Annals of Mathematics, **160** (2004), 1–26

Real polynomial diffeomorphisms with maximal entropy: Tangencies

By Eric Bedford and John Smillie*

Introduction

This paper deals with some questions about the dynamics of diffeomorphisms of $\mathbf{R}^2$. A "model family" which has played a significant historical role in dynamical systems and served as a focus for a great deal of research is the family introduced by Hénon, which may be written as

$$f_{a,b}(x,y) = (a - by - x^2, x) \qquad b \neq 0.$$

When $b \neq 0$, $f_{a,b}$ is a diffeomorphism. When $b = 0$ these maps are essentially one dimensional, and the study of dynamics of $f_{a,0}$ reduces to the study of the dynamics of quadratic maps

$$f_a(x) = a - x^2.$$

Like the Hénon diffeomorphisms of $\mathbf{R}^2$, the quadratic maps of $\mathbf{R}$, have also played a central role in the field of dynamical systems.

These two families of dynamical systems fit together naturally, which raises the question of the extent to which the dynamics is similar. One difference is that our knowledge of these quadratic maps is much more thorough than our knowledge of these quadratic diffeomorphisms. Substantial progress in the theory of quadratic maps has led to a rather complete theoretical picture of their dynamics and an understanding of how the dynamics varies with the parameter. Despite significant recent progress in the theory of Hénon diffeomorphisms, due to Benedicks and Carleson and many others, there are still many phenomena that are not nearly so well understood in this two-dimensional setting as they are for quadratic maps.

One phenomenon which illustrates the difference in the extent of our knowledge in dimensions one and two is the dependence of the complexity of the system on parameters. In one dimension the nature of this dependence is understood, and the answer is summarized by the principle of monotonicity. Loosely stated, this is the assertion that the complexity of f_a does not

*Research supported in part by the NSF.

decrease as the parameter a increases. The notion of complexity used here can be made precise either in terms of counting periodic points or in terms of entropy. The paper [KKY] shows that monotonicity is a much more complicated phenomenon for diffeomorphisms. In this paper we will focus on one end of the complexity spectrum, the diffeomorphisms of maximal entropy, and we will show to what extent the dynamics in the two-dimensional case are similar to the dynamics in the one-dimensional case. In the case of quadratic maps, complex techniques proved to be an important tool for developing the theory. In this paper we apply complex techniques to study quadratic (and higher degree) diffeomorphisms.

Topological entropy is a measure of dynamical complexity that can be defined either for maps or diffeomorphisms. By Friedland and Milnor [FM] the topological entropy of Hénon diffeomorphisms satisfies: $0 \le h_{\rm top}(f_{a,b}) \le \log 2$. We will say that f has maximal entropy if the topological entropy is equal to $\log 2$. The notion of maximal entropy makes sense for polynomial maps of $\mathbf{R}$ as well as polynomial diffeomorphisms of $\mathbf{R}^2$ of degree greater than two. In either of these cases we say that f has maximal entropy if $h_{\rm top}(f) = \log(d)$ where d is the algebraic degree of f and $d \ge 2$. We will see that this condition is equivalent to the assumption that f^n has d^n (real) fixed points for all n.

The quadratic maps f_a of maximal entropy are those with $a \ge 2$. These maps are hyperbolic (that is to say expanding) for $a > 2$, whereas the map f_2, the example of Ulam and von Neumann, is not hyperbolic. Examples of maps of maximal entropy in the Hénon family were given by Devaney and Nitecki [DN] (see also [HO] and [O]), who showed that for certain parameter values $f_{a,b}$ is hyperbolic and a model of the Smale horseshoe. Examples of maximal entropy polynomial diffeomorphisms of degree $d \ge 2$ are given by the d-fold horseshoe mappings of Friedland and Milnor (see [FM, Lemma 5.1]).

We will see that all polynomial diffeomorphisms of maximal entropy (whether or not they are hyperbolic) exhibit a certain form of expansion. Hyperbolic diffeomorphisms have uniform expansion and contraction which implies uniform expansion and contraction for periodic orbits. To be precise, let p be a point of period n for a diffeomorphism f. We say that p is a saddle point if $Df^n(p)$ has eigenvalues $\lambda^{s/u}$ with $|\lambda^s| < 1 < |\lambda^u|$. If f is hyperbolic then for some $\kappa > 1$ independent of p we have $|\lambda^u| \ge \kappa^n$ and $|\lambda^s| \le \kappa^n$. On the other hand it is not true that uniform expansion/contraction for periodic points implies hyperbolicity. A one-dimensional example of a map with expansion at periodic points which is not hyperbolic is given by the Ulam-von Neumann map. This map is not expanding because the critical point, 0, is contained in the nonwandering set, $[-2, 2]$. The map satisfies the inequalities above with $\kappa = 2$. In fact for every periodic point of period n except the fixed point $p = -2$ we have $|Df^n(p)| = 2^n$. At $p = -2$ we have $n = 1$ yet $|Df^n(p)| = 4$.

THEOREM 1. *If f is a maximal entropy polynomial diffeomorphism, then*

(1) *Every periodic point is a saddle point.*

(2) *Let p be a periodic point of period n. Then $|\lambda^s(p)| < 1/d^n$, and $|\lambda^u(p)| > d^n$.*

(3) *The set of fixed points of f^n has exactly d^n elements.*

Let K be the set of points in $\mathbf{R}^2$ with bounded orbits. In Theorem 5.2 (below) we show that K is a Cantor set for every maximal entropy diffeomorphism. By [BS8, Prop. 4.7] this yields the strictness of the inequalities in (2). Note that the situation for maps of maximal entropy in one variable is different. In the case of the Ulam-von Neuman map, K is a connected interval, and the strict inequalities do not hold.

We note that by [BLS], condition (3) implies that f has maximal entropy. Thus we see that condition (3) provides a way to characterize the class of maximal entropy diffeomorphisms which makes no explicit reference to entropy. As was noted above, we can define the set of maximal entropy diffeomorphisms using either notion of complexity: they are the polynomial diffeomorphisms for which entropy is as large as possible, or equivalently those having as many periodic points as possible.

For the Ulam-von Neumann map the fixed point $p = -2$ which is the left-hand endpoint of K is distinguished as was noted above. This distinction has an analog in dimension two. Let p be a saddle point. Let $W^{s/u}(p)$ denote the stable and unstable manifolds of p. These sets are analytic curves. We say a periodic point p is s/u one-sided if only one component of $W^{s/u} - \{p\}$ meets K. For one-sided periodic points the estimates of Theorem 1 (2) can be improved. If p is s one-sided, then $|\lambda^s(p)| < 1/d^{2n}$; and if p is u one-sided, then $|\lambda^u(p)| > d^{2n}$.

The set of parameter values corresponding to diffeomorphisms of maximal entropy is closed, while the set of parameter values corresponding to hyperbolic diffeomorphisms is open. It follows that not all maximal entropy diffeomorphisms are hyperbolic. We now address the question: which properties of hyperbolicity fail in these cases.

THEOREM 2. *If f has maximal entropy, but K is not a hyperbolic set for f, then*

(1) *There are periodic points p and q in K (not necessarily distinct) so that $W^u(p)$ intersects $W^s(q)$ tangentially with order 2 contact.*

(2) *p is s one-sided, and q is u one-sided.*

(3) *The restriction of f to K is not expansive.*

Condition (1) is incompatible with K being a hyperbolic set. Thus this theorem describes a specific way in which hyperbolicity fails. Condition (3), which is proved in [BS8, Corollary 8.6], asserts that for any $\varepsilon > 0$ there are points x and y in K such that for all $n \in \mathbf{Z}$, $d(f^n(x), f^n(y)) \le \varepsilon$. Condition (3) is a topological property which is not compatible with hyperbolicity. We conclude that when f is not hyperbolic it is not even *topologically* conjugate to any hyperbolic diffeomorphism.

The proofs of the stated theorems owe much to the theory of quasi-hyperbolicity developed in [BS8]. In [BS8] we show that maximal entropy diffeomorphisms are quasi-hyperbolic. We also define a singular set $\mathcal{C}$ for any quasi-hyperbolic diffeomorphism. Much of the work of this paper is devoted to showing that in the maximal entropy case $\mathcal{C}$ is finite and consists of one-sided periodic points. Further analysis allows us to show that these periodic points have period either 1 or 2. In the case of quadratic mappings we can say exactly which points are one-sided.

We say that a saddle point is nonflipping if λ^u and λ^s are both positive.

Theorem 3. *Let $f_{a,b}$ be a quadratic mapping with maximal entropy. If $f_{a,b}$ preserves orientation, then the unique nonflipping fixed point of f is doubly one-sided. If f reverses orientation, then one of its fixed points is stably one-sided, and the other is unstably one-sided. There are no other one-sided points in either case.*

We can use our results to describe how hyperbolicity is lost on the boundary of the horseshoe region for Hénon diffeomorphisms. We focus on the orientation-preserving case here, but our results allow us to treat the orientation-reversing case as well. The parameter space for orientation-preserving Hénon diffeomorphisms is the set $\{(a, b) : b > 0\}$. Let us define the horseshoe region to be the largest connected open set containing the Devaney-Nitecki horseshoes and consisting of hyperbolic diffeomorphisms. Let $f = f_{a_0,b_0}$ be a point on the boundary of the horseshoe region. It follows from the continuity of entropy that f has maximal entropy. Theorem 1 tells us that f has the same number of periodic points as the horseshoes and that they are all saddles. In particular no bifurcations of periodic points occur at a_0, b_0. Let p_0 be the unique nonflipping fixed point for f. It follows from Theorem 2 that the stable and unstable manifolds of p_0 have a quadratic homoclinic tangency.

Figure 0.1 shows computer-generated pictures of mappings $f_{a,b}$ with $a = 6.0$, $b = 0.8$ on the left and $a = 4.64339843$, $b = 0.8$ on the right.[1] The curves pictured are the stable/unstable manifolds of the nonflipping saddle point p_0, which is the point marked by a disk in each picture at the lower leftmost point

[1]We thank Vladimir Veselov for using a computer program that he wrote to generate this second set of parameter values for us.

of intersection of the stable and unstable manifolds. The manifolds themselves are connected; the apparent disconnectedness is a result of clipping the picture to a viewbox. There are no tangential intersections evident on the left, while there appears to be a tangency on the right. This is consistent with the analysis above.

