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Removability of point singularities of Willmore surfaces

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Abstract

We investigate point singularities of Willmore surfaces, which for example appear as blowups of the Willmore flow near singularities, and prove that closed Willmore surfaces with one unit density point singularity are smooth in codimension one. As applications we get in codimension one that the Willmore flow of spheres with energy less than 8π exists for all time and converges to a round sphere and further that the set of Willmore tori with energy less than $8\pi - \delta$ is compact up to Möbius transformations.

1. Introduction

For an immersed closed surface $f : \Sigma \to \mathbb{R}^n$ the Willmore functional is defined by

$$\mathcal{W}(f) = \frac{1}{4} \int\limits_\Sigma |H|^2 \, \mathrm{d}\mu_g,$$

where H denotes the mean curvature vector of f, $g = f^* g_{\mathrm{euc}}$ the pull-back metric and μ_g the induced area measure on Σ. The Gauss equations and the Gauss-Bonnet Theorem give rise to equivalent expressions

$$\mathcal{W}(f) = \frac{1}{4} \int\limits_\Sigma |A|^2 \, \mathrm{d}\mu_g + \pi\chi(\Sigma) = \frac{1}{2} \int\limits_\Sigma |A^\circ|^2 \, \mathrm{d}\mu_g + 2\pi\chi(\Sigma),$$

where A denotes the second fundamental form, $A^\circ = A - \frac{1}{2} g \otimes H$ its trace-free part and χ the Euler characteristic. The Willmore functional is scale invariant and moreover invariant under the full Möbius group of $\mathbb{R}^n$. Critical points of $\mathcal{W}$ are called Willmore surfaces or more precisely Willmore immersions.

We always have $\mathcal{W}(f) \geq 4\pi$ with equality only for round spheres; see [Wil] in codimension one, that is $n = 3$. On the other hand, if $\mathcal{W}(f) < 8\pi$

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then f is an embedding by an inequality of Li and Yau in [LY]; for the reader's convenience see also (A.17) in our appendix. Bryant classified in [Bry] all Willmore spheres in codimension one.

In [KuSch 2], we studied the L^2 gradient flow of the Willmore functional up to a factor, the Willmore flow for short, which is the fourth order, quasilinear geometric evolution equation

$$\partial_t f + \Delta_g H + Q(A^0)H = 0$$

where the Laplacian of the normal bundle along f is used and $Q(A^0)$ acts linearly on normal vectors along f by

$$Q(A^0)\phi := g^{ik}g^{jl}A^0_{ij}\langle A^0_{kl}, \phi\rangle.$$

There we estimated the existence time of the Willmore flow in terms of the concentration of local integrals of the squared second fundamental form. These estimates enable us to perform a blowup procedure near singularities, see [KuSch 1], which yields a compact or noncompact Willmore surface as blowup. In contrast to mean curvature flow, the blowup is stationary as the Willmore functional is scale invariant. In case the blowup is noncompact, its inversion is again a smooth Willmore surface, but with a possible point singularity at the origin.

The purpose of this article is to study unit density point singularities of general Willmore surfaces in codimension one. Our first main result, Lemma 3.1, states that the Willmore surface extends $C^{1,\alpha}$ for all $\alpha < 1$ into the point singularity. This cannot be improved to $C^{1,1}$ as one sheet of an inverted catenoid shows. For the proof, we establish that the integral of the squared mean curvature over an exterior ball around the point singularity decays in a power of the radius; that is,

$$\int\limits_{[|f|<\varrho]} |H|^2 \, \mathrm{d}\mu_g \leq C\varrho^\beta \quad \text{for some } \beta > 0. \tag{1.1}$$

(1.1) implies the regular extension of the Willmore surface by standard technics in geometric measure theory, when we take into account our assumption of unit density. In codimension one, we can choose a smooth normal ν and define the scalar mean curvature $H_{\mathrm{sc}} := H\nu$ up to a sign. Observing for the normal Laplacian that $\Delta_g H = (\Delta_g H_{\mathrm{sc}})\nu$, the Euler-Lagrange equation satisfied on the Willmore surface simplifies in codimension one to

$$\Delta_g H_{\mathrm{sc}} + |A^0|^2 H_{\mathrm{sc}} = 0. \tag{1.2}$$

The decisive point in order to make (1.2) applicable, more precisely to control the metric near the point singularity, is to introduce conformal coordinates by the work [MuSv] of Müller and Sverak, again using our assumption of unit density. Considering (1.2) as a scalar second order linear elliptic equation in H_{sc}, conformal changes result in multiplying the Laplacian with a factor, and the equation transforms to a linear elliptic equation in a punctered disc involving the euclidean Laplacian. Using interior $L^\infty - L^2-$estimates for the second fundamental form of Willmore surfaces, as proved in [KuSch 1], we obtain

$$\Delta H_{\mathrm{sc}} + q H_{\mathrm{sc}} = 0 \quad \text{in } B_1^2(0) - \{0\},$$

$$|y|^2 q(y) \to 0 \text{ for } x \to 0,$$

$$\int \sup_{|y|=\varrho} |q(y)| \varrho \, \mathrm{d}\varrho < \infty.$$

In Section 2, we investigate this equation by introducing polar coordinates (r, φ) combined with an exponential change of variable $r = e^{-t}$. As the resulting function is periodic in φ, we derive ordinary differential equations for its Fourier modes from which we are able to conclude decay for the higher Fourier modes for $t \to \infty$. This yields (1.1).

Knowing $C^{1,\alpha}-$regularity, we can expand the mean curvature

$$H(x) = H_0 \log |x| + C^{0,\alpha}_{\mathrm{loc}}$$

around the point singularity where H_0 are normal vectors at 0 which we call the residue. The point singularity can be removed completely to obtain an analytic surface if and only if the residue vanishes. Inspired by the Noether principle for minimal surfaces, we get a closed 1-form by calculating the first variation of the Willmore functional with respect to a constant Killing field and observe that the residue can be computed as the limit of the line integral around the point singularity of this 1-form. From this we conclude in Lemma 4.2 that the residues of a closed Willmore surface with finitely many point singularities of unit density add up to zero. As inverted blowups have at most one singularity at zero, inverted blowups are smooth provided this singularity has unit density.

The final section is devoted for applications of our general removability results. Here, we will always verify the unit density condition for the possible point singularities by considering surfaces with Willmore energy $< 8\pi$ via the Li-Yau inequality; see (A.17). The main importance of the argument in our applications is that we are able to exclude topological spheres as blowups. Indeed, by our removability results we know that the inversions of blowups are smooth and by Bryant's classification of Willmore spheres in codimension one in [Bry], the only Willmore spheres with energy less than 16π are the round spheres. Now round spheres are excluded as inversions of blowups, since blowups are nontrivial in the sense that they are not planes.

As application we mention

THEOREM 5.2. *Let $f_0 : S^2 \to \mathbb{R}^3$ be a smooth immersion of a sphere with Willmore energy*

$$\mathcal{W}(f_0) \leq 8\pi.$$

Then the Willmore flow with initial data f_0 exists smoothly for all times and converges to a round sphere.

Actually this improves the smallness assumption of Theorem 5.1 in [KuSch 1] to $\varepsilon_0 = 8\pi$. This constant is optimal, as a numerical example of a singularity recently obtained in [MaSi] indicates.

Further we mention the following compactness result for Willmore tori.

THEOREM 5.3. *The set*

$$\mathcal{M}_{1,\delta} := \{\Sigma \subseteq \mathbb{R}^3 \text{ Willmore} \mid \text{genus}(\Sigma) = 1, \mathcal{W}(\Sigma) \leq 8\pi - \delta\ \}$$

is compact up to Möbius transformations under smooth convergence of compactly contained surfaces in $\mathbb{R}^3$.

2. Power-decay

We consider $\Omega := B_1^2(0) - \{0\} \subseteq \mathbb{R}^2, v \in C^\infty(\Omega), A$ measurable on Ω which satisfy

$$|\Delta v| \leq |A|^2\ |v| \quad \text{in } \Omega, \tag{2.1}$$

$$|v| \leq C|A| \quad \text{in } \Omega, \tag{2.2}$$

$$\parallel A \parallel_{L^\infty(B_\varrho)} \leq C\varrho^{-1} \parallel A \parallel_{L^2(B_{2\varrho})} \quad \text{for} \quad B_{2\varrho} \subseteq \Omega, \tag{2.3}$$

$$\int\limits_\Omega |A|^2 < \infty. \tag{2.4}$$

LEMMA 2.1 (Power-decay-lemma). *Under the assumptions* (2.1)–(2.4), $\forall\ \varepsilon > 0$, $\exists\ C_\varepsilon < \infty$, $\forall\ 0 < \varrho \leq 1$,

$$\int\limits_{B_\varrho(0)} |v|^2 \leq C_\varepsilon \varrho^{2-\varepsilon}. \tag{2.5}$$

Remark. From (2.1)–(2.4), we can conclude

$$\begin{aligned} \Delta v + qv = 0 \quad &\text{in } B_1^2(0) - \{0\}, \\ |y|^2 q(y) \to 0 \quad &\text{for } y \to 0. \end{aligned} \tag{2.6}$$

In [Sim 3] equations with this asymptotics were investigated, and Lemma 1.4 in [Sim 3] yields

$$\varrho^{-1} \parallel v \parallel_{L^2(B_\varrho(0)-B_{\varrho/2}(0))}= O(\varrho^{k+\varepsilon}) \Longrightarrow \varrho^{-1} \parallel v \parallel_{L^2(B_\varrho(0)-B_{\varrho/2}(0))}= O(\varrho^{k+1-\varepsilon})$$

for all $k \in \mathbb{Z}, \varepsilon > 0$. From (2.2) we only get $v(y) = o(|y|^{-1})$ which does not suffice to obtain the conclusion (2.5) from (2.6) as the example

$$v(y) = v(r(\cos\varphi, \sin\varphi)) := \frac{1}{r\log(2/r)}\cos\varphi$$

shows. For the proof of the power-decay-lemma it is decisive to observe that

$$\int_0^{1/2} \sup_{|y|=\varrho} |q(y)|\varrho \, \mathrm{d}\varrho < \infty$$

by (2.3) and (2.4), which yields integrability in Proposition 2.2 and (2.14) below.

We reformulate the problem by putting, for $0 < t < \infty$,

$$\begin{aligned} u(t,\varphi) &:= v(e^{-t+i\varphi}), \\ \omega(t,\varphi) &:= e^{-2t} \ |A(e^{-t+i\varphi})|^2. \end{aligned}$$

Introducing polar coordinates and $r = e^{-t}$, that is,

$$\begin{aligned} \tilde{v}(r,\varphi) &= v(re^{i\varphi}), \\ u(t,\varphi) &= \tilde{v}(e^{-t},\varphi), \end{aligned}$$

we calculate $\partial_t = -r\partial_r$ and

$$\Delta v = \frac{1}{r}\partial_r(r\partial_r\tilde{v}) + \frac{1}{r^2}\partial_\varphi^2\tilde{v} = \frac{1}{r^2}(\partial_t^2 u + \partial_\varphi^2 u) = e^{2t}\Delta u;$$

hence by (2.1)

$$|\Delta u| = e^{-2t}|\Delta v| \le e^{-2t}|A|^2|v| = |\omega u| \quad \text{in } \mathbb{R}_+ \times \mathbb{R}. \tag{2.7}$$

From (2.2)–(2.4), we see for $\varrho = e^{-t}$ that

$$\begin{aligned} \sup_\varphi |\omega(t,\varphi)| &\le \varrho^2 \parallel A \parallel^2_{L^\infty(\partial B_\varrho)} \\ &\le C \parallel A \parallel^2_{L^2(B_{2\varrho})} \to 0 \quad \text{for } \varrho \to 0, \text{ that is } t \to \infty. \end{aligned} \tag{2.8}$$

Then (2.2) yields

$$\sup_\varphi |e^{-t}u(t,\varphi)| \le C\varrho \parallel A \parallel_{L^\infty(\partial B_\varrho)} \to 0 \quad \text{for } \varrho \to 0, \text{ that is } t \to \infty. \tag{2.9}$$

The next proposition gives an integral bound on the supremum in (2.8).

Proposition 2.2.

$$\int_{t_0}^{\infty} \sup_{\varphi} |\omega(t,\varphi)| \, \mathrm{d}t < \infty \quad \forall t_0 > 0. \tag{2.10}$$

Proof. We calculate, using (2.2) and (2.3), that

$$\int_{\log 2}^{\infty} \sup_{\varphi} |\omega(t,\varphi)| \, \mathrm{d}t = \int_{0}^{1/2} \sup_{\varphi} |\omega(\log \frac{1}{\varrho}, \varphi)| \varrho^{-1} \, \mathrm{d}\varrho \leq \int_{0}^{1/2} \varrho \parallel A \parallel^2_{L^\infty(\partial B_\varrho)} \, \mathrm{d}\varrho$$

$$\leq \int_{0}^{1/2} C \varrho^{-1} \parallel A \parallel^2_{L^2(B_{2\varrho} - B_{\varrho/2})} \, \mathrm{d}\varrho = C \int_{0}^{1/2} \int_{\varrho/2}^{2\varrho} \int_{\partial B_1} |A(r\omega)|^2 r \varrho^{-1} \, \mathrm{d}\mathcal{H}^1(\omega) \, \mathrm{d}r \, \mathrm{d}\varrho$$

$$\leq C \int_{0}^{1} \int_{\partial B_1} \int_{r/2}^{2r} |A(r\omega)|^2 \, \mathrm{d}\varrho \, \mathrm{d}\mathcal{H}^1(\omega) \, \mathrm{d}r$$

$$\leq C \int_{0}^{1} \int_{\partial B_1} |A(r\omega)|^2 r \, \mathrm{d}\mathcal{H}^1(\omega) \, \mathrm{d}r = C \int_{B_1} |A|^2 < \infty$$

by (2.4). □

The power-decay-lemma is an easy consequence of the following PDE-lemma and (2.7) to (2.10).

