Annals of Mathematics, **160** (2004), 359–373

Grothendieck's problems concerning profinite completions and representations of groups

By Martin R. Bridson and Fritz J. Grunewald

Abstract

In 1970 Alexander Grothendieck [6] posed the following problem: let Γ_1 and Γ_2 be finitely presented, residually finite groups, and let $u : \Gamma_1 \to \Gamma_2$ be a homomorphism such that the induced map of profinite completions $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ is an isomorphism; does it follow that u is an isomorphism?

In this paper we settle this problem by exhibiting pairs of groups $u : P \hookrightarrow \Gamma$ such that Γ is a direct product of two residually finite, hyperbolic groups, P is a finitely presented subgroup of infinite index, P is not abstractly isomorphic to Γ, but $\hat{u} : \hat{P} \to \hat{\Gamma}$ is an isomorphism.

The same construction allows us to settle a second problem of Grothendieck by exhibiting finitely presented, residually finite groups P that have infinite index in their Tannaka duality groups $\mathrm{cl}_A(P)$ for every commutative ring $A \neq 0$.

1. Introduction

The profinite completion of a group Γ is the inverse limit of the directed system of finite quotients of Γ; it is denoted by $\hat{\Gamma}$. If Γ is residually finite then the natural map $\Gamma \to \hat{\Gamma}$ is injective. In [6] Grothendieck discovered a remarkably close connection between the representation theory of a finitely generated group and its profinite completion: if $A \neq 0$ is a commutative ring and $u : \Gamma_1 \to \Gamma_2$ is a homomorphism of finitely generated groups, then $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ is an isomorphism if and only if the restriction functor $u_A^* : \mathrm{Rep}_A(\Gamma_2) \to \mathrm{Rep}_A(\Gamma_1)$ is an equivalence of categories, where $\mathrm{Rep}_A(\Gamma)$ is the category of finitely presented A-modules with a Γ-action.

Grothendieck investigated under what circumstances $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ being an isomorphism implies that u is an isomorphism of the original groups. This led him to pose the celebrated problem:

Grothendieck's First Problem. *Let Γ_1 and Γ_2 be finitely presented, residually finite groups and let $u : \Gamma_1 \to \Gamma_2$ be a homomorphism such that*

$\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ *is an isomorphism of profinite groups. Does it follow that* u *is an isomorphism from* Γ_1 *onto* Γ_2?

A negative solution to the corresponding problem for finitely generated groups was given by Platonov and Tavgen [11] (also [12]). The methods used in [11] subsequently inspired Bass and Lubotzky's construction of finitely generated linear groups that are super-rigid but are not of arithmetic type [1]. In the course of their investigations, Bass and Lubotzky discovered a host of other finitely generated, residually finite groups such that $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ is an isomorphism but $u : \Gamma_1 \to \Gamma_2$ is not. All of these examples are based on a fibre product construction and it seems that none are finitely presentable. Indeed, as the authors of [1] note, "a result of Grunewald ([7, Prop. B]) suggests that [such fibre products are] rarely finitely presented."

In [13] L. Pyber constructed continuously many pairs of 4-generator groups $u : \Gamma_1 \to \Gamma_2$ such that $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ is an isomorphism but $\Gamma_1 \not\cong \Gamma_2$. Once again, these groups are not finitely presented.

The emphasis on finite presentability in Grothendieck's problem is a consequence of his original motivation for studying profinite completions: he wanted to understand the extent to which the topological fundamental group of a complex projective variety determines the algebraic fundamental group, and vice versa. Let X be a connected, smooth projective scheme over $\mathbb{C}$ with base point x and let X^{an} be the associated complex variety. Grothendieck points out that the profinite completion of the topological fundamental group $\pi_1(X^{\mathrm{an}}, x)$ (although defined by transcendental means) admits a purely algebraic description as the étale fundamental group of X. Since X^{an} is compact and locally simply-connected, its fundamental group $\pi_1(X^{\mathrm{an}}, x)$ is finitely presented.

In this article we settle Grothendieck's problem in the negative. In order to do so, we too exploit a fibre product construction; but it is a more subtle one that makes use of the techniques developed in [2] to construct unexpected finitely presented subgroups of direct products of hyperbolic groups. The key idea in this construction is to gain extra finiteness in the fibre product by presenting arbitrary finitely presented groups Q as quotients of 2-dimensional hyperbolic groups H rather than as quotients of free groups. One gains finiteness by ensuring that the kernel of $H \to Q$ is finitely generated; to do so one exploits ideas of Rips [14]. In the current setting we also need to ensure that the groups we consider are residually finite. To this end, we employ a refinement of the Rips construction due to Wise [15]. The first step in our construction involves the manufacture of groups that have aspherical balanced presentations and no proper subgroups of finite index (see Section 4).

In the following statement "hyperbolic" is in the sense of Gromov [5], and "dimension" is geometric dimension (thus H has a compact, 2-dimensional, classifying space $K(H, 1)$).

THEOREM 1.1. *There exist residually finite*, 2*-dimensional, hyperbolic groups H and finitely presented subgroups $P \hookrightarrow \Gamma := H \times H$ of infinite index, such that P is not abstractly isomorphic to* Γ, *but the inclusion $u : P \hookrightarrow \Gamma$ induces an isomorphism $\hat{u} : \hat{P} \to \hat{\Gamma}$.*

Explicit examples of such pairs $P \hookrightarrow \Gamma$ are described in Section 7. In Section 8 we describe an abundance of further examples by assigning such a pair $P \hookrightarrow \Gamma$ to every group that has a classifying space with a compact 3-skeleton.