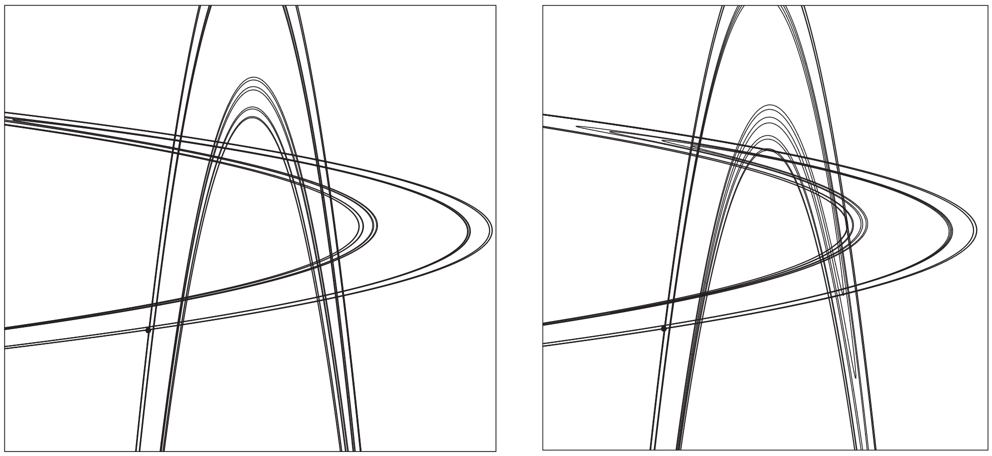


Figure 0.1

1. Background

Despite the fact that we study real polynomial diffeomorphisms, the proofs of the results of this paper depend on the theory of complex polynomial diffeomorphisms. In particular the theory of quasi-hyperbolicity which lies at the heart of much of what we do is a theory of complex polynomial diffeomorphisms. The notation we use in the paper is chosen to simplify the discussion of complex polynomial diffeomorphisms. A polynomial diffeomorphism of $\mathbf{C}^2$ will be denoted by $f_{\mathbf{C}}$, or simply f, when no confusion will result. Let $\tau(x,y) = (\overline{x},\overline{y})$ denote complex conjugation in $\mathbf{C}^2$. The fixed point set of complex conjugation in $\mathbf{C}^2$ is exactly $\mathbf{R}^2$. We say that f is real when $f : \mathbf{C}^2 \to \mathbf{C}^2$ has real coefficients, or equivalently, when f commutes with τ. When f is real we write $f_{\mathbf{R}}$ for the restriction of f to $\mathbf{R}^2$.

Let us consider mappings of the form $f = f_1 \circ \cdots \circ f_m$, where

$$f_j(x,y) = (y, p_j(y) - a_j x), \tag{1.1}$$

p_j is a polynomial of degree $d_j \geq 2$. If we set $d = d_1 \ldots d_m$, then it is easily seen that if f has the form 1.1 then the degree of f is d. The degree of f^{-1} is also d and, since $h(f_{\mathbf{R}}) = h(f_{\mathbf{R}}^{-1})$ it follows that f has maximal entropy if and only if f^{-1} does.

Proposition 1.1. *If a real polynomial diffeomorphism f has maximal entropy, then it is conjugate via a real polynomial diffeomorphism to a real polynomial diffeomorphism of the same degree in the form* (1.1).

Proof. According to [FM] a polynomial diffeomorphism $f_{\mathbf{R}}$ of $\mathbf{R}^2$ is conjugate via a polynomial diffeomorphism, g, to a diffeomorphism of the form $e(x,y) = (\alpha x + p(y), \beta y + \gamma)$ or to a diffeomorphism of the form (1.1). Since $f_{\mathbf{R}}$ has positive entropy it is not conjugate to a diffeomorphism of the form $e(x,y)$. In [FM] it is also shown that a diffeomorphism in the form (1.1) has minimal entropy among all elements in its conjugacy class so $\deg(g_{\mathbf{R}}) \le \deg(f_{\mathbf{R}})$. Since entropy is a conjugacy invariant we have:

$$\log\deg(g_{\mathbf{R}}) \le \log\deg(f_{\mathbf{R}}) = h(f_{\mathbf{R}}) = h(g_{\mathbf{R}}).$$

Again by [FM], $h(g_{\mathbf{R}}) \le \log\deg(g_{\mathbf{R}})$ and so we conclude that the inequalities are equalities and that $\deg(g_{\mathbf{R}}) = \deg(f_{\mathbf{R}})$.

Thus we may assume that we are dealing with maximal entropy polynomial diffeomorphisms written in form (1.1). The mapping $f_{a,b}$ in the introduction is not in the form (1.1), but the linear map $L(x,y) = (-y,-x)$ conjugates $f_{a,b}$ to

$$(x,y) \mapsto (y, y^2 - a - bx).$$

In Sections 1 through 4, we are dealing with polynomial diffeomorphisms of arbitrary degree, and we will assume that they are in the form (1.1).

We recall some standard notation for general polynomial diffeomorphisms of $\mathbf{C}^2$. The set of points in $\mathbf{C}^2$ with bounded forward orbits is denoted by K^+. The set of points with bounded backward orbits is denoted by K^-. The sets $J^\pm$ are defined to be the boundaries of $K^\pm$. The set J is $J^+ \cap J^-$ and the set K is $K^+ \cap K^-$. Let $\mathcal{S}$ denote the set of saddle points of f. For a general polynomial diffeomorphism of $\mathbf{C}^2$ the closure of $\mathcal{S}$ is denoted by J^*. For a real polynomial diffeomorphism of $\mathbf{C}^2$ each of these f-invariant sets is also invariant under τ. For a real maximal entropy mapping it is proved in [BLS] that $J^* = J = K$ and furthermore that this set is real; that is $K \subset \mathbf{R}^2$.

For $p \in \mathcal{S}$, there is a holomorphic immersion $\psi_p^u : \mathbf{C} \to \mathbf{C}^2$ such that $\psi_p^u(0) = p$ and $\psi_p^u(\mathbf{C}) = W^u(p)$. The immersion ψ_p^u is well defined up to multiplication by a nonzero complex scalar. By using a certain potential function we can choose distinguished parametrizations. Define G^+ by the formula

$$G^+(x,y) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(x,y)|.$$

Changing the parameter in the domain via a change of coordinates $\zeta' = \alpha\zeta$, $\alpha \ne 0$, we may assume that ψ_p^u satisfies

$$\max_{|\zeta|\le 1} G^+ \circ \psi_p^u(\zeta) = 1.$$

With this normalization, ψ_p^u is uniquely determined modulo rotation; that is, all such mappings are of the form $\zeta \mapsto \psi_p^u(e^{i\theta}\zeta)$.

When the diffeomorphism f is real and $p \in \mathbf{R}^2$ we may choose the parametrization of W_p^u so that it is real, which is to say that $\psi = \psi_p^u$ satisfies $\psi(\bar{\zeta}) = \tau \circ \psi(\zeta)$. In this case the set $\psi^{-1}(K) = \psi^{-1}(K^+)$ is symmetric with respect to the real axis in $\mathbf{C}$ and the parametrization is well defined up to multiplication by ± 1. In the real case $\psi(\mathbf{R}) \subset \mathbf{R}^2$, and the set $\psi(\mathbf{R})$ is equal to the unstable manifold of p with respect to the map $f_{\mathbf{R}}$.

When f is real and has maximal entropy more is true. In this case every periodic point is contained in $\mathbf{R}^2$. Let ψ be a real parametrization. Since ψ is injective, the inverse image of the fixed point set of τ in $\mathbf{C}^2$ is contained in the fixed point set of $\zeta \mapsto \bar{\zeta}$ in $\mathbf{C}$. Thus $\psi^{-1}(\mathbf{R}^2) = \mathbf{R}$, and $\psi^{-1}(K) \subset \mathbf{R}$. If p is a u one-sided periodic point then K meets only one component of $W^u(p, \mathbf{R})$ so that $\psi^{-1}(K)$ is contained in one of the rays $\{\zeta \in \mathbf{R} : \zeta \geq 0\}$ or $\{\zeta \in \mathbf{R} : \zeta \leq 0\}$.

We define the set of all such unstable parametrizations as $\psi_{\mathcal{S}}^u := \{\psi_p^u : p \in \mathcal{S}\}$. For $\psi \in \psi_p^u$ there exist $\lambda = \lambda_p^u \in \mathbf{R}$ and $\tilde{f}\psi \in \psi_{fp}^u$ such that

$$(\tilde{f}\psi)(\zeta) = f(\psi(\lambda^{-1}\zeta)) \tag{1.2}$$

for $\zeta \in \mathbf{C}$.

A consequence of the fact that $\psi^{-1}(K) \subset \mathbf{R}$ [BS8, Th. 3.6] is that

$$|\lambda_p| \geq d. \tag{1.3}$$

Furthermore if p is u one-sided then

$$|\lambda_p| \geq d^2.$$

The condition that $|\lambda_p|$ is bounded below by a constant greater than one is one of several equivalent conditions that can serve as definitions of the property of quasi-expansion defined in [BS8]. Thus, as in [BS8], we see that f and f^{-1} are quasi-expanding. A consequence of quasi-expansion is that $\psi_{\mathcal{S}}^u$ is a normal family (see [BS8, Th. 1.4]). In this case we define Ψ^u to be the set of normal (uniform on compact subsets of $\mathbf{C}$) limits of elements of $\psi_{\mathcal{S}}^u$. Let $\Psi_p^u := \{\psi \in \Psi^u : \psi(0) = p\}$. It is a further consequence of quasi-expansion that Ψ^u contains no constant mappings.

For $p \in J$, the mappings in Ψ_p^u have a common image which we denote by $V^u(p)$ ([BS8, Lemma 2.6]). Let $W^u(p)$ denote the "unstable set" of p. This consists of q such that

$$\lim_{n \to +\infty} \operatorname{dist}(f^{-n}p, f^{-n}q) = 0.$$

It is proved in [BS8, Prop. 1.4] that $V^u(p) \subset W^u(p)$. It follows that $V^u(p) \subset K^-$. In many cases the stable set is actually a one-dimensional complex manifold. When this is the case it follows that $V^u(p) = W^u(p)$.

Let $V^u_\varepsilon(p)$ denote the component of $V^u(p)\cap B(p,\varepsilon)$ which contains p. For ε sufficiently small $V^u_\varepsilon(p)$ is a properly embedded variety in $B(p,\varepsilon)$. Let E^u_p denote the tangent space to this variety at p. It may be that the variety $V^u_\varepsilon(p)$ is singular at p. In this case we define the tangent cone to be the set of limits of secants.

For $\psi\in\Psi^u$ we say that $\mathrm{Ord}(\psi)=1$ if $\psi'(0)\neq 0$; and if $k>1$, we say $\mathrm{Ord}(\psi)=k$ if $\psi'(0)=\cdots=\psi^{(k-1)}(0)=0$, $\psi^{(k)}(0)\neq 0$. Since Ψ^u contains no constant mappings, $\mathrm{Ord}(\psi)$ is finite for each ψ. If $\psi\in\Psi^{s/u}$, and if $\mathrm{Ord}(\psi)=k$, then there are $a_j\in\mathbf{C}^2$ for $k\le j<\infty$ such that

$$\psi(\zeta)=p+a_k\zeta^k+a_{k+1}\zeta^{k+1}+\dots\ .$$

It is easy to show that the tangent cone E^u_p to the variety $V^u_\varepsilon(p)$ is actually the complex subspace of the tangent space $T_p\mathbf{C}^2$ spanned by a_k. One consequence of this is that the span of the a_k term depends only on p and not on the particular mapping in Ψ^u_p. (It is possible however that different parametrizations give different values for k.) A second consequence is that even when the variety $V^u_\varepsilon(p)$ is singular the tangent cone is actually a complex line and, in what follows, we will refer to E^u_p as the tangent space. The mapping $\psi\mapsto\mathrm{Ord}(\psi)$ is an upper semicontinuous function on Ψ^u. For $p\in J$, we set $\tau^u(p)=\max\{\mathrm{Ord}(\psi):\psi\in\Psi^u_p\}$. The reality of ψ is equivalent to the condition that $a_j\in\mathbf{R}^2$.

