The space of embedded minimal surfaces of fixed genus in a 3-manifold I; Estimates off the axis for disks

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0. Introduction

This paper is the first in a series where we describe the space of all embedded minimal surfaces of fixed genus in a fixed (but arbitrary) closed Riemannian 3-manifold. The key for understanding such surfaces is to understand the local structure in a ball and in particular the structure of an embedded minimal disk in a ball in $\mathbb{R}^3$ (with the flat metric). This study is undertaken here and completed in [CM6]. These local results are then applied in [CM7] where we describe the general structure of fixed genus surfaces in 3-manifolds.

There are two local models for embedded minimal disks (by an embedded disk, we mean a smooth injective map from the closed unit ball in $\mathbb{R}^2$ into $\mathbb{R}^3$). One model is the plane (or, more generally, a minimal graph), the other is a piece of a helicoid. In the first four papers of this series, we will show that every embedded minimal disk is either a graph of a function or is a double spiral staircase where each staircase is a multi-valued graph. This will be done by showing that if the curvature is large at some point (and hence the surface is not a graph), then it is a double spiral staircase. To prove that such a disk is a double spiral staircase, we will first prove that it is built out of $N$-valued graphs where $N$ is a fixed number. This is initiated here and will be completed in the second paper. The third and fourth papers of this series will deal with how the multi-valued graphs fit together and, in particular, prove regularity of the set of points of large curvature – the axis of the double spiral staircase.

The reader may find it useful to also look at the survey [CM8] and the expository article [CM9] for an outline of our results, and their proofs, and how these results fit together. The article [CM9] is the best to start with.

*The first author was partially supported by NSF Grant DMS 9803253 and an Alfred P. Sloan Research Fellowship and the second author by NSF Grant DMS 9803144 and an Alfred P. Sloan Research Fellowship.
Our main theorem about embedded minimal disks is that every such disk can either be modelled by a minimal graph or by a piece of the helicoid depending on whether the curvature is small or not; see Theorem 0.2 below. This will be proven in [CM6] with the first steps taken here. The helicoid is the minimal surface in $\mathbb{R}^3$ parametrized by $(s \cos(t), s \sin(t), t)$ where $s, t \in \mathbb{R}$.

To be able to discuss the helicoid some more and in particular give a precise meaning to the fact that it is like a double spiral staircase, we will need the notion of a multi-valued graph; see Figure 1. Let $D_r$ be the disk in the plane centered at the origin and of radius $r$ and let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbb{C} \setminus \{0\}$ with global polar coordinates $(\rho, \theta)$ so that $\rho > 0$ and $\theta \in \mathbb{R}$. An $N$-valued graph of a function $u$ on the annulus $D_s \setminus D_r$ is a single valued graph over

$$\{ (\rho, \theta) \mid r \leq \rho \leq s, \ |\theta| \leq N \pi \}. \leqno{(0.1)}$$

The middle sheet $\Sigma^M$ (an annulus with a slit as in [CM3]) is the portion over

$$\{ (\rho, \theta) \in \mathcal{P} \mid r \leq \rho \leq s \text{ and } 0 \leq \theta \leq 2\pi \}. \leqno{(0.1)}$$

The multi-valued graphs that we will consider will never close up; in fact they will all be embedded. Note that embedded means that the separation never vanishes. Here the separation (see Figure 1) is the function given by

$$w(\rho, \theta) = u(\rho, \theta + 2\pi) - u(\rho, \theta).$$

If $\Sigma$ is the helicoid (see Figure 2), then $\Sigma \setminus x_3 - \text{axis} = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1, \Sigma_2$ are $\infty$-valued graphs. Also, $\Sigma_1$ is the graph of the function $u_1(\rho, \theta) = \theta$ and $\Sigma_2$ is the graph of the function $u_2(\rho, \theta) = \theta + \pi$. In either case the separation $w = 2\pi$. A multi-valued minimal graph is a multi-valued graph of a function $u$ satisfying the minimal surface equation.

Figure 1: The separation of a multi-valued graph.
Figure 2: The helicoid is obtained by gluing together two $\infty$-valued graphs along a line. The two multi-valued graphs are given in polar coordinates by $u_1(\rho, \theta) = \theta$ and $u_2(\rho, \theta) = \theta + \pi$. In either case $w(\rho, \theta) = 2\pi$.

Here, we have normalized so that our embedded multi-valued graphs have positive separation. This can be achieved after possibly reflecting in a plane.

Let now $\Sigma_i \subset B_{2R}$ be a sequence of embedded minimal disks with $\partial \Sigma_i \subset \partial B_{2R}$. Clearly (after possibly going to a subsequence) either (1) or (2) occur:

1. $\sup_{B_R \cap \Sigma_i} |A|^2 \leq C < \infty$ for some constant $C$.
2. $\sup_{B_R \cap \Sigma_i} |A|^2 \to \infty$.

In (1) (by a standard argument) the intrinsic ball $B_s(y_i)$ is a graph for all $y_i \in B_R \cap \Sigma_i$, where $s$ depends only on $C$. Thus the main case is (2) which is the subject of the next theorem.

Using the notion of multi-valued graphs, we can now state our main theorem:

**Theorem 0.2** (Theorem 0.1 in [CM6] (see Figure 3)). Let $\Sigma_i \subset B_{R_i} = B_{R_i}(0) \subset \mathbb{R}^3$ be a sequence of embedded minimal disks with $\partial \Sigma_i \subset \partial B_{R_i}$ where $R_i \to \infty$. If

$$\sup_{B_i \cap \Sigma_i} |A|^2 \to \infty,$$

then there exist a subsequence, $\Sigma_j$, and a Lipschitz curve $S : \mathbb{R} \to \mathbb{R}^3$ such that after a rotation of $\mathbb{R}^3$:

1. $x_3(S(t)) = t$. (That is, $S$ is a graph over the $x_3$-axis.)
2. Each $\Sigma_j$ consists of exactly two multi-valued graphs away from $S$ (which spiral together).
3. For each $1 > \alpha > 0$, $\Sigma_j \setminus S$ converges in the $C^\alpha$-topology to the foliation, $\mathcal{F} = \{x_3 = t\}_t$, of $\mathbb{R}^3$. 
(4) \(\sup_{B_r(S(t) \cap \Sigma_j)} |A|^2 \to \infty\) for all \(r > 0, t \in \mathbb{R}\). (The curvatures blow up along \(S\)).

In (2), (3) that \(\Sigma_j \setminus S\) are multi-valued graphs and converge to \(F\) means that for each compact subset \(K \subset \mathbb{R}^3 \setminus S\) and \(j\) sufficiently large, \(K \cap \Sigma_j\) consists of multi-valued graphs over (part of) \(\{x_3 = 0\}\) and \(K \cap \Sigma_j \to K \cap F\) in the sense of graphs.

Theorem 0.2 — the singular set, \(S\), and the two multi-valued graphs.

Theorem 0.2 (like many of the other results discussed below) is modelled by the helicoid and its rescalings. Take a sequence \(\Sigma_i = a_i \Sigma\) of rescaled helicoids where \(a_i \to 0\). The curvatures of this sequence are blowing up along the vertical axis. The sequence converges (away from the vertical axis) to a foliation by flat parallel planes. The singular set \(S\) (the axis) then consists of removable singularities.

Before we proceed, let us briefly describe the strategy of the proof of Theorem 0.2.

The proof has the following three main steps; see Figure 4:

A. Fix an integer \(N\) (the “large” of the curvature in what follows will depend on \(N\)). If an embedded minimal disk \(\Sigma\) is not a graph (or equivalently if the curvature is large at some point), then it contains an \(N\)-valued minimal graph which initially is shown to exist on the scale of \(1/\max |A|\). That is, the \(N\)-valued graph is initially shown to be defined on an annulus with both inner and outer radii inversely proportional to \(\max |A|\).

B. Such a potentially small \(N\)-valued graph sitting inside \(\Sigma\) can then be seen to extend as an \(N\)-valued graph inside \(\Sigma\) almost all the way to the boundary. That is, the small \(N\)-valued graph can be extended to an \(N\)-valued graph defined on an annulus where the outer radius of the annulus is proportional to \(R\). Here \(R\) is the radius of the ball in \(\mathbb{R}^3\) in which the boundary of \(\Sigma\) is contained.

C. The \(N\)-valued graph not only extends horizontally (i.e., tangent to the initial sheets) but also vertically (i.e., transversally to the sheets). That is, once there are \(N\) sheets there are many more and, in fact, the disk \(\Sigma\) consists of two multi-valued graphs glued together along an axis.
A will be proved in [CM4], B will be proved in this paper, and C will be proved in [CM5] and [CM6], where we also will establish the regularity of the “axis.”

We will now return to the results proved in this paper, i.e., the proof of B above. We show here that if such an embedded minimal disk in $\mathbb{R}^3$ starts off as an almost flat multi-valued graph, then it will remain so indefinitely.

**Theorem 0.3** (see Figure 5). Given $\tau > 0$, there exist $N, \Omega, \varepsilon > 0$ so that the following hold:

Let $\Sigma \subset B_{R_0} \subset \mathbb{R}^3$ be an embedded minimal disk with $\partial \Sigma \subset \partial B_{R_0}$. If $\Omega r_0 < 1 < R_0/\Omega$ and $\Sigma$ contains an $N$-valued graph $\Sigma_g$ over $D_1 \setminus D_{r_0}$ with gradient $\leq \varepsilon$ and

$$\Sigma_g \subset \{x_3^2 \leq \varepsilon^2(x_1^2 + x_2^2)\},$$

then $\Sigma$ contains a 2-valued graph $\Sigma_d$ over $D_{R_0/\Omega} \setminus D_{r_0}$ with gradient $\leq \tau$ and $(\Sigma_g)^M \subset \Sigma_d$. 

Figure 5: Theorem 0.3 — extending a small multi-valued graph in a disk.
Figure 6: Theorem 0.4— finding a small multi-valued graph in a disk near a point of large curvature.

Theorem 0.3 is particularly useful when combined with a result from [CM4] asserting that an embedded minimal disk with large curvature at a point contains a small, almost flat, multi-valued graph nearby. Namely, we prove in [CM4] the following theorem:

**Theorem 0.4** ([CM4] (see Figure 6)). Given $N, \omega > 1$, and $\varepsilon > 0$, there exists $C = C(N, \omega, \varepsilon) > 0$ so that the following holds:

Let $0 \in \Sigma^2 \subset B_R \subset \mathbb{R}^3$ be an embedded minimal disk with $\partial \Sigma \subset \partial B_R$. If for some $0 < r_0 < R$,

$$\sup_{B_{r_0} \cap \Sigma} |A|^2 \leq 4 |A|^2(0) = 4C^2 r_0^{-2},$$

then there exist $\bar{R} < r_0/\omega$ and (after a rotation of $\mathbb{R}^3$) an $N$-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega \bar{R}} \setminus D_{\bar{R}}$ with gradient $\leq \varepsilon$, and $\text{dist}_{\Sigma}(0, \Sigma_g) \leq 4 \bar{R}$.

Combining Theorem 0.3 and Theorem 0.4 with a standard blow-up argument gives the following theorem:

**Theorem 0.5** ([CM4]). Given $N \in \mathbb{Z}_+, \varepsilon > 0$, there exist $C_1, C_2 > 0$ so that the following holds:

Let $0 \in \Sigma^2 \subset B_R \subset \mathbb{R}^3$ be an embedded minimal disk with $\partial \Sigma \subset \partial B_R$. If for some $R > r_0 > 0$,

$$\max_{B_{r_0} \cap \Sigma} |A|^2 \geq 4C^2_1 r_0^{-2},$$

then there exists (after a rotation of $\mathbb{R}^3$) an $N$-valued graph $\Sigma_g$ over $D_{R/C_2} \setminus D_{2r_0}$ with gradient $\leq \varepsilon$ and contained in $\Sigma \cap \{x_3^2 \leq \varepsilon^2 (x_1^2 + x_2^2)\}$.

The multi-valued graphs given by Theorem 0.5 should be thought of (see [CM6]) as the basic building blocks of an embedded minimal disk. In fact, one should think of such a disk as being built out of such graphs by stacking them on top of each other. It will follow from Proposition II.2.12 that the separation between the sheets in such a graph grows sublinearly.
An important component of the proof of Theorem 0.3 is a version of it for stable minimal annuli with slits that start off as multi-valued graphs. Another component is a curvature estimate “between the sheets” for embedded minimal disks in $\mathbb{R}^3$; see Figure 7. We will think of an axis for such a disk $\Sigma$ as a point or curve away from which the surface locally (in an extrinsic ball) has more than one component. With this weak notion of an axis, our estimate is that if one component of $\Sigma$ is sandwiched between two others that connect to an axis, then the one that is sandwiched has curvature estimates; see Theorem I.0.8. The example to keep in mind is a helicoid and the components are “consecutive sheets” away from the axis. These separate sheets can be connected along the axis of the helicoid and every component between them must then be graphical and hence have bounded curvature.

Theorems 0.3, 0.4, 0.5 are local and are for simplicity stated and proved only in $\mathbb{R}^3$ although they can with only very minor changes easily be seen to hold for minimal disks in a sufficiently small ball in any given fixed Riemannian 3-manifold.

The paper is divided into 4 parts. In Part I, we show the curvature estimate “between the sheets” when the disk is in a thin slab. In Part II, we show that certain stable disks with interior boundaries starting off as multi-valued graphs remain very flat (cf. Theorem 0.3). This result will be needed, together with Part I, in Part III to generalize the results of Part I to when the disk is not anymore assumed to lie in a slab. Part II will also be used together with Part III, in Part IV to show Theorem 0.3.

Let $x_1, x_2, x_3$ be the standard coordinates on $\mathbb{R}^3$ and $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ orthogonal projection to $\{x_3 = 0\}$. For $y \in S \subset \Sigma \subset \mathbb{R}^3$ and $s > 0$, the extrinsic and intrinsic balls and tubes are

$$B_s(y) = \{x \in \mathbb{R}^3 \mid |x - y| < s\}, \quad T_s(S) = \{x \in \mathbb{R}^3 \mid \text{dist}_{\mathbb{R}^3}(x, S) < s\},$$

$$B_s(y) = \{x \in \Sigma \mid \text{dist}_{\Sigma}(x, y) < s\}, \quad T_s(S) = \{x \in \Sigma \mid \text{dist}_{\Sigma}(x, S) < s\}.$$  

$D_s$ denotes the disk $B_s(0) \cap \{x_3 = 0\}$. $K_\Sigma$ the sectional curvature of a smooth compact surface $\Sigma$ and when $\Sigma$ is immersed $A_\Sigma$ will be its second fundamental form. When $\Sigma$ is oriented, $n_\Sigma$ is the unit normal. We will often consider
the intersection of curves and surfaces with extrinsic balls. We assume that these intersect transversely since this can be achieved by an arbitrarily small perturbation of the radius.

**Part I: Minimal disks in a slab**

Let $\gamma_{p,q}$ denote the line segment from $p$ to $q$ and $\overline{p,q}$ the ray from $p$ through $q$. A curve $\gamma$ is $h$-almost monotone if given $y \in \gamma$, then $B_{4h}(y) \cap \gamma$ has only one component which intersects $B_{2h}(y)$. Our curvature estimate “between the sheets” is (see Figure 8):

**Theorem I.0.8.** There exist $c_1 \geq 4$ and $2c_2 < c_4 < c_3 \leq 1$ so that the following holds:

Let $\Sigma^2 \subset B_{c_1 r_0}$ be an embedded minimal disk with $\partial \Sigma \subset \partial B_{c_1 r_0}$ and $y \in \partial B_{2r_0}$. Suppose that $\Sigma_1$, $\Sigma_2$, and $\Sigma_3$ are distinct components of $B_{r_0}(y) \cap \Sigma$ and

$$\gamma \subset (B_{r_0} \cup T_{c_2 r_0}(\gamma_{0,y})) \cap \Sigma$$

is a curve with $\partial \gamma = \{y_1, y_2\}$ where $y_i \in B_{c_2 r_0}(y) \cap \Sigma_i$ and each component of $\gamma \setminus B_{r_0}$ is $c_2 r_0$-almost monotone.