In Section 3.1 of [6] Grothendieck considers the category C' of those groups K which have the property that, given any homomorphism $u : G_1 \to G_2$ of finitely presented groups, if $\hat{u} : \hat{G}_1 \to \hat{G}_2$ is an isomorphism then the induced map $f \mapsto f \circ u$ gives a bijection $\mathrm{Hom}(G_2, K) \to \mathrm{Hom}(G_1, K)$. He notes that his results give many examples of groups in C' and asks whether there exist finitely presented, residually finite groups that are not in C'. The groups Γ that we construct in Theorem 1.1 give concrete examples of such groups.

In Section 3.3 of [6] Grothendieck described an idea for reconstructing a residually finite group from the tensor product structure of its representation category $\mathrm{Rep}_A(\Gamma)$. He encoded this tensor product structure into a Tannaka duality group $\mathrm{cl}_A(\Gamma)$ (as explained in Section 10) and posed the following problem.

GROTHENDIECK'S SECOND PROBLEM. *Let* Γ *be a finitely presented, residually finite group. Is the natural monomorphism from* Γ *to* $\mathrm{cl}_A(\Gamma)$ *an isomorphism for every nonzero commutative ring A, or at least some suitable commutative ring $A \neq 0$?*

From our examples in Theorem 1.1 and the functoriality properties of the Tannaka duality group, it is obvious that there cannot be a commutative ring A so that the natural map $\Gamma \to \mathrm{cl}_A(\Gamma)$ is an isomorphism for all residually finite groups Γ. In Section 10 we prove the following stronger result.

THEOREM 1.2. *If P is one of the* (*finitely presented, residually finite*) *groups constructed in Theorem* 1.1, *then P is of infinite index in* $\mathrm{cl}_A(P)$ *for every commutative ring $A \neq 0$.*

In 1980 Lubotzky [9] exhibited finitely presented, residually finite groups Γ such that $\Gamma \to \mathrm{cl}_{\mathbb{Z}}(\Gamma)$ is not surjective, thus providing a negative solution of Grothendieck's Second Problem for the fixed ring $A = \mathbb{Z}$.

2. Fibre products and the 1-2-3 theorem

Associated to any short exact sequence of groups
$$1 \to N \to H \xrightarrow{\pi} Q \to 1$$

one has the *fibre product* $P \subset H \times H$,

$$P := \{(h_1, h_2) \mid \pi(h_1) = \pi(h_2)\}.$$

Let $N_1 = N \times \{1\}$ and $N_2 = \{1\} \times N$. It is clear that $P \cap (H \times \{1\}) = N_1$, that $P \cap (\{1\} \times H) = N_2$, and that P contains the diagonal $\Delta = \{(h,h) \mid h \in H\} \cong H$. Indeed $P = N_1 \cdot \Delta = N_2 \cdot \Delta \cong N \rtimes H$, where the action in the semi-direct product is simply conjugation.

LEMMA 2.1. *If H is finitely generated and Q is finitely presented, P is finitely generated.*

Proof. Since Q is finitely presented, $N \subset H$ is finitely generated *as a normal subgroup*. To obtain a finite generating set for P, one chooses a finite normal generating set for N_1 and then appends a generating set for $\Delta \cong H$. □

The question of when P is finitely presented is much more subtle. If N is not finitely generated as an abstract group, then in general one expects to have to include infinitely many relations in order to force the generators of N_1 to commute with the generators of N_2. Even when N is finitely generated, one may still encounter problems. These problems are analysed in detail in Sections 1 and 2 of [2], where the following "1-2-3 *Theorem*" is established.

Recall that a discrete group Γ is said to be of *type F_n* if there exists an Eilenberg-Maclane space $K(\Gamma, 1)$ with only finitely many cells in the n-skeleton.

THEOREM 2.2. *Let $1 \to N \to H \xrightarrow{\pi} Q \to 1$ be an exact sequence of groups. Suppose that N is finitely generated, H is finitely presented, and Q is of type F_3. Then the fibre product*

$$P := \{(h_1, h_2) \mid \pi(h_1) = \pi(h_2)\} \ \subseteq \ H \times H$$

is finitely presented.

We shall apply this theorem first in the case where the group Q has an aspherical presentation. In this setting, the process of writing down a presentation of P in terms of π and Q is much easier than in the general case — see Theorem 2.2 of [2]. The process becomes easier again if the aspherical presentation of Q is obtained from a presentation of H by simple deletion of all occurrences of a set of generators of N. The effective nature of the process in this case will be exemplified in Section 7.

3. A residually finite version of the Rips construction

In [14], E. Rips described an algorithm that, given a finite group presentation, will construct a short exact sequence of groups $1 \to N \to H \to Q \to 1$, where Q is the group with the given presentation, H is a small-cancellation group (a certain type of hyperbolic group with an aspherical presentation), and N is a 2-generator group. There have since been a number of refinements of Rips's original construction, engineered so as to ensure that the group H has additional desirable properties; the price that one must pay for such desirable properties is an increase in the number of generators of N. The variant that we require is due to Wise [15], who refined the Rips construction so as to ensure that the small-cancellation group obtained is residually finite.

THEOREM 3.1. *There is an algorithm that associates to every finite group presentation $\mathcal{P}$ a short exact sequence of groups*

$$1 \to N \to H \to Q \to 1,$$

where Q is the group presented by $\mathcal{P}$, the group N is generated by three elements, and the group H is a torsion-free, residually-finite, hyperbolic group (satisfying the small cancellation condition $C'(\frac{1}{6})$).

The explicit nature of the Rips-Wise construction will be demonstrated in Section 7.