Lemma 2.3 (PDE-lemma). *Let $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ be periodic,*

$$u(t, \varphi + 2\pi) = u(t, \varphi),$$

and $\omega \geq 0$ measurable on $\mathbb{R}_+$ satisfying

$$|\Delta u| \leq \omega |u| \quad \text{in } \mathbb{R}_+ \times \mathbb{R}, \tag{2.11}$$

$$\sup_{\varphi} |e^{-t} u(t,\varphi)| \to 0 \quad \text{for } t \to \infty, \tag{2.12}$$

$$\omega(t) \to 0 \quad \text{for } t \to \infty, \tag{2.13}$$

$$\int_{0}^{\infty} \omega(t) \, \mathrm{d}t < \infty. \tag{2.14}$$

Then for any $\varepsilon > 0$

$$\lim_{t \to \infty} e^{-\varepsilon t} \parallel u(t,.) \parallel_{L^2(0,2\pi)} = 0. \tag{2.15}$$

Proof that the (PDE-*lemma* $\Rightarrow$ *power-decay-lemma*). From (2.7) to (2.10), we see that $u(.+t_0,.), \sup_\varphi |\omega(.+t_0,\varphi)|$ satisfy (2.11) to (2.14). Then (2.15) yields

$$\int_{B_\varrho} |v|^2 = \int_0^\varrho \int_0^{2\pi} |v(re^{i\varphi})|^2 r \,\mathrm{d}\varphi\,\mathrm{d}r = \int_{\log(1/\varrho)}^\infty \int_0^{2\pi} |u(t,\varphi)|^2 e^{-2t}\,\mathrm{d}\varphi\,\mathrm{d}t$$

$$\leq C_\varepsilon \int_{\log(1/\varrho)}^\infty e^{-(2-\varepsilon)t}\,\mathrm{d}t \leq \Big[C_\varepsilon (2-\varepsilon)^{-1}\ e^{-(2-\varepsilon)t}\Big]_{\log(1/\varrho)}^\infty = C_\varepsilon \varrho^{2-\varepsilon}$$

which is (2.5). □

To prove the PDE-lemma, we carry out a Fourier-transform. We put, for $k \in \mathbb{Z}$,

$$u_k(t) := \frac{1}{2\pi} \int_0^{2\pi} u(t,\varphi) e^{-ik\varphi}\,\mathrm{d}\varphi.$$

Clearly

$$u_k \in C^\infty([0,\infty[),$$
$$u(t,\varphi) = \sum_{k\in\mathbb{Z}} u_k(t) e^{ik\varphi},$$
$$\frac{1}{2\pi} \parallel u(t,.) \parallel^2_{L^2(0,2\pi)} = \sum_{k\in\mathbb{Z}} |u_k(t)|^2.$$

Further,

$$\Delta u = \sum_{k\in\mathbb{Z}} (u_k'' - k^2 u_k) e^{ik\cdot},$$

and (2.11) implies

$$\sum_{k\in\mathbb{Z}} |u_k'' - k^2 u_k|^2 \leq \frac{1}{2\pi} \parallel \omega u \parallel^2_{L^2(0,2\pi)} = \omega^2 \sum_{k\in\mathbb{Z}} |u_k|^2. \tag{2.16}$$

For $m \in \mathbb{N}_0, 0 < \delta \leq 1$, we put

$$J_m := \sum_{|k|\geq m} |u_k|^2,$$
$$I_m := \sum_{|k|\leq m} |u_k|^2,$$
$$a_m^\delta := \sum_{|k|\leq m} \Big(\delta^2 |u_k|^2 + |u_k'|^2\Big).$$

Denoting the real part by Re, we calculate

$$J'_m = \sum_{|k|\geq m} (u_k \bar{u}'_k + \bar{u}_k u'_k) = \mathrm{Re} \sum_{|k|\geq m} 2u_k \bar{u}'_k,$$
$$J''_m = \sum_{|k|\geq m} \Big(2|u'_k|^2 + \mathrm{Re}(2u_k \bar{u}''_k) \Big).$$

Then (2.16) yields

(2.17)
$$\begin{aligned} J''_m &\geq \mathrm{Re} \sum_{|k|\geq m} 2u_k \Big(k^2 \bar{u}_k + (\bar{u}''_k - k^2 \bar{u}_k) \Big) \geq 2m^2 J_m - 2\omega J_m^{1/2} J_0^{1/2} \\ &= 2m^2 J_m - 2\omega J_m^{1/2} (I_{m-1} + J_m)^{1/2} \geq 2m^2 J_m - 2\omega J_m^{1/2} I_{m-1}^{1/2} - 2\omega J_m \\ &\geq 2(m^2 - \omega) J_m - 2\omega J_m^{1/2} I_{m-1}^{1/2}. \end{aligned}$$

Next,

(2.18)
$$\begin{aligned} \big|(a_m^\delta)'\big| &= \Big| \mathrm{Re} \sum_{|k|\leq m} 2(\delta^2 u_k + u''_k) \bar{u}'_k \Big| \\ &= \Big| \mathrm{Re}\ 2 \sum_{|k|\leq m} \Big[(k^2 + \delta^2) u_k + (u''_k - k^2 u_k) \Big] \bar{u}'_k \Big| \\ &\leq 2 \Big(\sum_{|k|\leq m} |u'_k|^2 \Big)^{1/2} \Big[(m^2 + \delta^2) \Big(\sum_{|k|\leq m} |u_k|^2 \Big)^{1/2} + \omega J_0^{1/2} \Big] \\ &\leq 2(m^2 + \delta^2 + \omega) \Big(\sum_{|k|\leq m} |u_k|^2 \Big)^{1/2} \Big(\sum_{|k|\leq m} |u'_k|^2 \Big)^{1/2} \\ &\quad + 2 \Big(\sum_{|k|\leq m} |u'_k|^2 \Big)^{1/2} \omega J_{m+1}^{1/2} \\ &\leq (m^2 + \delta^2 + \omega) \Big(\delta \sum_{|k|\leq m} |u_k|^2 + \delta^{-1} \sum_{|k|\leq m} |u'_k|^2 \Big) + 2\omega (a_m^\delta)^{1/2} J_{m+1}^{1/2} \\ &\leq (m^2 + \delta^2 + \omega) \delta^{-1} a_m^\delta + 2\omega (a_m^\delta)^{1/2} J_{m+1}^{1/2}. \end{aligned}$$

For $m = 0$,

$$|(a_0^\delta)'| \leq (\delta + \delta^{-1}\omega) a_0^\delta + 2\omega (a_0^\delta)^{1/2} J_1^{1/2}. \tag{2.19}$$

For $m = 1$ and $a_1 = a_1^1$,

$$|a'_1| \leq (2 + \omega) a_1 + 2\omega (a_1)^{1/2} J_2^{1/2}. \tag{2.20}$$

To proceed we need the following ODE-lemma.

LEMMA 2.4 (ODE-lemma). *Let* $J, a \in C^\infty([0,\infty[)$, $\omega \in L^1(0,\infty)$, $J, a, \omega \geq 0$, $J + a \not\equiv 0$ *on* $[t,\infty[$ *for some large* t *and* $0 < q < p$ *satisfy*

$$\begin{aligned} J'' &\geq (p^2 - \omega)J - \omega J^{1/2}a^{1/2}, \\ |a'| &\leq (q+\omega)a + \omega J^{1/2}a^{1/2}, \\ \omega(t) &\to 0 \quad \textit{for } t \to \infty. \end{aligned} \tag{2.21}$$

Then

$$\textit{either } \lim_{t\to\infty} e^{-p_0 t}J(t) = \infty, \forall\, p_0 < p, \tag{2.22}$$

$\lim_{t\to\infty} \frac{a(t)}{J(t)} = 0$ *and*

$$\textit{or } \lim_{t\to\infty} e^{p_0 t}J(t) = 0, \forall\, p_0 < p, \tag{2.23}$$

or

$$\lim_{t\to\infty} \frac{a(t)}{J(t)} = \infty \quad \textit{and} \quad \limsup_{t\to\infty} e^{-qt}a(t) < \infty. \tag{2.24}$$

Proof. First, we fix $q < p_0 < p$ and consider $\mu \in\,]0,\infty[$ satisfying

$$\exists t_j \uparrow \infty : \mu^2 J(t_j) > a(t_j), J'(t_j) \geq -p_0 J(t_j), \tag{2.25}$$

and define

$$\mu_0 := \inf\{\mu \in\,]0,\infty[\text{ satisfying } (2.25)\}$$

where we set $\inf\emptyset := +\infty$.

Let $\mu_0 < \mu < \infty$ and choose $p_0 < \tilde{p} < p$ and $1 < \Gamma = \Gamma(p_0,\tilde{p})$ large below. We fix j large and put

$$T := \inf\{t \in [t_j,\infty[\ |\ \Gamma^2\mu^2 J(t) \leq a(t)\ \} \in\,]t_j,\infty],$$

where we observe $\Gamma^2\mu^2 J(t_j) \geq \mu^2 J(t_j) > a(t_j)$ since $J \geq 0, \Gamma \geq 1$. Then

$$\Gamma^2\mu^2 J > a \quad \text{on } [t_j, T[; \tag{2.26}$$

hence by (2.21)

$$J'' \geq (p^2 - \omega(1+\Gamma\mu))J \quad \text{on } [t_j, T[.$$

For t_j large enough depending on $\mu, p_0, \tilde{p}, p$ and ω, we see

$$J'' \geq \tilde{p}^2 J \quad \text{on } [t_j, T[. \tag{2.27}$$

We calculate

$$(e^{p_0 t}J)' = e^{p_0 t}(J' + p_0 J)$$

and by (2.27)

$$(e^{p_0 t}J)'' = e^{p_0 t}(J'' + 2p_0 J' + p_0^2 J) \geq 2p_0 e^{p_0 t}(J' + p_0 J) = 2p_0(e^{p_0 t}J)' \quad \text{on } [t_j, T[.$$

By (2.25), we know

$$(e^{p_0 t} J)'_{t=t_j} \geq 0;$$

hence

$$J' \geq -p_0 J \quad \text{on } [t_j, T[. \tag{2.28}$$

Now, (2.27), (2.28) yield for $t' < t'' \in [t_j, T[$ that

$$\begin{aligned} J(t'') &\geq J(t') \cosh \tilde{p}(t'' - t') + \frac{J'(t')}{\tilde{p}} \sinh \tilde{p}(t'' - t') \\ &\geq J(t') \big[\cosh \tilde{p}(t'' - t') - \frac{p_0}{\tilde{p}} \sinh \tilde{p}(t'' - t') \big] \\ &\geq J(t')(1 - \frac{p_0}{\tilde{p}}) \cosh \tilde{p}(t'' - t') \geq \frac{\tilde{p} - p_0}{2\tilde{p}} J(t') e^{\tilde{p}(t'' - t')}. \end{aligned} \tag{2.29}$$

We claim

$$T = \infty. \tag{2.30}$$

Indeed if $T < \infty$, we see from (2.29) that

$$J(T) > 0,$$

since $\mu^2 J(t_j) > a(t_j) \geq 0$, and

$$\mu^2 J(T) < \Gamma^2 \mu^2 J(T) = a(T). \tag{2.31}$$

We put

$$t' := \sup\{ t \in [t_j, T[\mid \mu^2 J(t) \geq a(t) \} \in [t_j, T[.$$

Next,

$$\mu^2 J(t') = a(t'). \tag{2.32}$$

By (2.26),

$$\mu^2 J \leq a \leq \Gamma^2 \mu^2 J \quad \text{on } [t', T[.$$

From (2.21), we calculate

$$a' \leq (q + \omega(1 + \mu^{-1}))a \text{ on } [t', T[.$$

Hence by (2.29), (2.31), (2.32),

$$\begin{aligned} 0 < \Gamma^2 \mu^2 J(T) = a(T) &\leq a(t') \exp\Big(\int_{t_j}^{\infty} \omega(1 + \mu^{-1}) \Big)\ e^{q(T - t')} \\ &= \mu^2 J(t')\ \exp\Big((1 + \mu^{-1}) \int_{t_j}^{\infty} \omega \Big)\ e^{q(T - t')} \\ &\leq \frac{2\tilde{p}}{\tilde{p} - p_0} \exp\Big((1 + \mu^{-1}) \int_{t_j}^{\infty} \omega \Big)\ \exp\Big((q - \tilde{p})(T - t') \Big)\ \mu^2 J(T). \end{aligned}$$

Since,

$$\Gamma = \Gamma(p_0, \tilde{p}) > \sqrt{\frac{2\tilde{p}}{\tilde{p} - p_0}} \geq \sqrt{2} > 1,$$

this is impossible for t_j large as $J(T) > 0$ and

$$\lim_{t \to \infty} \int_t^\infty \omega = 0.$$

This proves (2.30).

Therefore by (2.26)

$$a < \Gamma^2 \mu^2 J \quad \text{on } [t_j, \infty[;$$

hence

$$\limsup_{t \to \infty} \frac{a}{J} \leq \Gamma^2 \mu^2$$

and by (2.29)

$$\lim_{t \to \infty} e^{-p_0 t} J(t) = \infty.$$

By definition of μ_0, this yields

$$\limsup_{t \to \infty} \frac{a}{J} \leq \Gamma^2 \mu_0^2 \tag{2.33}$$

and

$$\lim_{t \to \infty} e^{-p_0 t} J(t) = \infty \quad \text{if } \mu_0 < \infty. \tag{2.34}$$

Now we consider

$$0 < \mu < \mu_0.$$

By definition of μ_0,

$$\mu^2 J(t) > a(t) \Rightarrow J'(t) \leq -p_0 J(t) \quad \text{for large } t. \tag{2.35}$$

Therefore by (2.21)

$$\max(\mu^2 J, a)' \leq a' \chi_{[\mu^2 J \leq a]} \leq (q + \omega(1 + \mu^{-1})) \max(\mu^2 J, a) \quad \text{for large } t$$

and

$$\max(\mu^2 J, a) > 0 \quad \text{for large } t,$$

since $J + a \not\equiv 0$ for large t by assumption. If $0 < \log \frac{\mu^2 J}{a} < \infty$, we conclude from (2.21) and (2.35) for large t that

$$\left(\log \frac{\mu^2 J}{a}\right)' = \frac{J'}{J} - \frac{a'}{a} \leq -p_0 + q + \omega + \omega \ \left(\frac{J}{a}\right)^{1/2}.$$

We infer for $\Lambda > 0$ that $\min((\log \frac{\mu^2 J}{a})_+, \Lambda)$ is locally lipschitz and by (2.21) that

$$\min\left(\left(\log \frac{\mu^2 J}{a}\right)_+, \Lambda\right)' \leq 0 \quad \text{for large } t.$$

If

$$\liminf_{t\to\infty} \frac{J}{a} < \infty, \tag{2.36}$$

we choose

$$\log\left(\liminf_{t\to\infty} \frac{\mu^2 J}{a}\right)_+ < \Lambda < \infty$$

and see that

$$\left(\log \frac{\mu^2 J}{a}\right)_+ \leq \Lambda \quad \text{for large } t;$$

hence

$$\left(\log \frac{\mu^2 J}{a}\right)' \leq -\varepsilon$$

for some $\varepsilon > 0$, for large t, if $\log \frac{\mu^2 J}{a} > 0$. This implies

$$\mu^2 J \leq a \quad \text{for large } t. \tag{2.37}$$

Again from (2.21), we get

$$a' \leq (q + \omega(1+\mu^{-1}))a \quad \text{for large } t;$$

hence

$$a(t) \leq Ce^{qt}, \tag{2.38}$$

since $\int_0^\infty \omega < \infty$. From (2.36), (2.37),(2.38), we see that if

$$\limsup_{t\to\infty} \frac{a}{J} > 0 \tag{2.39}$$

then

$$\liminf_{t\to\infty} \frac{a}{J} \geq \mu_0^2, \tag{2.40}$$

and if further $\mu_0 > 0$ then

$$\limsup_{t\to\infty} e^{-qt} a(t) < \infty. \tag{2.41}$$

If $\mu_0 = 0$, then (2.22) is satisfied for the fixed p_0 by (2.33), (2.34).

If $0 < \mu_0 < \infty$, then by (2.34)

$$\lim_{t\to\infty} e^{-p_0 t} J(t) = \infty.$$

We claim that

$$\lim_{t\to\infty} \frac{a}{J} = 0$$

and hence (2.22) is satisfied for p_0.

Indeed if $\limsup_{t\to\infty} \frac{a}{J} > 0$, then we get from (2.41)

$$\limsup_{t\to\infty} e^{-qt}a(t) < \infty;$$

hence

$$\limsup_{t\to\infty} \frac{a}{J} \leq \limsup_{t\to\infty} e^{-qt}a(t) \ \limsup_{t\to\infty} \frac{1}{e^{-p_0 t}J(t)} \ \limsup_{t\to\infty} e^{(q-p_0)t} = 0.$$

If $\mu_0 = \infty$ and $\limsup_{t\to\infty} \frac{a}{J} > 0$, then (2.24) is satisfied by (2.40) and (2.41).

If $\mu_0 = \infty$ and $\lim_{t\to\infty} \frac{a}{J} = 0$, then

$$J(t) > a(t) \quad \text{for large } t.$$

As $\mu_0 = \infty$, (2.25) is not satisfied for $\mu = 1$; hence

$$J'(t) \leq -p_0 J(t) \quad \text{for large } t$$

which yields

$$\lim_{t\to\infty} e^{(p_0-\varepsilon)t}J(t) = 0 \quad \forall \varepsilon > 0.$$

Now for any $q < p_0 < p$, exactly one of the three statements (2.22), (2.23), (2.24) is satisfied. This implies (2.22)–(2.24) for any $q < \tilde{p} < p_0$; hence exactly one of the statements (2.22)–(2.24) is satisfied for all $q < p_0 < p$. □

Now we are ready to prove the PDE-lemma.