If $\Sigma'_3$ is a component of $B_{c_4 r_0}(y) \cap \Sigma_3$ with $y_1, y_2$ in distinct components of $B_{c_4 r_0}(y) \setminus \Sigma'_3$, then $\Sigma'_3$ is a graph.

![Figure 8: $y_1, y_2, \Sigma_1, \Sigma_2, \Sigma'_3$, and $\gamma$ in Theorem I.0.8.](image)

The idea for the proof of Theorem I.0.8 is to show that if this were not the case, then we could find an embedded stable disk that would be almost flat and would lie in the complement of the original disk. In fact, we can choose the stable disk to be sandwiched between the two components as well. The flatness would force the stable disk to eventually cross the axis in the original disk, contradicting that they were disjoint.
In this part, we prove Theorem I.0.8 when the surface is in a slab, illustrating the key points (the full theorem, using the results of this part, will be proved later). Two simple facts about minimal surfaces in a slab will be used:

- Stable surfaces in a slab must be graphical away from their boundary (see Lemma I.0.9 below).
- The maximum principle, and catenoid foliations in particular, force these surfaces to intersect a narrow cylinder about every vertical line (see the appendix).

**Lemma I.0.9.** Let \( \Gamma \subset \{ |x_3| \leq \beta h \} \) be a stable embedded minimal surface. There exist \( C_g, \beta_s > 0 \) so that if \( \beta \leq \beta_s \) and \( E \) is a component of \( R^2 \setminus T_h(\Pi(\partial \Gamma)) \), then each component of \( \Pi^{-1}(E) \cap \Gamma \) is a graph over \( E \) of a function \( u \) with \( |\nabla_{R^2} u| \leq C_g \beta \).

**Proof.** If \( B_h(y) \subset \Gamma \), then the curvature estimate of [Sc] gives

\[
\sup_{B_h/2(y)} |A| \leq C_s h^{-2}.
\]

Since \( \Delta_{\Gamma} x_3 = 0 \), the gradient estimate of [ChY] yields

\[
(I.0.10) \sup_{B_h/4(y)} |\nabla_{\Gamma} x_3| \leq \tilde{C}_g h^{-1} \sup_{B_h/2(y)} |x_3| \leq \tilde{C}_g \beta,
\]

where \( \tilde{C}_g = \tilde{C}_g(C_s) \). Since

\[
|\nabla_{R^2} u|^2 = |\nabla_{\Gamma} x_3|^2 / (1 - |\nabla_{\Gamma} x_3|^2),
\]

(I.0.10) gives the lemma. \(\square\)

The next lemma shows that if an embedded minimal disk \( \Sigma \) in the intersection of a ball with a thin slab is not graphical near the center, then it contains a curve \( \gamma \) coming close to the center and connecting two boundary points which are close in \( R^3 \) but not in \( \Sigma \). The constant \( \beta_A \) is defined in (A.6).

**Lemma I.0.11.** Let \( \Sigma^2 \subset B_{60h} \cap \{ |x_3| \leq \beta_A h \} \) be an embedded minimal disk with \( \partial \Sigma \subset \partial B_{60h} \) and let \( z_b \in \partial B_{50h} \). If a component \( \Sigma' \) of \( B_{5h} \cap \Sigma \) is not a graph, then there are:

- **Distinct components** \( S_1, S_2 \) of \( B_{8h}(z_b) \cap \Sigma \).
- **Points** \( z_1 \) and \( z_2 \) with \( z_i \in B_{h/4}(z_b) \cap S_i \).
- **A curve** \( \gamma \subset (B_{30h} \cup T_h(\gamma_{q,z})) \cap \Sigma \) with \( \partial \gamma = \{ z_1, z_2 \} \) and \( \gamma \cap \Sigma' \neq \emptyset \). Here \( q \in B_{50h}(z_b) \cap \partial B_{30h} \).
proof. See Figure 9. Since $\Sigma'$ is not graphical, we can find $z \in \Sigma'$ with $\Sigma$ vertical at $z$, i.e.,

$$|\nabla_{\Sigma} x_3|(z) = 1.$$  

Fix a point $y \in \partial B_{4h}(z)$ so that $\gamma_{y,z}$ is normal to $\Sigma$ at $z$. Then $f_y(z) = 4h$ (see (A.5)). Let $y'$ be given such that $y' \in \partial B_{10h}(y)$ and $z \in \gamma_{y,y'}$. The first step is to use the catenoid foliation $f_y$ to build the desired curve on the scale of $h$; see Figure 10. The second and third steps will bring the endpoints of this curve out near $z_b$.

Any simple closed curve $\sigma \subset \Sigma \setminus \{f_y > 4h\}$ bounds a disk $\Sigma_{\sigma} \subset \Sigma$. By Lemma A.8, $f_y$ has no maxima on $\Sigma_{\sigma} \cap \{f_y > 4h\}$ so that we conclude $\Sigma_{\sigma} \cap \{f_y > 4h\} = \emptyset$. On the other hand, by Lemma A.7, we get a neighborhood $U_z \subset \Sigma$ of $z$ where $U_z \cap \{f_y = 4h\} \setminus \{z\}$ is the union of $2n \geq 4$ disjoint embedded arcs meeting at $z$. Moreover, $U_z \setminus \{f_y \geq 4h\}$ has $n$ components $U_1, \ldots, U_n$ with

$$U_i \cap U_j = \{z\} \text{ for } i \neq j.$$
If a simple curve $\tilde{z} \subset \Sigma \setminus \{f_y \geq 4h\}$ connects $U_1$ to $U_2$, then connecting $\partial \tilde{z}$ by a curve in $U_2$ gives a simple closed curve $\sigma_z \subset \Sigma \setminus \{f_y \geq 4h\}$ with $\tilde{z} \subset \sigma_z$ and $\sigma_z \cap \{f_y \geq 4h\} = \{z\}$. Hence, $\sigma_z$ bounds a disk $\Sigma_{\sigma_z} \subset \Sigma \setminus \{f_y \geq 4h\}$. By construction, we have

$$U_2 \cap \Sigma_{\sigma_z} \setminus \cup_i U_i \neq \emptyset.$$ 

This is a contradiction, so we conclude that $U_1, U_2$ are contained in components $\Sigma_1 \neq \Sigma_2$ of $\Sigma \setminus \{f_y \geq 4h\}$ with $z \in \Sigma_1 \cap \Sigma_2$. For $i = 1, 2$, Lemma A.8 and (A.6) give $y_i^a \in B_{4h}(y) \cap \Sigma_i$. Corollary A.10 gives $\nu_i \subset T_h(\gamma_{y,y'}) \cap \Sigma$ with $\partial \nu_i = \{y_i^a, y_i^b\}$ where $y_i^b \in B_{4h}(y')$. There are now two cases:

- If $y_i^a$ and $y_i^b$ do not connect in $B_{4h}(y') \cap \Sigma$, then take $\gamma_0 \subset B_{5h}(y) \cap \Sigma$ from $y_i^a$ to $y_i^b$ and set $\gamma_0 = \nu_1 \cup \gamma_0 \cup \nu_2$ and $y_i = y_i^a$.

- Otherwise, if $\gamma_0 \subset B_{4h}(y') \cap \Sigma$ connects $y_i^a$ and $y_i^b$, set $\gamma_0 = \nu_1 \cup \gamma_0 \cup \nu_2$ and $y_i = y_i^a$.

In either case, after possibly switching $y$ and $y'$, we get a curve

$$\gamma_a \subset (T_h(\gamma_{y,y'}) \cup B_{5h}(y')) \cap \Sigma$$

with $\partial \gamma_a = \{y_1, y_2\} \subset B_{4h}(y)$ and $y_i \in S_i^a$ for components $S_i^a \neq S_i^b$ of $B_{4h}(y) \cap \Sigma$. This completes the first step.

![Figure 11: Proof of Lemma I.0.11: Step 2: $y_1$ and $y_2$ cannot connect in the half-space $H$ since this would give a point in $\Sigma_{1,2}$ far from $\partial \Sigma_{1,2}$, contradicting Corollary A.10.](image)

Second, we use the maximum principle to restrict the possible curves from $y_1$ to $y_2$; see Figure 11. Set

$$(I.0.12) \quad H = \{x \mid \langle y - y', x - y \rangle > 0\}.$$ 

If $\eta_{1,2} \subset T_h(H) \cap \Sigma$ connects $y_1$ and $y_2$, then $\eta_{1,2} \cup \gamma_a$ bounds a disk $\Sigma_{1,2} \subset \Sigma$. Since $\eta_{1,2} \subset T_h(H)$, we get that $\partial B_{8h}(y') \cap \partial \Sigma_{1,2}$ consists of an odd number
of points in each $S_1^a$ and hence $\partial B_{8h}(y') \cap \Sigma_{1,2}$ contains a curve from $S_1^a$ to $S_2^b$. However, $S_1^a$ and $S_2^b$ are distinct components of $B_{4h}(y) \cap \Sigma$, so that we conclude this curve contains a point

\[(I.0.13) \quad y_{1,2} \in \partial B_{4h}(y) \cap \partial B_{8h}(y') \cap \Sigma_{1,2} .\]

By construction, $\Pi(y_{1,2})$ is in an unbounded component of $\mathbf{R}^2 \setminus T_{h/4}(\Pi(\partial \Sigma_{1,2}))$, contradicting Corollary A.11. This contradiction shows that $y_1$ and $y_2$ cannot be connected in $T_h(H) \cap \Sigma$.

Third, we extend $\gamma_a$. There are two cases:

(A) If $z_b \in H$, Corollary A.10 gives

\[(I.0.14) \quad \tilde{\nu}_1, \tilde{\nu}_2 \subset T_h(\gamma_{y,z_a}) \cap \Sigma \subset T_h(H) \cap \Sigma \]

from $y_1, y_2$ to $z_1, z_2 \in B_{h/4}(z_b)$, respectively.

(B) If $z_b \not\in H$, then fix $z_c \in \partial B_{20h}(y) \cap \Pi(\partial H)$ on the same side of $\Pi(y,y')$ as $\Pi(z_b)$ and fix $z_d \in \partial B_{10h}(z_c) \setminus H$ with $\gamma_{z_c,z_d}$ orthogonal to $\partial H$ (so the four points $\Pi(y'), \Pi(y), z_c, z_d$ form a 10 $h$ by 20 $h$ rectangle). Corollary A.10 gives curves

\[(I.0.15) \quad \tilde{\nu}_1, \tilde{\nu}_2 \subset T_h(\gamma_{y,z_c} \cup \gamma_{z_c,z_d} \cup \gamma_{z_d,z_b}) \cap \Sigma \]

from $y_1, y_2$ to $z_1, z_2 \in B_{h/4}(z_b)$, respectively.

In either case, set $\gamma = \tilde{\nu}_1 \cup \gamma_a \cup \tilde{\nu}_2$. Set $q = \partial B_{30h}(y) \cap \gamma_{y,z_b}$ (in (A)) or $q = \partial B_{30h}(y) \cap \gamma_{z_c,z_b}$ (in (B)). By Corollary A.11 as above, $z_1, z_2$ are in distinct components of $B_{8h}(z_b) \cap \Sigma$.

The next result illustrates the main ideas for Theorem I.0.8 in the simpler case where $\Sigma$ is in a slab. Set

$$\beta_3 = \min\{\beta_A, \beta_s, \tan \theta_0/(2 C_g)\};$$

$C_g, \beta_s$ are defined in Lemma I.0.9, $\theta_0$ in (A.3), and $\beta_A$ in (A.6).

**Proposition I.0.16.** Let $\Sigma \subset B_{4r_0} \cap \{|x_3| \leq \beta_3 h\}$ be an embedded minimal disk with $\partial \Sigma \subset \partial B_{4r_0}$ and let $y \in \partial B_{2r_0}$. Suppose that $\Sigma_1, \Sigma_2, \Sigma_3$ are distinct components of $B_{r_0}(y) \cap \Sigma$ and

$$\gamma \subset (B_{r_0} \cup T_h(\gamma_{0,y})) \cap \Sigma$$

is a curve with $\partial \gamma = \{y_1, y_2\}$ where $y_i \in B_{h}(y) \cap \Sigma_i$ and each component of $\gamma \setminus B_{r_0}$ is $h$-almost monotone.

If $\Sigma_3'$ is a component of $B_{5h_{-80h}}(y) \cap \Sigma_3$ for which $y_1, y_2$ are in distinct components of $B_{5h}(y) \setminus \Sigma_3'$, then $\Sigma_3'$ is a graph.
**Proof.** We will suppose that \( \Sigma' \) is not a graph and deduce a contradiction.

Fix a vertical point \( z \in \Sigma' \). Define \( z_0, y_0, y_b \) on the ray \( \overline{0, y} \) by
\[
\begin{align*}
z_0 &= \partial B_{3r_0 - 21h} \cap \overline{0, y}, \\
y_0 &= \partial B_{3r_0 - 10h} \cap \overline{0, y}, \\
y_b &= \partial B_{4r_0} \cap \overline{0, y}.
\end{align*}
\]

Set \( z_b = \partial B_{50h}(z) \cap \gamma_{z, z_0} \). Define the half-space
\[
(I.0.17) \quad H = \{ x \mid \langle x - z_0, z_0 \rangle > 0 \}.
\]

The first step is to find a simple curve
\[
\gamma_3 \subset (B_{r_0 - 20h}(y) \cup T_h(\gamma_{y, y_b})) \cap \Sigma
\]
which can be connected to \( \Sigma' \) in \( B_{r_0 - 20h}(y) \cap \Sigma \), with \( \partial \gamma_3 \subset \partial \Sigma \), such that \( \partial B_{r_0 - 10h}(y) \cap \gamma_3 \) consists of an odd number of points in each of two distinct components of \( H \cap \Sigma \). To do that, we begin by applying Lemma I.0.11 to get \( q \in B_{50h}(z_b) \cap \partial B_{30h}(z) \), distinct components \( S_1, S_2 \) of \( B_{8h}(z_b) \cap \Sigma \) with \( z_i \in B_{h/4}(z_b) \cap S_i \), and a curve
\[
(I.0.18) \quad \gamma_3^* \subset (B_{30h}(z) \cup T_h(\gamma_{q, z_b})) \cap \Sigma, \partial \gamma_3^* = \{ z_1, z_2 \}, \gamma_3^* \cap \Sigma_3' \neq \emptyset.
\]

Corollary A.10 gives \( h \)-almost monotone curves
\[
\nu_1, \nu_2 \subset T_h(\gamma_{z_1, z_0} \cup \gamma_{z_2, z_b}) \cap \Sigma
\]
from \( z_1, z_2 \), respectively, to \( \partial \Sigma \). Then \( \gamma_3 = \nu_1 \cup \gamma_3^* \cup \nu_2 \) extends \( \gamma_3^* \) to \( \partial \Sigma \). Fix points
\[
\begin{align*}
z^+ &\in B_h(y_0) \cap \nu_1, \\
z^- &\in B_h(y_0) \cap \nu_2.
\end{align*}
\]

We will show that \( z^+, z^- \) do not connect in \( H \cap \Sigma \). If \( \eta_+^- \subset H \cap \Sigma \) connects \( z^+ \) and \( z^- \), then \( \eta_+^- \) together with the portion of \( \gamma_3 \) from \( z^+ \) to \( z^- \) bounds a disk \( \Sigma_3^- \subset \Sigma \). Using the almost monotonicity of each \( \nu_i \), we get that \( \partial B_{50h}(z) \cap \partial \Sigma_3^- \) consists of an odd number of points in each \( S_i \). Consequently, a curve \( \sigma_+^- \subset \partial B_{50h}(z) \cap \Sigma_3^- \) connects \( S_1 \) to \( S_2 \) and so \( \sigma_+^- \cap B_{8h}(z_b) \neq \emptyset \). This would contradict Corollary A.11 and we conclude that there are distinct components \( \Sigma_3^+ \) and \( \Sigma_H^1 \) of \( H \cap \Sigma \) with \( z^+ \in \Sigma_3^+ \). Finally, removing any loops in \( \gamma_3 \) (so it is simple) gives the desired curve.