4. Some seed groups

In this section we describe the group presentations used as our initial input to the constructions in the preceding sections. We remind the reader that associated to any finite group-presentation one has the compact combinatorial 2-complex that has one vertex, one directed edge $e(a)$ corresponding to each generator a of the presentation, and one 2-cell corresponding to each relator; the boundary of the 2-cell corresponding to the relator $r = a_1 \dots a_l$ is attached to the 1-skeleton by the loop $e(a_1) \dots e(a_l)$. The presentation is said to be *aspherical* if this presentation complex has a contractible universal covering. A presentation is said to be *balanced* if it has the same number of generators as relators.

PROPOSITION 4.1. *There exist infinite groups Q, given by finite, aspherical, balanced presentations, such that Q has no nontrivial finite quotients.*

Explicit examples will be given in Sections 4.1 and 4.2. The balanced nature of the presentations we construct will be used in the following way.

LEMMA 4.2. *If Q has a finite, balanced presentation and $H_1(Q, \mathbb{Z}) = 0$, then $H_2(Q, \mathbb{Z}) = 0$.*

Proof. Writing F for the free group on the given generators of Q, and R for the normal closure of the given relations, we have a short exact sequence $1 \to R \to F \to Q \to 1$. From this we obtain an exact sequence of abelian groups:

$$0 \to \frac{R \cap [F,F]}{[R,F]} \to \frac{R}{[R,F]} \to \frac{F}{[F,F]} \to \frac{F}{R\,[F,F]} \to 0.$$

Hopf's formula identifies the first group in this sequence as $H_2(Q, \mathbb{Z})$. We have assumed that $H_1(Q, \mathbb{Z}) = 0$, and $F/[F,F]$ is *free* abelian, of rank n say. Thus, splitting the middle arrow, we get $R/[R,F] \cong H_2(Q, \mathbb{Z}) \oplus \mathbb{Z}^n$.

The abelian group $R/[R,F]$ is generated by the images of the given relations of Q, of which there are only n. Thus $H_2(Q, \mathbb{Z})$ must be trivial. □

There are many groups of the type described in the above proposition. We shall describe one family of famous examples and one family that is more novel. In both cases one sees that the presentations are aspherical by noting that they are built-up from infinite cyclic groups by repeatedly forming amalgamated free products and HNN extensions along free subgroups. The natural presentations of such groups are aspherical.[1] Explicitly:

LEMMA 4.3. *Suppose that for $i = 1, 2$ the presentation $G_i = \langle A_i \mid R_i \rangle$ is aspherical, and suppose that the words $u_{i,1}, \dots, u_{i,n}$ generate a free subgroup of rank n in G_i. Then*

$$\langle A_1, A_2 \mid R_1, R_2, u_{1,1}u_{2,1}^{-1}, \dots, u_{1,n}u_{2,n}^{-1} \rangle$$

is an aspherical presentation of the corresponding amalgamated free product $G_1 *_{F_n} G_2$.

Similarly, if $v_{1,1}, \dots, v_{1,n}$ generate a free subgroup of rank n in G_1, then

$$\langle A_1, t \mid R_1,\ t^{-1}u_{1,1}tv_{1,1}^{-1}, \dots, t^{-1}u_{1,n}tv_{1,n}^{-1} \rangle$$

is an aspherical presentation of the corresponding HNN *extension $G_1 *_{F_n}$.*

4.1. *The Higman groups.* Graham Higman [8] constructed the following group and showed that it has no proper subgroups of finite index.

$$J_4 = \langle a_1, a_2, a_3, a_4 \mid a_2^{-1}a_1a_2a_1^{-2},\ a_3^{-1}a_2a_3a_2^{-2},\ a_4^{-1}a_3a_4a_3^{-2},\ a_1^{-1}a_4a_1a_4^{-2} \rangle.$$

One can build this group as follows. First note that $B = \langle x, y \mid y^{-1}xyx^{-2} \rangle$ is aspherical, by Lemma 4.3. Take two pairs of copies of B and amalgamate each pair by identifying the letter x in one copy with the letter y in the other copy. In each of the resulting amalgams, G_1 and G_2, the unidentified copies

[1] For example, suppose X is an aspherical presentation complex for A and Y is an aspherical presentation complex for B, and injections $i : F \to A$ and $j : F \to B$ are given, where F is a finitely generated free group. One can realise i and j by cellular maps $I : Z \to X$ and $J : Z \to Y$ where Z is a compact graph with one vertex v. An aspherical presentation complex for $A *_F B$ is then obtained as $X \cup (Z \times [0,1]) \cup Y$ modulo the equivalence relation generated by $(z, 0) \sim I(z), (z, 1) \sim J(z)$ and $(v, t) \sim (v, 1)$ for all $z \in Z$ and $t \in [0, 1]$.

of x and y generate a free group (by Britton's Lemma). The group J_4 is obtained from G_1 and G_2 by amalgamating these free subgroups. Lemma 4.3 assures us that the resulting presentation (i.e. the one displayed above) is aspherical. The group is clearly infinite since we have constructed it as a nontrivial amalgamated free product.

Entirely similar arguments apply to the group

$$J_n = \langle a_1 \ldots, a_n \mid a_i^{-1} a_{i-1} a_i a_{i-1}^{-2} \ (i = 2, \ldots, n);\ a_1^{-1} a_n a_1 a_n^{-2} \rangle$$

for each integer $n \geq 4$.

Higman [8] provides an elementary proof that these groups have no nontrivial finite quotients. In particular, $H_1(J_n, \mathbb{Z}) = 0$; hence $H_2(J_n, \mathbb{Z}) = 0$, by Lemma 4.2.

