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Determination of the algebraic relations among special Γ-values in positive characteristic

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Abstract

We devise a new criterion for linear independence over function fields. Using this tool in the setting of dual t-motives, we find that all algebraic relations among special values of the geometric Γ-function over $\mathbb{F}_q[T]$ are explained by the standard functional equations.

Contents

1. Introduction

1.1. *Background on special* Γ*-values.*

1.1.1. *Notation.* Let $\mathbb{F}_q$ be a field of q elements, where q is a power of a prime p. Let $A := \mathbb{F}_q[T]$ and $k := \mathbb{F}_q(T)$, where T is a variable. Let $A_+ \subset A$ be the subset of monic polynomials. Let $|\cdot|_\infty$ be the unique valuation of k for which $|T|_\infty = q$. Let $k_\infty := \mathbb{F}_q((1/T))$ be the $|\cdot|_\infty$-completion of k, let $\overline{k_\infty}$ be an algebraic closure of k_∞, let $\mathbb{C}_\infty$ be the $|\cdot|_\infty$-completion of $\overline{k_\infty}$, and let $\bar{k}$ be the algebraic closure of k in $\mathbb{C}_\infty$.

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1.1.2. *The geometric* Γ*-function.* In [Th], Thakur studied the geometric Γ-function over $\mathbb{F}_q[T]$,

$$\Gamma(z) := \frac{1}{z} \prod_{n \in A_+} \left(1 + \frac{z}{n}\right)^{-1} \quad (z \in \mathbb{C}_\infty),$$

which is a meromorphic function on $\mathbb{C}_\infty$. Notably, it satisfies several natural functional equations, which are the analogues of the translation, reflection and Gauss multiplication identities satisfied by the classical Euler Γ-function, and to which we refer as the *standard functional equations* (see §5.3.5).

1.1.3. *Special* Γ*-values and the fundamental period of the Carlitz module.* We define the set of *special* Γ*-values* to be

$$\{\Gamma(z) \mid z \in k \setminus (-A_+ \cup \{0\})\} \subset k_\infty^\times.$$

Up to factors in $k^\times$ a special Γ-value $\Gamma(z)$ depends only on z modulo A. In connection with special Γ-values it is natural also to consider the number

$$\varpi := T \sqrt[q-1]{-T} \prod_{i=1}^{\infty} \left(1 - T^{1-q^i}\right)^{-1} \quad \in k_\infty \left(\sqrt[q-1]{-T}\right)^\times$$

where $\sqrt[q-1]{-T}$ is a fixed $(q-1)^{st}$ root of $-T$ in $\mathbb{C}_\infty$. The number ϖ is the fundamental period of the Carlitz module (see §5.1) and hence deserves to be regarded as the $\mathbb{F}_q[T]$-analogue of $2\pi i$. The transcendence of ϖ over k was first shown in [Wa]. (See §3.1.2 for a new proof.) Our goal in this paper to determine all Laurent polynomial relations with coefficients in $\bar{k}$ among special Γ-values and ϖ.

1.1.4. *Transcendence of special* Γ*-values.* For all $z \in A$ the value $\Gamma(z)$, when defined, belongs to k. However, it is known that for all $z \in k \setminus A$ the value $\Gamma(z)$ is transcendental over k. A short history of this transcendence result is as follows. Isolated results on the transcendence of special Γ-values were obtained in [Th]; in particular, it was observed that when $q = 2$, all values $\Gamma(z)$ with $z \in k \setminus A$ are $\bar{k}$-multiples of the Carlitz period ϖ. The first transcendence result for a general class of values of the Γ-function was obtained in [Si a]. Namely, Sinha showed that the values $\Gamma(\frac{a}{f} + b)$ are transcendental over k for all a, $f \in A_+$ and $b \in A$ such that $\deg a < \deg f$. Sinha's results were obtained by representing the Γ-values in question as periods of t-modules defined over $\bar{k}$ and then invoking a transcendence criterion of Gelfond-Schneider type from [Yu a]. Subsequently all the values $\Gamma(z)$ for $z \in k \setminus A$ were represented in [BrPa] as periods of t-modules defined over $\bar{k}$ and thus proved transcendental.

1.1.5. Γ*-monomials and the diamond bracket criterion.* An element of the subgroup of $\mathbb{C}_\infty^\times$ generated by ϖ and the special Γ-values will for brevity's sake be called a Γ*-monomial.* By adapting the Deligne-Koblitz-Ogus criterion

[De] to the function field setting along lines suggested in [Th], we have at our disposal a *diamond bracket criterion* (see Corollary 6.1.8) capable of deciding in a mechanical way whether between a given pair of Γ-monomials there exists a $\bar{k}$-linear relation explained by the standard functional equations. We call the two-term $\bar{k}$-linear dependencies thus arising *diamond bracket relations*.

1.1.6. *Cautionary example.* In order to deduce certain $\bar{k}$-linear relations between Γ-monomials from the standard functional equations, root extraction cannot be avoided. Consider the following example concerning the classical Γ-function taken from [Da]. The relation

$$\frac{\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{1}{5}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{15}\right)} = \sqrt{\left\{\begin{array}{l} \frac{\Gamma(\frac{1}{5})}{\Gamma(\frac{1}{15})\Gamma(\frac{2}{5})\Gamma(\frac{11}{15})} \times \frac{\Gamma(\frac{2}{5})}{\Gamma(\frac{2}{15})\Gamma(\frac{4}{5})\Gamma(\frac{7}{15})} \\[2ex] \times \frac{\Gamma(\frac{1}{15})\Gamma(\frac{4}{15})\Gamma(\frac{7}{15})\Gamma(\frac{2}{3})\Gamma(\frac{13}{15})}{\Gamma(\frac{1}{3})} \\[2ex] \times \frac{\Gamma(\frac{4}{15})\Gamma(\frac{11}{15})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \times \frac{\Gamma(\frac{1}{5})\Gamma(\frac{4}{5})}{\Gamma(\frac{2}{15})\Gamma(\frac{13}{15})} \end{array}\right.} = 3^{-\frac{1}{5}} 5^{\frac{1}{12}} \sqrt{\frac{\sin\frac{\pi}{3}\cdot\sin\frac{2\pi}{15}}{\sin\frac{4\pi}{15}\cdot\sin\frac{\pi}{5}}}$$

confirms the Deligne-Koblitz-Ogus criterion but decomposes into instances of the standard functional equations only after the terms are squared. The results of [Ku b] imply the existence of such peculiar examples in great abundance. See [Da] for a method by which essentially all such examples can be constructed explicitly. The analogous phenomena occur in the function field situation. For a discussion of the latter, see [BaGeKaYi]. For a simple example in the case $q = 3$, which was in fact discovered before all the others mentioned in this paragraph, see [Si b, §4].

1.1.7. *Linear independence.* It was shown in [BrPa] that the only relations of $\bar{k}$-linear dependence among 1, ϖ, and special Γ-values are those following from the diamond bracket relations. This result was obtained by carefully analyzing t-submodule structures and then invoking Yu's powerful theorem of the t-Submodule [Yu c].

1.2. *The main result.* We prove:

THEOREM 1.2.1 (cf. Theorem 6.2.1). *A set of* Γ-*monomials is $\bar{k}$-linearly dependent exactly when some pair of* Γ-*monomials is. Pairwise $\bar{k}$-linear (in)dependence of* Γ-*monomials is entirely decided by the diamond bracket criterion.*

In other words, all $\bar{k}$-linear relations among Γ-monomials are $\bar{k}$-linear combinations of the diamond bracket relations. The theorem has the following implication concerning transcendence degrees:

Corollary 1.2.2 (cf. Corollary 6.2.2). *For all $f \in A_+$ of positive degree, the extension of $\bar{k}$ generated by the set*

$$\{\varpi\} \cup \left\{\Gamma(x) \,\middle|\, x \in \frac{1}{f} A \setminus (\{0\} \cup -A_+)\right\}$$

is of transcendence degree $1 + \frac{q-2}{q-1} \cdot \#(A/f)^\times$ over $\bar{k}$.

In fact the corollary is equivalent to the theorem (see Proposition 6.2.3).

1.3. *Methods.* We outline the proof of Theorem 1.2.1, emphasizing the new methods introduced here, and compare our techniques to those used previously.

1.3.1. *A new linear independence criterion.* We develop a new method for detecting $\bar{k}$-linear independence of sets of numbers in $\overline{k_\infty}$, culminating in a quite easily stated criterion. Let t be a variable independent of T. Given $f = \sum_{i=0}^\infty a_i t^i \in \mathbb{C}_\infty[\![t]\!]$ and $n \in \mathbb{Z}$, put $f^{(n)} := \sum_{i=0}^\infty a_i^{q^n} t^i$ and extend the operation $f \mapsto f^{(n)}$ entrywise to matrices. Let $\mathcal{E} \subset \bar{k}[\![t]\!]$ be the subring consisting of power series $\sum_{i=0}^\infty a_i t^i$ such that $[k_\infty(\{a_i\}_{i=0}^\infty) : k_\infty] < \infty$ and $\lim_{i\to\infty} \sqrt[i]{|a_i|_\infty} = 0$. We now state our criterion (Theorem 3.1.1 is the verbatim repetition; see also Proposition 4.4.3):

Theorem 1.3.2. *Fix a matrix $\Phi = \Phi(t) \in \mathrm{Mat}_{\ell\times\ell}(\bar{k}[t])$ such that $\det \Phi$ is a polynomial in t vanishing (if at all) only at $t = T$. Fix a (column) vector $\psi = \psi(t) \in \mathrm{Mat}_{\ell\times 1}(\mathcal{E})$ satisfying the functional equation $\psi^{(-1)} = \Phi\psi$. Evaluate ψ at $t = T$, thus obtaining a (column) vector $\psi(T) \in \mathrm{Mat}_{\ell\times 1}\left(\overline{k_\infty}\right)$. For every (row) vector $\rho \in \mathrm{Mat}_{1\times\ell}(\bar{k})$ such that $\rho\psi(T) = 0$ there exists a (row) vector $P = P(t) \in \mathrm{Mat}_{1\times\ell}(\bar{k}[t])$ such that $P(T) = \rho$ and $P\psi = 0$.*

In other words, in the situation of this theorem, every $\bar{k}$-linear relation among entries of the specialization $\psi(T)$ is explained by a $\bar{k}[t]$-linear relation among entries of ψ itself.

1.3.3. *Dual t-motives.* The category of dual t-motives (see §4.4) provides a natural setting in which we can apply Theorem 1.3.2. Like t-motives in [An a], dual t-motives are modules of a certain sort over a certain skew polynomial ring. From a formal algebraic perspective dual t-motives differ very little from t-motives, and consequently most t-motive concepts carry over naturally to the dual t-motive setting. In particular, the concept of rigid analytic triviality carries over (see §4.4). Crucially, to give a rigid analytic trivialization of a dual t-motive is to give a square matrix with columns usable as input to Theorem 1.3.2 (see Lemma 4.4.12).

1.3.4. *Position of the new linear independence criterion with respect to Yu's Theorem of the t-Submodule.* We came upon Theorem 1.3.2 in the process of searching for a t-motivic translation of Yu's Theorem of the t-Submodule [Yu c]. Our discovery of a direct proof of Theorem 1.3.2 was a happy accident, but it was one for which we were psychologically prepared by close study of the proof of Yu's theorem.

Roughly speaking, the points of view adopted in the two theorems correspond as follows. If $H = \mathrm{Hom}(\mathbb{G}_a, E)$ is the dual t-motive defined over $\bar{k}$ corresponding canonically to a uniformizable abelian t-module E defined over $\bar{k}$, and $\Psi = \Psi(t)$ is a matrix describing a rigid analytic trivialization of H as in Lemma 4.4.12, then it is possible to express the periods of E in a natural way as $\bar{k}$-linear combinations of entries of $\Psi(T)^{-1}$ and *vice versa*. Thus it becomes at least plausible that Theorem 1.3.2 and Yu's theorem provide similar information about $\bar{k}$-linear independence. A detailed comparison of the two theorems is not going to be presented here; indeed, such has yet to be worked out. But we are inclined to believe that at the end of the day the theorems differ insignificantly in terms of ability to detect $\bar{k}$-linear independence.

In any case, it is clear that both theorems are strong enough to handle the analysis of $\bar{k}$-linear relations among Γ-monomials. Ultimately Theorem 1.3.2 is our tool of choice just because it is the easier to apply. Theorem 1.3.2 allows us to carry out our analysis entirely within the category of dual t-motives, which means that we can exclude t-modules from the picture altogether at a considerable savings of labor in comparison to [Si a] and [BrPa].

1.3.5. *Linking* Γ*-monomials to dual t-motives via Coleman functions.* In order to generalize beautiful examples in [Co] and [Th], *solitons* over $\mathbb{F}_q[T]$ were defined and studied in [An b]. In turn, in order to obtain various results on transcendence and algebraicity of special Γ-values, variants of solitons called *Coleman functions* were defined and studied in [Si a] and [Si b].

We present in this paper a self-contained elementary approach to Coleman functions producing new simple explicit formulas for them (see §5, §6.3). From the Coleman functions we then construct dual t-motives with rigid analytic trivializations described by matrices with entries specializing at $t = T$ to $\bar{k}$-linear combinations of Γ-monomials (see §6.4), thus putting ourselves in a position where Theorem 1.3.2 is at least potentially applicable.

Our method for attaching dual t-motives to Coleman functions is straightforwardly adapted from [Si a]. But our method for obtaining rigid analytic trivializations is more elementary than that of [Si a] because the explicit formulas for Coleman functions at our disposal obviate sophisticated apparatus from rigid analysis.

1.3.6. *Geometric complex multiplication.* The dual t-motives engendered by Coleman functions are equipped with extra endomorphisms and are examples of dual t-motives with geometric complex multiplication, GCM for short (see §4.6). We extend a technique developed in [BrPa] for analyzing t-modules with complex multiplication to the setting of dual t-motives with GCM, dubbing the generalized technique the *Dedekind-Wedderburn trick* (see §4.5). We determine that rigid analytically trivial dual t-motives with GCM are semisimple up to isogeny. In fact each such object is isogenous to a power of a simple dual t-motive.

1.3.7. *End of the proof.* Combining our general results on the structure of GCM dual t-motives with our concrete results on the structure of the dual t-motives engendered by Coleman functions, we can finally apply Theorem 1.3.2 (in the guise of Proposition 4.4.3) to rule out all $\bar{k}$-linear relations among Γ-monomials not following from the diamond bracket relations (see §6.5), thus proving Theorem 1.2.1.

1.4. *Comments on the classical case.* In the classical situation various people have formed a clear picture about what algebraic relations should hold among special Γ-values. Those ideas stimulated our interest and guided our intuition in the function-field setting. We discuss these ideas in more detail below.

1.4.1. *Temporary notation and terminology.* For the duration of §1.4, let $\Gamma(s)$ be the classical Γ-function, call $\{\Gamma(s) \mid s \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}\}$ the set of *special* Γ*-values*, and let a Γ*-monomial* be any element of the subgroup of $\mathbb{C}^\times$ generated by the special Γ-values and $2\pi i$.

1.4.2. *Rohrlich's conjecture.* Rohrlich in the late 1970's made a conjecture which in rough form can be stated thus: all multiplicative algebraic relations among special Γ-values and $2\pi i$ are explained by the standard functional equations. See [La b, App. to §2, p. 66] for a more precise formulation of the conjecture in the language of distributions. In language very similar to that we have used above, Rohrlich's conjecture can also be formulated as the assertion that the Deligne-Koblitz-Ogus criterion for a Γ-monomial to belong to $\overline{\mathbb{Q}}^\times$ is not only sufficient, but necessary as well.

1.4.3. *Lang's conjecture.* Lang subsequently strengthened Rohrlich's conjecture to a conjecture which in rough form can be stated thus: all polynomial algebraic relations among special Γ-values and $2\pi i$ with coefficients in $\overline{\mathbb{Q}}$ are explained by the standard functional equations. See [La b, loc. cit.] for a formulation of this conjecture in the language of distributions. In language very similar to that we have used above, Lang's conjecture can also be formulated as the assertion that all $\overline{\mathbb{Q}}$-linear relations among Γ-monomials follow linearly from the two-term relations provided by the Deligne-Koblitz-Ogus criterion.

Yet another formulation of Lang's conjecture is the assertion that for every integer $n > 2$ the transcendence degree of the extension of $\overline{\mathbb{Q}}$ generated by the set $\{2\pi i\} \cup \left\{\Gamma(x) \,\middle|\, x \in \frac{1}{n}\mathbb{Z} \setminus \mathbb{Z}_{\leq 0}\right\}$ is equal to $1 + \phi(n)/2$, where $\phi(n)$ is Euler's totient. In fact, as is underscored by the direct analogy between the numbers

$$1 + \phi(n)/2 = 1 + \left(1 - \frac{1}{\#\mathbb{Z}^\times}\right) \cdot \#(\mathbb{Z}/n)^\times$$

and

$$1 + \frac{q-2}{q-1} \cdot \#(A/f)^\times = 1 + \left(1 - \frac{1}{\#A^\times}\right) \cdot \#(A/f)^\times,$$

Corollary 1.2.2 is the precise analogue of the last version of Lang's conjecture mentioned above.

1.4.4. *Evidence in the classical case.* There are very few integers $n > 1$ such that all Laurent polynomial relations among elements of the set $\{2\pi i\} \cup \left\{\Gamma\left(\frac{1}{n}\right), \ldots, \Gamma\left(\frac{n-1}{n}\right)\right\}$ with coefficients in $\overline{\mathbb{Q}}$ can be ruled out save those following from the two-term relations provided by the Deligne-Koblitz-Ogus [De] criterion, to wit:

- $n = 2$ (Lindemann 1882, since $\Gamma(1/2) = \sqrt{\pi}$).
- $n = 3, 4$ (Chudnovsky 1974, cf. [Wal]).

The only other evidence known for Lang's conjecture is indirect, and it is contained in a result of [WoWü]: all $\overline{\mathbb{Q}}$-linear relations among the special beta values

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (a, b \in \mathbb{Q}, \quad a, b, a+b \notin \mathbb{Z}_{\leq 0})$$

and $2\pi i$ follow from the two-term relations provided by the Deligne-Koblitz-Ogus criterion.

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2. Notation and terminology

2.1. *Table of special symbols.*

T, t, z	:=	independent variables
$\mathbb{F}_q$	:=	a field of q elements
k	:=	$\mathbb{F}_q(T)$
$\lvert\cdot\rvert_\infty$	:=	the unique valuation of k such that $\lvert T\rvert_\infty = q$

k_∞	$:=$	$\mathbb{F}_q(\!(1/T)\!) =$ the $\lvert\cdot\rvert_\infty$-completion of k
$\overline{k_\infty}$	$:=$	an algebraic closure of k_∞
$\mathbb{C}_\infty$	$:=$	the $\lvert\cdot\rvert_\infty$-completion of $\overline{k_\infty}$
$\bar{k}$	$:=$	the algebraic closure of k in $\mathbb{C}_\infty$
$\widetilde{T}$	$:=$	a fixed choice in $\bar{k}$ of a $(q-1)^{st}$ root of $-T$
$\mathbb{C}_\infty\{t\}$	$:=$	the subring of the power series ring $\mathbb{C}_\infty[\![t]\!]$ consisting of power series convergent in the "closed" unit disc $\lvert t\rvert_\infty \leq 1$
$\#S$	$:=$	the cardinality of a set S
$\mathrm{Mat}_{r\times s}(R)$	$:=$	the set of r by s matrices with entries in a ring or module R
$R^\times$	$:=$	the group of units of a ring R with unit
$\mathrm{GL}_n(R)$	$:=$	$\mathrm{Mat}_{n\times n}(R)^\times$, where R is a ring with unit
$\mathbf{1}_n$	$:=$	the n by n identity matrix
A	$:=$	$\mathbb{F}_q[T]$
deg	$:=$	$(a \mapsto$ degree of a in $T) : A \to \mathbb{Z}\cup\{-\infty\}$
A_+	$:=$	the set of elements of A monic in T
D_N	$:=$	$\prod_{i=0}^{N-1}(T^{q^N} - T^{q^i}) \in A_+$
Res	$:=$	$\left(\sum_i a_i T^i \mapsto a_{-1}\right) : k_\infty \to \mathbb{F}_q$

2.2. *Twisting.* Fix $n \in \mathbb{Z}$. Given a formal power series $f = \sum_{i=0}^\infty a_i t^i \in \mathbb{C}_\infty[\![t]\!]$ we define the *n-fold twist* by the rule $f^{(n)} := \sum_{i=0}^\infty a_i^{q^n} t^i$. The n-fold twisting operation is an automorphism of the power series ring $\mathbb{C}_\infty[\![t]\!]$ stabilizing various subrings, e. g., $\bar{k}[\![t]\!]$, $\bar{k}[t]$, and $\mathbb{C}_\infty\{t\}$. More generally, for any matrix F with entries in $\mathbb{C}_\infty[\![t]\!]$ we define the n-fold twist $F^{(n)}$ by the rule $\left(F^{(n)}\right)_{ij} := (F_{ij})^{(n)}$. In particular, for any matrix X with entries in $\mathbb{C}_\infty$ we have $\left(X^{(n)}\right)_{ij} = (X_{ij})^{q^n}$. The n-fold twisting operation commutes with matrix addition and multiplication.

2.3. *Norms.* For any matrix X with entries in $\mathbb{C}_\infty$ we put $\lvert X\rvert_\infty := \max_{ij}\lvert X_{ij}\rvert_\infty$. Now $\left\lvert X^{(n)}\right\rvert_\infty = \lvert X\rvert_\infty^{q^n}$ for all $n \in \mathbb{Z}$ and

$$\lvert U+V\rvert_\infty \leq \max(\lvert U\rvert_\infty, \lvert V\rvert_\infty), \quad \lvert XY\rvert_\infty \leq \lvert X\rvert_\infty \cdot \lvert Y\rvert_\infty$$

for all matrices U, V, X, Y with entries in $\mathbb{C}_\infty$ such that $U+V$ and XY are defined.

2.4. *The ring $\mathcal{E}$.* We define $\mathcal{E}$ to be the ring consisting of formal power series

$$\sum_{n=0}^\infty a_n t^n \in \bar{k}[\![t]\!]$$

such that

$$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert_\infty} = 0, \quad [k_\infty(a_0, a_1, a_2, \ldots) : k_\infty] < \infty.$$

The former condition guarantees that such a series has an infinite radius of convergence with respect to the valuation $|\cdot|_\infty$. The latter condition guarantees that for any $t_0 \in \overline{k_\infty}$ the value of such a series at $t = t_0$ belongs again to $\overline{k_\infty}$. Note that the ring $\mathcal{E}$ is stable under the n-fold twisting operation $f \mapsto f^{(n)}$ for all $n \in \mathbb{Z}$.

2.5. *The Schwarz-Jensen formula.* Fix $f \in \mathcal{E}$ not vanishing identically. It is possible to enumerate the zeroes of f in $\mathbb{C}_\infty$ because there are only finitely many zeroes in each disc of finite radius. Put

$$\{\omega_i\} := \text{an enumeration (with multiplicity) of the zeroes of } f \text{ in } \mathbb{C}_\infty$$

and

$$\lambda := \text{the leading coefficient of the Maclaurin expansion of } f.$$

The *Schwarz-Jensen* formula

$$\sup_{\substack{x\in\mathbb{C}_\infty\\ |x|\leq r}} |f(x)|_\infty = |\lambda|_\infty \cdot r^{\#\{i|\omega_i=0\}} \cdot \prod_{i:\, 0<|\omega_i|_\infty<r} \frac{r}{|\omega_i|_\infty} \qquad (r \in \mathbb{R}_{>0})$$

relates the growth of the modulus of f to the distribution of the zeroes of f. This fact is an easily deduced corollary to the Weierstrass Preparation Theorem over a complete discrete valuation ring.

3. A linear independence criterion

3.1. *Formulation and discussion of the criterion.*

THEOREM 3.1.1. *Fix a matrix*

$$\Phi = \Phi(t) \in \mathrm{Mat}_{\ell\times\ell}(\bar{k}[t]),$$

such that $\det\Phi$ *is a polynomial in t vanishing (if at all) only at $t = T$. Fix a (column) vector*

$$\psi = \psi(t) \in \mathrm{Mat}_{\ell\times 1}(\mathcal{E})$$

satisfying the functional equation

$$\psi^{(-1)} = \Phi\psi.$$

Evaluate ψ at $t = T$, thus obtaining a (column) vector

$$\psi(T) \in \mathrm{Mat}_{\ell\times 1}\left(\overline{k_\infty}\right).$$

For every (row) vector

$$\rho \in \mathrm{Mat}_{1\times\ell}(\bar{k})$$

such that

$$\rho\psi(T) = 0$$

there exists a (row) vector

$$P = P(t) \in \mathrm{Mat}_{1\times\ell}(\bar{k}[t])$$

such that

$$P(T) = \rho, \quad P\psi = 0.$$

The proof commences in §3.3 and takes up the rest of Section 3. We think of the $\bar{k}[t]$-linear relation P among the entries of ψ produced by the theorem as an "explanation" or a "lifting" of the given $\bar{k}$-linear relation ρ among the entries of $\psi(T)$.

3.1.2. *The basic application.* Consider the power series

$$\Omega = \Omega(t) := \tilde{T}^{-q} \prod_{i=1}^{\infty} \left(1 - t/T^{q^i}\right) \in k_\infty(\widetilde{T})[\![t]\!] \subset \mathbb{C}_\infty[\![t]\!].$$

The power series $\Omega(t)$ has an infinite radius of convergence and satisfies the functional equation

$$\Omega^{(-1)} = (t-T)\cdot\Omega.$$

Consider the Maclaurin expansion

$$\Omega(t) = \sum_{i=0}^{\infty} a_i t^i.$$

The functional equation satisfied by Ω implies the recursion

$$\sqrt[q]{a_i} + Ta_i = \begin{cases} a_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0. \end{cases}$$

Therefore Ω belongs to $\bar{k}[\![t]\!]$ and hence to $\mathcal{E}$. Suppose now that there exists a nontrivial $\bar{k}$-linear relation

$$\sum_{i=0}^{n} \rho_i \Omega(T)^i = 0 \quad (\rho_i \in \bar{k}, \quad n > 0, \quad \rho_0\rho_n \neq 0)$$

among the powers of the number

$$\Omega(T) = \tilde{T}^{-q} \prod_{i=1}^{\infty} (1 - T^{1-q^i}) \in k_\infty(\widetilde{T}).$$

Theorem 3.1.1 provides a $\bar{k}[t]$-linear "explanation"

$$\sum_{i=0}^{n} P_i \Omega^i = 0 \quad (P_i \in \bar{k}[t], \;\; P_i(T) = \rho_i).$$

But the polynomial P_0 must vanish at all the zeroes $t = T^{q^i}$ of the function Ω. Thus P_0 vanishes identically, contrary to our assumption that $\rho_0 = P_0(T) \neq 0$. We conclude that $\Omega(T)$ is transcendental over k.

