E^{cu}_{Y_T(y)}\| = \frac{1}{2} \, ,$$

and so there is $v \in E^{cu}_{Y_T(y)}$ such that (26) holds, and we are done. Otherwise, if for any $T_0 \leq T < t_y$ we have

$$\|DY_T/E_y^s\| \, \|DY_{-T}/E^{cu}_{Y_T(y)}(v)\| \geq \frac{1}{2} \, , \tag{27}$$

we argue as follows. Observe that, by Theorem 3.6(a1) and (a2), we have that

$$\|DY_{t_y}/E^s_y\|\,\|DY_{-t_y}/E^u_{Y_{t_y}(y)}\| < \lambda^{2t_y} < \frac{1}{2}\,.$$

Hence, we get that for $T < t_y$ and close enough to t_y, the inequality above holds when we replace t_y by T. Thus, using (27) and the fact that $E^u_{Y_T(y)} \subset E^{cu}_{Y_T(y)}$, we conclude that there is a vector $v \in E^{cu}_{Y_T(y)}$ such that

$$\|DY_T/E^s_y\|\,\|DY_{-T}(Y_T(y))(v)\| = \frac{1}{2}\,,$$

proving (26).

Now, let A_t, for $0 \le t \le T$, be the one-parameter family of linear maps defined by

$$A_t/E^s_y = 2^{\frac{t}{T}}\ DY_t/E^s_y, \quad A_t/E^{cu}_y = DY_t/E^{cu}_y\,.$$

Observe that

$$\begin{aligned}\partial_s A_{t+s}A_t^{-1}\big|_{s=0}/E^s_{Y_t(y)} &= \partial_s DY_{t+s}(y)DY_t^{-1}/E^s_{Y_t(y)}\,(2^{\frac{s}{T}})\big|_{s=0}\\ &= DY/E^s_{Y_t(y)} + \frac{\log(2)}{T}.\end{aligned}$$

Thus

$$\|\partial_s A_{t+s}A_t^{-1}\big|_{s=0}/E^s_{Y_t(y)} - DY/E^s_{Y_t(y)}\| = \Big|\frac{\log(2)}{T}\Big|\,.$$

Since $A_t/E^{cu}_y = DY_t/E^{cu}_y$, we get

$$\|\partial_s A_{t+s}A_t^{-1}\big|_{s=0} - DY\| = \Big|\frac{\log(2)}{T}\Big|\,.$$

As T can be chosen arbitrarily big we have that $\frac{\log(2)}{T}$ is arbitrarily near 0. Moreover, since the flow direction is contained in E^{cu} and $A_t/E^{cu}_y = DY_t/E^{cu}_y$ we obtain that A_t preserves the flow direction of X. Furthermore, as $Y \in \mathcal{X}^\infty$, the family A_t is C^2 with respect to the parameter t. Thus A_t satisfies the hypotheses of Lemma 3.1.

Hence, there is another C^1 vector field, which we continue to denote by Y, C^1 near X, satisfying $DY_t(y) = A_t$. On the other hand, using (26) and the definition of A_t we get $\|A_T/E^s_y\|\,\|A_T^{-1}(v)\| = 1$ which implies

$$\|DY_T/E^s_y\|\,\|DY_{-T}(Y_T(y))(v)\| = 1\,.$$

If v is not collinear to $Y(y)$ then we are done. Otherwise we take anothe C^∞ vector field that we continue to denote by Y, C^1-close to X, and $\tilde{v}$ near v such that the product

$$\|DY_T/E^s_y\|\,\|DY_{-T}(Y_T(y))(\tilde{v})\| = b_T \quad \text{is arbitrarily close to } 1\,,$$

and define the one parameter family of linear maps B_t by

$$B_t/E^s_y = b_T^{-\frac{t}{T}}\,DY_t/E^s_y\,, \quad B_t/E^{cu}_y = DY_t/E^{cu}_y\,, \quad 0 \le t \le T\,.$$

Reasoning as above, we obtain that this family satisfies the hypothesis of Lemma 3.1 and, as before, we find a new C^∞ vector field that we continue to denote by Y, C^1 near X, and $w \in E^{cu}_{Y_T(y)}$ not collinear to $Y(Y_T(y))$, and $T < t_y$ arbitrarily large, such that

$$\|DY_T/E^s_y\| \, \|DY_{-T}(Y_T(y))(w)\| = 1\,.$$

This completes the proof of Claim 3.11.

Claim 3.12. *There are $Z \in \mathcal{X}^\infty(M)$, C^1 near Y, and $y \in \mathrm{Per}_Z(\Lambda_Z(U))$, such that $\alpha(E^{s,Z}_y, E^{cu,Z}_y)$ is arbitrarily small.*

Proof. Fix T arbitrarily large and let $Y \in \mathcal{U}_U \cap \mathcal{X}^\infty(M)$ C^1 arbitrarily close to X, $y \in \mathrm{Per}_Y(\Lambda_Y(U))$ with period $t_y > T$, and $v \in E^{cu}_{Y_T(y)}$ not collinear to $Y(Y_T(y))$ be given by Claim 3.11. Then

$$\|DY_T/E^s_y\| \, \|DY_{-T}(Y_T(y))(v)\| = 1\,.$$

Let

$$w = \frac{DY_{-T}(Y_T(y))(v)}{\|DY_{-T}(Y_T(y))(v)\|}\,.$$

Observe that $w \in E^{cu}_y$ and it is not collinear to $Y(y)$. Set in T_yM the basis

$$\mathcal{B}_y = \left\{ Y(y)/\|Y(y)\|, w, e^s_y \right\}\,,$$

where e^s_y is the unitary generator of E^s_y. For each $0 \le r \le T < t_y$ set, in $T_{Y_r(y)}M$, the basis

$$\mathcal{B}_{Y_r(y)} = \left\{ Y(Y_r(y))/\|Y(Y_r(y))\|, w_r, e^s_{Y_r(y)} \right\}\,,$$

where $w_r = \frac{DY_r(y)(w)}{\|DY_r(y)(w)\|}$ and $e^s_{Y_r(y)}$ is the unitary generator of $E^s_{Y_r(y)}$.

Since, by Theorem 3.7, $\alpha(E^s_{Y_r(y)}, E^{cu}_{Y_r(y)}) > C$ for all $y \in \mathrm{Per}_Y(\Lambda_Y(U))$ and all r, there is $K = K(C)$ such that for all $Y \in \mathcal{V}$ and all $y \in \mathrm{Per}(\Lambda_Y(U))$, there is a metric $\|.\|_{(Y,y)}$ depending on Y and y, such that $E^s_{Y_r(y)}$ and $E^{cu}_{Y_r(y)}$ are orthogonal for all r, and

$$\frac{1}{K}\,\|.\| \,\le\, \|.\|_{(Y,y)} \,\le\, K\,\|.\|.$$

Then, the matrix of $DY_s(y)$, for all s, with respect to the basis $\mathcal{B}_{Y_s(y)}$ in the metric $\|.\|_{(Y,y)}$ is given by

$$DY_s(y) = \begin{bmatrix} Y(s) & * & 0 \\ 0 & a(s) & 0 \\ 0 & 0 & b(s) \end{bmatrix}, \text{ where}$$

$$Y(s) = \|Y(Y_s(y))\|_{(Y,y)}, \;\; a(s) = \|DY_s(y)(w)\|_{(Y,y)}, \;\; b(s) = \|DY_s(y)(e^s_y)\|_{(Y,y)}.$$

Observe that in this basis, $Y(Y_s(y)) = (1,0,0)$, and $a(T) = b(T)$, by the choice of v and w.

Let $A_{s,y}$ be the restriction of $DY_s(y)$ to $[w, e^s_y]$. Observe that any perturbation on $A_{s,y}$ does not affect the direction of Y.

For each $\delta > 0$, let

$$A^+_{s,y} = \begin{bmatrix} a(s) & \delta a(s) \int_0^s \frac{b(r)}{a(r)} dr \\ 0 & b(s) \end{bmatrix},$$

and

$$A^-_{s,y} = \begin{bmatrix} a(s) & 0 \\ \delta b(s) \int_0^s \frac{a(r)}{b(r)} dr & b(s) \end{bmatrix}.$$

Observe that for any $h \geq 0$,

$$A^+_{s+h,y}(A^+_{s,y})^{-1} = \begin{bmatrix} a(s,h) & c(s,s+h) \\ 0 & b(s,h) \end{bmatrix}$$

where

$$a(s,h) = \frac{a(s+h)}{a(s)}, \quad b(s,h) = \frac{b(s+h)}{b(s)}, \quad \text{and}$$

$$c(s,s+h) = \delta \frac{a(s+h)}{b(s)} \int_s^{s+h} \frac{b(r)}{a(r)} dr.$$

A similar formula holds for $A^-_{s+h}(A^-_s)^{-1}$.

We claim that

$$\|\partial_h A^+_{s+h,y}(A^+_{s,y})^{-1}|_{h=0} - DY(Y_s(y))\|_{(Y,y)} \leq \delta. \tag{28}$$

Indeed, we have

$$\partial_h A^+_{s+h,y}(A^+_{s,y})^{-1}|_{h=0} = \begin{bmatrix} \frac{a'(s)}{a(s)} & \partial_h c(s,s+h)|_{h=0} \\ 0 & \frac{b'(s)}{b(s)} \end{bmatrix}$$

and

$$DY(Y_s(y)) = \begin{bmatrix} \frac{a'(s)}{a(s)} & 0 \\ 0 & \frac{b'(s)}{b(s)} \end{bmatrix}.$$

So, all we need is $\|\partial_h c(s,s+h)|_{h=0}\|_{(Y,y)} \leq \delta$. But

$$c(s,s+h) = \delta \frac{a(s+h)}{b(s)} \int_s^{s+h} \frac{b(r)}{a(r)} dr = \delta \frac{a(s+h)}{b(s)} \frac{b(\eta)}{a(\eta)} h,$$

for some $\eta \in [s, s+h]$. From this it follows that

$$\partial_h c(s,s+h)|_{h=0} = \lim_{h \to 0} \frac{c(s,s+h)}{h} = \delta\,.$$

This implies (28). A similar result holds for $A^-_{s+h}(A^-_s)^{-1}$.