Proof of the PDE-*lemma*. We apply the ODE-lemma to $J = J_1$, $a = a_0^\delta$, $p = \sqrt{2}, q = \delta \leq 1$, by (2.13), (2.14), (2.17), (2.19). If $J_1 + a_0^\delta \equiv 0$ for large t, or (2.23) or (2.24) of the ODE-lemma is satisfied then we put $a_0 = a_0^1$,

$$J_0 \leq a_0 + J_1 \leq \delta^{-2}a_0^\delta + J_1 \leq C_\delta e^{\delta t},$$

which implies (2.15) as $J_0(t) = \frac{1}{2\pi} \parallel u(t,.) \parallel^2_{L^2(0,2\pi)}$. Therefore it suffices to consider that (2.22) of the ODE-lemma is satisfied; that is,

$$\lim_{t\to\infty} \frac{a_0(t)}{J_1(t)} = 0. \tag{2.42}$$

Next, we apply the ODE-lemma to $J = J_2, a = a_1, p = 2\sqrt{2} > 2 = q$ by (2.13), (2.14), (2.17), (2.20). From (2.12) we see that

$$J_2(t) \leq \frac{1}{2\pi} \parallel u(t,.) \parallel^2_{L^2(0,2\pi)} \leq Ce^{2t}.$$

Therefore (2.22) of the ODE-lemma is not satisfied. If $J_2 + a_1 \equiv 0$ for large t or (2.23) of the ODE-lemma is satisfied, then

$$J_0(t) \leq a_1(t) + J_2(t) \leq C$$

which implies (2.15).

Therefore it remains to consider that (2.24) of the ODE-lemma is satisfied; hence

$$\lim_{t\to\infty} \frac{a_1(t)}{J_2(t)} = \infty. \tag{2.43}$$

We put

$$b := \sum_{|k|=1} (|u_k|^2 + |u_k'|^2).$$

Clearly

$$a_1 = b + a_0 \quad \text{and} \quad J_1 \leq b + J_2.$$

From (2.42), (2.43), we see that

$$\frac{a_0}{b+J_2} \leq \frac{a_0}{J_1} \to 0, \qquad \frac{a_0+b}{J_2} \to \infty.$$

Therefore

$$\frac{a_0}{a_0+b} = \frac{a_0}{b+J_2}\,\frac{b+J_2}{a_0+b} \leq \frac{a_0}{b+J_2}\left(1+\frac{J_2}{a_0+b}\right) \to 0;$$

hence

$$\frac{b}{a_0} = \frac{a_0+b}{a_0} - 1 \to \infty. \tag{2.44}$$

Further

$$\frac{a_0+b}{b} = 1 + \frac{a_0}{b} \to 1;$$

hence

$$\frac{b}{J_2} = \frac{a_0+b}{J_2}\,\frac{b}{a_0+b} \to \infty. \tag{2.45}$$

This implies

$$\liminf_{t\to\infty} \frac{b}{J_0} \geq \lim_{t\to\infty} \frac{b}{a_0+b+J_2} = 1. \tag{2.46}$$

From (2.16) and (2.46), we conclude for $|k| = 1$ that

$$|u_k'' - u_k| \leq \frac{1}{2\pi} \parallel \omega \parallel_{L^2(0,2\pi)} \parallel u \parallel_{L^2(0,2\pi)} \leq C\omega b^{1/2} \tag{2.47}$$

and

$$b' = \operatorname{Re} \sum_{|k|=1} 2(u_k + u_k'')\bar{u}_k' = \operatorname{Re} \sum_{|k|=1} 4u_k\bar{u}_k' + \operatorname{Re} \sum_{|k|=1} 2(u_k'' - u_k)\bar{u}_k'.$$

Therefore

$$|b' - \operatorname{Re} 4u_k\bar{u}_k'| \leq C\omega b \tag{2.48}$$

and

$$b(t) \le b(0) \exp\Big(\int_0^\infty C\omega\Big) \exp\Big(\int_0^t \mathrm{Re}\frac{4u_k\bar{u}_k'}{b}\Big) \tag{2.49}$$

$$\le C \exp\Big(\int_0^t \mathrm{Re}\frac{4u_k\bar{u}_k'}{b}\Big).$$

Now,

$$c = \mathrm{Re} \sum_{|k|=1} 2u_k\bar{u}_k'$$

and we see that

$$|c| \le b. \tag{2.50}$$

We calculate

$$c' = 2|u_k'|^2 + \mathrm{Re}\ 2u_k\bar{u}_k'' = 2(|u_k'|^2 + |u_k|^2) + \mathrm{Re}\ 2u_k(\bar{u}_k'' - \bar{u}_k);$$

hence by (2.47)

$$|c' - 2b| \le C\omega b \tag{2.51}$$

and (2.48) is rewritten

$$|b' - 2c| \le C\omega b. \tag{2.52}$$

Now, (2.49) shows

$$b(t) \le C \exp(2 \int_0^t \frac{c}{b}). \tag{2.53}$$

Next, using (2.48), (2.50) and (2.51), we get

$$\Big(\frac{c}{b}\Big)' = \frac{c'b - cb'}{b^2} = \frac{2b^2 + (c' - 2b)b - c(b' - 2c) - 2c^2}{b^2}$$
$$\ge 2 - C\omega - C\omega\Big|\frac{c}{b}\Big| - 2\Big|\frac{c}{b}\Big|^2 \ge -C\omega.$$

This yields

$$\inf_{t\in[t_0,\infty[} \frac{c}{b}(t) \ge \frac{c}{b}(t_0) - \int_{t_0}^\infty C\omega$$

and

$$\liminf_{t\to\infty} \frac{c}{b}(t) \ge \limsup_{t\to\infty} \frac{c}{b}(t),$$

since

$$\int_0^\infty \omega < \infty.$$

This means that
$$\alpha := \lim_{t\to\infty} \frac{c}{b}(t) \in [-1,1]$$
exists. We claim
$$\alpha \leq 0. \tag{2.54}$$
Indeed if $\alpha > 0$ then
$$c \geq \frac{\alpha}{2} b > 0 \text{ for large } t.$$
We put
$$\gamma := b + c \geq b > 0 \text{ for large } t$$
and see by (2.51) and (2.52) that
$$\gamma' = b' + c' = (b' - 2c) + (c' - 2b) + 2(c+b) \geq 2\gamma - C\omega b \geq (2 - C\omega)\gamma.$$
Hence
$$2b(t) \geq \gamma(t) \geq \gamma(0) \exp\Big(- C \int_0^\infty \omega \Big) e^{2t} \geq c_0 e^{2t}. \tag{2.55}$$
From (2.12), we know
$$\limsup_{t\to\infty} e^{-2t} \sum_{|k|=1} |u_k(t)|^2 \leq \limsup_{t\to\infty} e^{-2t} J_0(t) = 0;$$
hence by (2.55)
$$\liminf_{t\to\infty} \sum_{|k|=1} |u_k'(t)|^2 e^{-2t} = \liminf_{t\to\infty} b(t) e^{-2t} > 0x$$
and
$$\lim_{t\to\infty} \frac{\sum_{|k|=1} |u_k|^2}{\sum_{|k|=1} |u_k'|^2} = 0.$$
Then
$$0 < \alpha = \lim_{t\to\infty} \frac{c}{b}(t) = \lim_{t\to\infty} \frac{\operatorname{Re} \sum_{|k|=1} 2 u_k \bar{u}_k'(t)}{\sum_{|k|=1} (|u_k(t)|^2 + |u_k'(t)|^2)}$$
$$\leq \lim_{t\to\infty} \frac{\sum_{|k|=1} (\varepsilon^{-1} |u_k|^2 + \varepsilon |u_k'|^2)}{\sum_{|k|=1} (|u_k|^2 + |u_k'|^2)} \leq \varepsilon,$$
which is a contradiction and (2.54) is proved.

From (2.54), we conclude for any $\varepsilon > 0$ that

$$\lim_{t\to\infty}\Big(\int_0^t 2\frac{c}{b} - \varepsilon t\Big) = -\infty;$$

hence by (2.53)

$$\limsup_{t\to\infty} e^{-\varepsilon t} b(t) \leq \limsup_{t\to\infty} C \exp\Big(\int_0^t 2\frac{c}{b} - \varepsilon t\Big) = 0.$$

From (2.46), we get

$$\limsup_{t\to\infty} e^{-\varepsilon t} \parallel u(t,\,) \parallel^2_{L^2(0,2\pi)} = 2\pi \limsup_{t\to\infty} e^{-\varepsilon t} J_0(t) = 0$$

which implies (2.15). $\square$

3. $C^{1,\alpha}$-regularity for point singularities

Let Σ be an open surface and $f : \Sigma \to \mathbb{R}^3$ be a smooth immersion with pull-back metric $g = f^* g_{\mathrm{euc}}$ and induced area-measure μ_g. Its image as varifold is given by

$$\mu := f(\mu_g) = (x \mapsto \mathcal{H}^0(f^{-1}(x)))\, \mathcal{H}^2 \lfloor f(\Sigma)$$

which is an integral 2-varifold in $\mathbb{R}^3$; see [Sim 1, §15], if μ is locally finite, for example, when Σ is closed.

LEMMA 3.1. *Let Σ be an open surface and $f : \Sigma \to \mathbb{R}^3$ be a smooth Willmore immersion that satisfies*

$$0 \in \mathrm{spt}\ \mu, \tag{3.1}$$

$$\theta^2_*(\mu, 0) < 2, \tag{3.2}$$

(3.3) *where μ has square integrable weak mean curvature in $B_\delta(0) - \{0\}$ for some $\delta > 0$,*

$$\int_\Sigma |A|^2 \,\mathrm{d}\mu_g < \infty. \tag{3.4}$$

Then μ is a $C^{1,\alpha}$-embedded, unit density surface at 0 for all $0 < \alpha < 1$, and the second fundamental form A satisfies the estimate

$$|A(x)| \leq C_\varepsilon |x|^{-\varepsilon} \quad \forall \varepsilon > 0. \tag{3.5}$$

Proof. By (3.2), (3.3), (A.1) and (A.2), we see that

$$\mu \text{ has square integrable weak mean curvature in } B_\delta(0). \tag{3.6}$$

From (3.1), (3.2), (A.7) and (A.10), we get

$$1 \leq \theta^2(\mu, 0) < 2. \tag{3.7}$$

Hence by (3.6), we see from [Sim 1, §42] that tangent cones exist; that is,

$$\mu_{\varrho_m} := \zeta_{\varrho_m,\#}\mu \to \mu_C,$$

where $\zeta_\varrho(x) := \varrho^{-1}x$, converge for subsequences $\varrho_m \downarrow 0$ weakly as varifolds to stationary, integral cones C, depending on the subsequence, with

$$\frac{\mu_C(B^3_\varrho(x))}{\omega_2 \varrho^2} \leq \theta^2(\mu_C, 0) = \theta^2(\mu, 0) < 2 \quad \text{for all } x \in \mathbb{R}^3. \tag{3.8}$$

Invoking [KuSch 1, Th. 2.10], as f is a Willmore immersion and by (3.4), we obtain that also the convergence $\mu_{\varrho_m} \to \mu_C$ is smooth in compact subsets of $\mathbb{R}^3 - \{0\}$ and $A_C = 0$ in $\mathbb{R}^3 - \{0\}$. Hence C is a union of integral planes and, by (3.8), C is a single density plane through 0 and $\theta^2(\mu, 0) = 1$.

Further spt μ is a smooth graph over some plane in $B^3_\varrho(0) - B^3_{\varrho/2}(0)$ for small ϱ, and hence it is a smooth embedded, unit-density Willmore surface in $B^3_\delta(0) - \{0\}$ for δ small enough which is diffeomorphic to an annulus

$$\text{spt } \mu \cap (B^3_\delta(0) - \{0\}) \cong B^2_1(0) - \{0\}.$$

Since the conclusion of the lemma is local near 0, we can identify Σ with its image and modify Σ and f outside $B^3_\delta(0)$ so that Σ is a smooth, embedded surface in $\mathbb{R}^3 - \{0\}$ which is Willmore in $B^3_\delta(0) - \{0\}$ and can be parametrised by

$$f : \mathbb{R}^2 \to \Sigma \subseteq \mathbb{R}^3 - \{0\}$$

such that $f(y) \to 0$ for $y \to \infty$.

We consider the inversion $I(x) := |x|^{-2}\, x$, which is a conformal diffeomorphism with conform factor $\lambda(x)^2 := |\partial_i I(x)|^2 = |x|^{-4}$ on $\mathbb{R}^3 - \{0\}$, put $\bar{f} := I \circ f, \bar{\Sigma} = I(\Sigma), \bar{\mu} := \mathcal{H}^2 \lfloor \bar{\Sigma}$ and consider the pull-back metric

$$\bar{g} := \bar{f}^* g_{\text{euc}} = (\lambda^2 \circ f) f^* g_{\text{euc}} = (\lambda^2 \circ f) g.$$

$\bar{\Sigma}$ is a smooth, complete surface in $\mathbb{R}^3$.

Now we use the conformal invariance of the Willmore functional; more precisely this means that $|A^0|^2 \mu_g$, where A^0 denotes the trace-free second fundamental, remains invariant under conformal changes of the ambient metric; see [Ch]. This yields, by (3.4),

$$\int_{\bar{\Sigma}} |A^0_{\bar{\Sigma}}|^2 \, \mathrm{d}\bar{\mu} \leq \int_{\Sigma} |A_\Sigma|^2 \, \mathrm{d}\mu < \infty. \tag{3.9}$$

Next we abbreviate $\bar{\Sigma}_R := \bar{\Sigma} \cap B_R(0)$ for large R and see from Gauss-Bonnet's theorem that

$$\int\limits_{\bar{\Sigma}_R} K_{\bar{\Sigma}} \, \mathrm{d}\bar{\mu} + \int\limits_{\partial\bar{\Sigma}_R} \kappa_{\partial\bar{\Sigma}_R} \, \mathrm{d}\mathcal{H}^1 = 2\pi\chi(\bar{\Sigma}_R) = 2\pi,$$

where $K_{\bar{\Sigma}}$ and $\kappa_{\partial\bar{\Sigma}_R}$ denote the Gaussian- and geodesic curvature on $\bar{\Sigma}$ and $\partial\bar{\Sigma}_R$. By smooth convergence for subsequences around $R^{-1}\partial\bar{\Sigma}_R$ to flat annuli, we see

$$\lim_{R\to\infty} \int\limits_{\partial\bar{\Sigma}_R} \kappa_{\partial\bar{\Sigma}_R} \, \mathrm{d}\mathcal{H}^1 = 2\pi$$

and obtain

$$\lim_{R\to\infty} \int\limits_{\bar{\Sigma}_R} K_{\bar{\Sigma}} \, \mathrm{d}\bar{\mu} = 0.$$

As

$$|A^0|^2 = \frac{1}{2}|H|^2 - 2K = |A|^2 - \frac{1}{2}|H|^2,$$

we see, using (3.9) first, that $H_{\bar{\Sigma}} \in L^2(\bar{\mu})$; then

$$\int\limits_{\bar{\Sigma}} |A_{\bar{\Sigma}}|^2 \, \mathrm{d}\bar{\mu} < \infty, \tag{3.10}$$

$K \in L^1(\bar{\mu})$, and

$$\int\limits_{\bar{\Sigma}} K_{\bar{\Sigma}} \, \mathrm{d}\bar{\mu} = 0. \tag{3.11}$$