4.2. *Amalgamating non-Hopfian groups.* Fix $p \geq 2$ and consider

$$G = \langle a_1, a_2 \mid a_1^{-1} a_2^p a_1 = a_2^{p+1} \rangle.$$

This group admits the noninjective epimorphism $\phi(a_1) = a_1$, $\phi(a_2) = a_2^p$. The nontrivial element $c = [a_2, a_1^{-1} a_2 a_1]$ lies in the kernel of ϕ. Britton's Lemma tells us that a_1 and c generate a free subgroup of rank 2.

Observe that if $\pi : G \to R$ is a homomorphism to a finite group, then $\pi(c) = 1$. Indeed, if $\pi(c) \neq 1$ then we would have infinitely many distinct maps $G \to R$, namely $\pi \circ \phi^n$, contradicting the fact that there are only finitely many homomorphisms from any finitely generated group to any finite group.

We amalgamate two copies G' and G'' of G by setting $c' = a_1''$ and $a_1' = c''$. Lemma 4.3 tells us that the natural presentation of the resulting amalgam is aspherical. Under any homomorphism from this amalgam $G' *_{F_2} G''$ to a finite group, $c'(= a_1'')$ and $c''(= a_1')$ must map trivially, which forces the whole group to have trivial image.

Thus for each $p \geq 2$ we obtain the following aspherical presentation of a group with no nontrivial finite quotients.

$$B_p = \langle a_1, a_2, b_1, b_2 \mid a_1^{-1} a_2^p a_1 a_2^{-p-1},\ b_1^{-1} b_2^p b_1 b_2^{-p-1},\ a_1^{-1}[b_2, b_1^{-1} b_2 b_1],\ b_1^{-1}[a_2, a_1^{-1} a_2 a_1] \rangle.$$

5. The Platonov-Tavgen criterion

As noted in [1], one can abstract the following criterion from the arguments in [11].

THEOREM 5.1. *Let $1 \to N \to H \to Q \to 1$ be a short exact sequence of groups and let $P \subset H \times H$ be the associated fibre product. If H is finitely generated, Q has no finite quotients, and $H_2(Q, \mathbb{Z}) = 0$, then the inclusion $u : P \hookrightarrow H \times H$ induces an isomorphism $\hat{u} : \hat{P} \to \hat{H} \times \hat{H}$.*

For the sake of completeness, we include a proof of this criterion, distilled from [11].

Proof. Let $\Gamma = H \times H$. The surjectivity of $\hat{u}$ is equivalent to the statement that there is no proper subgroup of finite index $G \subset \Gamma$ that contains P. If there were such a subgroup, then we would have $N \times N \subset P \subset G$, and $G/(N \times N)$ would be a proper subgroup of finite index in $(H/N) \times (H/N)$, of which we have supposed there are none.

In order to show that $\hat{u}$ is injective, it is enough to prove that given any normal subgroup of finite index $R \subset P$, there exists a subgroup of finite index $S \subset \Gamma$ such that $S \cap P \subseteq R$. Note that $L_1 := R \cap (N \times \{1\})$, which is normal in P and of finite index in $N_1 = (N \times \{1\})$, is also normal in $H_1 = H \times \{1\}$, because the action of $(h, 1) \in H_1$ by conjugation on L_1 is the same as the action of $(h, h) \in P$. Similar considerations apply to $N_2 = (\{1\} \times N)$ and $L_2 = R \cap N_2$.

Lemma 5.2. *Let H be a finitely generated group, and let $L \subset N$ be normal subgroups of H. Assume N/L is finite, $Q = H/N$ has no finite quotients and $H_2(Q, \mathbb{Z}) = 0$. Then there exists a subgroup $S_1 \subset H$ of finite index such that $S_1 \cap N = L$.*

Proof. Let M be the kernel of the action $H \to \mathrm{Aut}(N/L)$ by conjugation. Since M has finite index in H, it maps onto Q. Thus we have a central extension

$$1 \to (N/L) \cap (M/L) \to M/L \to Q \to 1.$$

Because Q perfect, it has a universal central extension. Because $H_2(Q, \mathbb{Z}) = 0$, this extension is trivial. Thus every central extension of Q splits. In particular M/L retracts onto $(N/L) \cap (M/L)$. We define S_1 to be the kernel of the resulting homomorphism $M \to (N/L) \cap (M/L)$. $\square$

Returning to the proof of Theorem 5.1, we now have subgroups of finite index $S_1 \subset H \times \{1\}$ and $S_2 \subset \{1\} \times H$ such that $S_i \cap N_i = L_i$ for $i = 1, 2$. Thus $S := S_1 S_2$ intersects $N_1 N_2$ in $L_1 L_2 \subseteq R \cap N_1 N_2$.

Consider $p \in P \smallsetminus R$. Since P and $R \cap S$ have the same image in $Q \times Q = \Gamma/N_1 N_2$ (namely the diagonal) there exists $r \in R \cap S$ such that $pr \in N_1 N_2 \smallsetminus R$. Since $N_1 N_2 \cap S \subseteq N_1 N_2 \cap R$, we conclude $p \notin S$. Hence $P \cap S \subseteq R$. $\square$

6. Proof of the Main Theorem

We begin with a finite aspherical presentation for one of the seed groups J_n or B_p constructed in Section 4; let Q be such a group. By applying the Rips-Wise construction from Section 3 we obtain a short exact sequence

$$1 \to N \to H \to Q \to 1$$

with H a residually finite (2-dimensional) hyperbolic group and N a finitely generated subgroup. The 1-2-3 Theorem (Section 2) tells us that the fibre product $P \subset H \times H$ associated to this sequence is finitely presented. Since Q is infinite, P is a subgroup of infinite index. The sequence $1 \to N \to H \to Q \to 1$ satisfies the Platonov-Tavgen criterion (Section 5), and hence the inclusion $u : P \hookrightarrow H \times H$ induces an isomorphism $\hat{u} : \hat{P} \to \hat{H} \times \hat{H}$.