The second step is to find disjoint stable disks
\[
\Gamma_1, \Gamma_2 \subset B_{r_0 - 2h}(y) \setminus \Sigma
\]
with \( \partial \Gamma_i \subset \partial B_{r_0 - 2h}(y) \) and graphical components \( \Gamma_i' \) of \( B_{r_0 - 4h}(y) \cap \Gamma_i \) so that \( \Sigma_3' \) is between \( \Gamma_1', \Gamma_2' \) and \( y_1, y_2, \Sigma_3' \) are each in its own component of
\[
B_{r_0 - 4h}(y) \setminus (\Gamma_1' \cup \Gamma_2').
\]
To achieve this, we will solve two Plateau problems using $\Sigma$ as a barrier and then use the fact that $\Sigma'_3$ separates $y_1, y_2$ near $y$ to get that these are in different components. Let $\Sigma'_1, \Sigma'_2$ be the components of $B_{r_0 - 2h}(y) \cap \Sigma$ with $y_1 \in \Sigma'_1, y_2 \in \Sigma'_2$. By the maximum principle, each of these is a disk. Let $\Sigma_y$ be the component of $B_3(y) \cap \Sigma$ with $y_2 \in \Sigma_y$. Since $y_1 \notin \Sigma_y$, Lemma A.8 gives $y'_2 \in \Sigma_y \setminus \mathcal{N}_{\theta_0}(y)$ with $\theta_0 > 0$ from (A.3). Hence, the vector $y_1 - y'_2$ is nearly orthogonal to the slab, i.e.,

$$ (I.0.19) \quad |\Pi(y'_2 - y_1)| \leq |y'_2 - y_1| \cos \theta_0. $$

Since $\Sigma'_3$ separates $y_1, y_2$ in $B_{3h}(y)$, we get $y_3 = \gamma_{y_1, y_2} \cap \Sigma'_3$. Fix a component $\Omega_1$ of $B_{r_0 - 2h}(y) \setminus \Sigma$ containing a component of $\gamma_{y_1, y_2} \setminus \Sigma$ with exactly one endpoint in $\Sigma'_1$. By [MeYa], we get a stable embedded disk $\Gamma_1 \subset \Omega_1$ with $\partial \Gamma_1 = \partial \Sigma'_1$. Similarly, let $\Omega_2$ be a component of $B_{r_0 - 2h}(y) \setminus (\Sigma \cup \Gamma_1)$ containing a component of $\gamma_{y_1, y_2} \setminus (\Sigma \cup \Gamma_1)$ with exactly one endpoint in $\Sigma'_2$. Again by [MeYa], we get a stable embedded disk $\Gamma_2 \subset \Omega_2$ with $\partial \Gamma_2 = \partial \Sigma'_2$. Since $\partial \Gamma_1, \partial \Gamma_2$ are linked in $\Omega_1, \Omega_2$ with (segments of) $\gamma_{y_1, y_2}$, respectively, we get components $\Gamma'_i$ of $B_{r_0 - 4h}(y) \cap \Gamma_i$ with $z'_1 \in \Gamma'_1 \cap \gamma_{y_1, y_2}$ and $z'_2 \in \Gamma'_2 \cap \gamma_{y_1, y_2}$. By Lemma I.0.9, each $\Gamma'_i$ is a graph of a function $u_i$ with $|\nabla u_i| \leq C \beta_3$. Hence, since $1 + C_\beta \beta_3^2 < 1/\cos^2 \theta_0$, we have

$$ (I.0.20) \quad \Gamma'_i \setminus \{z'_i\} \subset \mathcal{N}_{\theta_0}(z'_i). $$

By (I.0.19), we have $\gamma_{y_1, y_2} \cap \mathcal{N}_{\theta_0}(z'_i) = \emptyset$, so that (I.0.20) implies $\Gamma'_i \cap \gamma_{y_1, y_2} = \{z'_i\}$. In particular, $y_1, y_2, y_3$ are in distinct components of

$$ B_{r_0 - 4h} \setminus (\Gamma'_1 \cup \Gamma'_2). $$

This completes the second step.

Set $\hat{y} = \partial B_{r_0 + 10h} \cap \gamma_{0, y}$. Let $\hat{\gamma}$ be the component of $B_{r_0 + 10h} \cap \gamma$ with $\partial \gamma_0 \cap \hat{\gamma} \neq \emptyset$. Then $\hat{\gamma} = \{\hat{y}_1, \hat{y}_2\}$ with $\hat{y}_i \in B_{h}(\hat{y}) \cap \Sigma'_i$.

The third step is to solve the Plateau problem with $\gamma_3$ together with part of $\partial \Sigma \subset \partial B_{4r_0}$ as the boundary to get a stable disk $\Gamma_3 \subset B_{4r_0} \setminus \Sigma$ passing between $\hat{y}_1, \hat{y}_2$. To do this, note that the curve $\gamma_3$ divides the disk $\Sigma$ into two sub-disks $\Sigma_3^+, \Sigma_3^-$. Let $\Omega^+, \Omega^-$ be the components of $B_{4r_0} \setminus (\Sigma \cup \Gamma_1 \cup \Gamma_2)$ with $\gamma_3 \subset \partial \Omega^+ \cup \partial \Omega^-$. Note that $\Omega^+, \Omega^-$ are mean convex in the sense of [MeYa] since $\partial \Gamma_1 \cup \partial \Gamma_2 \subset \Sigma$ and $\partial \Sigma \subset \partial B_{4r_0}$. Using the first step, we can label $\Omega^+, \Omega^-$ so that the $z^+, z^-$ do not connect in $H \cap \Omega^+$. By [MeYa], we get a stable embedded disk $\Gamma_3 \subset \Omega^+$ with $\partial \Gamma_3 = \partial \Sigma_3^+$. By the almost monotonicity, $\partial B_{r_0 - 10h}(y) \cap \partial \Gamma_3$ consists of an odd number of points in each of $\Sigma_3^+, \Sigma_3^-$. Hence, there is a curve

$$ \gamma_+^- \subset \partial B_{r_0 - 10h}(y) \cap \Gamma_3 $$

from $\Sigma_3^+$ to $\Sigma_3^-$. By construction, $\gamma_+^- \setminus B_{8h}(y_0) \neq \emptyset$. Hence, since

$$ \partial B_{r_0 - 10h}(y) \cap T_h(\partial \Gamma_3) \subset B_{3h}(y_0), $$
Lemma I.0.9 gives \( \hat{z} \in B_{h}(\hat{y}_1) \cap \gamma_- \). By the second step, \( \Gamma_3 \) is between \( \Gamma_1' \) and \( \Gamma_2' \).

Let \( \hat{\Gamma}_3 \) be the component of \( B_{r_{0}+19h} \cap \Gamma_3 \) with \( \hat{z} \in \hat{\Gamma}_3 \). By Lemma I.0.9, \( \hat{\Gamma}_3 \) is a graph. Finally, since \( \hat{\gamma} \subset B_{r_{0}+10h} \) and \( \hat{\Gamma}_3 \) passes between \( \partial \hat{\gamma} \), this forces \( \hat{\Gamma}_3 \) to intersect \( \hat{\gamma} \). This contradiction completes the proof. \( \square \)

**Part II. Estimates for stable annuli with slits**

In this part, we will show that certain stable disks starting off as multi-valued graphs remain the same (see Theorem II.0.21 below). This is needed in Part III when we generalize the results of Part I to when the surface is not anymore in a slab and in Part IV when we show Theorem 0.3.

**Theorem II.0.21.** Given \( \tau > 0 \), there exist \( N_1, \Omega_1, \varepsilon > 0 \) so that the following holds:

Let \( \Sigma \subset B_{R_0} \) be a stable embedded minimal disk with \( \partial \Sigma \subset B_{r_0} \cup \partial B_{R_0} \cup \{x_1 = 0\} \) where \( \partial \Sigma \setminus \partial B_{R_0} \) is connected. If \( \Omega_1 r_0 < 1 < R_0 / \Omega_1 \) and \( \Sigma \) contains an \( N_1 \)-valued graph \( \Sigma_g \) over \( D_1 \setminus D_{r_0} \) with gradient \( \leq \varepsilon \),

\[
\Pi^{-1}(D_{r_0}) \cap \Sigma^M \subset \{ |x_3| \leq \varepsilon r_0 \},
\]

and a curve \( \eta \) connects \( \Sigma_g \) to \( \partial \Sigma \setminus \partial B_{R_0} \) where

\[
\eta \subset \Pi^{-1}(D_{r_0}) \cap \Sigma \setminus \partial B_{R_0},
\]

then \( \Sigma \) contains a 2-valued graph \( \Sigma_d \) over \( D_{R_0} / \Omega_1 \setminus D_{r_0} \) with gradient \( \leq \tau \).

Two analytical results go into the proof of this extension theorem. First, we show that if an almost flat multi-valued graph sits inside a stable disk, then the outward defined intrinsic sector from a curve which is a multi-valued graph over a circle has a subsector which is almost flat (see Corollary II.1.23 below). As the initial multi-valued graph becomes flatter and the number of sheets in it go up, the subsector becomes flatter. The second analytical result that we will need is that in a multi-valued minimal graph the distance between the sheets grows sublinearly (Proposition II.2.12).

After establishing these two facts, the first application (Corollary II.3.1) is to extend the middle sheet as a multi-valued graph. This is done by dividing the initial multi-valued graph (or curve in the graph that is itself a multi-valued graph over the circle) into three parts where the middle sheet is the second part. The idea is then that the first and third parts have subsectors which are almost flat multi-valued graphs and the middle part (which has curvature estimates since it is stable) is sandwiched between the two others. Hence its sector is also almost flat.

The proof of the extension theorem is somewhat more complicated than suggested in the above sketch since we must initially assume a bound for the
ratio between the size of the initial multi-valued graph and how far out it extends. This is because the flatness of the subsector comes from a total curvature estimate which is in terms of this ratio (see (II.1.2)) and can only be made small by looking at a fixed large number of rotations for the graph. This forces us to successively extend the multi-valued graph. The issue is then to make sure that as we move out in the sector and repeat the argument we have essentially not lost sheets. This is taken care of by using the sublinear growth of the separation between the sheets together with the Harnack inequality (Lemma II.3.8) and the maximum principle (Corollary II.3.1). (The maximum principle is used to make sure, as we try to recover sheets after we have moved out that we do not hit the boundary of the disk before we have recovered essentially all of the sheets that we started with.) The last statement is a result from [CM3] to guarantee as we patch together these multi-valued graphs coming from different scales that the surface obtained is still a multi-valued graph over a fixed plane.

Unless otherwise stated in this part, $\Sigma$ will be a stable embedded disk. Let $\gamma \subset \Sigma$ be a simple curve with unit normal $n_\gamma$ and geodesic curvature $k_g$ (with respect to $n_\gamma$). We will always assume that $\gamma'$ does not vanish. Given $R_1 > 0$, we define the intrinsic sector (see Figure 12),

$$S_{R_1}(\gamma) = \bigcup_{x \in \gamma} \gamma_x,$$

where $\gamma_x$ is the (intrinsic) geodesic starting at $x \in \gamma$, of length $R_1$, and initial direction $n_\gamma(x)$. For $0 < r_1 < R_1$, set

$$S_{r_1, R_1}(\gamma) = S_{R_1}(\gamma) \setminus S_{r_1}(\gamma),$$

$$\rho(x) = \text{dist}_{S_{r_1}(\gamma)}(x, \gamma).$$

For example, if $\gamma = \partial D_{r_1} \subset \mathbb{R}^2$ and $n_\gamma(x) = x/|x|$, then $S_{r_1, R_1}$ is the annulus $D_{R_1+r_1} \setminus D_{r_1+r_1}$.

Note that if $k_g > 0$, $S_{R_1}(\gamma) \cap \partial \Sigma = \emptyset$, and there is a simple curve $\gamma^\partial \subset \Sigma$ with $\gamma \subset \gamma^\partial$, $\partial \gamma^\partial \subset \partial \Sigma$, and $\gamma_x \cap \gamma^\partial = \{x\}$ for any $\gamma_x$ as above (see Figure 13),
then the normal exponential map from $\gamma$ (in direction $n_\gamma$) gives a diffeomorphism to $S_{R_1}(\gamma)$. Namely, by the Gauss-Bonnet theorem, an $n$-gon in $\Sigma$ with concave sides and $n$ interior angles $\alpha_i > 0$ has

$$ (n - 2) \pi \geq \sum_{i=1}^{n} \alpha_i - \int k_g \geq \sum_{i=1}^{n} \alpha_i. $$

(II.0.23)

In particular, $n > 2$ always and if $\sum_i \alpha_i > \pi$, then $n > 3$. Fix $x, y \in \gamma$ and geodesics $\gamma_x, \gamma_y$ as above. If $\gamma_x$ had a self-intersection, then it would contain a simple geodesic loop, contradicting (II.0.23). Similarly, if $\gamma_x$ were to intersect $\gamma_y$, then we would get a concave triangle with $\alpha_1 = \alpha_2 = \pi/2$ (since $\gamma_x, \gamma_y$ do not cross $\gamma_\partial$), contradicting (II.0.23).

Note also that $S_{r_1, R_1}(\gamma) = S_{R_1 - r_1}(S_{r_1, r_1}(\gamma))$ for $0 < r_1 < R_1$.

II.1. Almost flat subsectors

We will next show that certain stable sectors contain almost flat subsectors.

**Lemma II.1.1.** Let $\gamma \subset \Sigma$ be a curve with $\text{Length}(\gamma) \leq 3 \pi m r_1$, geodesic curvature $k_g$ satisfying $0 < k_g < 2/r_1$, and

$$ \text{dist}_\Sigma(S_{R_1}(\gamma), \partial \Sigma) \geq r_1/2, $$

where $R_1 > 2r_1$. If there is a simple curve $\gamma^\partial \subset \Sigma$ with $\gamma \subset \gamma^\partial$, $\partial \gamma^\partial \subset \partial \Sigma$, and so

$$ \gamma_x \cap \gamma^\partial = \{x\} \text{ for each } x \in \gamma, $$

then for any $\Omega > 2$ and $t$ satisfying $2r_1 \leq t \leq 3R_1/4$,

$$ \int_{S_{r_1, R_1/\Omega}(\gamma)} |A|^2 \leq C_1 R_1/r_1 + C_2 \frac{m}{\log \Omega}, $$

(II.1.2)

$$ t \int_{\gamma} k_g \leq \text{Length}(\{\rho = t\}) \leq C_3 (m + R_1/r_1) t. $$

(II.1.3)

**Proof.** The boundary of $S_{R_1} = S_{R_1}(\gamma)$ has four pieces:

$$ \gamma, \{\rho = R_1\}, \text{ and the sides } \gamma_a, \gamma_b. $$

Define the functions $\ell(t)$ and $K(t)$ by

$$ \ell(t) = \text{Length}(\{\rho = t\}), $$

(II.1.4)

$$ K(t) = \int_{S_t} |A|^2. $$

(II.1.5)

Since the exponential map is an embedding, an easy calculation gives

$$ \ell'(t) = \int_{\{\rho = t\}} k_g > 0. $$

(II.1.6)
Let $d\mu$ be 1-dimensional Hausdorff measure on the level sets of $\rho$. The Jacobi equation gives

\begin{equation}
\frac{d}{dt}(k_g\, d\mu) = |A|^2/2\, d\mu.
\end{equation}

(II.1.7)

Define $\bar{K}(t)$ to be the integral of $K(t)$, i.e., set

\[ \bar{K}(t) = \int_0^t K(s) \, ds. \]

Integrating (II.1.7) twice, we see that (II.1.6) yields

\[ \ell(t) = \ell(0) + \int_0^t \left( \int_{\gamma} k_g + K(s)/2 \right) \, ds \]

(II.1.8)

This gives the first inequality in (II.1.3). Again by the co-area formula, (II.1.8) gives

\[ R^{-2}_1 \text{Area}(S_{R_1}) = R^{-1}_1 \int_0^{R_1} \ell(t) \leq R^{-1}_1 \text{Length}(\gamma) + \int_{\gamma} k_g/2 + R^{-2}_1 \int_0^{R_1} \bar{K}(t)/2 \]

(II.1.9)

where the last inequality used $k_g < 2/r_1$ on $\gamma$, $\text{Length}(\gamma) \leq 3\pi m r_1$, and $R_1 > 2 r_1$.