Since f^{-1} is also quasi-expanding, we may repeat the definitions above with f replaced by f^{-1} and unstable manifolds replaced by stable manifolds; and in this case we replace the superscript u by s. We set

$$J_{j,k}=\{p\in J:\tau^s(p)=j,\tau^u(p)=k\},$$

and define

$$\lambda^{s/u}(p,n)=\lambda^{s/u}_p\cdots\lambda^{s/u}_{f^{n-1}p}.$$

Iterating the mapping $\tilde f$ defined above, we have mappings $\tilde f^n:\Psi^{s/u}_p\to\Psi^{s/u}_{f^np}$ defined by

$$\tilde f^n(\psi^{s/u}(\zeta))=f^n\circ\psi^{s/u}(\lambda^{s/u}(p,n)^{-1}\zeta). \tag{1.5}$$

By (1.3),

$$|\lambda^s(p,n)|\le d^{-n},\qquad |\lambda^u(p,n)|\ge d^n. \tag{1.6}$$

We will give here the proof of item (3) of Theorem 1. Since f and f^{-1} are quasi-expanding it follows that every periodic point in J^* is a saddle. Since every periodic point is contained in K and $K=J^*$ it follows that every periodic point is a saddle. According to [FM] the number of fixed points of $f^n_{\mathbf{C}}$ counted with multiplicity is d^n. Since all periodic points are saddles they all have multiplicity one (multiplicity is computed with respect to $\mathbf{C}^2$ rather than $\mathbf{R}^2$). Thus the set of fixed points of f^n has cardinality d^n. Since $K\subset\mathbf{R}^2$ all of these points are real.

2. The maximal entropy condition and its consequences

Let us return to our discussion of the maximal entropy condition. The argument that $\psi^{-1}(\mathbf{R}^2) = \mathbf{R}$ depended on the injectivity of ψ. Even though elements of Ψ^u are obtained by taking limits of elements of $\psi^u_{\mathcal{S}}$ it does not follow that $\psi \in \Psi^u$ is injective. In fact it need not be the case that $\psi^{-1}(\mathbf{R}^2) \subset \mathbf{R}$, but the following proposition shows that a related condition still holds.

PROPOSITION 2.1. *For $\psi \in \Psi^u$, $\psi^{-1}(K) \subset \mathbf{R}$.*

Proof. The image of ψ is contained in K^-, it follows that $\psi^{-1}(K^+) = \psi^{-1}(K)$ for $\psi \in \psi^u_{\mathcal{S}}$. Since G^+ is harmonic on $\mathbf{C}^2 - K^+$, it follows that $G^+ \circ \psi$ is harmonic on $\mathbf{C} - \mathbf{R} \subset \mathbf{C} - \psi^{-1}K$. By Harnack's principle, $G^+ \circ \psi$ is harmonic on $\mathbf{C} - \mathbf{R}$ for any limit function $\psi \in \Psi^u$. If $G^+ \circ \psi$ is zero at some point $\zeta \in \mathbf{C} - \mathbf{R}$ with, say, $\Im(\zeta) > 0$, then it is zero on the upper half plane by the minimum principle. By the invariance under complex conjugation, it is zero everywhere. But this means that $\psi(\mathbf{C}) \subset \{G^+ = 0\} = K^+$. By (1.4), this means that $\psi(\mathbf{C}) \subset K \subset \mathbf{R}^2$. Since K is bounded, ψ must be constant. But this is a contradiction because Ψ^u contains no constant mappings. $\square$

Our next objective is to find a bound on $\mathrm{Ord}(\psi)$ for $\psi \in \Psi^u$. Set $m^u = \max_J \tau^u$ and consider the maximal index j so that J_{j,m^u} is nonempty. Thus J_{j,m^u} is a maximal index pair in the language of [BS8]. By [BS8, Prop. 5.2], J_{j,m^u} is a hyperbolic set with stable/unstable subspaces given by $E^{s/u}_p$.

The notion of a homogeneous parametrization was defined in [BS8, §6]. A homogeneous parametrization of order m, $\psi : \mathbf{C} \to \mathbf{C}^2$, is one that can be written as $\psi(\zeta) = \phi(a\zeta^m)$ for some $a \in \mathbf{C} - \{0\}$ and some nonsingular $\phi : \mathbf{C} \to \mathbf{C}^2$. It follows from [BS8, Lemma 6.5] that for every p in a maximal index pair such as J_{j,m^u} there is a homogeneous parametrization in Ψ^u_p with order m^u.

PROPOSITION 2.2. *Suppose that $\psi \in \Psi^u$, is a homogeneous parametrization of order m. Then it follows that $m \leq 2$.*

Proof. By Proposition 2.1, $\psi^{-1}(J) \subset \mathbf{R}$. And from the condition $\psi(\zeta) = \phi(\zeta^m)$ it follows that $\psi^{-1}(J)$ is invariant under rotation by m-th roots of unity. Now $\psi^{-1}(J)$ is nonempty (containing 0) and a nonpolar subset of $\mathbf{C}$, since it is the zero set of the continuous, subharmonic function $G^+ \circ \psi$. Since a polar set contains no isolated points it follows that $\psi^{-1}(J)$ contains a point $\zeta_0 \neq 0$. Since the rotations of ζ_0 by the m-th roots of unity must lie in $\mathbf{R}$, it follows that $m \leq 2$. $\square$

COROLLARY 2.4. $J = J_{1,1} \cup J_{2,1} \cup J_{1,2} \cup J_{2,2}$.

There are three possibilities to consider.

(1) $J_{2,1} \cup J_{1,2} \cup J_{2,2}$ is empty. In this case it follows from [BS8] that f is hyperbolic.

(2) $J_{2,2}$ is empty and $J_{2,1} \cup J_{1,2}$ is nonempty. In this case $J_{2,1}$ and $J_{1,2}$ are maximal index pairs and both are hyperbolic sets.

(3) $J_{2,2}$ is nonempty but $J_{2,1} \cup J_{1,2}$ is empty. In this case $J_{2,2}$ is a maximal set and is hyperbolic.

(4) $J_{2,2}$ is nonempty and $J_{2,1} \cup J_{1,2}$ is nonempty. This is the only case in which we do not know *a priori* that points in $J_{2,1} \cup J_{1,2}$ are regular.

PROPOSITION 2.5. *For $p \in J$, let $\psi \in \Psi_p^{s/u}$ be given. Then $\zeta \mapsto \psi(\zeta)$ is at most two-to-one. If ψ is two-to-one, then it has one critical point, which must be real.*

Proof. This follows from Proposition 2.2 and [BS8, Lemma 4.6]. □

PROPOSITION 2.6. *Let p be in $J_{*,2}$, and let $V_\varepsilon^u(p)$ be regular. If $\psi \in \Psi_p^u$ has order 2, there is an embedding $\phi : \mathbf{C} \to \mathbf{C}^2$ such that $\psi(\zeta) = \phi(\zeta^2)$.*

Proof. By Proposition 2.5, ψ has at most one critical point, which must be $\zeta = 0$. Thus all points of $\psi(\mathbf{C}) - \{p\}$ are regular. Since $V_\varepsilon^u(p)$ is regular, it follows that $\psi(\mathbf{C})$ is regular, so there is an embedding $\phi : \mathbf{C} \to \mathbf{C}^2$ with $\phi(\mathbf{C}) = \psi(\mathbf{C})$. By Proposition 2.3, $\tau^u \le 2$, and so $J_{*,2}$, being a set of maximal order, is compact. Thus $\alpha(p) \subset J_{*,2}$, and so the result follows from [BS8, Prop. 4.4]. □

If $\psi \in \Psi_p^u$ is one-to-one, then $\psi(\mathbf{C}) \cap \mathbf{R}^2 = \psi(\mathbf{R})$. (For if there is a point $\zeta \in \mathbf{C} - \mathbf{R}$ with $\psi(\zeta) \in \mathbf{R}^2$, then we would also have $\psi(\bar{\zeta}) \in \mathbf{R}^2$. But $\zeta \ne \bar{\zeta}$, contradicting the assumption that ψ is one-to-one.) If ψ is 2-to-1, then ψ has a critical point $t_0 \in \mathbf{R}$. Let us suppose that ψ has a quadratic singularity at $\zeta = 0$, i.e. $\psi(\zeta) = p + a_2\zeta^2 + O(|\zeta|^3)$. If $\psi(\mathbf{C}) \cap \mathbf{R}^2$ is a smooth curve, then p divides this curve into two pieces: in Figure 2.1 the image of $\mathbf{R}$ under ψ is drawn dark, and the image of $i\mathbf{R}$ is shaded. By Proposition 2.1, the shaded region is disjoint from J.

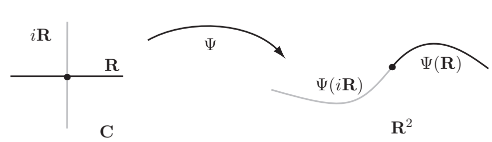


Figure 2.1

Recall that the tangent space to $V_\varepsilon^{s/u}(p)$ at p is $E_p^{s/u}$. We say that $V_\varepsilon^u(p)$ and $V_\varepsilon^s(p)$ intersect tangentially at p if $E_p^s = E_p^u$. We recall that $\alpha(p)$, the α-limit set of p, is the set of limit points of $\{f^{-n}p : n \geq 0\}$, and the ω-limit set, $\omega(p)$, is the set of limit points of $\{f^n p : n \geq 0\}$. Compactness of J implies that $\alpha(p)$ and $\omega(p)$ are nonempty. The following are consequences of Theorem 7.3 of [BS8].