To see that P is not abstractly isomorphic to $\Gamma = H \times H$, we appeal to the fact that centralizers of nontrivial elements in torsion-free hyperbolic groups are cyclic [5]. Indeed this observation allows us to characterize $H \times \{1\}$ and $\{1\} \times H$ as the only non-abelian subgroups of Γ that are the centralizers of noncyclic subgroups of $\Gamma \smallsetminus \{1\}$. The subgroups $\{1\} \times N$ and $N \times \{1\}$ of P are characterized in the same way. Thus if P were abstractly isomorphic to Γ, then H would be isomorphic to N. But H is finitely presented whereas N is not [3].

7. An explicit example

In this section we give explicit presentations for a pair of groups $P \hookrightarrow H \times H$ satisfying the conclusion of Theorem 1.1. The fact that we are able to do so illustrates the constructive nature of the proof of the 1-2-3 Theorem (in the aspherical case) and the Rips-Wise construction.

Although they are explicit, our presentations are not small: the presentation of P has ten generators and seventy seven relations, and the sum of the lengths of the relations is approximately eighty thousand.

As seed group we take

$$J_4 = \langle a_1, a_2, a_3, a_4 \mid a_2^{-1}a_1a_2a_1^{-2},\ a_3^{-1}a_2a_3a_2^{-2},\ a_4^{-1}a_3a_4a_3^{-2},\ a_1^{-1}a_4a_1a_4^{-2} \rangle.$$

In general, given a presentation of a group that has r generators and m relations, the hyperbolic group produced by the Rips-Wise construction will have $r + 3$ generators and $m + 6r$ relations.

7.1. *A Presentation of H.* There are seven generators,

$$a_1, a_2, a_3, a_4, x_1, x_2, x_3,$$

subject to the relations

$$(S_1) \qquad a_i^{\varepsilon} x_j a_i^{-\varepsilon} = V_{ij\varepsilon}(\underline{x}) \quad \text{for } i \in \{1,2,3,4\},\ j \in \{1,2,3\},\ \varepsilon = \pm 1,$$

and

$$(S_2) \qquad \begin{aligned} &a_2^{-1}a_1a_2a_1^{-2} = U_1(\underline{x}),\ a_3^{-1}a_2a_3a_2^{-2} = U_2(\underline{x}), \\ &a_4^{-1}a_3a_4a_3^{-2} = U_3(\underline{x}),\ a_1^{-1}a_4a_1a_4^{-2} = U_4(\underline{x}), \end{aligned}$$

where $V_{ij\varepsilon}(\underline{x}) = v_{ij\varepsilon}x_3v'_{ij\varepsilon}x_3^{-1}$ for $j = 1, 2$, and $V_{i3\varepsilon}(\underline{x}) = v_{i3\varepsilon}x_3v'_{i3\varepsilon}$, and $U_i(\underline{x}) = u_ix_3u'_ix_3^{-1}$, with the 56 words $u_i, u'_i, v_{ij\varepsilon}, v'_{ij\varepsilon}$ being (in any order)

$$\{x_1x_2^{5n}x_1x_2^{5n+1}x_1x_2^{5n+2}x_1x_2^{5n+3}x_1x_2^{5n+4} \mid n = 1, \dots, 56\}.$$

The point about this last set of words is that it satisfies the $c(5)$ small cancellation condition; any other such set of words would serve the same purpose (see [15]).

The relations (S_1) ensure that the subgroup $N := \langle x_1, x_2, x_3 \rangle$ is normal in H, and the relations (S_2) ensure that H/N is isomorphic to J_4 via the map $a_i \mapsto a_i$ implicit in the notation. □

In order to present the fibre product $P \subset H \times H$ associated to the short exact sequence $1 \to N \to H \to J_4 \to 1$, we need the following notation.

For each $j \in \{1, 2, 3\}$ we introduce generators x_j^L to represent $(x_j, 1)$ and x_j^R to represent $(1, x_j)$. Given a word $W(\underline{x})$ in the letters $\underline{x} = \{x_1, x_2, x_3\}$, we write $W(\underline{x}^L)$ for the word obtained by making the formal substitutions $x_j \mapsto x_j^L$; and likewise for $W(\underline{x}^R)$. We introduce generators A_i to represent $(a_i, a_i) \in H$.

The following presentation is a special case of Theorem 2.2 of [2]. Our notation (S_1) and (S_2) agrees with that of [2]; the additional sets of relations (S_3) and Z_σ of [2] are empty in the current setting.

7.2. *A Presentation of P.* There are ten generators,

$$A_1, A_2, A_3, A_4, x_1^L, x_2^L, x_3^L, x_1^R, x_2^R, x_3^R,$$

subject to the relations

$$\begin{aligned} A_{i+1}^{-1} A_i A_{i+1} A_i^{-2} &= U_i(\underline{x}^L) U_i(\underline{x}^R) \text{ (for } i = 1, 2, 3) \text{ and} \\ A_1^{-1} A_4 A_1 A_4^{-2} &= U_4(\underline{x}^L) U_4(\underline{x}^R), \end{aligned}$$

and

$$[x_j^L, x_k^R] = 1 \quad \text{for all} \quad j, k \in \{1, 2, 3\},$$

and

$$\begin{aligned} &A_i^\varepsilon x_j^L A_i^{-\varepsilon} = V_{ij\varepsilon}(\underline{x}^L) \text{ and} \\ &A_i^\varepsilon x_j^R A_i^{-\varepsilon} = V_{ij\varepsilon}(\underline{x}^R) \text{ for } i \in \{1, 2, 3, 4\},\ j \in \{1, 2, 3\},\ \varepsilon = \pm 1, \end{aligned}$$

where the words U_i and $V_{ij\varepsilon}$ are as above. □

8. An abundance of examples

In this section we describe a construction that associates to every group of type F_3 a pair of groups $P \hookrightarrow \Gamma$ satisfying the conclusion of Theorem 1.1.