Define a function $\psi$ on $S_{R_1}$ by

\[ \psi = \psi(\rho) = 1 - \rho/R_1 \]

and set $d_S = \text{dist}_\Sigma(\cdot, \gamma_a \cup \gamma_b)$. Define functions $\chi_1, \chi_2$ on $S_{R_1}$ by

\begin{align*}
(II.1.10) & \quad \chi_1 = \chi_1(d_S) = \begin{cases} 
  d_S/r_1 & \text{if } 0 \leq d_S \leq r_1, \\
  1 & \text{otherwise},
\end{cases} \\
(II.1.11) & \quad \chi_2 = \chi_2(\rho) = \begin{cases} 
  \rho/r_1 & \text{if } 0 \leq \rho \leq r_1, \\
  1 & \text{otherwise}.
\end{cases}
\end{align*}

Set $\chi = \chi_1 \chi_2$. Using the curvature estimate $|A|^2 \leq C r_1^{-2}$ (by [Sc]) and standard comparison theorems to bound the area of a tubular neighborhood of the boundary, we get

\begin{align*}
(II.1.12) & \quad \text{Area}(S_{R_1} \cap \{\chi < 1\}) \leq \tilde{C} (R_1 r_1 + m r_1^2), \\
(II.1.13) & \quad E(\chi_1) + \int_{S_{R_1} \cap (\chi_1 < 1)} |A|^2 \leq \tilde{C} R_1/r_1, \\
(II.1.14) & \quad E(\chi) + \int_{S_{R_1} \cap (\chi < 1)} |A|^2 \leq \tilde{C} (R_1/r_1 + m).
\end{align*}
Substitution of $\chi \psi$ into the stability inequality, the Cauchy-Schwarz inequality and (II.1.14) give

$$\int |A|^2 \chi^2 \psi^2 \leq \int |\nabla (\chi \psi)|^2 = \int (\chi^2 |\nabla \psi|^2 + 2 \chi \psi \langle \nabla \chi, \nabla \psi \rangle + \psi^2 |\nabla \chi|^2)$$

(II.1.15)

$$\leq 2 \int \chi^2 |\nabla \psi|^2 + 2 \tilde{C} (R_1/r_1 + m).$$

Using (II.1.14) and the co-area formula, we have

$$\int_{R_1}^0 \psi^2(t) K'(t) = \int_{S_{R_1}} |A|^2 \psi^2 \leq \int |A|^2 \chi^2 \psi^2 + \tilde{C} (R_1/r_1 + m).$$

(II.1.16)

Integration by parts twice in (II.1.16), (II.1.15) gives

$$2 R_1^{-2} \int_0^{R_1} \tilde{K}(t) = \int_0^{R_1} \tilde{K}(t) (\psi^2)' = - \int_0^{R_1} \tilde{K}(t) (\psi^2)' = \int_0^{R_1} \psi^2 K'(t) \leq 3 \tilde{C} (R_1/r_1 + m) + 2 R_1^{-2} \int_0^{R_1} \ell(t).$$

(II.1.17)

Note that all integrals in (II.1.17) are in one variable and there is a slight abuse of notation with regard to $\psi$ as a function on both $[0, R_1]$ and $S_{R_1}$. Substitution of (II.1.9), (II.1.17) gives

$$4 R_1^{-2} \int_0^{R_1} \ell(t) \leq 24 \pi m + 3 \tilde{C} (R_1/r_1 + m) + 2 R_1^{-2} \int_0^{R_1} \ell(t).$$

(II.1.18)

In particular, (II.1.18) gives

$$R_1^{-2} \text{Area}(S_{R_1}) \leq C (R_1/r_1 + m).$$

(II.1.19)

Since $\ell(t)$ is monotone increasing (by (II.1.6)), (II.1.19) gives the second inequality in (II.1.3) for $t = 3 R_1/4$. Since the above argument applies with $R_1$ replaced by $t$ where $2 r_1 < t < R_1$, we get (II.1.3) for $2 r_1 \leq t \leq 3 R_1/4$.

To complete the proof, we will use the stability inequality together with the logarithmic cutoff trick to take advantage of the quadratic area growth. Define a cutoff function $\psi_1$ by

$$\psi_1 = \psi_1(\rho) = \begin{cases} \log(\rho/r_1)/\log \Omega & \text{on } S_{r_1, \Omega r_1}, \\ 1 & \text{on } S_{\Omega r_1, R_1/\Omega}, \\ -\log(\rho/R_1)/\log \Omega & \text{on } S_{R_1/\Omega, R_1}, \\ 0 & \text{otherwise}. \end{cases}$$

(II.1.20)

Using (II.1.3) and (II.1.19), we get

$$E(\psi_1) \leq C (m + R_1/r_1)/\log \Omega.$$  

(II.1.21)
As in (II.1.15), we apply the stability inequality to $\chi_1\psi_1$ to get
\begin{equation}
\int |A|^2 \chi_1^2 \psi_1^2 \leq 2E(\psi_1) + 2E(\chi_1) \leq 2C(m + R_1/r_1)/\log \Omega + 2\tilde{C} R_1/r_1. \tag{II.1.22}
\end{equation}
Combination of (II.1.13) and (II.1.22) completes the proof. \hfill \square

The next corollary uses Lemma II.1.1 to show that large stable sectors have almost flat subsectors:

**Corollary II.1.23.** Given $\omega > 8, 1 > \varepsilon > 0$, there exist $m_1, \Omega_1$ so that the following holds:

Suppose $\gamma \subset B_{2r_1} \cap \Sigma$ is a curve with $1/(2r_1) < k_g < 2/r_1$, $\text{Length}(\gamma) = 32\pi m_1 r_1$, $\text{dist}_\Sigma(S_{\Omega_1^2 \omega r_1}(\gamma), \partial \Sigma) \geq r_1/2$. If there is a simple curve $\gamma^0 \subset \Sigma$ with $\gamma \subset \gamma^0$, $\partial \gamma^0 \subset \partial \Sigma$, and

$$\gamma_x \cap \gamma^0 = \{x\} \text{ for each } x \in \gamma,$$

then (after a rotation of $\mathbb{R}^3$) $S_{\Omega_1^2 \omega r_1}(\gamma)$ contains a 2-valued graph $\Sigma_d$ over $D_2 \omega_1 r_1 \setminus D_{\Omega_1^2 \omega r_1/2}$ with gradient $\leq \varepsilon/2$, $|A| \leq \varepsilon/(2r_1)$, and $\text{dist}_{S_{\Omega_1^2 \omega r_1}(\gamma)}(\gamma, \Sigma_d) < 2\Omega_1 r_1$.

**Proof.** We will choose $\Omega_1 > 12$ and then set $m_1 = \omega \Omega_1^2 \log \Omega_1$. By Lemma II.1.1 (with $\Omega = \Omega_1/6$, $R_1 = \Omega_1^2 \omega r_1$, and $m = 32 m_1/3$),

\begin{equation}
\int_{S_{\Omega_1^2 \omega r_1}(\gamma)} |A|^2 \leq C(\Omega_1^2 \omega + m_1/\log \Omega_1) = 2C m_1/\log \Omega_1. \tag{II.1.24}
\end{equation}

Fix $m_1$ disjoint curves $\gamma_1, \ldots, \gamma_{m_1} \subset \gamma$ with $\text{Length}(\gamma_i) = 32\pi r_1$. By (II.1.24) and since the $S_{\Omega_1^2 \omega r_1}(\gamma_i)$ are pairwise disjoint, there exists $\gamma_i$ with

\begin{equation}
\int_{S_{\Omega_1^2 \omega r_1}(\gamma_i)} |A|^2 \leq 2C/\log \Omega_1. \tag{II.1.25}
\end{equation}

To deduce the corollary from (II.1.25) we need a few standard facts. First, define a map

$$\Phi: [0, \Omega_1^2 \omega r_1] \times [0, \text{Length}(\gamma)] \to \Sigma$$

by $\Phi(\rho, x) = \gamma_x(\rho)$. By the Riccati comparison argument (using $K_{\Sigma} \leq 0$ and $k_g > 1/(2r_1)$ on $\gamma$),

\begin{equation}
\Phi \text{ is distance nondecreasing and } k_g > \frac{1}{\rho + 2r_1}. \tag{II.1.26}
\end{equation}

Second, let $\gamma_{i/2} \subset \gamma_i$ be the subcurve of length $16\pi r_1$ with $\text{dist}_\Sigma(\gamma_{i/2}, \partial \gamma_i) = 8\pi r_1$. Since $k_g > 1/(2r_1)$ on $\gamma$, we have $\int_{\gamma_{i/2}} k_g > 8\pi$. By (II.1.7),

$$\int_{S_{\Omega_1^2 \omega r_1}(\gamma_{i/2}) \cap \{\rho = t\}} k_g.$$
is a monotone nondecreasing function of \( t \). In particular, we can choose a curve \( \tilde{\gamma} \subset \gamma_{i/2} \) with

\[
(\text{II.1.27}) 
\int_{S_{\Omega_1 r_1}(\tilde{\gamma}) \cap \{ \rho = \Omega_1 r_1/3 \}} k_g = 8 \pi .
\]

Set \( S = S_{\Omega_1 r_1/3, \Omega_1} \omega r_1(\tilde{\gamma}) \) and \( \hat{\gamma} = S \cap \{ \rho = \Omega_1 r_1/3 \} \).

Third, by the Gauss-Bonnet theorem, (II.1.25), and (II.1.27) (for \( \Omega_1 \) large),

\[
(\text{II.1.28}) 
8 \pi \leq \int_{S \cap \{ \rho = t \}} k_g \leq 8 \pi + \int_{S} |A|^2/2 \leq 8 \pi + C/\log \Omega_1 \leq 9 \pi .
\]

Note also that, by (II.1.26) and (II.1.28),

\[
\text{Length}(S \cap \{ \rho = t \}) \leq 9 \pi (t + 2 r_1) \leq 14 \pi t .
\]

Finally, observe that, by stability, (II.1.25), and by (II.1.26), the mean value theorem gives for \( y \in S \)

\[
(\text{II.1.29}) 
\sup_{B_\rho(y)} |A|^2 \leq C_1 \rho^{-2}(y)/\log \Omega_1 .
\]

Integrating (II.1.29) along rays and level sets of \( \rho \), we get

\[
(\text{II.1.30}) 
\max_{x, y \in S} \text{dist}_{S^2}(n(x), n(y)) \leq C_2 (\log \omega + 1)/\sqrt{\log \Omega_1} .
\]

We can now combine these facts to prove the corollary. Choose \( \Omega_1 \) so that

\[
C_2 (\log \omega + 1)/\sqrt{\log \Omega_1} < C_3 \varepsilon .
\]

For \( C_3 \) small, after rotating \( \mathbb{R}^2 \), \( S \) is locally a graph over \( \{ x_3 = 0 \} \) with gradient \( \leq \varepsilon/2 \). Since \( \tilde{\gamma} \subset B_2 r_1 \) and \( \Omega_1 > 12 \),

\[
\tilde{\gamma} \subset B_2 r_1 + \Omega_1 r_1/3 \subset B_{\Omega_1 r_1/2} .
\]

Choosing \( \Omega_1 \) even larger and combining (II.1.26), (II.1.28), (II.1.29), and (II.1.30), we see that (the orthogonal projection) \( \Pi(\tilde{\gamma}) \) is a convex planar curve with total curvature at least \( 7 \pi \), so that its Gauss map covers \( S^1 \) three times.

Given \( x \in \tilde{\gamma} \), set \( \tilde{\gamma}_x = S \cap \gamma_x \). By (II.1.29), \( \tilde{\gamma}_x \) has total (extrinsic geodesic) curvature at most

\[
C_2 \log \omega/\sqrt{\log \Omega_1} < C_3 \varepsilon
\]

and hence \( \tilde{\gamma}_x \) lies in a narrow cone centered on its tangent ray at \( \tilde{x} = \tilde{\gamma}_x \cap \tilde{\gamma} \). For \( C_3 \) small, this implies that \( \tilde{\gamma}_x \) does not rotate and

\[
(\text{II.1.31}) 
|\Pi(\tilde{x}) - \Pi(\tilde{\gamma}_x \cap \{ \rho = t \})| \geq 9 (t - \Omega_1 r_1/3)/10 .
\]

Hence, \( \Pi(\partial \tilde{\gamma}_x \setminus \{ \tilde{x} \}) \notin D_{2\omega \Omega_1 r_1} \) which gives \( \Sigma_d \) and also \( \text{dist}_{S_{\Omega_1 r_1}(\gamma)}(\gamma, \Sigma_d) < 2 \Omega_1 r_1 \). \( \square \)
Remark II.1.32. For convenience, we assumed that $k_g < 2/r_1$ in Corollary II.1.23. This was used only to apply Lemma II.1.1 and it was used there only to bound $\int_{\gamma} k_g$ in (II.1.9).

Recall that a domain $\Omega$ is $1/2$-stable if and only if, for all $\phi \in C^0_0(\Omega)$, there exists the $1/2$-stability inequality:

\[(II.1.33) \quad 1/2 \int |A|^2 \phi^2 \leq \int |\nabla \phi|^2.\]

Note that the interior curvature estimate of [Sc] extends to $1/2$-stable surfaces.

In light of Remark II.1.32, it is easy to get the following analog of Corollary II.1.23:

**Corollary II.1.34.** Given $\omega > 8, 1 > \varepsilon > 0, C_0, \text{ and } N$, there exist $m_1, \Omega_1$ so that the following holds:

Suppose that $\Sigma$ is an embedded minimal disk and $\gamma \subset \partial B_{r_1}(y) \subset \Sigma$ is a curve with

$$\int_{\gamma} k_g < C_0 m_1 \text{ and } \text{Length}(\gamma) = m_1 r_1.$$ If $T_{r_1/8}(S_{\Omega_1^0 \omega r_1}(\gamma))$ is 1/2-stable, then (after rotating $\mathbb{R}^3$) $S_{\Omega_1^0 \omega r_1}(\gamma)$ contains an $N$-valued graph $\Sigma_N$ over $D_{\omega \Omega_1 r_1} \setminus D_{\Omega_1 r_1}$ with gradient $\leq \varepsilon$, $|A| \leq \varepsilon/r$, and

$$\text{dist}_{S_{\Omega_1^0 \omega r_1}(\gamma)}(\gamma, \Sigma_N) < 4 \Omega_1 r_1.$$ Note that, in Corollary II.1.34, both $k_g \geq 1/r_1$ and the injectivity of the exponential map follow immediately from comparison theorems.

**II.2. The sublinear growth**

This section gives an elementary gradient estimate for multi-valued minimal graphs which is applied to show that the separation between the sheets of certain minimal graphs grows sublinearly; see Figure 14. The example to keep in mind is the portion of a (rescaled) helicoid in a slab between two cylinders about the vertical axis. This gives (two) multi-valued graphs over an annulus; removal of a vertical half-plane through the axis cuts these into sheets which remain a bounded distance apart.

The next lemma and corollary construct the cutoff function needed in our gradient estimate.

**Lemma II.2.1.** Given $N > 36/(1 - e^{-1/3})^2$, there exists a function $0 \leq \phi \leq 1$ on $\mathcal{P}$ with $E(\phi) \leq 4\pi/\log N$ and

\[(II.2.2) \quad \phi = \begin{cases} 1 & \text{if } R/e \leq \rho \leq e R \text{ and } |\theta| \leq 3\pi, \\ 0 & \text{if } \rho \leq e^{-N} R \text{ or } e^N R \leq \rho \text{ or } |\theta| \geq \pi N. \end{cases}\]
GRAPHICAL OFF THE AXIS

Figure 14: The sublinear growth of the separation $u$ of the multi-valued graph $\Sigma$: $u(2R) \leq 2^\alpha u(R)$ with $\alpha < 1$.