Now $\bar{\Sigma}$ is a simply connected, complete, noncompact, oriented surface embedded in $\mathbb{R}^3$ with square integrable second fundamental form. By a theorem of Huber, see [Hu], it is conformally equivalent to $\mathbb{C} = \mathbb{R}^2$, say

$$\hat{f} : \mathbb{R}^2 \xrightarrow{\cong} \bar{\Sigma} \subseteq \mathbb{R}^3$$

with conformal factor $|\partial_i \hat{f}|^2 = e^{2\hat{u}}$. Taking (3.11) into account, more precise information is given in [MuSv, Th. 4.2.1 and Cor 4.2.5] which yield that $\bar{\Sigma}$ has a single end with multiplicity one, that is,

$$\hat{u} \in L^\infty(\mathbb{R}^2), \tag{3.12}$$

$$\lim_{y\to\infty} \frac{|\hat{f}(y)|}{|y|} \in]0, \infty[. \tag{3.13}$$

Composing $\hat{f}$ with I^{-1} and an inversion at 0 in $\mathbb{R}^2$, we get a conformal diffeomorphism $\tilde{f} : (\mathbb{R}^2 \cup \{\infty\}) - \{0\} \xrightarrow{\cong} \Sigma$ defined by

$$\tilde{f}(y) = (I^{-1} \circ \hat{f}) \left(\frac{y}{|y|^2} \right).$$

We calculate the conformal factor via the pull-back metric

$$\tilde g(y) = (\tilde f^* g_{\text{euc}})(y) = |y|^{-4}(\hat f^* |z|^{-4} g_{\text{euc}})\left(\frac{y}{|y|^2}\right)$$
$$= |y|^{-4} \hat f\left(\frac{y}{|y|^2}\right)^{-4} e^{2\hat u\left(\frac{y}{|y|^2}\right)} g_{\text{euc}} =: e^{2\tilde u(y)} g_{\text{euc}}$$

and see by (3.12) and (3.13) that it remains bounded as $y \to 0$. That is,

$$\tilde u \in L^\infty_{\text{loc}}(\mathbb{R}^2). \tag{3.14}$$

Further, by (3.13),

$$\lim_{y\to 0} \frac{|\tilde f(y)|}{|y|} = \lim_{y\to 0}\left(|y|\ |\hat f\left(\frac{y}{|y|^2}\right)|\right)^{-1} \in]0,\infty[; \tag{3.15}$$

in particular, there is $C < \infty$ such that

$$\Sigma \cap B^3_\varrho(0) \subseteq \tilde f(B^2_{C\varrho}(0)) \quad \text{for } \varrho > 0 \text{ small.} \tag{3.16}$$

Abbreviating, we delete the tildes and consider $\tilde f$ as our original embedding f. As f is a Willmore immersion near 0, say on $\Omega := B^2_1(0) - \{0\}$, it satisfies the Euler-Lagrange equation

$$\mathbf{W}(f) := \Delta_g H_{\text{sc}} + |A^0|^2 H_{\text{sc}} = 0 \quad \text{in } \Omega,$$

where H_{sc} denotes the scalar mean curvature and A^0 is again the trace-free second fundamental form, see [KuSch 1, (1.2)]. This is a linear, second order elliptic equation in the mean curvature H_{sc}. Since f is conformal, we can write this using the euclidean Laplace-operator in Ω:

$$\Delta H_{\text{sc}} + e^{2u}|A^0|^2 H_{\text{sc}} = 0 \quad \text{in } \Omega. \tag{3.17}$$

We want to apply the power-decay-Lemma 2.1 to $v = H_{\text{sc}}$. Clearly

$$|v|, e^u|A^0| \le C|A| \quad \text{in } \Omega,$$

and

$$A \in L^2(B^2_1(0)).$$

This verifies (2.1), (2.2) and (2.4). To verify (2.3), we use [KuSch 1, Th. 2.10, Rem. 2.11] after reparametrising so that $\int_\Omega |A|^2 \,\mathrm{d}\mu_g < \varepsilon_0(3)$. Since the euclidean distance in Ω and the intrinsic distance in $f(\Omega)$ compare by a bounded factor with (3.14) and $\mathbf{W}(f) = 0$, as f is a Willmore immersion, this yields

$$\| A \|_{L^\infty(B^2_\varrho)} \le C\varrho^{-1} \| A \|_{L^2(B^2_{2\varrho})} \quad \text{for any } B^2_{2\varrho} \subseteq \Omega. \tag{3.18}$$

This verifies (2.3), and the power-decay-Lemma 2.1 implies

$$\int_{B^2_\varrho(0)} |H_{\text{sc}}|^2 \,\mathrm{d}\mu_g \le C_\varepsilon \varrho^{2-\varepsilon} \quad \forall 0 < \varrho \le 1 : \forall \varepsilon > 0.$$

Using (3.16), we see

$$\int_{B^3_\varrho(0)} |H_\mu|^2 \, \mathrm{d}\mu \le \int_{B^2_{C\varrho}(0)} |H_{\mathrm{sc}}|^2 \, \mathrm{d}\mu_g \le C_\varepsilon \varrho^{2-\varepsilon} \quad \forall \varepsilon > 0. \tag{3.19}$$

Next we apply [Bra, Th. 5.6] in the version of the remark following its proof, recalling that μ has at least one tangent cone in 0 which is a single density plane, and obtain from (3.19) that for each $0 < \varrho < \delta$ there exists an unoriented 2-plane $T_\varrho \in G(3,2)$ such that

$$\text{height } ex_\mu(0,\varrho,T_\varrho) := \varrho^{-4} \int_{B^3_\varrho(0)} \mathrm{dist}(\xi, T_\varrho)^2 \, \mathrm{d}\mu(\xi) \le C_\varepsilon \varrho^{2-\varepsilon} \quad \forall \varepsilon > 0. \tag{3.20}$$

Using [Bra, Th. 5.5] or likewise [Sim 1, Lemma 22.2], we obtain again from (3.19) that

$$\text{tilt } ex_\mu(0,\varrho,T_\varrho) := \varrho^{-2} \int_{B^3_\varrho(0)} \| T_\xi\mu - T_\varrho \|^2 \; \mathrm{d}\mu(\xi) \le C_\varepsilon \varrho^{2-\varepsilon} \quad \forall \varepsilon > 0. \tag{3.21}$$

First we obtain from the densitiy bound (3.7) that

$$\| T_\varrho - T_{\varrho/2} \| \le C_\varepsilon \varrho^{1-\varepsilon} \quad \forall \varepsilon > 0;$$

hence $T_\varrho \to T_0$ and

$$\| T_\varrho - T_0 \| \le C_\varepsilon \varrho^{1-\varepsilon} \quad \forall \varepsilon > 0. \tag{3.22}$$

By (3.18), we see for $y', y'' \in B^2_{2\varrho}(0) - B^2_\varrho(0) \subseteq \Omega$ that

$$\| T_{f(y')}\mu - T_{f(y'')}\mu \| \le C|y' - y''| \, \| A \|_{L^\infty(B^2_{2\varrho}(0) - B^2_\varrho(0))} \le C \, \| A \|_{L^2(B^2_{3\varrho}(0))} \, .$$

Together with (3.22) this implies

$$\sup_{\xi \in B^3_\varrho(0) \cap \Sigma} \| T_\xi\mu - T_0 \| \to 0 \quad \text{for } \varrho \to 0; \tag{3.23}$$

hence for small enough $\varrho_0 > 0$, we see that μ, respectively Σ, can be written as a graph of a smooth function φ on $\overline{B^2_{\varrho_0}(0)} - \{0\}$ over the plane T_0. We infer from (3.13) and (3.23) that φ extends to a C^1−function on $B^2_{\varrho_0}(0)$ with $\varphi(0) = 0, D\varphi(0) = 0$ and by (3.20), (3.21) and (3.22)

$$\begin{aligned} \| \varphi \|_{L^2(B^2_\varrho(0))} &\le C_\varepsilon \varrho^{3-\varepsilon} \quad \forall \varepsilon > 0, \\ \| D\varphi \|_{L^2(B^2_\varrho(0))} &\le C_\varepsilon \varrho^{2-\varepsilon} \quad \forall \varepsilon > 0. \end{aligned} \tag{3.24}$$

Since $D\varphi$ is bounded, we get

$$|A_\mu(.,\varphi)| \le |D^2\varphi| \le C|A_\mu(.,\varphi)| \quad \text{in } B^2_{\varrho_0}(0) - \{0\}, \tag{3.25}$$

where A_μ denotes the second fundamental form on Σ. Therefore

$$\int\limits_{B^2_{\varrho_0}(0)} |D^2\varphi|^2 < \infty \tag{3.26}$$

and choosing a suitable cut-off function, we get by (3.24) that

$$\varphi \in W^{2,2}(B_{\varrho_0}(0)).$$

For the pull-pack metric $\bar{g} := (.,\varphi)^* g_{\mathrm{euc}}$, we see that

$$\partial_i(\bar{g}^{ij}\sqrt{\bar{g}}\ \partial_j\varphi) = \sqrt{\bar{g}}\ H_{3,\mu}(.,\varphi) =: h \quad \text{ weakly in } B^2_{\varrho_0}(0) - \{0\}$$

with

$$\| h \|_{L^2(B^2_\varrho(0))} \leq C_\varepsilon \varrho^{1-\varepsilon} \quad \forall \varepsilon > 0 \tag{3.27}$$

by (3.19). Putting $a_{ij}(D\varphi) := \bar{g}^{ij}\sqrt{\bar{g}}$ with $\bar{g}_{ij} = \bar{g}_{ij}(D\varphi) = \delta_{ij} + \partial_i\varphi\partial_j\varphi$, we calculate

$$a_{ij}(D\varphi)\partial_{ij}\varphi = h - \partial_{\partial_r\varphi} a_{ij}(D\varphi)\partial_j\varphi\partial_{ir}\varphi \quad \text{ in } B^2_{\varrho_0}(0);$$

hence

$$|a_{ij}(D\varphi)\partial_{ij}\varphi| \leq |h| + C|D\varphi|\ |D^2\varphi| \quad \text{ in } B^2_{\varrho_0}(0) \tag{3.28}$$

as $D\varphi(y)$ is bounded. Since $D\varphi$ is continuous and $D\varphi(0) = 0$, we obtain by Calderon-Zygmund estimates, (3.24), (3.26) and (3.27) that

$$\begin{aligned} &\| D^2\varphi \|_{L^2(B^2_\varrho(0))} \\ &\leq C\Big(\| h \|_{L^2(B^2_{2\varrho}(0))} + \| D\varphi \|_{L^\infty(B^2_{2\varrho}(0))}\ \| D^2\varphi \|_{L^2(B^2_{2\varrho}(0))} + \varrho^{-2} \| \varphi \|_{L^2(B^2_{2\varrho}(0))}\Big) \\ &\leq \tau \| D^2\varphi \|_{L^2(B^2_{2\varrho}(0))} + C_\varepsilon \varrho^{1-\varepsilon} \end{aligned}$$

for any $\tau, \varepsilon > 0$ and $0 < \varrho < \varrho_\tau$ small enough. Iterating, we get

$$\| D^2\varphi \|_{L^2(B^2_\varrho(0))} \leq C_\varepsilon \varrho^{1-\varepsilon} \quad \forall \varepsilon > 0.$$

Using (3.18) with extrinsic balls, see [KuSch 1, Th. 2.10], we get for any $x \neq 0$ with $\varrho := |x|/2$ small,

$$\begin{aligned} \| A_\mu \|_{L^\infty(B^3_\varrho(x))} &\leq C\varrho^{-1} \| A_\mu \|_{L^2(B^3_{2\varrho}(x))} \\ &\leq C\varrho^{-1} \| D^2\varphi \|_{L^2(B^2_{3\varrho}(0))} \leq C_\varepsilon \varrho^{-\varepsilon} \quad \forall \varepsilon > 0, \end{aligned}$$

which yields (3.5). This implies $A_\mu(.,\varphi) \in L^p(B_{\varrho_0}(0))$ for all $1 \leq p < \infty$; hence $\varphi \in W^{2,p}(B_{\varrho_0}(0))$ by (3.25) and finally $\varphi \in C^{1,\alpha}(B_{\varrho_0}(0))$ for all $0 < \alpha < 1$. $\square$

Remark. 1. The above lemma cannot be improved to get $C^{1,1}$-regularity. Indeed, the inverted catenoid is a Willmore surface as it is an inversion of a minimal surface. Like the catenoid, it has square integrable second fundamental form. It admits the parametrisation

$$f(t,\theta) = \frac{\cosh t}{\cosh(t)^2 + t^2}(\cos\theta, \sin\theta, 0) \pm \frac{t}{\cosh(t)^2 + t^2}\ e_3$$

and consists of two graphs near 0 which correspond to $\pm t > 0$. Therefore each of these graphs satisfies the assumptions of the lemma near 0. Writing $r = \sqrt{x^2 + y^2} = \frac{\cosh t}{\cosh(t)^2 + t^2}$, we see

$$\varphi(r) = \frac{\pm t}{\cosh(t)^2 + t^2} \approx \pm r^2 \log\frac{1}{r};$$

hence these graphs are not $C^{1,1}$ near 0.

2. If $\Sigma \subseteq \mathbb{R}^3$ is a smooth, embedded surface with

$$(\overline{\Sigma} - \Sigma) \cap B_\delta(0) = \{0\}$$

then (3.3) is immediately implied by (3.4).

3. If Σ is a closed surface, $p_0 \in \Sigma$ and $f : \Sigma - \{p_0\} \to \mathbb{R}^3$ is a smooth immersion which can continuously be extended on Σ and satisfies $\mathcal{W}(\Sigma) = \mathcal{W}(f) < 8\pi$ and $\theta^1_*(\mu, f(p_0)) = 0$, then by (A.2), we get $H_\mu \in L^2(\mu), \mathcal{W}(\mu) = \mathcal{W}(f) < 8\pi$ and obtain from the Li-Yau inequality (A.17)

$$\theta^2(\mu, f(p_0)) \leq \frac{1}{4\pi}\mathcal{W}(\mu) < 2.$$

4. Higher regularity for point singularities

Let Σ be an open surface and $f_t : \Sigma \to \mathbb{R}^n$ be a smooth family of immersions with

$$\partial_t f_t|_{t=0} = V =: N + Df.\xi$$

where $N \in N\Sigma$ is normal and $\xi \in T\Sigma$ is tangential. In [KuSch 2, §2], the first variation of the Willmore integrand with a different factor was calculated for normal variations $V = N$ to be

$$\begin{aligned} \partial_t\Big(\frac{1}{4}|H|^2\ \mathrm{d}\mu\Big) &= \frac{1}{2}\langle \Delta_g V + Q(A^0)V, H\rangle\ \mathrm{d}\mu \\ &= \frac{1}{2}\langle \Delta_g H + Q(A^0)H, N\rangle\ \mathrm{d}\mu \\ &\quad + \frac{1}{2}\nabla_{e_i}\Big(\langle \nabla_{e_i} N, H\rangle - \langle N, \nabla_{e_i} H\rangle\Big)\ \mathrm{d}\mu, \end{aligned} \tag{4.1}$$

where the Laplacian of the normal bundle along f is used, e_i is an orthonormal basis of $T\Sigma$ satisfying $\nabla e_i = 0$ in the point considered and

$$Q(A^0)H = A^0(e_i, e_j)\langle A^0(e_i, e_j), H\rangle = g^{ik}g^{jl}A^0_{ij}\langle A^0_{kl}, H\rangle. \tag{4.2}$$

For tangential variations $V = Df.\xi$, we consider the flow Φ_t of ξ, that is, $\Phi_0 = \mathrm{id}_\Sigma, \partial_t\Phi_t = \xi \circ \Phi_t$, and calculate for $t = 0$,

$$\begin{aligned}(4.3)\quad & \partial_t\Big(\frac{1}{4}|H_{f_t}|^2 \,\mathrm{d}\mu_{f_t}\Big)\\ &= \partial_t\Big(\frac{1}{4}|H_{f\circ\Phi_t}|^2 \,\mathrm{d}\mu_{f\circ\Phi_t}\Big) = \partial_t\Big(\frac{1}{4}|H\circ\Phi_t|^2\, \Phi_t^*(\mathrm{d}\mu)\Big)\\ &= \frac{1}{4}g(\mathrm{grad}_g|H|^2, \xi)\,\mathrm{d}\mu + \frac{1}{4}|H|^2\mathrm{div}_g(\xi)\,\mathrm{d}\mu = \mathrm{div}_g(\frac{1}{4}|H|^2\xi)\,\mathrm{d}\mu,\end{aligned}$$

where $\mathrm{grad}_g|H|^2 = g^{\cdot j}\partial_j|H|^2$ and $\mathrm{div}_g(\xi) := \sqrt{g}^{-1}\partial_i(\sqrt{g}\xi^i)$. Putting (4.1) and (4.4) together, we get

$$(4.4)\qquad \partial_t\Big(\frac{1}{4}|H|^2\,\mathrm{d}\mu\Big) = \frac{1}{2}\langle\Delta_g H + Q(A^0)H, N\rangle\,\mathrm{d}\mu + d\omega_V$$

where ω_V is the 1-form on Σ whose hodge with respect to g is given by

$$(4.5)\qquad (*\omega_V)(X) := \frac{1}{2}\langle\nabla_X N, H\rangle - \frac{1}{2}\langle N, \nabla_X H\rangle + \frac{1}{4}|H|^2 g(\xi, X).$$

Considering $V \equiv \mathrm{const} \in \mathbb{R}^n$ and a Willmore immersion $f : \Sigma \to \mathbb{R}^n$, we obtain for any open $\Omega \subseteq \Sigma$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_\Omega(f + tV) = \int_\Omega \mathrm{d}\omega_V;$$

hence ω_V is closed on Σ.