See §5.1 below for the interpretation of $-1/\Omega(T)$ as the fundamental period of the Carlitz module. The power series $\Omega(t)$ plays a key role in this paper.

Proposition 3.1.3. *Suppose*

$$\Phi \in \mathrm{Mat}_{\ell\times\ell}(\bar{k}[t]), \quad \psi \in \mathrm{Mat}_{\ell\times 1}\left(\mathbb{C}_\infty\{t\}\right)$$

such that

$$\det \Phi(0) \neq 0, \quad \psi^{(-1)} = \Phi\psi.$$

Then

$$\psi \in \mathrm{Mat}_{\ell\times 1}(\mathcal{E}).$$

The proposition simplifies the task of checking the hypotheses of Theorem 3.1.1.

Proof. Write

$$\Phi = \sum_{i=0}^{N} b_{(i)} t^i \quad (b_{(i)} \in \mathrm{Mat}_{\ell\times\ell}(\bar{k}),\ N:\ \text{positive integer}).$$

By hypothesis $b_{(0)} \in \mathrm{GL}_\ell(\bar{k})$. By the theory of Lang isogenies [La a] there exists $U \in \mathrm{GL}_{\ell\times\ell}(\bar{k})$ such that

$$U^{(-1)} b_{(0)} U^{-1} = \mathbf{1}_\ell \quad \left(\text{equivalently: } b_{(0)}^{(1)} = U^{-1}U^{(1)}\right).$$

After making the replacements

$$\psi \leftarrow U\psi, \quad \Phi \leftarrow U^{(-1)}\Phi U^{-1},$$

we may assume without loss of generality that $b_{(0)} = \mathbf{1}_\ell$. Now write

$$\psi = \sum_{i=0}^{\infty} a_{(i)} t^i \quad (a_{(i)} \in \mathrm{Mat}_{\ell\times 1}(\mathbb{C}_\infty)).$$

We have

$$a_{(n)}^{(-1)} - a_{(n)} = \sum_{i=1}^{\min(n,N)} b_{(i)} a_{(n-i)},$$

and hence

$$a_{(n)} \in \mathrm{Mat}_{\ell\times 1}(\bar{k})$$

for all integers $n \geq 0$. By hypothesis

$$\lim_{n\to\infty} |a_{(n)}|_\infty = 0,$$

and hence the series

$$\tilde{a}_{(n)} := \sum_{\nu=1}^{\infty} \left(\sum_{i=1}^{N} b_{(i)} a_{(n-i)}\right)^{(\nu)}$$

converges for all $n \gg 0$. Moreover,

$$\lim_{n\to\infty} |\tilde{a}_{(n)}|_\infty = 0.$$

Since

$$\left(\tilde{a}_{(n)} - a_{(n)}\right)^{(-1)} = \left(\tilde{a}_{(n)} - a_{(n)}\right)$$

for $n \gg 0$, it follows that $\tilde{a}_{(n)} - a_{(n)} = 0$ for $n \gg 0$ and hence that the collection of entries of all the vectors $a_{(n)}$ generates an extension of k_∞ of finite degree. Now fix $C > 1$ arbitrarily. From the fact that $\tilde{a}_{(n)} = a_{(n)}$ for $n \gg 0$, we have inequalities

$$\begin{aligned} C^n |a_{(n)}|_\infty &\leq \max_{i=1}^{N} C^n |b_{(i)}|_\infty^q |a_{(n-i)}|_\infty^q \\ &\leq \left(\max_{i=1}^{N} C^i |b_{(i)}|_\infty^q\right) \cdot \left(\max_{i=1}^{N} |a_{(n-i)}|_\infty\right)^{q-1} \cdot \left(\max_{i=1}^{N} C^{n-i} |a_{(n-i)}|_\infty\right) \\ &\leq \max_{i=0}^{n-1} C^i |a_{(i)}|_\infty \end{aligned}$$

for $n \gg 0$, and hence

$$\sup_{n=0}^{\infty} C^n |a_{(n)}|_\infty < \infty.$$

Therefore the radius of convergence of each entry of ψ is infinite. □

3.1.4. *Remark.* Theorem 3.1.1 is in essence the (dual) t-motivic translation of Yu's Theorem of the t-Submodule [Yu c, Thms. 3.3 and 3.4]. Once the setting is sufficiently developed, we expect that the class of numbers about which Theorem 3.1.1 provides $\bar{k}$-linear independence information is essentially the same as that handled by Yu's theorem of the t-Submodule, and the type of information provided is essentially the same, too. We omit discussion of the comparison.

3.2. *Specialized notation for making estimates.*

3.2.1. *Degree in t.* Given a polynomial $f \in \bar{k}[t]$ let $\deg_t f$ denote its degree in t (as usual $\deg 0 := -\infty$) and, more generally, given a matrix F with entries in $\bar{k}[t]$ put $\deg_t F := \max_{ij} \deg_t F_{ij}$. Now, $\deg_t F^{(n)} = \deg_t F$ for all $n \in \mathbb{Z}$ and we have

$$\deg_t(D+E) \leq \max\left(\deg_t D, \deg_t E\right), \quad \deg_t(FG) \leq \deg_t F + \deg_t G$$

for all matrices D, E, F, G with entries in $\bar{k}[t]$ such that $D+E$ and FG are defined.

3.2.2. *Size.* Given an algebraic number $x \in \bar{k}$ we set $\|x\| := \max_\tau |\tau x|_\infty$, where τ ranges over the automorphisms of $\bar{k}/k$, thereby defining the *size* of x.

More generally given a polynomial $f = \sum_i a_i t^i \in \bar{k}[t]$, we define $\|f\| := \max_i \|a_i\|$. Yet more generally, given a matrix F with entries in $\bar{k}[t]$ we define $\|F\| := \max_{ij} \|F_{ij}\|$. Then we have $\left\|F^{(n)}\right\| = \|F\|^{q^n}$ for all $n \in \mathbb{Z}$. Now,

$$\|D+E\| \leq \max(\|D\|, \|E\|), \quad \|FG\| \leq \|F\| \cdot \|G\|$$

for all matrices D, E, F, G with entries in $\bar{k}[t]$ such that $D+E$ and FG are defined.

3.3. *The basic estimates.*

3.3.1. *The setting.* Throughout §3.3 we fix fields

$$k \subset K_0 \subset K \subset \bar{k}$$

and rings

$$A \subset \mathcal{O}_0 \subset \mathcal{O} \subset K$$

such that

- K_0/k is a finite separable extension,
- K is the closure of K_0 in $\bar{k}$ under the extraction of q^{th} roots,
- $\mathcal{O}$ is the integral closure of A in K, and
- $\mathcal{O}_0 = \mathcal{O} \cap K_0$.

Note that $\bar{k}$ is the union of all its subfields of the form K.

3.3.2. *Lower bound from size.* We claim that

$$\|x\| \geq 1, \quad |x|_\infty \geq \|x\|^{1-[K_0:k]}$$

for all $0 \neq x \in \mathcal{O}$. Clearly these estimates hold in the case $0 \neq x \in \mathcal{O}_0$, because in that case x has at most $[K_0 : k]$ conjugates over k and the product of those conjugates is a nonzero element of A; but then, since we have

$$\mathcal{O} = \bigcup_{\nu=0}^{\infty} \mathcal{O}_0^{q^{-\nu}},$$

the claim holds in general.

Lemma 3.3.3 (Liouville Inequality). *Fix a polynomial*

$$f(z) := \sum_{i=0}^{n} a_i z^i \in \mathcal{O}[z]$$

not vanishing identically. For every nonzero root $\lambda \in \bar{k}$ of $f(z)$ of order ν,

$$|\lambda|_\infty^\nu \geq \left(\max_{i=0}^{n} \|a_i\|\right)^{-[K_0:k]}.$$

Proof. We may of course assume that $|\lambda|_\infty < 1$, for otherwise the claim is obvious. After factoring out a power of z we may also assume that $a_0 \neq 0$. Write

$$f(z+\lambda) = \sum_{i=\nu}^{n} b_i z^i \quad (b_i \in \mathcal{O}[\lambda]),$$

noting that

$$|b_i|_\infty \leq \max_{j=i}^{n} |a_j|_\infty \leq \max_{i=0}^{n} \|a_i\|.$$

Evaluate the displayed expression for $f(z+\lambda)$ at $z = -\lambda$, thus obtaining an estimate

$$|a_0|_\infty = |f(0)|_\infty \leq \max_{i=\nu}^{n} |b_i \lambda^i|_\infty \leq |\lambda|_\infty^\nu \max_{i=0}^{n} \|a_i\|.$$

Finally, apply the fundamental lower bound of §3.3.2 to a_0. □

LEMMA 3.3.4. *For all constants $C > 1$,*

$$\lim_{\nu\to\infty} \left(\#\left\{x \in \mathcal{O}_0^{q^{-\nu}} \,\middle|\, \|x\| \leq C\right\}\right)^{\frac{1}{q^\nu \cdot [K_0:k]}} = C.$$

The normalization $|T|_\infty = q$ was imposed to make this formula hold.

Proof. We may assume without loss of generality that C is of the form

$$C = q^\delta \qquad \left(\delta \in \bigcup_{\nu=0}^{\infty} q^{-\nu}\mathbb{Z}, \quad \delta > 0\right).$$

The Riemann-Roch theorem yields constants n_0 and n_1 such that

$$n > n_0 \Rightarrow \#\{x \in \mathcal{O}_0 | \|x\| \leq |T|_\infty^n\} = q^{[K_0:k]n + n_1}$$

for all $n \in \mathbb{Z}$. We then have

$$\#\left\{x \in \mathcal{O}_0^{q^{-\nu}} \,\middle|\, \|x\| \leq |T|_\infty^\delta\right\} = \#\left\{x \in \mathcal{O}_0 \,\middle|\, \|x\| \leq |T|_\infty^{q^\nu\delta}\right\} = q^{[K_0:k]q^\nu\delta + n_1}$$

for all integers $\nu \gg 0$, whence the result. □

LEMMA 3.3.5 (Thue-Siegel Analogue). *Fix parameters*

$$C > 1, \quad 0 < r < s \quad (C \in \mathbb{R}, \quad r, s \in \mathbb{Z}).$$

For each matrix

$$M \in \mathrm{Mat}_{r\times s}(\mathcal{O})$$

such that

$$\|M\| < C$$

there exists

$$x \in \mathrm{Mat}_{s\times 1}(\mathcal{O})$$

such that

$$x \neq 0, \quad Mx = 0, \quad \|x\| < C^{\frac{r}{s-r}}.$$

Proof. Choose $C' > 1$ and $\varepsilon > 0$ such that

$$\|M\| < C', \quad (1+\varepsilon)(C')^{\frac{r}{s-r}} < C^{\frac{r}{s-r}}.$$

For all $\nu \gg 0$ the cardinality of the set

$$\left\{ x \in \mathrm{Mat}_{s\times 1}\left(\mathcal{O}_0^{q^{-\nu}}\right) \middle| \, \|x\| \leq (1+\varepsilon)(C')^{\frac{r}{s-r}} \right\}$$

exceeds the cardinality of the set

$$\left\{ x \in \mathrm{Mat}_{r\times 1}\left(\mathcal{O}_0^{q^{-\nu}}\right) \middle| \, \|x\| \leq (1+\varepsilon)(C')^{\frac{s}{s-r}} \right\}$$

by Lemma 3.3.4. Further, for all $\nu \gg 0$ multiplication by M maps the former set to the latter. Therefore the desired vector x exists by the pigeonhole principle. □

LEMMA 3.3.6. *Again fix parameters*

$$C > 1, \quad 0 < r < s \quad (C \in \mathbb{R}, \ \ r, s \in \mathbb{Z}).$$

For each matrix

$$M \in \mathrm{Mat}_{r\times s}(\mathcal{O}[t])$$

such that

$$\|M\| < C$$

there exists

$$x \in \mathrm{Mat}_{s\times 1}(\mathcal{O}[t])$$

such that

$$x \neq 0, \quad Mx = 0, \quad \|x\| < C^{\frac{r}{s-r}}.$$

Proof. Let d and e be nonnegative integers presently to be chosen efficaciously large and put

$$r' := r(d+e+1), \quad s' := s(e+1).$$

Choose d large enough so that

$$\deg_t M \leq d,$$

and then choose e large enough so that

$$r' < s', \quad \|M\| < C' := C^{\frac{r}{s-r}/\frac{r'}{s'-r'}}.$$

Consider now the $\mathcal{O}$-linear map

$$\{x \in \mathrm{Mat}_{s\times 1}(\mathcal{O}[t]) | \deg_t x \leq e\} \to \{x \in \mathrm{Mat}_{r\times 1}(\mathcal{O}[t]) | \deg_t x \leq d+e\}$$

induced by multiplication by M. With respect to the evident choice of bases the map under consideration is represented by a matrix

$$M' \in \mathrm{Mat}_{r'\times s'}(\mathcal{O})$$

such that
$$\left\|M'\right\| < C'.$$
The existence of $x \in \mathrm{Mat}_{s\times 1}(\mathcal{O}[t])$ such that
$$x \neq 0, \quad Mx = 0, \quad \deg_t x \leq e, \quad \|x\| < (C')^{\frac{r'}{s'-r'}} = C^{\frac{r}{s-r}}$$
now follows by an application of the preceding lemma with the triple of parameters (C', r', s') in place of the triple (C, r, s). □

3.4. *Proof of the criterion.*

3.4.1. *The case $\ell = 1$.* Assume for the moment that $\ell = 1$. In this case we may assume without loss of generality that $\rho \neq 0$ and hence that
$$\psi(T) = 0,$$
in which case our task is to show that ψ vanishes identically. For any integer $\nu \geq 0$ we have
$$\left(\psi\left(T^{q^{-\nu}}\right)\right)^{q^{-1}} = \psi^{(-1)}\left(T^{q^{-(\nu+1)}}\right) = \Phi\left(T^{q^{-(\nu+1)}}\right)\psi\left(T^{q^{-(\nu+1)}}\right),$$
$$\Phi\left(T^{q^{-(\nu+1)}}\right) \neq 0$$
and hence,
$$\psi\left(T^{q^{-\nu}}\right) = 0 \quad (\nu = 0, 1, 2, \ldots).$$
Since ψ vanishes infinitely many times in the disc $|t|_\infty \leq |T|_\infty$, necessarily ψ vanishes identically. Thus the case $\ell = 1$ of Theorem 3.1.1 is dispatched.

3.4.2. *Reductions and further notation.* Assume now that $\ell > 1$. We may of course assume that
$$\rho \neq 0.$$
As in §3.3 let
$$k \subset K_0 \subset K \subset \bar{k}$$
be fields such that K_0/k is a finite separable extension and K is the closure of K_0 under the extraction of q^{th} roots. Since $\bar{k}$ is the union of fields of the form K we may assume without loss of generality that
$$\Phi \in \mathrm{Mat}_{\ell\times\ell}(K[t]), \quad \rho \in \mathrm{Mat}_{1\times\ell}(K).$$
As in §3.3 let $\mathcal{O}$ be the integral closure of A in K. After making replacements
$$\Phi \leftarrow a^{q-1}\Phi, \ \ \psi \leftarrow a^{-q}\psi, \quad \rho \leftarrow b\rho$$
for suitably chosen
$$a, b \in A, \quad ab \neq 0,$$

we may assume without loss of generality that

$$\Phi \in \mathrm{Mat}_{\ell\times\ell}(\mathcal{O}[t]), \quad \rho \in \mathrm{Mat}_{1\times\ell}(\mathcal{O}).$$

Fix a matrix

$$\vartheta \in \mathrm{Mat}_{\ell\times(\ell-1)}(\mathcal{O})$$

of maximal rank such that

$$\rho\vartheta = 0.$$

Then the K-subspace of $\mathrm{Mat}_{1\times\ell}(K)$ annihilated by right multiplication by ϑ is the K-span of ρ. Let

$$\Theta \in \mathrm{Mat}_{\ell\times\ell}(\mathcal{O}[t])$$

be the transpose of the matrix of cofactors of Φ. Then,

$$\Phi\Theta = \Theta\Phi = \det\Phi \cdot \mathbf{1}_\ell = c(t-T)^s \cdot \mathbf{1}_\ell$$

for some $0 \neq c \in \mathcal{O}$ and integer $s \geq 0$. Let N be a parameter taking values in the set of positive integers divisible by 2ℓ.

3.4.3. *Construction of the auxiliary function E.* We claim there exists

$$h = h(t) \in \mathrm{Mat}_{1\times\ell}(\mathcal{O}[t])$$

depending on the parameter N such that

- $\|h\| = O(1)$ as $N \to \infty$

and with the following properties for each value of N:

- $h \neq 0$.
- $\deg_t h < \left(1 - \frac{1}{2\ell}\right) N$.
- $E\left(T^{q^{-(N+\nu)}}\right) = 0$ for $\nu = 0, \ldots, N-1$, where $E := h\psi \in \mathcal{E}$.
(We call E the *auxiliary function*.)

Before proving the claim, we note first that the auxiliary function E figures in the following identity:

$$\begin{aligned} &h\Theta^{(-0)} \cdots \Theta^{(-(N+\nu-1))}\psi^{(-(N+\nu))} \\ &\qquad = h\Theta^{(-0)} \cdots \Theta^{(-(N+\nu-1))}\Phi^{(-(N+\nu-1))} \cdots \Phi^{(-0)}\psi \\ &\qquad = c^{q^{-(N+\nu-1)}+\cdots+q^0}\left(t - T^{q^{-(N+\nu-1)}}\right)^s \cdots \left(t - T^{q^{-0}}\right)^s E. \end{aligned}$$

This identity is useful again below and so for convenient reference we dub it the *key identity*. By the key identity, the hypothesis

$$\rho\psi(T) = 0 \quad \left(\text{equivalently:} \quad \rho^{(-(N+\nu))}\psi^{(-(N+\nu))}\left(T^{q^{-(N+\nu)}}\right) = 0\right),$$

and the definition of ϑ, the following condition forces the desired vanishing of E:

- $h\Theta^{(0)}\cdots\Theta^{(-(N+\nu-1))}\vartheta^{(-(N+\nu))}\big|_{t=T^{q^{-(N+\nu)}}} = 0$ for $\nu = 0,\ldots,N-1$.

Put

$$r := (\ell-1)\,N, \quad s := \left(\ell - \frac{1}{2}\right)N.$$

With respect to the evident choices of bases, the homogeneous system of linear equations that we need to solve is described by a matrix

$$M \in \mathrm{Mat}_{r\times s}(\mathcal{O})$$

depending on N such that

$$\|M\| \le |T|_\infty^{q^{-N}((1-\frac{1}{2\ell})N+2N\cdot\deg_t\Theta)} \cdot \|\Theta\|^{\frac{q}{q-1}} \cdot \|\vartheta\| = O(1) \quad \text{as} \quad N\to\infty,$$

and the solution we need to find is described by a vector

$$x \in \mathrm{Mat}_{s\times 1}(\mathcal{O})$$

depending on N such that

$$x \ne 0, \quad Mx = 0, \quad \|x\| = O(1) \quad \text{as} \quad N\to\infty.$$

Lemma 3.3.5 now proves our claim.

3.4.4. *A functional equation for E.* We claim there exist polynomials

$$a_0,\ldots,a_\ell \in \mathcal{O}[t]$$

depending on the parameter N such that

- $\max\limits_{i=0}^{\ell}\|a_i\| = O(1)$ as $N\to\infty$

and with the following properties for each value of N:

- Not all the a_i vanish identically.
- $a_0E + a_1E^{(-1)} + \cdots + a_\ell E^{(-\ell)} = 0.$

Since

$$E^{(-\nu)} = h^{(-\nu)}\Phi^{(-(\nu-1))}\cdots\Phi^{(-0)}\psi$$

for any integer $\nu \geq 0$, the functional equation we want E to satisfy is implied by the following condition:

- $a_0 h^{(0)} + a_1 h^{(-1)}\Phi^{(-0)} + \cdots + a_\ell h^{(-\ell)}\Phi^{(-(\ell-1))}\cdots\Phi^{(-0)} = 0.$

The latter system of homogeneous linear equations for $a_0, \ldots, a_\ell$ is with respect to the evident choice of bases described by a matrix

$$M \in \mathrm{Mat}_{\ell\times(\ell+1)}(\mathcal{O}[t])$$

depending on N such that

$$\|M\| = O(1) \quad \text{as} \quad N \to \infty,$$

and the solution we have to find is described by a vector

$$x \in \mathrm{Mat}_{(\ell+1)\times 1}(\mathcal{O}[t])$$

depending on N such that

$$x \neq 0, \quad Mx = 0, \quad \|x\| = O(1) \quad \text{as} \quad N \to \infty.$$

Lemma 3.3.6 now proves our claim. After dividing out common factors of t we may further assume that for each value of N:

- Not all the constant terms $a_i(0)$ vanish.

3.4.5. *Vanishing of E.* We claim that E vanishes identically for some N. Suppose that this is not the case. Let λ be the leading coefficient of the Maclaurin expansion of E. We have

$$a_0(0)\lambda^{q^0} + \cdots + a_\ell(0)\lambda^{q^{-\ell}} = 0,$$

and hence

$$1/|\lambda|_\infty = O(1) \text{ as } N \to \infty$$

by Lemma 3.3.3. But we also have

$$|\lambda|_\infty \cdot |T|_\infty^{N-\frac{q}{q-1}} \leq \sup_{\substack{x\in\mathbb{C}_\infty \\ |x|_\infty\leq|T|_\infty}} |E(x)|_\infty \leq \sup_{\substack{x\in\mathbb{C}_\infty \\ |x|_\infty\leq|T|_\infty}} |\psi(x)|_\infty \cdot \|h\| \cdot |T|_\infty^{N\left(1-\frac{1}{2\ell}\right)},$$

for all N, the inequality on the left by the Schwarz-Jensen formula, and hence

$$|\lambda|_\infty = O\left(|T|_\infty^{-\frac{N}{2\ell}}\right) \quad \text{as } N \to \infty.$$

These bounds for $|\lambda|_\infty$ are contradictory for $N \gg 0$. Our claim is proved.

3.4.6. *The case* $E = 0$. Now fix a value of N such that the auxiliary function E vanishes identically. Since the entries of the vector h are polynomials in t of degree $< N$, not all vanishing identically, there exists some $0 \leq \nu < N$ such that

$$h^{(N+\nu)}(T) = h\left(T^{q^{-(N+\nu)}}\right)^{q^{N+\nu}} \neq 0.$$

Put

$$P = P(t) := h^{(N+\nu)}\Theta^{(N+\nu)} \cdots \Theta^{(1)} \in \mathrm{Mat}_{1\times\ell}(\mathcal{O}[t]).$$

Since

$$\det\left(\Theta^{(N+\nu)} \cdots \Theta^{(1)}\right)\Big|_{t=T} \neq 0,$$

we have

$$P(T) \neq 0.$$

We also have

$$P(T)\vartheta = \left(h\Theta^{(-0)} \cdots \Theta^{(-(N+\nu-1))}\vartheta^{(-(N+\nu))}\Big|_{t=T^{q^{-(N+\nu)}}}\right)^{q^{N+\nu}} = 0,$$

and hence

$$P(T) \in (K\text{-span of } \rho) \subset \mathrm{Mat}_{1\times\ell}(K).$$

Finally, we have

$$P\psi = c^{q+\cdots+q^{N+\nu}}(t - T^q)^s \cdots \left(t - T^{q^{N+\nu}}\right)^s E^{(N+\nu)} = 0$$

by the key identity. Therefore (up to a nonzero correction factor in K) the vector P is the vector we want, and the proof of Theorem 3.1.1 is complete. □

4. Tools from (non)commutative algebra

4.1. *The ring* $\bar{k}[\sigma]$.

4.1.1. *Definition.* Let $\bar{k}[\sigma]$ be the ring obtained by adjoining a noncommutative variable σ to $\bar{k}$ subject to the commutation relations

$$\sigma x = x^{q^{-1}}\sigma \quad (x \in \bar{k}).$$

Every element of $\bar{k}[\sigma]$ has a unique presentation of the form

$$\sum_{i=0}^{\infty} a_i\sigma^i \quad (a_i \in \bar{k},\ \ a_i = 0 \text{ for } i \gg 0),$$

and in terms of such presentations the multiplication law in $\bar{k}[\sigma]$ takes the form

$$\left(\sum_i a_i\sigma^i\right)\left(\sum_j b_j\sigma^j\right) = \sum_i\sum_j a_i b_j^{q^{-i}}\sigma^{i+j}.$$

Given

$$\phi = \sum_{i=0}^{\infty} a_i \sigma^i \in \bar{k}[\sigma] \quad (a_i \in \bar{k}, \;\; a_i = 0 \text{ for } i \gg 0),$$

we define

$$\deg_\sigma \phi := \max(\{-\infty\} \cup \{i | a_i \neq 0\}).$$

Clearly we have

$$\deg_\sigma \phi\psi = \deg_\sigma \phi + \deg_\sigma \psi \quad (\phi, \psi \in \bar{k}[\sigma]).$$

The ring $\bar{k}[\sigma]$ admits interpretation as the opposite of the ring of $\mathbb{F}_q$-linear endomorphisms of the additive group over $\bar{k}$. This interpretation is not actually needed in the sequel but might serve as a guide to the intuition of the reader.

4.1.2. *Division algorithms and their uses.* The ring $\bar{k}[\sigma]$ has a left (resp., right) division algorithm:

- For all $\psi, \phi \in \bar{k}[\sigma]$ such that $\phi \neq 0$ there exist unique $\theta, \rho \in \bar{k}[\sigma]$ such that $\psi = \phi\theta + \rho$ (resp., $\psi = \theta\phi + \rho$) and $\deg_\sigma \rho < \deg_\sigma \phi$.

Some especially useful properties of $\bar{k}[\sigma]$ and of left modules over it readily deducible from the existence of left and right division algorithms are as follows:

- Every left ideal of $\bar{k}[\sigma]$ is principal.
- Every finitely generated left $\bar{k}[\sigma]$-module is noetherian.
- $\dim_{\bar{k}} \bar{k}[\sigma]/\bar{k}[\sigma]\phi = \deg_\sigma \phi < \infty$ for all $0 \neq \phi \in \bar{k}[\sigma]$.
- For every matrix $\phi \in \mathrm{Mat}_{r\times s}(\bar{k}[\sigma])$ there exist matrices $\alpha \in \mathrm{GL}_r(\bar{k}[\sigma])$ and $\beta \in \mathrm{GL}_s(\bar{k}[\sigma])$ such that the product $\alpha\phi\beta$ vanishes off the main diagonal.
- A finitely generated free left $\bar{k}[\sigma]$-module has a well-defined rank; i.e., all $\bar{k}[\sigma]$-bases have the same cardinality.
- A $\bar{k}[\sigma]$-submodule of a free left $\bar{k}[\sigma]$-module of rank $s < \infty$ is free of rank $\leq s$.
- Every finitely generated left $\bar{k}[\sigma]$-module is isomorphic to a finite direct sum of cyclic left $\bar{k}[\sigma]$-modules.

