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# On contact Anosov flows

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**Abstract**

Exponential decay of correlations for $\mathcal{C}^4$ contact Anosov flows is established. This implies, in particular, exponential decay of correlations for all smooth geodesic flows in strictly negative curvature.

## 1. Introduction

The study of decay of correlations for hyperbolic systems goes back to the work of Sinai [36] and Ruelle [32]. While many results were obtained through the years for maps, some positive results have been established for Anosov flows only recently. Notwithstanding the proof of ergodicity, and mixing, for geodesic flows on manifolds of negative curvature [15], [1], [35], the first quantitative results consisted in the proof of exponential decay of correlations for geodesic flows on manifolds of constant negative curvature in two [4], [23], [30] and three [26] dimensions. The proof there is group theoretical in nature and therefore ill suited to generalizations of the nonconstant curvature case.[1] The conjecture that all Axiom A mixing flows exhibit exponential decay of correlations had already been proven false by Ruelle [34], [27] who produced piecewise constant ceiling suspensions with arbitrarily slow rates of decay.

The next advance was due to Chernov [3] who put forward the first *dynamical* proof showing sub-exponential decay of correlations for geodesic flows on surfaces of variable negative curvature. The basic idea was to construct a suitable stochastic approximation of the flow (see also [20] for a generalization of such a point of view).

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[1]Although some partial results for slowly varying curvature were obtained by perturbative techniques [4].

The last substantial advance in the field is due to the work of Dolgopyat [7], [8], [9]. He was able to use the thermodynamics formalism [36], [33], [28] and elaborate the necessary estimate on the Perron-Frobenius operator to control the Laplace transform of the correlation function. As a consequence he established exponential decay of correlations for all Anosov flows with $\mathcal{C}^1$ strong stable and unstable foliations. He also gave conditions for fast decay of correlations (for $\mathcal{C}^\infty$ observable) in more general cases.

Unfortunately, $\mathcal{C}^1$ strong stable and unstable foliations seem to be a quite rare phenomenon for higher dimensional Anosov flows [29], [10], [37]. One is therefore led to think that, unless some further geometrical structure is present, Anosov flows decay typically slower than exponentially.

The simplest geometrical structure that can be considered is certainly a contact structure, geodesic flows in particular. In this case an explicit formula by Katok and Burns [16] provides an approximation to the *temporal function* which is the real quantity on which some smoothness is required. An improvement on the error term for the above formula, that can be found in this paper (Appendix B, Lemma B.7), shows that, for a contact Anosov flow, if the strong foliations are $\tau$-Hölder, with $\tau > \sqrt{3}-1$, then the temporal function is likely to be $\mathcal{C}^1$ (see Remark B.8). On the other hand, geodesic flows that are $a$-pinched[2] have foliations that are $\mathcal{C}^{2\sqrt{a}}$ ([18] and Appendix B; see also [13], [11] for more complete results on such an issue). Dolgopyat's results would then, at best, imply that any geodesic flow in negative curvature which is $a$-pinched, with $a > 1 - \frac{\sqrt{3}}{2}$, enjoys exponential decay of correlations.

Given the fact that the above numbers do not look particularly inspiring it is then natural to guess that all Anosov contact flows exhibit exponential decay of correlations. This is exactly what is proved in the present paper (Theorem 2.4).

To obtain such a result I built on Dolgopyat's work and on the results in [2] where a functional space is introduced over which the Perron-Frobenius operator can be studied directly, without any coding, contrary to the previous approaches by Dolgopyat, Chernov and Pollicott.

Over such a space all the thermodynamics quantities studied by Dolgopyat have a particularly simple analogy with a specially transparent interpretation. It is then possible to establish a spectral gap for the generator of the flow and this, in turn, implies exponential decay of correlations.

The simplification of the approach is considerable as is testified by the length of the (self-contained) proof. In addition, the transparency of the relevant quantities allows us to recognize that in certain cases the results of

[2]That is, such that there exists $C > 0$ for which $-C \leq$ sectional curvatures $< -aC$; clearly it must be $a \in (0, 1)$. Recall that here we are considering higher dimensional manifolds, geodesic flows on surfaces always have $\mathcal{C}^1$ foliations.

Dolgopyat can be dramatically improved. To keep the exposition as simple as possible I have chosen to restrict it to the main case in which new results can be obtained: spectral properties of contact Anosov flows with respect to the contact volume. This allows choice of a function space simpler than the one needed in the general case (see [2] for a more general choice of the Banach space that would accommodate any Anosov flow with respect to any equilibrium measure).

The plan of the paper is as follows. Section 2 starts by describing the type of flows under consideration and the key objects used in the proof. Then the main result is stated precisely (Theorem 2.4). After that a proof of the result is presented. The proof is complete provided one assumes Lemma 2.7, Lemma 2.9 and Proposition 2.12. Lemma 2.7 is proven in Section 3 as is Lemma 2.9. Section 5 contains the proof of Proposition 2.12 modulo an inequality, Lemma 5.2, which is proven in Section 6.

Finally, for the reader's convenience, the paper contains three appendices. Appendix A contains a collection of needed–but already well established–facts on Anosov flows. Appendix B is devoted to the discussion of known–and less known–properties of Contact flows. Appendix C contains a few technical facts about averages that will certainly not surprise the experts but needed to be proven somewhere.

## 2. Statements and results

We will consider a $\mathcal{C}^4$, $2d+1$ dimensional, connected compact Riemannian manifold $\mathcal{M}$ and a $\mathcal{C}^4$ flow[3] $T_t : \mathcal{M} \to \mathcal{M}$ defined on it which satisfies the following conditions.

*Condition* 1. At each point $x \in \mathcal{M}$ there exists a splitting of the tangent space $\mathcal{T}_x\mathcal{M} = E^s(x) \oplus E^c(x) \oplus E^u(x)$. The splitting is invariant with respect to $T_t$, $E^c$ is one dimensional and coincides with the flow direction; in addition there exists $A, \mu > 0$ such that

$$\begin{aligned} &\|dT_t v\| \le A e^{-\mu t}\|v\| && \text{for each } v \in E^s \text{ and } t \ge 0,\\ &\|dT_t v\| \ge A e^{\mu t}\|v\| && \text{for each } v \in E^u \text{ and } t \le 0. \end{aligned}$$

That is, the flow is *Anosov*.

*Condition* 2. There exists a $\mathcal{C}^2$ one-form $\alpha$ on $\mathcal{M}$, such that $\alpha \wedge (d\alpha)^d$ is nowhere zero, which is left invariant by $T_t$ (that is $\alpha(dT_t v) = \alpha(v)$ for each $t \in \mathbb{R}$ and tangent vector $v \in \mathcal{T}\mathcal{M}$). In other words $T_t$ is a *contact flow.*

[3] That is, $T_0 = \mathbf{Id}$ and $T_{t+s} = T_t \circ T_s$ for each $t, s \in \mathbb{R}$.

*Remark* 2.1. From now on I will assume $\mathcal{M}$ to be a Riemannian manifold with the Riemannian volume being the same as the contact volume $\alpha \wedge (d\alpha)^d$. This is not really necessary, yet it is convenient and can be done without loss of generality.

With a slight abuse of notation let us define on $\mathcal{C}^1(\mathcal{M}, \mathbb{C})$ the following group of operators

$$T_t\varphi := \varphi \circ T_t \; ; \quad \mathcal{L}_t f := f \circ T_{-t}. \tag{2.1}$$

The operator $\mathcal{L}_t$ specifies the evolution of the densities and therefore should determine the statistical properties of the system. Unfortunately, the spectral properties of $\mathcal{L}_t$ on $\mathcal{C}^1(\mathcal{M}, \mathbb{C})$ are not well connected to the statistical properties of the map. To establish such a connection it is necessary to enlarge the space. In order to do so we must define weaker norms. Clearly such norms will need to have a relation with the dynamical properties of the system.

The simplest way to embed the dynamics of a system into the topology is to introduce a *dynamical distance*. In our case several natural possibilities are available: for each $\sigma \in \mathbb{R}$ let

$$d_\sigma^+(x, y) := \int_0^\infty e^{\sigma t} d(T_t x, T_t y)\, dt; \quad d_\sigma^-(x, y) := \int_{-\infty}^0 e^{-\sigma t} d(T_t x, T_t y)\, dt, \tag{2.2}$$

where $d(\cdot, \cdot)$ is the Riemannian metric of $\mathcal{M}$.

*Remark* 2.2. Note that $d_\sigma^+$ and $d_\sigma^-$ are distances only if $\sigma$ is sufficiently small (that is, negative and larger, in absolute value, than the absolute values of all the Lyapunov exponents); otherwise they are only *pseudo-distances*.[4]

In the present article we are interested only in the special cases of (2.2) considered in the following lemma (the trivial proof is left to the reader).

LEMMA 2.3. *Choose $\lambda \in (0, \mu)$ and let $d_s := d_\lambda^+$ and $d_u := d_\lambda^-$. Then $d_u$ is a pseudo-distance on $\mathcal{M}$ and $d_u(T_{-t}x, T_{-t}y) \leq e^{-\lambda t} d_u(x, y)$. In addition, $d_u$, restricted to any strong-unstable manifold, is a smooth function and it is equivalent to the restriction of the Riemannian metric, while points belonging to different unstable manifolds are at an infinite distance. The analogous properties hold for $d_s$.*

We can now start to describe the spaces on which we will consider the operators $T_t$ and $\mathcal{L}_t$. First of all let us fix $\delta > 0$ so that it will be sufficiently small (how small will be specified later in the paper) and define

$$H_{s,\beta}(\varphi) := \sup_{d_s(x,y) \leq \delta} \frac{|\varphi(x) - \varphi(y)|}{d_s(x, y)^\beta}; \quad |\varphi|_{s,\beta} := |\varphi|_\infty + H_{s,\beta}(\varphi). \tag{2.3}$$

[4]That is, they can attain the value $+\infty$.

*Definition* 1. In the following by the Banach space $\mathcal{C}^\beta_s(\mathcal{M},\mathbb{C}) \subset \mathcal{C}^0(\mathcal{M},\mathbb{C})$ we will mean the *closure of* $\mathcal{C}^1(\mathcal{M},\mathbb{C})$ *with respect to the norm* $|\cdot|_{s,\beta}$. Similar definitions hold with respect to the metric $d_u$ and the Riemannian metric $d$ (given the space of Hölder function $\mathcal{C}^\beta$).

Let us also define the unit ball $\mathcal{D}_\beta := \{\varphi \in \mathcal{C}^\beta_s(\mathcal{M},\mathbb{C}) \mid |\varphi|_{s,\beta} \leq 1\}$. For a given $\beta < 1$, and $f \in \mathcal{C}^1(\mathcal{M},\mathbb{C})$, let

$$\begin{aligned} &\|f\|_w := \sup_{\varphi\in\mathcal{D}_1} \int_{\mathcal{M}} \varphi f \ ; \quad \|f\| := \|f\|_s + \|f\|_u \ ; \\ &\|f\|_s := \sup_{\varphi\in\mathcal{D}_\beta} \int_{\mathcal{M}} \varphi f \ ; \quad \|f\|_u := H_{u,\beta}(f). \end{aligned} \tag{2.4}$$

Let $\mathcal{B}(\mathcal{M},\mathbb{C})$ and $\mathcal{B}_w(\mathcal{M},\mathbb{C})$ be the completion of $\mathcal{C}^1(\mathcal{M},\mathbb{C})$ with respect to the norms $\|\cdot\|$ and $\|\cdot\|_w$ respectively. Note that such spaces are separable by construction and are all contained in $(\mathcal{C}^\beta)^*$, the dual of the $\beta$-Hölder functions.

It is well known that the strong stable and unstable foliations for an Anosov flow are $\tau$-Hölder (see Appendices A, B for quantitative estimates of $\tau$ and Remark B.4 for the use of $\tau$ in this paper). Moreover the Jacobian of the holonomies associated to the stable and unstable foliations are $\tau$-Hölder. From now on we will assume[5]

$$\beta < \tau^2. \tag{2.5}$$

The *main result* of the paper is the following.

THEOREM 2.4. *For a* $\mathcal{C}^4$ *Anosov contact flow* $T_t$ *satisfying Conditions* 1 *and* 2 *the operators* $\mathcal{L}_t$ *form a strongly continuous group on* $\mathcal{B}(\mathcal{M},\mathbb{C})$.[6] *In addition, there exists* $\sigma, C_1 > 0$ *such that, for each* $f \in \mathcal{C}^1$, $\int f = 0$, *the following holds true*

$$\|\mathcal{L}_t f\| \leq C_1 e^{-\sigma t} |f|_{\mathcal{C}^1}.$$

Clearly the above theorem implies exponential decay of correlations for $\mathcal{C}^1$ functions:

$$\int f\varphi\circ T_t = \int \mathcal{L}_t\left[f - \int f\right]\varphi + \int f \int \varphi\mathcal{L}_t 1 = \int f \int \varphi + O(e^{-\sigma t}|f|_{\mathcal{C}^1}|\varphi|_{s,\beta}).$$

In fact, a standard approximation argument extends the result to all Hölder functions.

[5] The square is needed only in Lemma 4.3. In fact, employing the strategy used in [2, §3.6], and refining Lemma B.7, it may be possible to replace $\tau^2$ by $\tau$. I do not pursue this possibility since it would complicate the proofs without any substantial addition to the present results.

[6] In fact the only place in which the $\mathcal{C}^4$ hypothesis is used is in the estimate (C.5). With a bit more work, adoption of the alternative approach used in [2, Sub-lemma 3.1.3], it is possible to reduce the needed smoothness to $\mathcal{C}^3$, possibly $\mathcal{C}^{2+\alpha}$, but to reduce it further some new ideas seem to be needed.

Corollary 2.5. *For each* $\alpha \in (0,1)$ *there exists* $C_\alpha > 0$ *such that, for each* $f, \varphi \in \mathcal{C}^\alpha$,

$$\left| \int f\varphi \circ T_t - \int f \int \varphi \right| \leq C_\alpha |f|_{\mathcal{C}^\alpha} |\varphi|_{\mathcal{C}^\alpha} e^{-\frac{\alpha\sigma}{2-\alpha}t}.$$

*Remark* 2.6. Note that Theorem 2.4 does not imply that $\mathcal{L}_1$ is a quasi-compact operator nor that it enjoys a spectral gap. This is a reflection of the impossibility, with the ideas at hand, to investigate directly the time one map and indicates that the result must be pursued in a more roundabout way.