After these preliminary remarks, we turn to the following lemma.

Lemma 4.1. *Let $\Sigma = \mathrm{graph}\,\varphi$ be a $C^{1,\alpha}$-graph, $\varphi \in C^{1,\alpha}(B_1^2(0))$, $0 < \alpha < 1, \varphi(0) = 0$, in $\mathbb{R}^3$ with $\int |A|^2\,\mathrm{d}\mu_g < \infty$,*

$$(4.6)\qquad |A(x)| \le C_\varepsilon|x|^{-\varepsilon} \quad \forall\varepsilon > 0$$

and $\Sigma - \{0\}$ is a smooth Willmore surface.

Then there is the expansion

$$(4.7)\qquad H(x) = H_0\log|x| + C^{0,\alpha}_{\mathrm{loc}}, \quad \nabla H(x) = \frac{H_0 x^T}{|x|^2} + O(|x|^{\alpha-1}),$$

for some $H_0, h_0 \in N_0\Sigma \subseteq \mathbb{R}^3$, is called the residue

$$\mathrm{Res}_\Sigma(0) := H_0$$

of Σ at 0.

The residue can be calculated with the use of the closed 1-form ω_V on $\Sigma - \{0\}$ for any $V \in \mathbb{R}^3$ by

$$(4.8)\qquad \int_{\partial\Sigma_\varrho} \omega_V \to -\pi\langle V, \mathrm{Res}_\Sigma(0)\rangle \quad \text{for } \varrho \to 0,$$

where $\Sigma_\varrho := B^3_\varrho(0) \cap \Sigma$.

If $\mathrm{Res}_\Sigma(0) = 0$ then Σ is a smooth Willmore surface.

Proof. Since the induced metric of the chart $(y \mapsto (y, \varphi(y)))$ is $C^{0,\alpha}$, we get a conformal $C^{1,\alpha}-$parametrisation $f : B^2_{2\varrho}(0) \cong \Sigma \cap U(0)$ of Σ in a neighbourhood $U(0)$ of 0 with conformal factor $|\partial_i f|^2 =: e^{2u}$ by standard elliptic theory. Without loss of generality, we may assume $Df(0) = i : \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$.

Further let $\nu \in C^{0,\alpha}(B^2_{2\varrho}(0))$ be the normal, defined up to a sign. From the Weingarten equations $\langle \partial_i \nu, \partial_j f\rangle = -\langle \nu, A_{ij}\rangle$ and (4.6), we see

$$|\nabla \nu(y)| \leq C_\varepsilon |y|^{-\varepsilon} \quad \forall \varepsilon > 0. \tag{4.9}$$

Since $\Sigma - \{0\}$ is Willmore, we get the Euler-Lagrange equation (1.2)

$$\Delta_g H_{\mathrm{sc}} + |A^0|^2 H_{\mathrm{sc}} = 0 \quad \text{in } B^2_{2\varrho}(0) - \{0\}.$$

In the above conformal coordinates, this reads, by (4.2) and (4.6),

$$\Delta H_{\mathrm{sc}} = -e^{2u}|A^0|^2 H_{\mathrm{sc}} \in L^p(B^2_\varrho(0)) \quad \forall p < \infty,$$

where Δ denotes the euclidean Laplacian. Hence the solution of the Dirichlet problem

$$\Delta w = -e^{2u} Q(A^0) H \quad \text{in } B^2_\varrho(0), \quad w = 0 \quad \text{on } \partial B^2_\varrho(0) \tag{4.10}$$

lies in $w \in W^{2,p}(B^2_\varrho(0)) \hookrightarrow C^{1,\alpha}(B^2_\varrho(0))$.

We see that $H_{\mathrm{sc}} - w$ is harmonic in $B^2_\varrho(0) - \{0\}$, and as $|H_{\mathrm{sc}}(y) - w(y)| \leq C_\varepsilon |y|^{-\varepsilon}$, the only singular contribution can be a logarithm; hence

$$H_{\mathrm{sc}}(y) = a \log|y| + C^{1,\alpha}_{\mathrm{loc}}$$

for some $a \in \mathbb{R}$. As $H = H_{\mathrm{sc}}\nu, \nu \in C^{0,\alpha}$ and by (4.9), we get the expansion

$$H(y) = H_0 \log|y| + C^{0,\alpha}_{\mathrm{loc}}, \quad \nabla H(y) = \frac{H_0 y^T}{|y|^2} + O(|y|^{\alpha-1})$$

where clearly $H_0 = a\nu(0) \in N_0\Sigma$. Recall that $f \in C^{1,\alpha}$ and $Df(0) = i : \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$. Now $x = f(y) = y + O(|y|^{1+\alpha})$, and we arrive at (4.7).

When the residue H_0 vanishes, we see that $H - w$ is harmonic in $B^2_\varrho(0)$; hence $H \in C^{1,\alpha}_{\mathrm{loc}}(B^2_\varrho(0))$. In general, we see from the equation

$$\Delta f = e^{2u} H \quad \text{weakly in } B^2_{2\varrho}(0)$$

and the facts that $f \in C^{1,\alpha}(B^2_{2\varrho}(0))$ and $e^{2u} = |\partial_i f|^2$ that $H \in C^{k,\alpha}_{\mathrm{loc}}, k \geq 0$ implies $f \in C^{k+2,\alpha}_{\mathrm{loc}}$ and $A = \nabla^2 f \in C^{k,\alpha}_{\mathrm{loc}}$. This in turn yields $w \in C^{k+2,\alpha}_{\mathrm{loc}}$ and $H \in C^{k+2,\alpha}_{\mathrm{loc}}$. Then the bootstrap proceeds proving that f and Σ are smooth.

Finally, we calculate the residue with the help of ω_V. For $0 < \varrho \ll 1$ small, we see that $\Sigma_\varrho := B^3_\varrho(0) \cap \Sigma$ is a disk whose boundary $\partial\Sigma_\varrho = \partial B^3_\varrho(0) \cap \Sigma$ is a smooth curve converging when rescaled to a planar circle as $\Sigma \in C^{1,\alpha}$. More precisely, we get for the unit outward normal at $\partial\Sigma_\varrho$ in Σ

$$n_\varrho(x) = \frac{x}{|x|} + O(|x|^\alpha). \tag{4.11}$$

As $*n_\varrho$ is the positive oriented tangent of $\partial\Sigma_\varrho$,

$$\int\limits_{\partial\Sigma_\varrho} \omega_V = \int\limits_{\partial\Sigma_\varrho} \omega_V(*n_\varrho)\ \mathrm{d}\mathcal{H}^1 = \int\limits_{\partial\Sigma_\varrho} (*\omega_V)(n_\varrho)\ \mathrm{d}\mathcal{H}^1.$$

Decomposing $V =: N + \xi, N \in N\Sigma, \xi \in T\Sigma$, in normal and tangential components, we calculate the terms in the definition (4.5) using (4.6),

$$\Big| \int\limits_{\partial\Sigma_\varrho} \frac{1}{4}|H|^2 g(\xi, n_\varrho)\ \mathrm{d}\mathcal{H}^1 \Big| \leq C\varrho C_\varepsilon \varrho^{-\varepsilon} \to 0,$$

and

$$|\langle \nabla_{n_\varrho} N, H\rangle| = |\langle \nabla_{n_\varrho}(V - \xi), H\rangle| = |\langle A(\xi, n_\varrho), H\rangle| \leq C_\varepsilon \varrho^{-\varepsilon}.$$

Hence

$$\Big| \int\limits_{\partial\Sigma_\varrho} \frac{1}{2}\langle \nabla_{n_\varrho} N, H\rangle\ \mathrm{d}\mathcal{H}^1 \Big| \to 0.$$

From (4.6), (4.7) and (4.11), we obtain

$$\begin{aligned}\langle N, \nabla_{n_\varrho} H\rangle &= \langle N, \Big(H_0 \frac{x^T}{|x|^2} + O(|x|^{\alpha-1})\Big)\Big(\frac{x}{|x|} + O(|x|^\alpha)\Big)\rangle| \\ &= \langle N, H_0\rangle \frac{1}{|x|} + O(|x|^{\alpha-1}).\end{aligned}$$

Hence

$$\int\limits_{\partial\Sigma_\varrho} \frac{1}{2}\langle N, \nabla_{n_\varrho} H\rangle\ \mathrm{d}\mathcal{H}^1 \to \pi\langle V, H_0\rangle,$$

and (4.8) follows. □

LEMMA 4.2. *Let Σ be an open surface and $f : \Sigma \to \mathbb{R}^3$ be a smooth Willmore immersion with pull-back metric $g = f^* g_{\mathrm{euc}}$, induced area-measure μ_g and $\mu = f(\mu_g)$. Assume for distinct points $p_1, \dots, p_N \in \mathbb{R}^3$ that*

$$\operatorname{spt}\ \mu = f(\Sigma) \cup \{p_1, \dots, p_N\} \quad \textit{is compact},$$

$$\theta^2_*(\mu, p_k) < 2,$$

$$\mu \textit{ has square integrable weak mean curvature in } \mathbb{R}^3 - \{p_1, \dots, p_N\},$$

$$\int\limits_\Sigma |A|^2\ \mathrm{d}\mu_g < \infty.$$

Then Σ, more precisely spt μ *is $C^{1,\alpha}$-embedded with unit-density near p_k and*

$$\sum_{k=1}^N \mathrm{Res}_\Sigma(p_k) = 0, \tag{4.12}$$

where the residue Res_Σ *is as defined in the development of Lemma* 4.1.

In particular, if $N = 1$ then Σ is a smooth, immersed Willmore surface.

Proof. By Lemma 3.1 spt μ is a $C^{1,\alpha}$-embedded, unit-density surface satisfying (4.6) near p_k, and the residue of Σ at p_k is well defined. Putting

$$\Omega_\varrho := f^{-1}(\mathbb{R}^3 - \cup_{k=1}^N B^3_\varrho(p_k)) \subset\subset \Sigma$$

and $\Sigma_\varrho(p_k) := B^3_\varrho(p_k) \cap \Sigma$ for small $\varrho > 0$, we obtain for any $V \in \mathbb{R}^3$ and the associated closed 1-form ω_V on Σ in (4.5) that

$$0 = \int\limits_{\Omega_\varrho} \mathrm{d}\omega_V = -\sum_{k=1}^N \int\limits_{\partial\Sigma_\varrho(p_k)} \omega_V \to \pi\langle V, \sum_{k=1}^N \mathrm{Res}_\Sigma(p_k)\rangle;$$

hence

$$\sum_{k=1}^N \mathrm{Res}_\Sigma(p_k) = 0,$$

as V is arbitrary.

When $N = 1$, this means $\mathrm{Res}_\Sigma(p_1) = 0$, and Σ is a smooth Willmore surface according to Lemma 4.1. $\square$

Remark. Lemma 4.2 applies in particular to smooth, embedded surfaces $\Sigma \subset\subset \mathbb{R}^3$ with

$$\overline{\Sigma} - \Sigma = \{p_1, \ldots, p_N\}$$

by Remark 2 following Lemma 3.1.

The following lemma removes point singularities at infinity.

LEMMA 4.3. *Let Σ be a smooth, noncompact Willmore surface satisfying*

$$\liminf_{R\to\infty} \frac{\mu_\Sigma(B_R(0))}{\omega_2 R^2} < 2, \tag{4.13}$$

$$\int\limits_\Sigma |A_\Sigma|^2 \, \mathrm{d}\mu_\Sigma < \infty. \tag{4.14}$$

Then for any $x_0 \notin \Sigma$ and the inversion $I(x) := |x - x_0|^{-2}\,(x - x_0)$, there is $\bar{\Sigma} := I(\Sigma) \cup \{0\}$, a smooth Willmore surface,

$$\mathcal{W}(\bar{\Sigma}) = \mathcal{W}(\Sigma) + 4\pi. \tag{4.15}$$

Proof. As Σ is noncompact, we obtain from (4.13) and (A.22)

$$1 \le \lim_{R\to\infty} \frac{\mu_\Sigma(B_R(0))}{\omega_2 R^2} < 2. \tag{4.16}$$

Further we can perform a blowdown; that is,

$$R_m^{-1}\Sigma \to T \quad \text{weakly as varifolds}$$

for subsequences $R_m \to \infty$ where T depends on the subsequence. We get, for almost all $\varrho > 0$,

$$\varrho^{-2}\mu_T(B_\varrho(0)) = \lim_{m\to\infty} \varrho^{-2}\mu_{R_m^{-1}\Sigma}(B_\varrho(0)) = \lim_{m\to\infty} (R_m\varrho)^{-2}\mu(B_{R_m\varrho}(0));$$

hence by (4.16),

$$1 \leq \frac{\mu(B_\varrho(0))}{\omega_2\varrho^2} < 2 \quad \forall \varrho > 0, \tag{4.17}$$

in particular

$$0 \in \text{spt } \mu_T \tag{4.18}$$

and $\mu_T \neq 0$.

From (4.14) and since Σ is Willmore, we see by [KuSch 1, Th. 2.10] that the convergence to T is smooth in compact subsets of $\mathbb{R}^3 - \{0\}$ and $A_T = 0$ in $\mathbb{R}^3 - \{0\}$. Hence T is a union of integral planes, and, by (4.17), (4.18), it is a single density plane through 0.