These facts are quite well known. The proofs run along lines very similar to the proofs of the analogous statements for, say, the commutative ring $\bar{k}[t]$.

4.1.3. *The functors* mod σ *and* mod $(\sigma-1)$. Given a homomorphism

$$f : H_0 \to H_1$$

of left $\bar{k}[\sigma]$-modules, let

$$f \bmod \sigma : \frac{H_0}{\sigma H_0} \to \frac{H_1}{\sigma H_1}, \quad f \bmod (\sigma-1) : \frac{H_0}{(\sigma-1)H_0} \to \frac{H_1}{(\sigma-1)H_1}$$

be the corresponding induced maps.

LEMMA 4.1.4. *Let*

$$f : H_0 \to H_1$$

be an injective homomorphism of free left $\bar{k}[\sigma]$-modules of finite rank such that

$$n := \dim_{\bar{k}} \operatorname{coker}(f) < \infty.$$

Now,

$$\# \ker(f \bmod (\sigma-1)) \leq q^n$$

with equality if and only if f mod σ is bijective.

Proof. We may assume without loss of generality that

$$H_0 = \operatorname{Mat}_{1\times r}(\bar{k}[\sigma]), \quad H_1 = \operatorname{Mat}_{1\times s}(\bar{k}[\sigma]), \quad f = (x \mapsto x\phi) \quad (\phi \in \operatorname{Mat}_{r\times s}(\bar{k}[\sigma])).$$

After replacing ϕ by $\alpha\phi\beta$ for suitably chosen $\alpha \in \mathrm{GL}_r(\bar{k}[\sigma])$ and $\beta \in \mathrm{GL}_s(\bar{k}[\sigma])$, we may assume without loss of generality that ϕ vanishes off the main diagonal, in which case clearly ϕ vanishes nowhere on the main diagonal and $r = s$. We might as well assume now also that $r = s = 1$. Write

$$\phi = \sum_{i=0}^{n} a_i \sigma^i \quad (a_i \in \bar{k}, \quad a_n \neq 0, \quad a_0 \neq 0 \Leftrightarrow f \bmod \sigma \text{ is bijective}).$$

Now,

$$\bar{k}[\sigma] = \bar{k} \oplus (\sigma - 1)\cdot \bar{k}[\sigma]$$

and

$$x\phi \equiv \sum_{i=0}^{n} a_i^{q^i} x^{q^i} \bmod (\sigma-1)\cdot\bar{k}[\sigma]$$

for all $x \in \bar{k}$, whence the result. □

LEMMA 4.1.5. *For $i = 1, 2$ let*

$$f_i : H_0 \to H_i$$

be a homomorphism of free, left $\bar{k}[\sigma]$-modules of finite rank. Assume that H_0, H_1 and H_2 are all of the same rank over $\bar{k}[\sigma]$. Assume further that f_1 mod σ is bijective and that

$$\ker(f_1 \bmod (\sigma-1)) \subset \ker(f_2 \bmod (\sigma-1)).$$

Then f_2 factors uniquely through f_1; i.e., there exists a unique homomorphism

$$g : H_1 \to H_2$$

of left $\bar{k}[\sigma]$-modules such that

$$g \circ f_1 = f_2.$$

Proof. (Cf. [Yu c, Lemma 1.1].) We may assume without loss of generality that

$$H_0 = H_1 = H_2 = \mathrm{Mat}_{1\times s}(\bar{k}[\sigma]),$$
$$f_1 = (x \mapsto x\phi), \quad f_2 = (x \mapsto x\psi), \quad (\phi, \psi \in \mathrm{Mat}_{s\times s}(\bar{k}[\sigma])).$$

After making replacements

$$\phi \leftarrow \alpha\phi\beta, \quad \psi \leftarrow \alpha\psi$$

for suitably chosen $\alpha, \beta \in \mathrm{GL}_s(\bar{k}[\sigma])$, we may assume without loss of generality that ϕ vanishes off the main diagonal. Since $f_1 \bmod \sigma$ is bijective, no diagonal entry of ϕ vanishes. We might as well assume now also that $s = 1$. Use the left division algorithm to find $\theta, \rho \in \bar{k}[\sigma]$ such that

$$\psi = \phi\theta + \rho, \quad \deg_\sigma \rho < \deg_\sigma \phi.$$

Put

$$g := (x \mapsto x\theta), \quad h := (x \mapsto x\rho).$$

Then

$$f_2 = g \circ f_1 + h.$$

If $h = 0$ we are done. Suppose instead that $h \neq 0$. We then have

$$\ker(f_1 \bmod (\sigma - 1)) \subset \ker(h \bmod (\sigma - 1)), \quad \dim_{\bar{k}} \mathrm{coker}(f_1) > \dim_{\bar{k}} \mathrm{coker}(h).$$

But the latter relations are contradictory in view of Lemma 4.1.4 and our hypothesis that $f_1 \bmod \sigma$ is bijective. □

4.2. *The ring $\bar{k}[[\sigma]]$.*

4.2.1. *Definition.* We define $\bar{k}[[\sigma]]$ to be the completion of $\bar{k}[\sigma]$ with respect to the system of two-sided ideals

$$\left\{\sigma^n \bar{k}[\sigma]\right\}_{n=0}^{\infty}.$$

Every element of $\bar{k}[[\sigma]]$ has a unique presentation of the form

$$\sum_{i=0}^{\infty} a_i \sigma^i \quad (a_i \in \bar{k}).$$

In terms of such presentations the multiplication law in $\bar{k}[[\sigma]]$ takes the form

$$\left(\sum_i a_i \sigma^i\right)\left(\sum_j b_j \sigma^j\right) = \sum_i \sum_j a_i b_j^{q^{-i}} \sigma^{i+j}.$$

The ring $\bar{k}[\![\sigma]\!]$ contains $\bar{k}[\sigma]$ as a subring. The ring $\bar{k}[\![\sigma]\!]$ is a domain.

4.2.2. *The operation ∂.* Given

$$\phi = \sum_{i=0}^{\infty} a_{(i)}\sigma^i \in \mathrm{Mat}_{r\times s}(\bar{k}[\![\sigma]\!]) \quad (a_{(i)} \in \mathrm{Mat}_{r\times s}(\bar{k})),$$

put

$$\partial\phi := a_{(0)}.$$

The operation ∂ thus defined is $\bar{k}$-linear and satisfies

$$\partial(\phi\psi) = (\partial\phi)(\partial\psi)$$

for all matrices ϕ and ψ with entries in $\bar{k}[\![\sigma]\!]$ such that the product $\phi\psi$ is defined.

LEMMA 4.2.3. (i) *For all $\phi \in \mathrm{Mat}_{s\times s}(\bar{k}[\![\sigma]\!])$, if $\partial\phi \in \mathrm{GL}_s(\bar{k})$, then $\phi \in \mathrm{GL}_s(\bar{k}[\![\sigma]\!])$.* (ii) *Every nonzero left ideal of $\bar{k}[\![\sigma]\!]$ is generated by a power of σ.*

Proof. (i) After replacing ϕ by $\alpha\phi$ for suitably chosen $\alpha \in \mathrm{GL}_s(\bar{k})$ we may assume $\partial\phi = \mathbf{1}_s$. Now write $\phi = \mathbf{1}_s - X$. The series $\mathbf{1}_s + \sum_{n=1}^{\infty} X^n$ converges to a two-sided inverse to ϕ. (ii) Let $I \subset \bar{k}[\![\sigma]\!]$ be a nonzero left ideal. Let $\phi = \alpha\sigma^n$ be a nonzero element of I where $\partial\alpha \neq 0$ and n is a nonnegative integer taken as small as possible. Then we have $\alpha \in \bar{k}[\![\sigma]\!]^\times$ by (i); hence $\sigma^n \in I$, and hence σ^n generates I. □

LEMMA 4.2.4. *Let*

$$\theta \in \mathrm{Mat}_{r\times r}(\bar{k}[\![\sigma]\!]), \quad a \in \mathrm{Mat}_{r\times r}(\bar{k}), \quad e \in \mathrm{Mat}_{r\times s}(\bar{k}), \quad b \in \mathrm{Mat}_{s\times s}(\bar{k})$$

be given such that

$$\partial\theta = a, \quad (a - T\cdot\mathbf{1}_r)^r = 0, \quad ae = eb, \quad (b - T\cdot\mathbf{1}_s)^s = 0.$$

Then there exists unique

$$E \in \mathrm{Mat}_{r\times s}(\bar{k}[\![\sigma]\!])$$

such that

$$\theta E = Eb, \quad \partial E = e.$$

Proof. (Cf. [An a, Prop. 2.1.4].) Write

$$\theta = \sum_{i=0}^{\infty} a_{(i)}\sigma^i \in \mathrm{Mat}_{r\times r}(\bar{k}[\![\sigma]\!]) \;\; (a_{(i)} \in \mathrm{Mat}_{r\times r}(\bar{k}),\; a_{(0)} = a)$$

and

$$E = \sum_{i=0}^{\infty} e_{(i)}\sigma^i \in \mathrm{Mat}_{r\times s}(\bar{k}[\![\sigma]\!]) \;\; (e_{(i)} \in \mathrm{Mat}_{r\times s}(\bar{k}),\; e_{(0)} = e).$$

Then,

$$\theta E=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{(i)}e_{(j)}^{(-i)}\sigma^{i+j}$$

and hence $\theta E = Eb$ if and only if the system of coefficients $\left\{e_{(n)}\right\}$ satisfies the recursion

$$e_{(n)}b^{(-n)}-ae_{(n)}=\sum_{0<i\leq n}a_{(i)}e_{(n-i)}^{(-i)}\qquad(n=1,2,\dots).$$

Now for each $n>0$ the $\bar{k}$-linear map

$$\left(z\mapsto zb^{(-n)}-az\right):\mathrm{Mat}_{r\times s}(\bar{k})\to\mathrm{Mat}_{r\times s}(\bar{k})$$

is invertible because all of its eigenvalues equal $T^{q^{-n}}-T$; indeed, for $q^i\geq\max\{r,s\}$ the q^i-th iteration sends z to $(T^{q^{-n}}-T)^{q^i}z$. It follows that the recursion satisfied by the coefficients $e_{(n)}$ has a unique solution with $e_{(0)}=e$. □

LEMMA 4.2.5. *Let matrices*

$$\theta,\phi,\rho\in\mathrm{Mat}_{s\times s}(\bar{k}[\sigma])$$

be given such that

$$(\partial\theta-T\cdot\mathbf{1}_s)^s=0,\quad\theta\phi=\phi\rho,\quad(\partial\rho-T\cdot\mathbf{1}_s)^s=0,$$

Assume further that the map

$$(x\mapsto x\phi):\mathrm{Mat}_{1\times s}(\bar{k}[\sigma])\to\mathrm{Mat}_{1\times s}(\bar{k}[\sigma])$$

is injective. Then

$$\det\partial\phi\neq0.$$

Proof. (Cf. [Yu b, Lemma 1.3].) After making replacements

$$\theta\leftarrow\alpha\theta\alpha^{-1},\quad\phi\leftarrow\alpha\phi\beta,\quad\rho\leftarrow\beta^{-1}\rho\beta,$$

for suitably chosen $\alpha,\beta\in\mathrm{GL}_s(\bar{k}[\sigma])$ we may assume without loss of generality that ϕ vanishes off the main diagonal, in which case it is clear that no entry of ϕ on the main diagonal vanishes. By Lemma 4.2.4 there exist unique matrices

$$D,E,F\in\mathrm{Mat}_{s\times s}(\bar{k}[\![\sigma]\!])$$

such that

$$\theta D=D\cdot\partial\theta,\quad\theta E=E\cdot\partial\rho,\quad\rho F=F\cdot\partial\rho,$$

and

$$\partial D=\mathbf{1}_r,\quad\partial E=\partial\phi,\quad\partial F=\mathbf{1}_s.$$

By the uniqueness asserted in Lemma 4.2.4 we have

$$\phi F = E = D \cdot \partial\phi.$$

Further,

$$D, F \in \mathrm{GL}_s(\bar{k}[\![\sigma]\!])$$

by Lemma 4.2.3 (i). Now consider the quotient M (resp., N) of the free left $\bar{k}[\![\sigma]\!]$-module $\mathrm{Mat}_{s\times 1}(\bar{k}[\![\sigma]\!])$ by the $\bar{k}[\![\sigma]\!]$-submodule generated by the rows of ϕ (resp., $\partial\phi$). Since ϕ is diagonal with no vanishing diagonal entries, we have $\dim_{\bar{k}} M < \infty$ by Lemma 4.2.3 (ii). Since $D^{-1}\phi F = \partial\phi$, the left $\bar{k}[\![\sigma]\!]$-modules M and N are isomorphic and hence we have $\dim_{\bar{k}} N < \infty$. Under the latter condition it is impossible for any diagonal entry of $\partial\phi$ to vanish. □

4.3. *The ring* $\bar{k}[t,\sigma]$.

4.3.1. *Definition.* Let $\bar{k}[t,\sigma]$ be the ring obtained by adjoining the commutative variable t to $\bar{k}[\sigma]$. Every element of $\bar{k}[t,\sigma]$ has a unique presentation of the form

$$\sum_{i=0}^{\infty} \alpha_i t^i \quad (\alpha_i \in \bar{k}[\sigma],\ \alpha_i = 0 \text{ for } i \gg 0).$$

In terms of such presentations the multiplication law in $\bar{k}[t,\sigma]$ takes the form

$$\left(\sum_i \alpha_i t^i\right)\left(\sum_j \beta_j t^j\right) = \sum_i \sum_j \alpha_i \beta_j t^{i+j}.$$

Every element of $\bar{k}[t,\sigma]$ also has a unique presentation of the form

$$\sum_{i=0}^{\infty} a_i \sigma^i \quad (a_i \in \bar{k}[t],\ \ a_i = 0 \text{ for } i \gg 0).$$

In terms of such presentations the multiplication law in $\bar{k}[t,\sigma]$ takes the form

$$\left(\sum_i a_i \sigma^i\right)\left(\sum_j b_j \sigma^j\right) = \sum_i \sum_j a_i b_j^{(-i)} \sigma^{i+j}.$$

The ring $\bar{k}[t,\sigma]$ contains both the noncommutative ring $\bar{k}[\sigma]$ and the commutative ring $\bar{k}[t]$ as subrings. The ring $\mathbb{F}_q[t]$ is contained in the center of the ring $\bar{k}[t,\sigma]$. The ring $\bar{k}[t,\sigma]$ is a domain.

PROPOSITION 4.3.2. *Let M be a left $\bar{k}[t,\sigma]$-module finitely generated over both $\bar{k}[\sigma]$ and $\bar{k}[t]$. Let M_σ (resp., M_t) be the sum of all $\bar{k}[\sigma]$- (resp., $\bar{k}[t]$-) submodules $N \subset M$ such that $\dim_{\bar{k}} N < \infty$. Then $M_\sigma = M_t$, $\dim_{\bar{k}} M_\sigma = \dim_{\bar{k}} M_t < \infty$ and the quotient $M/M_\sigma = M/M_t$ is free of finite rank over both $\bar{k}[\sigma]$ and $\bar{k}[t]$.*

In particular it follows that a left $\bar{k}[t,\sigma]$-module finitely generated over both $\bar{k}[\sigma]$ and $\bar{k}[t]$ is free over $\bar{k}[\sigma]$ if and only if free over $\bar{k}[t]$.

Proof (cf. [An a, Lemma 1.4.5, p. 463]) . As a $\bar{k}[\sigma]$-module M decomposes as a finite direct sum of cyclic left $\bar{k}[\sigma]$-modules. Therefore $\dim_{\bar{k}} M_\sigma < \infty$ and M/M_σ is free of finite rank over $\bar{k}[\sigma]$. Similarly $\dim_{\bar{k}} M_t < \infty$ and M/M_t is free of finite rank over $\bar{k}[t]$. Now for any $\bar{k}[t]$-submodule $N \subset M$ of finite dimension over $\bar{k}$, again σN is a $\bar{k}[t]$-submodule of M of finite dimension over $\bar{k}$ and hence $\sigma M_t \subset M_t$. Similarly we have $tM_\sigma \subset M_\sigma$. Therefore each of the modules M_σ and M_t contains the other. □

4.3.3. *Saturation.* Let $N \subset M$ be left $\bar{k}[t,\sigma]$-modules. Assume that M is finitely generated over both $\bar{k}[\sigma]$ and $\bar{k}[t]$. By Proposition 4.3.2 there exists a unique $\bar{k}[t,\sigma]$-submodule $\tilde{N} \subset M$ such that $\tilde{N} = \sum N' = \sum N''$ where N' ranges over $\bar{k}[\sigma]$-submodules such that $\dim_{\bar{k}}(N'+N)/N < \infty$ and N'' ranges over $\bar{k}[t]$-submodules such that $\dim_{\bar{k}}(N''+N)/N < \infty$. Clearly we have $\tilde{N} \supset N$. We call $\tilde{N}$ the *saturation* of N in M. By Proposition 4.3.2 the quotient $M/\tilde{N}$ is free of finite rank over both $\bar{k}[\sigma]$ and $\bar{k}[t]$ and moreover $\dim_{\bar{k}} \tilde{N}/N < \infty$. If $N = \tilde{N}$ we say that N is *saturated* in M. A necessary and sufficient condition for N to be saturated in M is that M/N be torsion-free over $\bar{k}[t]$ or torsion-free over $\bar{k}[\sigma]$.

4.4. *Dual t-motives.*

4.4.1. *Definition.* A *dual t-motive* H is a left $\bar{k}[t,\sigma]$-module with the following three properties:

- H is free of finite rank over $\bar{k}[t]$.
- H is free of finite rank over $\bar{k}[\sigma]$.
- $(t-T)^n H \subset \sigma H$ for $n \gg 0$.

A *morphism* of dual t-motives is by definition a homomorphism of left $\bar{k}[t,\sigma]$-modules. Thus dual t-motives form a category. For any dual t-motive H there exist

$$\mathbf{g} \in \mathrm{Mat}_{r\times 1}(H), \quad \mathbf{h} \in \mathrm{Mat}_{s\times 1}(H), \quad \Phi \in \mathrm{Mat}_{r\times r}(\bar{k}[t]), \quad \theta \in \mathrm{Mat}_{s\times s}(\bar{k}[\sigma])$$

such that

- the entries of $\mathbf{g}$ form a $\bar{k}[t]$-basis for H and $\sigma\mathbf{g} = \Phi\mathbf{g}$,
- the entries of $\mathbf{h}$ form a $\bar{k}[\sigma]$-basis for H and $t\mathbf{h} = \theta\mathbf{h}$,

in which case

- $\det \Phi = c(t-T)^s$ for some nonzero $c \in \bar{k}$ and
- $(\partial\theta - T \cdot \mathbf{1}_s)^s = 0$.

The latter two assertions both follow from the assumption that the quotient $H/\sigma H$ is killed by a power of $t - T$.

4.4.2. *Basic stability properties of the class of dual t-motives.* Let

$$H_0 \subset H_1$$

be left $\bar{k}[t,\sigma]$-modules. The following statements hold:

- If H_1 is a dual t-motive and $(t-T)^n H_0 \subset \sigma H_0$ for $n \gg 0$, then H_0 is again a dual t-motive.
- If H_1 is a dual t-motive and H_0 is saturated in H_1, then both H_0 and H_1/H_0 are again dual t-motives.
- If H_0 and H_1/H_0 are both dual t-motives, then H_1 is again a dual t-motive.

Using the background on module theory over $\bar{k}[\sigma]$ and $\bar{k}[t,\sigma]$ provided above, the reader should have no difficulty verifying these statements.

Proposition 4.4.3. *Let*

$$\Phi \in \mathrm{Mat}_{\ell\times\ell}(\bar{k}[t]), \quad \psi \in \mathrm{Mat}_{\ell\times 1}(\mathcal{E})$$

be as in Theorem 3.1.1. *Suppose further that there exist a dual t-motive H and a vector*

$$\mathbf{g} \in \mathrm{Mat}_{\ell\times 1}(H)$$

with entries forming a $\bar{k}[t]$-basis of H such that

$$\sigma\mathbf{g} = \Phi\mathbf{g}.$$

Equip $\mathcal{E}$ with left $\bar{k}[t,\sigma]$-module structure by the rule

$$\sigma e := e^{(-1)} \quad (e \in \mathcal{E}).$$

Put

$$\begin{aligned} H_0 &:= \bar{k}[t]\text{-span in } \mathcal{E} \text{ of the entries of the vector } \psi, \\ V &:= \bar{k}\text{-span in } \overline{k_\infty} \text{ of the entries of the vector } \psi(T). \end{aligned}$$

Then the following statements hold:

- *H_0 is a $\bar{k}[t,\sigma]$-submodule of $\mathcal{E}$.*
- *H_0 is a dual t-motive admitting presentation as a quotient of H.*
- $\mathrm{rk}_{\bar{k}[t]} H_0 = \dim_{\bar{k}} V$.

The proposition positions Theorem 3.1.1 in the setting of dual t-motives.

Proof. Consider the exact sequence

$$0 \to H_1 \subset H \to H_0 \to 0$$

of left $\bar{k}[t]$-modules, where

$$H_1 := \{P\mathbf{g} \in H \mid P \in \mathrm{Mat}_{1\times\ell}(\bar{k}[t]),\ P\psi = 0\}$$

and the projection $H \to H_0$ is given by the rule

$$P\mathbf{g} \mapsto P\psi \quad (P \in \mathrm{Mat}_{1\times\ell}(\bar{k}[t])).$$

A straightforward calculation verifies that the exact sequence in question is in fact an exact sequence of $\bar{k}[t,\sigma]$-modules. Therefore H_0 is a dual t-motive admitting presentation as a quotient of H. Since every $\bar{k}[t]$-basis for H_1 can be completed to a $\bar{k}[t]$-basis of H, the number of $\bar{k}$-linearly independent relations of $\bar{k}$-linear dependence among the entries of $\psi(T)$ is at least as great as $\mathrm{rk}_{\bar{k}[t]} H_1$ and hence we have

$$\mathrm{rk}_{\bar{k}[t]} H_0 \geq \dim_{\bar{k}} V.$$

But we also have

$$\mathrm{rk}_{\bar{k}[t]} H_0 \leq \dim_{\bar{k}} V$$

because by Theorem 3.1.1 every relation of $\bar{k}$-linear dependence among the entries of $\psi(T)$ lifts to a $\bar{k}[t]$-linear relation among the entries of ψ. □

THEOREM 4.4.4. *For all dual t-motives H_0 and H_1 the natural map*

$$\bar{k} \otimes_{\mathbb{F}_q} \mathrm{Hom}_{\bar{k}[t,\sigma]}(H_0, H_1) \to \mathrm{Hom}_{\bar{k}[t]}(H_0, H_1)$$

is injective.

Proof. The proof of [An a, Thm. 2, p. 464] can easily be modified to prove this result. □

4.4.5. *Isogenies.* An injective morphism $f : H_0 \to H_1$ of dual t-motives with cokernel finite-dimensional over $\bar{k}$ is called an *isogeny*. We say that dual t-motives H_0 and H_1 are *isogenous* if there exists an isogeny $f : H_0 \to H_1$.

LEMMA 4.4.6. *Let $f : H_0 \to H_1$ be an isogeny of dual t-motives. Then the induced map f mod σ is bijective.*

Proof. Without loss of generality, we think of f as an inclusion. Let s be the common rank of H_0 and H_1 over $\bar{k}[\sigma]$. For $i = 0, 1$ select

$$\mathbf{h}_{(i)} \in \mathrm{Mat}_{s\times 1}(H_i), \quad \theta_{(i)} \in \mathrm{Mat}_{s\times s}(\bar{k}[\sigma])$$

such that the entries of $\mathbf{h}_{(i)}$ form a $\bar{k}[\sigma]$-basis of H_i and

$$t\mathbf{h}_{(i)} = \theta_{(i)}\mathbf{h}_{(i)}.$$

Let

$$\phi \in \mathrm{Mat}_{s\times s}(\bar{k}[\sigma])$$

be the unique solution of the equation

$$\mathbf{h}_{(0)} = \phi \mathbf{h}_{(1)},$$

with

$$\theta_{(0)}\phi\mathbf{h}_{(1)} = \theta_{(0)}\mathbf{h}_{(0)} = t\mathbf{h}_{(0)} = t\phi\mathbf{h}_{(1)} = \phi t\mathbf{h}_{(1)} = \phi\theta_{(1)}\mathbf{h}_{(1)}.$$

By Lemma 4.2.5 we have $\det \partial\phi \neq 0$ and hence $f \bmod \sigma$ is bijective. □

THEOREM 4.4.7. *Let $H_0 \subset H_1$ be dual t-motives such that $\dim_{\bar{k}} H_1/H_0 < \infty$. Then there exists $0 \neq a \in \mathbb{F}_q[t]$ such that $aH_1 \subset H_0$.*

It follows that the isogeny relation is not only reflexive and transitive, but symmetric as well and hence an equivalence relation.

Proof. Let $f_1 : H_0 \to H_1$ be the inclusion. By Lemma 4.4.6 the induced map $f_1 \bmod \sigma$ is bijective. By Lemma 4.1.4 the kernel of the induced map $f_1 \bmod (\sigma - 1)$ is finite, and, since the latter group naturally has the structure of a finite $\mathbb{F}_q[t]$-module, there exists $0 \neq a \in \mathbb{F}_q[t]$ killing it. Let $f_2 : H_0 \to H_0$ be the morphism of dual t-motives induced by multiplication by a. Then

$$\ker(f_1 \bmod (\sigma - 1)) \subset \ker(f_2 \bmod (\sigma - 1)).$$

By Lemma 4.1.5 there exists a unique $\bar{k}[\sigma]$-module homomorphism $g\colon H_1 \to H_0$ such that $f_2 = g \circ f_1$. We have

$$\ker(f_1 \bmod (\sigma - 1)) \subset \ker(tf_2 \bmod (\sigma - 1))$$

and

$$tg(f_1(h)) = tf_2(h) = f_2(th) = g(f_1(th)) = g(tf_1(h)) \quad (h \in H_0).$$

By the uniqueness asserted in Lemma 4.1.5, it follows that g commutes with t and hence is a morphism of dual t-motives. Now

$$\ker(f_1 \bmod (\sigma - 1)) \subset \ker(f_1 \circ g \circ f_1 \bmod (\sigma - 1))$$

and

$$f_1(g(f_1(h))) = f_1(f_2(h)) = f_1(ah) = af_1(h) \quad (h \in H_0).$$

By the uniqueness asserted in Lemma 4.1.5, it follows that $f_1 \circ g$ coincides with multiplication by a. Therefore $aH_1 \subset H_0$. □

COROLLARY 4.4.8. *For all dual t-motives H_0 and H_1 the module*

$$\mathrm{Hom}_{\bar{k}[t,\sigma]}(H_0, H_1)$$

is free over $\mathbb{F}_q[t]$ of finite rank and moreover its rank over $\mathbb{F}_q[t]$ depends only on the isogeny classes of H_0 and H_1.