THEOREM 2.7. *Suppose the varieties $V_\varepsilon^u(p)$ and $V_\varepsilon^s(p)$ intersect tangentially at $p \in J$ (i.e. suppose $E_p^s = E_p^u$). Then the α- and ω-limit sets satisfy $\alpha(p) \subset J_{2,*}$ and $\omega(p) \subset J_{*,2}$. Further, p belongs to $J_{1,1}$, and the varieties of $V_p^{s/u}$ are regular at p.*

THEOREM 2.8. *If $V_\varepsilon^s(p)$ and $V_\varepsilon^u(p)$ are tangent at $p \in J$, then the tangency is at most second order; i.e., $V_\varepsilon^s(p)$ and $V_\varepsilon^u(p)$ have different curvatures at p.*

3. Finiteness of singular points

Let us consider a point $p \in J$ where the varieties $V_\varepsilon^s(p)$ and $V_\varepsilon^u(p)$ are nonsingular and intersect transversally. We may perform a real, affine change of coordinates so that in the new coordinate (x, y) we have $p = (0,0)$, $V_\varepsilon^u(p)$ is tangent to the x-axis at p, and $V_\varepsilon^s(p)$ is tangent to the y-axis at p. let $\pi_s(x,y) = y$ and $\pi_u(x,y) = x$. For $\varepsilon > 0$ let $\Delta(\varepsilon) = \{\zeta \in \mathbf{C} : |\zeta| < \varepsilon\}$. For $q \in \Delta^2(\varepsilon) \cap J$ let $V^{s/u}(q, \varepsilon)$ denote the connected component of $V_q^{s/u} \cap \pi_{s/u}^{-1}\Delta(\varepsilon)$ containing q. For $\varepsilon > 0$ small,

$$\pi_s : V^u(p,\varepsilon) \subset \Delta(\varepsilon/2), \quad \pi_u : V^s(p,\varepsilon) \subset \Delta(\varepsilon/2), \tag{3.1}$$

and

$$\pi_{s/u} : V^{s/u}(p,\varepsilon) \to \Delta(\varepsilon) \text{ are proper maps of degree 1.} \tag{3.2}$$

By [BS8, Lemmas 2.1 and 2.2] the varieties $V_\varepsilon^{s/u}(q)$ depend continuously on q. Thus for $\delta > 0$ small, (3.1) will hold for the varieties at q if $q \in \Delta^2(\delta) \cap J$, and the projections $\pi_{s/u} : V^{s/u}(q,\varepsilon) \to \Delta(\varepsilon)$ will be proper.

Let us define $\mathcal{V}^s$ as the set of varieties $V^s(q,\varepsilon)$ for $q \in \overline{\Delta}^2(\delta) \cap J$. Further, we define $\mathcal{V}_j^s$ as the set of varieties $V^s \in \mathcal{V}^s$ such that the projection $\pi_s|_{V^s} : V^s \to \Delta(\varepsilon)$ has mapping degree j. In a similar way, we define $\mathcal{V}^u$ and $\mathcal{V}_j^u$. It is evident that elements of $\mathcal{V}_1^{s/u}$ are represented as graphs of analytic functions, and so $\mathcal{V}_1^{s/u}$ is a compact family of varieties.

LEMMA 3.1. *If $V^s \in \mathcal{V}_j^s$, $V^u \in \mathcal{V}_k^u$, then the intersection $V^s \cap V^u$ consists of jk points (counted with "intersection" multiplicity). If ε and δ are sufficiently small, then $\mathcal{V}^s = \mathcal{V}_1^s \cup \mathcal{V}_2^s$.*

Proof. If V^s is a j-fold branched cover over $\Delta(\varepsilon)$, then it is homologous to j times the class of $\{0\} \times \Delta(\varepsilon)$ in $H_2(\Delta^2(\varepsilon), \Delta(\varepsilon) \times \partial\Delta(\varepsilon))$. Similarly, V^u is homologous to k times the class of $\Delta(\varepsilon) \times \{0\}$ in $H_2(\Delta^2(\varepsilon), \partial\Delta(\varepsilon) \times \Delta(\varepsilon))$. Thus the intersection number of the classes $[V^s]$ and $[V^u]$ is jk times the intersection number of $\{0\} \times \Delta(\varepsilon)$ and $\Delta(\varepsilon) \times \{0\}$, which is 1.

For $q \in J \cap \Delta^2(\delta)$, we let $j = j_q$ denote the branching degree of $\pi_u : V^u(q, \varepsilon) \to \Delta(\varepsilon)$. Let us take a sequence $q_k \to p$ such that $j = j_{q_k}$ is constant and $\psi^u_{q_k} \to \psi^u \in \Psi^u_p$. Let $\omega_k \subset \mathbf{C}$ denote the connected component of $\psi^{-1}_{q_k}(V^u(q_k, \varepsilon))$ containing 0. For each $x_0 \in \Delta(\varepsilon)$ and each k we have $\#\{\zeta \in \omega_k : \pi_u \circ \psi^u_{q_k}(\zeta) = x_0\} = j$. By [BS8, Lemma 2.1] there exists $r > 0$ such that $\omega_k \subset \{|\zeta| < r\}$ for all k. It follows that $\#\{|\zeta| \le r : \pi_u \circ \psi^u(\zeta) = x_0\} \ge j$. By (3.2) we have a holomorphic map $\pi_u^{-1} : \Delta(\varepsilon) \to V^u(p, \varepsilon)$, so we conclude that $\pi_u^{-1}\pi_u\psi_p = \psi_p$ is at least j-to-1. It follows from Proposition 2.3 that $j \le 2$. $\square$

The sets $S := \Delta^2(\varepsilon) \cap \mathbf{R}^2$ and $S_0 := \Delta^2(\delta) \cap \mathbf{R}^2$ are squares in $\mathbf{R}^2$. We define the *vertical boundary* $\partial_v S$ (resp. the *horizontal boundary* $\partial_h S$) as the portion of (the square) ∂S which is vertical (resp. horizontal) with respect to the coordinate system given by the projections (π_s, π_u). For $q \in J \cap S_0$, we define γ^s_q as the intersection $V^s(q, \varepsilon) \cap \mathbf{R}^2$. We define Γ^s to be the set of curves γ^s_q with $q \in S_0$ and Γ^s_j as the set of curves $\gamma^s \cap V^s$ with $V^s \in \mathcal{V}^s_j$. The layout of this configuration is illustrated in Figure 3.1: $\gamma^s_p \in \Gamma^s_1$, and $\gamma^s_q, \gamma^s_r \in \Gamma^s_2$. By the reality condition, $\gamma^{s/u}_p \in \Gamma^{s/u}$ is a one-dimensional set, and so $\gamma^{s/u}_p$ is regular if and only if $V^{s/u}_\varepsilon(p)$ is regular.

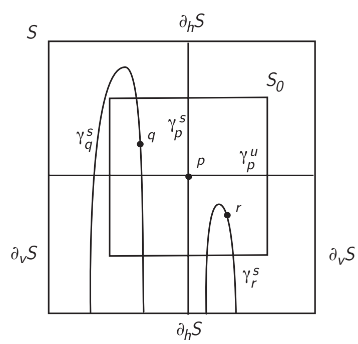


Figure 3.1

Corollary 3.2. *If $\gamma^s \in \Gamma^s_j$ and $\gamma^u \in \Gamma^u_k$, then the number of points of $\gamma^s \cap \gamma^u$ counted with multiplicity, is equal to jk.*

Proof. This is a direct consequence of Lemma 3.1 and the fact that $V^s \cap V^u \subset \mathbf{R}^2$. $\square$

If $\psi \in \Psi_p^u$ has order 2, and if γ_p^u is regular, then by Proposition 2.6 there is an embedding ϕ such that $\psi(\zeta) = \phi(\zeta^2)$. It follows that $\psi(\mathbf{C}) \cap \mathbf{R}^2 = \psi(\mathbf{R}) \cup \psi(i\mathbf{R})$. Working inside a box $\mathcal{B}$, we write $(\gamma_p^u)^r := \psi(\mathbf{R}) \cap \gamma_p^u$ and $(\gamma_p^u)^i := \psi(i\mathbf{R}) \cap \gamma_p^u$. The phantom gray region $(\gamma_p^u)^i$, as in Figure 2.1, is disjoint from J. We state this observation as follows.

Lemma 3.3. *If $p \in J_{*,2}$ and γ_p^u is regular, then p is u one-sided; if $p \in J_{2,*}$ and γ_p^s is regular, then p is s one-sided.*

Let q be a point of J for which V_q^s is regular For $\psi \in \Psi_q^s$, we define the set ω_ψ as the connected component of $\psi^{-1}V^s(q,\varepsilon)$ containing the origin. Since $\pi_u \circ \psi$ is an entire function, ω_ψ is simply connected. By the reality condition on ψ, ω_ψ is invariant under complex conjugation. Thus $\omega_\psi \cap \mathbf{R}$ is a (connected) interval $(-a, b)$. It follows that γ_q^s is a connected submanifold of S. We refer to $\psi(-a)$ and $\psi(b)$ as the endpoints of γ_1^s. Since $\partial V^s(q,\varepsilon) \subset \Delta(\varepsilon) \times \partial\Delta(\varepsilon)$, it follows that the endpoints of γ_q^s lie in $\partial_h S$.

Lemma 3.4. *If $\gamma \in \Gamma^s$, then $\gamma \in \Gamma_1^s$ if the endpoints of γ lie in different components of $\partial_h S$. Otherwise (if the endpoints lie in the same component of $\partial_h S$), $\gamma \in \Gamma_2^s$.*

Proof. The horizontal boundary $\partial_v S$ consists of fibers of the projection π_s intersected with $\mathbf{R}^2$. By Lemma 3.1, the multiplicity of the projection is no greater than 2. If γ^s intersects one of the fibers in two points, then the multiplicity is in fact equal to 2.

Now we prove the first assertion of the lemma. Suppose that γ^s has one endpoint in each component of $\partial_h S$. Then for each point $t \in \Delta(\varepsilon) \cap \mathbf{R}$, there is a point $s \in (-a, b)$ such that $\psi(s) = t$. Now there cannot be a point $\zeta \in \mathbf{C} - \mathbf{R}$ with $\psi(\zeta) = t$, for by the reality condition we would have $\psi(\overline{\zeta}) = t$, which would give three solutions. Finally, we cannot have the situation where $\pi_s^{-1}(t) \cap \gamma^s$ consists of exactly two points. For, in this case, we may assume that $\pi_s(\psi(-a)) = -\varepsilon$ and $\pi_s(\psi(b)) = \varepsilon$. Then, as in calculus, there must be a nearby $t' \in (-\varepsilon, \varepsilon)$ for which $\pi_s^{-1}(t')$ consists of three points. $\square$

Lemma 3.5. *When $\delta$is shrunk, if necessary, it follows that if $q \in J_{2,*} \cap S_0$ and if γ_q^s is regular, then $\gamma_q^s \in \Gamma_1^s$. Similarly, if $q \in J_{*,2} \cap S_0$, and if γ_q^u is regular, then $\gamma_q^u \in \Gamma_1^u$.*

Proof. It follows from Lemma 3.1 and Corollary 3.2 that γ_q^s belongs to Γ_1^s or Γ_2^s. If there is no δ satisfying the conclusion of the lemma, then there is a sequence $q_j \to 0$, $q_j \in J_{2,*}$, $\gamma_{q_j}^s$ regular, and $\gamma_{q_j}^s \in \Gamma_2^s$. Since $\gamma_{q_j}^s$ is regular, there exists $\psi_j^s \in \Psi_{q_j}^s$ and a holomorphic embedding ϕ_j such that $\psi_j^s(\zeta) = \phi_j(\zeta^2)$. We may extract a subsequence such that there is a limit $\psi_j^s \to \psi \in \Psi_p^s$.