The following lemma is a special case of the general phenomenon that if a class of groups $\mathcal{G}$ is closed under the formation of HNN extensions and amalgamated free products along finitely generated free groups, then one can embed groups $G \in \mathcal{G}$ into groups $\overline{G} \in \mathcal{G}$ that have no finite quotients, preserving desirable geometric properties of G; see [4].

Let $\mathcal{F}_3$ denote the class of groups of type F_3. The mapping cylinder construction sketched in Section 4 shows that $\mathcal{F}_3$ is closed under the formation of HNN extensions and amalgamated free products along finitely generated free groups.

LEMMA 8.1. *Every $G \in \mathcal{F}_3$ can be embedded in a group $\overline{G} \in \mathcal{F}_3$ that has no proper subgroups of finite index.*

Proof. We may assume that G is generated by elements $\{a_1, \dots, a_n\}$ of infinite order, for if necessary we can replace G by $G * \mathbb{Z}$ and $\{a_1, \dots, a_n\}$ by $\{a_1 t, \dots, a_n t, t\}$, where $\mathbb{Z} = \langle t \rangle$.

Let Q be an aspherical group with no nontrivial finite quotients, as described in Section 4. We fix a nontrivial element $q \in Q$ and modify G by repeatedly forming amalgamated free products with copies of Q as follows. Let $G_1 = G *_{\mathbb{Z}} Q$, where $a_1 \in G$ is identified with $q \in Q$. Then, for $i = 2, \dots, n$, let $G_i = G_{i-1} *_{\mathbb{Z}} Q$, where $a_i \in G \subset G_{i-1}$ is identified with $q \in Q$. Let $\overline{G} = G_n$.

Since Q has no nontrivial finite quotients, any homomorphism from $\overline{G}$ to a finite group must kill each copy of Q and hence each of the generators a_i. $\square$

The group $\overline{G}$ produced by the above construction is perfect and hence has a universal central extension (see Chapter 5 of [10], for example):

$$1 \to H_2(\overline{G}, \mathbb{Z}) \to \tilde{G} \to \overline{G} \to 1,$$

where $\tilde{G}$ is superperfect, i.e. $H_1(\tilde{G}, \mathbb{Z}) = H_2(\tilde{G}, \mathbb{Z}) = 0$. Since $\overline{G}$ and $H_2(\overline{G}, \mathbb{Z})$ lie in $\mathcal{F}_3$, so does $\tilde{G}$. Moreover, since $\overline{G}$ has no nontrivial finite quotients, neither does $\tilde{G}$: since $\tilde{G}$ is perfect, such a quotient could not be abelian, so factoring out the image of $H_2(\overline{G}, \mathbb{Z})$ would yield a nontrivial finite quotient of $\overline{G}$.

We now proceed as in Section 6: Wise's modification of the Rips construction yields a short exact sequence $1 \to N \to H \to \tilde{G} \to 1$ with H a residually finite hyperbolic group; the 1-2-3 Theorem assures us that the associated fibre product P is finitely presented (this time with a less obvious presentation); and the Platonov-Tavgen criterion tells us that the inclusion $P \hookrightarrow \Gamma := H \times H$ induces an isomorphism $\hat{P} \to \hat{\Gamma}$.

9. Varying the subgroup

Let Γ be a residually finite group. We record two remarks concerning the number of subgroups $u_P : P \hookrightarrow \Gamma$ for which $\hat{u}_P$ is an isomorphism.

We fix an infinite group $\tilde{G}$ as in the previous section and let H and N be constructed accordingly.

PROPOSITION 9.1. *The direct sum H^{2n} of $2n$ copies of H contains at least n nonisomorphic, finitely presented subgroups P such that $P \hookrightarrow H^{2n}$ induces an isomorphism $\hat{P} \to \hat{H}^{2n}$.*

Proof. We have a short exact sequence $1 \to N \to H \xrightarrow{\pi} \tilde{G} \to 1$. For each integer $r = 1, \ldots, n$ we consider the epimorphism $\pi_r : H^n \to \tilde{G}^r$ that maps the first r factors by π and maps the remaining $n - r$ factors trivially. The kernel of this map is $N^r \times H^{n-r}$, which is finitely generated. $\tilde{G}^r \in \mathcal{F}_3$ is superperfect and has no nontrivial finite quotients. Thus, as in Section 6, we conclude that the fibre product associated to π_r is a finitely presented group whose inclusion $P_r \hookrightarrow H^{2n}$ induces an isomorphism on profinite completions.

As in Section 6, one can see that P_r is not isomorphic to P_s when $r \neq s$ by considering the structure of centralizers in P_r. □

PROPOSITION 9.2. *If F is a free group of rank at least three, then there exist infinitely many nonisomorphic finitely generated subgroups P of infinite index such that $u_P : P \hookrightarrow F \times F$ induces an isomorphism $\hat{u}_P : \hat{P} \to \hat{F} \times \hat{F}$.*

Proof. In Section 4 we constructed infinitely many nonisomorphic finitely presented groups B_r, each of which can be generated by three elements; these groups have no finite quotients and $H_2(B_r, \mathbb{Z}) = 0$.