Proof. Theorem 4.4.4 already proves that the module in question is free of finite rank over $\mathbb{F}_q[t]$. Now let $r(H_0, H_1)$ denote the rank over $\mathbb{F}_q[t]$ of the module in question. For $i = 0, 1$ let H_i' be a dual t-motive isogenous to H_i and without loss of generality assume that $H_i' \subset H_i$ and $\dim_{\bar{k}} H_i/H_i' < \infty$. Choose $0 \neq a \in \mathbb{F}_q[t]$ such that $aH_i \subset H_i'$ for $i = 0, 1$. We have

$$r(H_0, H_1) \leq r(H_0', H_1) \leq r(aH_0, H_1) = r(H_0, H_1)$$

and

$$r(H_0, H_1) = r(H_0, aH_1) \leq r(H_0, H_1') \leq r(H_0, H_1),$$

where each inequality is justified by the existence of a suitably constructed injective $\mathbb{F}_q[t]$-linear map. □

4.4.9. *Simplicity.* We say that a dual t-motive H is *simple* if $H \neq \{0\}$ and there exist no saturated $\bar{k}[t, \sigma]$-submodules of H other than $\{0\}$ and H.

PROPOSITION 4.4.10. (i) *A dual t-motive isogenous to a simple dual t-motive is again simple.* (ii) *A nonzero morphism of dual t-motives with simple source and target is automatically an isogeny.* (iii) *Let $\{H_i\}$ be a family of simple dual t-motives each embedded as a $\bar{k}[t, \sigma]$-submodule of a dual t-motive H. If*

$$\dim_{\bar{k}} H/\left(\sum_i H_i\right) < \infty,$$

then H is isogenous to a finite direct sum of dual t-motives of the family $\{H_i\}$.

Proof. (i) Let $H_0 \subset H_1$ be dual t-motives with $\dim_{\bar{k}} H_1/H_0 < \infty$ and H_0 simple. It suffices to show that H_1 is simple. Let $M \subset H_1$ be a $\bar{k}[t, \sigma]$-submodule saturated in H_1 and hence free over $\bar{k}[\sigma]$ and $\bar{k}[t]$. Then $M \cap H_0$ is saturated in H_0 and hence $M \cap H_0 = \{0\}$ or $M \cap H_0 = H_0$ by the simplicity of H_0. In the former case, M injects into H_1/H_0, so that $\dim_{\bar{k}} M < \infty$ and hence $M = 0$ since M is a free $\bar{k}[\sigma]$-module. In the latter case $\dim_{\bar{k}} H_1/M < \infty$ and hence $M = H_1$ since M is saturated in H_1. Therefore H_1 is indeed simple.

(ii) Let $f : H_0 \to H_1$ be a nonzero morphism of dual t-motives with simple source and target. The kernel of f is a saturated $\bar{k}[t, \sigma]$-submodule of H_0 distinct from H_0 and hence equal to $\{0\}$. The saturation of the image of f is a saturated $\bar{k}[t, \sigma]$-submodule of H_1 distinct from $\{0\}$, hence equal to H_1, and hence the cokernel of f is of finite dimension over $\bar{k}$. Therefore f is indeed an isogeny.

(iii) Since H is noetherian over $\bar{k}[t, \sigma]$, there exists a finite set I of indices such that

$$\dim_{\bar{k}} H/\left(\sum_{i \in I} H_i\right) < \infty.$$

Fix such a set I now with $\#I$ minimal. Consider the exact sequence

$$0 \to K \to \bigoplus_{i \in I} H_i \to \sum_{i \in I} H_i \to 0$$

of left $\bar{k}[t,\sigma]$-modules. It suffices to show that $K = 0$; suppose instead that $K \neq 0$. In any case K is a saturated $\bar{k}[t,\sigma]$-submodule of a dual t-motive, and hence K is a dual t-motive. Let M be a nonzero saturated $\bar{k}[t,\sigma]$-submodule of K of minimal rank over $\bar{k}[t]$. Then M is a simple dual t-motive. For some index $i_0 \in I$ the evident map $M \to H_{i_0}$ is nonzero and hence an isogeny, in which case $\sum_{i \in I \setminus \{i_0\}} H_i$ is of finite $\bar{k}$-codimension in H in contradiction to the minimality of $\#I$. This contradiction proves that $K = 0$. □

4.4.11. *Rigid-analytic triviality.* Given a dual t-motive H, put

$$\tilde{H} := \mathbb{C}_\infty\{t\} \otimes_{\bar{k}[t]} H,$$

equip $\tilde{H}$ with an action of σ by the rule

$$\sigma(f \otimes h) := f^{(-1)} \otimes h,$$

and put

$$H^{\mathrm{Betti}} := \{\sigma\text{-invariant elements of } \tilde{H}\}.$$

We say that H is *rigid analytically trivial* if the natural map

$$\mathbb{C}_\infty\{t\} \otimes_{\mathbb{F}_q[t]} H^{\mathrm{Betti}} \to \tilde{H}$$

is bijective, cf. [An a, p. 474].

Lemma 4.4.12. *Let H be a dual t-motive. Select*

$$\mathbf{g} \in \mathrm{Mat}_{r \times 1}(H), \quad \Phi \in \mathrm{Mat}_{r \times r}(\bar{k}[t])$$

such that the entries of $\mathbf{g}$ form a $\bar{k}[t]$-basis of H and $\sigma \mathbf{g} = \Phi \mathbf{g}$. (i) *A necessary and sufficient condition for H to be rigid analytically trivial is that there exists a solution*

$$\Psi \in \mathrm{GL}_r(\mathbb{C}_\infty\{t\})$$

of the equation

$$\Psi^{(-1)} = \Phi \Psi.$$

(ii) *For any such solution Ψ the entries of the column vector $\Psi^{-1}\mathbf{g}$ form an $\mathbb{F}_q[t]$-basis for H^{Betti}.*

In particular if H is rigid analytically trivial, then H^{Betti} is free over $\mathbb{F}_q[t]$ of rank equal to the rank of H over $\bar{k}[t]$. Note that $\Psi \in \mathrm{Mat}_{r \times r}(\mathcal{E})$ by Proposition 3.1.3.

Proof. Note that by hypothesis and definition the diagrams

$$\begin{tikzcd} \mathrm{Mat}_{1\times r}(\bar{k}[t]) \arrow[r, "\times\mathbf{g}"] \arrow[d, "P\mapsto P^{(-1)}\Phi"'] & H \arrow[d, "\sigma\times"] \\ \mathrm{Mat}_{1\times r}(\bar{k}[t]) \arrow[r, "\times\mathbf{g}"] & H \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty\{t\}) \arrow[r, "\times\mathbf{g}"] \arrow[d, "P\mapsto P^{(-1)}\Phi"'] & \tilde{H} \arrow[d, "\sigma\times"] \\ \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty\{t\}) \arrow[r, "\times\mathbf{g}"] & \tilde{H} \end{tikzcd}$$

commute and have bijective horizontal arrows. (i)($\Rightarrow$) There exists by hypothesis a matrix $\Theta \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty\{t\})$ such that the entries of the column vector $\Theta\mathbf{g}$ are at once a $\mathbb{C}_\infty\{t\}$-basis for $\tilde{H}$ and an $\mathbb{F}_q[t]$-basis for H^{Betti}. Such a matrix Θ necessarily belongs to $\mathrm{GL}_r(\mathbb{C}_\infty\{t\})$ and satisfies the functional equation $\Theta^{(-1)}\Phi = \Theta$. The matrix $\Psi := \Theta^{-1}$ has then the desired properties. (i)($\Leftarrow$)&(ii) The entries of the column vector $\Psi^{-1}\mathbf{g}$ form a $\mathbb{C}_\infty\{t\}$-basis for $\tilde{H}$ and are also an $\mathbb{F}_q[t]$-linearly independent collection of elements of H^{Betti}. Every element of H^{Betti} is of the form $P\mathbf{g}$ for unique $P \in \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty\{t\})$ such that $P = P^{(-1)}\Phi$, and we have

$$(P\Psi)^{(-1)} = P^{(-1)}\Phi\Psi = P\Psi \in \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty\{t\} \cap \mathbb{F}_q[\![t]\!]) = \mathrm{Mat}_{1\times r}(\mathbb{F}_q[t]).$$

Therefore the entries of the column vector $\Psi^{-1}\mathbf{g}$ span H^{Betti} over $\mathbb{F}_q[t]$ and hence form an $\mathbb{F}_q[t]$-basis of H^{Betti}. □

LEMMA 4.4.13. *For all $0 \neq a \in \mathbb{F}_q[t]$, we have $\mathbb{F}_q[t] \cap a\cdot\mathbb{C}_\infty\{t\} = a\cdot\mathbb{F}_q[t]$.*

Proof. View $\mathbb{C}_\infty\{t\}$ as a subring of the Laurent series field $\mathbb{C}_\infty(\!(t)\!)$. We have

$$a^{-1}\mathbb{F}_q[t] \cap \mathbb{C}_\infty\{t\} \subset \mathbb{F}_q(\!(t)\!) \cap \mathbb{C}_\infty\{t\} = \mathbb{F}_q[t],$$

whence the result. □

THEOREM 4.4.14. *For all rigid analytically trivial dual t-motives H_0 and H_1, the natural map*

$$\mathrm{Hom}_{\bar{k}[t,\sigma]}(H_0, H_1) \to \mathrm{Hom}_{\mathbb{F}_q[t]}\left(H_0^{\mathrm{Betti}}, H_1^{\mathrm{Betti}}\right)$$

is injective and its cokernel is without $\mathbb{F}_q[t]$-torsion.

Proof. After replacing both H_0 and H_1 by $H_0 \oplus H_1$ we may assume without loss of generality that $H_0 = H_1$, in which case we might as well drop subscripts and simply write $H = H_0 = H_1$. Let $\mathbf{g}$, Φ and Ψ be as in Lemma 4.4.12. Fix

$$e \in \mathrm{End}_{\bar{k}[t]}(H)$$

arbitrarily, let

$$\tilde{e} \in \mathrm{End}_{\mathbb{C}_\infty\{t\}}(\tilde{H})$$

be the unique $\mathbb{C}_\infty\{t\}$-linear extension of e, and let

$$E \in \mathrm{Mat}_{r\times r}(\bar{k}[t])$$

be the representation of e with respect to the $\bar{k}[t]$-basis $\mathbf{g}$, i.e., the unique solution of the equation

$$e\mathbf{g} = E\mathbf{g}.$$

In this last and in analogous expressions below, in order to avoid having to manipulate indices and summations, e (resp., $\tilde{e}$) is understood to be applied entrywise to any column vector with entries in H (resp., $\tilde{H}$) that it precedes. We have

$$\sigma e\mathbf{g} = \sigma E\mathbf{g} = E^{(-1)}\sigma\mathbf{g} = E^{(-1)}\Phi\mathbf{g}, \quad e\sigma\mathbf{g} = e\Phi\mathbf{g} = \Phi e\mathbf{g} = \Phi E\mathbf{g},$$

and hence

$$\begin{aligned} e \in \mathrm{End}_{\bar{k}[t,\sigma]}(H) \quad &\Leftrightarrow \quad E^{(-1)}\Phi = \Phi E \\ &\Leftrightarrow \quad (\Psi^{-1}E\Psi)^{(-1)} = \Psi^{-1}E\Psi \\ &\Leftrightarrow \quad \Psi^{-1}E\Psi \in \mathrm{Mat}_{r\times r}(\mathbb{F}_q[\![t]\!] \cap \mathbb{C}_\infty\{t\}) = \mathrm{Mat}_{r\times r}(\mathbb{F}_q[t]). \end{aligned}$$

We have

$$\left.\begin{array}{l} e \in \mathrm{End}_{\bar{k}[t,\sigma]}(H) \\ \\ \tilde{e}(H^{\mathrm{Betti}}) = 0 \end{array}\right\} \Rightarrow 0 = \tilde{e}\Psi^{-1}\mathbf{g} = \Psi^{-1}e\mathbf{g} \Rightarrow e\mathbf{g} = 0 \Rightarrow e = 0.$$

Therefore the map in question is injective. Now fix $0 \neq a \in \mathbb{F}_q[t]$. We have

$$\begin{aligned} ae \in \mathrm{End}_{\bar{k}[t,\sigma]}(H) \quad &\Rightarrow \quad a\Psi^{-1}E\Psi \in \mathrm{Mat}_{r\times r}(\mathbb{F}_q[t] \cap a\cdot\mathbb{C}_\infty\{t\}) \\ &\Rightarrow \quad \Psi^{-1}E\Psi \in \mathrm{Mat}_{r\times r}(\mathbb{F}_q[t]) \\ &\Rightarrow \quad e \in \mathrm{End}_{\bar{k}[t,\sigma]}(H) \end{aligned}$$

where the second implication is justified by Lemma 4.4.13. Thus we rule out the possibility of $\mathbb{F}_q[t]$-torsion in the cokernel of the map in question. □

4.5. *The Dedekind-Wedderburn trick.*

4.5.1. *Dedekind domains.* Recall that a ring with unit is called a *Dedekind domain* if commutative, noetherian, entire ($1 \neq 0$ and no proper zero-divisors), one-dimensional (nonzero primes exist and every such is maximal), and integrally closed. Now let a Dedekind domain $\mathbf{K}$ be given. The following hold:

- Every finitely generated $\mathbf{K}$-module without $\mathbf{K}$-torsion is projective.

- Every projective $\mathbf{K}$-module of finite rank is a direct sum of projective $\mathbf{K}$-modules of rank one.

We take these basic facts for granted.

LEMMA 4.5.2. *For every Dedekind domain* $\mathbf{K}$ *and projective* $\mathbf{K}$*-module* $\mathbf{M}$ *of rank one the map* $(x \mapsto (m \mapsto xm)) : \mathbf{K} \to \mathrm{End}_{\mathbf{K}}(\mathbf{M})$ *is bijective.*

Proof. Injectivity is clear. Failure of surjectivity would entail failure of $\mathbf{K}$ to be integrally closed. □

PROPOSITION 4.5.3. *For* $i = 0, 1$ *let* $\mathbf{L}_i$ *be a Dedekind domain. Let* $\mathbf{M}$ *be an abelian group equipped with right* $\mathbf{L}_0$*-module structure and left* $\mathbf{L}_1$*-module structure in such fashion that* $(a_1 m)a_0 = a_1(m a_0)$ *for all* $a_1 \in \mathbf{L}_1$, $m \in \mathbf{M}$ *and* $a_0 \in \mathbf{L}_0$. *Assume further that* $\mathbf{M}$ *is projective of rank one both as an* $\mathbf{L}_0$*-module and as an* $\mathbf{L}_1$*-module. Then there exists a unique ring isomorphism* $\theta : \mathbf{L}_0 \to \mathbf{L}_1$ *such that* $\theta(a)m = ma$ *for all* $a \in \mathbf{L}_0$ *and* $m \in \mathbf{M}$.

Proof. This is a trivial (but quite useful) consequence of Lemma 4.5.2. □

LEMMA 4.5.4. *Let* $\mathbf{K} \supset \mathbf{F}$ *be an integral extension of entire rings. For every* $0 \neq b \in \mathbf{K}$ *there exists* $0 \neq a \in \mathbf{F}$ *such that* $a/b \in \mathbf{K}$.

It follows that in this setting any $\mathbf{K}$-module is without $\mathbf{K}$-torsion if (and of course only if) without $\mathbf{F}$-torsion.

Proof. Since $\mathbf{K}$ is integral over $\mathbf{F}$, there exists a polynomial $f(z) \in \mathbf{F}[z]$ monic in z such that $f(b) = 0$. Moreover, since $\mathbf{K}$ is entire, after factoring out a power of z, we may assume without loss of generality that $a := f(0) \neq 0$. Now write $f(z) = a - zg(z)$ with $g(z) \in \mathbf{F}[z]$. Then $a/b = g(b) \in \mathbf{K}$. □

PROPOSITION 4.5.5. *Let* $\mathbf{L} \supset \mathbf{K} \supset \mathbf{F}$ *be a tower of rings where* $\mathbf{L}$ *and* $\mathbf{F}$ *are Dedekind domains and* $\mathbf{L}$ *is a finitely generated projective* $\mathbf{F}$*-module. Then* $\mathbf{K}$ *is a Dedekind domain if and only if the quotient* $\mathbf{L}/\mathbf{K}$ *is a projective* $\mathbf{F}$*-module.*

Proof. ($\Rightarrow$) For all $x \in \mathbf{L}$ and $0 \neq a \in \mathbf{F}$ such that $ax \in \mathbf{K}$ we must have $x \in \mathbf{K}$ since $\mathbf{L}$ is integral over $\mathbf{K}$ and $\mathbf{K}$ is integrally closed. Therefore the quotient $\mathbf{L}/\mathbf{K}$ is without $\mathbf{F}$-torsion and, since finitely generated over $\mathbf{F}$, must be projective over $\mathbf{F}$.

($\Leftarrow$) Clearly $\mathbf{K}$ is commutative, noetherian and entire. Let $P \subset \mathbf{K}$ be a nonzero prime of $\mathbf{K}$. By Lemma 4.5.4 the prime $P \cap \mathbf{F}$ is nonzero and hence a maximal ideal. Since $\mathbf{K}$ is integral over $\mathbf{F}$ the prime P has to be maximal, too. Therefore $\mathbf{K}$ is one-dimensional. Since $\mathbf{L}$ is integrally closed, every element of the integral closure of $\mathbf{K}$ belongs to $\mathbf{L}$ and hence gives rise to a $\mathbf{K}$-torsion element of the quotient $\mathbf{L}/\mathbf{K}$. But by Lemma 4.5.4 and hypothesis, the quotient $\mathbf{L}/\mathbf{K}$ is without $\mathbf{K}$-torsion. Therefore $\mathbf{K}$ is integrally closed. Therefore $\mathbf{K}$ is a Dedekind domain. □

4.5.6. *Wedderburn's theorem.* Let R be a (possibly noncommutative) ring with unit. Let M be a simple faithful left R-module and put $K := \mathrm{End}_R(M)$. View M as a left K-module. Assume that M is finitely generated over K. Then according to *Wedderburn's theorem* (a special case of the *Jacobson density theorem*) every element of $\mathrm{End}_K(M)$ is of the form $m \mapsto rm$ for unique $r \in R$.

PROPOSITION 4.5.7 ("The Dedekind-Wedderburn Trick"). *Let $\mathbf{L} \supset \mathbf{F}$ be an extension of Dedekind domains such that $\mathbf{L}$ is a projective $\mathbf{F}$-module of finite rank. Let $\mathbf{M}$ be an $\mathbf{L}$-module finitely generated and projective over $\mathbf{F}$. Assume that $\mathbf{L}$ and $\mathbf{M}$ are of the same rank over $\mathbf{F}$. Let $\mathbf{R}$ be a subring of $\mathrm{End}_{\mathbf{F}}(\mathbf{M})$ such that*

$$\mathbf{R} \supset \{(m \mapsto xm) \in \mathrm{End}_{\mathbf{F}}(\mathbf{M}) \mid x \in \mathbf{L}\}$$

and the quotient $\mathrm{End}_{\mathbf{F}}(\mathbf{M})/\mathbf{R}$ is without $\mathbf{F}$-torsion. Put

$$\mathbf{K} := \{x \in \mathbf{L} \mid (m \mapsto xm) \in \mathrm{End}_{\mathbf{R}}(\mathbf{M})\}.$$

Then

$$\mathbf{R} = \mathrm{End}_{\mathbf{K}}(\mathbf{M}).$$

The following trivial consequences of the definition of $\mathbf{K}$ bear emphasis since they are crucial in applications:

- $\mathbf{K}$ is a Dedekind domain equipped with $\mathbf{F}$-algebra structure.
- $\mathbf{K}$ is a projective $\mathbf{F}$-module of finite rank.
- $\mathbf{M}$ is a finite direct sum of projective $\mathbf{K}$-modules of rank one.

Proof. By Lemma 4.5.4 and hypothesis the module $\mathbf{M}$ is projective over $\mathbf{L}$ of rank one and by Lemma 4.5.2 the evident map $\mathbf{L} \to \mathrm{End}_{\mathbf{L}}(\mathbf{M})$ is bijective. To simplify notation we now identify $\mathbf{L}$ with the subring $\mathrm{End}_{\mathbf{L}}(\mathbf{M})$ of $\mathrm{End}_{\mathbf{F}}(\mathbf{M})$. Note that

$$\mathbf{L} = \mathrm{End}_{\mathbf{L}}(\mathbf{M}) \supset \mathrm{End}_{\mathbf{R}}(\mathbf{M}) = \mathbf{K}.$$

Now put

$$F := \text{fraction field of } \mathbf{F}, \quad M := F \otimes_{\mathbf{F}} \mathbf{M},$$

$$\mathrm{End}_F(M) \supset R := F \otimes_{\mathbf{F}} \mathbf{R} \supset L := F \otimes_{\mathbf{F}} \mathbf{L} = \text{fraction field of } \mathbf{L},$$

$$K := F \otimes_{\mathbf{F}} \mathbf{K} = \text{fraction field of } \mathbf{K} = \mathrm{End}_R(M) \subset \mathrm{End}_L(M) = L.$$

By Wedderburn's theorem $R = \mathrm{End}_K(M)$ and hence

$$\mathbf{R} \subset R \cap \mathrm{End}_{\mathbf{F}}(\mathbf{M}) = \mathrm{End}_K(M) \cap \mathrm{End}_{\mathbf{F}}(\mathbf{M}) = \mathrm{End}_{\mathbf{K}}(\mathbf{M}).$$

Finally, we have $\mathbf{R} = \mathrm{End}_{\mathbf{K}}(\mathbf{M})$ because the quotient $\mathrm{End}_{\mathbf{K}}(\mathbf{M})/\mathbf{R}$ is a torsion $\mathbf{F}$-submodule of $\mathrm{End}_{\mathbf{F}}(\mathbf{M})/\mathbf{R}$ and hence by hypothesis vanishes. □

4.6. *Geometric complex multiplication* (GCM).

4.6.1. GCM $\mathbb{F}_q[t]$-*algebras.* Let $\mathbf{L}$ be an $\mathbb{F}_q[t]$-algebra satisfying the following conditions:

- $\mathbf{L}$ is a free $\mathbb{F}_q[t]$-module of finite rank.
- $\mathbf{L}$ is a Dedekind domain.
- $\bar{k} \otimes_{\mathbb{F}_q} \mathbf{L}$ is also a Dedekind domain.

Under these conditions we say that $\mathbf{L}$ is a GCM $\mathbb{F}_q[t]$-algebra. (GCM is short for *geometric complex multiplications.*) By Proposition 4.5.5 every $\mathbb{F}_q[t]$-subalgebra $\mathbf{K} \subset \mathbf{L}$ such that $\mathbf{K}$ is a Dedekind domain is again a GCM $\mathbb{F}_q[t]$-algebra.

4.6.2. *Constructions functorial in* GCM $\mathbb{F}_q[t]$-*algebras.* For any GCM $\mathbb{F}_q[t]$-algebra $\mathbf{L}$ put

$$\overline{\mathbf{L}} := \bar{k} \otimes_{\mathbb{F}_q} \mathbf{L}.$$

For any integer $n \in \mathbb{Z}$ let

$$\left(a \mapsto a^{(n)}\right) : \overline{\mathbf{L}} \to \overline{\mathbf{L}}$$

be the unique $\mathbf{L}$-linear extension of the automorphism

$$(x \mapsto x^{q^n}) : \bar{k} \to \bar{k}.$$

Let $\overline{\mathbf{L}}[\sigma]$ be the ring obtained by adjoining a noncommutative variable σ to $\overline{\mathbf{L}}$ subject to the relations

$$\sigma a = a^{(-1)}\sigma \quad (a \in \overline{\mathbf{L}}).$$

Every element of $\overline{\mathbf{L}}[\sigma]$ has a unique presentation of the form

$$\sum_{i=0}^{\infty} a_i \sigma^i \quad (a_i \in \overline{\mathbf{L}}, \quad a_i = 0 \text{ for } i \gg 0)$$

and in terms of such presentations the multiplication law in $\overline{\mathbf{L}}[\sigma]$ takes the form

$$\left(\sum_i a_i \sigma^i\right)\left(\sum_j b_j \sigma^j\right) = \sum_i \sum_j a_i b_j^{(-i)} \sigma^{i+j}.$$

Note that $\mathbf{L}$ is contained in the center of $\overline{\mathbf{L}}[\sigma]$. If $\mathbf{L} = \mathbb{F}_q[t]$, then $\overline{\mathbf{L}}[\sigma] = \bar{k}[t, \sigma]$.

4.6.3. *Geometric complex multiplications.* Let $\mathbf{L}$ be a GCM $\mathbb{F}_q[t]$-algebra. Let H be a left $\overline{\mathbf{L}}[\sigma]$-module such that the $\overline{\mathbf{L}}$-module underlying H is projective of rank one and the $\bar{k}[t, \sigma]$-module underlying H is a dual t-motive. We call

H a dual t-motive with *geometric complex multiplications* by $\mathbf{L}$ (for short: GCM by $\mathbf{L}$). We call the $\bar{k}[t,\sigma]$-module underlying H the *bare* dual t-motive underlying H. We define the GCM *type* of H with respect to $\mathbf{L}$ to be the ideal of $\overline{\mathbf{L}}$ annihilating the quotient $H/\sigma H$. If the natural map $\mathbf{L} \to \mathrm{End}_{\bar{k}[t,\sigma]}(H)$ is bijective, we say that H has *tight* GCM by $\mathbf{L}$. We say that dual t-motives H_0 and H_1 both with GCM by $\mathbf{L}$ are $\mathbf{L}$-*linearly isogenous* if there exists an injective homomorphism $H_0 \to H_1$ of $\overline{\mathbf{L}}[\sigma]$-modules with cokernel of finite dimension over $\bar{k}$. By Theorem 4.4.7 the relation of $\mathbf{L}$-linear isogeneity is an equivalence relation and by Lemma 4.4.6 the GCM type of a dual t-motive with GCM by $\mathbf{L}$ depends only on its $\mathbf{L}$-linear isogeny class.

THEOREM 4.6.4. *Let H be a dual t-motive with* GCM *by* $\mathbf{L}$. *Let H_0 be any simple dual t-motive embedded in the bare dual t-motive underlying H, e.g., any nonzero saturated $\bar{k}[t,\sigma]$-submodule of minimum possible rank over $\bar{k}[t]$.* (i) *The bare dual t-motive underlying H is isogenous to a finite direct sum of copies of the dual t-motive H_0.* (ii) *If H has tight* GCM *by* $\mathbf{L}$ *then the bare dual t-motive underlying H is simple.*

Proof. (i) Consider the family

$$\{aH_0\}_{0\neq a\in\mathbf{L}}$$

of isomorphic copies of H_0 embedded $\bar{k}[t,\sigma]$-linearly in H. The sum $\sum_a aH_0$ is a nonzero $\overline{\mathbf{L}}[\sigma]$-submodule of H and *a fortiori* a $\bar{k}[t,\sigma]$-submodule of H of finite codimension over $\bar{k}$. Therefore the bare dual t-motive underlying H is isogenous to a direct sum of simple dual t-motives isomorphic to H_0 by Proposition 4.4.10.