Let

$$A^+_{T,y}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\delta a(T)\int_0^T\frac{b(r)}{a(r)}dr\\ b(T)\end{bmatrix} \text{ and } A^-_{T,y}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}a(T)\\ \delta b(T)\int_0^T\frac{a(r)}{b(r)}dr\end{bmatrix}.$$

We shall prove next that shrinking δ results in either

$$\frac{b(T)}{\delta a(T)\int_0^T\frac{b(r)}{a(r)}dr} \quad \text{is arbitrarily small or} \tag{29}$$

$$\delta\frac{b(T)}{a(T)}\int_0^T\frac{a(r)}{b(r)}dr \quad \text{is arbitrarily large.} \tag{30}$$

The meaning of this is that either

$$A^+_{T,y}\begin{bmatrix}0\\1\end{bmatrix} \quad \text{is near a horizontal vector or}$$

$$A^-_{T,y}\begin{bmatrix}1\\0\end{bmatrix} \quad \text{is near a vertical vector.}$$

If (29) holds we consider the family

$$B^+_{s,y}=\begin{bmatrix}Y(s) & * & 0\\ 0 & a(s) & \delta a(s)\int_0^s\frac{b(r)}{a(r)}dr\\ 0 & 0 & b(s)\end{bmatrix}$$

and if (30) holds we take

$$B^-_{s,y}=\begin{bmatrix}Y(s) & * & 0\\ 0 & a(s) & 0\\ 0 & \delta b(s)\int_0^s\frac{a(r)}{b(r)}dr & b(s)\end{bmatrix}.$$

Assume (29). Observe that (28) together with the fact that $Y\in C^\infty$ imply that $B^+_{s,y}$ satisfies Lemma 3.1. So there is $Z\in\mathcal{U}_U\cap\mathcal{X}^1(M)$ C^1 near Y, and so C^1 near X, such that y is a periodic point of Z with period T, $Z_t(y)=Y_t(y)$ for every t, and $DZ_s(y)=B^+_{s,y}$. In particular, the restriction of $DZ_s(y)$ to $[w,e^s_y]$ is equal to $A^+_{s,y}$ for every s. Hence, $E^{cu,Z}_{Z_s(y)}=E^{cu,Y}_{Y_s(y)}$ for all s. Moreover, Theorem 3.7 combined with the fact that the metric $\|.\|_{(Y,y)}$ is comparable with $\|.\|$, give $\alpha(E^{cu,Z}_y,E^{s,Z}_y)>C'$, $C'=C'(C)>0$, in the metric $\|.\|_{(Y,y)}$. Thus, in the basis $\mathcal{B}_{Y_s(y)}$ fixed above, we obtain $E^{s,Z}_y=(g,d,1)$ with $|g|$ and $|d|$ upper bounded by a constant depending only on C. Thus,

$$E^{s,Z}_{Z_T(y)}=DZ_{Z_T(y)}\begin{bmatrix}g\\d\\1\end{bmatrix}=\begin{bmatrix}gY(T)+*d\\ da(T)+\delta a(T)\int_0^T\frac{b(r)}{a(r)}dr\\ b(T)\end{bmatrix}.$$

As $a(T) = b(T)$, we obtain that the ratio between the third and the second coordinate of $E^{s,Z}_{Z_T(y)}$ is equal to

$$\left(d + \delta \int_0^T \frac{b(r)}{a(r)} dr\right)^{-1} .$$

Now, (29) implies $\delta \int_0^T \frac{b(r)}{a(r)} dr > K_1$, with K_1 arbitrarily large. Hence,

$$d + \delta \int_0^T \frac{b(r)}{a(r)} dr > K_1 + d \quad \text{and so} \quad \left(d + \delta \int_0^T \frac{b(r)}{a(r)} dr\right)^{-1} < \frac{1}{K_1 + d} .$$

As $(K_1+d)^{-1}$ is arbitrarily small, we obtain that $\alpha(E^{s,Z}_{Z_T(y)}, E^{cu,Z}_{Z_T(y)})$ is arbitrarily small in the metric $\|.\|_{(Y,y)}$. And since $\|.\|$ and $\|.\|_{(Y,y)}$ are comparable, we obtain that $\alpha(E^{s,Z}_{Z_T(y)}, E^{cu,Z}_{Z_T(y)})$ is also arbitrarily small in the original metric, contradicting Theorem 3.7.

Assuming (30) and reasoning analogously, we obtain that $\alpha(E^{cu,Z}_{Z_T(y)}, E^{s,Z}_{Z_T(y)})$ is arbitrarily small in the original metric.

All of these facts together prove Claim 3.12, which contradicts Theorem 3.7.

Thus, to conclude the proof of Theorem F, all that is left to prove is that we have either (29) or (30). For this, set $\delta = T^{-1/2}$. Note that as T is arbitrarily large, δ can be taken arbitrarily small. Since $a(s) > 0$ and $b(s) > 0$ for every $s \in [0, \tau]$, we can write

$$T = \int_0^T dY = \int_0^T \sqrt{\frac{a(Y)}{b(Y)}} \sqrt{\frac{b(Y)}{a(Y)}} dY \leq \sqrt{\int_0^T \frac{a(Y)}{b(Y)} dY} \sqrt{\int_0^T \frac{b(Y)}{a(Y)} dY}$$

and so

$$T^2 \leq \int_0^T \frac{a(Y)}{b(Y)} dY \int_0^T \frac{b(Y)}{a(Y)} dY ,$$

implying that

$$\frac{T}{\delta \int_0^T \frac{b(Y)}{a(Y)} dY} = \frac{T^2 \delta^2}{\delta \int_0^T \frac{b(Y)}{a(Y)} dY} \leq \delta \int_0^T \frac{a(Y)}{b(Y)} dY .$$

Thus, if $(\delta \int_0^T \frac{b(Y)}{a(Y)} dY)^{-1} > T^{-1/2}$ then $T \ (\delta \int_0^T \frac{b(Y)}{a(Y)} dY)^{-1} > T\ T^{-1/2} = \sqrt{T}$, which implies

$$\delta \int_0^T \frac{a(Y)}{b(Y)} dr \geq \sqrt{T} .$$

Since T is arbitrarily large, we obtain that either (29) or (30) holds. The proof of Theorem F is complete, and we conclude that the splitting $E^s \oplus E^{cu}$ over $\mathrm{Per}^{T_0}_Y(\Lambda_Y(U))$ given by Definition 3.5 is an invariant uniformly dominated splitting. $\square$

4. Proofs of Theorems 3.6 and 3.7

Here we prove Theorems 3.6 and 3.7, used in the proofs of the results in the previous section.

Proof of Theorem 3.6(a). Suppose, by contradiction, that given $\delta > 0$ small, there is $Y \in \mathcal{X}^\infty(M)$ C^1 arbitrarily close to X, and a periodic orbit y of Y with period t_y, such that $\|DY_{t_y}/E^s_y\| \geq (1-\delta)^{t_y}$.

Let A_t, $0 \leq t \leq t_y$, be the one-parameter family of linear maps given by

$$A_t = DY_t(y)\,(1-2\delta)^{-t}.$$

By construction, A_t preserves the direction of the flow and the eigenspaces of DY_{t_y}. Moreover,

$$\|\partial_h A_{t+h} A_t^{-1}|_{h=0} - DY(Y_t(y))\| < -\log(1-\delta)\,.$$

As $(1-\delta)$ is as near 1 as we wish, the inequality above together with the fact that $Y \in C^\infty$ imply that A_t satisfies Lemma 3.1.

So, there is $Z \in C^1$, C^1-near Y such that y is a periodic point of Z with period t_y, and $DZ_t(Z_t(y)) = A_t$ for $0 \leq t \leq t_y$. But, by construction, we get that $\|DZ_{t_y}/E^s_y\| > 1$, implying that y is a source for Z, contradicting Lemma 2.10.

By the same argument we prove (a.2). This finishes the proof of (a). $\square$

Proof of (b). By contradiction, assume that for every $\gamma > 0$, there exist $Y \in \mathcal{X}^\infty$ C^1-close to X and $p \in \mathrm{Per}_Y(\Lambda_Y(U))$, such that $\alpha(E^s_p, E^u_p) < \gamma$.

Let t_p be the period of p and λ_s, λ_u be the stable and unstable eigenvalues of $DY_{t_p}(p)$. Then $\lambda_s < \lambda^{t_p}$ and $\lambda_u > \lambda^{-t_p}$, where λ is given by (a). Observe that there is t_0 such that $t_p > t_0$ and thus, $|1 - \frac{\lambda_s}{\lambda_u}|$ is uniformly bounded away from 0, and if $\lambda_s\,\lambda_u > 0$ then there is $D_1 > 0$ such that

$$D_1^{-1} < \left|\frac{2\sqrt{\lambda_s\lambda_u} - \lambda_s - \lambda_u}{\lambda_u - \lambda_s}\right| < D_1, \tag{31}$$

and if $\lambda_s\,\lambda_u < 0$ then there is $D_2 > 0$ such that

$$D_2^{-1} < \left|\frac{-(\lambda_s + \lambda_u)}{(\lambda_u - \lambda_s)}\right| < D_2\ . \tag{32}$$

Let $\hat{\gamma}$ be the slope between E^s_p and E^u_p. Observe that $\hat{\gamma}$ is small if the angle $\alpha(E^s_p, E^u_p)$ is small. In the case that $\lambda_s\,\lambda_u > 0$, we set $\delta = \left|\frac{2\sqrt{\lambda_s\lambda_u} - \lambda_s - \lambda_u}{\lambda_u - \lambda_s}\right| \hat{\gamma}$. Otherwise, $\delta = \left|\frac{-(\lambda_s+\lambda_u)}{(\lambda_u-\lambda_s)}\right| \hat{\gamma}$. By hypothesis, $\hat{\gamma}$ can be taken arbitrarily small, so (31) and (32) imply that δ also can be taken arbitrarily small.

Now, let $\mathcal{B}_t$, $0 \leq t \leq t_p$, be a continuous positive oriented basis in $T_{Y_t(p)}M$ defined as

$$\mathcal{B}_t(p) = \left\{\frac{Y_t(p)}{\|Y_t(p)\|}, v_2(t), v_3(t)\right\},$$

where $v_2(t) \in E^{cs}_{Y_t(p)}$ and is orthonormal to $Y(Y_t(p))$, and $v_3(t)$ is orthonormal to $E^{cs}_{Y_t(p)}$.

In this basis we have

$$DY_{t_p}(p) = \begin{bmatrix} 1 & * & * \\ 0 & \lambda_s & \frac{\lambda_u - \lambda_s}{\hat{\gamma}} \\ 0 & 0 & \lambda_u \end{bmatrix}.$$

For each δ as above, let

$$A(\delta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \delta & 1 \end{bmatrix},$$

and consider $B(\delta) = A(\delta)DY_{t_p}(p)$. Since δ is arbitrarily small, $B(\delta)$ is arbitrarily near $DY_{t_p}(p)$. Moreover, a straightforward calculation shows that $B(\delta)$ has one eigenvalue equal to 1, and the other two eigenvalues having modulus equal to $\sqrt{|\lambda_s \, \lambda_u|}$, which is either bigger than 1 or smaller or equal to 1.