The proof of Theorem 2.4 is achieved via a careful study of the spectral properties of the generator of the group. The first step consists in the following result proven in Section 3.

Lemma 2.7. *The operators* $\mathcal{L}_t$ *extend to a group of bounded operators on* $\mathcal{B}(\mathcal{M},\mathbb{C})$ *and* $\mathcal{B}_w(\mathcal{M},\mathbb{C})$; *they form a strongly continuous group. In addition, for each* $\beta' < \beta$ *there exists a constant* $B \geq 0$ *such that, for each* $f \in \mathcal{B}_w(\mathcal{M},\mathbb{C})$, $t \geq 0$,

$$\|\mathcal{L}_t f\|_w \leq \|f\|_w$$

*and, for each* $f \in \mathcal{B}(\mathcal{M},\mathbb{C})$, $t \geq 0$,

$$\|\mathcal{L}_t f\| \leq \|f\|; \quad \|\mathcal{L}_t f\| \leq 3e^{-\lambda\beta' t}\|f\| + B\|f\|_w.$$

From now on let $\beta'$ be fixed.

Accordingly the spectral radius of $\mathcal{L}_t$, $t \geq 0$, is bounded by one. In addition, it is possible to define the generator $X$ of the group. Clearly, the domain $D(X) \supset \mathcal{C}^2(\mathcal{M},\mathbb{C})$ and, restricted to $\mathcal{C}^2(\mathcal{M},\mathbb{C})$, it is nothing but the action of the vector field defining the flow.

The spectral properties of the generator depend on the resolvent $R(z) = (z\mathbf{Id} - X)^{-1}$. It is well known (e.g. see [5]) that for all $z \in \mathbb{C}$, $\Re(z) > 0$, the following holds:

$$R(z)f = \int_0^\infty e^{-zt}\mathcal{L}_t f dt. \tag{2.6}$$

Thanks to (2.6) it is possible to obtain the analogue of Lemma 2.7 for the resolvent.

Lemma 2.8. *For each* $z \in \mathbb{C}$, $\Re(z) = a > 0$,

$$\|R(z)\|_w \leq a^{-1}\ ; \quad \|R(z)\| \leq a^{-1}\ ; \quad \|R(z)^n f\| \leq \frac{3}{(a+\lambda\beta')^n}\|f\| + a^{-n}B\|f\|_w.$$

*Proof.* The first two inequalities follow directly from formula (2.6) and the first two inequalities of Lemma 2.7:

$$\|R(z)f\| \le \int_0^\infty e^{-at}\|\mathcal{L}_t f\|\,dt \le a^{-1}\|f\|.$$

By induction one easily obtains the formula

$$R(z)^n f = \frac{1}{(n-1)!}\int_{\mathbb{R}^+} t^{n-1}e^{-zt}\mathcal{L}_t f dt. \tag{2.7}$$

Again, by Lemma 2.7

$$\begin{aligned}\|R(z)^n f\| &\le \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-at}(3e^{-\lambda\beta' t}\|f\| + B\|f\|_w)\\ &\le \frac{3\|f\|}{(a+\lambda\beta')^n} + a^{-n}B\|f\|_w.\end{aligned}$$

□

The next basic result (proven in Section 4) is a compactness property for the operators $R(z)$.

LEMMA 2.9. *For each $a = \Re(z) > 0$ the operator $R(z)$, seen as an operator from $\mathcal{B}(\mathcal{M},\mathbb{C})$ to $\mathcal{B}_w(\mathcal{M},\mathbb{C})$, is compact.*

PROPOSITION 2.10. *For each $a = \Re(z) > 0$ the operator $R(z)$, seen as an operator on $\mathcal{B}(\mathcal{M},\mathbb{C})$, is quasi-compact, has spectral radius $a^{-1}$ and essential spectral radius bounded by $(a+\lambda\beta')^{-1}$.*

*Proof.* The bound on the spectral radius of $R(z)$ follows trivially from the second inequality of Lemma 2.8. By the third inequality of Lemma 2.8, Lemma 2.9 and the usual Hennion's argument [12] based on Nussbaum's formula [25], it follows that the essential spectral radius is bounded by $(a+\lambda\beta')^{-1}$. Let us recall the argument. Nussbaum's formula asserts that if $r_n$ is the inf of the $r$ such that $\{R(z)^n f\}_{\|f\|\le 1}$ can be covered by a finite number of balls of radius $r$, then the essential spectral radius of $R(z)$ is given by $\liminf_{n\to\infty}\sqrt[n]{r_n}$. Let $B_1 := \{f \in \mathcal{B} \mid \|f\| \le 1\}$. By Lemma 2.9, $R(z)B_1$ is relatively compact in $\mathcal{B}_w$. Thus, for each $\epsilon > 0$ there are $f_1, \dots, f_{N_\epsilon} \in R(z)B_1$ such that $R(z)B_1 \subseteq \bigcup_{i=1}^{N_\epsilon} U_\epsilon(f_i)$, where $U_\epsilon(f_i) = \{f \in \mathcal{B} \mid \|f - f_i\|_w < \epsilon\}$. For $f \in R(z)B_1 \cap U_\epsilon(f_i)$, Lemma 2.8 implies that

$$\begin{aligned}\|R(z)^{n-1}(f-f_i)\| &\le \frac{3}{(a+\lambda\beta')^{n-1}}\,\|f-f_i\| + \frac{B}{a^{n-1}}\|f-f_i\|_w\\ &\le a^{-n+1}\left\{\frac{3}{(1+\lambda\beta' a^{-1})^{n-1}} + B\epsilon\right\}\ .\end{aligned}$$

Choosing $\epsilon = (1+\lambda\beta' a^{-1})^{-n+1}$ we can conclude that for each $n \in \mathbb{N}$ the set $R(z)^n(B_1)$ can be covered by a finite number of $\|\cdot\|$-balls of radius $(3+B)(a+\lambda\beta')^{-n+1}$. □

For each $\zeta \in \mathbb{R}^+$ let $U_\zeta := \{z \in \mathbb{C} \mid \Re(z) > -\zeta\}$. Proposition 2.10 implies the following corollary.[7]

COROLLARY 2.11. *The spectrum $\sigma(X)$ of the generator is contained in the left half-plane. The set $\sigma(X) \cap U_{\lambda\beta'}$ consists of, at most, countably many isolated points of point spectrum with finite multiplicity. Zero is the only eigenvalue on the imaginary axis and has multiplicity one.*

*Proof.* If $F_z(w) := z - w^{-1}$, then $\sigma(X) = F_z(\sigma(R(z)))$. Thus the essential spectrum of $X$ must lie outside $\bigcup_{\Re(z)>0}\{w \in \mathbb{C} \mid |z-w| \leq a + \lambda\beta'\}$. This is exactly $U_{\lambda\beta'}$.

Since $\mathcal{L}_t 1 = 1$, and the space $V_0 := \overline{\{f \in \mathcal{C}^1(\mathcal{M},\mathbb{C}); \mid \int f = 0\}}^{\mathcal{B}(\mathcal{M},\mathbb{C})}$ is invariant, it follows that $\sigma(X) = \{0\} \cup \sigma(X|_{V_0})$. Next, suppose $Xf = ibf$ for some $b \in \mathbb{R}$ and $f \in V_0$, $f \neq 0$; then $R(z)f = (z+ib)^{-1}f$; thus for $z = a - ib$ (see equation (3.2)),

$$\|f\|_u \leq \frac{|z+ib|}{a+\beta\lambda}\|f\|_u = \frac{a}{a+\beta\lambda}\|f\|_u.$$

That is, $\|f\|_u = 0$. Let $\{f_n\} \subset \mathcal{C}^1$ be an approximating sequence for $f$, $\varphi \in \mathcal{D}_\beta$, and $t \in \mathbb{R}^+$; then

$$\left|\int f\varphi\right| = \left|e^{-ibt}\int fT_t\varphi\right| \leq \left|\int f_nT_t\varphi\right| + \|f - f_n\|.$$

Contact Anosov flows are mixing (see Corollary B.6); hence $\lim_{t\to\infty}\int f_nT_t\varphi = 0$. The arbitrariness of $t$ and $n$ implies then $\int f\varphi = 0$; that is, $\|f\|_s = 0$, which implies the contradiction $f \equiv 0$. □

The above result, although rather interesting, does not suffice to investigate the statistical properties of the system. To do so it is necessary to exclude the presence of the spectrum near the imaginary axis (apart from 0). This follows from the next result proven in Sections 5, 6.

PROPOSITION 2.12. *There exists $b_* > 0$, $\bar{c} > 1$ and $\nu \in (0,1)$ such that for each $z = a + ib$, $a \in [\bar{c}^{-1}, \bar{c}]$, $|b| \geq b_*$, the spectral radius of $R(z)$ is bounded by $\nu a^{-1}$. More precisely, there exists $c^* > 0$ such that, for $\bar{n} = \lceil c^* \ln|b|\rceil$,*

$$\|R(z)^{\bar{n}}\| \leq \left(\frac{\nu}{a}\right)^{\bar{n}}.$$

COROLLARY 2.13. *There exists $\zeta_1 < 0$ such that $\sigma(X) \cap U_{\zeta_1} = \{0\}$.*

[7]This is the equivalent of the statement that the Laplace transform of the correlation function can be extended to a meromorphic function in a neighborhood of the imaginary axes; see [28].

*Proof.* By the same argument from the beginning of Corollary 2.11, with $\zeta_0 = \min\{\lambda\beta', \nu^{-1} - 1\}$, we see that $U_{\zeta_0} \cap \sigma(X) \subset \{z \in \mathbb{C} \mid \Re(z) \in [-\zeta_0, 0], |\Im(z)| \leq b_*\}$. By Corollary 2.11 it follows that $U_{\zeta_0} \cap \sigma(X)$ contains only finitely many points and from this the result follows. □

To conclude we need to transfer the knowledge gained on the spectrum of $X$ into an estimate on the behavior of the semigroup. A typical way to do so would be to use the Weak Spectral Mapping Theorem ([24, p. 91]) stating that, for all $t \in \mathbb{R}$, $\sigma(T_t) = \overline{\exp(t\sigma(X))}$, provided the semigroup is polynomially bounded for all times. Unfortunately, our semigroup grows exponentially in the past. Thus we need to argue directly. For this purpose a silly preliminary fact is needed.

LEMMA 2.14. *For each $z \in \rho(X)$ (the resolvent set) and $f \in D(X^2)$ the following holds true*:

$$\|R(z)f - z^{-1}f - z^{-2}Xf\| \leq |z|^{-2}\|R(z)\| \, \|X^2 f\|.$$

*Proof.* This follows from the identity $R(z)f = z^{-1}f + z^{-2}Xf + z^{-2}R(z)X^2 f$, for all $f \in D(X^2)$. □

Next notice that, for each $a > 0$ and $f \in D(X^2) \cap \mathcal{C}^0(\mathcal{M}, \mathbb{C})$,[8]

$$\mathcal{L}_t f = \frac{1}{2\pi} \lim_{w \to \infty} \int_{-w}^{w} db \, e^{at+ibt} R(a+ib)f. \tag{2.8}$$

We can now conclude the section with the proof of Theorem 2.4.

*Proof of Theorem* 2.4. Let $\nu_1 = \max\{\nu, \frac{4c^*}{3+4c^*}\}$ and

$$3\omega = \min\{\zeta_1, (\nu_1^{-1} - 1)\bar{c}\}.\text{[9]}$$

First of all by equation (3.2) it follows that

$$\|\mathcal{L}_t f\|_u \leq e^{-\lambda\beta t}\|f\|_u, \tag{2.9}$$

and so we need only worry about the stable part of the norm.

[8]Just notice that, for $f \in D(X^2)$, $\|R(z)f\|_\infty \leq |z|^{-1}(\|X^2 f\| + \|Xf\| + \|f\|)$ (see Lemma 2.14). Hence for each $x \in \mathcal{M}$, $a > 0$, $R(a+ib)f(x)$ is in $L^2$ as a function of $b$. This means that for $f \in D(X^2)$ and $x \in \mathcal{M}$ one can apply the inverse Laplace transform formula and obtain the formula (2.8) point-wise. Note that this implies only that the limit in (2.8) takes place in the $L^2([0,\infty], e^{-at}dt)$ sense as a function of $t$. On the other hand $\mathcal{L}_t f$ is a continuous function of $t$ and, again by Lemma 2.14, $R(a+ib)f - \frac{1}{a+ib}f$ is in $L^1(\mathbb{R}, \mathcal{B})$, as a function of $b$. From this it follows that the limit in (2.8) converges in the $\mathcal{B}$ norm for each $t \in \mathbb{R}^+$.

[9]The constants $\nu, c^*, \bar{c}$ are defined in Proposition 2.12; $\zeta_1$ is defined in Corollary 2.13.