Now since $\psi^s_{q_j} \in \Gamma^s_2$ it follows that for $x_0 \in \Delta(\varepsilon)$, $\pi_s^{-1}(x_0) \cap V^s_{p_j}$ consists of two points. Thus

$$\#(\pi_s \circ \psi^s_j)^{-1}(x_0) = \#\{\zeta \in \mathbf{C} : \pi_s \phi(\zeta^2) = x_0\} \geq 4.$$

By [BS8, Lemma 2.1] this set is contained in a disk $\{|\zeta| < r\}$, independent of j. Letting $j \to \infty$ we obtain $\#(\pi_x \circ \psi)^{-1}(x_0) \geq 4$. By (3.2), there exists $a_0 \in V^s_p$ such that $V^s_p \cap \pi_s^{-1}(x_0) = \{a_0\}$ is a single point so that $\#(\pi_s \psi)^{-1}(x_0) = \#\psi^{-1}(a_0) \geq 4$, which contradicts Lemma 2.3. □

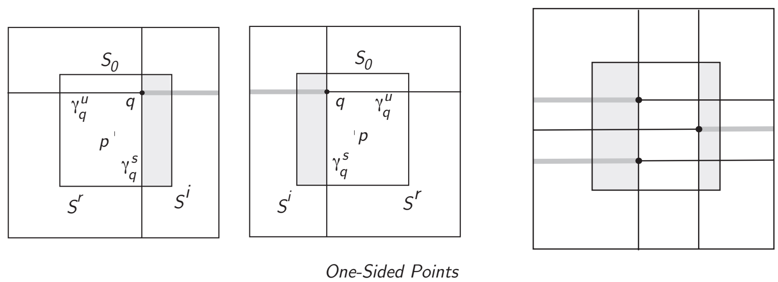


Figure 3.2

We define a *regular box* $\mathcal{B}$ about p to be a pair $(\Delta^2(\varepsilon), \Delta^2(\delta))$ with $\varepsilon, \delta > 0$ chosen such that the conclusions of Lemmas 3.1 through 3.5 hold. Let $\mathcal{B}$ be a regular box, and let $q \in S_0 \cap J_{*,2}$ be a point with $\gamma^s_q \in \Gamma^s_1$. Then $S - \gamma^s_q$ consists of two components, which we may label S^r and S^i, as in Figure 3.2. That is, S^r contains the variety of $\psi(\mathbf{R})$ at q, and S^i contains the local variety of the phantom region $\psi(i\mathbf{R})$ at q.

LEMMA 3.6. *For $p \in J_{2,2}$, a regular box $\mathcal{B}$ about p may be constructed. For $q \in S_0 \cap J_{2,2}$, γ^s_q belongs to Γ^s_1. If we split $S - \gamma^s_q = S^r \cup S^i$ as above, then $S_0 \cap S^i \cap J = \emptyset$. The corresponding statement holds for $S - \gamma^u_q$.*

Proof. By hypothesis, $J_{2,2} \neq \emptyset$. Thus $2 = \max \tau^u = \max \tau^s$, so that by [BS8, Prop. 5.2] $J_{2,2}$ is a hyperbolic set. It follows that $\gamma^{s/u}_q$ are nonsingular and transversal. In particular, for $q = p$, we may construct a regular box $\mathcal{B}$ about p. If $\mathcal{B}$ is sufficiently small, then it follows by hyperbolicity that $\gamma^{s/u} \in \Gamma^{s/u}_1$ for all $q \in S_0 \cap J_{2,2}$.

To complete the proof, we must show that $S_0 \cap S^i \cap J = \emptyset$. For otherwise, if there exists $r \in S_0 \cap S^i \cap J$, then by Corollary 3.2 $\gamma^s_r \cap \gamma^u_q \neq \emptyset$. Since γ^s_r cannot intersect γ^s_q, it follows that $\gamma^s_r \cap \gamma^u_q$ must lie inside the phantom region of γ^u_q, which is forbidden. □

THEOREM 3.7. *$J_{2,2}$ is finite.*

Proof. By Lemma 3.6 we may construct a regular box $\mathcal{B}$ about any $p \in J_{2,2}$. Let us select a finite family of boxes $\mathcal{B}$ such that the corresponding sets S_0 cover $J_{2,2}$. Let us fix one of these sets S_0. If $q \in J_{2,2} \cap S_0$, then q corresponds to one of the four types of doubly one-sided points pictured on the left hand side of Figure 3.3. For each of these four cases, it follows from Lemma 3.6 that the set $J \cap S$ must lie in the quadrant bounded by the solid lines.

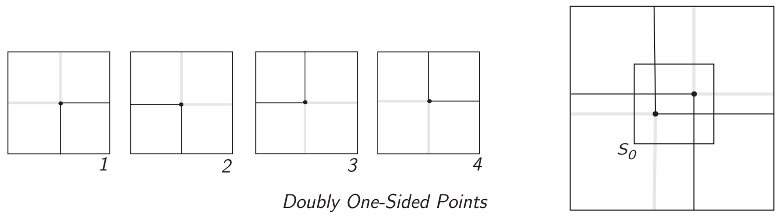


Figure 3.3

Now we consider the possibility that $S_0 \cap J_{2,2}$ might consist of more than one point. Let us start by supposing that a pair of points $p_1, p_2 \in J$ belong to the same box S_0. The only way that two types of box can both occupy the same set S is if they are of type 1 and 3 or type 2 and 4. The situation where points of type 2 and 4 occupy the same box S is pictured on the right-hand side of Figure 3.3, and by Lemma 3.6 $J \cap S_0$ is contained in the shaded region. It follows that $J \cap S_0$ cannot contain a third point r. If there were, it would lie in the shaded portion; but as r is one-sided, then the phantom (gray prong) region would necessarily intersect the sides of the shaded region.

Since we have covered $J_{2,2}$ by finitely many sets S_0 and $\#(J_{2,2} \cap S) \leq 2$, it follows that $J_{2,2}$ is finite. □

THEOREM 3.8. *If $p \in J_{2,1} \cup J_{1,2}$, then $V_p^{s/u}$ is regular at p.*

Proof. Without loss of generality we may assume that $p \in J_{1,2}$. If $\alpha(p) \cap J_{1,2} \neq \emptyset$ it follows from [BS8, Th. 5.5] that $V_\varepsilon^u(p)$ is regular. The other possibility is that $\alpha(p) \subset J_{2,2}$ by Corollary 2.4. Since $J_{2,2}$ is hyperbolic and finite, it is an isolated hyperbolic set. By [R, p. 380] there exists $q \in J_{2,2}$ such that $p \in W^u(q)$. By (1.4) it follows that $V_p^u \subset W^u(q)$, and so $V_\varepsilon^u(p)$ is regular. □

For $p \in J_{1,2}$ it follows from Theorem 3.8 that γ_p^u is regular. By Theorem 2.7, γ_p^s is regular and transverse to γ_p^u. If $p \in J_{2,2}$, then as in the proof of Theorem 3.7 γ_p^s and γ_p^u are regular and transverse at p. Thus if $p \in J_{*,2}$ we may construct a regular box $\mathcal{B}$ centered at p. The following result will involve shrinking this box $\mathcal{B}$. Before giving the proof, we make an observation concerning the relationship between shrinking and the multiplicities of varieties. Let

π_s and π_u be the projections associated with $\mathcal{B}$. If $V \in \mathcal{V}_m^u$, then the projection $\pi_u : V \to \Delta(\varepsilon)$ has mapping degree m. This is equivalent to the statement that the total multiplicities of the critical points of π_u is $m-1$. It follows that if we shrink the box $\mathcal{B}$ to $\mathcal{B}' := \{q \in \mathcal{B} : \pi_u(q) \in \Delta(\varepsilon_1'), \pi_s(q) \in \Delta(\varepsilon_2')\}$ for some $\varepsilon' < \varepsilon$, then each component of $V \cap \mathcal{B}'$ belongs to $\mathcal{V}_j^u(\mathcal{B}')$ for $j \le m$. Now, if $S' = \mathbf{R}^2 \cap \mathcal{B}'$ is a regular box obtained by shrinking $\mathcal{B}$ in this way, then for each $\gamma \in \Gamma_1^u(S)$, $\gamma \cap S' \in \Gamma_1^u(S')$. And for each $\gamma \in \Gamma_2^u(S)$, we have that either $\gamma \cap S'$ is connected and belongs to Γ_2^u; or $\gamma \cap S'$ consists of two components, each of which belongs to $\Gamma_1^u(S')$. In this sense, $\Gamma_1^{s/u}(S)$ is preserved under shrinking.

LEMMA 3.9. *For $p \in J_{*,2}$ there is a regular box $\mathcal{B}$ centered at p with the properties*:

1. *For all $\gamma \in \Gamma_2^u$, $\overline{\gamma} \cap \partial_v S \subset \overline{S}^r$.*
2. *For all $\gamma \in \Gamma_2^u$, $\gamma \cap S^r$ consists of two components γ_1 and γ_2, as in Figure* 3.4.
3. *For all $\sigma \in \Gamma_2^s$, $\#(\gamma_1 \cap \sigma) = \#(\gamma_2 \cap \sigma) = 2$.*

Proof. As noted in the previous paragraph, we may construct a regular box $\mathcal{B}$ centered at p. Now we show that we may shrink S_0 and S so that 1, 2, and 3 hold. Since $p \in J_{*,2}$, γ_p^u will have a phantom region, which we may assume extends to the right-hand side, as in Figure 3.4. In order to establish 1, we must show that for $\gamma \in \Gamma_2^u$, the endpoints of γ lie in the left-hand side of the vertical boundary of S. Let $\gamma' \in \Gamma_2^u$ denote any unstable arc whose endpoints lie in the right-hand side of $\partial_v S$. For $r \in S_0 \cap J$, $\gamma_r^s \cap \gamma$ consists of two points, which means that any γ' must loop around to the left of γ_r^s. If we shrink S in the unstable direction, i.e., replace it with $S \cap \pi^{-1}\Delta(\varepsilon')$ with $\varepsilon' > 0$ small enough that there exists $r \in S_0 \cap J$ with $\gamma_r^s \cap S' = \emptyset$, then all γ' become simple in S'. That is, $\gamma' \cap S' \in \Gamma_1^u(S')$.

Assertion 2 follows from assertion 1, as is illustrated in the left-hand side of Figure 3.4.

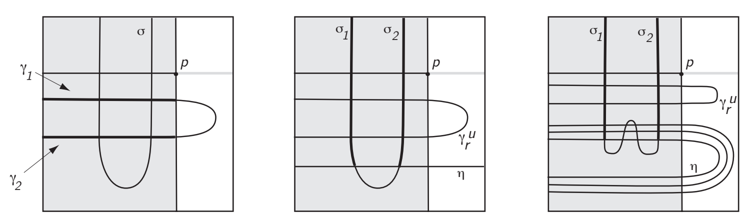


Figure 3.4

To prove assertion 3, we consider first the case where there is an $\eta \in \Gamma_1^u$ lying below γ_p^u, as in the central picture in Figure 3.4. (The case where there η lies above γ_p^u is analogous.) We may shrink S_0 so that for all $r \in J \cap S_0$, γ_r^u

lies between η and γ_p^u. In this case we consider $\gamma \in \Gamma_2^u$ lying between η and γ_p^u and $\sigma \in \Gamma_2^s$. The case drawn in the central picture in Figure 3.4 shows the endpoints of σ in the top of the boundary $\partial_h S$. (The other case, where the endpoints are in the bottom portion of $\partial_h S$ is analogous.) As is pictured, η cuts off two pieces σ_1 and σ_2, and each of these intersects γ_j, $j = 1, 2$. Thus $\#(\sigma \cap \gamma_j) = 2$. This proves assertion 3 in this case.