Since δ can be taken arbitrarily small, there is a nonnegative C^2 real function $\delta(t)$ such that $\delta(0) = 0$, $\delta(t_p) = \delta$, $|\delta'(t)| < 2\delta$, and $|\delta(t)| < 2\delta$. Now, define the one parameter family A_t, $0 \le t \le t_p$, of linear maps whose matrix, in the basis $\mathcal{B}_t$ is

$$A_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \delta(t) & 1 \end{bmatrix}.$$

Let $C_t = A_t DY_t(p)$, $0 \le t \le t_p$. Note that, by construction, C_t preserves the flow direction along the Y-orbit of p. Moreover, the choice of $\delta(t)$ implies that A_t, $0 \le t \le t_p$, is a small perturbation of the identity map $I_t : T_{Y_t(p)}M \to T_{Y_t(p)}M$, and so C_t satisfies Lemma 3.1. So, there is $Z \in C^1$, C^1 near Y, and $p \in \mathrm{Per}_Z(\Lambda_Z(U))$ such that $DZ_t(p) = C_t = A_t DY_t(p)$, for $0 \le t \le t_p$. On the other hand, $DZ_{t_p} = A_{t_p} DY_{t_p}(p) = B(\delta)$, where $B(\delta)$ is as defined above. Thus, taking δ sufficiently small, we get that there is a C^1 vector field Z nearby Y exhibiting a periodic point p which is either a sink or a source, contradicting Lemma 2.10. The proof of Theorem 3.6 is complete. □

Proof of Theorem 3.7. We shall prove that if Theorem 3.7 fails then we can create periodic points with angle between the stable and unstable directions arbitrarily small, leading to a contradiction to the second part of Theorem 3.6.

Theorem 3.7 is an immediate consequence of Propositions 4.1 and 4.2 below. The first one establishes that for periodic points close to a singularity, the stable direction remains close to the strong stable direction of the singularity, and the central unstable direction is close to the central unstable direction of the singularity. This result gives the compatibility between the splitting proposed for the periodic points in Definition 3.5 and the local partially hyperbolic splitting at the singularities. The second one says that far from singularities, the angle between the stable direction and the central unstable direction of any periodic point is uniformly bounded away from zero.

Before we state Propositions 4.1 and 4.2, let us fix some notation. Given a singularity σ of $X \in \mathcal{U}_U$, we know that σ is hyperbolic and so, for Y close to X, it is defined as the unique continuation of σ, which is denoted by σ_Y. Again, as all singularities of X are hyperbolic we conclude that the singularities of Y nearby X are the continuations of the ones of X. So, we can assume that for any Y close to X, the singularities of Y in $\Lambda_Y(U)$ coincide with the ones of X in $\Lambda_X(U)$. Even with this assumption, we denote these singularities of Y close to X as σ_Y.

By Theorem B, for all $Y \in \mathcal{U}_U$, the eigenvalues λ_i^Y, $0 \le i \le 3$, of $DY(\sigma_Y)$ are real and satisfy $\lambda_2^Y < \lambda_3^Y < 0 < -\lambda_3^Y < \lambda_1^Y$. We denote by $\hat{E}^{ss,Y}_{\sigma_Y}$ the eigenspace associated to the strongest contracting eigenvalue λ_2^Y and by $\hat{E}^{cu,Y}_{\sigma_Y}$ the two-dimensional eigenspace associated to $\{\lambda_3^Y, \lambda_1^Y\}$. Without loss of generality, we can assume that for Y close to X, the eigenvalues of $DY(\sigma_Y)$ are the same as the ones of $DX(\sigma)$.

Since M is a Riemannian manifold, given $x \in M$ there is a normal neighborhood V such that $x \in V$; i.e., for any two distinct points of V there is a unique geodesic contained in V connecting them. Thus, using the parallel transport, we get that the angle between two vectors at different points in V is well defined. In the sequel, any neighborhood of the singularities is a normal one.

Proposition 4.1. *Given $X \in \mathcal{U}_U$, $\varepsilon > 0$ and $\sigma \in \mathrm{Sing}(\Lambda_X(U))$, there exist a neighborhood $\mathcal{V} \subset \mathcal{U}_U$ of X and $\delta > 0$ such that for all $Y \in \mathcal{V}$, if $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ satisfies $\mathrm{dist}(p, \sigma_Y) < \delta$ then*

(a) $\alpha(E_p^{s,Y}, \hat{E}^{ss,Y}_{\sigma_Y}) < \varepsilon$ *and*

(b) $\alpha(E_p^{cu,Y}, \hat{E}^{cu,Y}_{\sigma_Y}) < \varepsilon$.

Proposition 4.2. *Given $X \in \mathcal{U}_U$ and $\delta > 0$, there are a neighborhood $\mathcal{V} \subset \mathcal{U}$ of X and a positive constant $C = C(\delta)$ such that if $Y \in \mathcal{V}$ and $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ satisfies $\mathrm{dist}(p, \mathrm{Sing}_Y(\Lambda_Y(U))) > \delta$ then*

$$\alpha(E_p^{s,Y}, E_p^{cu,Y}) > C\,.$$

To prove Propositions 4.1 and 4.2, we use the following results.

The first one establishes that any compact invariant set $\Gamma \subset \Lambda_X(U)$ containing no singularities is hyperbolic.

Lemma 4.3. *Let $X \in \mathcal{U}_U$ and $\Gamma \subset \Lambda_X(U)$ be a compact invariant set without singularities. Then Γ is hyperbolic.*

Given $X \in \mathcal{U}_U$ and $\delta > 0$, set $C_\delta(\mathrm{Sing}_X(\Lambda_X(U))) = \cup_{\sigma \in \mathrm{Sing}_X(\Lambda_X(U))} B_\delta(\sigma)$, where $B_\delta(\sigma)$ is the ball of radius δ centered at σ, and denote $U_\delta = \mathrm{Cl}(U \setminus C_\delta(\mathrm{Sing}_X(\Lambda_X))$. Set

$$\Omega_X(U_\delta) = \{x \in \Omega(X) : \mathcal{O}(x) \subset U_\delta\}\,.$$

As a matter-of-fact, we shall use the following application of Lemma 4.3:

COROLLARY 4.4. *For any $\delta > 0$, $\Omega_X(U_\delta)$ is hyperbolic.*

Given a regular point $x \in M$ recall that we set N_x^Y for the orthogonal complement of $[Y(x)]$ in T_xM, $\Lambda_Y^*(U) = \Lambda_Y(U) \setminus \mathrm{Sing}_Y(\Lambda_Y(U))$ and

$$N_{\Lambda_Y^*(U)} = N^{s,Y} \oplus N^{u,Y}$$

denotes the splitting for the linear Poincaré flow P_t^Y of Y; see Theorem 2.15. So, for $x \in \Lambda_Y^*(U)$, we have the splittings $E_x^{cs,Y} = N_x^{s,Y} \oplus [Y(x)]$ and $E_x^{cu,Y} = N_x^{u,Y} \oplus [Y(x)]$.

Recall also that for Y near X and $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ we denote by $E_p^{s,Y} \oplus E_p^{cu,Y}$ the splitting given by Definition 3.5. In this case, we have that $E_p^{cu,Y} = N_p^{u,Y} \oplus [Y(p)]$ and $E_p^{s,Y} \subset E_p^{cs,Y} = N_p^{s,Y} \oplus [Y(p)]$.

Using the fact that a hyperbolic set locally has a unique continuation for flows close to the initial one, we get the following result.

LEMMA 4.5. *Let $X \in \mathcal{U}_U$ and Γ be a compact invariant set without singularities. Then, there are neighborhoods $\mathcal{V}$ of X, V of Γ and $\gamma > 0$ such that for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that if $Y \in \mathcal{V}$, $y \in V \cap \Lambda_Y(U)$ with $Y_s(y) \in V$ for $0 \le s \le t$, $t \ge T$, and $v \in N_y^{s,Y} \oplus [Y(y)]$ with $\alpha(v, Y(y)) < \gamma$, then*

$$\alpha(DY_t(y)(v), Y(Y_t(y))) < \varepsilon\ .$$

The next result gives angle estimates through the passage near a singularity. It says that for a point y in $\Lambda_Y(U)$ and vectors v with angle bounded away from zero with the strong stable bundle at the singularity, after the passage through the singularity, $DY_t(v)$ lands in the direction of the central unstable bundle at $Y_t(y)$. Before we state it, let us introduce some notation and definitions.

Given $\sigma_Y \in \mathrm{Sing}_Y(\Lambda_Y(U))$, $W^s_{\mathrm{loc}}(\sigma_Y)$ ($W^u_{\mathrm{loc}}(\sigma_Y)$ respectively) stands for the local stable (unstable respectively) manifold at σ_Y. To simplify notations we set $\hat W^s_{\mathrm{loc}}(\sigma_Y) = W^s_{\mathrm{loc}}(\sigma_Y) \setminus \{\sigma_Y\}$, and $\hat W^u_{\mathrm{loc}}(\sigma_Y) = W^u_{\mathrm{loc}}(\sigma_Y) \setminus \{\sigma_Y\}$. Since σ_Y is Lorenz-like, there is a unique bundle in $TW^s_{\mathrm{loc}}(\sigma_Y)$, $\hat E^{ss,Y}$, that is strongly contracted by the derivative of the flow. For each $y \in W^s_{\mathrm{loc}}(\sigma_Y)$, $\hat E_y^{ss,Y}$ is the fiber of $\hat E^{ss,Y}$ at y.

For further references, let us define cross sections at points in the unstable bundle at a singularity.

Definition 4.6. Given $\sigma \in \mathrm{Sing}_X(\Lambda_X(U))$ and $\delta > 0$, we can define a *compact cross section* $\Sigma^u_\delta \subset B_\delta(\sigma)$ to X satisfying the following properties:

(1) Σ^u_δ is a cross section to all Y near X;

(2) $\Sigma^u_\delta \cap \hat{W}^u_{\mathrm{loc}}(\sigma_Y) \neq \emptyset$, for all Y near X;

(3) For all Y near X and all $y \in \Lambda_Y(U) \cap \hat{W}^u_{\mathrm{loc}}(\sigma_Y)$ there is t such that $Y_t(y) \in \Sigma^u_\delta$;

(4) For all $y \in \Lambda_Y(U) \cap B_\delta(\sigma_Y)$ there is t such that $Y_t(y) \in \Sigma^u_\delta$, and $Y_s(y) \in B_\delta(\sigma)$ for all $0 \leq s \leq t$.

Definition 4.7. If $y \in B_\delta(\sigma_Y)$ we let y_* be the point in $\hat{W}^s_{\mathrm{loc}}(\sigma_Y)$ such that $\mathrm{dist}(y, \hat{W}^s_{\mathrm{loc}}(\sigma_Y)) = \mathrm{dist}(y, y_*)$.

LEMMA 4.8. *Let $X \in \mathcal{U}_U$, $\sigma \in \mathrm{Sing}(\Lambda_X(U))$, and $\delta > 0$. There is a neighborhood $\mathcal{V}$ of X such that given $\gamma > 0$ and $\varepsilon > 0$ there is $r = r(\varepsilon, \gamma)$ such that for $Y \in \mathcal{V}$, $y \in B_\delta(\sigma) \cap \Lambda_Y(U)$ verifying $\mathrm{dist}(y, \hat{W}^s_{\mathrm{loc}}(\sigma_Y)) < r$ and $v \in T_yM$ with $\alpha(v, \hat{E}^{ss,Y}_{y_*}) > \gamma$ then*

$$\alpha(DY_{s_y}(y)(v), E^{cu,Y}_{Y_{s_y}(y)}) < \varepsilon$$

where s_y is the first positive time such that $Y_{s_y}(y) \in \Sigma^u_\delta$.