Now, we consider any $x_0 \notin \Sigma$ and the inversion $I(x) := |x - x_0|^{-2}\ (x - x_0)$. $\bar{\Sigma} := I(\Sigma) \cup \{0\}$ is a smooth Willmore surface outside 0. Since $R_m^{-1}\Sigma$ converges for subsequences weakly as varifolds to single density planes T through 0, we conclude that

$$\varrho_m^{-1}\bar{\Sigma} = I(x_0 + \varrho_m(\Sigma - x_0)) \to I(x_0 + T) = T \quad \text{for } \varrho_m := R_m^{-1} \to 0,$$

and therefore

$$\theta^2(\mu_{\bar{\Sigma}}, 0) = 1. \tag{4.19}$$

Moreover, as the convergence $x_0 + \varrho_m(\Sigma - x_0) \to x_0 + T$ is smooth in compact subsets of $\mathbb{R}^3 - \{x_0\}$, we see that $\varrho_m^{-1}\bar{\Sigma} \to T$ smoothly in compact subsets of $\mathbb{R}^3 - \{0\}$. Therefore $\partial B_\varrho(0)$ intersects $\bar{\Sigma}$ in a single closed, smooth curve for small $\varrho = R^{-1} > 0$, and

$$(\Sigma - B_R(x_0)) \cup \{\infty\} \cong \bar{\Sigma} \cap B_\varrho(0) \cong B_1(0) \quad \text{under homeomorphy.} \tag{4.20}$$

We see that $\Sigma \cup \{\infty\} \cong \bar{\Sigma}$ are topological manifolds and, putting $\Sigma_R := \Sigma \cap B_R(x_0), \bar{\Sigma}_\varrho := \bar{\Sigma} - B_\varrho(0)$, we get

$$\chi(\Sigma) = \chi(\Sigma_R) = \chi(\bar{\Sigma}_\varrho) = \chi(\bar{\Sigma}) - 1.$$

Gauss-Bonnet's Theorem yields

$$\int_{\Sigma_R} K_\Sigma \, \mathrm{d}\mu_\Sigma + \int_{\partial\Sigma_R} \kappa_{\partial\Sigma_R} \, \mathrm{d}\mathcal{H}^1 = 2\pi\chi(\Sigma_R) = 2\pi\chi(\Sigma),$$

$$\int_{\bar{\Sigma}_\varrho} K_{\bar{\Sigma}} \, \mathrm{d}\mu_{\bar{\Sigma}} + \int_{\partial\bar{\Sigma}_\varrho} \kappa_{\partial\bar{\Sigma}_\varrho} \, \mathrm{d}\mathcal{H}^1 = 2\pi\chi(\bar{\Sigma}_\varrho) = 2\pi\chi(\bar{\Sigma}) - 2\pi.$$

By smooth convergence for subsequences around $R^{-1}\partial\Sigma_R$ and $\varrho^{-1}\partial\bar{\Sigma}_\varrho$ to flat annuli, we see

$$\lim_{R\uparrow\infty} \int\limits_{\partial\Sigma_R} \kappa_{\partial\Sigma_R}\, \mathrm{d}\mathcal{H}^1 = 2\pi = -\lim_{\varrho\downarrow 0} \int\limits_{\partial\bar{\Sigma}_\varrho} \kappa_{\partial\bar{\Sigma}_\varrho}\, \mathrm{d}\mathcal{H}^1$$

and obtain

$$\int\limits_{\Sigma} K_\Sigma\, \mathrm{d}\mu_\Sigma = 2\pi\chi(\Sigma) - 2\pi = 2\pi\chi(\bar{\Sigma}) - 4\pi = \lim_{\varrho\downarrow 0} \int\limits_{\bar{\Sigma}_\varrho} K_{\bar{\Sigma}}\, \mathrm{d}\mu_{\bar{\Sigma}} - 4\pi.$$

From (4.14) and the conformal invariance of $|A^0|^2\mu_\Sigma$, see [Ch], we get

$$\int\limits_{\bar{\Sigma}} |A^0_{\bar{\Sigma}}|^2\, \mathrm{d}\mu_{\bar{\Sigma}} = \int\limits_{\Sigma} |A^0_\Sigma|^2\, \mathrm{d}\mu_\Sigma < \infty,$$

and, since $|A^0|^2 = \frac{1}{2}|H|^2 - 2K = |A|^2 - \frac{1}{2}|H|^2$, we conclude

$$\begin{aligned}\mathcal{W}(\bar{\Sigma}) &= \lim_{\varrho\downarrow 0} \frac{1}{4} \int\limits_{\bar{\Sigma}_\varrho} |H_{\bar{\Sigma}}|^2\, \mathrm{d}\mu_{\bar{\Sigma}} = \frac{1}{2} \int\limits_{\bar{\Sigma}} |A^0_{\bar{\Sigma}}|^2\, \mathrm{d}\mu_{\bar{\Sigma}} + \lim_{\varrho\downarrow 0} \int\limits_{\bar{\Sigma}_\varrho} K_{\bar{\Sigma}}\, \mathrm{d}\mu_{\bar{\Sigma}} \\ &= \frac{1}{2} \int\limits_{\Sigma} |A^0_\Sigma|^2\, \mathrm{d}\mu_\Sigma + \int\limits_{\Sigma} K_\Sigma\, \mathrm{d}\mu_\Sigma + 4\pi = \mathcal{W}(\Sigma) + 4\pi < \infty,\end{aligned}$$

which establishes (4.15), and $H_{\bar{\Sigma}}, A_{\bar{\Sigma}} \in L^2(\mu_{\bar{\Sigma}})$.

In particular, $\bar{\Sigma}$ has square integrable weak mean curvature in $\mathbb{R}^3 - \{0\}$ and by (4.19), (4.20), we finally obtain from Lemma 4.2 with $N = 1$ that $\bar{\Sigma}$ is a smooth embedded Willmore surface, concluding the proof. □

5. Convergence and compactness results

In this section, we derive several applications of the removability of point singularities for Willmore surface. We start with a convergence result for bounded surfaces.

THEOREM 5.1. *Let $\Sigma_j \subseteq \mathbb{R}^3$ be a sequence of smooth, closed Willmore surfaces satisfying*

$$\mathcal{H}^2(\Sigma_j) = 1, \qquad \int\limits_{\Sigma_j} |A_{\Sigma_j}|^2\, \mathrm{d}\mu_{\Sigma_j} \le C,$$
$$\mathcal{W}(\Sigma_j) \le 8\pi - \delta, \qquad \Sigma_j \to \Sigma \neq 0 \quad \textit{weakly as varifolds.}$$

Then Σ is a smooth, closed Willmore surface.

Proof. Any connected component C of Σ_j satisfies $\mathcal{W}(C) \geq 4\pi$ by (A.18). Therefore Σ_j are connected and by [Sim 2, Lemma 1.1], the diameter of Σ_j is uniformly bounded; hence Σ is compact and has square integrable weak mean curvature with

$$\mathcal{W}(\Sigma) = \mathcal{W}(\mu_\Sigma) \leq 8\pi - \delta.$$

From (A.17), we get

$$\theta^2(\mu) \leq \frac{1}{4\pi}\mathcal{W}(\Sigma) \leq \frac{8\pi - \delta}{4\pi} < 2 \quad \text{in } \mathbb{R}^3. \tag{5.1}$$

Since $\int_{\Sigma_j} |A_{\Sigma_j}|^2 \, d\mu_{\Sigma_j}$ is uniformly bounded, we see from [KuSch 1, Th. 2.10] that the convergence is smooth outside finitely many distincts points $p_1, \dots, p_N \in \Sigma$. Moreover

$$\int_\Sigma |A_\Sigma|^2 \, d\mu_\Sigma < \infty.$$

From Lemma 3.1 and (5.1), we see that Σ is an embedded $C^{1,\alpha}$-surface in $\mathbb{R}^3$ satisfying (3.5), respectively (4.6), near p_i.

We calculate the residues of Σ in p_i as defined in Lemma 4.1. For ϱ small enough, $\partial B_\varrho(p_i)$ intersects Σ and Σ_j for j large enough depending on ϱ in a single, closed, smooth curve. By smooth convergence,

$$\int\limits_{\partial B_\varrho(p_i)\cap\Sigma} \omega_\Sigma \leftarrow \int\limits_{\partial B_\varrho(p_i)\cap\Sigma_j} \omega_{\Sigma_j} = -\int\limits_{\Sigma_j - B_\varrho(p_i)} d\omega_{\Sigma_j} = 0$$

as ω_{Σ_j} is closed and Σ_j is smooth. Therefore by Lemma 4.1,

$$\mathrm{Res}_\Sigma(p_i) = \lim_{\varrho \downarrow 0} \int\limits_{\partial B_\varrho(p_i)\cap\Sigma} \omega_\Sigma = 0,$$

and Σ is a smooth, closed Willmore surface. $\square$

Remark. The assumption $\Sigma \neq 0$ is equivalent to the assumption that

$$\operatorname{spt} \Sigma_j \not\to \infty.$$

Clearly, if $\operatorname{spt} \Sigma_j \to \infty$ then $\Sigma = 0$.

On the other hand, if there exists $x_j \in \operatorname{spt} \Sigma_j$ with $\limsup_{j\to\infty} |x_j| < \infty$, then $\operatorname{spt} \Sigma_j \subseteq B_R(0)$ for some large R, as the diameter of Σ_j is uniformly bounded by [Sim 2, Lemma 1.1]. Then

$$\mu_\Sigma(\overline{B_R(0)}) \geq \limsup_{j\to\infty} \mu_{\Sigma_j}(B_R(0)) = \limsup_{j\to\infty} \mathcal{H}^2(\Sigma_j) = 1,$$

and $\Sigma \neq 0$.

In the following, we will perform several blowup procedures. The next lemma gives the necessary convergence properties.

LEMMA 5.1. *Let Σ_j be a sequence of closed surfaces satisfying*

$$\mathcal{W}(\Sigma_j) \le 8\pi - \delta, \tag{5.2}$$

$$\int_{\Sigma_j} |A_{\Sigma_j}|^2 \, \mathrm{d}\mu_{\Sigma_j} \le C, \tag{5.3}$$

$$\Sigma_j \to \Sigma \quad \textit{smoothly in compact subsets of } \mathbb{R}^3, \tag{5.4}$$

where Σ is a smooth, noncompact Willmore surface.

Then for any $x_0 \notin \Sigma$ and the inversion $I(x) := |x - x_0|^{-2}\,(x - x_0)$

$$\bar{\Sigma} := I(\Sigma) \cup \{0\} \quad \textit{is a smooth Willmore surface},$$

$$\mathcal{W}(\Sigma) + 4\pi = \mathcal{W}(\bar{\Sigma}) \le \liminf_{j\to\infty} \mathcal{W}(\Sigma_j), \tag{5.5}$$

$$g(\bar{\Sigma}) \le \liminf_{j\to\infty} g(\Sigma_j). \tag{5.6}$$

If there are $r_j \downarrow 0$, $x_j \to 0$ such that for $\bar{\Sigma}_j := I(\Sigma_j)$ and a subsequence

$$\Sigma'_j := r_j^{-1}(\bar{\Sigma}_j - x_j) \to \Sigma' \quad \textit{smoothly in compact subsets of } \mathbb{R}^3 \tag{5.7}$$

and Σ' is a noncompact Willmore surface then for any $x'_0 \notin \Sigma$ and the inversion $I'(x) := |x - x'_0|^{-2}\,(x - x'_0)$,

$$\bar{\Sigma}' := I'(\Sigma') \cup \{0\} \quad \textit{is a smooth Willmore surface},$$

$$\mathcal{W}(\bar{\Sigma}) + \mathcal{W}(\bar{\Sigma}') \le \liminf_{j\to\infty} \mathcal{W}(\Sigma_j) + 4\pi, \tag{5.8}$$

$$g(\bar{\Sigma}) + g(\bar{\Sigma}') \le \liminf_{j\to\infty} g(\Sigma_j). \tag{5.9}$$

Proof. Since the Σ_j are compact, we get from (5.2) and (A.23)

$$\lim_{R\to\infty} \frac{\mu(B_R(0))}{\omega_2 R^2} + \frac{1}{4\pi}\mathcal{W}(\Sigma) \le \liminf_{j\to\infty} \frac{1}{4\pi}\mathcal{W}(\Sigma_j) \le \frac{8\pi - \delta}{4\pi} < 2.$$

Clearly, from (5.3),

$$\int_{\Sigma} |A_\Sigma|^2 \, \mathrm{d}\mu_\Sigma < \infty.$$

This verifies (4.13) and (4.14), and Lemma 4.3 implies that $\bar{\Sigma}$ is a smooth Willmore surface. From the convergence in (5.4), we see that $\mathrm{dist}(x_0, \Sigma_j) \to \mathrm{dist}(x_0, \Sigma) > 0$; hence $x_0 \notin \bar{\Sigma}_j$ for large j, and by conformal invariance, we see that Σ'_j satisfies (5.2) and (5.3); hence $\bar{\Sigma}'$ is a smooth Willmore surface by what we have just proved.

Further

$$\bar{\Sigma}_j := I(\Sigma_j) \to \bar{\Sigma} \quad \text{smoothly in compact subsets of } \mathbb{R}^3 - \{0\}. \tag{5.10}$$

First this implies with (4.15) that

$$\mathcal{W}(\Sigma) + 4\pi = \mathcal{W}(\bar{\Sigma}) \le \liminf_{j\to\infty} \mathcal{W}(\bar{\Sigma}_j) = \liminf_{j\to\infty} \mathcal{W}(\Sigma_j),$$

which is (5.5).

Secondly, as $\bar{\Sigma}$ is smooth near 0, (5.10) yields for $\varrho > 0$ small enough and j large enough depending on ϱ that $\partial B_\varrho(0)$ intersects $\bar{\Sigma}_j$ in a single closed, smooth curve. We put

$$\begin{aligned} \bar{\Sigma}_\varrho &:= \bar{\Sigma} - B_\varrho(0),\\ \bar{\Sigma}_{j,\varrho,+} &:= \bar{\Sigma}_j - B_\varrho(0). \end{aligned}$$

Considering homology or appropriate triangulations, we see

$$\chi(\Sigma_j) = \chi(\bar{\Sigma}_{j,\varrho,+}) + \chi(\bar{\Sigma}_j \cap B_\varrho(0)) \le \chi(\bar{\Sigma}_{j,\varrho,+}) + 1, \tag{5.11}$$

as $\chi(\bar{\Sigma}_j \cap B_\varrho(0)) \le 1$. By smooth convergence $\bar{\Sigma}_{j,\varrho,+} \to \bar{\Sigma}_\varrho$, we get

$$\chi(\bar{\Sigma}) = \chi(\bar{\Sigma}_\varrho) + 1 = \lim_{j\to\infty} \chi(\bar{\Sigma}_{j,\varrho,+}) + 1 \ge \lim_{j\to\infty} \chi(\Sigma_j), \tag{5.12}$$

or likewise

$$g(\bar{\Sigma}) \le \liminf_{j\to\infty} g(\Sigma_j),$$

which is (5.6).

Next, we extend $\bar{\Sigma}_j \cap B_\varrho(0)$ outside $B_\varrho(0)$ to a smooth surface in $\mathbb{R}^3$ which is a plane near infinity and whose Willmore energy exceeds that of $\bar{\Sigma}_j \cap B_\varrho(0)$ only by $\omega(\varrho) \to 0$ for $\varrho \to 0$. Then replacing the plane by a large, slightly deformed sphere, we get a smooth, closed surface $\bar{\Sigma}_{j,\varrho,-} \subset\subset \mathbb{R}^3$ satisfying

$$\liminf_{j\to\infty} \mathcal{W}(\bar{\Sigma}_{j,\varrho,-}) \le \liminf_{j\to\infty} \frac{1}{4} \int\limits_{\bar{\Sigma}_j \cap B_\varrho(0)} |H_{\bar{\Sigma}_j}|^2 \, d\mu_{\bar{\Sigma}_j} + 4\pi + \omega(\varrho), \tag{5.13}$$

$$\chi(\bar{\Sigma}_{j,\varrho,-}) = \chi(\bar{\Sigma}_j \cap B_\varrho(0)) + 1, \tag{5.14}$$

$$\bar{\Sigma}_{j,\varrho,-} \cap B_\varrho(0) = \bar{\Sigma}_j \cap B_\varrho(0). \tag{5.15}$$

As $r_j \downarrow 0, x_j \to 0$, we see from (5.7) that

$$r_j^{-1}(\bar{\Sigma}_{j,\varrho,-} - x_j) \to \Sigma' \quad \text{smoothly in compact subsets of } \mathbb{R}^3.$$

As we already know that $\bar{\Sigma}'$ is smooth near 0, we get as in (5.12) that

$$\chi(\bar{\Sigma}') \ge \limsup_{j\to\infty} \chi(\bar{\Sigma}_{j,\varrho,-}) \ge \limsup_{j\to\infty} \chi(\bar{\Sigma}_j \cap B_\varrho(0)) + 1 \tag{5.16}$$

when recalling (5.14).