Since $\int f = 0$, Corollary 2.13 implies that the function $R(z)f$ is analytic in the domain $\{\Re(z) \geq -\zeta_1\}$. Then $M := \sup_{a\in[-2\omega,0];\,|b|\leq b_*} \|R(a+ib)f\| < \infty$; moreover, for $a \in [-2\omega, 0]$ and $|b| \geq b_*$,

$$R(a+ib) = [\mathbf{Id} + (a-\bar{c})R(\bar{c}+ib)]^{-1}\, R(\bar{c}+ib).$$

To see that the above formula is well defined consider that, by hypothesis and Lemma 2.8,

$$\|(a-\bar{c})R(\bar{c}+ib)\| \leq (1+\frac{|a|}{\bar{c}}) \leq 1/3 + 2/3\nu_1^{-1}.$$

In addition, for $\bar{n} = \lceil c^* \ln|b|\rceil$ Proposition 2.12 implies

$$\|(a-\bar{c})^{\bar{n}} R(\bar{c}+ib)^{\bar{n}}\| \leq \left[\nu_1\left(1+\frac{|a|}{\bar{c}}\right)\right]^{\bar{n}} \leq \left[\frac{2}{3}+\frac{\nu_1}{3}\right]^{\bar{n}}.$$

Accordingly,

$$\begin{aligned}
&\|\,[\mathbf{Id} + (a-\bar{c})R(\bar{c}+ib)]^{-1}\,\| \\
&\qquad\leq \sum_{n=0}^{\infty} \|(a-\bar{c})^n R(\bar{c}+ib)^n\| \\
&\qquad\leq \sum_{k=0}^{\infty} \|[(a-\bar{c})^{\bar{n}} R(\bar{c}+ib)^{\bar{n}}]^k\| \sum_{j=0}^{\bar{n}-1} \|[(a-\bar{c})R(\bar{c}+ib)]^j\| \\
&\qquad\leq \frac{9}{2(1-\nu_1)^2}|b|^{c^*\ln[\frac{1}{3}+\frac{2}{3\nu_1}]} \leq \frac{9}{2(1-\nu_1)^2}|b|^{1/2}.
\end{aligned}$$

Thus there exists $M_1 > 0$ such that, for $a \in [-2\omega, 0]$ and $b \in \mathbb{R}$,

$$\|R(a+ib)\| \leq M_1\sqrt{|b|} + M. \tag{2.10}$$

To conclude we use (2.8) and shift the contour of integration. For each $f \in D(X^2)\cap\mathcal{C}^0$,

$$\mathcal{L}_t f = \frac{1}{2\pi i}\int_{-2\omega+i\mathbb{R}} dz\, e^{zt}R(z)f = \frac{1}{2\pi i}\int_{-2\omega+i\mathbb{R}} dz\, e^{zt}\left(R(z)-\frac{1}{z}\right)f.$$

By Lemma 2.14 and (2.10) we have that for each $\varphi \in \mathcal{D}_\beta$ and $f \in D(X^2)\cap\mathcal{C}^0$,

$$\begin{aligned}
\left|\int_{\mathcal{M}} \mathcal{L}_t f\varphi\right| &\leq \frac{1}{2\pi}\int_{\mathbb{R}} db\left\|R(-2\omega+ib)f - \frac{1}{-2\omega+ib}f\right\| e^{-2\omega t}\\
&\leq C\left\{\|X^2 f\| + \|Xf\| + \|f\|\right\} e^{-2\omega t}.
\end{aligned}$$

We have thus completed the proof for all $f \in D(X^2)\cap\mathcal{C}^0$; to obtain the announced result for $f \in \mathcal{C}^1$ a standard approximation argument suffices. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a $\mathcal{C}^\infty$ function such that $\mathrm{supp}(\phi) \subset (0,1)$ and $\int\phi = 1$. For each $\varepsilon > 0$ define $\phi_\varepsilon(t) := \varepsilon^{-1}\phi(\varepsilon^{-1}t)$ and, for each $f \in \mathcal{B}(\mathcal{M},\mathbb{C})$,

$$f_\varepsilon := \int_0^\infty \phi_\varepsilon(t)\mathcal{L}_t f.$$

Clearly $f_\varepsilon \in D(X^n)\cap\mathcal{C}^1$ for each $n\in\mathbb{N}$. More to the point,

$$\|X^2 f_\varepsilon\| \le \int |\phi_\varepsilon''(t)|\|\mathcal{L}_t f\| \le \varepsilon^{-2}|\phi''|_{L^1}|f|_{\mathcal{C}^1}.$$

In addition, if $f\in\mathcal{C}^1$,

$$\|f_\varepsilon - f\| \le \int \phi_\varepsilon(t)|f\circ T_{-t} - f|_{\mathcal{C}^\beta} \le \varepsilon^{1-\beta}|f|_{\mathcal{C}^1}\sup_{t\in[0,1]}|T_{-t}|_{\mathcal{C}^1}.$$

Accordingly, for each $f\in\mathcal{C}^1(\mathcal{M},\mathbb{C})$, $\int f=0$, we have

$$\|\mathcal{L}_t f\| \le \|\mathcal{L}_t f_\varepsilon\| + \|f-f_\varepsilon\| \le C_1 e^{-2\omega t}\varepsilon^{-2}|f|_{\mathcal{C}^1} + C_2\varepsilon^{1-\beta}|f|_{\mathcal{C}^1},$$

and the desired results follow by choosing $\varepsilon = e^{-2\omega(3-\beta)^{-1}t}$; hence $\sigma = 2\omega(1-\beta)(3-\beta)^{-1}$. □

## 3. Proofs: the Lasota-Yorke inequality

*Proof of Lemma* 2.7. By Lemma 2.3, for each $\alpha\in(0,1]$

$$|T_t\varphi|_\infty = |\varphi|_\infty;\;\; H_{s,\alpha}(T_t\varphi) \le e^{-\lambda\alpha t}H_{s,\alpha}(\varphi). \tag{3.1}$$

The first inequalities of Lemma 2.7 are immediate since, for $f\in\mathcal{C}^1(\mathcal{M},\mathbb{C})$ and $\varphi\in\mathcal{D}_\beta$ or $\varphi\in\mathcal{D}_1$,

$$\int_{\mathcal{M}}\varphi\mathcal{L}_t\, f = \int_{\mathcal{M}} fT_t\varphi.$$

In addition, again by Lemma 2.3

$$\|\mathcal{L}_t f\|_u = H_{u,\beta}(\mathcal{L}_t f) \le e^{-\beta\lambda t}H_{u,\beta}(f) = e^{-\lambda\beta t}\|f\|_u. \tag{3.2}$$

To conclude the argument we need the averaging operator[10]

$$\mathbb{A}^s_\delta\varphi(x) := \frac{1}{m^s(W^s_\delta(x))}\int_{W^s_\delta(x)}\varphi(z)m^s(dz). \tag{3.3}$$

The basic properties of such an operator consist in the following

Sub-lemma 3.1. *There exists $C>0$ such that for each $\varphi\in\mathcal{D}_\beta$,*

$$\begin{aligned}
|\mathbb{A}^s_\delta\varphi - \varphi|_\infty &\le C\delta^\beta|\varphi|_{s,\beta},\\
H_{s,\beta}(\mathbb{A}^s_\delta\varphi-\varphi) &\le (2+C\delta)H_{s,\beta}(\varphi) + C\delta^{1-\beta}|\varphi|_\infty,\\
H_{s,1}(\mathbb{A}^s_\delta\varphi) &\le C\delta^{-1}|\varphi|_\infty.
\end{aligned}$$

[10]By $W^s_\delta(x)$ we mean a ball of radius $\delta$, centered at $x$, with respect to the metric obtained by restricting the Riemannian metric to $W^s(x)$. By $m^s$ we designate the corresponding volume form.

The above sub-lemma is hardly surprising, yet its proof is a bit technical and it is postponed to Appendix C. By Sub-Lemma 3.1 it follows that, given $\varphi \in \mathcal{D}_\beta$ and $f \in \mathcal{C}^1$,

$$\begin{aligned}\int_{\mathcal{M}} f\varphi &= \int_{\mathcal{M}} f\{\varphi - \mathbb{A}^s_\delta\varphi\} + \int_{\mathcal{M}} f\mathbb{A}^s_\delta\varphi \leq |\varphi - \mathbb{A}^s_\delta\varphi|_{s,\beta}\|f\|_s + |\mathbb{A}_\delta\varphi|_{s,1}\|f\|_w \\ &\leq (C(\delta^\beta + \delta^{1-\beta})|\varphi|_\infty + (2 + C\delta)H_{s,\beta}(\varphi))\|f\|_s + C\delta^{-1}\|f\|_w.\end{aligned}$$

Accordingly, remembering (3.1), for each $\varphi \in \mathcal{D}_\beta$,

$$\begin{aligned}\int_{\mathcal{M}} \mathcal{L}_t f\varphi &= \int_{\mathcal{M}} fT_t\varphi \leq (C(\delta^\beta + \delta^{1-\beta})|\varphi|_\infty \\ &\quad + (2 + C\delta)H_{s,\beta}(\varphi \circ T_t))\|f\|_s + C\delta^{-1}\|f\|_w \\ &\leq (C(\delta^\beta + \delta^{1-\beta})|\varphi|_\infty + (2 + C\delta)e^{-\lambda\beta t}H_{s,\beta}(\varphi))\|f\|_s + C\delta^{-1}\|f\|_w.\end{aligned}$$

We start by requiring $2 + C\delta \leq 3$, then let $T_0 \in \mathbb{R}^+$ be such that $3e^{-\lambda\beta T_0} \leq e^{-\lambda\beta' T_0}$; at last we choose $\delta$ so that $C(\delta^\beta + \delta^{1-\beta}) \leq e^{-\lambda\beta' T_0}$. Thus, for each $t \leq T_0$,

$$\begin{aligned}\|\mathcal{L}_t f\|_s &\leq 3e^{-\lambda\beta' t}\|f\|_s + C\delta^{-1}\|f\|_w, \\ \|\mathcal{L}_{T_0} f\|_s &\leq e^{-\lambda\beta' T_0}\|f\|_s + C\delta^{-1}\|f\|_w.\end{aligned} \tag{3.4}$$

For each $t \in \mathbb{R}^+$ we write $t = kT_0 + s$, $k \in \mathbb{N}$, $s \in (0, T_0)$, and we use (3.4) iteratively to obtain

$$\|\mathcal{L}_t f\|_s \leq 3e^{-\lambda\beta' t}\|f\|_s + B\|f\|_w \tag{3.5}$$

with $B = C\delta^{-1}(1 - e^{-\lambda\beta' T_0})^{-1}$.

The strong continuity of the group follows trivially since, for each $f \in \mathcal{C}^1(\mathcal{M}, \mathbb{C})$,[11]

$$\lim_{t\to 0} \|\mathcal{L}_t f - f\| = 0$$

and $\mathcal{C}^1(\mathcal{M}, \mathbb{C})$ is dense in $\mathcal{B}(\mathcal{M}, \mathbb{C})$ and $\mathcal{B}_w(\mathcal{M}, \mathbb{C})$ by construction. □

## 4. Proofs: Quasi-compactness of the resolvent

*Proof of Lemma* 2.9. The idea is to introduce approximate operators $R_\varepsilon(z)$ (close in norm to $R(z)$ as operators from $\mathcal{B}(\mathcal{M}, \mathbb{C})$ to $\mathcal{B}_w(\mathcal{M}, \mathbb{C})$) and then consider the following sequence of maps (for some $\tau^2 \geq \beta_* > \beta > 0$):

$$\mathcal{B}(\mathcal{M}, \mathbb{C}) \xmapsto{\mathrm{Id}} \mathcal{C}^\beta(\mathcal{M}, \mathbb{C})^* \xhookrightarrow{\mathrm{Id}} \mathcal{C}^{\beta_*}(\mathcal{M}, \mathbb{C})^* \xmapsto{R_\varepsilon(z)} \mathcal{B}_w(\mathcal{M}, \mathbb{C}). \tag{4.1}$$

[11] Indeed, $|f \circ T_{-t} - f|_\infty + H_{u,\beta}(f \circ T_{-t} - f) \to 0$ as $t \to 0$.

The first map is clearly continuous since for each $\varphi \in \mathcal{C}^\beta(\mathcal{M}, \mathbb{C})$ and $f \in \mathcal{B}(\mathcal{M}, \mathbb{C})$ one has

$$\int_{\mathcal{M}} f\varphi \leq \|f\| \, |\varphi|_{s,\beta} \leq \|f\| \, |\varphi|_{\mathcal{C}^\beta}$$

and thus $\|f\|_{(\mathcal{C}^\beta)^*} \leq \|f\|$. The second is well known to be compact. Hence it suffices to prove that the last map is continuous; the compactness of $R_\varepsilon(z)$ as an operator from $\mathcal{B}$ to $\mathcal{B}_w$ immediately follows. Let us postpone the proof of this fact to Lemma 4.4.

To define the approximate operators we introduce the averaging operator

$$\mathbb{A}^u_\varepsilon f(x) := Z_\varepsilon(x) \int_{W^u_\varepsilon(x)} f(\xi) m^u(d\xi), \tag{4.2}$$

where $Z_\varepsilon(x)$ is determined by the equation $\mathbb{A}^u_\varepsilon 1 = 1$. We set $R_\varepsilon(z) := R(z)\mathbb{A}^u_\varepsilon$.

Sub-lemma 4.1. *The operators $R_\varepsilon(z)$ satisfy*[12]

$$|||R(z) - R_\varepsilon(z)||| \leq C\varepsilon^\beta.$$

*Proof.* For each $f \in \mathcal{C}^1(\mathcal{M}, \mathbb{C})$ and $\varphi \in \mathcal{C}^0(\mathcal{M}, \mathbb{C})$, we have

$$\begin{aligned}\left|\int_{\mathcal{M}} \mathbb{A}^u_\varepsilon f\varphi - \int_{\mathcal{M}} f\varphi\right| &\leq |\varphi|_\infty \int_{\mathcal{M}} dx Z_\varepsilon(x) \int_{W^u_\varepsilon(x)} d\xi |f(\xi) - f(x)| \\ &\leq C\varepsilon^\beta \|f\|_u |\varphi|_\infty.\end{aligned}$$

Accordingly, $\|\mathbb{A}^u_\varepsilon f - f\|_w \leq C\varepsilon^\beta \|f\|$; that is, $|||\mathbb{A}^u_\varepsilon - \mathbf{Id}||| \leq C\varepsilon^\beta$. From Lemma 2.8 it follows that $|||R_\varepsilon(z) - R(z)||| \leq Ca^{-1}\varepsilon^\beta$. □

From Sub-Lemma 4.1 and the compactness of $R_\varepsilon(z)$ the compactness of $R(z) : \mathcal{B}(\mathcal{M}, \mathbb{C}) \to \mathcal{B}_w(\mathcal{M}, \mathbb{C})$ is obvious since the compact operators form a closed set. □

In the previous lemma we postponed the proof of Lemma 4.4. Before giving such a proof some preparatory work is needed.

*Definition* 2. Given an operator $B : \mathcal{B} \to \mathcal{B}$ we define $B^* : \mathcal{B}^* \to \mathcal{B}^*$ as usual. Notice that if $\varphi \in \mathcal{D}_\beta \subset \mathcal{B}^*$ and $B^*\varphi \subset L^\infty$ then, for each $f \in \mathcal{C}^1$, $Bf \in L^1$, one has

$$\int Bf\varphi = \int fB^*\varphi. \tag{4.3}$$

Similar definitions hold for $\mathcal{B}_w$ and $\mathcal{D}_1$.