**Proof.** After rescaling, we may assume that $R = 1$. Since energy is conformally invariant on surfaces, composing with $z^3 N$ implies that (II.2.2) is equivalent to $E(\phi) \leq 4 \pi / \log N$ and

$$
\phi = \begin{cases} 
1 & \text{if } |\log \rho| < 1/(3N) \text{ and } |\theta| \leq \pi/N, \\
0 & \text{if } |\log \rho| > 1/3 \text{ or } |\theta| \geq \pi/3.
\end{cases}
$$

This is achieved (with $E(\phi) = 2 \pi / \log[N(1 - e^{-1/3})/6]$) by setting

$$
\phi = \begin{cases} 
1 & \text{on } B_{6/N}(1,0), \\
\frac{\log[\text{dist}_P((1,0),\cdot)/6]}{\log[N(1-e^{-1/3})/6]} & \text{on } B_{1-e^{-1/3}}(1,0) \setminus B_{6/N}(1,0), \\
0 & \text{otherwise}.
\end{cases}
$$

Given an $N$-valued graph $\Sigma$, let $\Sigma^{\theta_1,\theta_2}_{r_3,r_4} \subset \Sigma$ be the subgraph (cf. (0.1)) over

$$
\{(\rho, \theta) \mid r_3 \leq \rho \leq r_4, \theta_1 \leq \theta \leq \theta_2\}.
$$

Transplanting the cutoff function from Lemma II.2.1 to a multi-valued graph gives the next corollary:

**Corollary II.2.6.** Given $\varepsilon_0, \tau > 0$, there exists $N > 0$ so if $\Sigma \subset \mathbb{R}^3$ is an $N$-valued graph over $D_{e^N} \setminus D_{e^{-N}}$ with gradient $\leq \tau$, then there is a cutoff function $0 \leq \phi \leq 1$ on $\Sigma$ with $E(\phi) \leq \varepsilon_0$, $\phi|_{\partial \Sigma} = 0$, and

$$
\phi \equiv 1 \text{ on } \Sigma_{R/2.5R/2}.
$$

**Proof.** Since $\Sigma_{R/2.5R/2} \subset \Sigma_{R/e,eR}$ and the projection from $\Sigma$ to $\mathcal{P}$ is bi-Lipschitz with bi-Lipschitz constant bounded by $\sqrt{1 + \tau^2}$, the corollary follows from Lemma II.2.1. \qed
If \( u > 0 \) is a solution of the Jacobi equation \( \Delta u = -|A|^2 u \) on \( \Sigma \), then \( w = \log u \) satisfies

\[
\Delta w = -|\nabla w|^2 - |A|^2.
\]

(II.2.8)

The Bochner formula, (II.2.8), \( K_\Sigma = -|A|^2/2 \), and the Cauchy-Schwarz inequality give

\[
\Delta |\nabla w|^2 = 2 |\text{Hess}_w|^2 + 2 \langle \nabla w, \Delta \nabla w \rangle - |A|^2 |\nabla w|^2 \\
\geq 2 |\text{Hess}_w|^2 - 4 |\nabla w|^2 |\text{Hess}_w| - 4 |\nabla w| |A| |\nabla A| - |A|^2 |\nabla w|^2
\]

(II.2.9)

Since the Jacobi equation is the linearization of the minimal graph equation over \( \Sigma \), analogs of (II.2.8) and (II.2.9) hold for solutions of the minimal graph equation over \( \Sigma \). In particular, standard calculations give the following analog of (II.2.8):

**Lemma II.2.10.** There exists \( \delta_g > 0 \) so that if \( \Sigma \) is minimal and \( u \) is a positive solution of the minimal graph equation over \( \Sigma \) (i.e., \( \{x + u(x) n_\Sigma(x) \mid x \in \Sigma\} \) is minimal) with

\[
|\nabla u| + |u| |A| \leq \delta_g,
\]

then \( w = \log u \) satisfies, on \( \Sigma \),

\[
\Delta w = -|\nabla w|^2 + \text{div}(a \nabla w) + \langle \nabla w, a \nabla w \rangle + \langle b, \nabla w \rangle + (c - 1)|A|^2,
\]

for functions \( a_{ij}, b_j, c \) on \( \Sigma \) with \( |a|, |c| \leq 3 |A| |u| + |\nabla u| \) and \( |b| \leq 2 |A| |\nabla u| \).

The following gives an improved gradient estimate, and consequently an improved bound for the growth of the separation between the sheets, for multi-valued minimal graphs:

**Proposition II.2.12.** Given \( \alpha > 0 \), there exist \( \delta_p > 0, N_g > 5 \) so that the following holds:

If \( \Sigma \) is an \( N_g \)-valued minimal graph over \( D_{e^{N_g} R} \setminus D_{e^{-N_g} R} \) with gradient \( \leq 1 \) and \( 0 < u < \delta_p R \) is a solution of the minimal graph equation over \( \Sigma \) with \( |\nabla u| \leq 1 \), then for \( R \leq s \leq 2R \)

\[
\sup_{\Sigma_{s/R : R}} |A| + \sup_{\Sigma_{s/R : R}} |\nabla u|/u \leq \alpha/(4R),
\]

(II.2.13)

\[
\sup_{\Sigma_{s/R : R}} u \leq (s/R)^\alpha \sup_{\Sigma_{s/R : R}} u.
\]

(II.2.14)

**Proof.** Fix \( \varepsilon_E > 0 \) (to be chosen depending only on \( \alpha \)). Corollary II.2.6 gives \( N \) (depending only on \( \varepsilon_E \)) and a function \( 0 \leq \phi \leq 1 \) with compact support on \( \Sigma_{s/N, sN} \) given by

\[
E(\phi) \leq \varepsilon_E \quad \text{and} \quad \phi \equiv 1 \text{ on } \Sigma_{R/2, 5R/2}.
\]

(II.2.15)
Set $N_g = N + 1$, so that $\text{dist}(\Sigma e^{-N R} \cup \partial \Sigma) > e^{-N} R/2$ and hence $|A| \leq Ce^{-N}/R$ on $\Sigma e^{-N R}$. Now fix $x \in \Sigma^{0.2 \pi}$. When we substitute $\phi$ into the stability inequality, (II.2.15) bounds the total second fundamental form of $\Sigma^{0.3 \pi}$ by $\varepsilon E$. Hence, by elliptic estimates for the minimal graph equation, (II.2.16)

$$\sup_{B_{3R/8}(x)} (R^2 |\nabla A\Sigma|^2 + |A\Sigma|^2) \leq C\varepsilon R^{-2}.$$  

Since $\Sigma$ and the graph of $u$ are (locally) graphs with bounded gradient, it is easy to see that (II.2.17)

$$\sup_{\Sigma e^{-N R}} |\nabla u| \leq C e^N \sup_{\Sigma} |u|/R \leq C e^N \delta_p.$$  

Applying Stokes’ theorem to $\text{div}(\phi^2 \nabla w - \phi^2 a \nabla w)$ and using the absorbing inequality, we see that (II.2.18)

$$\int_{B_{R/2}} |\nabla w|^2 \leq \int \phi^2 |\nabla w|^2 \leq C E(\phi) \leq C \varepsilon E.$$  

When we combine (II.2.11) and (II.2.16), an easy calculation (as in (II.2.9)) shows that on $B_{3R/8}(x)$ (II.2.19)

$$\Delta |\nabla w|^2 \geq -C |\nabla w|^4 - C \varepsilon R^{-2} |\nabla w|^2 - C \varepsilon R^{-4}.$$  

By the rescaling argument of [CiSc] (and by the meanvalue inequality), (II.2.18) and (II.2.19) imply a pointwise bound for $|\nabla w|^2$ on $B_{R/4}(x)$; combining this with (II.2.16) gives (II.2.13) for $\varepsilon E$ small. Integrating (II.2.13) and using the elementary inequality

$$(s - R)/R \leq 2 \log(s/R),$$

we get (II.2.14).

\section*{II.3. Extending multi-valued graphs in stable disks}

Throughout this section $\Sigma \subset B_{R_0}$ is a stable embedded minimal disk with

$$\partial \Sigma \subset B_{R_0} \cup \partial B_{R_0} \cup \{x_1 = 0\}$$

and $\partial \Sigma \setminus \partial B_{R_0}$ connected. Fix a constant $\tau_k$ with $0 < \tau_k < 1/4$ so that if $\Sigma_g$ is a multi-valued minimal graph over $D_{2R} \setminus D_{R/2}$ with gradient $\leq \tau_k$, then $\Pi^{-1}(\partial D_R) \cap \Sigma_g$ has geodesic curvature $k_g$ satisfying

$$1/(2 R) < k_g < 2/R$$

(with respect to the outward normal).
The next corollary shows that for certain such $\Sigma$ containing multi-valued graphs, the middle sheet $\Sigma^M$ extends to a larger scale. The main point is to apply Corollary II.1.23 to get two 2-valued graphs on a larger scale with $\Sigma^M$ pinched between them. We first use the convex hull property to construct the curves $\gamma^2_j$ needed for Corollary II.1.23.

**Corollary II.3.1.** Given $\omega, m > 1$, $1/4 \geq \varepsilon > 0$, there exist $\Omega_1, m_0, \delta$ so that for $r_0, r_2, R_2, R_0$ with $4 \Omega_1 r_0 \leq 4 \Omega_1 r_2 < R_2 < R_0/(4 \Omega_1 \omega)$ the following holds:

Let $\Sigma_g \subset \Sigma$ be an $m_0$-valued graph over $D_{R_2} \setminus D_{r_2}$ with gradient $\leq \tau_k$, separation between the top and bottom sheets $\leq \delta R_2$ over $\partial D_{R_2}$, and

$$\Pi^{-1}(D_{r_2}) \cap \Sigma_g \subset \{|x_3| \leq r_2/2\}.$$ 

If a curve $\eta \subset \Pi^{-1}(D_{r_2}) \cap \Sigma \setminus \partial B_{R_0}$ connects $\Sigma_g$ to $\partial \Sigma \setminus \partial B_{R_0},$ then $\Sigma^M$ extends to an $m$-valued graph over $D_\omega R_2 \setminus D_{r_2}$ with gradient $\leq 1$ and $|A| \leq \varepsilon/r$ over $D_\omega R_2 \setminus D_{r_2}$.

**Proof.** First, we set up the notation. Let $\Omega_1, m_1 > 1$ be given by Corollary II.1.23. Assume that $\Omega_1^2 \omega, m, m_1 \in \mathbb{Z}$. Set

$$m_0 = 24 \Omega_1^2 \omega + 32 m_1 + m + 1,$$

$$\gamma = \Pi^{-1}(\partial D_{R_2/\Omega_1}) \cap \Sigma_g.$$ 

Since $\Pi^{-1}(D_{r_2}) \cap \Sigma_g \subset \{|x_3| \leq r_2/2\}$, the gradient bound gives, for $r_2 \leq R \leq R_2$,

$$\max_{\Pi^{-1}(\partial D_\omega) \cap \Sigma_g} |x_3| \leq r_2/2 + \tau_k (R - r_2) \leq R/2,$$

so that $\gamma \subset B_{2R_2/\Omega_1}$. By the definition of $\tau_k$, we have $\Omega_1/(2 R_2) < k_g < 2 \Omega_1/R_2$ on $\gamma$. Arguing on part of $\Sigma$ itself, by the convex hull property, we see that there are $m_0$ components of $\gamma \cap \{x_1 \geq R_2/(2 \Omega_1)\}$ which are in distinct components of $\Sigma \cap \{x_1 \geq R_2/(2 \Omega_1)\}$. Hence, see Figure 15, there are $m_0$ distinct $y_i \in \gamma$ and (nodal) curves

$$\sigma_0, \ldots, \sigma_{m_0 - 1} \subset \{x_1 = R_2/\Omega_1\} \cap \Sigma$$

with $\partial \sigma_i \equiv \{y_i, z_i\}, \sigma_i \cap \gamma = \{y_i\}, z_i \in \partial \Sigma \cap \{x_1 = R_2/\Omega_1\} \subset \partial B_{R_0}$, and for $i \neq j$

$$\text{dist}_\Sigma(\sigma_i, \sigma_j) > R_2/\Omega_1.$$ 

Order the $\sigma_i$’s using the ordering of the $y_i$’s in $\gamma$ and set $i_1 = 0$, $i_2 = 8 \Omega_1^2 \omega + 16 m_1$, $i_3 = 16 \Omega_1^2 \omega + 16 m_1 + m$, and $i_4 = m_0 - 1$. Let $\gamma_1, \gamma_2, \gamma_3 \subset \gamma$ be the curves
Consequently, (II.3.6) implies that \( \gamma \) satisfies (II.3.4)

\[
\text{Since } \Sigma \text{ is a disk, let } \Sigma(\sigma_1, \sigma_2) \text{ be the union of } \sigma_1, \sigma_2, \text{ and the curve in } \gamma \text{ from } y_{k_1} \text{ to } y_{k_2}.
\]

Hence, \( \gamma_1, \gamma_2, \gamma_3 \subset \gamma \) are 16 \( m_1 \)-, \( m_- \), \( m_1 \)-valued graphs, respectively, with \( \gamma_2 \) centered on \( \Sigma^M \), each \( \gamma_j \) between \( y_{j_1} \) and \( y_{j_1+1} \), and for \( j = 1, 2, 3 \)

\[
(\text{II.3.4}) \quad \min_{\{k \mid y_k \in \gamma_j\}} \{|i_j - k|, |i_{j+1} - k|\} \geq 4 \Omega_1^2 \omega.
\]

Next, we construct the curves \( \gamma_j^\delta \) needed to apply Corollary II.1.23 to each \( \gamma_j \). We will also use (II.3.3) and (II.3.4) to separate the \( \gamma_j \)'s. For \( k_1 < k_2 \), let \( \gamma(k_1, k_2) \subset \Sigma \) be the union of \( \sigma_{k_1}, \sigma_{k_2} \), and the curve in \( \gamma \) from \( y_{k_1} \) to \( y_{k_2} \).

Since \( \Sigma \) is a disk, \( \partial \gamma(k_1, k_2) \subset \partial \Sigma \), and \( \partial \Sigma \setminus \partial B_{R_\omega} \) is connected, one component \( \Sigma(k_1, k_2) \) of \( \Sigma \setminus \gamma(k_1, k_2) \) has \( \partial \Sigma(k_1, k_2) \cap \partial \Sigma \subset \partial B_{R_\omega} \). By the fact that the \( \sigma_i \)'s do not cross \( \eta \), it is easy to see that \( n_\gamma \) points into \( \Sigma(k_1, k_2) \) and

\[
(\text{II.3.5}) \quad \Sigma(j_1, j_2) \cap \Sigma(k_1, k_2) = \Sigma(\max\{j_1, k_1\}, \min\{j_2, k_2\}),
\]

where, by convention, \( \Sigma(k_1, k_2) = \emptyset \) if \( k_1 > k_2 \). Set \( \gamma_j^\delta = \gamma(i_j, i_{j+1}) \) and note that \( \gamma_j \subset \gamma_j^\delta \) and \( \partial \gamma_j \subset \partial \Sigma \). Set \( S_j = S_{\Omega_1 \omega} R_\omega(\gamma_j) \). By (II.3.4) and (II.3.5), any curve \( \tilde{\eta} \subset \Sigma(i_j, i_{j+1}) \) from \( \gamma_j \) to \( \gamma_j^\delta \setminus (\gamma \cup \partial B_{R_\omega}) \) hits at least \( 4 \Omega_1^2 \omega \) of the \( \sigma_i \)'s and so, by (II.3.3), Length(\( \tilde{\eta} \)) > \( 2 \Omega_1 \omega R_2 \). Combining this with \( R_0 > 4 \Omega_1 \omega R_2 \), we get

\[
(\text{II.3.6}) \quad \text{dist}_{\Sigma(i_j, i_{j+1})}(\gamma_j, \partial \Sigma(i_j, i_{j+1}) \setminus \gamma_j) > 2 \Omega_1 \omega R_2.
\]

Fix \( x \in \gamma_j \) and \( \gamma_x \) (the geodesic normal to \( \gamma_j \) at \( x \) and of length \( \Omega_1 \omega R_2 \)). By (II.0.23), the first point (after \( x \)) where \( \gamma_x \) hits \( \partial \Sigma(i_j, i_{j+1}) \) cannot be in \( \gamma \).