Lemma 2.1 shows that the fibre product $P_r \subset F \times F$ associated to each short exact sequence $1 \to N \to F \to B_r \to 1$ is finitely generated, and the Platonov-Tavgen criterion applies to the inclusion $P_r \hookrightarrow F \times F$. As in previous arguments, one sees that the subgroup $N \times N \subset P_r$ is uniquely determined by the structure of centralizers in P_r. And since $P_r/(N \times N) \cong B_r$, it follows that P_q is not isomorphic to P_r if $q \neq r$. □

It is an open problem to establish an analogous result for finitely presented groups. For this it would be enough to construct a finitely presented group Γ with the following property: there exist infinitely many nonisomorphic superperfect groups $Q \in \mathcal{F}_3$, each with no nontrivial finite quotients, and short exact sequences $1 \to N_Q \to \Gamma \to Q \to 1$ with N_Q finitely generated.

10. Grothendieck's Tannaka duality groups

In this section we explain our negative solution to Grothendieck's Second Problem.

We began this paper by recalling the principal result of Grothendieck's paper [6]: a homomorphism $u : \Gamma_1 \to \Gamma_2$ between residually finite groups induces an isomorphism $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ if and only if the restriction functor $u_A^* : \mathrm{Rep}_A(\Gamma_2) \to \mathrm{Rep}_A(\Gamma_1)$ is an equivalence of categories for every (or even one) nonzero commutative ring A.

After proving this theorem and posing the first of the problems we stated in our introduction, Grothendieck outlined an idea for answering that question in the affirmative. His idea is that one should try to reconstruct a residually finite group Γ from its representation category $\mathrm{Rep}_A(\Gamma)$ using the tensor product structure, as we shall now explain.

Let $\mathrm{Mod}(A)$ denote the category of all finitely generated A-modules and consider the forgetful functor

$$\mathcal{F} : \mathrm{Rep}_A(\Gamma) \to \mathrm{Mod}(A).$$

In resonance with his ideas on Tannaka duality, Grothendieck defined $\mathrm{cl}_A(\Gamma)$ to be the group of natural self-transformations of the functor $\mathcal{F}$ that are compatible with the tensor product $\otimes_A$. Thus an element $\alpha \in \mathrm{cl}_A(\Gamma)$ is a collection (α_M) of A-linear isomorphisms $\alpha_M : \mathcal{F}(M) \to \mathcal{F}(M)$, one for each $M \in \mathrm{Ob}(\mathrm{Rep}_A(\Gamma))$, satisfying the following two conditions:

(1) For all M, $N \in \mathrm{Ob}(\mathrm{Rep}_A(\Gamma))$ and all Γ-equivariant, A-linear maps $\varphi : M \to N$, the diagram

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{\mathcal{F}(\varphi)} & \mathcal{F}(N) \\ \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_N} \\ \mathcal{F}(M) & \xrightarrow{\mathcal{F}(\varphi)} & \mathcal{F}(N) \end{array}$$

is commutative.

(2) For all M, $N \in \mathrm{Ob}(\mathrm{Rep}_A(\Gamma))$ we have $\alpha_{M \otimes_A N} = \alpha_M \otimes_A \alpha_N$.

There is an obvious group homomorphism $t_A^\Gamma : \Gamma \to \mathrm{cl}_A(\Gamma)$ defined by $t_A^\Gamma(\gamma)_M := (\gamma|_M)$ for all $\gamma \in \Gamma$ and $M \in \mathrm{Ob}(\mathrm{Rep}_A(\Gamma))$. If Γ is residually finite and $A \neq 0$, this homomorphism is injective.

The assignment $\Gamma \mapsto \mathrm{cl}_A(\Gamma)$ extends to a covariant functor cl_A on the category of groups: cl_A assigns to a group homomorphism $u : \Gamma_1 \to \Gamma_2$, the homomorphism $\tilde{u}_A : \mathrm{cl}_A(\Gamma_1) \to \mathrm{cl}_A(\Gamma_2)$ that sends $\alpha = (\alpha_M) \in \mathrm{cl}_A(\Gamma_1)$ to $\tilde{u}_A(\alpha) \in \mathrm{cl}_A(\Gamma_2)$ according to the rule

$$\tilde{u}_A(\alpha)_N := \alpha_{u_A^*(N)} \qquad (N \in \mathrm{Ob}(\mathrm{Rep}_A(\Gamma_2))).$$

Note that $\tilde{u}_A(t_A^{\Gamma_1}(\Gamma_1)) = t_A^{\Gamma_2}(\Gamma_2)$.

If we restrict our attention to residually finite groups, then we may conflate Γ with $t_A^\Gamma(\Gamma)$. Grothendieck's Second Problem can now be stated as:

Grothendieck's Second Problem. *If Γ is finitely presented and residually finite, then is $\Gamma = \mathrm{cl}_A(\Gamma)$ for every commutative ring $A \neq 0$, or at least for a suitable ring A?*

This problem is closely related to Grothendieck's First Problem, as we shall explain now.

Let $u : \Gamma_1 \to \Gamma_2$ be a monomorphism of residually finite groups and suppose that $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ is an isomorphism. Grothendieck deduces from the main result of his paper [6] that if there is a commutative ring $A \neq 0$ such that $\Gamma_1 = \mathrm{cl}_A(\Gamma_1)$ and $\Gamma_2 = \mathrm{cl}_A(\Gamma_2)$, then $u : \Gamma_1 \to \Gamma_2$ is an isomorphism.