[12] By $|||\cdot|||$ we mean the norm of an operator viewed as an operator from $\mathcal{B}(\mathcal{M}, \mathbb{C})$ to $\mathcal{B}_w(\mathcal{M}, \mathbb{C})$.

*Remark* 4.2. In the following we will never need to investigate the duals $\mathcal{B}^*$, $\mathcal{B}_w^*$; it will suffice to consider elements of $\mathcal{D}_\beta$ and $\mathcal{D}_1$. Accordingly we will always use (4.3).

Next we isolate a result needed in the present argument but useful also in the following.

LEMMA 4.3. *There exists $c > 0$ such that for each $\alpha \in (0, \tau^2)$, $\varphi \in \mathcal{D}_1$, $z \in \mathbb{C}$ with $|b| = |\Im(z)| > 1$ and $a = \Re(z) > 0$, $\mathbb{A}_\varepsilon^{u*} R(z)^* \varphi \in \mathcal{C}^\alpha$. More precisely,*

$$|\mathbb{A}_\varepsilon^{u*} R(z)^* \varphi|_{\mathcal{C}^\alpha} \leq c(|b| + \varepsilon^{-1})|\varphi|_{s,1}.$$

*Proof.* Let $f \in \mathcal{C}^1(\mathcal{M}, \mathbb{C})$ and $\varphi \in \mathcal{D}_1$; then

$$\int_{\mathcal{M}} R_\varepsilon(z) f \varphi = \int_{\mathcal{M}} f R_\varepsilon(z)^* \varphi$$

where $R_\varepsilon(z)^* = \mathbb{A}_\varepsilon^{u*} R(z)^*$,

$$R(z)^* \varphi(x) = \int_0^\infty e^{-zt} T_t \varphi(x)\, dt \tag{4.4}$$

by the definition of $\mathcal{L}_t$. On the other hand in Appendix C it is shown that

$$\mathbb{A}_\varepsilon^{u*} \varphi(x) = \int_{W_\varepsilon^u(x)} \tilde{Z}_\varepsilon(x, \xi) \varphi(\xi) m^u(d\xi) \tag{4.5}$$

for some appropriate $\tau$-Hölder function $\tilde{Z}_\varepsilon$ (see Lemma C.2). Since by (4.4)

$$\frac{d}{dt}(R(z)^* \varphi) \circ T_t \bigg|_{t=0} = z R(z)^* \varphi - \varphi$$

it follows that $R(z)^* \varphi$ is Hölder along the strong stable direction and differentiable along the flow direction. Let us set $\varphi_* := R(z)^* \varphi$.

Let $x, y$ be two points on the same strong stable manifold, and let $\Psi$ be the stable holonomy between $W^{uc}(x)$ and $W^{uc}(y)$. According to Lemma C.1, $d(z, \Psi(z)) \leq C d(x, y)^\tau$ holds for each $z \in W_\delta^u(x)$. Moreover, $|1 - J\Psi(z)| \leq C d(x, y)^\tau$.

If $\delta^{\tau^2} \geq d(x, y)^{\tau^2} \geq \varepsilon$ then (see Lemma C.2)

$$|\mathbb{A}_\varepsilon^{u*} \varphi_*(x) - \mathbb{A}_\varepsilon^{u*} \varphi_*(y)| \leq 2\bar{c} |\varphi|_\infty d(x, y)^\alpha \varepsilon^{-\tau^2}.$$

Suppose instead $d(x, y)^{\tau^2} \leq \varepsilon$. Let $\hat{\Psi} : W^u(x) \to W^u(y)$ be the weak stable holonomy ($\{\hat{\Psi}(\xi)\} = W^{sc}(\xi) \cap W^u(y)$). The distance along the flow between $\hat{\Psi}(\xi)$ and $\Psi(\xi)$ is nothing but the temporal distance $\Delta(y, \xi)$ (see definition at the end of App. A or Figure 2, App. B). Accordingly, Lemma B.7 yields $d(\hat{\Psi}(\xi), \Psi(\xi)) \leq C d(x, y)^{\tau^2}$. In addition, $W^{uc}_{\varepsilon - c\varepsilon d(x,y)^{\tau^2}}(y) \subset \Psi(W_\varepsilon^{uc}(x)) \subset$

$W^{uc}_{\varepsilon+c\varepsilon d(x,y)^{\tau^2}}(y)$.[13] This, together with the uniform transversality between the unstable manifold and the flow direction, implies that the symmetric difference between $W^u_\varepsilon(y)$ and $\hat\Psi(W^u_\varepsilon(x))$ has a volume bounded by a $\varepsilon^{-1}d(x,y)^\alpha$ times the volume of $W^u_\varepsilon(x)$. Finally, it is easy to verify that $J\hat\Psi = J\Psi$. Hence, remembering Lemma C.2,

$$
\begin{aligned}
&|\mathbb{A}^{u*}_\varepsilon\varphi_*(x)-\mathbb{A}^{u*}_\varepsilon\varphi_*(y)|\\
&\quad\le C\left\{|\varphi|_\infty d(x,y)^\alpha(\varepsilon^{-1}+|b|)+\int_{W^u_\varepsilon(x)}\tilde Z_\varepsilon(x,\xi)|\varphi_*(\Psi(\xi))-\varphi_*(\xi)|m^u(d\xi)\right\}\\
&\quad\le C\left\{|\varphi|_\infty(\varepsilon^{-1}+|b|)+H_{s,1}(\varphi)\right\}d(x,y)^\alpha.
\end{aligned}
$$

To conclude note that the arguments in the proof of Sub-Lemma 3.1 hold unchanged for $\mathbb{A}^{u*}_\varepsilon$ instead of $\mathbb{A}^s_\varepsilon$. Accordingly,

$$H_{u,\alpha}(\mathbb{A}^{u*}_\varepsilon\varphi_*)\le C\varepsilon^{-1}|\varphi_*|_\infty\le C\varepsilon^{-1}|\varphi|_\infty.$$

A direct computation shows

$$\left|\frac{d}{dt}(\mathbb{A}^{u*}_\varepsilon\varphi_*)\circ T_t\Big|_{t=0}\right|_\infty\le C\left(|\varphi_*|_\infty+\left|\frac{d}{dt}(\varphi_*)\circ T_t\Big|_{t=0}\right|_\infty\right)\le C|b|\,|\varphi|_\infty.$$

Since any point in a $\delta$-neighborhood of $x$ can be reached by a path along the stable, unstable and flow directions of length less than const.$\delta$, the lemma follows. $\square$

We are finally able to prove the continuity of the operator $R_\varepsilon:\mathcal{C}^{\beta_*}(\mathcal{M},\mathbb{C})\to B_w(\mathcal{M},\mathbb{C})$.

LEMMA 4.4. *For each $\varepsilon>0$ and $z\in\mathbb{C}$, $\Re(z)>0$, the operators $R_\varepsilon(z)$ are bounded operators from $\mathcal{C}^{\beta_*}(\mathcal{M},\mathbb{C})^*$ to $\mathcal{B}_w(\mathcal{M},\mathbb{C})$.*

---

[13] By introducing a coordinate system in which $W^{uc}(x)$ and $W^s(x)$ are linear spaces one can represent $W^{uc}(y)$ as $\{(\xi,F(\xi))\}$ where, by the Hölder continuity of the unstable foliation and the fact that $U(\xi):=D_\xi F$, one has $\|U(\xi)\|\le c\|F(\xi)\|^\tau$. And, by the Hölder continuity of the unstable holonomy, $\|F(\xi)\|\le cd(x,y)^\tau$. Thus, setting $\gamma(t)=(vt,F(vt))$, with $v:=z-x$, and $z':=(z,F(z))$, one can estimate

$$
\begin{aligned}
\mathrm{dist}(y,z')&=\int_0^1\|\gamma'(t)\|dt=\int_0^1\sqrt{\langle(v,U(vt)v),g((vt,F(vt)))(v,U(vt)v)\rangle}dt\\
&=\int_0^1\sqrt{\langle(v,0),g((vt,0))(v,0)\rangle+O(d(x,z)^2d(x,y)^{\tau^2})}\,dt=d(x,z)(1+O(d(x,y)^{\tau^2})
\end{aligned}
$$

where $g$ is the matrix defining the Riemannian metric. On the other hand one can represent $W^s(z)$ as $\{(G(\zeta),\zeta)\}$, where $V(\zeta):=D_\zeta G$ is bounded in norm by $c\varepsilon^\tau$. Setting $\Psi(z)=:(a,b)=(a,F(a))=(G(b),b)$ we see that $\|b\|\le cd(x,y)^\tau$. Hence (provided $d(x,z)\ge d(x,y)^\tau$) $\mathrm{dist}(z',\Psi(z))\le c\,\mathrm{dist}((a,0),z)\le\int_0^1\|V(bt)b\|dt\le c\,d(x,z)^\tau d(x,y)^\tau\le c\,d(x,z)d(x,y)^{\tau^2}$.

*Proof.* By Lemma 4.3 it follows that, for each $f \in \mathcal{C}^1$ and $\varphi \in \mathcal{D}_1$,

$$\int_{\mathcal{M}} R_\varepsilon(z) f\varphi \le |f|_{(\mathcal{C}^{\beta_*})^*}\ |R_\varepsilon(z)^*\varphi|_{\mathcal{C}^{\beta_*}} \le C(|z|+\varepsilon^{-1})|\varphi|_{s,1}|f|_{(\mathcal{C}^{\beta_*})^*}$$

which means $\|R_\varepsilon(z)f\|_w \le C(|z|+\varepsilon^{-1})|f|_{(\mathcal{C}^{\beta_*})^*}$ and the required result follows by an obvious density argument. □

## 5. Proofs: Resolvent bounds for large $\Im(z)$

*Proof of Proposition* 2.12. Lemma 2.8 states that, for each $m, n \in \mathbb{N}$ and $f \in \mathcal{C}^1(\mathcal{M}, \mathbb{C})$,

$$\begin{aligned}\|R(z)^{n+m}f\| &\le \frac{3}{(a+\lambda\beta')^m}\|R(z)^n f\| + a^{-m}B\|R(z)^n f\|_w \\ &\le \frac{3}{(a+\lambda\beta')^m a^n}\|f\| + a^{-m}B\|R(z)^n f\|_w.\end{aligned} \tag{5.1}$$

Hence all we need is to estimate more precisely the weak norm of $R(z)^n f$.

By (4.2),

$$\int f\varphi = \int \mathbb{A}^u_\delta f\varphi + O(\delta^\beta\|f\|_u|\varphi|_\infty) = \int f\mathbb{A}^{u*}_\delta\varphi + O(\|f\|_u|\varphi|_\infty). \tag{5.2}$$

Thus, for each $k, l \in \mathbb{N}$, $k + l = n$, and $\varphi \in \mathcal{D}_1$ plus equation (5.2),

$$\begin{aligned}\int_{\mathcal{M}} R(z)^n f\varphi &= \int_{\mathcal{M}} R(z)^k f R(z)^{*l}\varphi \\ &= \int_{\mathcal{M}} R(z)^k f\mathbb{A}^{u*}_\delta R(z)^{*l}\varphi + a^{-l}O(\|R(z)^k f\|_u).\end{aligned}$$

To continue let

$$\Phi_l(\varphi) := \mathbb{A}^{u*}_\delta R(z)^{*l}\varphi.$$

Thus, taking into account (2.7) and (2.9), we obtain

$$\int_{\mathcal{M}} R(z)^n f\varphi = \int_{\mathcal{M}} R(z)^k f\Phi_l(\varphi) + a^{-n}(1+a^{-1}\lambda\beta)^{-k}O(\|f\|_u). \tag{5.3}$$

Lemma 5.1. *There exists $c > 0$ such that, for each $l \in \mathbb{N}$ and $\varphi \in \mathcal{D}_1$,*

$$H_{s,\beta}(\Phi_l(\varphi)) \le c|b|a^{-l}|\varphi|_{s,1}.$$

*Proof.* The proof follows immediately from Lemma 4.3 and formulae (2.7), (3.1). □

The above estimate is not particularly impressive and clearly it can have some interest only if we can get good bounds on $|\Phi_l(\varphi)|_\infty$. This can be achieved by using an inequality due to Dolgopyat.[14]

LEMMA 5.2 (the Dolgopyat inequality). *There exist $c_*, c_1, \gamma > 0$ such that, for each $\varphi \in R(z)^*(\mathcal{D}_1)$ and $l \geq \lceil c_* \ln |b| \rceil$, the following holds*:

$$a^l |\Phi_l(\varphi)|_\infty \leq c_1 |b|^{-\gamma} l |\varphi|_{s,1}.$$

The proof of the above lemma can be found in Section 6.

Since equation (3.1) implies that, for each $q \in \mathbb{N}$, $a^q R(z)^{*q}\varphi \in \mathcal{D}_{s,1}$ and $H_{s,\beta}(R(z)^{*q}\varphi) \leq (a+\beta\lambda)^{-q} H_{s,\beta}(\varphi)$ by Lemma 5.2 and Lemma 5.1, it follows that

$$|R(z)^{*k}\Phi_l(\varphi)|_{s,\beta} \leq c_4\{(1+a^{-1}\lambda\beta)^{-k}|b| + |b|^{-\gamma} l\} a^{-n} |\varphi|_{s,1}.$$

Choose $l := \lceil c_* \ln b \rceil$; then there exist $c' > 0$ and $\nu_0 \in (0,1)$ such that when $k = \lceil c' \ln b \rceil$, equation (5.3) yields

$$\|R(z)^n f\|_w \leq c_5 a^{-n} \nu_0^n \|f\|. \tag{5.4}$$

The proposition follows by (5.1), (5.4), when $m = n = \bar{n}/2$ (hence $c^* = 2(c_* + c'))$, $\bar{c} = 2$, $\nu \in (\sqrt{\nu_0}, 1)$ and $b_*$ such that $c_5(\nu_0 \nu^{-2})^n \leq 1$. □

## 6. Dolgopyat inequality

This section is devoted to the proof of Lemma 5.2. The strategy is based on the representation (2.7) (actually on the obvious adjoint representation obtained by (4.4)) and a careful estimate of the corresponding integral.