Combining (5.11), (5.12) and (5.16), we see

$$\chi(\bar{\Sigma}) + \chi(\bar{\Sigma}') \ge \lim_{j\to\infty} \chi(\bar{\Sigma}_{j,\varrho,+}) + \limsup_{j\to\infty} \chi(\bar{\Sigma}_j \cap B_\varrho(0)) + 2 = \limsup_{j\to\infty} \chi(\Sigma_j) + 2,$$

or likewise

$$g(\bar{\Sigma}) + g(\bar{\Sigma}') \leq \liminf_{j \to \infty} g(\Sigma_j),$$

which is (5.9).

Next we see

$$\mathcal{W}(\bar{\Sigma}) \leq \lim_{\varrho \downarrow 0} \liminf_{j \to \infty} \tfrac{1}{4} \int\limits_{\bar{\Sigma}_j - B_\varrho(0)} |H_{\bar{\Sigma}_j}|^2 \, \mathrm{d}\mu_{\bar{\Sigma}_j},$$

$$\mathcal{W}(\bar{\Sigma}') \leq \liminf_{j \to \infty} \mathcal{W}(\bar{\Sigma}_{j,\varrho,-}),$$

which yields (5.8) by (5.13). □

In the following applications, we will strongly use Bryant's result in [Bry] that Willmore spheres $M^2 \subseteq \mathbb{R}^3$, not round spheres, satisfy

$$\mathcal{W}(M^2) \geq 16\pi. \tag{5.17}$$

A more elementary proof of [Bry, Th. E] can be found in [Es, §6, Prop.]. When combined with a theorem of Osserman [Os, Th. 9.2], one obtains the estimate slightly weaker than (5.17) that Willmore spheres $M^2 \subseteq \mathbb{R}^3$ which are not round spheres satisfy

$$\mathcal{W}(M^2) \geq 8\pi.$$

Actually, this estimate suffices for all applications in this section, except that we have to assume the strict inequalities

$$\mathcal{W}(f_0) < 8\pi \quad \text{and} \quad \int\limits_\Sigma |A^0|^2 \, \mathrm{d}\mu_g < 8\pi$$

in (5.18) and (5.19) below, respectively.

We continue our applications with a long-time existence theorem for immersed spheres.

Theorem 5.2. *Let $f_0 : S^2 \to \mathbb{R}^3$ be a smooth immersion of a sphere with Willmore energy*

$$\mathcal{W}(f_0) \leq 8\pi. \tag{5.18}$$

Then the Willmore flow with initial data f_0 exists smoothly for all times and converges to a round sphere.

Remark. Using Gauss-Bonnet's theorem we can reformulate the above theorem:

COROLLARY. *Let Σ be a closed surface and $f_0 : \Sigma \to \mathbb{R}^3$ a smooth immersion satisfying*

$$\int_\Sigma |A^0|^2 \, \mathrm{d}\mu_g \leq 8\pi. \tag{5.19}$$

Then the Willmore flow with initial data f_0 exists smoothly for all times and converges to a round sphere.

A numerical example of a singularity which was recently obtained in [MaSi] indicates that one cannot improve 8π in the above statement. This determines $\varepsilon_0(3) = 8\pi$ as the optimal constant in the smallness assumption of Theorem 5.1 in [KuSch 1].

Proof. In case $\mathcal{W}(f_0) = 8\pi$, we see from (5.17) that f_0 is not a Willmore immersion. Since the statement of the theorem concerns only the asymptotic behaviour of the Willmore flow $f : S^2 \times [0, T[\to \mathbb{R}^3$ with initial data $f(0) = f_0$, we may assume $\mathcal{W}(f_0) < 8\pi$. Therefore

$$\mathcal{W}(f_t) \leq \mathcal{W}(f_0) < 8\pi \quad \forall t \in [0, T[, \tag{5.20}$$

and all f_t are embeddings by (A.17). We put $\Sigma_t := f_t(S^2)$ and assume that $[0, T[, 0 < T \leq \infty$ is the maximal existence interval of f.

As in [KuSch 1, §4], we define

$$\kappa(r, t) := \sup_{x \in \mathbb{R}^3} \int_{\Sigma_t \cap B_r(x)} |A_{\Sigma_t}|^2 \, \mathrm{d}\mu_{\Sigma_t}.$$

Clearly for fixed t,

$$\lim_{r \uparrow \varrho} \kappa(r, t) = \kappa(\varrho, t) \leq \liminf_{r \downarrow \varrho} \kappa(r, t)$$

and with a simple covering argument

$$\limsup_{r \downarrow \varrho} \kappa(r, t) \leq C \kappa(\varrho, t) \tag{5.21}$$

for some $C < \infty$. Hence for $\varepsilon > 0$ small enough, we can choose $r_t > 0$ with

$$\varepsilon < \kappa(r_t, t) \leq C\varepsilon.$$

As in [KuSch 1] any sequence $t \uparrow T$ has a subsequence $t_j \uparrow T$ and $x_j \in \mathbb{R}^3$ such that

$$\hat{\Sigma}_j := r_{t_j}^{-1}(\Sigma_{t_j + c_0 r_{t_j}^4} - x_j) \to \hat{\Sigma} \quad \text{smoothly in compact subsets of } \mathbb{R}^3.$$

As all the Σ_t are not only immersed, but embedded, this convergence procedure is much simpler than the general situation of [KuSch 1, Th. 4.2].

The limit $\hat{\Sigma}$ is a smooth, complete Willmore surface that satisfies

$$\mathcal{W}(\hat{\Sigma}) \leq \mathcal{W}(f_0) < 8\pi. \tag{5.22}$$

Actually, we want to select t_j, x_j in such a way that

$$\varepsilon \leq \int\limits_{\hat{\Sigma}} |A_{\hat{\Sigma}}|^2 \, \mathrm{d}\mu_{\hat{\Sigma}} < \infty. \tag{5.23}$$

This is certainly possible, if

$$\liminf_{t \uparrow T} \frac{r_{t+c_0 r_t^4}}{r_t} < \infty. \tag{5.24}$$

Indeed for $\liminf_{t\uparrow T} r_{t+c_0 r_t^4}/r_t > \Gamma > 0$, we can choose $t_0 < T$ such that $r_{t+c_0 r_t^4} \geq \Gamma r_t \ \forall t_0 \leq t < T$. Putting $t_{j+1} := t_j + c_0 r_{t_j}^4$, we see

$$r_{t_j} \geq r_{t_0} \Gamma^j > 0 \quad \forall j \in \mathbb{N}_0.$$

Clearly for ε small and by [Sim 2, Lemma 1.1],

$$r_t \leq \operatorname{diam}(\Sigma_t) \leq C \mathcal{H}^2(\Sigma_t)^{1/2}.$$

The first variation formula for the areas yields

$$\frac{d}{dt} \mathcal{H}^2(\Sigma_t) = -\int \langle H_{\Sigma_t}, \partial_t f \rangle \, \mathrm{d}\mu_{\Sigma_t} \leq C \Big(\int |\partial_t f|^2 \, \mathrm{d}\mu_{\Sigma_t} \Big)^{1/2},$$

and as f evolves as gradient flow of the Willmore functional up to a factor,

$$\mathcal{H}^2(\Sigma_t) \leq \mathcal{H}^2(\Sigma_0) + C t^{1/2} \Big(\int_0^T \int |\partial_t f|^2 \, \mathrm{d}\mu_{\Sigma_t} \, \mathrm{d}t \Big)^{1/2} \leq C(1 + t^{1/2}).$$

Hence

$$r_t^4 \leq C(1+t),$$
$$1 + t_{j+1} = 1 + t_j + c_0 r_{t_j}^4 \leq (1 + C c_0)(1 + t_j),$$

and

$$0 < r_{t_0}^4 \Gamma^{4j} \leq r_{t_j}^4 \leq C(1 + t_j) \leq C(1 + C c_0)^j (1 + t_0),$$

which yields $\Gamma \leq (1 + C c_0)$ for $j \to \infty$. This establishes (5.24), hence (5.23).

Now if

$$\hat{\Sigma} \quad \text{is compact,} \tag{5.25}$$

then the conclusion of the theorem easily follows. Indeed, in this case the convergence $\hat{\Sigma}_j \to \hat{\Sigma}$ is smooth, and $\hat{\Sigma}$ is a smooth Willmore sphere. By (5.22), it follows from (5.17) that $\hat{\Sigma}$ is a round sphere. This yields

$$\lim_{j \to \infty} \int\limits_{\Sigma_{t_j}} |A^0_{\Sigma_{t_j}}|^2 \, \mathrm{d}\mu_{t_j} = \lim_{j \to \infty} \int\limits_{\hat{\Sigma}_j} |A^0_{\hat{\Sigma}_j}|^2 \, \mathrm{d}\mu_{\hat{\Sigma}_j} = \int\limits_{\hat{\Sigma}} |A^0_{\hat{\Sigma}}|^2 \, \mathrm{d}\mu_{\hat{\Sigma}} = 0,$$

and the theorem follows immediately from [KuSch 1, Th. 5.1].

Therefore it remains to prove (5.25). If $\hat{\Sigma}$ is not compact, we get from Lemma 5.1 that for any $x_0 \notin \hat{\Sigma}$ and the inversion $I(x) := |x - x_0|^{-2} \, (x - x_0)$,

$$\bar{\Sigma} := I(\hat{\Sigma}) \cup \{0\} \quad \text{is a smooth Willmore surface.}$$

Moreover from (5.5) and (5.20), we get

$$\mathcal{W}(\bar{\Sigma}) < 8\pi,$$

and from (5.6), we see that $\bar{\Sigma}$ is a Willmore sphere which by (5.17) is a round sphere containing 0. Then $\hat{\Sigma} = I^{-1}(\bar{\Sigma})$ is a plane, hence $A_{\hat{\Sigma}} = 0$, contradicting (5.23), and (5.25) is proved. □

For tori we obtain the following compactness result.

THEOREM 5.3. *The set*

$$\mathcal{M}_{1,\delta} := \{\Sigma \subseteq \mathbb{R}^3 \text{ Willmore} \mid g(\Sigma) = 1, \mathcal{W}(\Sigma) \leq 8\pi - \delta\ \}$$

is compact up to Möbius transformations under smooth convergence of compactly contained surfaces in $\mathbb{R}^3$.

Proof. We consider $\Sigma_j \in \mathcal{M}_{1,\delta}$. To prove the compactness only up to Möbius transformations, we may assume

$$\mathcal{H}^2(\Sigma_j) = 1, \quad 0 \in \Sigma_j.$$

From the bounds on the Willmore energy and the fixed genus of Σ_j, we conclude with Gauss-Bonnet's theorem,

$$\int_{\hat{\Sigma}_j} |A_{\Sigma_j}|^2 \, \mathrm{d}\mu_{\Sigma_j} \leq 32\pi,$$

as $|A|^2 = |H|^2 - 2K$. A subsequence converges

$$\Sigma_j \to \Sigma \quad \text{weakly as varifolds.} \tag{5.26}$$

From Theorem 5.1 and its remark, we see that $\Sigma \neq 0$ and Σ is a smooth Willmore surface. Clearly,

$$\mathcal{W}(\Sigma) \leq 8\pi - \delta. \tag{5.27}$$

Now, define

$$\kappa_j(r) := \sup_{x \in \mathbb{R}^3} \int_{\Sigma_j \cap B_r(x)} |A_{\Sigma_j}|^2 \, \mathrm{d}\mu_{\Sigma_j},$$

and choose $r_j > 0$ recalling (5.21) with

$$\varepsilon < \kappa_j(r_j) \leq C\varepsilon$$

for $\varepsilon > 0$ small enough. If

$$\liminf_{j \to \infty} r_j > 0, \tag{5.28}$$

we see from [KuSch 1, Th. 2.10] that the convergence $\Sigma_j \to \Sigma$ is smooth and Σ is a torus; hence $\Sigma \in \mathcal{M}_{1,\delta}$ by (5.27), and the conclusion of the theorem follows.

Otherwise, $r_j \to 0$ for a subsequence and we choose $x_j \in \mathbb{R}^3$ so that

$$\int\limits_{\Sigma_j \cap B_{r_j}(x_j)} |A_{\Sigma_j}|^2 \, \mathrm{d}\mu_{\Sigma_j} \geq \varepsilon.$$

Again by [KuSch 1, Th. 2.10], we see that

$$(5.29) \qquad \hat{\Sigma}_j := r_j^{-1}(\Sigma_j - x_j) \to \hat{\Sigma} \quad \text{smoothly in compact subsets of } \mathbb{R}^3$$

and

$$(5.30) \qquad \int\limits_{\hat{\Sigma}} |A_{\hat{\Sigma}}|^2 \, \mathrm{d}\mu_{\hat{\Sigma}} \geq \varepsilon.$$

As $\hat{\Sigma}_j \cong T^2$ is connected and $r_j \to 0$, we see that $\hat{\Sigma}$ is not compact.

Then Lemma 5.1 yields for any $x_0 \notin \hat{\Sigma}$ and the inversion

$$I(x) := |x - x_0|^{-2} \, (x - x_0)$$

that

$$\bar{\Sigma} := I(\hat{\Sigma}) \cup \{0\} \quad \text{is a smooth Willmore surface.}$$

Moreover, by (5.5),

$$\mathcal{W}(\bar{\Sigma}) \leq 8\pi - \delta,$$

and by (5.6) $g(\bar{\Sigma}) \leq 1$. If $g(\bar{\Sigma}) = 0$, that is, $\bar{\Sigma}$ is a Willmore sphere, we get from (5.17) that $\bar{\Sigma}$ is a round sphere containing 0. Then $\hat{\Sigma} = I^{-1}(\bar{\Sigma})$ is a plane; hence $A_{\hat{\Sigma}} = 0$ contradicting (5.30). Therefore $\bar{\Sigma}$ is a torus; hence $\bar{\Sigma} \in \mathcal{M}_{1,\delta}$ and

$$(5.31) \qquad g(\bar{\Sigma}) = 1.$$

Clearly

$$(5.32) \qquad \bar{\Sigma}_j := I(\hat{\Sigma}_j) \to \bar{\Sigma} \quad \text{smoothly in compact subsets of } \mathbb{R}^3 - \{0\}.$$

Proceeding from (5.26) with Σ_j replaced by $\bar{\Sigma}_j$, we claim that (5.28) holds true for $\bar{\Sigma}_j$. Hence the convergence in (5.32) is smooth and the conclusion of the theorem follows.