Now, let us introduce compact cross sections at points in the stable bundle at a singularity satisfying some nice properties.

Definition 4.9. Given $\sigma \in \mathrm{Sing}_X(\Lambda_X(U))$ and $\delta > 0$, by Theorem B we can define a *compact cross section* to X, $\Sigma^s_\delta \subset B_\delta(\sigma)$, satisfying the following properties:

(1) Σ^s_δ is a cross section to all Y near X;

(2) $\Sigma^s_\delta \cap \hat{W}^s_{\mathrm{loc}}(\sigma_Y) \neq \emptyset$, for all Y near X;

(3) $\Sigma^s_\delta \cap \hat{W}^{\mathrm{ss}}(\sigma_Y) = \emptyset$ for all Y near X;

(4) For all Y near X and all $y \in \Lambda_Y(U) \cap \hat{W}^s_{\mathrm{loc}}(\sigma_Y)$ there is t such that $Y_t(y) \in \Sigma^s_\delta$;

(5) For all $y \in \Lambda_Y(U) \cap B_\delta(\sigma_Y)$ there is t such that $Y_t(y) \in \Sigma^s_\delta$, and $Y_s(y) \in B_\delta(\sigma)$ for all $t \leq s \leq 0$.

Given $\delta' > 0$ we define

$$\Sigma^s_{\delta,\delta'} = \{x \in \Sigma^s_\delta; \mathrm{dist}(x, \hat{W}^s_{\mathrm{loc}}(\sigma) \cap \Sigma^s_\delta) \leq \delta'\}. \tag{33}$$

Finally, the last result also gives angle estimates for a passage near a singularity: it says that for vectors v in the central direction with angle bounded away from zero with the flow direction then, through the passage near σ, $DX_t(v)$ lands in the direction of the flow.

LEMMA 4.10. *Let $X \in \mathcal{U}_U$, $\sigma \in \mathrm{Sing}(\Lambda_X(U))$, and $\delta > 0$. There is a neighborhood $\mathcal{V}$ of X such that given $\varepsilon > 0$, $k > 0$, $\delta > 0$ and cross sections Σ^s_δ, Σ^u_δ as above, there is $\delta' > 0$ such that for all $Y \in \mathcal{V}$, $p \in \Sigma^s_{\delta,\delta'}$, and $v \in N^{u,Y}_p \oplus [Y(p)]$, if $\alpha(v, Y(p)) > k$ then*

$$\alpha(DY_{s_p}(p)(v), Y(Y_{s_p}(p))) < \varepsilon,$$

where s_p is the first positive time such that $Y_{s_p}(p) \in \Sigma^u_\delta$.

We postpone the proof of Lemmas 4.3, 4.5, 4.8, and 4.10 to the end of this section.

Since we have only a finite number of singularities, we can assume that the estimates given by the previous lemmas hold for all singularities of Y in $\Lambda_Y(U)$, for all $Y \in \mathcal{V}$.

Proof of Proposition 4.1 (a). The proof goes by contradiction. Since the continuation of the singularities varies continuously with the vector field, we have that if (a) fails then there are a singularity σ of X, $\gamma > 0$, a sequence of vector fields Y^n converging to X and a sequence of periodic points $p_n \in \mathrm{Per}_{Y^n}(\Lambda_{Y^n}(U))$, $p_n \to \sigma$, such that

$$\alpha(E^{s,Y^n}_{p_n}, \hat{E}^{ss,Y^n}_{\sigma_{Y_n}}) > \gamma. \tag{34}$$

We will prove that (34) implies that both the stable and the unstable directions at some periodic point q_n of Y^n, with n large, will be close to the flow direction at q_n, and so, the stable and unstable directions at q_n will be close, contradicting Theorem 3.6(b). With this purpose, we will show that after a first passage through a neighborhood of a singularity, the stable and the flow direction become close. This property holds up to the next return to that neighborhood, and after the second passage through it, we obtain that the stable and the flow directions are close as well, the unstable and the flow directions, implying the closeness between the stable and the unstable directions, leading, as we said, to a contradiction to Theorem 3.6(b).

Fix a neighborhood $B_\delta(\sigma)$ and cross sections $\Sigma^{u(s)}_\delta$ contained in $B_\delta(\sigma)$ as in Definition 4.6 (4.9). Since $p_n \to \sigma$, we have that for each n sufficiently large, there is a first $t_n > 0$ such that $q_n = Y^n_{t_n}(p_n) \in \Sigma^u_\delta$.

Note that there is $q \in \hat{W}^u_{\mathrm{loc}}(\sigma) \cap \Lambda_X(U)$ such that $q_n \to q$.

CLAIM 4.11. *Formula* (34) *implies that $\alpha(E^{s,Y_n}_{q_n}, Y^n(q_n)) \to 0$ as $n \to \infty$.*

Proof. We will prove first that, as a consequence of (34), the stable direction at q_n is close to the central unstable direction at q_n. Using some properties of the splitting given by the Poincaré flow, we will show that if this happens then the stable direction at q_n is necessarily close to the flow direction at q_n, proving the claim. For this we proceed as follows.

Since (34) holds and $p_n \to \sigma$, by Lemma 4.8 we get that

$$\alpha(E^{s,Y^n}_{q_n}, N^{u,Y^n}_{q_n} \oplus [Y^n(q_n)]) \to 0 \quad \text{as} \quad n \to \infty\,. \tag{35}$$

To complete the proof of the claim we have to prove that (35) holds because $E^{s,Y^n}_{q_n}$ is leaning in the direction of the flow. Indeed, since $q_n \to q \in \Lambda^*_X(U)$, Theorem 2.15 implies that

$$\alpha(N^{s,Y^n}_{q_n}, N^{u,Y^n}_{q_n}) > 0.9 \;\; \alpha(N^{s,X}_q, N^{u,X}_q), \qquad \text{for } n \text{ big enough}\,.$$

As $N^{s(u)}_{q_n}$ is orthogonal to $Y^n(q_n)$, we obtain

$$\alpha(N^{s,Y^n}_{q_n} \oplus [Y^n(q_n)], N^{u,Y^n}_{q_n} \oplus [Y^n(q_n)]) = \alpha(N^{s,Y^n}_{q_n}, N^{u,Y^n}_{q_n})\,.$$

Thus, $\alpha(N^{s,Y^n}_{q_n} \oplus [Y^n(q_n)], N^{u,Y^n}_{q_n} \oplus [Y^n(q_n)])$ is uniformly bounded away from zero. Since $E^{s,Y^n}_{q_n} \subset N^{s,Y^n}_{q_n} \oplus [Y^n(q_n)]$, and $Y^n(q_n) = N^{s,Y^n}_{q_n} \oplus [Y^n(q_n)] \cap N^{u,Y^n}_{q_n} \oplus [Y^n(q_n)]$, by (35) we obtain

$$\alpha(E^{s,Y^n}_{q_n}, Y^n(q_n)) \to 0 \quad \text{as} \quad n \to \infty, \tag{36}$$

proving Claim 4.11. □

Now we will apply Lemma 4.10. For this, let δ be as above, $k = c$ where c is as in Theorem 3.6(b), and $\varepsilon < c/2$. Let δ' be given by Lemma 4.10.

Fix $\delta^* < \max\{\delta, \delta'\}$ and consider $U_{\delta^*} = \mathrm{Cl}(U \setminus C_{\delta^*}(\mathrm{Sing}_X(\Lambda_X(U)))$. Since the singularities of $Y \in \mathcal{V}$ are the continuation of the ones of X, we can assume that $U_{\delta^*} \cap \mathrm{Sing}_Y(\Lambda_Y(U)) = \emptyset$ for all $Y \in \mathcal{V}$.

Since σ is an accumulation point of $\mathcal{O}_{Y^n}(q_n)$ we have that for n large enough, there is a first positive time s_n such that

$$\tilde{q}_n = Y^n_{s_n}(q_n) \in C_{\delta^*}(\mathrm{Sing}_{Y^n}(\Lambda_{Y^n}(U))).$$

We can take s_n in such a way that $\tilde{q}_n \in \Sigma^s_{\delta,\delta'}$, where $\Sigma^s_{\delta,\delta'}$ is as in (33).

We assume, without loss of generality, that every $\tilde{q}_n$ belongs to the same connected component of $\Sigma^s_{\delta,\delta'}$ associated to the same singularity of Y^n. Note that from the choice of δ^*, we get that $Y^n_s(q_n) \in U_{\delta^*}$ for all $0 \le s \le s_n$.

Next we prove that (36) holds when we replace q_n by $\tilde{q}_n$. That is, we shall prove

$$\alpha(E^{s,Y^n}_{\tilde{q}_n}, Y^n(\tilde{q}_n)) \to 0 \quad \text{as} \quad n \to \infty. \tag{37}$$

Indeed, if there exists $S > 0$ such that for infinitely many n we have $s_n < S$ then (36) immediately implies (37).

Otherwise, let q be such that $Y^n_{s_n/2}(q_n) \to q$, with $s_n \to \infty$. Then $\mathrm{Cl}(\mathcal{O}_X(q)) \subset U_{\delta^*}$ which implies $\omega(\mathcal{O}_X(q)) \subset \Omega_X(U_{\delta^*})$. By Corollary 4.4, $\Omega_X(U_{\delta^*})$ is hyperbolic. Let V be a neighborhood of $\Omega_X(U_{\delta^*})$ given by Lemma 4.5. The next claim establishes that the time spent by the Y^n orbit segment $\{Y^n_t(q_n), 0 \le t \le s_n\}$ outside V is uniformly bounded.

Claim 4.12. *There is $S'' > 0$ such that for all n, there are $0 \le s^1_n < s^2_n \le s_n$, $s^1_n < S''$, $s_n - s^2_n < S''$, such that $Y^n_s(q_n) \in V$ for all $s^1_n \le s \le s^2_n$.*

Proof. It is enough to prove that if there is S' such that given q_n and $0 < s'_n < s_n$ with the property that $Y^n_{s'_n}(q_n) \notin V$, then either $s'_n < S'$ or $s_n - s'_n < S'$. If this were not the case, there would exist s'_n such that $Y^n_{s'_n}(q_n) \notin V$ and $s_n - s'_n \to \infty$, $s'_n \to \infty$. Then we could consider a sequence $Y^n_{s'_n}(q_n) \to q'$ with $q' \notin V$, and this would imply that $\mathrm{Cl}(\mathcal{O}_X(q')) \subset U_{\delta^*}$, and that $\omega(\mathcal{O}_X(q')) \subset \Omega_X(U_{\delta^*})$ which implies $\omega(\mathcal{O}_X(q')) \subset V$. Hence, for large n, we would get $Y^n_{s'_n}(q_n) \in V$, contradicting the assumption. This finishes the proof of Claim 4.12. □

Returning to the proof of (37), recall that $\alpha(E^{s,Y^n}_{q_n}, Y^n(q_n))$ is arbitrarily small for n large enough. Then, Lemma 4.5 combined with the fact that for n sufficiently large the time spent by the orbit segment $\{Y^n_s(q_n), 0 \le s \le s_n\}$ outside V is finite (Claim 4.12) give (37).