(ii) Making use of Proposition 4.4.10 in a more precise way, we obtain a positive integer n and $a_1,\dots,a_n \in \mathbf{L}$ such that the map

$$\phi := ((h_1,\dots,h_n) \mapsto a_1h_1 + \cdots + a_nh_n) : H_0^n \to H$$

is an isogeny of (bare) dual t-motives. By Theorem 4.4.4 there exists $0 \neq a \in \mathbb{F}_q[t]$ such that aH is contained in the image of ϕ. Put

$$\left.\begin{array}{rcl} \pi_i & := & ((h_1,\dots,h_n)\mapsto h_i) : H_0^n \to H_0 \\[1ex] e_i & := & \big(h \mapsto a_i\pi_i(\phi^{-1}(ah))\big) \in \mathrm{End}_{\bar{k}[t,\sigma]}(H) \end{array}\right\} \quad (i=1,\dots,n).$$

The endomorphisms of the bare dual t-motive underlying H thus constructed satisfy the relations

$$e_i \neq 0, \quad e_ie_j = a\delta_{ij}e_i \quad (i,j=1,\dots,n).$$

But unless $n=1$, such a system of relations is forbidden because the ring $\mathrm{End}_{\bar{k}[t,\sigma]}(H)$ is isomorphic as an $\mathbb{F}_q[t]$-algebra to the domain $\mathbf{L}$. $\square$

THEOREM 4.6.5. *For $i = 0, 1$ let H_i be a dual t-motive with tight* GCM *by $\mathbf{L}_i$ and let $I_i \subset \overline{\mathbf{L}}_i$ be the* GCM *type of H_i with respect to $\mathbf{L}_i$. Assume that the bare dual t-motives underlying H_0 and H_1 are isogenous.* (i) *Then there exists a unique $\mathbb{F}_q[t]$-algebra isomorphism*

$$\theta : \mathbf{L}_0 \xrightarrow{\sim} \mathbf{L}_1$$

such that all the diagrams

$$\begin{array}{ccc} H_0 & \xrightarrow{\phi} & H_1 \\ {\scriptstyle a\times}\downarrow & & \downarrow{\scriptstyle \theta(a)\times} \\ H_0 & \xrightarrow{\phi} & H_1 \end{array} \qquad \left(\phi \in \mathrm{Hom}_{\bar{k}[t,\sigma]}(H_0, H_1), \quad a \in \mathbf{L}_0\right)$$

commute. (ii) *Under the unique $\bar{k}$-linear isomorphism $\overline{\mathbf{L}}_0 \to \overline{\mathbf{L}}_1$ induced by θ the ideal I_0 maps bijectively to ideal I_1.*

Proof. (i) Put $\mathbf{M} := \mathrm{Hom}_{\bar{k}[t,\sigma]}(H_0, H_1)$, where $\mathbf{M}$ is a right $\mathbf{L}_0$-module and a left $\mathbf{L}_1$-module in the evident fashion. Since H_0, H_1 have tight GCM by $\mathbf{L}_0$, $\mathbf{L}_1$, Corollary 4.4.8 shows that the $\mathbb{F}_q[t]$-modules $\mathbf{M}$, $\mathbf{L}_0$ and $\mathbf{L}_1$ are free of the same finite rank over $\mathbb{F}_q[t]$. It follows by Lemma 4.5.4 that $\mathbf{M}$ is projective of rank one both over $\mathbf{L}_0$ and over $\mathbf{L}_1$. Existence and uniqueness of θ now follow by Proposition 4.5.3. (ii) For any isogeny $\phi : H_0 \to H_1$ the induced map $\phi \bmod \sigma$ is bijective by Lemma 4.4.6, whence the result. □

THEOREM 4.6.6. *Let H be a dual t-motive with* GCM *by $\mathbf{L}$. Assume that the bare dual t-motive underlying H is rigid analytically trivial. Put*

$$\mathbf{R} := \mathrm{End}_{\bar{k}[t,\sigma]}(H), \quad \mathbf{K} := \{x \in \mathbf{L} \mid (x \mapsto xh) \in \mathrm{End}_{\mathbf{R}}(H)\}.$$

Then $\mathbf{K}$ is a GCM *$\mathbb{F}_q[t]$-subalgebra of $\mathbf{L}$ and there exists a $\overline{\mathbf{K}}[\sigma]$-submodule*

$$H_0 \subset H$$

with the following properties:

- *H_0 is a dual t-motive with tight GCM by $\mathbf{K}$.*
- *The bare dual t-motive underlying H_0 is simple and rigid analytically trivial.*
- *The* GCM *type of H with respect to $\mathbf{L}$ is generated as an ideal of $\overline{\mathbf{L}}$ by the* GCM *type of H_0 with respect to $\mathbf{K}$.*

Proof. We claim that the natural map

$$\mathbf{R} = \mathrm{End}_{\bar{k}[t,\sigma]}(H) \to \mathrm{End}_{\mathbf{K}}\left(H^{\mathrm{Betti}}\right)$$

is bijective. In any case the natural map

$$\mathrm{End}_{\bar{k}[t,\sigma]}(H) \to \mathrm{End}_{\mathbb{F}_q[t]}\left(H^{\mathrm{Betti}}\right)$$

is injective and has cokernel without $\mathbb{F}_q[t]$-torsion by Theorem 4.4.14. Moreover, we have

$$\mathrm{rk}_{\mathbb{F}_q[t]}\,\mathbf{L} = \mathrm{rk}_{\bar{k}[t]}\,\overline{\mathbf{L}} = \mathrm{rk}_{\bar{k}[t]}\,H = \mathrm{rk}_{\mathbb{F}_q[t]}\,H^{\mathrm{Betti}},$$

where the first equality is trivial, the second holds by definition of GCM, and the third holds by Lemma 4.4.12 and hypothesis. Therefore the Dedekind-Wedderburn trick proves the claim. It follows that $\mathbf{K}$ is a GCM $\mathbb{F}_q[t]$-subalgebra of $\mathbf{L}$.

Let n be the rank of H^{Betti} as a projective $\mathbf{K}$-module and let

$$H^{\mathrm{Betti}} = \bigoplus_{i=0}^{n-1} \mathbf{M}_i$$

be a decomposition of H^{Betti} as a direct sum of projective $\mathbf{K}$-modules of rank one. Let $e_0 \in \mathbf{R}$ be the idempotent endomorphism of H inducing the projection of H^{Betti} to the direct summand $\mathbf{M}_0$, and put

$$H_0 := e_0 H.$$

Then H_0 is a rigid analytically trivial dual t-motive since it is a $\bar{k}[t,\sigma]$-linear direct summand of a rigid analytically trivial dual t-motive. The idempotent element $e_0 \in \mathbf{R}$ is $\mathbf{K}$-linear by definition of $\mathbf{K}$ and hence H_0 comes naturally equipped with $\overline{\mathbf{K}}[\sigma]$-module structure extending the $\bar{k}[t,\sigma]$-module structure. Moreover, we have

$$H_0^{\mathrm{Betti}} = \mathbf{M}_0, \quad \mathrm{rk}_{\bar{k}[t]}\,\overline{\mathbf{K}} = \mathrm{rk}_{\mathbb{F}_q[t]}\,\mathbf{K} = \mathrm{rk}_{\mathbb{F}_q[t]}\,\mathbf{M}_0 = \mathrm{rk}_{\bar{k}[t]}\,H_0,$$

the last equality above by Lemma 4.4.12, and hence H_0 is projective of rank one over $\overline{\mathbf{K}}$. By a repetition of the argument of the first paragraph of the proof we have a natural bijective map

$$\mathrm{End}_{\bar{k}[t,\sigma]}(H_0) \to \mathrm{End}_{\mathbf{K}}(\mathbf{M}_0) = \mathbf{K}.$$

Thus H_0 comes naturally equipped with tight GCM by $\mathbf{K}$. It follows by Theorem 4.6.4 that H_0 is simple as a bare dual t-motive. The natural map

$$(a \otimes h \mapsto ah) : \overline{\mathbf{L}} \otimes_{\overline{\mathbf{K}}} H_0 \to H$$

is injective with $\overline{\mathbf{L}}[\sigma]$-stable image. Call the image H'. The $\overline{\mathbf{L}}$-module underlying H' is projective of rank one and *a fortiori* of finite codimension over $\bar{k}$ in H. Clearly the $\bar{k}[t]$- and $\bar{k}[\sigma]$-modules underlying H' are free of finite rank. Since H' is generated over $\mathbf{L}$ by H_0 and $\mathbf{L}$ is central in $\overline{\mathbf{L}}[\sigma]$, we have

$$(t-T)^n H_0 \subset \sigma H_0 \Rightarrow (t-T)^n H' \subset \sigma H'$$

for all integers $n \geq 0$ and hence the $\bar{k}[t,\sigma]$-module underlying H' is a dual t-motive. The upshot is that H' is a dual t-motive with GCM by $\mathbf{L}$ and that H' is $\mathbf{L}$-linearly isogenous to H via the inclusion. Necessarily, the GCM types of H' and H with respect to $\mathbf{L}$ are equal. Finally, we have at our disposal an isomorphism

$$\overline{\mathbf{L}} \otimes_{\overline{\mathbf{K}}} \frac{H_0}{\sigma H_0} = \frac{H'}{\sigma H'}$$

of torsion $\overline{\mathbf{L}}$-modules, whence it follows that the GCM type of H' with respect to $\mathbf{L}$ is generated as an ideal of $\overline{\mathbf{L}}$ by the GCM type of H_0 with respect to $\mathbf{K}$. □

Corollary 4.6.7. *Let* $\mathbf{L}$ *be a GCM* $\mathbb{F}_q[t]$*-algebra. Assume that the fraction field of* $\mathbf{L}$ *is Galois over* $\mathbb{F}_q(t)$ *and let* G *be the associated Galois group. Extend the action of* G *on* $\mathbf{L}$ *to* $\overline{\mathbf{L}}$ *by* $\bar{k}$*-linearity. Let* H *be a dual* t*-motive with* GCM *by* $\mathbf{L}$ *and with rigid analytically trivial underlying bare dual* t*-motive. Let* H_0 *be a simple quotient of the bare dual* t*-motive underlying* H*. Let* $I \subset \overline{\mathbf{L}}$ *be the* GCM *type of* H *with respect to* $\mathbf{L}$*. Let* r *be the cardinality of the set of ideals of* $\overline{\mathbf{L}}$ *that are* G*-conjugate to* I*. Then* H_0 *is of rank* $\geq r$ *over* $\bar{k}[t]$*.*

Proof. By Theorem 4.6.6 there exist a GCM $\mathbb{F}_q[t]$-subalgebra $\mathbf{K} \subset \mathbf{L}$ and a $\overline{\mathbf{K}}[\sigma]$-submodule $H_0' \subset H$ with the following properties:

- H_0' has tight GCM by $\mathbf{K}$.
- The bare dual t-motive underlying H_0' is simple.
- The GCM type $I_0 \subset \overline{\mathbf{K}}$ of H_0' with respect to $\mathbf{K}$ generates $I \subset \overline{\mathbf{L}}$.

Further, by Theorem 4.6.4 the given quotient H_0 of the bare dual t-motive underlying H and the bare dual t-motive underlying H_0' are isogenous. Finally, we have

$$\mathrm{rk}_{\bar{k}[t]} H_0 = \mathrm{rk}_{\bar{k}[t]} H_0' = \mathrm{rk}_{\mathbb{F}_q[t]} \mathbf{K} \geq r$$

because $\gamma I = I$ for every $\gamma \in G$ acting as the identity on $\mathbf{K}$. □

Corollary 4.6.8. *Let* $\mathbf{L}$ *be a GCM* $\mathbb{F}_q[t]$*-algebra. Assume that the fraction field of* $\mathbf{L}$ *is Galois over* $\mathbb{F}_q(t)$ *and let* G *be the associated Galois group. Extend the action of* G *on* $\mathbf{L}$ *to* $\overline{\mathbf{L}}$ *by* $\bar{k}$*-linearity. For* $i = 0, 1$ *let* H_i *be a dual* t*-motive with* GCM *by* $\mathbf{L}$ *with rigid analytically trivial underlying bare dual* t*-motive, let* $I_i \subset \overline{\mathbf{L}}$ *be the* GCM *type of* H_i *with respect to* $\mathbf{L}$*, and let* H_{i0} *be a simple quotient of the bare dual* t*-motive underlying* H_i*. If* H_{00} *and* H_{10} *are isogenous, then the ideals* I_0 *and* I_1 *are* G*-conjugate.*

Proof. For $i = 0, 1$, there exist by Theorem 4.6.6 a GCM $\mathbb{F}_q[t]$-subalgebra $\mathbf{K}_i \subset \mathbf{L}$ and a $\overline{\mathbf{K}}_i[\sigma]$-submodule $H_{i0}' \subset H_i$ with the following properties:

- H_{i0}' has tight GCM by $\mathbf{K}_i$.
- The bare dual t-motive underlying H_{i0}' is simple.
- The GCM type $I_{i0} \subset \overline{\mathbf{K}}_i$ of H_{i0}' with respect to $\mathbf{K}_i$ generates $I_i \subset \overline{\mathbf{L}}$.

Further, for $i = 0, 1$ by Theorem 4.6.4 the bare dual t-motive underlying H'_{i0} is isogenous to H_{i0}, and hence the bare dual t-motives underlying H'_{00} and H'_{10} are isogenous. Finally, by Theorem 4.6.5 there exists some $\gamma \in G$ such that $\gamma\mathbf{K}_0 = \mathbf{K}_1$ and $\gamma I_{00} = I_{10}$, in which case necessarily $\gamma I_0 = I_1$. $\square$

4.6.9. *Remark.* The theory of rigid analytically trivial GCM dual t-motives worked out above is essentially just the dual t-motivic translation of a method introduced in [BrPa] for analyzing the t-submodule structure of the geometric t-modules of [Si a].

5. Special functions

5.1. *The Carlitz exponential and its fundamental period.* The reference [Go, Chap. 3] is a good source of background material on this topic.

5.1.1. *The Carlitz exponential.* Put

$$\exp_C z := \sum_{n=0}^{\infty} \frac{z^{q^n}}{D_n} \quad \left(D_n := \prod_{i=0}^{n-1}\left(T^{q^n} - T^{q^i}\right)\right)$$

thereby defining an $\mathbb{F}_q$-linear power series in z with coefficients in k called the *Carlitz exponential*. The Carlitz exponential satisfies the functional equation

$$\exp_C(Tz) = T\exp_C z + (\exp_C z)^q$$

by [Go, Prop. 3.3.1].

5.1.2. *The fundamental period.* The power series $\exp_C z$ has an infinite radius of convergence with respect to the valuation $|\cdot|_\infty$ and has a Weierstrass product expansion of the form

$$\exp_C z = z \prod_{\substack{a\in A\\ a\neq 0}} \left(1 - \frac{z}{\varpi a}\right)$$

for unique $\varpi \in k_\infty(\widetilde{T})$ such that

$$|\varpi - \widetilde{T}T|_\infty < |\widetilde{T}T|_\infty$$

by [Go, Cor. 3.2.9 and Rmks. 3.2.10]. It follows that the sequences

$$0 \to \varpi \cdot A \subset \overline{k_\infty} \xrightarrow{\exp_C} \overline{k_\infty} \to 0, \quad 0 \to \varpi \cdot A \subset \mathbb{C}_\infty \xrightarrow{\exp_C} \mathbb{C}_\infty \to 0$$

are exact. We call ϖ the *fundamental period* of the Carlitz exponential. Transcendence of ϖ over k was first proved in [Wa].

The next two results relate the Carlitz exponential and its fundamental period to the power series $\Omega(t)$ and transcendental number $\Omega(T)$ discussed in §3.1.2.

PROPOSITION 5.1.3. *The equality*

$$1/\Omega^{(-1)}(t) = \sum_{i=0}^{\infty} \exp_C\left(\varpi/T^{i+1}\right) t^i$$

holds.

Proof. Recall that

$$\Omega(t) = \widetilde{T}^{-q} \prod_{i=1}^{\infty} \left(1 - \frac{t}{T^{q^i}}\right), \quad (t-T)\cdot\Omega = \Omega^{(-1)}.$$

Temporarily denote the power series on the right side of the identity to be proved by $\Theta(t)$. Consider the Maclaurin expansion

$$\Omega^{(-1)}(t)\Theta(t) = \sum_{i=0}^{\infty} c_i t^i \qquad (c_i \in k_\infty(\widetilde{T})).$$

The functional equation noted in §5.1.1 implies that

$$t\Theta = T\Theta + \Theta^{(1)},$$

and hence

$$\Omega\Theta^{(1)} = \Omega\cdot(t-T)\cdot\Theta = \Omega^{(-1)}\Theta = \left(\Omega\Theta^{(1)}\right)^{(-1)}.$$

By this last we have $c_i = \sqrt[q]{c_i}$ and hence $c_i \in \mathbb{F}_q$ for all $i \geq 0$. By plugging into the Weierstrass product expansion of $\exp_C z$ we find that $|c_0 - 1| < 1$ and hence

$$c_0 = \widetilde{T}^{-1}\cdot\exp_C(\varpi/T) = 1.$$

Now write

$$\Omega^{(-1)}(t) = \sum_{i=0}^{\infty} a_i t^i, \quad \Theta(t) = \sum_{i=0}^{\infty} b_i t^i \quad (a_i, b_i \in k_\infty(\widetilde{T})).$$

We have, so we claim, bounds

$$|a_i|_\infty \leq |\widetilde{T}|^{-1}, \quad |b_i|_\infty \leq |\widetilde{T}|$$

with strict inequality for $i > 0$. The bound for $|a_i|_\infty$ is clear. The bound for $|b_i|_\infty$ is verified by plugging into the Weierstrass product expansion of $\exp_C z$. Thus the claim is proved. It follows that $|c_i|_\infty < 1$ for all $i > 0$; hence $c_i = 0$ for all $i > 0$, and hence $\Omega^{(-1)}\Theta = 1$. □

COROLLARY 5.1.4. *The statement*

$$\varpi = T\widetilde{T} \prod_{i=1}^{\infty} \left(1 - T^{1-q^i}\right)^{-1} = -1/\Omega(T)$$

holds.

Proof. (Cf. [AnTh, Cor. 2.5.8].) By Proposition 5.1.3 and a little high school algebra (summation of geometric series), we have

$$\frac{\frac{1}{\Omega(t)}+\varpi}{t-T}=\frac{1}{\Omega^{(-1)}(t)}-\sum_{n=0}^{\infty}\left(\frac{\varpi}{T^{n+1}}\right)t^n=\sum_{n=0}^{\infty}\left(\sum_{i=1}^{\infty}\frac{1}{D_i}\left(\frac{\varpi}{T^{n+1}}\right)^{q^i}\right)t^n.$$

The power series on the right is convergent in the disc $|t|_\infty<|T|_\infty^q$ and so is the power series $1/\Omega(t)$. Therefore we have $1/\Omega(T)+\varpi=0$. □

5.2. *The functions* $\mathbf{e}$ *and* $\mathbf{e}^*$ *and their division polynomials.*

5.2.1. *Definition of* $\mathbf{e}$. Put

$$\mathbf{e}:=(x\mapsto \exp_C \varpi x):k_\infty\to k_\infty(\widetilde{T}),$$

where ϖ is the fundamental period of the Carlitz exponential. The function $\mathbf{e}$ maps the quotient k_∞/A isomorphically to a compact additive subgroup of $k_\infty(\widetilde{T})$. The function $\mathbf{e}$ is in many respects analogous to the function $(x\mapsto e^{2\pi i x}):\mathbb{R}\to\mathbb{C}$ mapping the quotient $\mathbb{R}/\mathbb{Z}$ isomorphically to the unit circle in $\mathbb{C}$.

5.2.2. *Division polynomials for* $\mathbf{e}$. Given $a\in A$, write

$$a=Tb+\varepsilon\quad (b\in A,\ \ \varepsilon\in\mathbb{F}_q)$$

and put

$$C_a(t,z):=\begin{cases}0 & \text{if } a=0,\\ C_b(t,tz+z^q)+\varepsilon z & \text{if } a\neq 0,\end{cases}$$

thereby recursively defining a polynomial

$$C_a(t,z)\in\mathbb{F}_q[t,z].$$

For example we have

$$C_1(t,z)=z,\quad C_T(t,z)=tz+z^q,\quad C_{T^2}(t,z)=t^2z+(t+t^q)z^q+z^{q^2}.$$

For all $f=f(T)\in A_+$ of positive degree we have

$$C_f(t,z)=f(t)z+\left(\begin{array}{c}\mathbb{F}_q[t]\text{-linear combination of}\\ \text{terms } z^{q^i} \text{ with } 0<i<\deg f\end{array}\right)+z^{q^{\deg f}}$$

as can be verified by an evident induction. From the functional equation noted in §5.1.1 it follows that

$$T\mathbf{e}(x)+\mathbf{e}(x)^q=\mathbf{e}(Tx)$$

for all $x\in k_\infty$ and hence by an evident induction that

$$C_a\left(T,\mathbf{e}(x)\right)=\mathbf{e}(ax)$$

for all $a \in A$ and $x \in k_\infty$. From the latter identity it follows that $C_a(t,z)$ depends $\mathbb{F}_q$-linearly on a and that the *composition law*

$$C_a(t, C_b(t,z)) = C_{ab}(t,z)$$

holds for all $a, b \in A$. We call $C_a(t,z)$ the *division polynomial* for $\mathbf{e}$ indexed by a.

5.2.3. *Torsion values of* $\mathbf{e}$. A number of the form

$$\mathbf{e}(x) \quad (x \in k)$$

will be called a *torsion value* of $\mathbf{e}$. For every $f \in A_+$ we have

$$C_f(T,z) = \prod_{\substack{a \in A \\ \deg a < \deg f}} (z - \mathbf{e}(a/f))$$

and hence every torsion value of $\mathbf{e}$ is separably algebraic over k.

5.2.4. *Remark*. The ring homomorphism

$$(a \mapsto (x \mapsto C_a(T,x))) : A \to \mathrm{End}_{\text{alg. gp.}/A}(\mathbf{G}_a)$$

is called the *Carlitz module* according to [Go, Def. 3.3.5]. The torsion values of $\mathbf{e}$ admit interpretation as the torsion points of the Carlitz module defined over $\bar{k}$. The latter interpretation makes it clear that torsion values of $\mathbf{e}$ generate abelian extensions of k. The Carlitz module plays in explicit class field theory over k a role analogous to that played by the multiplicative group in explicit class field theory over $\mathbb{Q}$. See [Ha] for a treatment of explicit class field theory over k; another good source of information is [Ro]. We consider some of the more delicate properties of torsion values of $\mathbf{e}$ below in §6.3.1.

5.2.5. *Definition of* $\mathbf{e}^*$. Let

$$\mathrm{Res} : k_\infty \to \mathbb{F}_q$$

be the unique $\mathbb{F}_q$-linear functional such that

$$\ker \mathrm{Res} = \mathbb{F}_q[T] + (1/T^2)\mathbb{F}_q[[1/T]], \quad \mathrm{Res}(1/T) = 1.$$

Write

$$\Omega^{(-1)}(t) = \widetilde{T}^{-1} \prod_{i=0}^{\infty} \left(1 - \frac{t}{T^{q^i}}\right) = \sum_{i=0}^{\infty} a_i t^i \quad (a_i \in k_\infty(\widetilde{T})).$$

For each $x \in k_\infty$ put

$$\mathbf{e}^*(x) := \sum_{i=0}^{\infty} \mathrm{Res}(T^i x) a_i,$$

e. g.,

$$\mathbf{e}^*(1/T^{n+1}) = \begin{cases} a_n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

for any $n \in \mathbb{Z}$, thereby defining a function

$$\mathbf{e}^* : k_\infty \to k_\infty(\widetilde{T}).$$

A number of the form

$$\mathbf{e}^*(x) \quad (x \in k)$$

will be called a *torsion value* of $\mathbf{e}^*$.

LEMMA 5.2.6. *For all $x \in k_\infty$ such that $0 < |x|_\infty < 1$, $\mathbf{e}^*(x) \neq 0$.*

Proof. With the coefficients $a_i \in k_\infty(\widetilde{T})$ as defined in §5.2.5,

$$\left| a_i - (-T)^{-(q^0+\cdots+q^{i-1})}\widetilde{T}^{-1} \right|_\infty = |a_i - \widetilde{T}^{-q^i}|_\infty < |\widetilde{T}^{-q^i}|_\infty,$$

for all integers $i \geq 0$ and hence

$$|\mathbf{e}^*(x)|_\infty = \left| \widetilde{T}^{-q^{\min\{i \mid \operatorname{Res}(T^i x) \neq 0\}}} \right|_\infty > 0$$

for all $x \in k_\infty$ such that $0 < |x|_\infty < 1$. □

5.2.7. *Division polynomials for* $\mathbf{e}^*$. Put

$$\overline{C}_1(t,z) := z.$$

For each $f \in A_+$ of positive degree, write

$$f = Tg + \varepsilon \quad (g \in A_+, \ \ \varepsilon \in \mathbb{F}_q)$$

and put

$$\overline{C}_f(t,z) := \overline{C}_g(t^q, tz^q + z) + \varepsilon z^{q^{\deg f}}.$$

In this way we recursively define polynomials

$$\overline{C}_f(t,z) \in \mathbb{F}_q[t,z]$$

for all $f \in A_+$. For example,

$$\overline{C}_1(T,z) = z, \quad \overline{C}_T(t,z) = tz^q + z, \quad \overline{C}_{T^2}(t,z) = t^{2q}z^{q^2} + (t+t^q)z^q + z.$$

For all $f = f(T) \in A_+$ of positive degree,

$$\overline{C}_f(t,z) = z + \left(\begin{array}{l} \mathbb{F}_q[t]\text{-linear combination of} \\ \text{terms } z^{q^i} \text{ with } 0 < i < \deg f \end{array} \right) + f(t)^{q^{-1+\deg f}} z^{q^{\deg f}}$$

as can be verified by an evident induction. With the coefficients $a_i \in k_\infty(\widetilde{T})$ as defined in §5.2.5 and in view of the functional equation $\Omega^{(-1)} = (t-T)\cdot\Omega$,

$$Ta_i^q + a_i = \begin{cases} a_{i-1}^q & \text{if } i > 0 \\ 0 & \text{if } i = 0 \end{cases}$$

for all nonnegative integers i and hence

$$T\mathbf{e}^*(x)^q + \mathbf{e}^*(x) = \mathbf{e}^*(Tx)^q$$

for all $x \in k_\infty$. It follows by an evident induction that

$$\overline{C}_f(T, \mathbf{e}(x)) = \mathbf{e}(fx)^{q^{\deg f}}$$

for all $f \in A_+$ and $x \in k_\infty$. This last is the end to which the definition of $\overline{C}_f(T, z)$ was contrived. We call $\overline{C}_f(t, z)$ the *division polynomial* for $\mathbf{e}^*$ indexed by f.