Indeed, otherwise, as in (5.29), there are $\bar{r}_j \downarrow 0, \bar{x}_j \in \mathbb{R}^3$ and $\bar{x}_j \to 0$ by (5.32) such that

$$\Sigma'_j := (\bar{r}_j)^{-1}(\bar{\Sigma}_j - \bar{x}_j) \to \Sigma' \quad \text{smoothly in compact subsets of } \mathbb{R}^3,$$

and, since $\bar{\Sigma}' := I'(\Sigma') \cup \{0\}$ for an appropriate inversion I',

$$g(\bar{\Sigma}') = 1$$

as (5.31). On the other hand by (5.9)

$$g(\bar{\Sigma}) + g(\bar{\Sigma}') \leq \liminf_{j \to \infty} g(\Sigma_j) = 1,$$

which is a contradiction by (5.31). □

Remark. If $\beta_g^3 \geq 6\pi$ for $g = 1, \dots, g_0 - 1$, the proof above shows by (5.8) that the sets

$$\mathcal{M}_{g_0,\delta} := \{\Sigma \subseteq \mathbb{R}^3 \text{ Willmore} \mid g(\Sigma) = g_0, \mathcal{W}(\Sigma) \leq 8\pi - \delta \}$$

are compact up to Möbius transformations.

Clearly the Willmore conjecture, see [Schm], implies $\beta_1^3 = 2\pi^2 \geq 6\pi$, hence compactness of $\mathcal{M}_{2,\delta}$ up to Möbius transformations.

Appendix

In this appendix, we collect for the reader's convenience some results which are consequences or adaptions of results already known in literature.

A. Monotonicity formula

In this section, we review the arguments in [Sim 2] proving a monotonicity formula for surfaces with square integrable mean curvature and show that all arguments generalize to integral 2-varifolds, $\mu \neq 0$, in an open set $U \subseteq \mathbb{R}^n$ with square integrable weak mean curvature $H_\mu \in L^2(\mu)$.

We extend our definition of the Willmore functional and put

$$\mathcal{W}(\mu) = \frac{1}{4} \int_U |H_\mu|^2 \, \mathrm{d}\mu.$$

Actually to treat point singularities $y_0 \in U$, we only assume that $\mu \llcorner (U - \{y_0\})$ has square integrable weak mean curvature when considered as varifold in $U - \{y_0\}$, and we add the assumption

$$\theta_*^1(\mu, y_0) = 0, \tag{A.1}$$

which is certainly satisfied if $\theta_*^2(\mu, y_0) < \infty$. We consider a cut-off function $\xi_\delta \in C_0^1(\mathbb{R}^n - \{y_0\})$ with $\xi_\delta \equiv 1$ on $\mathbb{R}^n - B_\delta(y_0)$, $0 \leq \xi_\delta \leq 1$, $|\nabla \xi_\delta| \leq C\delta^{-1}\chi_{B_\delta(y_0)}$ and obtain for any $\eta \in C_0^1(U)$, observing $\eta\xi_\delta \in C_0^1(U - \{y_0\})$, that

$$\begin{aligned}
-\int H_\mu \eta \, \mathrm{d}\mu \leftarrow -\int H_\mu \eta \xi_\delta \, \mathrm{d}\mu &= \int \mathrm{div}_\mu(\eta\xi_\delta) \, \mathrm{d}\mu \\
&= \int \xi_\delta \, \mathrm{div}_\mu \eta \, \mathrm{d}\mu + \int \eta \, \nabla_\mu \xi_\delta \, \mathrm{d}\mu \rightarrow \int \mathrm{div}_\mu \eta \, \mathrm{d}\mu,
\end{aligned}$$

as by (A.1)

$$\left| \int \eta \, \nabla_\mu \xi_\delta \, \mathrm{d}\mu \right| \leq C(\eta)\delta^{-1}\mu(B_\delta(y_0)) \rightarrow 0 \quad \text{for a subsequence } \delta_j \rightarrow 0.$$

Therefore μ has square integrable weak mean curvature

$$H_\mu \in L^2(\mu), \tag{A.2}$$

when considered as a varifold in U.

Approximating the lipschitz test function which leads to [Sim 2, (1.2)], we obtain for any $B_{\varrho_0}(x_0) \subset\subset U$ the following monotonicity formula:

(A.3)
$$\begin{aligned}
&\sigma^{-2}\mu(B_\sigma(x_0)) + \int\limits_{B_\varrho(x_0)-B_\sigma(x_0)} |\frac{1}{4}H_\mu + \frac{(x-x_0)^\perp}{r^2}|^2 \, \mathrm{d}\mu \\
&= \varrho^{-2}\mu(B_\varrho(x_0)) + \frac{1}{16} \int\limits_{B_\varrho(x_0)-B_\sigma(x_0)} |H_\mu|^2 \, \mathrm{d}\mu \\
&\quad + \frac{1}{2} \int\limits_{B_\varrho(x_0)} \varrho^{-2}(x-x_0)\ H_\mu(x)\ \mathrm{d}\mu(x) - \frac{1}{2} \int\limits_{B_\sigma(x_0)} \sigma^{-2}(x-x_0)\ H_\mu(x)\ \mathrm{d}\mu(x)
\end{aligned}$$

where $r(x) := |x - x_0|$ and $.^\perp$ denotes the orthogonal projection onto the normal space $(T\mu)^\perp$. First we get (A.3) for almost all $0 < \sigma \le \varrho \le \varrho_0$, then for *all* after approximation.

When
$$R_{x_0,\varrho} := \frac{1}{2} \int\limits_{B_\varrho(x_0)} \varrho^{-2}(x-x_0)\ H_\mu(x)\ \mathrm{d}\mu(x)$$

and

$$\gamma(\varrho) := \varrho^{-2}\mu(B_\varrho(x_0)) + \frac{1}{16} \int\limits_{B_\varrho(x_0)} |H_\mu|^2 \, \mathrm{d}\mu + R_{x_0,\varrho}, \tag{A.4}$$

then γ is monotonically nondecreasing. We estimate

$$|R_{x_0,\varrho}| \le \frac{1}{2}\Big(\varrho^{-2}\mu(B_\varrho(x_0))\Big)^{1/2} \ \| H_\mu \|_{L^2(B_\varrho(x_0))} \tag{A.5}$$

and get for any $\delta > 0$

$$\begin{aligned}
&\sigma^{-2}\mu(B_\sigma(x_0)) \\
&\le \varrho^{-2}\mu(B_\varrho(x_0)) + \frac{1}{16} \int\limits_{B_\varrho(x_0)} |H_\mu|^2 \, \mathrm{d}\mu + \frac{1}{2}\Big(\varrho^{-2}\mu(B_\varrho(x_0))\Big)^{1/2} \ \| H_\mu \|_{L^2(B_\varrho(x_0))} \\
&\quad + \frac{1}{2}\Big(\sigma^{-2}\mu(B_\sigma(x_0))\Big)^{1/2} \ \| H_\mu \|_{L^2(B_\sigma(x_0))} \\
&\le (1+\delta)\varrho^{-2}\mu(B_\varrho(x_0)) + (\frac{1}{16} + C\delta^{-1}) \int\limits_{B_\varrho(x_0)} |H_\mu|^2 \, \mathrm{d}\mu + \delta\sigma^{-2}\mu(B_\sigma(x_0)).
\end{aligned}$$

In particular, for $0 < \varrho \le \varrho_0$,

$$\varrho^{-2}\mu(B_\varrho(x_0)) \le (1+\delta)\varrho_0^{-2}\mu(B_{\varrho_0}(x_0)) + C(1+\delta^{-1})\mathcal{W}(\mu) < \infty. \tag{A.6}$$

Then we get from (A.5) that

$$\lim_{\varrho \downarrow 0} R_{x_0,\varrho} = 0;$$

hence the density

$$\theta^2(\mu, x_0) \text{ exists} \tag{A.7}$$

and

$$\omega_2 \theta^2(\mu, x_0) \leq \varrho^{-2} \mu(B_\varrho(x_0)) + \frac{1}{16} \int\limits_{B_\varrho(x_0)} |H_\mu|^2 \, \mathrm{d}\mu + R_{x_0,\varrho}. \tag{A.8}$$

For $x_j \to x_0, 0 < \varrho < \varrho_0/2$, we get

$$\begin{aligned} \varrho^{-2}\mu(\overline{B_\varrho(x_0)}) &\geq \limsup_{j\to\infty} \varrho^{-2}\mu(B_\varrho(x_j)) \\ &\geq \limsup_{j\to\infty} \Big(\omega_2\theta^2(\mu, x_j) - \frac{1}{16} \int\limits_{B_\varrho(x_j)} |H_\mu|^2 \, \mathrm{d}\mu - R_{x_j,\varrho} \Big) \\ &\geq \limsup_{j\to\infty} \ \omega_2\theta^2(\mu, x_j) \\ &\quad -C\Big(\varrho_0^{-2}\mu(B_{\varrho_0}(x_0)) + \mathcal{W}(\mu)\Big)^{1/2} \parallel H_\mu \parallel_{L^2(B_{2\varrho}(x_0))} . \end{aligned}$$

When $\varrho \downarrow 0$ this yields

$$\theta^2(\mu, x_0) \geq \limsup_{j\to\infty} \theta^2(\mu, x_j), \tag{A.9}$$

and $\theta^2(\mu)$ is upper semicontinuous. In particular,

$$\theta^2(\mu) \geq 1 \quad \text{on spt } \mu. \tag{A.10}$$

Now we consider $U = \mathbb{R}^n$. If $\limsup_{\varrho\to\infty} \varrho^{-2}\mu(B_\varrho(0)) = \infty$ then by (A.6)

$$\lim_{\varrho\to\infty} \varrho^{-2}\mu(B_\varrho(0)) = \infty \tag{A.11}$$

and by (A.5)

$$\lim_{\varrho\to\infty} \gamma(\varrho) = \infty. \tag{A.12}$$

If $\limsup_{\varrho\to\infty} \varrho^{-2}\mu(B_\varrho(0)) < \infty$, we estimate, for $0 < \sigma < \varrho < \infty$,

$$\begin{aligned} |R_{0,\varrho}| &\leq \frac{1}{2\varrho} \int\limits_{B_\varrho(0)} |H_\mu| \, \mathrm{d}\mu \leq \frac{1}{2\varrho} \int\limits_{B_\sigma(0)} |H_\mu| \, \mathrm{d}\mu \\ &\quad + \frac{1}{2}\Big(\varrho^{-2}\mu(B_\varrho(0))\Big)^{1/2} \ \parallel H_\mu \parallel_{L^2(\mathbb{R}^n - B_\sigma(0))}; \end{aligned}$$

hence

$$\limsup_{\varrho\to\infty} |R_{0,\varrho}| \leq \frac{1}{2}\Big(\limsup_{\varrho\to\infty} \varrho^{-2}\mu(B_\varrho(0))\Big)^{1/2} \ \parallel H_\mu \parallel_{L^2(\mathbb{R}^n - B_\sigma(0))}$$

and as $\sigma \to \infty$,

$$\lim_{\varrho\to\infty} R_{x_0,\varrho} = 0. \tag{A.13}$$

As γ is monotonically nondecreasing, we see in any case by (A.11), (A.12) and (A.13) that

$$\lim_{\varrho\to\infty}\gamma(\varrho)=\lim_{\varrho\to\infty}\varrho^{-2}\mu(B_\varrho(0))+\frac{1}{4}\mathcal{W}(\mu)\in[0,\infty]\quad\text{exists.}\tag{A.14}$$

If

$$\lim_{\varrho\to\infty}\varrho^{-2}\mu(B_\varrho(0))=0,\tag{A.15}$$

which is certainly true when spt μ is compact, we get from (A.6)

$$\varrho^{-2}\mu(B_\varrho(x_0))\le C\mathcal{W}(\mu).\tag{A.16}$$

Letting $\varrho\to\infty$ in (A.8) and recalling (A.13) and $\omega_2=\pi$, we arrive at the Li-Yau inequality, see [LY],

$$\theta^2(\mu,x_0)\le\frac{1}{4\pi}\mathcal{W}(\mu).\tag{A.17}$$

Together with (A.10) and $\mu\neq 0$, this implies

$$\mathcal{W}(\mu)\ge 4\pi\tag{A.18}$$

and hence

$$\inf_{\Sigma\text{ smooth}}\mathcal{W}(\Sigma)=4\pi=\inf_{\mu\neq 0}\mathcal{W}(\mu).\tag{A.19}$$

If

$$\mathcal{W}(\mu)<8\pi\tag{A.20}$$

then (A.17) and the integrality of μ yield

$$\theta^2(\mu)=1,\quad \mu\text{-almost everywhere.}\tag{A.21}$$

Actually, the assumption (A.15) is equivalent to the compactness of the support of μ. More precisely, we show that

$$\lim_{\varrho\to\infty}\varrho^{-2}\mu(B_\varrho(0))\ge\pi\Longleftrightarrow\text{spt }\mu\text{ is not compact.}\tag{A.22}$$

By (A.14) the limit on the left always exists in $[0,\infty]$. The inclusion from left to the right is obvious.

Now we assume that spt μ is not compact. For any compact component C of spt μ, we see from (A.18) that $\mathcal{W}(\mu\llcorner C)\ge 4\pi$. Therefore spt μ has only finitely many compact components, and as spt μ is assumed to be non-compact, it has at least one unbounded component. Therefore we can select $x_\varrho\in$ spt μ with $2\varrho:=|x_\varrho|$ for any large ϱ . From (A.6) and (A.10), we get for any $\delta>0$

$$\begin{aligned}\pi&\le(1+\delta)\varrho^{-2}\mu(B_\varrho(x_\varrho))+C_\delta\int\limits_{B_\varrho(x_\varrho)}|H_\mu|^2\,\mathrm{d}\mu\\&\le 9(1+\delta)(3\varrho)^{-2}\mu(B_{3\varrho}(0))+C_\delta\int\limits_{\mathbb{R}^n-B_\varrho(0)}|H_\mu|^2\,\mathrm{d}\mu.\end{aligned}$$

Letting first $\varrho \to \infty$ and then $\delta \downarrow 0$, we get (A.22) with π replaced by $\pi/9$.

To prove the full strength of (A.22), we may assume that

$$\lim_{\varrho \to \infty} \varrho^{-2} \mu(B_\varrho(0)) < \infty.$$

Then the limit

$$\zeta_{\varrho_j, \#} \mu \to \nu \quad \text{exists weakly as varifolds}$$

for some sequence $\varrho_j \to \infty$. We see for any $\varrho > 0$ that

$$\frac{\nu(\overline{B_\varrho(0)})}{\omega_2 \varrho^2} \geq \liminf_{j \to \infty} \frac{\mu(B_{\varrho_j \varrho}(0))}{\omega_2 (\varrho_j \varrho)^2} \geq 1/9$$

and $0 \in \operatorname{spt} \nu$. Further, ν is a stationary integral varifold. Hence $H_\nu = 0$ and by (A.8) and (A.10)

$$\pi \leq \nu(B_1(0)) \leq \liminf_{j \to \infty} \varrho_j^{-2} \mu(B_{\varrho_j}(0)),$$

which establishes (A.22).

Finally, we consider μ_j with $\mathcal{W}(\mu_j) < \infty$ and $\mu_j \to \mu$ weakly as varifolds. For γ_j defined in (A.4) by μ_j, we see

$$\gamma(\varrho) \leq \liminf_{j \to \infty} \gamma_j(\varrho) \quad \text{for almost all } \varrho > 0,$$

hence by (A.14) and monotonicity of γ_j

$$\lim_{\varrho \to \infty} \varrho^{-2} \mu(B_\varrho(0)) + \frac{1}{4} \mathcal{W}(\mu) \leq \liminf_{j \to \infty} \Big(\lim_{\varrho \to \infty} \varrho^{-2} \mu_j(B_\varrho(0)) + \frac{1}{4} \mathcal{W}(\mu_j) \Big).$$

In particular, if the spt μ_j are compact,

$$\lim_{\varrho \to \infty} \varrho^{-2} \mu(B_\varrho(0)) + \frac{1}{4} \mathcal{W}(\mu) \leq \liminf_{j \to \infty} \frac{1}{4} \mathcal{W}(\mu_j). \tag{A.23}$$

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