The alternative to this case is that $\eta \in \Gamma_2^u$ for all curves $\eta \in \Gamma^u$ lying below γ_p^u. If this happens, we consider $G(\eta)$, which is the set of all γ_r^u such that $\gamma_r^u \in \Gamma_2^u$, and γ_r^u separates η from p. We claim that we may shrink S_0 such that for $r \in S_0$ below γ_p^u, γ_r^u lies between $G(\eta)$ and γ_p^u. If this happens, then we see that any $\sigma \in \Gamma_2^s$ has two components σ_1 and σ_2 as in the right-hand side of Figure 3.4. These components intersect γ as desired. The alternative is that there are points $r_j \in S_0 \cap J$, lying below γ_p^u, and such that $r_j \to p$. But then we have that $\gamma_{r_j}^u \to \gamma_p^u$ in the topology of the Hausdorff metric. In this case, we let S' denote an arbitrarily small shrinking of S, and it follows that $\gamma_{r_j}^u \cap S'$ is ultimately disconnected. Thus a component of $\gamma_{r_j}^u$ serves as the curve η as we considered at first. $\square$

THEOREM 3.10. *$J_{2,1} \cup J_{1,2}$ is finite.*

Proof. It suffices to show that $J_{1,2}$ is finite. Write $X = J_{1,2} \cup J_{2,2}$. For each $p \in X$, we may construct a regular box $\mathcal{B}$ centered at p, satisfying the conclusions of Lemma 3.9. Since X is compact, we may select a finite number of regular boxes $\mathcal{B}$ such that the sets S_0 cover X.

Let us fix one of these boxes. We claim: *There are (at most) two verticals, γ_p^s and γ_q^s, with the property that $S_0 \cap X \subset \gamma_p^s \cup \gamma_q^s$.* Since $p \in X$, it is u one-sided. Without loss of generality we may assume that the phantom region of γ_p^u is on the right, so that $S_0 \cap J$ lies to the left of γ_p^s. To establish the claim, we show that for any two points $q, r \in X \cap S_0$ such that $q, r \notin \gamma_p^u$, it follows that $\gamma_q^s = \gamma_r^s$.

Let us assume first that γ_r^u and γ_q^u both belong to Γ_1^u. If $r \notin \gamma_q^s$, then we have $\gamma_q^s \cap \gamma_r^u \neq \emptyset$, as pictured in Figure 3.5. There are three possibilities for the γ_r^s. The first (on the left of Figure 3.5) is that $\gamma_r^s \in \Gamma_1^s$. But this is not possible, since the phantom region of γ_q^u blocks γ_r^s from reaching the upper portion of $\partial_h S$. The next two possibilities are that $\gamma_r^s \in \Gamma_2^s$. In both cases, $\gamma_r^s \cap \gamma_q^s = \emptyset$, and γ_r^s cannot intersect the phantom region of γ_q^u. In the central picture of Figure 3.4, we see that γ_r^s must come out of the box bounded by γ_q^u and γ_q^s. But the portion that is drawn shows $\#(\gamma_r^s \cap \gamma_r^u) = 2$, and thus γ_r^s cannot intersect γ_r^u again. Thus $\gamma_r^s \cap \gamma_q^u = \emptyset$, which is a contradiction. In the last case, on the right of Figure 3.5, we again have $\#(\gamma_r^s \cap \gamma_r^u) = 2$ and so it is not possible for γ_r^s to cross γ_r^u again; thus, it cannot intersect $\partial_h S$, which is a contradiction.

Now let us suppose that one or both of γ_r^u, γ_q^u belongs to Γ_2^u. We will refer to these as γ_x^u. As in Lemma 3.9, $\gamma^u \cap S^r$ consists of two pieces, γ_1 and γ_2. One of these, say γ_1, contains the point x. By Lemma 3.9, we have $\#(\gamma^s \cap \gamma_x^u) = 2$ for any $\gamma^s \in \Gamma_2^s$. Thus we replace γ_x^u in Figure 3.5 by γ_1 and proceed as before. This proves the claim.

Since there are only finitely many regular boxes in our covering of X, it follows that $J_{1,2}$ is contained in the union of a finite set $\{\gamma_1^s, \dots, \gamma_N^s\}$ of segments of stable manifolds. We let Γ_j^s denote the closure of the union of all the arcs γ_i^s which are contained in the global stable manifold $W^s(r_j) \supset \gamma_j^s$. It follows that f permutes the finite family of sets $\{\Gamma_1^s, \dots, \Gamma_N^s\}$. Thus, passing to a power of f, we have $f^t(\Gamma_1^s \cap J_{1,2}) = \Gamma_1^s \cap J_{1,2}$. Now $\Gamma_1^s \cap J_{1,2}$ is an f-invariant, compact subset of $W^s(r_1)$, and f^t is contracting on $W^s(r_1)$, so $\Gamma_1^s \cap J_{1,2}$ must be a single point. We conclude that $J_{1,2}$ is finite. $\square$

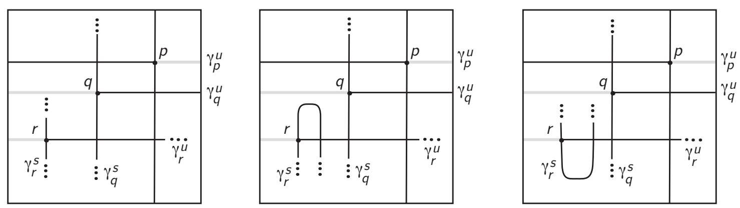


Figure 3.5

COROLLARY 3.11. *For any $p \in J$ and $\psi \in \Psi_p$, $\psi(\mathbf{C})$ is a nonsingular (complex) submanifold of $\mathbf{C}^2$, and $\psi(\mathbf{C}) \cap \mathbf{R}^2$ is a nonsingular (real) submanifold of $\mathbf{R}^2$.*

Proof. If ψ has no critical point, then $\psi(\mathbf{C})$ is nonsingular. And by our earlier discussion of the reality condition, it follows that if ψ has no critical point, then $\psi(\mathbf{C}) \cap \mathbf{R}^2$ is a nonsingular, real one-dimensional submanifold of $\mathbf{R}^2$.

If $\psi \in \Psi_p^u$ has a critical point, then by Proposition 2.5, ψ there is just one critical point ζ_0. The sequence $\tilde{f}^{-n}\psi = f^{-n}\psi(\lambda(p,-n)^{-1}\zeta) \in \Psi_{f^{-n}p}$ has a critical point at $\lambda(p,-n)\zeta_0$. For a subsequence $f^{-n_j}p \to q \in \alpha(p)$, we may pass to a further subsequence such that $\tilde{f}^{-n_j}\psi$ converges to $\hat{\psi} \in \Psi_q$. Since $\lambda(p,-n) \to 0$ as $n \to \infty$, $\hat{\psi}$ has a critical point at $\zeta = 0$. Thus $q \in J_{*,2}$. Since $J_{2,*} \cup J_{*,2}$ is a finite set of saddle points, we have $p \in W^u(p)$. Thus $\psi(\mathbf{C}) \subset W^u(q)$, and so this set and $\psi(\mathbf{C}) \cap \mathbf{R}^2$ are both regular. $\square$

4. Hyperbolicity and tangencies

In Section 3 we showed that $\mathcal{C} := J_{2,*} \cup J_{*,2}$ is a finite union of saddle points. We show next that all tangential intersections lie in stable manifolds of $J_{*,2}$ and unstable manifolds of $J_{2,*}$. In Theorem 4.2 we show that for $p \in J_{*,2}$, the stable manifold $W^s(p)$ contains a heteroclinic tangency. The condition for hyperbolicity is characterized (Theorem 4.4) in terms of the existence of heteroclinic tangencies.

THEOREM 4.1. *If* $p \in J$ *and* $E_p^s = E_p^u$, *then* $p \in W^s(J_{*,2}) \cap W^u(J_{2,*})$.

Proof. If p is a point of tangency, then by Theorem 2.7, $\alpha(p) \subset J_{2,*}$ and $\omega(p) \subset J_{*,2}$. Since $J_{2,*} \cup J_{*,2}$ is a finite set of saddle points, it follows that there exist $q \in J_{2,*}$ and $r \in J_{*,2}$ such that $p \in W^u(q) \cap W^s(r)$. $\square$

THEOREM 4.2. *If* $p \in J_{*,2}$ *then there exists* $q \in J_{2,*}$ *such that* $W^s(p)$ *intersects* $W^u(q)$ *tangentially.*

Proof. By Section 3, $\mathcal{C}$ is finite. For each point $p \in J_{*,2}$, there is a point $q \in \mathcal{C}$ such that $W^s(p)$ intersects $W^u(q)$ tangentially, by [BS8, Th. 8.10]. And by Theorem 2.7 we have $q \in J_{2,*}$. $\square$

COROLLARY 4.3. *If* $J_{1,2} \neq \emptyset$, *then* $J_{2,1} \neq \emptyset$.

THEOREM 4.4. *The following are equivalent for a real, polynomial mapping of maximal entropy*:

1. f *is not hyperbolic.*
2. $J_{2,*} \cup J_{*,2}$ *is nonempty.*
3. *There are saddle points* p *and* q *such that* $W^s(p)$ *intersects* $W^u(q)$ *tangentially.*

Remark. By Theorem 4.1, the saddle points p and q in condition 3 satisfy $p \in J_{*,2}$ and $q \in J_{2,*}$.

Proof. (2) $\Rightarrow$ (1). If $J_{2,*} \cup J_{*,2} \neq \emptyset$, then by Theorem 4.2 there is a tangency between $W^s(J_{*,2})$ and $W^u(J_{2,*})$. Thus f is not hyperbolic.

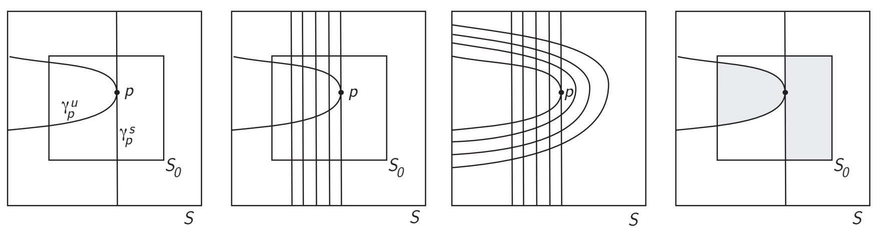
(1) $\Rightarrow$ (2). If $J_{2,*} \cup J_{*,2} = \emptyset$, then $J = J_{1,1}$. It follows that $J_{1,1}$ is compact, and so by [BS8, Prop. 5.2] $J_{1,1}$ is a hyperbolic set.