5.2.8. *Key properties of* $\mathbf{e}^*$. The function

$$\mathbf{e}^* : k_\infty \to k_\infty(\widetilde{T})$$

has, so we claim, the following properties:

- $\sum_{i=0}^\infty \mathbf{e}^*\left(1/T^{i+1}\right) t^i = \widetilde{T}^{-1} \prod_{i=0}^\infty \left(1 - t/T^{q^i}\right) = \Omega^{(-1)}(t)$.
- $\mathbf{e}^*$ is $\mathbb{F}_q$-linear and $|\cdot|_\infty$-continuous.
- $\ker \mathbf{e}^* = A$.
- $\mathbf{e}^*(x)\widetilde{T} \in \mathbb{F}_q[\![1/T]\!]$ for all $x \in k_\infty$.
- The torsion values of $\mathbf{e}^*$ are separably algebraic over k.

The function $\mathbf{e}^*$ has the first two of the claimed properties by definition. We have $A \subset \ker \mathbf{e}^*$ by definition and $\ker \mathbf{e}^* \subset A$ by Lemma 5.2.6. Therefore the third property holds. The first three properties imply the fourth. Each torsion value of $\mathbf{e}^*$ is a solution of an equation of the form $\overline{C}_f(T, z) = 0$ for some $f \in A_+$ and hence separably algebraic over k. Therefore the fifth property holds. Thus all claims concerning $\mathbf{e}^*$ are proved. Note that the first three properties above already determine the function $\mathbf{e}^*$ uniquely.

5.2.9. *Remark.* Given $f \in A_+$, write

$$C_f(t, z) = \sum_{i=0}^n f_i z^{q^i} \quad (f_i \in \mathbb{F}_q[t], \quad n = \deg f).$$

It is not hard to show that

$$\overline{C}_f(t, z) = \sum_{i=0}^n f_i^{q^{n-i-1}} z^{q^{n-i}}.$$

It follows that the torsion values of the function $\mathbf{e}^*$ admit interpretation as the q^{th} roots of the torsion points defined over $\bar{k}$ of the *adjoint Carlitz module*. For the definition of the latter, see [Go, §3.7]. For a general discussion of adjoints of Drinfeld modules and duality *à la* Elkies and Poonen, see [Go, §4.14].

5.2.10. *Key properties of* $\mathbf{e}$. The properties of the function $\mathbf{e}^*$ are strikingly parallel to those of the function

$$\mathbf{e} : k_\infty \to k_\infty(\widetilde{T})$$

as is made plain by the following summary of facts already proved:

- $\sum_{i=0}^\infty \mathbf{e}\left(1/T^{i+1}\right) t^i = \widetilde{T} \prod_{i=0}^\infty \left(1 - t/T^{q^i}\right)^{-1} = 1/\Omega^{(-1)}(t)$.
- $\mathbf{e}$ is $\mathbb{F}_q$-linear and $|\cdot|_\infty$-continuous.
- $\ker \mathbf{e} = A$.
- $\mathbf{e}(x)/\widetilde{T} \in \mathbb{F}_q[\![1/T]\!]$ for all $x \in k_\infty$.
- The torsion values of $\mathbf{e}$ are separably algebraic over k.

Note that the first three of the properties listed above determine $\mathbf{e}$ uniquely without any reference to the Carlitz exponential or its fundamental period.

5.2.11. *Remark.* Here is another way to see that the torsion values of $\mathbf{e}^*$ are algebraic. For each $n \in \mathbb{Z}_{\geq 0}$ let

$$n = \sum_i n_i q^i \quad (0 \leq n_i < q)$$

be the base q representation of n and put

$$\alpha(n) := \begin{cases} \sum_i n_i & \text{if } n_i \in \{0,1\} \text{ for all } i, \\ -\infty & \text{otherwise.} \end{cases}$$

By definition of $\mathbf{e}^*$ we have

$$\widetilde{T}\mathbf{e}^*(x) = \sum_{n=0}^\infty \operatorname{Res}\left((-T)^{\alpha(n)}\, x\right) T^{-n} \qquad \left(\text{by convention: } (-T)^{-\infty} = 0\right)$$

for all $x \in k_\infty$. Note that the very simple "program"

$$\alpha(n) = \begin{cases} 0 & \text{if } n = 0, \text{ else} \\[2ex] \alpha(n/q) & \text{if } n \equiv 0 \bmod q, \text{ else} \\[2ex] \alpha((n-1)/q) + 1 & \text{if } n \equiv 1 \bmod q, \text{ else} \\[2ex] -\infty & \end{cases}$$

computes $\alpha(n)$ recursively. Now fix $x \in k$. Then the sequence

$$\{\operatorname{Res}\left((-T)^n\, x\right)\}_{n=0}^\infty$$

is eventually periodic; hence the sequence

$$\left\{\mathrm{Res}\left((-T)^{\alpha(n)}x\right)\right\}_{n=0}^{\infty}$$

of coefficients is computable by a "q-automaton" and hence the number $\widetilde{T}\mathbf{e}^*(x)$ is algebraic over k by Christol's algebraicity criterion [Ch] (see also [ChKaMe-FrRa]).

5.3. *The geometric* Γ- *and* Π-*functions and their special values.*

5.3.1. *The Moore determinant identity.* Given an $\mathbb{F}_q$-algebra R and $x_1, \ldots, x_N \in R$, put

$$\mathrm{Moore}_q(x_1,\ldots,x_N) := \det_{i,j=1}^{N} x_j^{q^{N-i}} = \begin{vmatrix} x_1^{q^{N-1}} & \ldots & x_N^{q^{N-1}} \\ \vdots & & \vdots \\ x_1^{q^0} & \ldots & x_N^{q^0} \end{vmatrix} \in R$$

thereby defining the *Moore determinant* of $x_1, \ldots, x_N$. The *Moore determinant identity* reads

$$\mathrm{Moore}_q(x_1,\ldots,x_N) = \prod_{\substack{c\in\mathbb{F}_q^N \\ c:\mathrm{monic}}} \left(\sum_{i=1}^{N} c_i x_i\right)$$

where *ad hoc* we say that a vector $c = (c_1, \ldots, c_N) \in \mathbb{F}_q^N$ is *monic* if $c \neq 0$ and the leftmost nonzero entry of c equals 1. See [Go, Chap. 1, §3] for proof and further discussion of the identity.

5.3.2. *Definition of* $\Psi_N(z)$. For each integer $N \geq 0$, put

$$\Psi_N(z) := \left(\prod_{\substack{a\in A \\ \deg a<N}} (z-a)\right) \Bigg/ \left(\prod_{\substack{a\in A_+ \\ \deg a=N}} a\right)$$

thereby defining an $\mathbb{F}_q$-linear polynomial in z with coefficients in k. We have

$$\Psi_N(z) = \frac{\mathrm{Moore}_q\left(z, T^{N-1}, \ldots, 1\right)}{\mathrm{Moore}_q\left(T^N, \ldots, 1\right)} = \frac{z^{q^N}}{D_N} + \left(\text{terms of degree} < q^N \text{ in } z\right)$$

by combining the Moore and Vandermonde determinant identities. The definition of $\Psi_N(z)$ goes back to the paper [Ca]. A contemporary reference for this material is [Go, §3.5].

5.3.3. *Key relations satisfied by* $\Psi_N(z)$. Fix an integer $N \geq 0$. We have

$$1 + \Psi_N(z) = \prod_{\substack{a\in A_+ \\ \deg a=N}} \left(1 + \frac{z}{a}\right)$$

as can be verified by comparing zeroes and constant terms. Clearly we have

$$\deg a < N \Rightarrow \Psi_N(z+a) = \Psi_N(z)$$

for all $a \in A$. We have

$$N > 0 \Rightarrow \Psi_N(z) = \frac{\Psi_{N-1}(z)^q - \Psi_{N-1}(z)}{T^{q^N} - T}$$

as can be verified by comparing zeroes and leading terms; for convenient reference we dub this important fact the *fundamental recursion* for $\Psi_N(z)$. For any $f \in A_+$ we have

$$\prod_{\substack{a \in A \\ \deg a < \deg f}} \frac{1 + \Psi_N\left(\frac{z+a}{f}\right)}{1 + \Psi_N\left(\frac{a}{f}\right)} = 1 + \Psi_{N+\deg f}(z)$$

as can be verified by comparing zeroes and constant terms.

5.3.4. *Definitions of* $\Pi(z)$ *and* $\Gamma(z)$. Put

$$\Pi(z) := \prod_{a \in A_+} \left(1 + \frac{z}{a}\right)^{-1}, \quad \Gamma(z) := z^{-1}\Pi(z) = z^{-1} \prod_{a \in A_+} \left(1 + \frac{z}{a}\right)^{-1},$$

thereby defining the (one-variable) *geometric factorial* and *geometric* Γ-*function* attached to A, respectively. The basic references concerning these functions are [Go, §9.9] and [Th]. Note that $\Pi(z)^{-1}$ is the unique entire function of z taking the value 1 at the origin with no zeroes other than simple zeroes at each point of the set

$$-A_+ := \{-a | a \in A_+\}.$$

Similarly $\Gamma(z)^{-1}$ is the unique entire function of z normalized by the condition

$$\Gamma(z)^{-1} = z + O(z^2)$$

with no zeroes other than simple zeroes at each point of the set

$$-A_+ \cup \{0\}.$$

We obtain identities

$$\Pi(z) = \prod_{N=0}^{\infty} (1 + \Psi_N(z))^{-1}, \quad \Gamma(z) = z^{-1} \prod_{N=0}^{\infty} (1 + \Psi_N(z))^{-1}$$

of crucial importance in the sequel by grouping factors in the natural way by degree.

5.3.5. *The standard functional equations.* We have a *translation identity*

$$\frac{\Pi(z + a_0)}{\Pi(z)} = \prod_{i=0}^{\deg a_0} \frac{1 + \Psi_i(z)}{1 + \Psi_i(z + a_0)}$$

for each $0 \neq a_0 \in A$. The translation identity follows in evident fashion from the corresponding translation-invariance property of $\Psi_N(z)$ noted in §5.3.3. We have a *reflection identity*

$$\prod_{\varepsilon \in \mathbb{F}_q^\times} \Pi(\varepsilon z) = \frac{\varpi z}{\exp_C \varpi z}$$

which is essentially [Th, Th. 6.1.1]. To prove the reflection identity in the form stated here you have only to compare the Weierstrass product expansion of the reciprocal of the left side to that of the Carlitz exponential $\exp_C z$. For each $f \in A_+$ we have a *Gauss multiplication identity*

$$\prod_{\substack{a \in A \\ \deg a < \deg f}} \Pi\left(\frac{z+a}{f}\right) = \Pi(z) \cdot \prod_{0 \leq i < \deg f} (1 + \Psi_i(z)) \cdot \prod_{\substack{a \in A_+ \\ \deg a < \deg f}} \prod_{\varepsilon \in \mathbb{F}_q^\times} \Pi\left(\varepsilon \frac{a}{f}\right)$$

which is essentially [Th, Th. 6.2.1]. The Gauss multiplication identity in the form stated here can easily be recovered from the analogous identity satisfied by $\Psi_N(z)$ noted at the end of §5.3.3. We refer to the identities above as the *standard functional equations* satisfied by $\Pi(z)$. Of course we have analogous standard functional equations for $\Gamma(z)$. The latter we do not write out, referring the interested reader to [Th].

5.3.6. *Special* Γ- *and* Π-*values.* Numbers of the form

$$\Pi(x) \quad (x \in k \setminus -A_+)$$

will be called *special* Π-*values.* Numbers of the form

$$\Gamma(x) \quad (x \in k \setminus (-A_+ \cup \{0\}))$$

will be called *special* Γ-*values.* The goal of the paper is to determine all Laurent polynomial relations among ϖ and special Γ-values with coefficients in $\bar{k}$. From the point of view of transcendence theory over k it is clearly all the same whether we study special Γ-values or special Π-values. But the standard functional equations satisfied by $\Pi(z)$ take a slightly simpler form than do those satisfied by $\Gamma(z)$ and it is convenient also that $\Pi(0) = 1$. Accordingly, for the sake of convenience and for no deeper reason, we stress $\Pi(z)$ over $\Gamma(z)$ and special Π-values over special Γ-values in the sequel. Now in view of the translation identity of §5.3.5, a special Π-value $\Pi(x)$ up to factors in $k^\times$ depends only on x mod A. Accordingly, without any loss of generality, we stress special Π-values of the form $\Pi(x)$ with $|x|_\infty < 1$ in the sequel.

5.3.7. *Remark.* All $\bar{k}$-linear relations among 1, ϖ and the special Π-values were determined in [BrPa], building on the work of [Si a]. In particular, it is known that $\Pi(z)$ is transcendental for all $z \in k \setminus A$.

5.4. *Interpolation formulas.*

LEMMA 5.4.1. *For all integers N the identity*

$$\Omega^{(N)}/\Omega^{(-1)} = \begin{cases} -\sum_{n=0}^{\infty} \Psi_N\left(1/T^{n+1}\right) t^n & \textit{if } N \geq 0 \\ \prod_{i=1}^{|N|-1} \left(t - T^{q^{-i}}\right) & \textit{if } N < 0 \end{cases}$$

holds in the power series ring $\bar{k}[\![t]\!]$.

Proof. From the functional equation $\Omega^{(-1)} = \Omega \cdot (t-T)$ it follows that

$$\Omega^{(N)}/\Omega^{(-1)} = \begin{cases} \prod_{i=0}^{N} \left(t - T^{q^i}\right)^{-1} & \text{if } N \geq 0, \\ \prod_{i=1}^{|N|-1} \left(t - T^{q^{-i}}\right) & \text{if } N < 0. \end{cases}$$

Accordingly, we may assume without loss of generality for the rest of the proof that $N \geq 0$. The identity

$$(t-T)^{-1} = -\sum_{n=0}^{\infty} t^n/T^{n+1} = -\sum_{n=0}^{\infty} \Psi_0\left(1/T^{n+1}\right) t^n$$

dispatches the case $N = 0$. Assume now that $N > 0$. The recursion

$$\left(\prod_{i=0}^{N-1} \left(t - T^{q^i}\right)^{-1}\right)^{(1)} - \prod_{i=0}^{N-1} \left(t - T^{q^i}\right)^{-1} = (T^{q^N} - T) \prod_{i=0}^{N} \left(t - T^{q^i}\right)^{-1}$$

matches the fundamental recursion

$$\Psi_{N-1}(z)^q - \Psi_{N-1}(z) = \left(T^{q^N} - T\right) \Psi_N(z)$$

noted in §5.3.3. We are done by induction on N. □

LEMMA 5.4.2. *For all $x \in k_\infty$ and integers N such that*

$$|x|_\infty \leq |T|_\infty^{\min(-1,N)},$$

$$\sum_{i=0}^{\infty} \mathbf{e}^*(T^{-i-1})^{q^{N+1}} \mathbf{e}(T^i x) = \begin{cases} -\Psi_N(x) & \textit{if } N \geq 0, \\ \operatorname{Res}\left(T^{-N-1}x\right) & \textit{if } N < 0. \end{cases}$$

Proof. Both sides of the identity to be proved depend $\mathbb{F}_q$-linearly and $|\cdot|_\infty$-continuously on x. We may therefore assume without loss of generality that $x = T^{-n-1}$ for some integer n such that $-n-1 \leq \min(-1, N)$ (equivalently: $n \geq \max(0, -N-1)$). Then all we have to prove is the identity

$$\sum_{i=0}^{\infty} \mathbf{e}^*(T^{-i-1})^{q^{N+1}} \mathbf{e}(T^{i-n-1}) = \begin{cases} -\Psi_N(T^{-n-1}) & \text{if } N \geq 0, \\ \delta_{|N|-1,n} & \text{if } N < 0. \end{cases}$$

By Lemma 5.4.1, the formal properties of $\mathbf{e}^*$ noted in §5.2.8, and the formal properties of $\mathbf{e}$ noted in §5.2.10, both sides of the identity above admit interpretation as the coefficient with which t^n appears in the Maclaurin expansion of $\Omega^{(N)}/\Omega^{(-1)}$ in powers of t. □

5.4.3. *f-dual families.* Fix $f \in A_+$ of positive degree. We say that families

$$\{a_i\}_{i=1}^{\deg f}, \quad \{b_j\}_{j=1}^{\deg f} \quad (a_i, b_j \in A)$$

are *f-dual* if

$$\mathrm{Res}(a_i b_j / f) = \delta_{ij} \quad (i, j = 1, \ldots, \deg f).$$

Since the square matrix

$$\left\{\mathrm{Res}(T^{i+\deg f-j-1}/f)\right\}_{i,j=0}^{\deg f-1}$$

is lower triangular with 1's along the diagonal, the pairing

$$((a \bmod f, b \bmod f) \mapsto \mathrm{Res}(ab/f)) : A/f \times A/f \to \mathbb{F}_q$$

is perfect and hence f-dual families exist.

THEOREM 5.4.4. *Fix $f \in A_+$ of positive degree and f-dual families*

$$\{a_i\}_{i=1}^{\deg f}, \quad \{b_j\}_{j=1}^{\deg f} \quad (a_i, b_j \in A).$$

Fix $a \in A$ such that

$$\deg a < \deg f.$$

(i) *Now,*

$$\sum_{i=1}^{\deg f} \mathbf{e}^*(a_i/f)^{q^{N+1}} \mathbf{e}(b_i a/f) = -\Psi_N(a/f)$$

for all integers $N \geq 0$.

(ii) *Moreover, if $a \in A_+$, then*

$$\sum_{i=1}^{\deg f} \mathbf{e}^*(a_i/f) \mathbf{e}(b_i a/f)^{q^{\deg f - \deg a - 1}} = 1.$$

Proof (Cf. [An b, Thm. 2, p. 58]). For every $N \in \mathbb{Z}$ we have

$$\begin{aligned}
&\sum_{i=1}^{\deg f} \mathbf{e}^*(a_i/f)^{q^{N+1}} \mathbf{e}(b_i a/f) \\
&\qquad = \sum_{i=1}^{\deg f} \sum_{n=0}^{\infty} \mathbf{e}^*(1/T^{n+1})^{q^{N+1}} \operatorname{Res}(T^n a_i/f) \mathbf{e}(b_i a/f) \\
&\qquad = \sum_{n=0}^{\infty} \mathbf{e}^*(1/T^{n+1})^{q^{N+1}} \mathbf{e}(T^n a/f) \\
&\qquad = \begin{cases} -\Psi_N(a/f) & \text{if } N \geq 0 \\ 1 & \text{if } a \in A_+ \text{ and } N = \deg a - \deg f \end{cases}
\end{aligned}$$

by definition of the special function $\mathbf{e}^*$ and Lemma 5.4.2. □

5.4.5. *Remark.* Theorem 5.4.4 "interpolates" $\Psi_N(a/f)$ by an algebraic expression in which N figures as the power to which a Frobenius endomorphism is raised. We learned this seemingly strange but in fact fundamental notion of interpolation from examples in [Th, §9.3]; see [An b, §2] for an appreciation of Thakur's work. The possibility of such an interpolation was proved in [An b, Thm. 2, p. 58] without the interpolating expression being made explicit.

5.5. *Diamond brackets and L-functions.*

5.5.1. *Definitions.* For all $x \in k_\infty$ and integers $N \geq 0$, put

$$\langle x \rangle_N := \begin{cases} 1 & \text{if } \inf\limits_{a \in \mathbb{F}_q[T]} \left| x - a - T^{-N-1} \right|_\infty < |T|_\infty^{-N-1}, \\ & \\ 0 & \text{otherwise,} \end{cases}$$

and also set

$$\langle x \rangle := \sum_{N=0}^{\infty} \langle x \rangle_N.$$

The sum on the right makes sense because at most one of its terms is nonzero. We call the functions

$$\langle \cdot \rangle, \langle \cdot \rangle_N : k_\infty \to \{0, 1\}$$

thus defined the *diamond bracket* and the *generalized diamond bracket*, respectively. By definition these functions factor through the quotient k_∞/A. And further, if $|x|_\infty < 1$, then $\langle x \rangle_N$ equals 1 or 0 according to whether the leading term of the Laurent expansion of x in powers of $1/T$ is or is not equal to $1/T^{N+1}$, and consequently $\langle x \rangle$ equals 1 or 0 according to whether the leading coefficient of the Laurent expansion of x in powers of $1/T$ is or is not equal to 1.

5.5.2. *Remark.* The value assigned to the expression $\langle x\rangle$ here coincides with that assigned in [Si b], but with that assigned to $\langle -x\rangle$ in [Th, Def. 7.6.1]. The choice of sign is just a normalization not affecting the utility of the definition.

5.5.3. *Evaluation of L-functions.* Fix $f \in A_+$ of positive degree and a character $\chi : (A/f)^\times \to \mathbb{C}^\times$ *primitive* in the sense of not factoring through $(A/g)^\times$ for any divisor $g \in A_+$ of f, distinct from f. Note that under these hypotheses χ is not identically equal to 1. Put

$$\begin{aligned} L(s,\chi) &= \sum_{\substack{a\in A_+\\ (a,f)=1}} \chi(a \bmod f) q^{-s\deg a} \\ &= \sum_{\substack{a\in A_+\\ (a,f)=1\\ \deg a<\deg f}} \chi(a \bmod f) q^{-s\deg a} \\ &= \sum_{\substack{a\in A\\ \deg a<\deg f\\ (a,f)=1}} \chi(a \bmod f)\left(\sum_{N=0}^{\infty}\left\langle \frac{a}{f}\right\rangle_N q^{(N+1-\deg f)s}\right), \end{aligned}$$

noting in particular that

$$L(0,\chi) = \sum_{\substack{a\in A\\ \deg a<\deg f\\ (a,f)=1}} \chi(a \bmod f)\left\langle \frac{a}{f}\right\rangle.$$

The L-function $L(s,\chi)$ is the $\mathbb{F}_q[T]$-analogue of a Dirichlet L-function. As is well known, since χ is primitive and not identically equal to 1, we have $L(0,\chi) = 0$ if and only if $\chi(\varepsilon \bmod f) = 1$ for all $\varepsilon \in \mathbb{F}_q^\times$.

5.5.4. *Remark.* The connection between L-functions and diamond brackets recalled above is the chief motivation for the definition of diamond brackets.

5.5.5. *Diamond bracket relations.* We make the following claims:

- $\langle x + a\rangle = \langle x\rangle$ for all $x \in k_\infty$ and $a \in A$.
- $\displaystyle\sum_{\varepsilon\in\mathbb{F}_q^\times} \langle \varepsilon x\rangle = \begin{cases} 1 & \text{if } x \notin A \\ 0 & \text{if } x \in A \end{cases}$ for all $x \in k_\infty$.
- $\displaystyle\sum_{\substack{a\in A\\ \deg a<\deg f}} \left(\left\langle \frac{x+a}{f}\right\rangle - \left\langle \frac{a}{f}\right\rangle\right) = \langle x\rangle$ for all $x \in k_\infty$ and $f \in A_+$.

Clearly the first and second claims hold. To prove the third claim we may assume without loss of generality that $|x|_\infty < 1$. Then the only summand on the left possibly differing from 0 is the one indexed by $a = 0$, and we have $\langle x/f \rangle = \langle x \rangle$. Therefore the third claim holds.

6. Analysis of the algebraic relations among special Π-values

Throughout Section 6 we fix $f \in A_+$ of positive degree.

6.1. *The diamond bracket criterion.*

6.1.1. *The free abelian group $\mathcal{A}_f$.* Let $\mathcal{A}_f$ be the free abelian group on symbols of the form

$$[x] \quad \left(x \in f^{-1}A\right),$$

where symbols $[x]$ and $[x']$ are identified if $x \equiv x' \bmod A$. The group $\mathcal{A}_f$ is free abelian of rank $q^{\deg f}$. More precisely, every $\mathbf{a} \in \mathcal{A}_f$ has a unique expression of the form

$$\mathbf{a} = \sum_{\substack{a \in A \\ \deg a < \deg f}} m_a \left[\frac{a}{f}\right] \quad (m_a \in \mathbb{Z}).$$

If in the situation above all the coefficients m_a are nonnegative, we say that $\mathbf{a}$ is *effective*. Let

$$\mathrm{wt} : \mathcal{A}_f \to \mathbb{Z}\left[\frac{1}{q-1}\right]$$

be the unique homomorphism such that

$$\mathrm{wt}[x] = \begin{cases} \frac{1}{q-1} & \text{if } x \notin A \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in f^{-1}A$. Let $\mathcal{D}_f$ be the subgroup of $\mathcal{A}_f$ generated by all elements of the form

$$[x] - \sum_{\substack{a \in A \\ \deg a < \deg g}} \left[\frac{x+a}{g}\right] \quad \left(g \in A_+ \text{ dividing } f, \quad x \in \frac{g}{f} \cdot A\right).$$

The quotient $\mathcal{A}_f/\mathcal{D}_f$ is the analogue over $\mathbb{F}_q[T]$ of the *universal ordinary distribution* at a finite level, cf. [Ku a] or [La b, §2]. Let $\mathcal{R}_f$ be the subgroup of $\mathcal{A}_f$ generated by $\mathcal{D}_f$ along with all elements of the form

$$\sum_{\varepsilon \in \mathbb{F}_q^\times} [\varepsilon x] \quad \left(x \in f^{-1}A\right).$$

Let $\widetilde{\mathcal{R}}_f$ be the subgroup of $\mathcal{A}_f$ consisting of $\mathbf{a}$ such that $\mathrm{wt}\,\mathbf{a} = 0$ and $N\mathbf{a} \in \mathcal{R}_f$ for some positive integer N.

6.1.2. *The star action.* For each $a \in A$ prime to f there exists a unique automorphism

$$(\mathbf{a} \mapsto a \star \mathbf{a}) : \mathcal{A}_f \to \mathcal{A}_f$$

of free abelian groups such that

$$a \star [x] = [ax]$$

for all $x \in f^{-1}A$ and that this automorphism stabilizes $\mathcal{D}_f$, $\mathcal{R}_f$ and $\widetilde{\mathcal{R}}_f$ and that

$$a \star (b \star \mathbf{a}) = (ab) \star \mathbf{a}$$

for all $a, b \in A$ prime to f and $\mathbf{a} \in \mathcal{A}_f$. Thus, via the star operation, $\mathcal{A}_f$ is equipped with an action of $(A/f)^\times$ passing to the quotients $\mathcal{A}_f/\mathcal{D}_f$, $\mathcal{A}_f/\mathcal{R}_f$ and $\mathcal{A}_f/\widetilde{\mathcal{R}}_f$.