Consequently, (II.3.6) implies that \( \gamma_x \subset \Sigma(i_j, i_{j+1}) \) and so \( \gamma_x \cap \gamma_j^\delta = \{x\} \) and \( \gamma_j^\delta \) separates \( S_j \) from \( S_k \cup T_{R_2/(\Omega_1 \omega)} \)(\( \partial \Sigma \)) for \( j \neq k \).

The rest of the proof (see Figure 16) is to sandwich \( \Sigma^M \) between two graphs that will be given by Corollary II.1.23 and then deduce from stability that \( \Sigma^M \) itself extends to a graph. Namely, applying Corollary II.1.23 to

Figure 15: The proof of Corollary II.3.1: The nodal curves.
Let $\Omega$ and each other with separation sublinearly:

$$\delta$$

We will set $\delta$ and separations between the top and bottom sheets of $\Sigma$. Finally, since $\Sigma$ is a graph with gradient $\leq 3/4$ and $|A| \leq \epsilon/(2r)$, we get that $\Sigma_2$ is a graph with gradient $\leq 1$ and $|A| \leq \epsilon/r$ (cf. Lemma I.0.9).

Combining this and Proposition II.2.12, $\Sigma^M$ extends with separation growing sublinearly:

**Corollary II.3.7.** Given $1/4 \geq \epsilon > 0$, there exist $\Omega_0, m_0, \delta_0 > 0$ so that for any $r_0, r_2, R_2, R_0$ with $\Omega_0 r_0 \leq \Omega_0 r_2 < R_2 < R_0/\Omega_0$ the following holds:

Let $\Sigma_g \subset \Sigma$ be an $m_0$-valued graph over $D_{R_2} \setminus D_{r_2}$ with gradient $\leq \tau_1 \leq \tau_k$, separations between the top and bottom sheets of $\Sigma^M(\subset \Sigma_g)$ and $\Sigma_g$ are $\leq \delta_1 R_2$ and $\leq \delta_2 R_2$, respectively, over $\partial D_{R_2}$, and

$$\Pi^{-1}(D_{r_2}) \cap \Sigma_g \subset \{|x_3| \leq r_2/2\}.$$  

If a curve $\eta \subset \Pi^{-1}(D_{r_2}) \cap \Sigma \setminus \partial B_{R_0}$ connects $\Sigma_g$ to

$$\partial \Sigma \setminus \partial B_{R_0},$$

then $\Sigma^M$ extends as a graph over $D_{r_2} \setminus D_{R_2}$ with gradient $\leq \tau_1 + 3 \epsilon$, $|A| \leq \epsilon/r$ over $D_{R_2} \setminus D_{R_2}$, and, for $R_2 < s \leq 2 R_2$, separation $\leq (s/R_2)^{1/2} \delta_1 R_2$ over $D_s \setminus D_{R_2}$.

**Proof.** Let $\delta_0 > 0, N_0 > 5$ be given by Proposition II.2.12 with $\alpha = 1/2$.

Let $\Omega_1, m_0, \delta > 0$ be given by Corollary II.3.1 with $m = N_0 + 3$ and $\omega = 2 e^{N_0}$. We will set $\delta_0 = \delta_0(\delta, \delta_0, N_0)$ with $\delta > \delta_0 > 0$ and $\Omega_0 = 4 \Omega_1 e^{N_0}$. 

Figure 16: The proof of Corollary II.3.1: Sandwiching between two graphical pieces.
By Corollary II.3.1, $\Sigma^M$ extends to a graph

$$\Sigma^M_{r_2, 2e^{N_0} R_2}$$

of a function $v$ with $|\nabla v| \leq 1$ and $|A| \leq \varepsilon/r$ over $D_{2e^{N_0} R_2} \setminus D_{R_2}$. Integrating

$$|\nabla |\nabla v|| \leq |A|(1 + |\nabla v|^2)^{3/2} \leq 2^{3/2} \varepsilon/r,$$

we get that $|\nabla v| \leq \tau_1 + 4 \varepsilon \log 2 \leq \tau_1 + 3 \varepsilon$ on $D_{2e^{N_0} R_2} \setminus D_{R_2}$.

For $\delta_0 = \delta_0(N_0, \delta_0) > 0$, writing $\Sigma$ as a graph over itself and using the Harnack inequality, we get a solution $0 < u < \delta_0 R_2$ of the minimal graph equation on an $N_0$-valued graph over $D_{2e^{N_0} R_2} \setminus D_{e^{N_0} R_2}$. Applying Proposition II.2.12 to $u$ gives the last claim.

The next lemma uses the Harnack inequality to show that if $\Sigma^M$ extends with small separation, then so do the other sheets. The only complication is to keep track of $\partial \Sigma$.

**Lemma II.3.8.** Given $N \in \mathbb{Z}_+$, there exist $C_3, \delta_2 > 0$ so that for $r_0 \leq s < R_0/8$ the following holds:

Let $\Sigma_{g} \subset \Sigma \cap \{x_3 \leq 2s\}$ be an $N$-valued graph over $D_{2s} \setminus D_s$. If a curve $\eta \subset \Pi^{-1}(D_s) \cap \Sigma \setminus \partial B_{R_0}$ connects $\Sigma_{g}$ to $\partial \Sigma \setminus \partial B_{R_0}$, and $\Sigma^M$ extends graphically over $D_{4s} \setminus D_s$ with gradient $\leq \tau_2 \leq 1$ and separation

$$\leq \delta_3 s \leq \delta_2 s,$$

then $\Sigma_{g}$ extends to an $N$-valued graph over $D_{3s} \setminus D_s$ with gradient $\leq \tau_2 + C_3 \delta_3$ and separation between the top and bottom sheets $\leq C_3 \delta_3 s$.

**Proof.** Suppose $N$ is odd (the even case is virtually identical). Fix $y_{-N}, \ldots, y_N \in \Sigma_{g}$ with $y_j$ over $\{\rho = 2s, \theta = j \pi\}$. Let $\gamma_{0, \gamma_2} \subset \Sigma^M$ be the graphs over $\{2s \leq \rho \leq 3s, \theta = 0\}$ and $\{2s \leq \rho \leq 3s, \theta = 2 \pi\}$, respectively, with $\partial \gamma_{0} = \{y_0, z_0\}$ and $\partial \gamma_{2} = \{y_2, z_2\}$.

As in the proof of Corollary II.3.1, there are nodal curves

$$\sigma_{-N}, \ldots, \sigma_N \subset \{x_1 = -2s\} \cap \Sigma$$

from $y_j$ (for $j$ odd) to $\partial B_{R_0}$ so that

1. Any curve in $\Sigma \setminus \Pi^{-1}(\partial D_{2s})$ from $z_0$ to $\partial \Sigma \setminus \partial B_{R_0}$ hits either every $\sigma_j$ with $j > 0$ or every $\sigma_j$ with $j < 0$.

2. For $i < j$, $\sigma_i$ and $\sigma_j$ do not connect in $\Pi^{-1}(D_{4s}) \cap \{x_1 \leq -2s\} \cap \Sigma$.

3. $\text{dist}(\cup_j \sigma_j, \partial \Sigma \setminus \partial B_{R_0}) \geq s.$
Note that (2) follows easily from the convex hull property when \( i \neq -N \) or \( j \neq N \); the case \( i = -N \) and \( j = N \) follows since \( \Sigma \) separates \( y_{-N}, y_N \) in \( \Pi^{-1}(D_{1s}) \cap \{ x_1 \leq -2s \} \).

By the curvature estimate for stable surfaces of [Sc] and the Harnack inequality for the minimal graph equation, there exist \( C_4, \delta_4 > 0 \) so that if \( z_3, z_4 \in \Sigma \setminus T_{s/4} (\partial \Sigma), \Pi(z_3) = \Pi(z_4), \) and
\[
0 < |z_3 - z_4| \leq \delta_5 s \leq \delta_4 s,
\]
then \( B_{s/8}(z_4) \) is a graph over (a subset of) \( B_{s/7}(z_3) \) of a function \( u > 0 \) with
\[
|\nabla u| \leq \min \{ 1/2, C_4 \delta_5 \}.
\]
The lemma now follows easily by repeatedly applying this and using (1)–(3) to stay away from \( \partial \Sigma \) until we have recovered all \( N \) sheets.

II.4. Proof of Theorem II.0.21

Let again \( \Sigma \subset B_{R_0} \) be a stable embedded disk with
\[
\partial \Sigma \subset B_{r_0} \cup \partial B_{R_0} \cup \{ x_1 = 0 \}
\]
and \( \partial \Sigma \setminus \partial B_{R_0} \) connected. We will use the notation of (II.2.5), so that \( \Sigma^{0,2\pi}_{r_3,r_4} \) is an annulus with a slit as defined in [CM3]. The next lemma is an easy consequence of Theorem 3.36 of [CM3].

**Lemma II.4.1.** Given \( \tau_0 > 0 \), there exists \( 0 < \varepsilon_1 = \varepsilon_1(\tau_0) < 1/24 \) so that the following holds:

If \( 2r_0 \leq 1 < r_3 \leq R_0/2 \) and \( \Sigma^{0,2\pi}_{r_1,r_3} \subset \Sigma \) is the graph of a function \( u \) with
\[
|\nabla u| \leq 1/12, \quad \max_{z_i}^{r_0,2\pi} (|u| + |\nabla u|) \leq 2 \varepsilon_1, \quad |A| \leq \varepsilon_1/r, \quad \text{and for} \ 1 \leq t \leq r_3 \ \text{the separation over} \ \partial D_t \ \text{is}
\]
\[
\leq 4 \pi \varepsilon_1 t^{1/2},
\]
then \( |\nabla u| \leq \tau_0 \).

Lemma II.4.1 follows from Theorem 3.36 of [CM3] and two facts:

- Since \( \Sigma \) is a graph over a larger set in \( \mathcal{P} \) (by stability and the fact that \( \partial \Sigma \subset B_{r_0} \cup \partial B_{R_0} \cup \{ x_1 = 0 \} \)), the bound for the separation and estimates for the minimal graph equation over \( \Sigma \) give a bound for the difference in the two values of \( \nabla u \) along the slit (cf. Proposition II.2.12).

- Theorem 3.36 of [CM3] actually applies directly to \( B_{3r_3} / 4 \cap \Sigma^{0,2\pi}_{r_3,r_4} \setminus B_2 \) to get \( |\nabla u| \leq \tau_0/2 \) on \( D_{r_3/2} \setminus D_1 \); integrating
\[
|\nabla |\nabla u|| \leq |A| (1 + |\nabla u|^2)^{3/2} \leq 2 \varepsilon_1/r
\]
then gives \( |\nabla u| \leq \tau_0 \) on \( D_{r_3} \setminus D_1 \).
We will prove Theorem II.0.21 by repeatedly applying Corollary II.3.7 to extend $\Sigma^M$ as a graph, Lemma II.4.1 to get an improved gradient bound, and then Lemma II.3.8 to extend additional sheets.

Proof of Theorem II.0.21. Set $\tau_0 = \min\{\tau, \tau_k, 1/24\}/2$ and let $\epsilon_1 = \epsilon_1(\tau_0)$ with

$$0 < \epsilon_1 < 1/72$$

be given by Lemma II.4.1. The constants $\Omega_0, m_0, \delta_0$ are given by Corollary II.3.7 (depending on $\epsilon_1$) and $C_3, \delta_2 > 0$ are from Lemma II.3.8 with $N = m_0$. Set $N_1 = m_0$, $\Omega_1 = 2\Omega_0$, and choose $\epsilon > 0$ so the following three properties hold:

$$\Pi^{-1}(D_{r_0}) \cap \Sigma_g \subset \{|x_3| \leq r_0/2\},$$

$$|A| \leq \epsilon_1/r \text{ on } \Sigma^M \setminus B_{2r_0}.$$ To arrange the last condition, we use the gradient bound, stability, and second derivative estimates for the minimal graph equation (in terms of the gradient bound). Note that, integrating the bound gradient derivative estimates for the minimal graph equation (in terms of the gradient bound), we get $\Sigma$ holds for $1$ bound.

(II.4.2) \[ \epsilon < \min\left\{ \frac{\epsilon_1}{2}, \frac{\tau_0}{4\pi 2^{1/2} C_3}, \frac{\delta_0}{2\pi 2^{1/2} C_3}, \frac{\delta_0}{2\pi m_0}, \frac{\delta_2}{4\pi 2^{1/2}} \right\}, \]

$\Pi^{-1}(D_{r_0}) \cap \Sigma_g \subset \{|x_3| \leq r_0/2\}$,

$|A| \leq \epsilon_1/r$ on $\Sigma^M \setminus B_{2r_0}$.

To arrange the last condition, we use the gradient bound, stability, and second derivative estimates for the minimal graph equation (in terms of the gradient bound). Note that, integrating the bound gradient derivative estimates for the minimal graph equation (in terms of the gradient bound), we get $\Sigma$ holds for $1$ bound.

(1) Apply Corollary II.3.7 (with $r_2 = r_0, R_2 = 1, \tau_1 = 2\tau_0$) to extend $\Sigma^{0,2\pi}_{r_0,1}$ to a graph $\Sigma^{0,2\pi}_{r_0,1}$ with gradient $\leq 2\tau_0 + 3\epsilon_1 < 1/12$, $|A| \leq \epsilon_1/r$ over $D_2 \setminus D_1$, and, for $1 \leq t < 2$,

(II.4.3) \[ \text{separation } \leq 2\pi \epsilon t^{1/2} \text{ over } \partial D_t. \]

(2) By Lemma II.4.1 (with $r_3 = 2$), $\Sigma^{0,2\pi}_{1,2}$ and hence $\Sigma^{0,2\pi}_{r_0,2}$ have gradient $\leq \tau_0$.

(3) By Lemma II.3.8 (with $N = m_0, s = 1/2, \tau_2 = \tau_0, \delta_3 = 4\pi \epsilon 2^{1/2}$), $\Sigma^{0,2\pi}_{r_0,3/2}$ is contained in an $m_0$-valued graph $\Sigma^{0,2\pi}_{r_0,3/2} \subset \Sigma$ over $D_{3/2} \setminus D_{r_0}$ with gradient $\leq \tau_0 + C_3 4\pi \epsilon 2^{1/2} < 2\tau_0$ and separation $\leq C_3 2\pi \epsilon 2^{1/2} < \delta_0$.

Repeat (1)–(3) with: (1) $R_2 = 3/2$ to extend $\Sigma^{0,2\pi}_{r_0,3/2}$ to $\Sigma^{0,2\pi}_{r_0,3}$ with (II.4.3) holding for $1 \leq t < 3$, (2) $r_3 = 3$ so that $\Sigma^{0,2\pi}_{r_0,3}$ has gradient $\leq \tau_0$, (3) $s = 3/2$ to get $\Sigma^{0,2\pi}_{r_0,9/2} \subset \Sigma$, and then again (1) $R_2 = 9/2$, etc., giving the theorem.

$\square$
Part III. The general case of Theorem I.0.8

III.1. Constructing multi-valued graphs in disks in slabs

Using Part I, we show next that an embedded minimal disk in a slab contains a multi-valued graph if it is not a graph. We can therefore apply Part II to get almost flatness of a corresponding stable disk past the slab. This is needed when the minimal surface is not in a thin slab.