The implication (2) $\Rightarrow$ (3) follows from Theorem 4.2, and (3) $\Rightarrow$ (2) follows from Theorem 2.7. $\square$

Let $\mathcal{T}$ denote the set of points of tangential intersection between $W^s(a)$ and $W^u(b)$, for $a, b \in J$. By [BS8, Th. 8.10], $\mathcal{T}$ is a discrete subset of $J_{1,1}$. Since the parametrizations are nonsingular in $J_{1,1}$, the curves $\mathcal{W}^{s/u} = \{\gamma_r^{s/u} : r \in J_{1,1}\}$ form a lamination of a neighborhood of p. If $p \in J_{1,1} - \mathcal{T}$, the laminations $\mathcal{W}^s$ and $\mathcal{W}^u$ are transverse at p, and so they define a local product structure on J in a neighborhood of p.

Let us fix a point $p \in \mathcal{T}$. We cannot construct a regular box centered at p since it is a point of tangency, but we will construct a box with many of the same properties. We choose a real analytic coordinate system such that the square $S := \{|x|, |y| < \varepsilon\} \subset \mathbf{R}^2$ is centered at $p = (0,0)$, and has the properties that $\gamma^s_p = \{x = 0, |y| < \varepsilon\}$, and the projections $\pi_u : \gamma^u_p \to (-\varepsilon, \varepsilon)$ are proper, where as before $\pi_u(x,y) = x$, $\pi_s(x,y) = y$, and we use the notation $\gamma^{s/u}_q : V^{s/u}_q \cap S \cap \mathbf{R}^2$. By Theorem 2.8, the multiplicity of the intersection of γ^s_p and γ^u_p at p must be 2. Thus $\gamma^u_p \in \Gamma^u_2(S)$, and so by Lemma 3.4 γ^u_p lies to one side of γ^s_p, as pictured on the left-hand side of Figure 4.1. For $S_0 := \{|x|, |y| < \delta\}$ sufficiently small, we have (3.1), and $\pi_{s/u} : \gamma^{s/u}_q \to (-\varepsilon, \varepsilon)$ is proper for $q \in S_0 \cap J$.

The configuration of the curves in the third picture of Figure 4.1 follows from Lemma 3.4 and Corollary 3.2, since there must be two points of intersection between stable and unstable manifolds in S. This arrangement is associated with the failure of topological expansivity.



Stable/Unstable Laminations Near a Tangency

Figure 4.1

Corollary 4.5. *If $r \in \mathcal{T}$, then there is a neighborhood S_0 of r such that $J \cap S_0$ is disjoint from the region shaded in the right-hand picture in Figure* 4.1.

5. One-sided points

We have shown that the set of critical points $\mathcal{C}$ is a finite set of one-sided points. We use one-sided points to show (Theorem 5.2) that K is always a Cantor set. We analyze more carefully the possibilities for one-sided points, and obtain Propositions 5.8 and 5.9, which combine to prove Theorem 3 in the introduction.[2]

Theorem 5.1. *If f is hyperbolic, then there exist stably and unstably one-sided points.*

[2]We wish to thank André de Carvalho for a suggestion that resulted in this part of the paper.

Proof. The set K is saturated in the sense that $W^s(p) \cap W^u(q) \subset K$ for all $p, q \in K$. We use the following result of Newhouse and Palis [NP], as it is presented in [BL, Prop. 2.1.1, item 6]: If f does not have an unstably one-sided point, then K is a hyperbolic attractor. Since K is a basic set, it follows that if it is an attractor, then the set of points attracted to K is open. On the other hand, the set of points attracted to K is K^+, which is a closed, proper subset of $\mathbf{R}^2$, and is thus not open. Thus f has an unstably one-sided point. Repeating the argument for f^{-1} gives a stably one-sided point. □

THEOREM 5.2. *If f has maximal entropy, then K is a Cantor set.*

Proof. Since K is the zero set of a continuous, plurisubharmonic function G on $\mathbf{C}^2 \supset \mathbf{R}^2$, it follows that no point of K can be isolated. Thus it suffices to show that K is totally disconnected. Both $W^s(K)$ and $W^u(K)$ are laminations in a neighborhood of $\mathcal{J}_{1,1} = K - \mathcal{C}$. Let $\mathcal{T}$ denote the tangencies between $W^s(K)$ and $W^u(K)$. By [BS8, Th. 8.10] $\mathcal{T} \cup \mathcal{C}$ is a countable, closed set. Thus it suffices to show that K is totally disconnected in a neighborhood of each point of $K - (\mathcal{T} \cup \mathcal{C})$.

Now each point of $K - (\mathcal{T} \cup \mathcal{C})$ has a neighborhood R such that $R \cap K$ has local product structure. The local product structure means that for any $r \in K \cap U$, $R \cap K$ is homeomorphic to $(K \cap W^u_{R,\mathrm{loc}}(r)) \times (K \cap W^s_{R,\mathrm{loc}}(r))$.

By Theorem 5.1, there are an s one-sided periodic point p and a u one-sided point q. Let A denote the set of transverse intersections of $W^u(p)$ and $W^s(q)$. By [BLS, Th. 9.6] A is dense in $K \cap U$. It follows from the local product structure that the transverse intersections between $W^u(p)$ and $W^s_{R,\mathrm{loc}}(r)$ are dense in $W^s_{R,\mathrm{loc}}(r)$ for each $r \in K \cap U$. Since p is s one-sided, $K \cap W^s(p)$ lies to one side of p in $W^s(p)$. This one-sidedness propagates along the unstable manifold $W^u(p)$, and so for any point $b \in W^u(p) \cap W^s_{R,\mathrm{loc}}(r)$, $K \cap W^s_{R,\mathrm{loc}}(r)$ lies (locally) to one side of b in $W^s_{R,\mathrm{loc}}(r)$. Thus the set of disconnections of $K \cap W^s_{R,\mathrm{loc}}(r)$ is dense in $K \cap W^s_{R,\mathrm{loc}}(r)$, which means that $K \cap W^s_{R,\mathrm{loc}}(r)$ is totally disconnected. Similarly, $K \cap W^u_{R,\mathrm{loc}}(r)$ is totally disconnected. By the local product structure, $K \cap R$ is totally disconnected, and thus K is disconnected. □

By a (topological) attractor we will mean a compact, invariant S whose stable set $W^s(S) := \{q : \lim_{n \to +\infty} d(f^n q, S) = 0\}$ has nonempty interior.

COROLLARY 5.3. *If f has maximal entropy, then K^+ and K^- have no interior, and thus K contains no attractors or repellors.*

Proof. As in the proof of Theorem 5.2, we consider a local product neighborhood R of a point of $K - (\mathcal{T} \cup \mathcal{C})$. By the local product structure, $K^+ \cap R$ is homeomorphic to $(K \cap W^u_{R,\mathrm{loc}}(r)) \times W^s_{R,\mathrm{loc}}(r)$. Since $K \cap W^u_{R,\mathrm{loc}}(r)$ is totally disconnected, it contains no interior. Thus $R \cap K^+$ contains no interior.

If S is an attractor, it must be contained in K, and thus the basin $B(S)$ must be contained in K^+. However, since K^+ has no interior, $B(S)$ can have no interior. $\square$

Recall that if p is u one-sided, then $W^u(p) - \{p\}$ has a component which is disjoint from J. No point of $J \cap W^u(p)$ can be isolated, so only one of the components of $W^u(p) - \{p\}$ can be disjoint from J. We call this component the (unstable) separatrix associated with p.

Now let us note that if p is a saddle point of f with period n, then $Df^n(p)$ has eigenvalues $|\lambda^u| > 1 > |\lambda^s| > 0$. If p is u one-sided, f must preserve the unstable separatrix, and so $\lambda^u > 0$. Similarly, if p is s one-sided, we have $\lambda^s > 0$.

Lemma 5.4. *Let $p \in K$ be u one-sided, and let S be the separatrix which is associated with p and which is disjoint from K. Then S is properly embedded in $\mathbf{R}^2 - \{p\}$.*

Proof. Consider the uniformization $\psi^u : \mathbf{C} \to W^u(p)$ of the complex unstable manifold through p. Since p is u one-sided, we may assume that its separatrix S corresponds to the positive real axis in $\mathbf{C}$, and thus $G^+\psi^u(\zeta) > 0$ for $\zeta \in \mathbf{R}$, $\zeta > 0$. Now $\psi^u(0) = p$, and there is $\lambda^u > 1$ such that $G^+\psi^u(\lambda^u) = d \cdot G^+\psi^u(\zeta)$. Thus $\lim_{\zeta \to +\infty} G^+\psi^u(\zeta) = +\infty$. Since $\{G^+ \le c\} \cap J^-$ is compact, it follows that $\lim_{\zeta \to +\infty} \psi^u(\zeta) = +\infty$, and thus S is properly embedded in $\mathbf{R}^2$. $\square$

Let $\mathcal{O} = \{p_1, \ldots, p_j\}$ denote the set of u one-sided points. Let $S_1, \ldots S_j$ denote the corresponding separatrices. By Lemma 5.4, S_i is an arc in the sphere $S^2 = \mathbf{R}^2 \cup \{\infty\}$ which connects p_i to ∞. We let $\hat{S}_i$ denote the germ at infinity of S_i, i.e. $\hat{S}_i$ denotes the set of sub-arcs of S_i containing ∞. We consider a system of neighborhoods W of ∞ in S^2 with the property that ∂W is homeomorphic to S^1, and $\partial W \cap S_i$ consists of a unique point, for each i. For such a neighborhood, $W \cap (\mathbf{R}^2 - \bigcup_{i=1}^j S_i)$ consists of j open sets V, each containing ∞ in its closure. Let $\hat{V}$ denote the germ at infinity corresponding to V. We will define a graph $\mathcal{G}$ whose vertices are the classes $\hat{S}_i$, and whose edges are the germs $\hat{V}$. A vertex $\hat{S}_i$ is contained in an edge $\hat{V}$ if the germ of S_i is contained in ∂V. Now, $\mathcal{G}$ is homeomorphic to ∂W. Since f maps the separatrices S_i to themselves, the system of neighborhoods W is also preserved by f. Thus the structure of $\mathcal{G}$ is preserved. If f preserves/reverses orientation, fW has the same/opposite orientation as W. Thus we have:

Lemma 5.5. *$\mathcal{G}$ has the combinatorial structure of a simplicial circle, and f induces a simplicial homeomorphism $\hat{f}$ on $\mathcal{G}$. If f preserves/reverses orientation, then so does $\hat{f}$.*

Let us note that if f has the form (1.1), then

$$f(x,y) = (y, \varepsilon y^d + \cdots - ax). \tag{5.1}$$

We have $a > 0$ if f preserves orientation, and $a < 0$ if f reverses orientation. We conjugate by $\tau(x,y) = (\alpha x, \beta y)$ with $\alpha, \beta \in \mathbf{R}$, so that $\varepsilon = \pm 1$. If d is even, we require $\varepsilon = +1$. If d is odd, we define $\varepsilon(f) = \varepsilon$. If f_1 and f_2 are both of odd degree and in the form (5.1), then $\varepsilon(f_1 f_2) = \varepsilon(f_1)\varepsilon(f_2)$. Note that $f^{-1}(x,y) = (a^{-1}(\varepsilon x^d - y), x)$, and thus f^{-1} may be put in the form (5.1) after conjugation by $\nu(x,y) = (y,x)$ and a mapping of the form τ. If d is odd, then $\varepsilon(f^{-1}) = \pm\varepsilon(f)$, with the plus sign occurring if and only if f preserves orientation.