As before, since $\tilde{q}_n \in \Sigma^s_{\delta,\delta'}$, there is a first $r_n > 0$ such that $\tilde{\tilde{q}}_n = Y^n_{r_n}(\tilde{q}_n) \in \Sigma^u_\delta$. Next we prove that we also have $\alpha(E^{s,Y^n}_{\tilde{\tilde{q}}_n}, Y^n(\tilde{\tilde{q}}_n)) \to 0$ as $n \to \infty$.

If there is $S > 0$ such that $0 < r_n < S$ for infinitely many n, taking a subsequence, we get the assertion. Otherwise, taking a subsequence if necessary, we get that $\tilde{q}_n \to \hat{W}^s_{\mathrm{loc}}(\sigma) \cap \Sigma^s_{\delta,\delta'}$ and there exists $\tilde{\tilde{q}} \in \hat{W}^u_{\mathrm{loc}}(\sigma) \cap \Sigma^u_\delta$ such that $\tilde{\tilde{q}}_n \to q$. Observe that there is $d > 0$ such that for any $y \in \hat{W}^s_{\mathrm{loc}}(\sigma) \cap \Sigma^s_{\delta,\delta'}$ we have that $\alpha(X(y), \hat{E}^{\mathrm{ss}}_y) > d$ and so, provided n is large enough we get that

$$\alpha(Y^n(\tilde{q}_n), \hat{E}^{ss,Y^n}_{\tilde{q}_{n*}}) > d\,. \tag{38}$$

Combining (37) and (38) we obtain $\alpha(E^{s,Y^n}_{\tilde{q}_n}, \hat{E}^{ss,Y^n}_{\tilde{q}_{n*}}) > d$ for n large. Then, arguing as in Claim 4.11, replacing q_n by $\tilde{q}_n$, $n \ge 0$, we obtain

$$\lim_{n\to\infty} \alpha(E^{s,Y^n}_{\tilde{\tilde{q}}_n}, Y^n(\tilde{\tilde{q}}_n)) = 0\,. \tag{39}$$

Moreover, since (37) holds, Theorem 3.6(b) implies that

$$\alpha(E^{u,Y^n}_{\tilde{q}_n}, Y^n(\tilde{q}_n)) > c \quad \text{for } n \text{ big enough.} \tag{40}$$

As $E^{u,Y^n}_{\tilde{q}_n} \subset N^{u,Y^n}_{\tilde{q}_n} \oplus [Y^n(\tilde{q}_n)]$ and (40) holds, Lemma 4.10 implies

$$\alpha(DY^n_{r_n}(E^{u,Y^n}_{\tilde{q}_n}), Y^n(\tilde{\tilde{q}}_n)) < \varepsilon < c/2, \tag{41}$$

by the choice of ε.

Now, (39) and (41) combined with the fact that $E^{u,Y^n}_{\tilde{\tilde{q}}_n} = DY^n_{r_n}(E^{u,Y^n}_{\tilde{q}_n})$ give

$$\alpha(E^{u,Y^n}_{\tilde{\tilde{q}}_n}, E^{s,Y^n}_{\tilde{\tilde{q}}_n}) < c/2, \quad \text{for } n \text{ big enough.}$$

This contradicts Theorem 3.6(b), and proves Proposition 4.1(a). □

Proof of Proposition 4.1(b). We will prove that given Y near X and a periodic point p of Y close to σ_Y then $E^{cu,Y}_p$ is close to $\hat{E}^{cu,Y}_{\sigma_Y}$. For this we shall use the following claims.

First, given $\delta > 0$ and $\delta' > 0$, we will consider the cross sections Σ^s_δ and $\Sigma^s_{\delta,\delta'}$ as in Definition 4.9 and (33) respectively.

CLAIM 4.13. *Let $X \in \mathcal{U}_U$, $\sigma \in \mathrm{Sing}_X(\Lambda_X(U))$ and $\delta > 0$. There are a neighborhood $\mathcal{V}$ of X such that given $\gamma > 0$ and $\varepsilon > 0$, there is $r = r(\varepsilon, \gamma) > 0$ such that if $y \in \Sigma^s_\delta$ and $L_y \subset T_yM$ is a plane with $\alpha(L_y, \hat{E}^{\mathrm{ss}}_y) > \gamma$ then $\alpha(DY_{s_y}(y)(L_y), \hat{E}^{cu}_{\sigma_Y}) < \varepsilon$, where s_y is such that $Y_{s_y}(y) \in B_r(\sigma_Y)$ and $Y_s(y) \in B_\delta(\sigma_Y)$ for all $0 \le s \le s_y$.*

The proof of this claim is similar to the one for Claim 4.15, proved in the end of this section, and so we shall not do it here.

Given $y \in \Sigma^s_{\delta,\delta'}$ let y_* be as in Definition 4.7.

CLAIM 4.14. *Let $X \in \mathcal{U}_U$, $\sigma \in \mathrm{Sing}_X(\Lambda_X(U))$ and $\delta > 0$. There are a neighborhood $\mathcal{V}$ of X, $\gamma > 0$ and $\delta' > 0$ such that for all $Y \in \mathcal{V}$ and all $y \in \Lambda_Y(U) \cap \Sigma^s_{\delta,\delta'}$ there is $\alpha(E^{cu,Y}_y, \hat{E}^{ss,Y}_{y_*}) > \gamma$.*

Assuming the claims, let us finish the proof of the proposition.

Observe that for p close to σ_Y there is $s_p > 0$ such that $\tilde{p} = Y_{-s_p}(p) \in \Sigma^s_{\delta,\delta'}$, where δ and δ' are as in Claim 4.14. Let $\tilde{p}_*$ be as in Definition 4.7. By Claim 4.14, $\alpha(E^{cu,Y}_{\tilde{p}}, \hat{E}^{ss,Y}_{\tilde{p}_*}) > \gamma$, and hence, by Claim 4.13 we have that $\alpha(DY_t(\tilde{p})(E^{cu,Y}_{\tilde{p}}), \hat{E}^{cu,Y}_{\sigma_Y})$ is arbitrarily small, provided p is close enough to σ_Y, concluding the proof of Proposition 4.1(b).

Hence, we only need to prove Claim 4.14. First we show the claim for points $q \in \Sigma^s_{\delta,\delta'} \cap \Lambda_X(U) \cap \hat{W}^s_{\mathrm{loc}}(\sigma)$.

In this case, observe that $\alpha(E^{cu,X}_q, \hat{E}^{\mathrm{ss}}_q) \ge \alpha(E^{cu,X}_q, T_qW^s_{\mathrm{loc}}(\sigma))$. By [9, Prop. 2.2], $N^{s,X}_q = T_qW^s_{\mathrm{loc}}(\sigma) \cap N_y$ and since $X(q) \in T_qW^s_{\mathrm{loc}}(\sigma)$ we get that $T_qW^s_{\mathrm{loc}}(\sigma) = N^{s,X}_q \oplus [X(q)]$. Now, we conclude that

$$\alpha(E^{cu,X}_q, T_qW^s_{\mathrm{loc}}(\sigma)) = \alpha(E^{cu,X}_q, N^{s,X}_q \oplus [X(q)]) = \alpha(N^{u,X}_q, N^{s,X}_q)\,. \tag{42}$$

Since $\Sigma^s_{\delta,\delta'}$ is compact and does not contain singularities, by Theorem 2.15, there is $\gamma = \gamma(\delta, \delta')$ such that $\alpha(N^{u,X}_q, N^{s,X}_q) > \gamma$, for any $q \in \Sigma_{\delta,\delta'}$. Replacing this inequality in (42) we conclude the proof of the claim in this case. we have that $\mathrm{dist}(\tilde{p}, \Sigma^s_{\delta,\delta'} \cap \hat{W}^s_{\mathrm{loc}}(\sigma_Y))$ is arbitrarily small. So, using the continuity of the splitting $N^{s,X} \oplus N^{u,X}$ with the flow, Theorem 2.15, we get that the estimate (42) above still holds replacing q by $\tilde{p}$ and X by Y, concluding the proof of Claim 4.14.

All of these facts together give the proof of Proposition 4.1. □

Proof of Proposition 4.2. Assume, by contradiction, that there exists a sequence of periodic points $p_n \notin C_\delta(\mathrm{Sing}(\Lambda_X(U)))$ of flows $Y^n \to X$ such that

$$\alpha(E^{cu,Y^n}_{p_n}, E^{s,Y^n}_{p_n}) \to 0 \quad \text{as} \quad n \to \infty \ . \tag{43}$$

We claim that $\mathrm{Cl}(\cup_n \mathcal{O}_{Y^n}(p_n)) \cap \mathrm{Sing}(\Lambda_X(U)) \neq \emptyset$. Indeed, if this were not the case, we would get that there is δ^* such that $\mathrm{Cl}(\cup_n \mathcal{O}_{Y^n}(p_n)) \subset \Omega_X(U_{\delta^*})$.

By Corollary 4.4, $\Omega_X(U_{\delta^*})$ is hyperbolic, and so there are neighborhoods V and $\mathcal{V}$ of $\Omega_X(U_{\delta^*})$ and Y respectively, and $c > 0$ such that $\alpha(E^{s,Y}_p, E^{cu,Y}_p) > c$ for all $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ such that $\mathcal{O}_Y(p) \subset V$.

Since $Y^n \to X$ as $n \to \infty$, we have that $\mathcal{O}_{Y^n}(p_n) \subset V$ for n sufficiently large. Hence we conclude that $\alpha(E^{s,Y^n}_{p_n}, E^{cu,Y^n}_{p_n}) > c$, leading to a contradiction. Thus $\mathrm{Cl}(\cup_n \mathcal{O}_{Y^n}(p_n)) \cap \mathrm{Sing}(\Lambda_X(U)) \neq \emptyset$ as claimed.

Fix $\delta > 0$ and take cross sections $\Sigma^{s(u)}_\delta$ as in Definitions 4.6 and 4.9.

Since $\mathrm{Cl}(\cup_n \mathcal{O}_{Y^n}(p_n)) \cap \mathrm{Sing}(\Lambda_X(U)) \neq \emptyset$, we get that for each n there is s_n such that $\tilde{p}_n = Y^n_{s_n}(p_n) \in \Sigma^s_\delta(\sigma)$.