**Proposition III.1.1.** There exists $\beta > 0$ so that the following holds: Let $\Sigma^2 \subset B_{r_0} \cap \{ |x_3| \leq \beta h \}$ be an embedded minimal disk with $\partial \Sigma \subset \partial B_{r_0}$. If a component $\Sigma_1$ of $B_{10h} \cap \Sigma$ is not a graph, then $\Sigma$ contains an $N$-valued graph over $D_{r_0-2h} \setminus D_{(60+20N)h}$.

**Proof.** The proof has four steps. First we show, by using Lemma I.0.11 twice, that over a truncated sector in the plane, i.e., over

$$S_{s_1,s_2}(\theta_1,\theta_2) = \{(\rho, \theta) \mid s_1 \leq \rho \leq s_2, \theta_1 \leq \theta \leq \theta_2\},$$

we have three distinct components of $\Sigma$. Second, we separate these by stable disks and order them by height. Third, we use Proposition I.0.16 to show that the “middle” component is a graph over a large sector. Fourth, we repeatedly use the appendix to extend the top and bottom components around the annulus and then Proposition I.0.16 to extend the middle component as a graph. This will give the desired multi-valued graph.

![Figure 17: Proof of Proposition III.1.1: Step 1: Finding the three components.](image)

For $j = 1$ and 2, let $\Sigma_j$ be the component of $B_{20jh} \cap \Sigma$ containing $\Sigma_1$. By the maximum principle, each $\Sigma_j$ is a disk. Rado’s theorem (see, e.g., [CM1]) gives points

$$z_j \in \Pi^{-1}(\partial D_{(20j-10)h}) \cap \Sigma_j,$$

for $j = 1, 2$, where $\Sigma$ is not graphical. Rotate $\mathbb{R}^2$ so that $z_1, z_2 \in \{x_1 \geq 0\}$ and set $z = (r_0, 0, 0)$. Apply Lemma I.0.11 twice as in the first step of the proof of Proposition I.0.16 to get (see Figure 17):
(1) Disjoint curves $\gamma_1, \gamma_2 \subset \Sigma$ with $\partial \gamma_k \subset \partial B_{r_0/2}$,

(III.1.3) \hspace{1cm} \gamma_k \subset B_{5.5}(z_k) \cup T_h(\partial D_{(20k-10)} \cap \{x_1 \geq 0\}) \cup T_h(\gamma_{0,z/2}) ,

which are $C/\beta h$-almost monotone in $T_h(\gamma_{0,z/2}) \setminus B_{2k.5} h$.

(2) For $k = 1, 2$ and $y_0 \in \gamma_{0,z/2} \setminus B_{2k.5} h$, there are components $\Sigma'_{y_0,k,1} \neq \Sigma'_{y_0,k,2}$ of $B_{5.5}(y_0) \cap \Sigma$ each containing points of $B_h(y_0) \cap \gamma_k$.

It follows from (2) that, for $k = 1, 2$, there are components $\Sigma_{k,1}, \Sigma_{k,2}$ of

$$\Pi^{-1}(S_{4.2h.r_0 - 2h}(-3\pi/4, 3\pi/4)) \cap \Sigma$$

with $\Sigma'_{z/2,k,i} \subset \Sigma_{k,i}$. These components do not connect in

$$\Pi^{-1}(S_{4.0h.r_0}(-7\pi/8, 7\pi/8)) \cap \Sigma.$$

That is, $\Sigma$ would otherwise contain a disk violating the maximum principle (as in the second step of Lemma I.0.11). The same argument gives $\Sigma_{i,j,1}, \Sigma_{i,j,2}, \Sigma_{i,j,3}$ which do not connect in

(III.1.4) \hspace{1cm} \Pi^{-1}(S_{4.0h,r_0}(-7\pi/8, 7\pi/8)) \cap \Sigma.

By the second step of Proposition I.0.16, if $\Sigma_{i,j}, \Sigma_{k,\ell}$ do not connect in

$$\Pi^{-1}(S_{4.0h,r_0}(-7\pi/8, 7\pi/8)) \cap \Sigma,$$

then there is a stable embedded disk $\Gamma_\alpha$ with $\partial \Gamma_\alpha \subset \Sigma$, $\Gamma_\alpha \cap \Sigma = \emptyset$, and a graph $\Gamma'_\alpha \subset \Gamma_\alpha$ over $S_{4.1h,r_0 - h}(-13\pi/16, 13\pi/16)$ separating $\Sigma_{i,j}, \Sigma_{k,\ell}$. Applying this twice (and reordering the $k\ell, i\ell$), we get $\Gamma'_1 \subset \Gamma_1$ and $\Gamma'_2 \subset \Gamma_2$ so that each $\Sigma_{k_\ell, i\ell}$ is below $\Gamma'_\ell$ which is in turn below $\Sigma_{k_\ell, i\ell}$. Let $\gamma_{1\ell}^1$ and $\gamma_{1\ell}^b$ be top and bottom components of $\cup_j \gamma_j \setminus B_{4.0h}$ intersecting $\partial B_{r_0/2}$. Since $\Sigma_1 \subset \Sigma_2$, a curve $\gamma_{1\ell}^m \subset B_{4.0h} \cap \Sigma$ connects $\gamma_{1\ell}^1$ to $\gamma_{1\ell}^b$.

See Figure 18. By a slight variation of Proposition I.0.16 (with $\gamma = \gamma_{1\ell}^1 \cup \gamma_{1\ell}^b \cup \gamma_{1\ell}^m$), the middle component $\Sigma_{k_{\ell}, i_{\ell}}$ is a graph over $S_{4.2h,r_0 - 2h}(-3\pi/4, 3\pi/4)$. This variation follows from steps one and three of that proof (step two there constructs barriers $\Gamma_1$ which were constructed here above).

See Figure 19. Corollary A.10 gives curves

$$\gamma_{2\ell}^1, \gamma_{2\ell}^b \subset (B_{4.4h} \cup T_h(\gamma_{0,0,m}r_0, 0)) \setminus \Pi^{-1}(D_{4.2h}) \cap \Sigma$$

from $\partial B_{4.4h} \cap \gamma_{1\ell}^1$ and $\partial B_{4.4h} \cap \gamma_{1\ell}^m$, respectively, to $\partial B_{r_0/2}$. In particular, $\gamma_{2\ell}^1$ is below $\Sigma_{k_{2\ell}, i_{2\ell}}$ and $\gamma_{2\ell}^b$ is above $\Sigma_{k_{2\ell}, i_{2\ell}}$; i.e., $\Sigma_{k_{2\ell}, i_{2\ell}}$ is still a middle component.

Again by the maximum principle, this gives 3 distinct components of

$$\Pi^{-1}(S_{4.6h,r_0 - 2h}(-\pi/4, 5\pi/4)) \cap \Sigma$$

which do not connect in

$$\Pi^{-1}(S_{4.55h,r_0}(-3\pi/8, 11\pi/8)) \cap \Sigma.$$
TOBIAS H. COLDING AND WILLIAM P. MINICOZZI II

Extends since it is “between the sheets.”

Figure 18: Proof of Proposition III.1.1:
Step 3: Extending the middle component as a graph.

By Proposition I.0.16, \( \Sigma^2 \) further extends as a graph over \( S_{46,h,r_0 - 2h(-\pi/4,5\pi/4)} \), giving a graph \( \Sigma^{3\pi/4,5\pi/4} \) over \( S_{46,h,r_0 - 2h(-3\pi/4,5\pi/4)} \). By Rado’s theorem, this graph cannot close up. Repeating this with 
\[
\gamma_t^3, \gamma_b^3 \subset (B_{49h} \cup T_h(\gamma_0,(-r_0,0,0))) \setminus \Pi^{-1}(D_{47h}) \cap \Sigma,
\]
/etc., eventually gives the proposition. \qed

Extends by the maximum principle.

Figure 19: Proof of Proposition III.1.1:
Step 4: Extending the top and bottom components by the maximum principle. They stay disjoint since the middle component is a graph separating them.

III.2. Proof of Theorem I.0.8

In this section, we generalize Proposition I.0.16 to when the minimal surface is not in a slab; i.e., we show Theorem I.0.8. \( \Sigma^2 \subset B_{c_1r_0} \subset \mathbb{R}^3 \) will be an embedded minimal disk, \( \partial \Sigma \subset \partial B_{c_1r_0} \), \( c_1 \geq 4 \), and \( y \in \partial B_{2r_0} \). \( \Sigma_1, \Sigma_2, \Sigma_3 \) will be distinct components of \( B_{r_0}(y) \cap \Sigma \).

Lemma III.2.1. Given \( \tilde{\beta} > 0 \), there exist \( 2c_2 < c_4 < c_3 \leq 1 \) so that the following holds:
Let $\Sigma_3'$ be a component of $B_{c_3 r_0}(y) \cap \Sigma_3$ and $y_i \in B_{c_2 r_0}(y) \cap \Sigma_i$ for $i = 1, 2$. If $y_1, y_2$ are in distinct components of $B_{c_4 r_0}(y) \setminus \Sigma_3'$, then there are disjoint stable embedded minimal disks $\Gamma_1, \Gamma_2 \subset B_{r_0}(y) \setminus \Sigma$ with $\partial \Gamma_i = \partial \Sigma_i$, and (after a rotation) graphs $\Gamma_i' \subset \Gamma_i$ over $D_{3 c_3 r_0}(y)$ so that $y_1, y_2, \Sigma_3'$ are each in their own component of $\Pi^{-1}(D_{3 c_3 r_0}(y)) \setminus (\Gamma_1' \cup \Gamma_2')$ and $\Gamma_1', \Gamma_2' \subset \{ |x_3 - x_3(y)| \leq \bar{\beta} c_3 r_0 \}$.

Proof. This follows exactly as in the second step of the proof of Proposition I.0.16.

Proof of Theorem I.0.8. Let $N_1, \Omega_1, \varepsilon > 0$ be given by Theorem II.0.21 (with $\tau = 1$). Assume that $N_1$ is even. Let $\beta > 0$ be from Proposition III.1.1. Set

$$\bar{\beta} = \min \{ \beta_5, \varepsilon, \varepsilon/C_g, \beta/(6[60 + 20(N_1 + 3)]) \}/(5 \Omega_1),$$

where $\beta_5, C_g$ are as in Lemma I.0.9. Let $c_2, c_3, c_4$ and $\Gamma_i' \subset \Gamma_i$ be given by Lemma III.2.1. Set

$$c_5 = (60 + 20(N_1 + 3)) \bar{\beta} c_3 / \beta,$$

so that $c_5 \leq c_3/(30 \Omega_1)$. Finally, set $c_1 = 16 \Omega_1$.

We will suppose that $\Sigma_3'$ is not a graph at all. $z' \in \Sigma_3'$ and deduce a contradiction. Set $z = \Pi(z')$. Since $\Sigma_3'$ separates $y_1, y_2$, it is in the slab between $\Gamma_1', \Gamma_2'$. By Proposition III.1.1 (with $h = \tilde{\beta} c_3 r_0 / \beta$) and (III.2.2), $\Sigma$ contains an $(N_1 + 3)$-valued graph $\Sigma_g$ over $D_{c_3 r_0}(z) \setminus D_{c_5 r_0}(z)$ and $\Sigma_g$ is also in the slab. Let $\sigma_g \subset \Sigma_g$ be the $(N_1 + 2)$-valued graph over $\partial D_{c_3 r_0}(z)$ (see Figure 20). Let $E$ be the region in $\Pi^{-1}(D_{c_5 r_0/2}(z) \setminus D_{c_5 r_0/(2 \Omega_1)}(z))$ between the sheets of the (concentric) $(N_1 + 1)$-valued subgraph of $\Sigma_g$.

The first step is to find a curve $\gamma_3 \subset \Sigma$ containing $\sigma_g$ so that any stable disk with boundary $\gamma_3$ is forced to spiral. Also, $\gamma_3$ will have six pieces: $\sigma_g$, two segments, $\gamma^t, \gamma^b$, in $\Sigma_g$ which are graphs over a portion of the $\{ x_1 > x_1(z) \}$ part of the $x_1$-axis, two nodal curves, $\sigma^t, \sigma^b$, in $\{ x_1 = \text{constant} \}$, and a segment $\sigma^0$ in $\partial \Sigma$. Since $\Sigma_g$ is a graph, there are graphs $\gamma^t, \gamma^b \subset \Sigma_g$ over a portion of the $\{ x_1 > x_1(z) \}$ part of the $x_1$-axis from $\partial \sigma_g$ to

$$y^t, y^b \in \{ x_1 = x_1(z) + 3 c_5 r_0 \} \cap \Sigma.$$

By the maximum principle (as in the proof of Corollary II.3.1), there are nodal curves

$$\sigma^t, \sigma^b \subset \{ x_1 = x_1(z) + 3 c_5 r_0 \} \cap \Sigma.$$
from \(y_t, y_b\), respectively, to \(y_0^t, y_0^b\) \(\in \partial\Sigma\). Finally, connect \(y_0^t, y_0^b\) by a curve \(\sigma^0 \subset \partial\Sigma\) and set

\[\gamma_3 = \sigma^b \cup \gamma^b \cup \sigma_g \cup \gamma^t \cup \sigma^t \cup \sigma^0.\]

By [MeYa], there is a stable embedded disk \(\Gamma \subset B_{c_t r_0} \setminus \Sigma\) with \(\partial \Gamma = \gamma_3\). Note that \(\partial \Gamma \setminus \partial B_{r_0}\) is connected.

We claim that \(\sigma^t, \sigma^b\) do not intersect between any two of the components \(\{\sigma_i\}\) of

\[B_{(c_3-2c_5)r_0}(z) \cap \{x_1 = x_1(z) + 3c_5 r_0\} \cap \Sigma_g.\]

If not, we can assume that a curve \(\sigma \subset \sigma^t\) connects \(y^t\) to a point \(y_0\) between \(\sigma_i, \sigma_{i+1}\). By (a slight variation of) Proposition I.0.16, the portion \(\Sigma_{y_0}\) of \(\Sigma\) between the \(i\)-th and \((i+1)\)-st sheets of

\[B_{(c_3-2c_5)r_0}(z) \cap \{x_3 - x_3(z) \leq \varepsilon c_3 r_0/(2\Omega_1)\},\]

and a curve \(\eta \subset \Pi^{-1}(D_{c_3 r_0/(2\Omega_1)}(z) \cap \Gamma \setminus \partial B_{r_0}\) connects \(\Gamma_g\) to \(\partial \Gamma \setminus \partial B_{r_0}\). By the previous paragraph,

\[\text{dist}_\Gamma(E \cap \Gamma, \partial \Gamma) > c_3 r_0/(5\Omega_1).\]

By (the proof of) Lemma I.0.9 (with \(h = c_3 r_0/(5\Omega_1)\) and \(\beta = \tilde{\beta}, (\text{III.2.2}), \) and (\text{III.2.3}), we have that each component of \(E \cap \Gamma\) is a multi-valued graph.
Let $\sigma \subset E$ be a graph over $\partial D_{c_3 r_0/(2 \Omega_1)}(z)$. Since $\sigma$ separates $$\Pi^{-1}(\partial D_{c_3 r_0/(2 \Omega_1)}(z)) \cap \gamma^t \text{ and } \Pi^{-1}(\partial D_{c_3 r_0/(2 \Omega_1)}(z)) \cap \gamma^b$$ in the cylinder $\Pi^{-1}(\partial D_{c_3 r_0/(2 \Omega_1)}(z))$ (and the description of $\partial \Gamma$), there is a curve $$\eta \subset \Pi^{-1}(D_{c_3 r_0/(2 \Omega_1)}(z)) \cap \Gamma \setminus \partial B_{r_0}$$ from $\Gamma \cap \sigma$ to $\partial \Gamma \setminus \partial B_{r_0}$. Hence, since $E$ is between the sheets of an $(N_1 + 1)$-valued graph, we get the desired $\Gamma_{g}$.