We write $V^+ = \{(x,y) \in \mathbf{R}^2 : |y| \geq \max(R, |x|)\}$ as $V^+ = V_1^+ \cup V_2^+$, with $V_1^+ := V^+ \cap \{y > 0\}$ and $V_2^+ := V^+ \cap \{y < 0\}$. We choose R large enough that $V, V^\pm$ give a filtration, i.e. $fV^+ \subset V^+$ and $f(V \cup V^+) \subset V \cup V^+$. The condition $\varepsilon(f) = 1$ is equivalent to $f(V_1^+) \subset V_1^+$. In this case, it follows that $f(V_2^+) \subset V_1^+$ if the degree d is even, and $f(V_2^+) \subset V_2^+$ if d is odd.

Theorem 5.6. *If f preserves orientation, and if $\varepsilon(f) = +1$, then all the p_i are fixed. If $\varepsilon(f) = -1$, then all the p_i have period 2.*

Proof. Recall that for each unstably one-sided point, the separatrix S_i is unique. Let $S_{i_1}, \ldots, S_{i_m}$ denote the separatrices corresponding to V_1^+. That is, these are the separatrices whose germs at infinity are contained in V_1^+. Let $\mathcal{I}_1 \subset \mathcal{G}$ denote the subgraph whose vertices are the separatrices corresponding to V_1^+ and whose edges are the open sets V with $\hat{V} \subset V_1^+$. It follows that $\mathcal{I}_1$ is a proper subinterval of $\mathcal{G}$. Similarly, we let $\mathcal{I}_2$ denote the interval generated by the separatrices whose germs are contained in V_2^+. Note that all of the separatrices S_i are subsets of unstable manifolds. Thus their germs are contained in V^+, and so they belong to either $\mathcal{I}_1$ or $\mathcal{I}_2$.

If $\varepsilon(f) = 1$, then $fV_1^+ \subset V_1^+$. Thus $\hat{f}$ maps $\mathcal{I}_1$ to itself. If f preserves orientation, then $\hat{f} : \mathcal{I}_1 \to \mathcal{I}_1$ is an orientation-preserving simplicial homeomorphism. Now, $\hat{f}$ is the identity on $\mathcal{I}_1$, and so f is the identity on $\mathcal{G}$. Thus f maps each edge and each vertex to itself, or $f(p_i) = p_i$ for every i; and this completes the proof in this case.

The remaining case is $\varepsilon(f) = -1$, which implies that the degree is odd, and thus $fV_1^+ \subset V_2^+$ and $fV_2^+ \subset V_1^+$. This means that $\hat{f} : \mathcal{I}_1 \to \mathcal{I}_2$, and $\hat{f} : \mathcal{I}_2 \to \mathcal{I}_1$. Thus none of the separatrices can be fixed. On the other hand, f^2 preserves orientation and satisfies $\varepsilon(f^2) = 1$. By the argument above, these points are fixed for f^2, and so their periods are equal to 2. $\square$

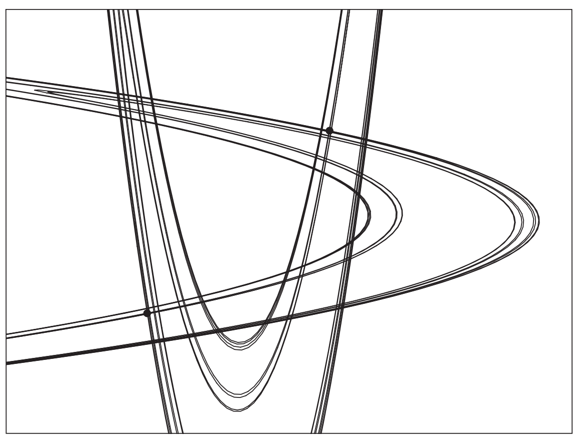
Corollary 5.7. *All one-sided points have period 1 or 2.*

Proof. Without loss of generality, we may consider only unstably one-sided points. The mapping f^2 is orientation-preserving, and $\varepsilon(f^2) = +1$. By Theorem 5.6, each one-sided point is fixed for f^2. Thus the period is 1 or 2. □

Let f be a real, quadratic diffeomorphism of maximal entropy. If f preserves orientation, then one of the saddle points, which we will call p_+, has positive multipliers $\lambda^u > \lambda^s > 0$. The other fixed point has negative multipliers.

PROPOSITION 5.8. *Suppose f is quadratic and orientation-preserving. Then the fixed point p_+ is both stably and unstably one-sided. No other point is one-sided.*

Proof. Let p denote an unstably one-sided point for f. By Lemma 5.6, p is a fixed point. If p is the saddle point with negative multipliers, it cannot be one-sided. Thus it must be the saddle point p_+, which has positive multipliers. Similarly, the stably one-sided point must also be p_+, so that p_+ is the only one-sided point, and it is doubly one-sided. □



Orientation-Reversing Quadratic

Figure 5.1

Let f be a real, quadratic diffeomorphism of maximal entropy which reverses orientation. Then the fixed points are a pair of saddles $p_\pm$ with the property $\pm\lambda^u(p_\pm) > 0 > \pm\lambda^s(p_\pm)$.

PROPOSITION 5.9. *If f is an orientation-reversing quadratic map, then the one-sided points are the fixed points $p_\pm$, and $p_\pm$ is u/s-one-sided.*

Proof. If p is a one-sided point, then the period of p must be 1 or 2. We show first that it cannot be 2. If q is a fixed point of f with multipliers $\lambda^{u/s}(q)$, then q is also a fixed point of f^2, and it has multipliers $(\lambda^{u/s}(q))^2 > 0$.

Now let h be an orientation preserving map of $\mathbf{R}^2$. For a saddle fixed point r of a mapping h, we define $\delta(h,r)$ to be $+1$ if the multipliers of h at r are both positive, and we set $\delta(h,r) = -1$ if the multipliers are both negative. If the degree of h is even, it follows from the Lefschetz Fixed Point Formula (cf. [F]) that $\sum \delta(h,r) = 0$, where we sum over all the fixed points of h. Now we apply this to the map $h = f^2$, which is orientation-preserving and degree 4. There are four fixed points of h, corresponding to the two fixed points of f and a 2-cycle for f. It follows that $\{p_1, p_2\}$ is the 2-cycle of f, and we must have $\delta(h, p_j) = -1$ for $j = 1, 2$.

On the other hand, suppose that p_1 is one-sided with separatrix S_1. Then p_2 is also one-sided, with separatrix S_2. Now f^2 maps the separatrix S_1 to itself, and thus the eigenvalue of $Df^2(p_1)$ must be positive. This means that $\delta(f^2, p_1) = +1$, which is a contradiction. Thus the only one-sided points are the fixed points of f.

Since f reverses orientation, each periodic point satisfies $\lambda^u\lambda^s < 0$, and thus λ^s, λ^u cannot both be positive. Thus p cannot be doubly one-sided.

Thus one of the fixed points must be stably one-sided with $\lambda^s > 0 > \lambda^u$ for this point. The other fixed point must be unstably one-sided and must have $\lambda^u > 0 > \lambda^s$. □

PROPOSITION 5.10. *Suppose that f reverses orientation and that $\varepsilon(f) = +1$. If d is even, then at most one u one-sided point can be a fixed point; if d is odd, then at most two u one-sided points can be fixed.*

Proof. We construct the subintervals $\mathcal{I}_1$ and $\mathcal{I}_2$ as in the proof of Theorem 5.6. If the degree of f is even, then $fV_2^+ \subset V_1^+$, and thus $\mathcal{I}_2$ is empty. The induced map $\hat{f} : \mathcal{I}_1 \to \mathcal{I}_1$ is an orientation-reversing simplicial homeomorphism. The simplicial map $\hat{f}$ can fix at most one vertex of $\mathcal{I}_1$ (which happens when the number of vertices in $\mathcal{I}_1$ is odd). If d is odd, then $fV_2^+ \subset V_2^+$, so that each of the restrictions $\hat{f}|\mathcal{I}_1$ and $\hat{f}|\mathcal{I}_2$ can have at most one fixed point. □

INDIANA UNIVERSITY, BLOOMINGTON, IN
E-mail address: bedford@indiana.edu

CORNELL UNIVERSITY, ITHACA, NY
E-mail address: smillie@math.cornell.edu

REFERENCES

[BLS] E. BEDFORD, M. LYUBICH, and J. SMILLIE, Polynomial diffeomorphisms of $\mathbf{C}^2$. IV. The measure of maximal entropy and laminar currents, *Invent. Math.* **112** (1993), 77–125.

[BS8] E. BEDFORD and J. SMILLIE, Polynomial diffeomorphisms of $\mathbf{C}^2$. VIII. Quasi-expansion, *Amer. J. Math.* **124** (2002), 221–271.

[BL] C. BONATTI and R. LANGEVIN, *Difféomorphismes de Smale des Surfaces*, *Astérisque* **250**, Soc. Math. France, Paris, 1998.

[DN] R. Devaney and Z. Nitecki, Shift automorphisms in the Hénon mapping, *Comm. Math. Phys.* **67** (1979), 137–146.

[F] J. M. Franks, *Homology and Dynamical Systems*, *CBMS* **49**, A.M.S., Providence, RI, 1982.

[FM] S. Friedland and J. Milnor, Dynamical properties of plane polynomial automorphisms, *Ergodic Theory Dynam. Systems* **9** (1989), 67–99.

[HO] J. H. Hubbard and R. Oberste-Vorth, Hénon mappings in the complex domain. II. Projective and inductive limits of polynomials, in: *Real and Complex Dynamical Systems* (Hillerød, 1993) (B. Branner and P. Hjorth, eds.), 89–132, Kluwer Acad. Publ., Dordrecht, 1995.

[KKY] I. Kan, H. Koçak, and J. A. Yorke, Antimonotonicity: concurrent creation and annihilation of periodic orbits, *Ann. of Math.* **136** (1992), 219–252.

[M] J. Milnor, Nonexpansive Hénon maps, *Adv. Math.* **69** (1988), 109–114.

[NP] S. Newhouse and J. Palis, Hyperbolic nonwandering sets on two-dimensional manifolds, in *Dyniamical Systems*m *Proc. Sympos*. (Univ. Bahia, Salvador, 1971), 293–301, Academic Press, New York, 1973.

[O] R. Oberste-Vorth, Complex horseshoes, Ph. D. Thesis, Cornell University, 1987.

[R] C. Robinson, *Dynamical Systems*, CRC Press, Boca Raton, FL, 1995.

(Received October 13, 2000)