The following simple preliminary lemma shows that we need to worry about only a part of the integral defining $\Phi_l(\varphi)$.

LEMMA 6.1. *There exists $\nu_* < 1$ such that*

$$\left| \frac{1}{(l-1)!} \int_0^{e^{-1}a^{-1}l} t^{l-1} e^{-zt} \mathbb{A}_\delta^{u*}(T_t\varphi) dt \right| \leq \nu_*^l a^{-l} |\varphi|_\infty.$$

The straightforward proof is left to the reader.

Thus we can limit ourselves to consideration of

$$\frac{1}{(l-1)!} \int_{e^{-1}a^{-1}l}^\infty t^{l-1} e^{-zt} \mathbb{A}_\delta^{u*}(T_t\varphi).$$

[14] Actually the original Dolgopyat estimate, [7], holds for the $L^2$ norm and it is done for a different operator in a different functional space, yet the key cancellation mechanism due to the oscillations of the exponential and the nonjoint integrability of the foliation remain substantially identical in the two settings.

To continue, it is useful to localize in time. To do so we introduce a $\mathcal{C}^\infty$ function $\mathbf{p}:\mathbb{R}\to\mathbb{R}$ such that $0\le\mathbf{p}\le 1$, $\operatorname{supp}(\mathbf{p})\subset[-1/2,3/2]$ and with the property that $\sum_{k=-\infty}^{\infty}\mathbf{p}(t-k)=1$ for each $t\in\mathbb{R}$. Using such a partition of unity and setting $p_0:=\lceil a^{-1}e^{-1}l\rceil$, we can write

$$
\begin{aligned}
&\left|\int_0^\infty t^{l-1}e^{-zt}\mathbb{A}_\delta^{u*}(T_t\varphi)\right|\\
&\qquad\le\left|\sum_{k=p_0}^{\infty}\int_{\mathbb{R}} t^{l-1}e^{-zt}\mathbf{p}(t-k)\mathbb{A}_\delta^{u*}(T_t\varphi)\right|+\nu_*^l a^{-l}(l-1)!|\varphi|_\infty.
\end{aligned}
$$

Let us analyze each of the above addenda separately.

For each $k\in\mathbb{N}$ (see (4.5)),

$$
\begin{aligned}
&\int_{\mathbb{R}} t^{l-1}e^{-zt}\mathbf{p}(t-k)\mathbb{A}_\delta^{u*}(T_t\varphi)\\
&\qquad=\int_{\mathbb{R}}\mathbf{p}(t-k)t^{l-1}e^{-zt}\int_{T_kW_\delta^u(x)}\tilde Z(x,T_{-k}\xi)\varphi(T_{t-k}\xi)J_uT_{-k}(\xi),
\end{aligned}
$$

where by $J_uT_t$ we designate the unstable Jacobian of the map $T_t$.

To compute the above quantity it is convenient to localize in space as well. To this end we fix a sequence of smooth partitions of unity. There exists $c_d>0$ such that, for each $r\in(0,1)$ one can consider a $\mathcal{C}^4$ partition of unity $\{\phi_{r,i}\}_{i=1}^{q(r)}$ enjoying the following properties:[15]

(i) For each $i\in\{1,\dots,q(r)\}$, there exists $x_i\in\mathcal{M}$ such that $\phi_{r,i}(\xi)=1$ for all $\xi\in B_r(x_i)$ (the ball of radius $r$ centered at $x_i$) and $\phi_{r,i}(\xi)=0$ for all $\xi\notin B_{c_dr}(x_i)$;

(ii) There exists a $K>0$ such that for each $r$, (i) holds[16]

$$
\|\phi'_{r,i}(x)\|\le Kr^{-1}\chi_{B_{c_dr}(x_i)}(x);
$$

(iii) There exists $C>0$ such that $q(r)\le Cr^{-2d-1}$.

Accordingly, we can write

$$
\begin{aligned}
\int_{\mathbb{R}} t^{l-1}e^{-zt}\mathbf{p}(t-k)\mathbb{A}_\delta^{u*}(T_t\varphi)=&\sum_{i=1}^{q(r)}e^{-zk}\int_{\mathbb{R}}\mathbf{p}(t)(t+k)^{l-1}e^{-zt}\\
&\times\int_{T_kW_\delta^u(x)}\phi_{r,i}(T_t\xi)\tilde Z(x,T_{-k}\xi)\varphi(T_t\xi)J_uT_{-k}(\xi).
\end{aligned}
$$

From now on we will assume $b>0$, the case $b<0$ being identical.

[15]It is an easy exercise to verify that partitions with the properties below do exist.

[16]Here, and in the following, $\chi_A$ is the characteristic function of the set $A$.

In the following we choose $\rho \in (0, \tau/8)$ and we fix

$$r := b^{-\varrho}; \quad \varrho := \frac{1-\tau+2\rho}{2-\tau}. \tag{6.1}$$

It is useful to partition $T^k W^u_\delta(x)$ into submanifolds. For each $x_i$ let us consider the connected pieces of $T^k W^u_\delta(x) \cap B_{\theta c_d r}(x_i)$ intersecting $B_{c_d r}(x_i)$ ($\theta$ is specified shortly). Call them $\{W^u_{k,i,m}\}$. Among such local manifolds discard the ones such that $\partial W^u_{k,i,m} \not\subset \partial B_{\theta c_d r}(x_i)$; see Figure 1. Clearly, if $W$ is a discarded manifold, then $T_{-k}W$ belongs to a $\theta c_d r \lambda^{-k}$-neighborhood of $\partial W^u_\delta(x)$; hence the total measure of the preimages of the discarded manifolds is bounded by const.$\lambda^{-k}$. The constant $\theta$ is chosen so that if $\xi \in W^u_{k,i,m} \cap B_{c_d r}(x_i)$, then $W^s_\delta(\xi) \cap W^u_{k,i,j} \neq \emptyset$, for all $j$.

Let us define $W^{uc}_{k,i,m} := \cup_{t\in[-2,2]} T_t W^u_{k,i,m}$.

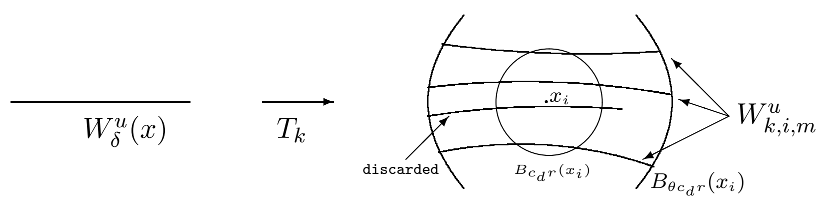


Figure 1: The manifolds $W^u_{k,i,m}$.

For each $\xi \in W^{uc}_{k,i,j}$ let $t(\xi)$ be such that $T_{t(\xi)}\xi \in W^u_{k,i,j}$ and let $u(\xi) := T_{t(\xi)}\xi$. Then

$$\begin{aligned}\int_{\mathbb{R}} t^{l-1} e^{-zt} \mathbf{p}(t-k) \mathbb{A}^{u*}_\delta(T_t\varphi) &= \sum_{ij} e^{-zk} k^{l-1} \int_{W^{uc}_{k,i,j}} \mathbf{p}(t(\xi)) \\ &\times \left(1+\frac{t(\xi)}{k}\right)^{l-1} e^{-zt(\xi)} \tilde{Z}(x, T_{-k}u(\xi))\varphi(\xi)\phi_{r,i}(\xi) J_u T_{-k}(u(\xi)) \\ &+ k^{l-1} e^{-ak} O(\lambda^{-k}|\varphi|_\infty).\end{aligned} \tag{6.2}$$

Next, for each $W^{uc}_{k,i,j}$ let $\Psi_{k,i,j}$ be the stable holonomy between $W^{uc}_{k,i,j}$ and $W^{uc}_{k,i,0}$. By the general theory of the holonomy maps (see Appendix A) it follows that $\Psi_{k,i,j}$ is a $\tau$-Hölder function with $\tau$-Hölder Jacobian $J\Psi_{k,i,j}$.

Notice that $T_{-k}W^u_{k,i,j}$ has size smaller than $2c_d\lambda^{-k}r$ and thus (see Lemma C.2)

$$\tilde{Z}(x, T_{-k}u(\xi)) = Z_{k,i,j} + O(\lambda^{-k\tau} r^\tau).$$

To simplify notation we introduce the functions

$$\begin{aligned}F_{k,i,j}(\xi) &= \mathbf{p}(t(\xi))\left(1+\frac{t(\xi)}{k}\right)^{l-1} \phi_{r,i}(\xi) J_u T_{-k}(u(\xi)) e^{-at(\xi)} Z_{k,i,j}, \\ \hat{F}_{k,i,j}(\xi) &= F_{k,i,j}(\Psi_{k,i,j}(\xi))) J\Psi_{k,i,j}(\xi).\end{aligned} \tag{6.3}$$

Using the above formulae we can rewrite (6.2) as

(6.4)
$$\int_{\mathbb{R}} t^{l-1}\mathbf{p}(t-k)e^{-zt}\mathbb{A}_\delta^{u*}(T_t\varphi)=\sum_{ij} e^{-zk}k^{l-1}\int_{W^{uc}_{k,i,0}} e^{-ibt(\Psi_{k,i,j}(\xi))}\hat F_{k,i,j}(\xi)\varphi(\xi)$$
$$+\,k^{l-1}e^{-ak}O(\lambda^{-k\tau}|\varphi|_\infty+rH_{s,1}(\varphi)).$$

The last preparatory step is to apply Schwartz inequality. More precisely, for each $k, i$, we can compute

$$\left|\int_{W^{uc}_{k,i,0}}\varphi(\xi)\left[\sum_j e^{-ibt(\Psi_{k,i,j}(\xi))}\hat F_{k,i,j}(\xi)\right]\right| \le C|\varphi|_\infty r^{(d+1)/2}$$
$$\times\left[\sum_{j,j'}\int_{W^{uc}_{k,i,0}} e^{-ibg^0_{j,j'}(\xi)}\hat F_{k,i,j}(\xi)\overline{\hat F_{k,i,j'}(\xi)}\right]^{\frac12}, \tag{6.5}$$

where

$$g^0_{j,j'}(\xi):=t(\Psi_{k,i,j}(\xi))-t(\Psi_{k,i,j'}(\xi)). \tag{6.6}$$

We are finally approaching the end of the story; to conclude we must only show that the above integral is small.

Let us perform the sum on $j'$ for each $j$. With fixed $j$ it is convenient to express the integral on the manifold $W^{uc}_{k,i,j}$:

$$\int_{W^{uc}_{k,i,0}} e^{-ibg^0_{j,j'}(\xi)}\hat F_{k,i,j}(\xi)\overline{\hat F_{k,i,j'}(\xi)}=\int_{W^{uc}_{k,i,j}} e^{-ibg_{j,j'}(\xi)}F_{k,i,j}(\xi)\overline{F_{k,i,j,j'}(\xi)}$$

where

$$\begin{aligned} g_{j,j'}(\xi)&:=t(\xi)-t(\Psi_{k,i,j,j'}(\xi)),\\ \Psi_{k,i,j,j'}(\xi)&:=\Psi_{k,i,j'}\circ\Psi^{-1}_{k,i,j}(\xi),\\ F_{k,i,j,j'}(\xi)&:=F_{k,i,j'}(\Psi_{k,i,j,j'}(\xi))J\Psi_{k,i,j'}\circ\Psi^{-1}_{k,i,j}(\xi);\end{aligned} \tag{6.7}$$

clearly $\Psi_{k,i,j,j'}$ is nothing less than the holonomy between $W^{uc}_{k,i,j}$ and $W^{uc}_{k,i,j'}$.

Finally, it is convenient to divide the sum over $j'$ into two parts: the sum over nearby manifolds and the sum over manifolds at a useful distance. Let us be more precise.

Let $y_{k,i,j}:=W^s_{2c_d r}(x_i)\cap W^{uc}_{k,i,j}$. We define the sets of indexes $A_{k,i,j}:=\{j'\mid d(y_{k,i,j},y_{k,i,j'})<b^{-\varsigma}\}$ and $B_{k,i,j}:=\{j'\mid d(y_{k,i,j},y_{k,i,j'})\ge b^{-\varsigma}\}$. In the following we choose

$$\varsigma:=\frac{1-4\rho}{2-\tau}. \tag{6.8}$$

Notice the the assumption $\rho < \tau/6$ implies that $b^{-\varsigma}$ is much smaller than $r$, as $b$ increases.

The first step is to estimate the sum with indexes in $A_{k,i,j}$. To do so we need the next lemma whose proof is postponed to the end of the section.

LEMMA 6.2. *For each $\epsilon > 0$, let $W^{uc}_\epsilon$ be an unstable disk of radius $\epsilon$. Then there exist constants $C, r_0 > 0$ such that for each $k \in \mathbb{N}$, $r_1 > 0$ and $x \in \mathcal{M}$, where $\{W_j\}$ are the connected components of $T_k W^{uc}_\epsilon \cap B_{2r_1}(x)$,*

$$\sum_{j\in\Omega} \sup_{\xi\in W_j} J^u_\xi T_{-k} \le C\, m^s(W^s_{r_1+\lambda^{-k}r_0}(x)),$$

*where $\Omega := \{j \mid W_j \cap W^s_{r_1}(x) \neq \emptyset\}$.*

We then require

$$\lambda^{-l} r_0 \le b^{-\varsigma}. \tag{6.9}$$

Using the above lemma and standard distortion arguments we readily obtain

$$\left|\sum_{j'\in A_{k,i,j}} \int_{W^{uc}_{k,i,j}} e^{-ibg_{j,j'}} F_{k,i,j}\overline{F_{k,i,j,j'}}\right| \le C J^u T_{-k}(y_{k,i,j}) b^{-d\varsigma} r^{d+1}. \tag{6.10}$$

We are then left with the estimate of the indexes in $B_{k,i,j}$. To this end it is useful to make a connection with the *temporal function* introduced at the end of Appendix A and shown pictorially in Figure 2, Appendix B. For each $\xi \in W^{uc}_{k,i,j}$,[17]

$$g_{j,j'}(\xi) = t(\Psi_{k,i,j,j'}(\xi)) - t(\xi) = \Delta(y_{k,i,j'}, T_{-t(y_{k,i,j})}u(\xi)) - t(y_{k,i,j}) + t(y_{k,i,j'}). \tag{6.11}$$

All the above work was just preparation to apply the following lemma (the proof can be found at the end of the section).