THEOREM 6.1.3. *The rational vector space*

$$\mathbb{Q} \otimes (\mathcal{A}_f/\mathcal{D}_f)$$

is $(A/f)^\times$-equivariantly isomorphic to the rational group ring

$$\mathbb{Q}\left[(A/f)^\times\right].$$

Proof. The classical model for the theorem is proved in [Ku a]; alternatively, see[La b, §2]. The methods of Kubert carry over to our function field situation without any difficulty. We omit the details. □

COROLLARY 6.1.4. *Fix a character $\chi : (A/f)^\times \to \mathbb{C}^\times$. The dimension over $\mathbb{C}$ of the χ-isotypical component of $\mathbb{C} \otimes (\mathcal{A}_f/\widetilde{\mathcal{R}}_f)$ is ≤ 1, with strict inequality if and only if χ is not identically equal to 1 but $\chi(\varepsilon \bmod f) = 1$ for all $\varepsilon \in \mathbb{F}_q^\times$.*

It follows that the free abelian group $\mathcal{A}_f/\widetilde{\mathcal{R}}_f$ is of rank $1 + \frac{q-2}{q-1} \cdot \#(A/f)^\times$.

Proof. We have

$$\mathcal{A}_f/\mathcal{R}_f = \text{module of } \mathbb{F}_q^\times\text{-coinvariants in } \mathcal{A}_f/\mathcal{D}_f.$$

The sequence of $(A/f)^\times$-modules

$$0 \to \mathbb{C} \cdot \left(\sum_{\substack{0 \neq a \in A \\ \deg a < \deg f}} \left[\frac{a}{f}\right] \bmod \widetilde{\mathcal{R}}_f \right) \to \mathbb{C} \otimes (\mathcal{A}_f/\widetilde{\mathcal{R}}_f) \to \mathbb{C} \otimes (\mathcal{A}_f/\mathcal{R}_f) \to 0$$

is exact. The result follows. □

6.1.5. *Extension of the definition of diamond brackets.* For all integers $N \geq 0$ let

$$(\mathbf{a} \mapsto \langle \mathbf{a} \rangle_N) : \mathcal{A}_f \to \mathbb{Z}$$

be the unique homomorphism such that

$$\langle [x] \rangle_N = \langle x \rangle_N$$

for all $x \in f^{-1}A$ and put

$$\langle \mathbf{a} \rangle = \sum_{N=0}^{\infty} \langle \mathbf{a} \rangle_N$$

for all $\mathbf{a} \in \mathcal{A}_f$. The sum on the right makes sense because only finitely many of its terms are nonzero. For all $\mathbf{a}, \mathbf{b} \in \mathcal{A}_f$ we write $\mathbf{a} \sim_f \mathbf{b}$ if and only if $\langle a \star \mathbf{a} \rangle = \langle a \star \mathbf{b} \rangle$ for all $a \in A$ such that $(a, f) = 1$ and $\deg a < \deg f$, thereby defining an equivalence relation $\sim_f$ in $\mathcal{A}_f$. When $\sim_f$ fails to hold between $\mathbf{a}$ and $\mathbf{b}$ we write $\mathbf{a} \not\sim_f \mathbf{b}$.

THEOREM 6.1.6. *For all* $\mathbf{a}, \mathbf{b} \in \mathcal{A}_f$, $\mathbf{a} \equiv \mathbf{b} \bmod \widetilde{\mathcal{R}}_f$ *if and only if* $\mathbf{a} \sim_f \mathbf{b}$.

Proof. ($\Rightarrow$) Since $\widetilde{\mathcal{R}}_f$ is stable under the action of $(A/f)^\times$ it is enough to prove

$$\mathbf{a} \in \widetilde{\mathcal{R}}_f \Rightarrow \langle \mathbf{a} \rangle = 0.$$

In turn it is enough to prove that

$$\mathbf{a} \in \mathcal{R}_f \Rightarrow \langle \mathbf{a} \rangle = \operatorname{wt} \mathbf{a}.$$

In order to prove the latter implication we may assume without loss of generality that $\mathbf{a}$ is one of the generators of $\mathcal{R}_f$ exhibited in §6.1.1. Then the diamond bracket relations of §5.5.5 do the job.

($\Leftarrow$) This is proved by an evident modification of the proof of the Deligne-Koblitz-Ogus criterion [De]. The connection between diamond brackets and values of L-functions at $s = 0$ noted in §5.5.3 is the essential point of the proof. We omit the details. □

6.1.7. Π-*monomials and their relationship with* Γ-*monomials.* Let

$$(\mathbf{a} \mapsto \Pi(\mathbf{a})) : \mathcal{A}_f \to \mathbb{C}_\infty^\times$$

be the unique homomorphism such that

$$\Pi([x]) = \Pi(x)$$

for all $x \in f^{-1}A$ such that $|x|_\infty < 1$. Numbers in the image of the homomorphism $\mathcal{A}_f \xrightarrow{\Pi} \mathbb{C}_\infty^\times$ we call Π-*monomials* of level f. The reflection identity satisfied by the Π-function and the hypothesis $\deg f > 0$ imply that up to a factor in $\bar{k}^\times$, the number ϖ is a Π-monomial of level f. Taking the translation identity satisfied by the Π-function also into account, as well as the simple relationship between Γ- and Π-functions, it is clear that up to a factor in $\bar{k}^\times$ every Γ-monomial belonging to the group of such generated by the set

$$\{\varpi\} \cup \left\{ \Gamma(x) \,\middle|\, x \in \frac{1}{f} A \setminus (\{0\} \cup -A_+) \right\}$$

is a Π-monomial of level f.

The following result is the direct analogue in our setting of the Deligne-Koblitz-Ogus criterion [De]. It allows us to decide in more or less mechanical fashion whether between a given pair of Γ-monomials there exists a relation of $\bar{k}$-linear dependence explained by the standard functional equations.

COROLLARY 6.1.8 (Diamond bracket criterion). *For all* $\mathbf{a}, \mathbf{b} \in \mathcal{A}_f$,

$$\mathbf{a} \sim_f \mathbf{b} \Rightarrow \Pi(\mathbf{a})/\Pi(\mathbf{b}) \in \bar{k}^\times.$$

Proof. It is enough to prove that

$$\mathbf{a} \in \mathcal{R}_f \Rightarrow \varpi^{-\operatorname{wt}\mathbf{a}}\,\Pi(\mathbf{a}) \in \bar{k}^\times.$$

In order to do so we may assume without loss of generality that $\mathbf{a}$ is one of the generators of $\mathcal{R}_f$ specified in §6.1.1. Then the standard functional equations stated in §5.3.5 do the job. □

6.1.9. *Remark.* Deligne's reciprocity law [DeMiOgSh, Thm. 7.15, p. 91] refines the Deligne-Koblitz-Ogus criterion by giving delicate information concerning the field to which an algebraic Γ-monomial belongs. An analogous refinement of Corollary 6.1.8 was proved in [Si b].

6.1.10. *Remark.* The converse to Corollary 6.1.8 is the $\mathbb{F}_q[T]$-analogue "at level f" of the conjecture of Rohrlich discussed in §1.4.2.

6.2. *Formulation and discussion of the main result.* The following theorem is the main result of this paper. It is the $\mathbb{F}_q[T]$-analogue "at level f" of the conjecture of Lang discussed in §1.4.3 above. It restates Theorem 1.2.1 in a precise way and allows us to determine all $\bar{k}$-linear relations among Γ-monomials.

THEOREM 6.2.1. *Let*

$$\mathbf{a}_1, \ldots, \mathbf{a}_N \in \mathcal{A}_f$$

be given. A necessary and sufficient condition for the corresponding Π-*monomials*

$$\Pi(\mathbf{a}_1), \ldots, \Pi(\mathbf{a}_N)$$

of level f *to be* $\bar{k}$-*linearly independent is that*

$$\mathbf{a}_i \not\sim_f \mathbf{a}_j$$

for all $1 \leq i < j \leq N$.

The proof of the theorem takes up almost all of the rest of the paper, concluding in §6.5.

COROLLARY 6.2.2. *Put*

$$\nu_f := 1 + \frac{q-2}{q-1} \cdot \#(A/f)^\times.$$

Let E_f be the subfield of $\mathbb{C}_\infty$ generated over $\bar{k}$ by the set

$$\{\Pi(a/f)\,|a \in A,\ \deg a < \deg f\}.$$

The transcendence degree of E_f over $\bar{k}$ equals ν_f.

The following proposition establishes equivalence of theorem and corollary.

PROPOSITION 6.2.3. *The transcendence degree of $E_f/\bar{k}$ is bounded above by ν_f. Moreover, a necessary and sufficient condition for ν_f strictly to exceed the transcendence degree of $E_f/\bar{k}$ is that for some integer $N \geq 2$ there exist $\mathbf{a}_1, \ldots, \mathbf{a}_N \in \mathcal{A}_f$ and nonzero $c_1, \ldots, c_N \in \bar{k}$ such that $\mathbf{a}_i \not\sim_f \mathbf{a}_j$ for all $1 \leq i < j \leq N$ and $c_1\Pi(\mathbf{a}_1) + \cdots + c_N\Pi(\mathbf{a}_N) = 0$.*

Proof. Let $\bar{k}[\mathcal{A}_f/\widetilde{\mathcal{R}}_f]$ be the group ring with coefficients in $\bar{k}$ of the finitely generated free abelian group $\mathcal{A}_f/\widetilde{\mathcal{R}}_f$. Let

$$\lambda : \mathcal{A}_f \to \bar{k}^\times$$

be any group homomorphism agreeing with Π on $\widetilde{\mathcal{R}}_f$ and let

$$\bar{k}[\mathcal{A}_f/\widetilde{\mathcal{R}}_f] \xrightarrow{\Lambda} E_f$$

be the unique $\bar{k}$-algebra homomorphism such that

$$\Lambda(\mathbf{a} \bmod \tilde{\mathcal{R}}_f) = \frac{\Pi(\mathbf{a})}{\lambda(\mathbf{a})}$$

for all $\mathbf{a} \in \mathcal{A}_f$. Put

$$I_f := \ker\left(\bar{k}[\mathcal{A}_f/\widetilde{\mathcal{R}}_f] \xrightarrow{\Lambda} E_f\right).$$

By Corollary 6.1.4 the free abelian group $\mathcal{A}_f/\widetilde{\mathcal{R}}_f$ is of rank ν_f and hence the ring $\bar{k}[\mathcal{A}_f/\widetilde{\mathcal{R}}_f]$ is isomorphic to the ring of Laurent polynomials in ν_f independent variables with coefficients in $\bar{k}$. By construction the ideal I_f is prime. Clearly the field E_f is isomorphic as a $\bar{k}$-algebra to the field of fractions of the ring $\bar{k}[\mathcal{A}_f/\widetilde{\mathcal{R}}_f]/I_f$. Therefore ν_f bounds the transcendence degree of E_f over $\bar{k}$. Moreover, a necessary and sufficient condition for ν_f strictly to exceed the transcendence degree of $E_f/\bar{k}$ is that $I_f \neq 0$. Note that every nonzero element of I_f has to be a formal $\bar{k}$-linear combination of at least two elements of $\mathcal{A}_f/\widetilde{\mathcal{R}}_f$ since $\Pi(\mathbf{a})$ is nonzero for all $\mathbf{a} \in \mathcal{A}_f$. In view of Theorem 6.1.6 and Corollary 6.1.8, it is clear that from any nonzero element of I_f we can produce a relation of $\bar{k}$-linear dependence among Π-monomials of the indicated form. $\square$

6.3. *Coleman functions.*

6.3.1. *A closer look at torsion values of* $\mathbf{e}$. In the course of our discussion of the special function $\mathbf{e}$ the following facts were verified:

- $k(\mathbf{e}(1/f))$ is separable over k.
- $k(\mathbf{e}(1/f))$ is a splitting field over k for $C_f(T,z)$.
- $\mathbf{e}(a/f)/\widetilde{T} \in \mathbb{F}_q[[1/T]]$ for all $a \in A$.

The following more delicate facts are well known:

- There exists for each $\gamma \in \mathrm{Gal}(k(\mathbf{e}(1/f))/k)$ unique $a \in A$ such that $(a,f)=1$, $\deg a < \deg f$ and $\gamma\mathbf{e}(b/f) = \mathbf{e}(ab/f) = C_a(T,\mathbf{e}(b/f))$ for all $b \in A$.
- The construction $\gamma \mapsto a$ induces an isomorphism $\mathrm{Gal}(k(\mathbf{e}(1/f))/k) \xrightarrow{\sim} (A/f)^\times$.
- $\mathrm{Gal}(k(\mathbf{e}(1/f))/k)$ is generated by its inertia subgroups.
- The integral closure of A in $k(\mathbf{e}(1/f))$ is $A[\mathbf{e}(1/f)]$.

See [Ha] or [Ro] for a treatment of the latter material.

6.3.2. *Definition of* $C_f^\star(t,z)$. There exists a unique factor

$$C_f^\star(t,z) \in \mathbb{F}_q[t,z]$$

of the division polynomial $C_f(t,z)$ such that

$$C_f^\star(T,z) = \prod_{\substack{a\in A\\ \deg a<\deg f\\ (a,f)=1}} (z - \mathbf{e}(a/f)).$$

The polynomial $C_f^\star(t,z)$ is the Carlitz analogue of a cyclotomic polynomial and has, so we claim, the following properties:

- The discriminant of $C_f^\star(t,z)$ with respect to z does not vanish identically.
- For all $a \in A$, $C_f^\star(t,z)$ divides $C_a(t,z)$ if and only if f divides a.
- $C_f^\star(t,z)$ is irreducible in $\mathbb{F}_q[t,z]$ and remains so in $\bar{k}[t,z]$.
- $C_f^\star$, $\partial C_f^\star/\partial t$ and $\partial C_f^\star/\partial z$ generate the unit ideal of $\mathbb{F}_q[t,z]$.

The first two properties are clear, as is irreducibility over $\mathbb{F}_q$. Irreducibility over any finite algebraic extension of $\mathbb{F}_q$ follows from the fact that $\mathrm{Gal}(k(\mathbf{e}(1/f))/k)$ is generated by inertia. Were $C_f^\star(t,z)$ to be reducible over $\bar{k}$, then $C_f^\star(t,z)$ would be reducible over some field L finite algebraic over k and hence reducible over the finite residue field of some discrete valuation of L, a contradiction. Therefore $C_f^\star(t,z)$ has the third property. Failure of the fourth property would imply failure of the ring $\mathbb{F}_q[t,z]/(C_f^\star(t,z))$ to be integrally closed: but the isomorphic ring $A[\mathbf{e}(1/f)]$ is known to be integrally closed. Therefore $C_f^\star(t,z)$ has the fourth property. Thus our claim is proved.

6.3.3. *The nonsingular projective curve $X/\mathbb{F}_q$.* Let $U/\mathbb{F}_q$ be the irreducible nonsingular plane algebraic curve in the affine (t,z)-plane$/\mathbb{F}_q$ defined by the equation $C_f^\star(t,z) = 0$ and let $X/\mathbb{F}_q$ be the nonsingular projective model of $U/\mathbb{F}_q$. We regard t and z as regular functions on U and meromorphic functions on X. We regard $X/\mathbb{F}_q$ as a covering of the projective t-line$/\mathbb{F}_q$. We say that the closed points of X in the complement of U are *at infinity.* For all $a \in A$ prime to f put

$$\xi_a := (T, \mathbf{e}(a/f)),$$

thereby defining a $\bar{k}$-valued point of U. A trivial but important remark to make here is that the set $\{\xi_a\}$ is the collection of $\bar{k}$-valued points of U above the $\bar{k}$-valued point $t = T$ of the affine t-line.

We make the following claims:

- X is a Galois covering of the t-line of degree $\#(A/f)^\times$.
- For each automorphism γ of X over the t-line there exists unique $a \in A$ such that $(a,f) = 1$, $\deg a < \deg f$ and $\gamma^* C_b(t,z) = C_{ab}(t,z)$ for all $b \in A$.
- Moreover, with γ and a as above, we have $\gamma\xi_b = \xi_{ab}$ for all $b \in A$ prime to f.
- Further, the construction $\gamma \mapsto a$ induces an isomorphism from the group of automorphisms of X over the t-line to $(A/f)^\times$.
- Each closed point of X at infinity has residue field $\mathbb{F}_q$ and is ramified of order $q-1$ over the point at infinity on the t-line.
- There are $\#(A/f)^\times/(q-1)$ closed points at infinity, and these are permuted transitively by the group of automorphisms of X over the t-line.
- For all $a \in A$ the function $C_a(t,z)$ has at each of the points of X at infinity no singularity worse than a simple pole.

These claims are verified by making a routine translation from arithmetical language to geometrical language. We can safely omit the details.

6.3.4. *The base-change $\overline{X}/\bar{k}$, deck transformations and n-fold twisting.* Put

$$\overline{U} := \bar{k} \otimes_{\mathbb{F}_q} U, \quad \overline{X} := \bar{k} \otimes_{\mathbb{F}_q} X,$$

thereby defining nonsingular irreducible curves over $\bar{k}$, the former being the curve in the affine (t, z)-plane/$\bar{k}$ defined by the equation $C_f^\star(t, z) = 0$, and the latter being the nonsingular projective model of the former. Closed points of $\overline{X}$ in the complement of $\overline{U}$ as before are said to be *at infinity*. No new closed points at infinity appear in the base-change $\overline{X}$ because all the closed points in X at infinity are already $\mathbb{F}_q$-rational.

By construction $\overline{X}/\bar{k}$ is a Galois covering of the projective t-line/$\bar{k}$. Just so as to have a convenient short turn of phrase at our disposal, we call an automorphism of $\overline{X}/\bar{k}$ over the projective t-line/$\bar{k}$ a *deck transformation*. By construction every deck transformation is the base-change of a unique automorphism of $X/\mathbb{F}_q$ over the projective t-line/$\mathbb{F}_q$, and hence the group of deck transformations is canonically isomorphic to $(A/f)^\times$.

We define the *n-fold twisting operation* on the function field of $\overline{X}$ to be the unique automorphism extending the $(q^n)^{\text{th}}$ power automorphism of $\bar{k}$ and fixing every element of the function field of X. We denote the result of applying the n-fold twisting operation to a function h by $h^{(n)}$. The n-fold twisting automorphism of the function field of $\overline{X}$ commutes with all pull-backs via deck transformations. For each $x \in X(\bar{k})$ we define the *n-fold twist* $x^{(n)} \in X(\bar{k})$ to be the point obtained by applying the $(q^n)^{\text{th}}$ power automorphism of $\bar{k}$ to the coordinates of x. Since each $\bar{k}$-rational point of X at infinity is already defined over $\mathbb{F}_q$, each such point is fixed by the n-fold twisting operation. Identifying closed points of $\overline{X}$ with $\bar{k}$-valued points of X in evident fashion, we extend the n-fold twisting operation to the group of divisors of $\overline{X}$ by $\mathbb{Z}$-linearity. The n-fold twisting operation on divisors commutes with the action of deck transformations. The operation of forming the divisor of a nonzero meromorphic function on $\overline{X}$ commutes with n-fold twisting; i.e., we have $(h^{(n)}) = (h)^{(n)}$ for all nonzero meromorphic functions h on $\overline{X}$.

6.3.5. *Definition of Coleman functions.* Fix

$$x \in f^{-1}A \setminus A.$$

Fix f-dual families

$$\{a_i\}_{i=1}^{\deg f}, \{b_j\}_{j=1}^{\deg f} \quad (a_i, b_j \in A)$$

and write

$$x = a_0/f \quad (a_0 \in A, \ \ a_0 \text{ not divisible by } f).$$

Put

$$g_x := 1 - \sum_{i=1}^{\deg f} \mathbf{e}^*(a_i/f) C_{a_0 b_i}(t,z),$$

thereby defining a meromorphic function on $\overline{X}$ regular on $\overline{U}$ with singularities at infinity no worse than simple poles. Now for a ranging over A, both the function $C_a(t,z)$ and the number $\mathbf{e}^*(a/f)$ depend $\mathbb{F}_q$-linearly on a and moreover depend only on a mod f. Therefore the function g_x depends only on x, not on the intervening choice of f-dual families $\{a_i\}$ and $\{b_j\}$. Moreover it is clear that g_x depends only on x mod A. We call g_x a *Coleman function.*

6.3.6. *The divisors of Coleman functions.* Let ∞_X be the formal sum of the $\bar{k}$-valued points of X at infinity, multiplied by $(q-1)$ and viewed as a divisor of $\overline{X}$. Now,

$$\deg \infty_X = \#(A/f)^\times, \quad (\infty_X)^{(1)} = \infty_X.$$

For every $x \in f^{-1}A \setminus A$ we have, so we claim, an equality

$$(g_x) = -\frac{1}{q-1} \cdot \infty_X + \sum_{\substack{a \in A \\ (a,f)=1 \\ \deg a < \deg f}} \sum_{N=0}^{\infty} \langle ax \rangle_N \cdot \xi_a^{(N)}$$

of divisors of $\overline{X}$. To see this, call the divisor on the right D. There appear only finitely many nonzero terms in the sum defining D and hence D is well-defined. Moreover we have

$$\deg D = -\frac{\#(A/f)^\times}{q-1} + \sum_{\substack{a \in A \\ (a,f)=1 \\ \deg a < \deg f}} \langle ax \rangle = 0.$$

Now let $\{a_i\}$, $\{b_j\}$ and a_0 be as in §6.3.5, and fix $a \in A$ prime to f such that

$$\langle ax \rangle = \sum_{N=0}^{\infty} \langle ax \rangle_N = 1.$$

Let b be the unique element of A_+ such that

$$a a_0 \equiv b \mod f, \quad \deg b < \deg f,$$

and put

$$N := \deg f - \deg b - 1,$$

noting that

$$\langle b/f \rangle_N = 1.$$

We have

$$\begin{aligned} g_x\left(\xi_a^{(N)}\right) &= 1-\sum_{i=1}^{\deg f}\mathbf{e}^*(a_i/f)C_{a_0b_i}\left(T^{q^N},\mathbf{e}(ax)^{q^N}\right)\\ &= 1-\sum_{i=1}^{\deg f}\mathbf{e}^*(a_i/f)\mathbf{e}(b_ib/f)^{q^N}\\ &= 0, \end{aligned}$$

the last equality by part (ii) of Theorem 5.4.4. Therefore g_x has at least as many zeroes in $\overline{U}$ as we claim for it. In any case g_x has no singularities at infinity worse than simple poles. Therefore the divisor $(g_x)-D$ is effective and of degree 0, so it vanishes identically. Thus our claim is proved.

6.3.7. *Interpolation properties of Coleman functions.* Fix

$$x\in f^{-1}A\setminus A,\quad a\in A,\quad N\in\mathbb{Z},\quad y\in k,$$

such that

$$(a,f)=1,\quad N\geq 0,\quad ax\equiv y \bmod A,\quad |y|_\infty<1.$$

We claim that

$$g_x^{(N+1)}(\xi_a)=1+\Psi_N(y).$$

Let $\{a_i\}$, $\{b_j\}$ and a_0 be as in §6.3.5. Now,

$$\begin{aligned} g_x^{(N+1)}(\xi_a) &= 1-\sum_{i=1}^{\deg f}\mathbf{e}^*(a_i/f)^{q^{N+1}}C_{a_0b_i}\left(T,\mathbf{e}(a/f)\right)\\ &= 1-\sum_{i=1}^{\deg f}\mathbf{e}^*(a_i/f)^{q^{N+1}}\mathbf{e}(b_iy)\\ &= 1+\Psi_N(y), \end{aligned}$$

the last equality by part (i) of Theorem 5.4.4. The claim is proved.

Remark 6.3.8. The notion of Coleman function was introduced in [Si a], building on the foundation of [An b]. The notion of Coleman function was in large part inspired by beautiful examples of [Co]; see [An b, §2] for an appreciation of Coleman's work. The approach to the theory of Coleman functions presented here is quite a bit simpler than previous approaches and was developed in an attempt more closely to approximate Coleman's own simple and very attractive point of view.

6.3.9. *Generalized Coleman functions.* Given effective $\mathbf{a}\in\mathcal{A}_f$ such that $\operatorname{wt}\mathbf{a}>0$, write

$$\mathbf{a}=\sum_{\substack{a\in A\\ \deg a<\deg f}} m_a\left[\frac{a}{f}\right]\quad(m_a\in\mathbb{Z},\quad m_a\geq 0)$$

and put

$$g_{\mathbf{a}} := \prod_{\substack{0\neq a\in A\\ \deg a<\deg f}} g_{\frac{a}{f}}^{m_a},$$

thereby defining a meromorphic function on $\overline{X}$ regular on $\overline{U}$. We call $g_{\mathbf{a}}$ a *generalized Coleman function*. We define effective divisors of $\overline{X}$ by the formulas

$$\xi_{\mathbf{a}} := \sum_{\substack{a\in A\\ (a,f)=1\\ \deg a<\deg f}} \langle a\star\mathbf{a}\rangle\cdot\xi_a,\quad W_{\mathbf{a}} := \sum_{\substack{a\in A\\ (a,f)=1\\ \deg a<\deg f}} \sum_{N=1}^{\infty} \langle a\star\mathbf{a}\rangle_N\cdot\left(\sum_{i=0}^{N-1}\xi_a^{(i)}\right).$$

The definition of $W_{\mathbf{a}}$ makes sense because only finitely many nonzero terms appear on the right side. By the divisor calculation of §6.3.6 we have

$$\begin{aligned}(g_{\mathbf{a}}) &= -(\mathrm{wt}\,\mathbf{a})\cdot\infty_X + \sum_{\substack{a\in A\\ \deg a<\deg f\\ (a,f)=1}} \sum_{N=0}^{\infty} \langle a\star\mathbf{a}\rangle_N \xi_a^{(N)}\\ &= -(\mathrm{wt}\,\mathbf{a})\cdot\infty_X + \xi_{\mathbf{a}} + W_{\mathbf{a}}^{(1)} - W_{\mathbf{a}}.\end{aligned}$$

By the interpolation formula of §6.3.7 we have

$$\Pi(a\star\mathbf{a})^{-1} = \prod_{N=1}^{\infty} g_{\mathbf{a}}^{(N)}(\xi_a)$$

for all $a\in A$ prime to f.