Combining all of this, Theorem II.0.21 gives a 2-valued graph $\Gamma_d \subset \Gamma$ over $D_{c_1 r_0/(2 \Omega_1)}(z) \setminus D_{c_3 r_0/(2 \Omega_1)}(z)$ with gradient $\leq 1$. Let $\hat{\gamma}$ be the component of $B_{(2 - 2 c_2)}r_0 \cap \gamma$ intersecting $B_{r_0}$. Note that since $\partial \gamma = \{y_1, y_2\}$ is separated by the slab between $\Gamma_1', \Gamma_2'$ and $\gamma \setminus B_{r_0}$ is $c_2 r_0$-almost monotone, $\Gamma_d$ separates the endpoints of $\partial \hat{\gamma}$. Finally, as in the proof of Proposition I.0.16, we must have $\Gamma_d \cap \hat{\gamma} \neq \emptyset$. This contradiction completes the proof.

Many variations of Theorem I.0.8 hold with almost the same proof. One of these is given in the following theorem:

**Theorem III.2.4.** There exist $d_1 \geq 8$ and $d_2 \leq 1$ so that the following holds:

Let $\Sigma^2 \subset B_{d_1 r_0} \subset \mathbb{R}^3$ be an embedded minimal disk with $\partial \Sigma \subset \partial B_{d_1 r_0}$ and let $y \in \partial D_{5 r_0}$. Suppose that $\Sigma_1, \Sigma_2 \subset \Sigma$ are disjoint graphs, over $D_{3 r_0}(y)$ with gradient $\leq d_2$, which intersect $B_{d_2 r_0}(y)$. If

$$\Sigma_1 \text{ and } \Sigma_2 \text{ can be connected in } B_{3 r_0} \cap \Sigma,$$

then any component of $B_{r_0}(y) \cap \Sigma$ which lies between them is a graph.

**Part IV. Extending multi-valued graphs off the axis**

In this section $\Sigma \subset B_{R_0} \subset \mathbb{R}^3$ will be an embedded minimal disk with $\partial \Sigma \subset \partial B_{R_0}$. In contrast to the results of Part II, $\Sigma$ is no longer assumed to be stable.

Note that, by [Sc], we can choose $d_3 > 4$ so that: If $\Gamma_0 \subset B_{d_3 s}$ with $\partial \Gamma_0 \subset \partial B_{d_3 s}$ is stable, then each component of $B_{4 s} \cap \Gamma_0$ is a graph (over some plane) with gradient $\leq 1/2$. 

with

$$\text{gradient } \leq 5 C_g \Omega_1 \beta \leq \varepsilon.$$
Proof of Theorem 0.3. The proof has two steps. First, the proofs of Theorem I.0.8 and Lemma II.3.8 give a stable disk $\Gamma \subset B_{R_0} \setminus \Sigma$ and a 4-valued graph $\Gamma_4 \subset \Gamma$ so that $\Sigma^M$ “passes between” $\Gamma_4$. Second, (a slight variation of) Theorem III.2.4 gives the 2-valued graph $\Sigma_d \subset \Sigma$.

Before proceeding, we choose the constants. Let $C_3, \delta_2$ be given by Lemma II.3.8 (with $N = 4$), $d_1, d_2$ be from Theorem III.2.4, and $C_g, \beta_s$ be from Lemma I.0.9. Set
\[
\tau_1 = \min\{\tau/(5C_g), \beta_s/5, d_2/10\},
\tau_2 = \min\{\delta_2/3, \tau_1/(1 + 3C_3)\}.
\]
Let $N_1, \Omega_1, \varepsilon$ be given by Theorem II.0.21 (with $\tau$ there equal to $\tau_2$). For convenience, assume that $N_1 \geq 16$ is even, $\Omega_1 > 4$, and rename this $\varepsilon$ as $\varepsilon_1$. Set $N = N_1 + 3$, $\Omega = \max\{d_1, 8 d_3 \Omega_1\}$, and
\[
\varepsilon = \min\{\varepsilon_1, \varepsilon_1/(5C_g), \beta_s/5, 1/4, d_2/10\}.
\]
For $N_2 \leq N$ and $r_0 \leq r_2 < r_3 \leq 1$, let $E_{r_2, r_3}^{N_2}$ be the region in $\Pi^{-1}(D_r \setminus D_{r_2})$ between the sheets of the (concentric) $N_2$-valued subgraph of $\Sigma_g$. Note that
\[
E_{r_2, r_3}^{N_2} \subset \{x_3^2 \leq \varepsilon^2 (x_1^2 + x_2^2)\}.
\]
As in the proof of Theorem I.0.8, let $\sigma_g \subset \Sigma_g$ be an $(N_1 + 2)$-valued graph over $\partial D_{r_0}$ and let $\gamma_3 \subset \Sigma$ be a curve with six pieces: $\sigma_g$, two segments, $\gamma^l, \gamma^b$, in $\Sigma_q$ which are graphs over a portion of the positive part of the $x_1$-axis, two nodal curves, $\sigma^l, \sigma^b$, in $\{x_1 = 2 d_3 r_0\}$, and $\sigma^d \subset \partial \Sigma$. By [MeYa], there is a stable embedded disk $\Gamma \subset B_{R_0} \setminus \Sigma$ with $\partial \Gamma = \gamma_3$.

Let $\{\sigma_i\}$ be the components of $B_{5/8} \cap \{x_1 = 2 d_3 r_0\} \cap \Sigma_g$ and suppose that a curve $\sigma \subset \sigma_i$ connects $\gamma^l$ to a point $y_0$ between $\sigma_i, \sigma_{i+1}$. By Theorem III.2.4, the portion $\Sigma_{y_0}$ with $y_0 \in \Sigma_{y_0}$ of $E_{3 r_0, 5/8}^{N_1 + 5/2} \cap \Sigma$ is a graph. Note that $B_{d_3 r_0} \cap \Sigma_{y_0}$ and $B_{3 r_0} \cap \Sigma_g$ are in the same component of $B_{d_3 r_0} \cap \Sigma$, since otherwise the stable disk between them given by [MeYa] would intersect $\Sigma_g$ (by [Sc]). Applying the maximum principle as before gives the desired contradiction. Hence, $\sigma^l, \sigma^b$ do not intersect between any of the $\sigma_i$’s. Therefore, if $z \in E_{4 d_3 r_0, 1/2}^{N_1 + 1} \cap \Gamma$, then
\[
\text{dist}_\Gamma(z, \partial \Gamma) \geq |\Pi(z)|/4.
\]
By the same linking argument as before, $E_{4 d_3 r_0, 1/2}^{N_1 + 1} \cap \Gamma$ contains an $N_1$-valued graph $\Gamma_g$ over $D_{1/2} \setminus D_{4 d_3 r_0}$ with gradient $\leq 5 C_g \varepsilon$,
\[
\Pi^{-1}(\partial D_{4 d_3 r_0}) \cap \Gamma_g \subset \{|x_3| \leq 4 \varepsilon d_3 r_0\},
\]
and a curve $\eta \subset \Pi^{-1}(D_{4 d_3 r_0}) \cap \Gamma \setminus \partial B_{R_0}$ connects $\Gamma_g$ to $\partial \Gamma \setminus \partial B_{R_0}$. Since $\Omega_1 < 1/(8 d_3 r_0)$, Theorem II.0.21 implies that $\Gamma$ contains a 2-valued graph $\Gamma_d$ over $D_{R_0}/\Omega_1 \setminus D_{4 d_3 r_0}$ with gradient $\leq \tau_2 < 1$. In particular,
\[
\Gamma_d \subset \{x_3^2 \leq \tau_2^2 (x_1^2 + x_2^2)\}.
\]
Next, we apply Lemma II.3.8 to extend \( \Gamma_d \) to a 4-valued graph \( \Gamma_4 \) over \( D_{5R_0/(6\Omega_1)} \setminus D_{5d_3r_0} \) with gradient \( \leq \tau_2 + 3C_{g\tau_2} \leq \tau_1 \). Let \( E_\Gamma \) be the region in \( \Pi^{-1}(D_{R_0/(2\Omega_1)} \setminus D_{15d_3r_0}) \) between the sheets of the (concentric) 3-valued subgraph of \( \Gamma_4 \), so that \( E_\Gamma \subset \{ x^2 \leq \tau_1^2 (x_1^2 + x_2^2) \} \).

If \( z \in E_\Gamma \cap \Sigma \), then there is a curve \( \gamma_z \subset \Gamma_4 \) with each component of \( \gamma_z \setminus \Pi^{-1}(D_{5d_3r_0}) \) a graph over the segment \( \gamma_{0,z}, \partial \gamma_z = \{ y_2^z, y_4^z \} \), and \( y_2^z, y_4^z \) are in distinct components of \( B_{3\Pi(z)}/5(\Pi(z)) \cap \Gamma \) with \( z \) between these components. By (a slight variation of) Theorem III.2.4 (with \( \Sigma \) nonempty since \( \Sigma^\gamma \) is a subgraph of \( \Gamma \)), the portion of \( \Sigma \) inside \( B_{R_0/d_3} \cap E_\Gamma \) is a graph over \( \Gamma_4 \). This is nonempty since \( (\Sigma_g)^M \) begins in \( E_\Gamma \), so we get the desired 2-valued graph \( \Sigma_d \) with gradient \( \leq 5C_g \tau_1 \leq \tau \) (by Lemma I.0.9).

\[ \square \]

**Appendix A: Catenoid foliations**

We recall here some consequences of the maximum principle for an embedded minimal surface \( \Sigma \) in a slab. Let \( \text{Cat}(y) \) be the vertical catenoid centered at \( y = (y_1, y_2, y_3) \) given by

\[
\text{Cat}(y) = \{ x \in \mathbb{R}^3 \mid \cosh^2(x_3 - y_3) = (x_1 - y_1)^2 + (x_2 - y_2)^2 \}.
\]

Given an angle \( 0 < \theta < \pi/2 \), let \( \partial N_\theta(y) \) be the cone

\[
\{ x \mid (x_3 - y_3)^2 = |x - y|^2 \sin^2 \theta \}.
\]

Since \( \cosh(t) > t \) for \( t \geq 0 \), it follows that \( \partial N_{\pi/4}(y) \cap \text{Cat}(y) = \emptyset \). Set

\[
\theta_0 = \inf \{ \theta \mid \partial N_\theta(y) \cap \text{Cat}(y) = \emptyset \},
\]

so that \( \partial N_{\theta_0}(y) \) and \( \text{Cat}(y) \) intersect tangentially in a pair of circles. Let \( \text{Cat}_0(y) \) be the component of \( \text{Cat}(y) \setminus \partial N_{\theta_0}(y) \) containing the neck

\[
\{ x \mid x_3 = y_3, (x_1 - y_1)^2 + (x_2 - y_2)^2 = 1 \}.
\]

If \( x \in \text{Cat}_0(y) \), then \( \overline{x, x} \cap \text{Cat}_0(y) = \{ x \} \) since \( \cosh \) is convex and \( \cosh'(0) = 0 \); i.e., \( \text{Cat}_0(y) \) is a radial graph. In particular, the dilations of \( \text{Cat}_0(y) \) about \( y \) are all disjoint and, consequently (see Figure 21), give a minimal foliation of the solid (open) cone

\[
N_{\theta_0}(y) = \{ x \mid (x_3 - y_3)^2 < |x - y|^2 \sin^2 \theta_0 \}.
\]

The leaves of this foliation have boundary in \( \partial N_{\theta_0}(y) \) and are level sets of the function \( f_y \) given by

\[
y + (x - y)/f_y(x) \in \text{Cat}_0(y).
\]

Choose \( \beta_A > 0 \) small so that

\[
\{ x \mid |x_3 - y_3| \leq 2\beta_A h \} \setminus B_{h/8}(y) \subset N_{\theta_0}(y)
\]
The intersection of two embedded minimal surfaces is locally given by 2n embedded arcs meeting at equal angles as in Figure 22, i.e., an “n-prong singularity” (e.g., the set where \((x+iy)^n\) is real); see Claim 1 in Lemma 4 of [HoMe]. This immediately implies the next lemma:

**Lemma A.7.** If \(z \in \Sigma \subset N_{\theta_0}(y)\) is a nontrivial interior critical point of \(f_y|\Sigma\), then \(\{x \in \Sigma \mid f_y(x) = f_y(z)\}\) has an n-prong singularity at \(z\) with \(n \geq 2\).

As a consequence, we get a version of the usual strong maximum principle:

**Lemma A.8.** If \(\Sigma \subset N_{\theta_0}(y)\), then \(f_y|\Sigma\) has no nontrivial interior local extrema.

In particular, we can use \(f_y\) to show that a minimal surface in a narrow slab either stays near its boundary or comes close to the center of the slab:

**Corollary A.9.** If \(\partial \Sigma \subset \partial B_h(y), B_{3h/4}(y) \cap \Sigma \neq \emptyset\), and

\[
\Sigma \subset B_h(y) \cap \{x \mid |x_3 - y_3| \leq 2 \beta_A h\},
\]

then \(B_{h/4}(y) \cap \Sigma \neq \emptyset\).

**Proof.** Scaling (A.6) by 4, we get

\[
\{x \in \Sigma \mid f_y(x) = 3h/16\} \cap \{x \mid |x_3 - y_3| \leq 2 \beta_A h\} \subset B_{7h/32}(y).
\]

By Lemma A.8, \(f_y\) has no interior minima in \(\Sigma\) so that the corollary now follows from

\[
f_y(x) \leq |x - y|.
\]
Iterating Corollary A.9 along a chain of balls gives the next corollary:

**Corollary A.10.** If \( \Sigma \subset \{ |x_3| \leq 2\beta_A h \} \), points \( p, q \in \{ x_3 = 0 \} \) satisfy \( T_h(\gamma_{p,q}) \cap \partial \Sigma = \emptyset \), and \( y_p \in B_{h/4}(p) \cap \Sigma \), then a curve \( \nu \subset T_h(\gamma_{p,q}) \cap \Sigma \) connects \( y_p \) to \( B_{h/4}(q) \cap \Sigma \).

**Proof.** Choose points
\[
y_0 = p, y_1, y_2, \ldots, y_n = q \in \gamma_{p,q}
\]
with \( |y_{i-1} - y_i| = h/2 \) for \( i < n \) and \( |y_{n-1} - y_n| \leq h/2 \). Repeatedly applying Corollary A.9 for \( 1 \leq i \leq n \), gives curves
\[
\nu_i : [0, 1] \to B_h(y_i) \cap \Sigma
\]
with \( \nu_1(0) = y_p \), \( \nu_i(1) \in B_{h/4}(y_i) \cap \Sigma \), and \( \nu_{i+1}(0) = \nu_i(1) \). Set \( \nu = \cup_{i=1}^n \nu_i \).

This produces curves which are “\( h \)-almost monotone” in the sense that if \( y \in \nu \), then \( B_{1/2}(y) \cap \nu \) has only one component which intersects \( B_{2h}(y) \).

**Corollary A.11.** If \( \Sigma \subset \{ |x_3| \leq 2\beta_A h \} \) and \( E \) is an unbounded component of
\[
\mathbb{R}^2 \setminus T_{h/4}(\Pi(\partial \Sigma)),
\]
then \( \Pi(\Sigma) \cap E = \emptyset \).

**Proof.** Given \( y \in E \), choose a curve
\[
\gamma : [0, 1] \to \mathbb{R}^2 \setminus T_{h/4}(\Pi(\partial \Sigma))
\]
with \( |\gamma(0)| > \sup_{x \in \Sigma} |x| + h \) and \( \gamma(1) = y \). Set
\[
\Sigma_t = \{ x \in \Sigma | f_{\gamma(t)}(x) = 3h/16 \}.
\]
By (A.6), we have \( \Sigma_t \subset B_{7/32}(\gamma(t)) \), so that \( \Sigma_0 = \emptyset \) and \( \Sigma_t \cap \partial \Sigma = \emptyset \). By Lemma A.8, either:

- \( \Sigma_t = \emptyset \), or
- \( \Sigma_t \) contains an arc of transverse intersection.

In particular, there cannot be a first \( t > 0 \) with \( \Sigma_t \neq \emptyset \), which gives the corollary. \( \square \)
References


[CM7] ———, The space of embedded minimal surfaces of fixed genus in a 3-manifold V; Fixed genus, in preparation.


(Received June 30, 2001)