Grothendieck identifies suitable rings A in startling generality. For example, he proves that $\Gamma = \mathrm{cl}_{\mathbb{Z}}(\Gamma)$ for all arithmetic groups Γ that have the congruence subgroup property.

In contrast, we prove:

THEOREM 10.1. *If P is one of the (finitely presented, residually finite) groups constructed in Theorem 1.1, then P is of infinite index in $\mathrm{cl}_A(P)$ for every commutative ring $A \neq 0$.*

Proof. Let $A \neq 0$ be a commutative ring. The inclusion $u : P \to \Gamma$ constructed in Theorem 1.1 induces an isomorphism $\hat{u} : \hat{P} \to \hat{\Gamma}$ and hence (by Grothendieck's result) an equivalence of categories $u_A^* : \mathrm{Rep}_A(\Gamma) \to \mathrm{Rep}_A(P)$. Since u_A^* is an equivalence, $\tilde{u}_A : \mathrm{cl}_A(P) \to \mathrm{cl}_A(\Gamma)$ is an isomorphism.

$$
\begin{array}{ccc}
P & \xrightarrow{u} & \Gamma \\
\Big\downarrow t_A^P & & \Big\downarrow t_A^\Gamma \\
\mathrm{cl}_A(P) & \xrightarrow{\tilde{u}_A} & \mathrm{cl}_A(\Gamma)
\end{array}
$$

The index of P in Γ is infinite, and so consideration of the above commutative diagram shows that the index of P in $\mathrm{cl}_A(P)$ is infinite. □

The only previous progress on Grothendieck's Second Problem was achieved by Alex Lubotzky in [9]. The following is a somewhat rough description of his results (see [9] for details). Note that in these examples, in contrast to our Theorem 10.1, the rings A are chosen in relation to the groups Γ.

Lubotzky first extended the result of Grothendieck mentioned above by proving that if Γ is an arithmetic group having the weak congruence subgroup property (i.e. the congruence kernel $C_\Gamma \subset \Gamma$ is finite and abelian), then $\mathrm{cl}_{\mathbb{Z}}(\Gamma) \cong C_\Gamma \times \Gamma$. Thus $\mathrm{cl}_{\mathbb{Z}}(\Gamma)$ can be a nontrivial finite extension of Γ.

Lubotzky also proved that if Γ is an S-arithmetic group where the finite set of primes S enters nontrivially into the definition of Γ, then $\mathrm{cl}_{\mathbb{Z}}(\Gamma) = \hat{\Gamma}$. Thus he discovered examples in which $\mathrm{cl}_{\mathbb{Z}}(\Gamma)$ is an uncountable extension of Γ. Moreover, such examples show that the index of Γ in $\mathrm{cl}_A(\Gamma)$ can depend on the ring A, because in certain cases there exists a ring of S-arithmetic integers A such that $\mathrm{cl}_A(\Gamma) = \Gamma$.

Mathematisches Institut, Heinrich-Heine-Universität Düsseldorf, Düsseldorf, Germany
E-mail address: grunewald@math.uni-duesseldorf.de

Imperial College London, London, U.K.
E-mail address: m.bridson@imperial.ac.uk

References

[1] H. Bass and A. Lubotzky, Nonarithmetic superrigid groups: counterexamples to Platonov's conjecture, *Ann. of Math.* **151** (2000), 1151–1173.

[2] G. Baumslag, M. R. Bridson, C. F. Miller III, and H. Short, Fibre products, non-positive curvature, and decision problems, *Comment. Math. Helv.* **75** (2000), 457–477.

[3] R. Bieri, Normal subgroups in duality groups and in groups of cohomological dimension 2, *J. Pure and Appl. Algebra* **7** (1976), 35–51.

[4] M. R. Bridson, Controlled embeddings into groups that have no nontrivial finite quotients, *Geom. Topol. Monogr.* **1** (1998), 99–116.

[5] M. Gromov, Hyperbolic groups, in *Essays in Group Theory* (S. M. Gersten, ed.), *Math. Sci. Res. Inst. Publ.* **8**, 75–263, Springer-Verlag, New York, 1987.

[6] A. Grothendieck, Représentations linéaires et compactification profinie des groupes discrets, *Manuscripta Math.* **2** (1970), 375–396.

[7] F. J. Grunewald, On some groups which cannot be finitely presented, *J. London Math. Soc.* (2) **17** (1978), 427–436.

[8] G. Higman, A finitely generated infinite simple group, *J. London Math. Soc.* **26** (1951), 61–64.

[9] A. Lubotzky, Tannaka duality for discrete groups, *Amer. J. Math.* **102** (1980), 663–689.

[10] J. Milnor, *Introduction to Algebraic K-Theory*, *Ann. of Math. Studies* **72**, Princeton Univ. Press, Princeton, NJ, 1971.

[11] V. Platonov and O. I. Tavgen, Grothendieck's problem on profinite completions of groups, *Soviet Math. Dokl.* **33** (1986), 822–825.

[12] ———, Grothendieck's problem on profinite completions and representations of groups, *K-Theory* **4** (1990), 89–101.

[13] L. Pyber, Groups of intermediate subgroup growth and a problem of Grothendieck, *Duke Math. J.* **121** (2004), 169–188.

[14] E. Rips, Subgroups of small cancellation groups, *Bull. London Math. Soc.* **14** (1982), 45–47.

[15] D. Wise, A residually finite version of Rips's construction, *Bull. London Math. Soc.* **35** (2003), 23–29.

(Received July 7, 2003)