LEMMA 6.3. *For each function $G \in \mathcal{C}^\alpha(W^{uc}_{k,i,j})$, $0 < \alpha < 1$, $j' \in B_{k,i,j}$, and with $\bar\phi(u) := \phi_{i,r}(u)\phi_{i,r}(\Psi_{k,i,j,j'}(u))$, the following holds:*

$$\left|\int_{W^{uc}_{k,i,j}} du\, e^{-ibg_{j,j'}(u)} G(u)\bar\phi(u)\right| \le C b^{-\alpha\rho} r^{d+1} |G|_{\mathcal{C}^\alpha}.$$

[17] To apply Figure 2 to the present case set: $y = y_{k,i,j'}$, $x = y_{k,i,j}$ and $y' = T_{-t(y_{k,i,j})}u(\xi)$.

Remembering (6.3), (6.4), (6.5), (6.7), using (6.10) with Lemma 6.3 and taking (A.3) into account we obtain[18]

(6.12)
$$\begin{aligned}\left|\int_{\mathbb{R}} t^{l-1}\mathbf{p}(t-k)e^{-zt}\mathbb{A}_\delta^{u*}(T_t\varphi)(x)\right| &\leq Ck^{l-1}e^{-ak}\sum_i |\varphi|_\infty r^{\frac{d+1}{2}}\\ &\quad\times\left[\sum_j J_uT_{-k}(y_{k,i,j})\left\{r^{d+1}b^{-d\varsigma}+r^{2d+1}b^{-\alpha\rho}l\right\}\right]^{\frac12}+Ck^{l-1}e^{-ak}b^{-\rho}|\varphi|_{s,1}\\ &\leq Ck^{l-1}e^{-ak}\left\{\sum_i |\varphi|_\infty r^{\frac{d+1}{2}}[r^{3d+1}b^{-\alpha\rho}l]^{\frac12}+b^{-\rho}|\varphi|_{s,1}\right\}\\ &\leq Ck^{l-1}e^{-ak}b^{-\frac{\alpha\rho}{2}}l^{\frac12}|\varphi|_{s,1}.\end{aligned}$$

We can finally sum over $k$ and the result follows.

We are left with the postponed proofs.

*Proof of Lemma* 6.2. Note that the Jacobian of $T_k$ must be equal to one; on the other hand it must also be equal to the product of the stable and unstable Jacobian times a function $\theta$ which expresses the "angle" between the stable and unstable manifold (and hence it is Hölder). Thus, with $\{\xi_j\} = W_j \cap W^s_{2r_1}(x)$,

$$\sum_{j\in\Omega} J^u_{\xi_j}T_{-k} = \sum_{j\in\Omega}\theta(T_{-k}\xi_j)J^s_{T_{-k}\xi_j}T_k.$$

Now, consider $T_{-k}W^s_{r_1}(x)$, which clearly will intersect $W^{uc}$ at the points $T_{-k}\xi_j$, $j\in\Omega$. Obviously, the disks $D_j\subset T_{-k}W^s_{r_1}(x)$, centered at $T_{-k}\xi_j$ and with radius $r_0$ sufficiently small, but depending only on $T$, will all be disjoint. Moreover the diameter of each $T_kD_j$ must be smaller than $\lambda^{-k}r_0$. This means that $\cup_{j\in\Omega}T_kD_j$ is a collection of disjoint sets contained in the disk $W^s_{r_1+\lambda^{-k}r_0}(x)$. In addition, by the usual distortion arguments, there exists $c>0$ such that

$$J^s_{T_{-k}\xi_j}T_k \leq c\, m^s(T_kD_j).$$

Again, by distortion,

$$\sum_{j\in\Omega}\sup_{\xi\in W_j} J^u_\xi T_{-k} \leq c^2|\theta|_\infty \sum_{j\in\Omega} m^s(T_kD_j) \leq Cm^s(D_{r_1+\lambda^{-k}r_0}(x)). \qquad \square$$

[18]If we choose $\alpha < \tau^2$, then

$$\begin{aligned}\sum_{j'\in B_{k,i,j}} |\bar\phi^{-1}F_{k,i,j}F_{k,i,j,j'}|_{\mathcal{C}^\alpha} &\leq Cl\sum_{j'\in B_{k,i,j}} |J_uT_{-k}J_uT_{-k}\circ\Psi_{k,i,j,j'}|_{\mathcal{C}^\alpha}\\ &\leq Cl\sum_{j'\in B_{k,i,j}} |J_uT_{-k}J_uT_{-k}\circ\Psi_{k,i,j,j'}|_\infty \leq Clr^dJ_uT_{-k}(y_{k,i,j}).\end{aligned}$$

*Proof of Lemma* 6.3. The lemma rests on smoothness estimates for $g_{j,j'}$ which, in turn, are obtained by estimates on $\Delta$. Indeed, by Figure 2 again, it follows that for each $\xi, \eta \in W^{uc}_{k,i,j}$ (see also footnote 17),

$$
\begin{aligned}
(6.13)\qquad g_{j,j'}(\xi) - g_{j,j'}(\eta) &= \Delta(\Psi_{k,i,j,j'}(T_{-t(y_{k,i,j})}u(\eta)), T_{-t(y_{k,i,j})}u(\xi)) \\
&= \Delta(\Psi_{k,i,j,j'}(\eta), T_{t(\eta)-t(\xi)}\xi).
\end{aligned}
$$

For each $y \in W^{uc}_{k,i,j}$ define $w_{j'}(y) \in E^s(y)$ by $\exp_y(w_{j'}(y)) = \Psi_{k,i,j,j'}(y)$. Then the normalized vectors $\hat{w}_{j'}(y) := w_{j'}(y)|w_{j'}(y)|^{-1}$ are uniformly continuous functions. It follows that there exists a uniformly smooth coordinate system $\{u_1, u_2, \ldots, u_{d+1}\} := \{u_1, \bar{u})\}$ for $W^{uc}_{k,i,j}$ such that $d\alpha(\partial_{u_1}, \hat{w}_{j'}(y)) \geq c_-\|\partial_{u_1}\|$ for all $y \in W^{uc}_{k,i,j}$. Without loss of generality we can assume $t(u) = u_{d+1}$. Let $v(u) := \|\partial_{u_1}\|^{-1}\partial_{u_1}$. All that is needed in the following are bounds on the dependence of $g_{jj'}$ from the coordinate $u_1$ with the other coordinates fixed.

For each $\bar{u}$, let us consider a partition $\{[a_q, a_{q+1}]\}$ of $[-2c_d r, 2c_d r]$, such that

$$
b(a_{q+1} - a_q)\, d\alpha(w_{j'}(a_q, \bar{u}), v((a_q, \bar{u}))) = 2\pi.
$$

This implies

$$
(6.14)\qquad 2\pi c_+ b^{-1}|w_{j'}(a_q, \bar{u})|^{-1} \geq a_{a+1} - a_q \geq 2\pi c_- b^{-1}|w_{j'}(a_q, \bar{u})|^{-1}.
$$

Now, since $j' \in B_{k,i,j}$ it follows that, by the Hölder continuity of the foliation,

$$
(6.15)\qquad |w_{j'}(u_q)| \geq b^{-\varsigma} - Cb^{-\varsigma\tau}r \geq b^{-\varsigma} - Cb^{-\frac{1-2\rho}{2-\tau}} \geq \frac{1}{2}b^{-\varsigma}
$$

provided $b$ is large enough. Hence, our choices imply $|a_{q+1} - a_q| << r$ provided $b$ is large.

Accordingly, if $u_q = (a_q, \bar{u})$ and $u' = (u_1, \bar{u})$, with $u_1 \in [a_q,\, a_{q+1}]$, where $\delta_q = a_{q+1} - a_q$, by Lemma B.7 and (6.13) the following holds:

$$
(6.16)\qquad |g_{j,j'}(u') - g_{j,j'}(u_q) - (u_1 - a_q)d\alpha(w_{j'}(u_q), v(u_q))| \leq C\left(|w_{j'}(u_q)|^2\delta_q^\tau + |w_{j'}(u_q)|^\tau\delta_q^2\right).
$$

Indeed, $|w_{j'}(u_q)|^{\tau_-} \geq \delta_q$ and $\delta_q^{\tau_-} \geq |w_{j'}(u_q)|$. This follows readily from (6.14) and $\theta c_d r \geq |w_{j'}(u_q)| \geq \frac{1}{2}b^{-\varsigma}$. By (6.14), $\delta_q \leq Cb^{-\frac{1-\tau+4\rho}{2-\tau}}$; therefore (6.1), (6.15) and (6.16) yield

$$
(6.17)\qquad |g_{j,j'}(u') - g_{j,j'}(u_q) - (u_1 - a_q)d\alpha(w_{j'}(u_q), v(u_q))| \leq Cb^{-1-2\rho}.
$$

Hence,[19]

$$
\begin{aligned}
&\int_{W^{uc}_{k,i,j}} du\; m(u)e^{-ibg_{j,j'}(u)}G(u)\bar\phi(u)\\
&=\int d\bar u\sum_q\int_{a_q}^{a_{q+1}} du_1\; m(u_1,\bar u)e^{-ibg_{j,j'}(u_1,\bar u)}G(u_1,\bar u)\bar\phi(u_1,\bar u)\\
&=\int d\bar u\sum_q\Bigg\{m(u_q)G(u_q)\bar\phi(u_q)e^{-ibg_{j,j'}(u_q)}\int_{a_q}^{a_{q+1}} du_1\; e^{-ib(u_1-a_q)d\alpha(w_{jj'}(u_q),v(u_q))}\\
&\quad+O\left((|G|_\infty b^{-2\rho}+|G|_{\mathcal{C}^\alpha}\delta_q^\alpha)\int_{a_q}^{a_{q+1}}\bar\phi+|G|_\infty\delta_q r^{-1}\int_{a_q}^{b_q}\chi_{B_{c_d r}(x_i)}\right)\Bigg\}
\end{aligned}
$$

where we have used the fact that, for each $|h|\le\delta_q$,

$$d(\Psi_{k,i,j,j'}(u_q),\Psi_{k,i,j,j'}(a_q+h,\bar u))\le C\delta_q$$

thanks to the Hölder continuity of the stable foliation, our choice of the parameters and since the maximal distance between $u$ and $\Psi_{k,i,j,j'}(u)$ is bounded by a constant times $r$.[20] Continuing the above chain of inequalities yields

$$
\begin{aligned}
&=\int d\bar u\sum_q\left\{m(u_q)G(u_q)\bar\phi(u_q)e^{-ibg_{j,j'}(u_q)}\int_0^{\delta_q} du_1\; e^{-ibu_1 d\alpha(w_{jj'}(u_q),v(u_q))}\right\}\\
&\quad+|G|_{\mathcal{C}^\alpha}r^{d+1}O(b^{-\alpha\rho}+b^{-\frac{2\rho}{2-\tau}})\\
&=\int d\bar u\sum_q\left\{m(u_q)G(u_q)\bar\phi(u_q)e^{-ibg_{j,j'}(u_q)}\delta_q\int_0^1 ds\; e^{-2\pi is}\right\}\\
&\quad+|G|_{\mathcal{C}^\alpha}r^{d+1}O(b^{-\alpha\rho}).
\end{aligned}
$$

Since the inner integral equals zero exactly, the lemma is proved. □

## Appendix A. Basic facts (Anosov flows)

In this appendix we collect, for the reader's convenience, some information on the smoothness properties of the invariant foliations in Anosov flows that are used in the paper.

[19] Let $m^{uc}(du) =: m(u)du$ be the measure on the manifold; clearly $m$ is uniformly smooth.

[20] Here is a more detailed argument: consider a coordinate chart based at $u_q$ in which $W^{uc}_{k,i,j}$ and $W^s(u_q)$ are linear spaces. Then $W^s(u')$, $u'=(a_q+h,\bar u)$, can be represented as $\{(G(\xi),\xi\}_{\xi\in\mathbb{R}^d}$ and if $\Psi_{k,i,j,j'}(u')=:(a,b)$, then $G(b)=a$. With $d(t):=\|G(bt)\|$, and by the Hölder continuity of the foliation,

$$|d'(t)|\le C\|b\|d(t)^\tau.$$

The above differential inequality yields $\|a\|\le[\delta_q^{1-\tau}+C\|b\|]^{\frac{1}{1-\tau}}\le\delta_q[1+C\delta_q^{\tau-1}r]^{\frac{1}{1-\tau}}$. The result follows since $r<\delta_q^{1-\tau}$, the maximal "angle" between $W^{uc}_{k,i,j}$ and $W^{uc}_{k,i,j'}$ is bounded by $Cr^\tau$, and the metric in the chart is equivalent to the Riemannian metric.

First of all, as already mentioned, for $\mathcal{C}^2$ Anosov flows the invariant distributions (sometimes called *splittings*) are known to be uniformly Hölder continuous. Let us be more precise.

For each invertible linear map $L$ let $\theta(L) := \|L^{-1}\|^{-1}$. We define $\|d_x^s T_t\| = \|d_x T_t|_{E_x^s}\|$, $\|d_x^u T_t\| = \|d_x T_t|_{E_x^u}\|$, $\theta(d_x^s T_t) = \theta(d_x T_t|_{E_x^s})$ and $\theta(d_x^u T_t) = \theta(d_u T_t|_{E_x^u})$. Then the following holds ([10], [37]):

(A.1)

- If there exists $\tau_d$ such that, for each $x \in \mathcal{M}$, and some $t \in \mathbb{R}^+$, $\|d_x^s T_t\| \, \|d_x^u T_t\|^{\tau_d} < \theta(d_x^u T_t)$, then $E^{sc} \in \mathcal{C}^{\tau_d}$.

- If there exists $\tau_d > 0$ such that, for each $x \in \mathcal{M}$, and some $t \in \mathbb{R}^+$, $\theta(d_x^s T_t)^{\tau_d} \, \theta(d_x^u T_t) > \|d_x^s T_t\|^{-1}$, then $E^{uc} \in \mathcal{C}^{\tau_d}$.