Now, we take $k = c$ where c is as in Theorem 3.6(b), $\varepsilon < c/2$ and δ' as in Lemma 4.10.

Fix $\delta^* < \delta, \delta'$ and consider $U_{\delta^*} = \mathrm{Cl}(U \setminus C_{\delta^*}(\mathrm{Sing}(\Lambda_X(U)))$. By Corollary 4.4 $\Omega_X(U_{\delta^*})$ is hyperbolic.

From the choice of δ^*, we get that $Y^n_s(p_n) \in U_{\delta^*}$ for any $0 \le s \le s_n$. We assume, without loss of generality, that every $\tilde{p}_n$ belongs to the same connected component of $\Sigma^s_{\delta,\delta'}$, associated to the same singularity σ, where $\Sigma^s_{\delta,\delta'}$ is as in (33). Reasoning as in Claim 4.11 we prove that (43) implies

$$\alpha(E^{s,Y^n}_{\tilde{p}_n}, Y^n(\tilde{p}_n)) \to 0 \quad \text{as} \quad n \to \infty. \tag{44}$$

Once (44) is settled, the proof is similar to that in the previous proposition, and we leave the details for the reader. □

We close this section presenting the proof of Lemmas 4.3, 4.5, 4.8, and 4.10.

Proof of Lemma 4.3. We will prove, with the help of the Ergodic Closing Lemma and Theorem 3.6(b), that the linear Poincaré flow restricted to Γ is hyperbolic. Applying [9, Prop. 1.1] we obtain the result.

To prove that the splitting $N^s \oplus N^u$ for the linear Poincaré flow P^X_t restricted to Γ is hyperbolic we proceed as follows. First observe that $N^s \oplus N^u$ is defined over Γ everywhere and is continuous. So, all we need is to show that

N^s is forward contractive and N^u is backward expansive by the derivative of P_t^X. To prove that the bundle N^s is uniformly contracting it is enough to prove that

$$\lim_{t\to\infty} \inf \|P_t^X/N_x^s\| = 0,$$

for every $x \in \Gamma$. Suppose, by contradiction, that there is $x \in \Gamma$ such that

$$\lim_{t\to\infty} \inf \|P_t^X/N_x^s\| > 0.$$

Then there is $s_n \to \infty$ as $n \to \infty$, such that

$$\lim_{s_n\to\infty} \frac{1}{s_n} \log \|P_{s_n}^X/N_x^s\| \geq 0. \tag{45}$$

Let $C^0(\Gamma)$ be the set of real continuous functions defined on Γ with the C^0 topology, and define the sequence of continuous operators

$$\begin{aligned} \Psi_n : C^0(\Gamma) &\to \mathbb{R}, \\ \varphi &\longmapsto \frac{1}{s_n}\int_0^{s_n} \varphi(P_s^X(x))ds. \end{aligned}$$

There exists a convergent subsequence of Ψ_n, which we still denote by Ψ_n, converging to a continuous map $\Psi : C^0(\Gamma) \to \mathbb{R}$. Let $\mathcal{M}(\Gamma)$ be the space of measures with support on Γ. By the Theorem of Riez, there exists $\mu \in \mathcal{M}(\Gamma)$ such that

$$\int_\Gamma \varphi d\mu = \lim_{s_n\to\infty} \frac{1}{s_n}\int_0^{s_n} \varphi(P_s^X(x))ds = \Psi(\varphi), \tag{46}$$

for every continuous map φ defined on Γ. It is clear that such μ is invariant by the flow P_t^X.

Define $\varphi_{P^X} : C^0(\Gamma) \longrightarrow \mathbb{R}$ by

$$\varphi_{P^X}(p) = \partial_h(\log \|P_h^X/N_p^s\|)_{h=0} = \lim_{h\to 0} \frac{1}{h} \log \|P_h^X/N_p^s\|.$$

This map is continuous, and so it satisfies (46).

On the other hand, for any $T \in \mathbb{R}$,

$$\begin{aligned} \frac{1}{T}\int_0^T \varphi_{P^X}(P_s^X(p))ds &= \frac{1}{T}\int_0^T \partial_h(\log \|P_h^X/N_{P_s^X(p)}^s\|)_{h=0}ds \\ &= \frac{1}{T}\log \|P_T^X/N_p^s\|. \end{aligned} \tag{47}$$

Combining (45), (46), and (47) we get

$$\int_\Gamma \varphi_{P^X} d\mu \geq 0.$$

By the Ergodic Theorem of Birkhoff,

$$\int_\Gamma \varphi_{P^X} d\mu = \int_\Gamma \lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi_{P^X}(P_s^X(y))dsd\mu(y).$$

Let Σ_X be the set of strongly closed points. Since μ is invariant and $\mathrm{Supp}(\mu) \subset \Gamma$, Theorem 3.4 implies $\mu(\Gamma \cap (\mathrm{Sing}(X) \cup \Sigma_X)) = 1$. As there are no singularities in Γ we conclude that $\mu(\Gamma \cap \mathrm{Sing}(X)) = 1$.

By the ergodic decomposition for invariant measures, we can suppose that μ is ergodic. Then there exists $y \in \Gamma \cap \Sigma_X$ such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi_{P^X}(P_s^X(y))ds \geq 0. \tag{48}$$

Then there are $\delta_n \to 0$ as $n \to \infty$, $Y^n \in \mathcal{U}_U$, $p_n \in \mathrm{Per}_{Y^n}(\Lambda_{Y^n}(U))$ with period t_n such that

$$\|Y^n - X\| < \delta_n, \quad \text{and} \quad \mathrm{dist}(Y_s^n(p_n), X_s(y)) < \delta_n,\ 0 \leq s \leq t_n,$$

where Y_s^n is the flow induced by Y^n. Observe that $t_n \to \infty$ as $n \to \infty$. Otherwise, $y \in \mathrm{Per}_X(\Gamma)$ and if t_y is the period of y, combining the continuity of the splitting $N^s \oplus N^u$ with the flow, (47) and (48) we get that $P_t^{X_{t_y}}/N_y^{s,X}$ expands. Then, taking $\gamma < 0$ arbitrarily small, we obtain that $\|P_t^{X_{t_y}}/N_y^{s,X}\| > e^{\gamma t_y}$.

Now consider the eigenspace $E_y^{cs} = N_y^s \oplus [X(y)]$ of $DX_{t_y}(y)$.

Let $n_y^s \in N_y^s$, $\|n_y^s\| = 1$. Then $DX_{t_y}(y)(n_y^s) = aX(y) + bn_y^s$. Since n_y^s is orthogonal to $X(y)$ and $DX_{t_y}(X(y)) = X(y)$ we conclude that $b = \det(DX_{t_y}(y))$.

On the other hand, if $e_y^s \in E_y^s$, $\|e_y^s\| = 1$, by Theorem 3.6(a) we have $DX_{t_y}(y)(e_y^s) = \tilde{\lambda} e_y^s$ with $|\tilde{\lambda}| < \lambda^{t_y}$, $\lambda < 1$. Then,

$$e^{\gamma t_y} < \|P_{t_y}^X(n_y^s)\| = \|P_N DX_{t_y}(n_y^s)\| = |b| < K\ \lambda^{t_y},$$

for some positive constant K, leading to a contradiction. Thus $t_n \to \infty$ as $n \to \infty$.

Let $\gamma < 0$ be arbitrarily small. By (48) again, there is T_γ such that for $t \geq T_\gamma$

$$\frac{1}{t} \int_0^t \varphi_{P^X}(P_s^X(y))ds \geq \gamma.$$

Since $t_n \to \infty$ as $n \to \infty$, we can assume that $t_n > T_\gamma$ for n sufficiently large. Using the continuity of the splitting $N^s \oplus N^u$ with the flow, we can take Y^n and p_n such that

$$\frac{1}{t_n} \int_0^{t_n} \varphi_{P^{Y^n}}(P_s^{Y^n}(p_n))ds \geq \gamma.$$

This implies that $\|P_{t_n}^{Y^n}/N_{p_n}^s\| > e^{\gamma t_n}$.

Considering the eigenspace $E_{p_n}^{cs}$ of $DY_{t_n}^n(p_n)$, and reasoning as above, replacing X by Y^n and y by p_n, we obtain

$$e^{\gamma t_n} < K\ \lambda^{t_n},$$

with $\lambda < 1$, leading again to a contradiction.

The proof that N^u_X is backward expansive by the derivative of P^X_t is similar and we leave the details to the reader.

The proof of Lemma 4.3 is finished. □

Proof of Lemma 4.5. Since Γ is hyperbolic, there are $0 < \lambda_\Gamma < 1$ and $c > 0$ such that $N^{s,X}_\Gamma = E^{s,X}_\Gamma \oplus [X]$ with $\|DX_t/E^{s,X}\| < c\,\lambda^t_\Gamma$, and $c^{-1} < \|X/\Gamma\| < c$. Changing uniformly the metric in a neighborhood of Γ, we can assume that for all $x \in \Gamma$, $E^{s,X}_x$ is orthogonal to $[X(x)]$, and $\|X(x)\| = 1$. In other words, in this new metric, $E^{s,X}_\Gamma$ coincides with the stable bundle $N^{s,X}_\Gamma$ for the linear Poincaré flow restricted to Γ.

For each $x \in \Gamma$, let $n^{s,X}_x \in N^{s,X}_x$ with $\|n^{s,X}_x\| = 1$, and consider the orthogonal basis $\mathcal{B}_x = \{X(x), n^{s,X}_x\}$ of $[X(x)] \oplus N^{s,X}_x$. In this basis, the matrix of $DX_t(x)$ restricted to $[X(x)] \oplus N^{s,X}_x$ is given by

$$DX_t(x)/[X(x)] \oplus N^{s,X}_x = \begin{bmatrix} 1 & 0 \\ 0 & n^{s,X}_{x,t} \end{bmatrix},$$

where $\|n^{s,X}_{x,t}\| < c\,{\lambda_\Gamma}^t$.

Fix t_0 such that $\|n^{s,X}_{x,t_0}\| < 1/2$ for all $x \in \Gamma$. Moreover, there is $c' > 0$ such that $\|n^{s,X}_{x,t_0}\| > c'$ for all $x \in \Gamma$.

Taking a neighborhood V of Γ and a neighborhood $\mathcal{V} \subset \mathcal{U}_U$ of X, both sufficiently small, and a continuous change of metric with the flow, we get that for all $Y \in \mathcal{V}$, and for all $y \in \Lambda_Y(U)$, $\|Y(y)\| = 1$. Thus, the matrix of $DY_{t_0}(y)$ restricted to $[Y(y)] \oplus N^{s,Y}_y$ with respect to the basis $\mathcal{B}_y = \{Y(y), n^{s,Y}_y\}$ is given by

$$DY_{t_0}(y)/[Y(y)] \oplus N^{s,Y}_y = \begin{bmatrix} 1 & \delta^Y_y \\ 0 & n^{s,Y}_{y,t_0} \end{bmatrix},$$

where $\delta^Y_y < \delta_0$, δ_0 small for Y sufficiently close to X, and $\|n^{s,Y}_{y,t_0}\| < 1/2$. Thus,

$$DY_{n\,t_0}(y)/[Y(y)] \oplus N^{s,Y}_y = \begin{bmatrix} 1 & \delta^Y_{y,n} \\ 0 & n^{s,Y}_{y,n\,t_0} \end{bmatrix},$$

with $\delta^Y_{y,n} << 2\delta_0$.