PROPOSITION 6.3.10. *Fix effective* $\mathbf{a},\mathbf{b}\in\mathcal{A}_f$ *such that* $\mathrm{wt}\,\mathbf{a},\mathrm{wt}\,\mathbf{b}>0$. *For any deck transformation* γ *and* $a\in A$ *prime to* f *corresponding canonically one to the other in the sense that* $\gamma^* z = C_a(t,z)$,

$$\gamma^{-1}\xi_{\mathbf{a}} = \xi_{\mathbf{b}} \Leftrightarrow a\star\mathbf{a}\sim_f \mathbf{b}.$$

Proof. Clearly $\xi_{\mathbf{a}}=\xi_{\mathbf{b}}\Leftrightarrow \mathbf{a}\sim_f\mathbf{b}$ and further

$$\gamma^{-1}\xi_{\mathbf{a}} = \sum_{\substack{b\in A\\ (b,f)=1\\ \deg b<\deg f}} \langle b\star\mathbf{a}\rangle\cdot\xi_{cb} = \sum_{\substack{b\in A\\ (b,f)=1\\ \deg b<\deg f}} \langle ab\star\mathbf{a}\rangle\cdot\xi_b = \xi_{a\star\mathbf{a}}$$

where $c\in A$ satisfies the congruence $ac\equiv 1 \bmod f$, whence the result. □

6.4. *A construction of rigid analytically trivial* GCM *dual t-motives.* For convenience we again put $\ell := \#(A/f)^\times$ and also fix effective $\mathbf{a}\in\mathcal{A}_f$ such that $\mathrm{wt}\,\mathbf{a}>0$. Put

$$\mathbf{L} := \mathbb{F}_q[t,z]/(C_f^\star(t,z)),\quad \overline{\mathbf{L}} := \bar{k}[t,z]/(C_f^\star(t,z)).$$

The rings $\mathbf{L}$ and $\overline{\mathbf{L}}$ are the coordinate rings of the nonsingular irreducible affine curves $U/\mathbb{F}_q$ and $\overline{U}/\bar{k}$, respectively. Clearly $\mathbf{L}$ qualifies as a GCM $\mathbb{F}_q[t]$-algebra.

We are going to use the generalized Coleman function $g_{\mathbf{a}} \in \overline{\mathbf{L}}$ to create a nice dual t-motive $H(\mathbf{a})$ with GCM by $\mathbf{L}$. To do so we translate to the dual t-motivic setting a basic construction that in the t-motivic setting was originally given in [Si a].

LEMMA 6.4.1. *Let $\tilde{H}(\mathbf{a})$ be the left $\overline{\mathbf{L}}[\sigma]$-module obtained by equipping $\overline{\mathbf{L}}$ with an action of σ by the rule*

$$\sigma h := g_{\mathbf{a}} h^{(-1)}.$$

Then the $\bar{k}[\sigma]$-module underlying $\tilde{H}(\mathbf{a})$ is free, of finite rank.

Proof. By Proposition 4.3.2, because the $\bar{k}[t]$-module underlying $\tilde{H}(\mathbf{a})$ is free, of finite rank, we have only to prove that the $\bar{k}[\sigma]$-module underlying $\tilde{H}(\mathbf{a})$ is finitely generated. Temporarily put

$$D := \mathrm{wt}\,\mathbf{a} \cdot \infty_X.$$

Since the multiplicity of D at each point of $\overline{X}$ at infinity is the order of the pole of the generalized Coleman function $g_{\mathbf{a}}$ at that point according to the divisor formula of §6.3.9, the induced maps

$$\frac{\mathcal{O}_{\overline{X}}(nD)}{\mathcal{O}_{\overline{X}}((n-1)D)} \xrightarrow{g_{\mathbf{a}}\times} \frac{\mathcal{O}_{\overline{X}}((n+1)D)}{\mathcal{O}_{\overline{X}}(nD)}$$

of skyscraper sheaves are bijective for all $n \in \mathbb{Z}$. By the Riemann-Roch theorem there exists an integer n_0 such that the natural maps

$$\frac{H^0\left(\overline{X}, \mathcal{O}_{\overline{X}}(nD)\right)}{H^0\left(\overline{X}, \mathcal{O}_{\overline{X}}((n-1)D)\right)} \to H^0\left(\overline{X}, \frac{\mathcal{O}_{\overline{X}}(nD)}{\mathcal{O}_{\overline{X}}((n-1)D)}\right)$$

are bijective for all $n \geq n_0$. Clearly we have

$$H^0\left(\overline{X}, \mathcal{O}_{\overline{X}}(nD)\right)^{(-1)} = H^0\left(\overline{X}, \mathcal{O}_{\overline{X}}\left(nD^{(-1)}\right)\right) = H^0\left(\overline{X}, \mathcal{O}_{\overline{X}}(nD)\right)$$

for all $n \in \mathbb{Z}$. Therefore

$$H^0\left(\overline{X}, \mathcal{O}_{\overline{X}}(nD)\right) + g_{\mathbf{a}} \cdot H^0\left(\overline{X}, \mathcal{O}_{\overline{X}}(nD)\right)^{(-1)} = H^0\left(\overline{X}, \mathcal{O}_{\overline{X}}((n+1)D)\right)$$

for all $n \geq n_0$ and hence the vector space

$$H^0\left(\overline{X}, \mathcal{O}_{\overline{X}}(n_0 D)\right)$$

is finite-dimensional over $\bar{k}$ and generates

$$\tilde{H}(\mathbf{a}) = H^0(\overline{U}, \mathcal{O}_{\overline{X}}) = \bigcup_n H^0\left(\overline{X}, \mathcal{O}_{\overline{X}}(nD)\right)$$

over $\bar{k}[\sigma]$. □

6.4.2. *Construction of the dual t-motive $H(\mathbf{a})$.* Recall now the divisor formula

$$(g_{\mathbf{a}}) = -(\mathrm{wt}\,\mathbf{a}) \cdot \infty_X + \xi_{\mathbf{a}} + W_{\mathbf{a}}^{(1)} - W_{\mathbf{a}}$$

of §6.3.9 and recall also that the divisors $\xi_{\mathbf{a}}$ and $W_{\mathbf{a}}$ figuring in this formula are effective. Put

$$H(\mathbf{a}) := H^0\left(\overline{U}, \mathcal{O}_{\overline{X}}\left(-W_{\mathbf{a}}^{(1)}\right)\right) \subset H^0\left(\overline{U}, \mathcal{O}_{\overline{X}}\right) = \tilde{H}(\mathbf{a})$$

thereby defining an $\overline{\mathbf{L}}$-submodule of $\tilde{H}(\mathbf{a})$. It is easy to verify that $H(\mathbf{a})$ is σ-stable and hence an $\overline{\mathbf{L}}[\sigma]$-submodule of $\tilde{H}(\mathbf{a})$. It is clear that $H(\mathbf{a})$ is projective over $\overline{\mathbf{L}}$ of rank one and free, of finite rank over $\bar{k}[t]$. Moreover $H(\mathbf{a})$ is a $\bar{k}[\sigma]$-submodule of a $\bar{k}[\sigma]$-module free, of finite rank by Lemma 6.4.1 and hence a free $\bar{k}[\sigma]$-module of finite rank. Since we have

$$\frac{H(\mathbf{a})}{\sigma H(\mathbf{a})} = \frac{H^0(\overline{U}, \mathcal{O}_{\overline{X}}(-W_{\mathbf{a}}^{(1)}))}{g_{\mathbf{a}} \cdot H^0(\overline{U}, \mathcal{O}_{\overline{X}}(-W_{\mathbf{a}}^{(1)}))^{(-1)}} = \frac{H^0(\overline{U}, \mathcal{O}_{\overline{X}}(-W_{\mathbf{a}}^{(1)}))}{H^0(\overline{U}, \mathcal{O}_{\overline{X}}(-\xi_{\mathbf{a}} - W_{\mathbf{a}}))},$$

and all the points in the support of the divisor $\xi_{\mathbf{a}}$ lie above the point $t = T$ on the t-line, it follows that $H(\mathbf{a})/\sigma H(\mathbf{a})$ is annihilated by a sufficiently high power of $t - T$. Therefore $H(\mathbf{a})$ is a dual t-motive with GCM by $\mathbf{L}$. Note that the ideal

$$I_{\mathbf{a}} := H^0\left(\overline{U}, \mathcal{O}_{\overline{X}}(-\xi_{\mathbf{a}})\right) \subset \overline{\mathbf{L}}$$

is the GCM type of $H(\mathbf{a})$ with respect to $\mathbf{L}$.

Lemma 6.4.3. (i) *Let*

$$\Phi_{\mathbf{a}} \in \mathrm{Mat}_{\ell\times\ell}(\bar{k}[t])$$

be the unique solution of the congruence

$$g_{\mathbf{a}} \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{\ell-1} \end{bmatrix} \equiv \Phi_{\mathbf{a}} \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{\ell-1} \end{bmatrix} \bmod C_f^{\star}(t,z).$$

Then with respect to the Banach norm on $\mathbb{C}_\infty\{t\}$ given by the rule

$$\left\| \sum_{i=0}^{\infty} a_i t^i \right\|_\infty := \sup_{i=0}^{\infty} |a_i|_\infty$$

and extended to $\mathrm{Mat}_{\ell\times\ell}(\mathbb{C}_\infty\{t\})$ by the rule

$$\|X\|_\infty = \max_{i,j=1}^{\ell} \|X_{ij}\|_\infty,$$

the infinite product

$$\Psi_{\mathbf{a}} := \prod_{N=1}^{\infty} \Phi_{\mathbf{a}}^{(N)}$$

converges to an element of

$$\mathrm{GL}_\ell(\mathbb{C}_\infty\{t\}) \cap \mathrm{Mat}_{\ell\times\ell}(\mathcal{E})$$

satisfying the functional equation

$$\Psi_{\mathbf{a}}^{(-1)} = \Phi_{\mathbf{a}}\Psi_{\mathbf{a}}.$$

(ii) *Consider now the matrix*

$$\Psi_{\mathbf{a}}(T) \in \mathrm{Mat}_{\ell\times\ell}(\overline{k_\infty})$$

obtained by evaluating $\Psi_{\mathbf{a}}$ at $t = T$. The sets

$$\{\Psi_{\mathbf{a}}(T)_{ij} \mid i,j = 1,\ldots,\ell\}, \quad \{\Pi(a \star \mathbf{a})^{-1} \mid a \in A,\ (a,f) = 1\}$$

span the same $\bar{k}$-subspace of $\overline{k_\infty}$.

Proof. (i) From the construction of the generalized Coleman function $g_{\mathbf{a}}$ it is clear that

$$g_{\mathbf{a}} \equiv 1 + \sum_i \sum_j c_{ij} t^i z^j \bmod C_f^\star(t,z)$$

for some constants $c_{ij} \in \bar{k}$, all but finitely many of which vanish and all of which satisfy the bound

$$|c_{ij}|_\infty \le \sup_{x\in k_\infty} |\mathbf{e}^*(x)|_\infty = |1/\widetilde{T}|_\infty < 1.$$

Now let

$$Z = Z(t) \in \mathrm{Mat}_{\ell\times\ell}(\mathbb{F}_q[t])$$

be the unique solution of the congruence

$$z\begin{bmatrix} 1 \\ z \\ \vdots \\ z^{\ell-1}\end{bmatrix} \equiv Z \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{\ell-1}\end{bmatrix} \bmod C_f^\star(t,z).$$

Clearly,

$$\Phi_{\mathbf{a}}^{(N)} = \mathbf{1}_\ell + \sum_i \sum_j c_{ij}^{q^N} t^i Z^j$$

for all integers N. It follows that the infinite product defining $\Psi_{\mathbf{a}}$ converges to an element of $\mathrm{GL}_\ell(\mathbb{C}_\infty\{t\})$ satisfying the desired functional equation. It follows also that $\det \Phi_{\mathbf{a}}(0) \neq 0$ and hence by Proposition 3.1.3 that the matrix $\Psi_{\mathbf{a}}$ has entries in $\mathcal{E}$.

(ii) Note that by construction the roots of the equation $C_f^\star(T,z) = 0$ give the ℓ eigenvalues of the matrix

$$Z(T) \in \mathrm{GL}_\ell(\bar{k}).$$

Now choose any matrix

$$M \in \mathrm{GL}_\ell(\bar{k})$$

such that

$$MZ(T)M^{-1} = \begin{bmatrix} \mathbf{e}(a_1/f) & & \\ & \ddots & \\ & & \mathbf{e}(a_\ell/f) \end{bmatrix}$$

where

$$\{a_1, \ldots, a_\ell\} = \{a \in A \mid \deg a < \deg f,\ (a, f) = 1\}.$$

Then

$$\left(M\Phi_{\mathbf{a}}^{(N)}(T)M^{-1}\right)_{ij} = g^{(N)}(\xi_{a_i}) \cdot \delta_{ij},$$

and hence

$$(M\Psi_{\mathbf{a}}(T)M^{-1})_{ij} = \Pi(a_i \star \mathbf{a})^{-1} \cdot \delta_{ij},$$

which proves the result. □

Proposition 6.4.4. *Let*

$$\mathbf{g} \in \mathrm{Mat}_{\ell\times 1}(H(\mathbf{a})), \quad \Phi \in \mathrm{Mat}_{\ell\times\ell}(\bar{k}[t])$$

be given such that the entries of $\mathbf{g}$ *form a* $\bar{k}[t]$*-basis of* $H(\mathbf{a})$ *and*

$$\sigma\mathbf{g} = \Phi\mathbf{g}.$$

There exists

$$\Psi \in \mathrm{GL}_\ell(\mathbb{C}_\infty\{t\}) \cap \mathrm{Mat}_{\ell\times\ell}(\mathcal{E})$$

satisfying the functional equation

$$\Psi^{(-1)} = \Phi\Psi$$

and with the further property that the sets

$$\{\Psi(T)_{ij} \mid i, j = 1, \ldots, \ell\}, \quad \{\Pi(a \star \mathbf{a})^{-1} \mid a \in A,\ (a, f) = 1\}$$

span the same $\bar{k}$*-subspace of* $\overline{k_\infty}$.

In particular $H(\mathbf{a})$ is rigid analytically trivial by Lemma 4.4.12.

Proof. Let

$$Q \in \mathrm{Mat}_{\ell\times\ell}(\bar{k}[t])$$

be the unique solution of the congruence

$$\mathbf{g} \equiv Q \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{\ell-1} \end{bmatrix} \bmod C_f^\star(t, z).$$

Now since the effective divisor $W_{\mathbf{a}}^{(1)}$ is supported in the set of $\bar{k}$-valued points of X lying above the points

$$t = T^q, T^{q^2}, \dots$$

on the t-line, the module

$$\frac{\tilde{H}(\mathbf{a})}{H(\mathbf{a})} = \frac{H^0(\overline{U}, \mathcal{O}_{\overline{X}})}{H^0(\overline{U}, \mathcal{O}_{\overline{X}}(-W_{\mathbf{a}}^{(1)}))}$$

is annihilated by

$$\prod_{i=1}^{N} (t - T^{q^i})^N$$

for $N \gg 0$. Hence,

$$Q \in \mathrm{GL}_\ell(\mathbb{C}_\infty\{t\}), \quad \det Q(T) \neq 0.$$

With notation as in Lemma 6.4.3,

$$\Phi Q \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{\ell-1} \end{bmatrix} \equiv \sigma \mathbf{g} \equiv Q^{(-1)} \Phi_{\mathbf{a}} \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{\ell-1} \end{bmatrix} \bmod C_f^\star(t, z);$$

hence

$$Q^{(-1)} \Phi_{\mathbf{a}} = \Phi Q,$$

and the matrix

$$\Psi := Q \Psi_{\mathbf{a}} \in \mathrm{GL}_\ell(\mathbb{C}_\infty\{t\}) \cap \mathrm{Mat}_{\ell \times \ell}(\mathcal{E})$$

has all the desired properties. □

Proposition 6.4.5. *Let r be the cardinality of the orbit of the coset* $\mathbf{a} \bmod \widetilde{\mathcal{R}}_f$ *under the action of $(A/f)^\times$. Then any simple quotient of the bare dual t-motive underlying $H(\mathbf{a})$ is of rank $\geq r$ over $\bar{k}[t]$.*

Proof. We temporarily denote the group of deck transformations by G. We have

$$\#\{a \star \mathbf{a} \bmod \widetilde{\mathcal{R}}_f \mid a \in A,\ (a, f) = 1\} = \#\{\gamma \xi_{\mathbf{a}} \mid \gamma \in G\} = \#\{\gamma^* I_{\mathbf{a}} \mid \gamma \in G\},$$

where the first equality holds by Proposition 6.3.10 and the second equality is trivial. The quantity on the right by Corollary 4.6.7 is a lower bound for the rank over $\bar{k}[t]$ of any simple quotient of the bare dual t-motive underlying $H(\mathbf{a})$. □

PROPOSITION 6.4.6. *For all* $\mathbf{b} \in \mathcal{A}_f$ *such that* $\operatorname{wt}\mathbf{b} > 0$, *if the bare dual* t*-motives underlying* $H(\mathbf{a})$ *and* $H(\mathbf{b})$ *have isogenous simple quotients*, *then* $a \star \mathbf{a} \sim_f \mathbf{b}$ *for some* $a \in A$ *prime to* f.

Proof. Once again let G denote the group of deck transformations. Consider the following sets:

$$\begin{gathered}\{\gamma^* I_{\mathbf{a}} = I_{\mathbf{b}} \mid \gamma \in G\},\\ \{\gamma\xi_{\mathbf{a}} = \xi_{\mathbf{b}} \mid \gamma \in G\},\\ \{a \star \mathbf{a} \equiv \mathbf{b} \bmod \widetilde{\mathcal{R}}_f \mid a \in A,\ (a,f)=1\}.\end{gathered}$$

The first set is nonempty by Corollary 4.6.8. Nonemptiness of the first set trivially implies that of the second. Nonemptiness of the second set implies that of the third by Proposition 6.3.10. □

6.5. *Proof of Theorem* 6.2.1 *and Corollary* 6.2.2. We need only prove the sufficiency asserted in the theorem because the diamond bracket criterion takes care of necessity and Proposition 6.2.3 takes care of the corollary.

6.5.1. *Easy reductions.* By hypothesis:

- The cosets $\mathbf{a}_1 \bmod \widetilde{\mathcal{R}}_f, \dots, \mathbf{a}_N \bmod \widetilde{\mathcal{R}}_f$ are distinct.

After enlarging the set $\{\mathbf{a}_1, \dots, \mathbf{a}_N\}$ suitably we may assume that:

- The finite set $\{\mathbf{a}_i \bmod \widetilde{\mathcal{R}}_f \mid i = 1, \dots, N\} \subset \mathcal{A}_f/\widetilde{\mathcal{R}}_f$ is $(A/f)^\times$-stable.

After relabeling the $\mathbf{a}_i$ we may assume that for some integer $1 \le n \le N$:

- The set $\{\mathbf{a}_i \bmod \widetilde{\mathcal{R}}_f \mid i = 1, \dots, n\}$ forms a set of representatives for the $(A/f)^\times$-orbits in $\{\mathbf{a}_i \bmod \widetilde{\mathcal{R}}_f \mid i = 1, \dots, N\}$.

For $i = 1, \dots, n$ put

$$\begin{aligned} r_i &:= \#\{a \star \mathbf{a}_i \bmod \widetilde{\mathcal{R}}_f \mid a \in A,\ (a,f)=1\},\\ V_i &:= \big(\bar{k}\text{-span of } \{\Pi(a \star \mathbf{a}_i)^{-1} \mid a \in A,\ (a,f)=1\}\big) \subset \overline{k_\infty}.\end{aligned}$$

We have $N = \sum_i r_i$. Moreover the $\bar{k}$-span of $\{\Pi(-\mathbf{a}_i) \mid i = 1, \dots, N\}$ equals $\sum_i V_i$ by the diamond bracket criterion. It therefore suffices to show that

$$\sum_{i=1}^n r_i \le \dim_{\bar{k}} \sum_{i=1}^n V_i.$$

After adding a fixed positive integral multiple of

$$\sum_{\substack{a \in A\\ \deg a < \deg f}} \left[\frac{a}{f}\right] \in \mathcal{A}_f$$

to $\mathbf{a}_i$ for all i, we may further assume that

- $\mathbf{a}_i$ is effective and $\operatorname{wt}\mathbf{a}_i > 0$ for $i = 1, \dots, n$.

6.5.2. *Further reductions.* For $i = 1, \ldots, n$ we make the following constructions. Put

$$H_i := H(\mathbf{a}_i).$$

Choose

$$\mathbf{g}_{(i)} \in \mathrm{Mat}_{\ell\times 1}(H_i), \quad \Phi_{(i)} \in \mathrm{Mat}_{\ell\times\ell}(\bar{k}[t]) \quad (\ell := \#(A/f)^\times)$$

such that the entries of $\mathbf{g}_{(i)}$ form a $\bar{k}[t]$-basis of H_i and

$$\sigma\mathbf{g}_{(i)} = \Phi_{(i)}\mathbf{g}_{(i)}.$$

By Proposition 6.4.4 there exists

$$\Psi_{(i)} \in \mathrm{GL}_\ell(\mathbb{C}_\infty\{t\}) \cap \mathrm{Mat}_{\ell\times\ell}(\mathcal{E})$$

such that the functional equation

$$\Psi_{(i)}^{(-1)} = \Phi_{(i)}\Psi_{(i)}$$

holds. Let $\psi_{(i)}$ be the first column of $\Psi_{(i)}$. Put

$$\begin{aligned} H_{i0} &:= \bar{k}[t]\text{-span in } \mathcal{E} \text{ of the entries of } \psi_{(i)}, \\ V_{i0} &:= \bar{k}\text{-span in } \overline{k_\infty} \text{ of the entries of } \psi_{(i)}(T). \end{aligned}$$

By Proposition 6.4.4,

- $V_{i0} \subset V_i$ for $i = 1, \ldots, n$.

By Proposition 4.4.3,

- $\mathrm{rk}_{\bar{k}[t]} \sum_{i=1}^n H_{i0} = \dim_{\bar{k}} \sum_{i=1}^n V_{i0}$.

It therefore suffices to prove that

$$\sum_{i=1}^n r_i \leq \mathrm{rk}_{\bar{k}[t]} \sum_{i=1}^n H_{i0}.$$

6.5.3. *Endgame.* In a sense made precise by Proposition 4.4.3:

- H_{i0} is a nonzero dual t-motive admitting presentation as a quotient of the bare dual t-motive underlying H_i for $i = 1, \ldots, n$.

We have

- $r_i \leq \mathrm{rk}_{\bar{k}[t]} H_{i0} = \dim_{\bar{k}} V_{i0} \leq \dim_{\bar{k}} V_i \leq r_i$, for $i = 1, \ldots, n$, by Proposition 6.4.5 (inequality at the extreme left), Proposition 4.4.3 (equality at second juncture) and the diamond bracket criterion (inequality at the extreme right). It follows that

- H_{i0} is simple and of rank r_i over $\bar{k}[t]$ for $i = 1, \ldots, n$.

The simple dual t-motives H_{i0} belong to distinct isogeny classes by Proposition 6.4.6 and hence:

- The natural map $\displaystyle\bigoplus_{i=1}^{n} H_{i0} \to \sum_{i=1}^{n} H_{i0}$ is bijective.

Therefore we have

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} \mathrm{rk}_{\bar{k}[t]}\, H_{i0} = \mathrm{rk}_{\bar{k}[t]} \sum_{i=1}^{n} H_{i0}$$

and the proof of sufficiency in Theorem 6.2.1 is finished. As noted above, with sufficiency proved, the proofs of Theorem 6.2.1 and Corollary 6.2.2 are complete.

6.6. *Remarks concerning transcendence bases.* Giving an explicit transcendence basis for the field E_f over $\bar{k}$ is in general not as straightforward as one might suspect from the statements of Theorem 6.2.1 and Corollary 6.2.2. In principle transcendence bases and systems of relations could be constructed explicitly by a translation to our context of the methods of [Ku], but we do not attempt such a construction here. We just work out the special case in which f is a power of an irreducible polynomial and then give a cautionary example.

PROPOSITION 6.6.1. *Let $f_1 \in A_+$ be irreducible, and suppose $f = f_1^s$ for some s. Let*

$$B_f = \{\Pi(a/f) \mid a \in A \setminus A_+, \deg a < \deg f, (a, f) = 1\} \cup \{\varpi\}.$$

Then $E_f = \bar{k}(B_f)$, and the numbers in B_f are algebraically independent over $\bar{k}$.

Proof. Consider $\kappa f_1^e/f$, where $(\kappa, f_1) = 1$ and $\deg \kappa < (s-e)\deg f_1$. As an element of $\mathcal{D}_f$,

$$\left[\frac{\kappa f_1^e}{f}\right] - \sum_{\deg a < e \deg f_1} \left[\frac{\kappa f_1^e/f + a}{f_1^e}\right] = \left[\frac{\kappa f_1^e}{f}\right] - \sum_{\deg a < e \deg f_1} \left[\frac{\kappa f_1^e + af}{f_1^e f}\right]$$

$$= \left[\frac{\kappa f_1^e}{f}\right] - \sum_{\deg a < e \deg f_1} \left[\frac{\kappa + a f_1^{s-e}}{f}\right].$$

Notice now that the terms on the right have numerators congruent to κ modulo a power of f_1. Thus the numerators are relatively prime to f. According to the diamond bracket criterion we can therefore express every special Π-value $\Pi(b/f)$ as a $\bar{k}$-multiple of a product of $\Pi(a/f)$, with a relatively prime to f, divided by a power of ϖ. Finally, for every $a \in A_+$, the reflection identity dictates that

$$\varpi^{-1} \prod_{c \in \mathbb{F}_q^\times} \Pi(ca/f) \in \bar{k}^\times,$$

and so we conclude that $E_f = \bar{k}(B_f)$ as claimed. Since $\#B_f = \nu_f$ is the transcendence degree of E_f over $\bar{k}$, the rest follows. □

6.6.2. *Cautionary example.* In light of Proposition 6.6.1, it would not be far-fetched to imagine that

$$B_f = \{\Pi(a/f) \mid a \in A \setminus A_+, \deg a < \deg f, (a, f) = 1\} \cup \{\varpi\}.$$

would provide a trancendence basis for E_f over $\bar{k}$ for *all* $f \in A_+$. That however is not always the case.

Consider the example of $q = 3$ and $f = T^2 - T$. By Corollary 6.2.2 the transcendence degree of E_f over $\bar{k}$ is 3. In §4.2 of [Si b] it is shown that

$$\Pi\left(\frac{1}{T^2 - T}\right) \Big/ \Pi\left(\frac{1}{T}\right) \in \bar{k}^\times,$$

and

$$\Pi\left(\frac{T+1}{T^2 - T}\right) \Big/ \Pi\left(\frac{1}{T}\right) \in \bar{k}^\times,$$

by applying the diamond bracket criterion. Consequently, in view of the reflection identity satisfied by the Π-function, $\bar{k}(B_f)$ has transcendence degree at most 2 over $\bar{k}$. But by Corollary 6.2.2 and the diamond bracket criterion, we know that E_f is the rational function field

$$E_f = \bar{k}\left(\varpi, \Pi\left(\frac{1}{T}\right), \Pi\left(\frac{1}{T-1}\right)\right).$$

6.6.3. *Remark.* Continuing in the case $q = 3$ considered in the preceding paragraph, we can show that the Π-monomial $\Pi\left(\frac{1}{T^2-T}\right)/\Pi\left(\frac{1}{T}\right)$ and its companion are examples of the sort considered above in §1.1.6.

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