Moreover the Hölder continuity is uniform (that is the $\tau_d$-Hölder norm of the distributions is bounded). The above conditions are often called *$\tau_d$-pinching* or *bunching* conditions.

The next relevant fact is that the above splittings are integrable. The integral manifolds are the stable and unstable manifolds, respectively. Clearly, this implies the existence of the weak stable and weak unstable manifolds as well. They form invariant continuous foliations. Each leaf of such foliations is as smooth as the map and it is tangent, at each point, to the corresponding distribution, [17]. In addition, for $\mathcal{C}^r$ maps, the $\mathcal{C}^r$ derivatives of such manifolds (viewed as graphs over the corresponding distributions) are uniformly bounded, [14]. Finally, the foliations are uniformly transversal and $\mathcal{C}^{\tau_d}$.

In the case in which both distribution are $\mathcal{C}^{1+\alpha}$, it follows by Frobenius' theorem that the holonomy maps are $\mathcal{C}^{1+\alpha}$ (§6 of [29]). If the splitting is only Hölder the situation is more subtle.

We will call *stable holonomy* any holonomy constructed via the strong stable foliations and *unstable holonomy* those constructed by the strong unstable foliation. The basic result on holonomies is given by the following; see [29].

(A.2)

- If there exists $\tau_h > 0$ such that for some $t \in \mathbb{R}^+$ and for each $x \in \mathcal{M}$ $\|d_x^s T_t\| \, \|d_x^u T_t\|^{\tau_h} < 1$, then the stable holonomies are uniformly $\mathcal{C}^{\tau_h}$.

- If there exists $\tau_h > 0$ such that for some $t \in \mathbb{R}^+$ and for each $x \in \mathcal{M}$ $\theta(d_x^s T_t)^{\tau_h} \, \theta(d_x^u T_t) > 1$, then the unstable holonomies are uniformly $\mathcal{C}^{\tau_h}$.

The relation between smoothness of holonomies and smoothness of the foliation (in the sense that the local foliation charts are smooth) is discussed in detail in [29, §6]. Here we restrict ourselves to what is needed in this paper.

This is not yet enough for our purposes: we need to talk about the smoothness of the Jacobian of the holonomies between two manifolds $W^{uc}(x)$ and $W^{uc}(y)$.[21]

(A.3)
- The stable and unstable holonomies are absolutely continuous.
- There exists $\tau_j > 0$ such that $|1 - J\Psi|_\infty \leq Cd(x,y)^{\tau_j}$.
- There exists $\tau_j > 0$ such that for each $x \in \mathcal{M}$ the Jacobians of the stable holonomies are uniformly $\mathcal{C}^{\tau_j}$.
- There exists $\tau_j > 0$ such that for each $x \in \mathcal{M}$ the Jacobian of the unstable holonomies are uniformly $\mathcal{C}^{\tau_j}$.

The last, but not least important, object for which we need smoothness information is the so-called *temporal distance*.

Fix any point $x \in \mathcal{M}$ and a small neighborhood $B_\delta(x)$. Consider a smooth $2d$ dimensional manifold $\mathcal{W}$ containing $W^u(x)$ and $W^s(x)$; clearly the flow is transverse to such a manifold. On $\mathcal{W}$ choose a smooth coordinate system $(u,s)$ such that $\{(u,0)\} = W^u(x)$, and $\{(0,s)\} = W^s(x)$. Although it is not necessary, for further convenience we can assume that the coordinate system, restricted to the stable and unstable manifolds, is the one given by the exponential map (corresponding to the metric restricted to such manifolds). Define then a coordinate system $(u,t,s)$ in $B_\delta(x)$ as follows: $T_{-t}(\xi) \in \mathcal{W}$ and $(u,s)$ are the coordinates of $T_{-t}(\xi)$; clearly such coordinates locally trivialize the flow. Let $y \in B_\delta(x) \cap W^s(x)$ and $y' \in B_\delta(x) \cap W^u(x)$. Moreover let $z' = W^u(y) \cap W^{sc}(y')$ and $z = W^s(y') \cap W^{uc}(y)$. By construction $z$ and $z'$ are on the same flow orbit. Thus there exists $\Delta(y,y')$ such that $T_{\Delta(y,y')}z = z'$. The function $\Delta(y,y')$ is called *temporal distance*, see Figure 2 for a pictorial description.

In general the only thing that can be said is that the temporal distance is as smooth as the strong stable and unstable foliation (see (A.2)), but we will see in Appendix B that, if some geometric structure is present, more can be said.

## Appendix B. Basic facts (contact flows)

Given an odd dimensional (say $2d+1$) connected compact manifold $\mathcal{M}$, a contact form is a $\mathcal{C}^1$ differential 1-form such that the $(2d+1)$-form $\alpha \wedge (d\alpha)^d$ is nonzero at every point.

[21] These are a direct consequence of the formula [22]

$$J\Psi(x) = \prod_{n=0}^{\infty} \frac{J^u T_{-1}(T_n \Psi(x))}{J^u T_{-1}(T_n x)}.$$

Given a flow $T_t$ on $\mathcal{M}$ we call it *contact flow* if its associated vector field $V$ ($V(x) := \frac{dT_t x}{dt}\big|_{t=0}$) is such that $d\alpha(V, v) = 0$ for all vector fields $v$ and $\alpha(V) = 1$, for some contact form $\alpha$.

Clearly the contact flow preserves the contact form and hence also the contact volume.

Let us start with some trivial facts showing that, for contact flows, a bit more can be said about the quantities introduced in the previous appendix.

LEMMA B.1. *For a contact flow there exists a constant $C > 0$ such that, for each $x \in \mathcal{M}$,*

$$C_0^{-1} \leq \|d_x^s T\| \theta(d_x^u T) \leq C_0 \; ; \quad C_0^{-1} \leq \|d_x^u T\| \theta(d_x^s T) \leq C_0.$$

*Proof.* When $v \in E^u(x)$, $|v| = 1$, clearly there must exist $w \in E^s(x)$, $|w| = 1$, such that $|d\alpha(v, w)| \geq c_- |v|\,|w|$. Accordingly,

$$c_-|v|\,|w| \leq |d\alpha(d_xT_tv, d_xT_tw)| \leq c_+|d_xT_tv|\,|d_xT_tw| \leq c_+|d_xT_tv|\,\|d_x^sT_t\|.$$

Taking the infimum on $v$, we have

$$\theta(d_x^uT_t)\,\|d_x^sT_t\| \geq c_-c_+^{-1}. \tag{B.1}$$

On the other hand, given $w \in E^s(x)$, $|w| = 1$, there must be $v \in E^u(x)$, $|v| = 1$, such that $|d\alpha(d_xT_tw, d_xT_tv)| \geq c_-|d_xT_tw|\,|d_xT_tv|$. Hence,

$$c_+ \geq c_-|d_xT_tw|\,|d_xT_tv| \geq c_-|d_xT_tw|\,\theta(d_x^uT_t).$$

Taking the supremum over $w$, we have

$$\|d_x^sT_t\|\,\theta(d_x^uT_t) \leq c_-^{-1}c_+. \tag{B.2}$$

The first inequality of the lemma is then obtained when we put together (B.1) and (B.2). The second inequality follows similarly. □

Another trivial, but helpful, property of contact flows is the following.

LEMMA B.2. *The contact form $\alpha$ restricted to a stable or unstable manifold must be identically zero. In addition, the form $d\alpha$ is identically zero when restricted to a weak stable or weak unstable manifold.*

*Proof.* The first statement is a consequence of the invariance of $\alpha$; for example if $v$ is a stable vector then $\alpha(v) = \lim_{t\to+\infty} \alpha(dT_tv) = 0$. The second statement is proved again by invariance. Let $v, w$ be weak stable vectors and write them as $v = v' + aV$ and $w = w' + bV$ where $v'$ and $w'$ are stable vectors. Then $d\alpha(v, w) = \lim_{t\to+\infty} d\alpha(dT_tv, dT_tw) = ab\, d\alpha(V, V) = 0$. □

COROLLARY B.3. *The distributions are smoother than indicated in Appendix* A: $E^u, E^s \in \mathcal{C}^{\tau_d}$.

*Proof.* Since $E^{uc} \in \mathcal{C}^{\tau_d}$ and $E^u = \{v \in E^{uc} \mid \alpha(v) = 0\}$ the result follows trivially. $\square$

*Remark* B.4. A bit more work should show that A.2 and A.3 hold with $\tau_d$ instead of $\tau_h$ and $\tau_j$. This is not important for the task at hand and we will ignore it. Throughout the paper $\tau$ will designate the best constant (less or equal one) for which the properties in A.1, A.2 and A.3 hold.

The first really interesting fact concerning contact flow is given by the following result proved in [16, Th. 3.6].

THEOREM B.5 (Katok-Burns). *Let $\mathcal{M}$ be a contact manifold as above. Let $E$ be an ergodic component of the contact flow $T$ which has positive measure and nonzero Lyapunov exponents except in the flow direction. Then the flow on $E$ is Bernoulli.*

Accordingly, by the usual Hopf argument [15], [21], the theorem is proved.

COROLLARY B.6. *Let $\mathcal{M}$ be a connected, compact, contact manifold as above and let $T_t$ be an Anosov contact flow. Then the flow is Bernoulli (and hence mixing).*

The proof of Theorem B.5 is based, among other things, on a lemma concerning the temporal function (see the definition at the end of the previous appendix) which, at least for us, has an interest in itself. Since we need it in a slightly different, stronger and more explicit form we will state and prove it here again.

LEMMA B.7. *Assume $\alpha \in \mathcal{C}^2$ and conditions* (A.1), (A.2) *for some $t > 0$. Let $\bar{v} \in E^u(x)$, $\bar{w} \in E^s(x)$ be such that $\exp_x(\bar{v}) = y'$ and $\exp_x(\bar{w}) = y$.*[22] *Then*

$$\Delta(y, y') = d\alpha(\bar{v}, \bar{w}) + O(\|\bar{v}\|^{\tau^2}\|\bar{w}\|^2 + \|\bar{w}\|^{\tau^2}\|\bar{v}\|^2).$$

*In addition,*

$$\Delta(y, y') = d\alpha(\bar{v}, \bar{w}) + O(\|\bar{v}\|^{\tau}\|\bar{w}\|^2 + \|\bar{w}\|^{\tau}\|\bar{v}\|^2),$$

*provided* $\|\bar{v}\|^{\frac{1}{\tau_-}} \leq \|\bar{w}\| \leq \|\bar{v}\|^{\tau_-}$, $\tau_- := \min\{\tau, (1-\tau)\}$.[23]

[22] The exponential function is with respect to the restriction of the metric to $W^u(x)$ and $W^s(x)$, respectively.

[23] The latter limitation–although compatible with our needs– is certainly excessive and, possibly, completely redundant. Yet, as will be clear from the proof, to remove it effectively it would be necessary to have some information on the Hölder continuity of the foliation in $\mathcal{C}^r$ topology, which seems not to be readily available in the literature. But it does hold true–at least to some extent; see footnotes 24 and 28.

*Proof.* Consider the coordinate system introduced at the end of Appendix A to define the temporal distance. Notice that the Euclidean metric in such coordinates gives the right measure for the temporal distance and the distance from $x$ of points in $W^u(x)$ or $W^s(x)$; at the same time it is uniformly equivalent to the Riemannian metric. We can then use it without any further comment.

Let $y = (0, 0, w)$ and $y' = (v, 0, 0)$. In coordinates the manifold $W^u(x)$ has the form $\{(u, 0, 0)\}$, the manifold $W^s(x)$ $\{(0, 0, s)\}$ and the manifolds $W^{uc}(y)$, $W^{sc}(y')$ have the form $\{(u, t, F(u))\}$, $\{(G(s), t, s\}$, respectively. In addition, on the one hand the smoothness of the holonomies implies $\|F\|_\infty \le C\|w\|^\tau$ and $\|G\|_\infty \le C\|v\|^\tau$. On the other hand the smoothness of the distributions implies $\|D_uF\| \le C\|F(u)\|^\tau$ and $\|D_sG\| \le C\|G(s)\|^\tau$. Finally, the uniform smoothness of the manifolds implies $\|F(u) - w - D_0Fu\| \le C\|u\|^2$, $\|G(s) - v - D_0Gs\| \le C\|s\|^2$.[24]

Our aim is to introduce a two-dimensional manifold that captures the essential geometric features related to $\Delta$. To do so we introduce two smooth foliations: $\mathcal{W}_u := \{W_u(b) \mid b \in [0, 1]\}$, $W_u(b) := \{(u, 0, bF(u))\}$, and $\mathcal{W}_s := \{W_s(a) \mid a \in [0, 1]\}$, $W_s(a) := \{(aG(s), 0, s)\}$.[25] Notice that the above two foliations are transversal, hence for all $(a, b) \in \Sigma_0 := [0, 1]^2$ the point $\{\Xi(a, b)\} := W_u(b) \cap W_s(a)$ is uniquely defined. In fact, if we define the function $\Phi : \mathbb{R}^{2d+2} \to \mathbb{R}^{2d}$ by $\Phi(u, s, a, b) := (u - aG(s), s - bF(u))$, then $\Phi(\Xi(a, b), a, b) \equiv 0$. Since

$$\frac{\partial \Phi}{\partial(u, s)} = \begin{pmatrix} \mathbf{Id} & -aDG \\ -bDF & \mathbf{Id} \end{pmatrix} =: \mathbf{Id} - \Lambda, \tag{B.3}$$

$\|\Lambda\| < 1$, provided the coordinate neighborhood has been chosen small enough, it follows that we can apply the implicit function theorem. Accordingly $\Xi$ is a uniformly $\mathcal{C}^4$ chart for the surface $\Sigma := \Xi(\Sigma_0)$. Such a surface is bounded by the curves $\gamma_1 := \{\Xi(a, 0)\} = \{(av, 0, 0)\}$, $\gamma_2 := \{\Xi(1, b)\}$ that belong to $W^{sc}(y')$, $\gamma_3 := \{\Xi(a, 1)\}$ that belongs to $W^{uc}(y)$ and $\gamma_4 := \{\Xi(0, b)\} = \{(0, 0, bw)\}$. Moreover, when $\hat{z} := \Xi(1, 1)$, clearly $\hat{z}$ lies on the same flow orbit of $z$ and $z'$. At last, consider the curves $\gamma \subset W^u(y)$ and $\gamma' \subset W^s(y')$ obtained by transporting $\gamma_3$ and $\gamma_2$ respectively along the flow direction.[26] Clearly $\gamma$, $\gamma_3$ and the flow line between $\hat{z}$ and $z'$ bound a two-dimensional manifold (contained in $\bigcup_{t\in\mathbb{R}} T_t\gamma_3$);

[24]Actually, here we use a very rough bound on the second derivative, but one can certainly do better. For example, since $F(u) = F(0) + D_0F(0)u + \frac{1}{2}D_0^2F(u, u) + O(\|u\|^3) \le C\|w\|^\tau$, it must be, at least, $|D_0^2F| \le C\|w\|^{\frac{\tau}{3}}$.