Let $\varepsilon > 0$, and n_0 be such that $|(1/2)^n| < \varepsilon$ for all $n \geq n_0$.

Given $v \in [Y(y)] \oplus N^{s,Y}_y$, in the basis B_y, we can write $v = (1, \gamma'_0)$. Then, for $m \in \mathbb{N}$ we get

$$\text{slope}(DY_{n_0\,m}(y)(v), (1,0)) \leq \frac{n^{s,Y}_{y,n_0\,m}\gamma'_0}{1 - \delta^Y_{y,n_0} n^{s,Y}_{y,n_0\,m}} < \frac{(1/2)^{n_0\,m}}{1 - 2\delta_0}.$$

For $t > n_0$, we write $t = m\,n_0 + s$ with $0 \leq s \leq n_0$. Then, $\alpha(DY_t(y)(v), Y(Y_t(y)) < K\,\varepsilon$, for some small positive constant K, proving Lemma 4.5. □

Proof of Lemma 4.8. Let $y \in B_\delta(\sigma)$ and s_y be such that $Y_{s_y}(y) \in \Sigma^u_\delta$, and $Y_s(y) \in B_\delta(\sigma)$ for all $0 \le s \le s_y$. Observe that if y is close to $W^s_{\text{loc}}(\sigma)$ then $Y_{s_Y}(y)$ is close to $W^u_{\text{loc}}(\sigma)$. Regarding this property, it is easy to see that the proof of Lemma 4.8 follows immediately from the next two claims. Before we state them, let us introduce some notation and definitions.

Given $\sigma \in \text{Sing}_X(\Lambda_X(U))$ consider the local central unstable manifold $W^{cu}_{\text{loc}}(\sigma_Y)$ at σ_Y for any Y close to X. Observe that $W^{cu}_{\text{loc}}(\sigma_Y)$ is not uniquely defined, but it has the property that $W^u_{\text{loc}}(\sigma_Y) \subset W^{cu}_{\text{loc}}(\sigma_Y)$, and for all $y \in W^u_{\text{loc}}(\sigma_Y)$, the tangent bundle $T_y W^{cu}_{\text{loc}}(\sigma_Y)$ does not depend on the choice of the central unstable manifold. So, for all $y \in W^u_{\text{loc}}(\sigma_Y)$ we define $\hat{E}^{cu,Y}_y = T_y W^{cu}_{\text{loc}}(\sigma_Y)$. In other words, for points in $W^u_{\text{loc}}(\sigma_Y)$ we get a central bundle induced by the central unstable manifold of the singularity.

The first claim gives angle estimates through the passage near a singularity. It says that for a point y in $B_\delta(\sigma)$ and vectors v with angle bounded away from zero with the strong stable bundle at the singularity, after the passage through a neighborhood of the singularity, $DY_t(v)$ lands in the direction of the central unstable bundle induced by the central unstable manifold of the singularity. Given $y \in B_\delta(\sigma_Y)$ such that $Y_{s_y}(y) \in \Sigma^u_\delta$ and $Y_s(y) \in B_\delta(\sigma_Y)$ for all $0 \le s \le s_y$, let y_* be as in Definition 4.7 and set y_{**} for the point in $\hat{W}^u_{\text{loc}}(\sigma_Y) \cap \Sigma^u_\delta$ such that $\text{dist}(Y_{s_y}(y), \hat{W}^u_{\text{loc}}(\sigma_Y) \cap \Sigma^u_\delta) = \text{dist}(Y_{s_y}(y), y_{**})$.

CLAIM 4.15. *Let $X \in \mathcal{U}_U$, $\sigma \in \text{Sing}(\Lambda_X(U))$, and $\delta > 0$. There is a neighborhood $\mathcal{V}$ of X such that given $\gamma > 0$ and $\varepsilon > 0$ there is $r = r(\varepsilon, \gamma)$ such that for $Y \in \mathcal{V}$, $y \in B_\delta(\sigma)$ with $\text{dist}(y, \hat{W}^s_{\text{loc}}(\sigma_Y)) < r$ and $v \in T_y M$ with $\alpha(v, \hat{E}^{ss}_{y_*}) > \gamma$. Then*

$$\alpha(DY_{s_y}(y)(v), \hat{E}^{cu,Y}_{y_{**}}) < \varepsilon$$

where s_y is the first positive time such that $Y_{s_y}(y) \in \Sigma^u_\delta$.

The next claim relates the splitting for the linear Poincaré flow and the local splitting at a singularity for points in the unstable manifold of the singularity. More precisely, the lemma shows that the central unstable bundle for the linear Poincaré flow coincides with the one given by the central unstable manifold of the singularity.

CLAIM 4.16. *Let $X \in \mathcal{U}_U$, $\sigma \in \text{Sing}(\Lambda_X(U))$ and $y \in \hat{W}^u_{\text{loc}}(\sigma) \cap \Lambda_X(U)$. Then there is a neighborhood $\mathcal{V}$ of X such that for any ε there exists $r = r(\varepsilon)$ such that for any $Y \in \mathcal{V}$ if $p \in \Lambda_Y(U)$ and $\text{dist}(p, y) < r$ then $\alpha(E^{cu,Y}_p, \hat{E}^{cu,Y}_y) < \varepsilon$.*

We will prove the first claim introducing linearizable coordinates in a normal neighborhood V of the singularity. For this, to simplify notations, we shall assume that there is a neighborhood V of σ where all Y sufficiently near X are linearizable. We fix $\delta > 0$ small such that $B_\delta(\sigma) \subset V$. We shall also assume that $\sigma_Y = \sigma$ and the eigenvalues of $DY(\sigma_Y)$ are the same as the ones of $DX(\sigma)$. Let $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$ be the eigenvalues of $DX(\sigma)$. Then, in local coordinates $\bar{x}, \bar{y}, \bar{z}$, we have that Y/V can be written as

$$Y(\bar{x}, \bar{w}, \bar{z}) = \begin{cases} \dot{\bar{x}} = \lambda_1 \bar{x} \\ \dot{\bar{y}} = \lambda_2 \bar{y} \\ \dot{\bar{z}} = \lambda_3 \bar{z}\,. \end{cases}$$

Note that in this case $W^s_{\text{loc}}(\sigma) = [(0,1,0),(0,0,1)] \cap V$, $W^u_{\text{loc}}(\sigma) = [(1,0,0)] \cap V$, $W^{cu}_{\text{loc}}(\sigma) = [(1,0,0),(0,0,1)] \cap V$, $\hat{E}^{ss,Y}_y = [(0,1,0)] \cap V$ for any $y \in W^s_{\text{loc}}(\sigma)$, $\hat{E}^{cu,Y}_y = [(1,0,0),(0,0,1)]$ for any $y \in W^u_{\text{loc}}(\sigma)$, and $\Sigma^u_\delta \cap W^u_{\text{loc}}(\sigma) = \{(\pm 1,0,0)\}$. For $y \in V$ and for $v = (v_1, v_2, v_3)$, if $t > 0$ is such that $Y_s(y) \in V$ for all $0 \le s \le t$ then $DY_t(y)(v) = (e^{\lambda_1 t} v_1, e^{\lambda_2 t} v_2, e^{\lambda_3} v_3)$.

Given two vectors v and w we set slope(v,w) for the slope between v and w.

Proof of Claim 4.15. Let $r > 0$ and $y \in B_\delta(\sigma)$ be such that $\text{dist}(y, \hat{W}^s_{\text{loc}}(\sigma)) < r$, $v = (v_1, v_2, v_3) \in T_y M$ and $t > 0$ such that $Y_s(y) \in V$ for all $0 \le s \le t$. Then,

$$\text{slope}(DY_t(y)(v), \hat{E}^{cu,X}_\sigma) = \frac{|e^{\lambda_2 t} v_2|}{\sqrt{(e^{\lambda_1 t} v_1)^2 + (e^{\lambda_3 t} v_3)^2}}.$$

On the other hand, assuming that $\alpha(\hat{E}^{ss,X}_\sigma, v) = \alpha((0,1,0), v) > \gamma$ we get that there is $0 < \hat{\gamma} < 1$ such that $0 \le |v_2| < \hat{\gamma}$. Hence $v_1^2 + v_3^2 > 1 - \hat{\gamma}^2$. This implies that either $v_1 > \sqrt{(1-\hat{\gamma}^2)/2}$ or $v_3 > \sqrt{(1-\hat{\gamma}^2)/2}$. Thus,

$$\text{slope}(DY_t(y)(v), \hat{E}^{cu,X}_\sigma) \le \frac{|e^{\lambda_2 t} v_2|}{|e^{\lambda_i t} v_i|} \le \frac{\hat{\gamma}}{\sqrt{(1-\hat{\gamma}^2)/2)}} e^{(\lambda_2 - \lambda_i)t}\,,$$

where i is chosen in such a way that v_i satisfies $v_i^2 > \sqrt{(1-\gamma^2)/2}$. As both $\lambda_2 - \lambda_3$ and $\lambda_2 - \lambda_1$ are strictly smaller than 0, there is $T = T(\varepsilon, \gamma) > 0$ such that if $t > T$ then the right side member of the inequality above is smaller than ε. Now, with r sufficiently small, for all $y \in (B_\delta(\sigma) \setminus \hat{W}^s_{\text{loc}}(\sigma))$ if $Y_t(y) \in \Sigma^u_\delta$ then $t > T$. These last two facts combined complete the proof. $\square$

Proof of Claim 4.16. Given Y, p, and y as in the statement, recall that $E^{cu,Y}_p = N^{u,Y}_p \oplus [Y(p)]$. By the continuity of the normal bundle for the linear Poincaré flow far from singularities, we get that given $\varepsilon > 0$ there are a neighborhood $\mathcal{V}$ of X and $r > 0$ such that if $Y \in \mathcal{V}$ and $\text{dist}(p, y) < r$ then

$$N^{u,Y}_p \oplus [Y(p)] \text{ is close to } N^{u,X}_y \oplus [X(y)]\,.$$

Now, to prove the claim it is enough to prove that the central unstable bundle for the linear Poincaré flow coincides with the one given by the central unstable manifold of the singularity; i.e.,

$$N^{u,X}_y \oplus [X(y)] = \hat{E}^{cu,X}_y.$$

On the other hand, since $X(y) \in \hat{E}^{cu}_y$ and $N^{u,}$ is orthogonal to $X(y)$, to obtain the equality above, we only need to show that $N^{u,X}_y \subset \hat{E}^{cu}_y$. For this we proceed as follows. Take $y_n \to y$ such that $y_n \in \Lambda_X(U)$ for all n. Again, by the continuity of the splitting for the linear Poincaré flow we get that $N^{u,X}_{y_n} \to N^{u,X}_y$. Thus, to conclude the result it suffices to prove that

$$\alpha(N^{u,X}_{y_n}, \hat{E}^{cu,X}_y) \to 0. \tag{49}$$

Since $y_n \to y \in \hat{W}^u_{\mathrm{loc}}(\sigma)$, we can assume first that $y_n \in B_\delta(\sigma)$ for all n. Second, for each n there is $t_n > 0$ such that $\hat{y}_n = X_{-t_n}(y_n) \in \Sigma^s_\delta$, where Σ^s_δ is a cross section as in Definition 4.9. Then, there is $\hat{y} \in \hat{W}^s_{\mathrm{loc}}(\sigma)$ such that $\hat{y}_n \to \hat{y}$.

For each n let $\hat{y}_{n_*} \in \hat{W}^s_{\mathrm{loc}}(\sigma)$ be as in Definition 4.7. We assert that there is some positive constant γ such that

$$\alpha(N^{u,X}_{\hat{y}_n}, \hat{E}^{ss,X}_{\hat{y}_{n_*}}) > \gamma. \tag{50}$$

Assuming (50), we apply Claim 4.15 to obtain (49) and complete the proof of Claim 4.16.

Thus, we are left to prove (50). For this, observe that $N^{u,X}_{\hat{y}} \subset N_{\hat{y}}$, and so, (50) follows from Claim 4.14.

All of these facts together prove Lemma 4.8. □

Proof of Lemma 4.10. For the proof of this lemma we shall use local linearisable coordinates in a neighborhood of σ as in the proof of Lemma 4.8.

Let $\delta > 0$ be small such that $B_\delta(\sigma) \subset V$, and consider $\Sigma^{s(u)}_\delta$ as in Definitions 4.6 and 4.9 respectively . Let $\delta' > 0$ and consider $\Sigma^s_{\delta,\delta'}$ as in (33). Let $p \in \Sigma_{\delta,\delta'} \cap \Lambda_Y(U)$ and $v \in N^{u,Y}_p \oplus Y(p)$ with $\alpha(v, Y(p)) > k$, where $k > 0$. Write $v = a(1,0,0) + b(0,1,0) + c(0,0,1)$ with $a^2 + b^2 + c^2 = 1$.

Claim 4.17. *There are $R > 0$ and δ' such that if p and v are as above then $|a| > R$.*

Proof. By the continuity of the flow direction and the normal bundle splitting far from singularities, it suffices to verify the claim for $p \in W^s_{\mathrm{loc}}(\sigma) \setminus \{\sigma\}$. In this case $E^{cs,Y}_p = \Pi_0$, where $\Pi_0 = [(0,1,0),(0,0,1)]$. Thus, all we need to prove is that $\alpha(v, E^{cs,Y}_p) > \kappa$ for some $\kappa > 0$. For this, observe that since $\mathrm{dist}(p,\sigma) > \delta$, by Theorem 2.15, there is $k' = k'(\delta)$ such that $\alpha(N^{s,Y}_p, N^{u,Y}_p) > k'$. As $\alpha(E^{cu,Y}_p, E^{cs,Y}_p) = \alpha(N^{s,Y}_p, N^{u,Y}_p)$, we conclude that

$$\alpha(E^{cu,Y}_p, E^{cs,Y}_p) > k' \, . \tag{51}$$

On the other hand, $v \in [Y(p)] \oplus N^{u,Y}_p = E^{cu,Y}_p$ and

$$\alpha(v, Y(p)) = \alpha(v, E^{cs,Y}_p \cap E^{cu,Y}_p) > k$$

by hypothesis. This fact combined with (51) give the proof of the claim. □

Returning to the proof of Lemma 4.10, let t_p be such that $Y_{t_p}(p) \in \Sigma^u_\delta$. Next we will prove that for δ' small we get that

(1) $\alpha(Y(Y_{t_p}(p)), (1,0,0))$ is small and,

(2) $\alpha(DY_{t_p}(p)(v), (1,0,0))$ is small.

Observe that if $\delta' \to 0$ then $t_p \to \infty$ and $Y_{t_p}(p)$ converges to a point in $\hat{W}^u_{\mathrm{loc}}(\sigma)$, where the flow direction is $(1,0,0)$. Thus, the continuity of the flow direction implies (1).

To prove (2), recall that $DY_{t_p}(p)(v) = (ae^{\lambda_1 t_p}, be^{\lambda_2 t_p}, ce^{\lambda_3 t_p})$. So, by Claim 4.17,

$$\left|\frac{be^{\lambda_2 t_p}}{ae^{\lambda_1 t_p}}\right| < e^{(\lambda_2-\lambda_1)t_p}\,|b|\,R^{-1}\,.$$

Similarly,

$$\left|\frac{ce^{\lambda_3 t_p}}{ae^{\lambda_1 t_p}}\right| < e^{(\lambda_3-\lambda_1)t_p}\,|c|\,R^{-1}\,.$$

Since $t_p \to \infty$ as $\delta' \to 0$, both $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ are negative numbers, and $R > 0$, we obtain that the right side member of both inequalities above go to 0 as $\delta' \to 0$, which concludes the proof of Lemma 4.10. □

5. Proof of the results in Section 1.4

In this section we present the proof of Propositions 1.8, 1.9 and 1.10.

Proof of Proposition 1.8. The proof of Corollary 1.8 relies on the fact that the intersection of the dominated splitting $E^s \oplus E^{cu}$ with the normal bundle N_Γ over Γ induces a hyperbolic splitting for the linear Poincaré flow defined over Γ. Thus, by [9, Prop. 1.1] we conclude the proof. To see this we proceed as follows.

From the fact that Γ does not contain singularities, there exists $K = K(\Gamma)$ such that $1/K < \|X(x)\| < K$ for every $x \in \Gamma$. Consider the following splitting on the normal bundle N_Γ: for $x \in \Gamma$, set

$$N^u_x = E^{cu}_x \cap N_x \quad \text{and} \quad N^s_x = E^{cs}_x \cap N_x,$$

where $E^{cs}_x = [X(x)] \oplus E^s_x$.

Next we show that this splitting is hyperbolic for the linear Poincaré flow P_t restricted to Γ. For this, note that for any $t \in \mathbb{R}$, and any $n^u_x \in N^u_x$ with $\|n^u_x\| = 1$,

$$|\det(DX_t/E^{cu}_x)| = \sin(\alpha(DX_t(x)(n^u_x), X(X_t(x))))\ \|DX_t(x)(n^u_x)\|\ \frac{\|X(X_t(x))\|}{\|X(x)\|}.$$

The second term of the equality above is equal to

$$\|P_{N_{X_t(x)}}(DX_t(x)(n_x^u))\| \,\frac{\|X(X_t(x))\|}{\|X(x)\|},$$

where $P_{N_{X_t(x)}}$ denotes the orthogonal projection onto $N_{X_t(x)}$. Thus,

$$|\det(DX_t/E_x^{cu})| = \|P_{N_{X_t(x)}}(DX_t(x)(n_x^u))\| \,\frac{\|X(X_t(x))\|}{\|X(x)\|}. \tag{52}$$

Since the central direction is (c,λ)-volume-expanding, $|\det(DX_t/E_x^{cu})| > c\, e^{t\lambda}$. Combining this last fact with (52) we get that

$$\|P_{N_{X_t(x)}}(DX_t(x)(n_x^u))\| > \frac{c}{K}\, e^{t\lambda},$$

for any $t \geq 0$, proving that N^u is uniformly expanded by P_t.
To see that N^s is uniformly contracted by the linear Poincaré flow, first note that since $E^s \oplus E^{cu}$ is partially hyperbolic along Γ, there is $a_0 > 0$ such that $\alpha(E_x^s, X(x)) \geq a_0$ for every $x \in \Gamma$. Then, there is a_0' such that for any $x \in \Gamma$ and $v \in N_x^s$ with $\|v\| = 1$, there is $w \in E_x^s$ with $\|w\| = 1$ such that $v = aw + b\,\frac{X(x)}{\|X(x)\|}$ with $|a| < a_0$. Hence

$$\begin{aligned}\|P_{N_{X_t(x)}}(DX_t(x)(v))\| &= \|P_{N_{X_t(x)}}(DX_t(x)(aw + b\frac{X(x)}{\|X(x)\|}))\| \\ &= \|P_{N_{X_t(x)}}(DX_t(x)(aw))\| \leq \|DX_t(x)(aw)\| \leq a_0'\, c\, e^{t\,\lambda},\end{aligned}$$

for $0 < \lambda < 1$ (recall that E^s is the contractive direction). Thus N^s is uniformly contracted by P_t.

All of these facts together prove Proposition 1.8. □

Proof of Proposition 1.9. Let Λ be as in the statement of Corollary 1.9. Given $x \in \Lambda$, if x is a singularity then the result follows from the fact that x is Lorenz-like for X. So, let us assume $X(x) \neq 0$ and take $v \in E_x^{cu}$, $\|v\| = 1$, orthogonal to $X(x)$. We have

$$|\det(DX_t(x))| \leq \|DX_t(x)v\| \,\frac{\|DX_t(x)X(x)\|}{\|X(x)\|}. \tag{53}$$

Since $v \in E_x^{cu}$, $|\det(DX_t(x)| \geq c\, e^{\lambda t}$ with $\lambda > 1$. Combining this fact with (53) one obtains that $\gamma(x,v) \geq \lambda > 0$. □

Proof of Proposition 1.10. Let Λ be a robust transitive set with singularities of $X \in \mathcal{X}^1(M)$. By Theorems A and C we can assume that Λ is a partially hyperbolic attractor for X. Residually, Λ has a hyperbolic period orbit p.

As Λ is an attractor, the unstable manifold $W^u(p)$ of a periodic point p is contained in Λ. In particular, the closure $\mathrm{Cl}(W^u(p))$ of $W^u(p)$ is contained in Λ. We shall prove that Λ is contained in $\mathrm{Cl}(W^u(p))$. Using the fact that

Λ is transitive, we can take $q \in \Lambda$ such that $\Lambda = \omega(q)$. Let $\mathcal{V}$ be a small neighborhood of p. As the orbit of q is dense in Λ, we can assume that $q \in \mathcal{V}$. On the other hand, since Λ is partially hyperbolic, projecting q into $W^u(p)$ through the stable manifold of q, we can actually assume that q is contained in $W^u(p)$. Indeed, being in the same stable manifold, q and its projection have the same ω-limit sets. Thus, since $W^u(p)$ is invariant by the flow of X, $\omega(q) \subset \mathrm{Cl}(W^u(p))$, and the result follows. $\square$

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