[25]These are just linear interpolations between the manifolds at $x$ and the manifolds at $y$ and $y'$, respectively.

[26]The smoothness of $W^u(y)$ and $\gamma_3$ imply trivially the smoothness of $\gamma$. The same considerations apply to $\gamma'$.

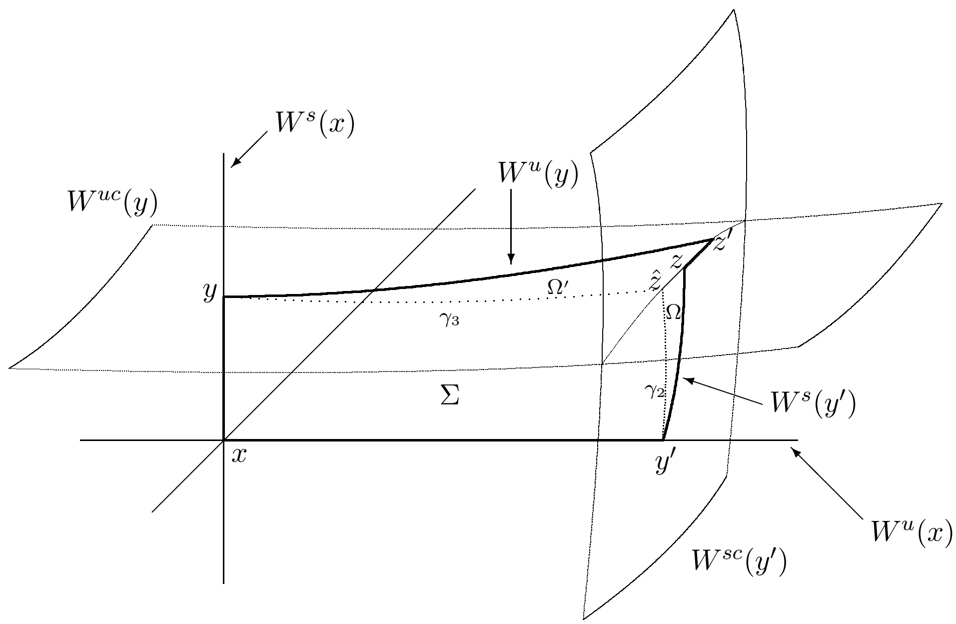


Figure 2: Definition of the *temporal function* $\Delta(y, y')$ and related quantities

let us call it $\Omega' \subset W^{uc}(y)$; analogously we define $\Omega$. See Figure 2 for a visual description.[27]

We can now compute the required quantity. Consider the closed curve $\Gamma$ following $\gamma_1$, $\gamma'$ then going from $z$ to $z'$ along the flow direction and finally coming back to $x$ via $\gamma$ and $\gamma_4$ (the bold path in Figure 2). Then

$$\int_\Gamma \alpha = \Delta(y, y'). \tag{B.4}$$

This is because $\alpha$ is identically zero when restricted to a stable or unstable manifold (see Lemma B.2). On the other hand

$$\int_\Gamma \alpha = \int_{\partial\Sigma} \alpha + \int_{\partial\Omega} \alpha + \int_{\partial\Omega'} \alpha = \int_\Sigma d\alpha \tag{B.5}$$

where we have used Stokes theorem and the fact that $d\alpha$ is identically zero when restricted to a weak stable or unstable manifold (see Lemma B.2). To continue, it is better to change coordinates.

$$\begin{aligned} \int_\Sigma d\alpha &= \int_{\Sigma_0} \Xi^* d\alpha = \int_{\Sigma_0} d_{\Xi(a,b)}\alpha(D\Xi e_1, D\Xi e_2) da db \\ &= \int_{\Sigma_0} d_x\alpha(D\Xi e_1, D\Xi e_2) da db + O(\|\Xi\|_\infty \|D\Xi e_1\|_\infty \|D\Xi e_2\|_\infty), \end{aligned} \tag{B.6}$$

[27]Of course the picture is a bit misleading due to a lack of dimensions. For example, the picture does not differentiate between the $d$-dimensional manifold $W^u(y)$ and the curve $\gamma$.

where we have used the fact that $\alpha$ is $\mathcal{C}^2$. By the implicit function theorem,

$$D\Xi = -(\mathbf{Id} - \Lambda)^{-1}\frac{\partial\Phi}{\partial(a,b)} = \sum_{k=0}^{\infty}\Lambda^k\begin{pmatrix} G & 0\\ 0 & F\end{pmatrix}. \tag{B.7}$$

Since all the following arguments are restricted to the hypersurface $t \equiv 0$, from now on we will forget the $t$ coordinate. Accordingly,

(B.8)

$$\begin{aligned}
D\Xi e_1 &= (\mathbf{Id} - \Lambda)^{-1}(G,0) = (G,0) + \sum_{k=0}^{\infty}\Lambda^{2k}\{\Lambda(G,0) + \Lambda^2(G,0)\}\\
&= (G,0) + b(\mathbf{Id} - \Lambda^2)^{-1}(aDG\,DF\,G, DF\,G)\\
&= (G,0) + b(a(\mathbf{Id} - abDG\,DF)^{-1}DG\,DF\,G, (\mathbf{Id} - abDF\,DG)^{-1}DF\,G)\\
&=: (v,0) + (\Delta_u v, \Delta_s v) =: \bar v + \Delta v\\
D\Xi e_2 &= (0,F) + a((\mathbf{Id} - abDG\,DF)^{-1}DG\,F, b(\mathbf{Id} - abDF\,DG)^{-1}DF\,DG\,F)\\
&=: (0,w) + (\Delta_u w, \Delta_s w) =: \bar w + \Delta w.
\end{aligned}$$

Since $d_x\alpha$ is identically zero on the weak stable and weak unstable manifold of $x$, we have

(B.9)

$$\begin{aligned}
d_x\alpha(D\Xi e_1, D\Xi e_2) &= d_x\alpha(\bar v, \bar w) + d_x\alpha(\bar v, \Delta w) + d_x\alpha(\Delta v, \bar w) + d_x\alpha(\Delta v, \Delta w)\\
&= d_x\alpha(\bar v, \bar w) + O((\|v\| + \|\Delta_u v\|)\|\Delta_s w\|\\
&\quad + \|w\|\|\Delta_u v\| + \|\Delta_s v\|\|\Delta_u w\|).
\end{aligned}$$

The last needed estimate concerns the variation of the functions $F, G$.

$$\begin{aligned}
\Delta G(a,b) &:= G(\Xi_s(a,b)) - v = G(\Xi_s(a,b)) - G(\Xi_s(a,0))\\
&= \int_0^b DG D\Xi_s e_2 = \int_0^b DGw + DG\Delta_s w;\\
\Delta F(a,b) &:= F(\Xi_u(a,b)) - w = F(\Xi_u(a,b)) - F(\Xi_u(0,b))\\
&= \int_0^a DF D\Xi_u e_1 = \int_0^a DFv + DF\Delta_u v.
\end{aligned}$$

Remembering (B.8) we can estimate

(B.10)

$$\begin{aligned}
\|\Delta_u v\| &\le \|\Delta G\| + Cab\|DG\,DF(v + \Delta G)\|_\infty \le C\|\Delta G\|_\infty + Cab\|DG\,DFv\|_\infty,\\
\|\Delta_s v\| &\le Cb\|DF\,G\|_\infty \le Cb\|DFv\|_\infty + Cb\|DF\Delta G\|_\infty,\\
\|\Delta_u w\| &\le Ca\|DGw\|_\infty + Ca\|DG\Delta F\|_\infty,\\
\|\Delta_s w\| &\le C\|\Delta F\|_\infty + Cab\|DF\,DGw\|_\infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Delta G\|_\infty &\leq \|DG\|_\infty \|w\| + C\|DG\|_\infty \|\Delta F\|_\infty + C\|DG\|_\infty \|DF\, DGw\|_\infty \\ &\leq C\|DG\|_\infty \|w\| + C\|DG\|_\infty \|\Delta F\|_\infty, \\ \|\Delta F\|_\infty &\leq C\|DF\|_\infty \|v\| + C\|DF\|_\infty \|\Delta G\|_\infty.\end{aligned}$$

Substituting the first in the second yields

$$\|\Delta F\|_\infty \leq C\|DF\|_\infty \|v\| + C\|DF\|_\infty \|DG\|_\infty \|w\| + C\|DF\|_\infty \|DG\|_\infty \|\Delta F\|_\infty,$$

that is

$$\begin{aligned}\|\Delta F\|_\infty &\leq C\|DF\|_\infty \|v\| + C\|DF\|_\infty \|DG\|_\infty \|w\|, \\ \|\Delta G\|_\infty &\leq C\|DG\|_\infty \|w\| + C\|DG\|_\infty \|DF\|_\infty \|v\|.\end{aligned} \tag{B.11}$$

Using estimates (B.11) and (B.10) in (B.9) yields

$$d_x\alpha(D\Xi e_1, D\Xi e_2) = d_x\alpha(\bar{v}, \bar{w}) + O(\|DF\|_\infty \|v\|^2 + \|DG\|_\infty \|w\|^2).$$

Remembering that, by definition, $\Xi_u = aG \circ \Xi_s$ and $\Xi_s = bF \circ \Xi_u$ we can use the above estimate in (B.6), (B.4) and finally obtain

$$\Delta(y, y') = d_x\alpha(\bar{v}, \bar{w}) + O(\|v\|^2\|w\| + \|w\|^2\|v\| + \|DF\|_\infty \|v\|^2 + \|DG\|_\infty \|w\|^2). \tag{B.12}$$

Since $\|DF\|_\infty \leq C\|F\|_\infty^\tau \leq C\|w\|^{\tau^2}$ and $\|DG\|_\infty \leq C\|v\|^{\tau^2}$ the first inequality of the lemma is proved. To prove the second let us assume $\|w\| \geq \|v\|$, the other situation being symmetric with respect to the exchange of the stable and unstable directions (which corresponds to a time reversal). Remember that $DF \in \mathcal{C}^1$; hence $\|DF\|_\infty \leq C\|w\|^\tau + C\|\Xi_u\|_\infty$; thus

$$\begin{aligned}\|DF\|_\infty &\leq C\|w\|^\tau + C\|v\| + C\|DG\|_\infty \|w\|, \\ \|DG\|_\infty &\leq C\|v\|^\tau + C\|w\| + C\|DF\|_\infty \|v\|\end{aligned}$$

which yields $\|DF\|_\infty \leq C\|w\|^\tau + C\|v\|$; $\|DG\|_\infty \leq C\|v\|^\tau + C\|w\|$.[28] This proves the lemma provided $\|v\|^\tau \geq \|w\|$. Clearly this condition is less stringent as $\tau$ decreases, while such a situation should be the worst case. Obviously the previous estimates must have been inefficient for "large" $\tau$. Indeed, it is possible to do a different estimate for $\Delta F$, $\Delta G$. Suppose $\|v\|^\tau \leq \|w\|$.

$$\frac{d\|F \circ \Xi_u\|}{da} = \frac{\langle DF D\Xi_u e_1, F\rangle}{\|F\|} \leq \|DF\|\|D\Xi_u e_1\| \leq C\|F\|^\tau \|G\|.$$

Integrating the above differential inequality (and the analogous one for $G$) yields

$$\begin{aligned}\left[\|w\|^{1-\tau} - C\|G\|_\infty\right]^{\frac{1}{1-\tau}} &\leq \|F\|_\infty \leq \left[\|w\|^{1-\tau} + C\|G\|_\infty\right]^{\frac{1}{1-\tau}}, \\ \left[\|v\|^{1-\tau} - C\|F\|_\infty\right]^{\frac{1}{1-\tau}} &\leq \|G\|_\infty \leq \left[\|v\|^{1-\tau} + C\|F\|_\infty\right]^{\frac{1}{1-\tau}}.\end{aligned}$$

[28]Here again a better knowledge of the size of the second derivative would improve the result; see footnote 24.

Clearly the above equations imply $\|\Delta F\| \leq \|w\|^\tau \|v\|$ and $\|\Delta G\| \leq \|v\|^\tau \|w\|$, provided $\|v\|^{1-\tau} \geq \|w\|$. This implies again $\|DF\|_\infty \leq C(\|w\|^\tau + \|v\|)$ and $\|DG\|_\infty \leq C(\|v\|^\tau + \|w\|)$. In addition, $\|F\|_\infty \leq C\|w\|$, $\|G\|_\infty \leq C\|v\|$ and $\|D\Xi e_1\|_\infty \leq C\|G\|_\infty \leq C\|v\|$, $\|D\Xi e_2\|_\infty \leq C\|w\|$. Using such estimates in (B.10), (B.9) and (B.6) yields

$$\Delta(y, y') = d_x\alpha(\bar{v}, \bar{w}) + O(\|v\|^2\|w\|^\tau + \|w\|^2\|v\|^\tau). \qquad \square$$

*Remark* B.8. It may be possible to optimize Lemma B.7 by pushing forward (or backward) the picture until $d(T_k x, T_k y) = d(T_k x, T_k y')$; of course one would need to be rather careful by properly estimating distortion. At any rate, the best result one can hope for is that if $\tau > \sqrt{3} - 1$, then $\Delta(y, y') = d\alpha(v, w) + o(|v|)$. That is, $\Delta$ is differentiable with respect to $y'$ and the derivative is $\mathcal{C}^\tau$. We do not push matters in such a direction since it is not necessary for the purpose at hand.

## Appendix C. Averages

We start with a long overdue proof.

*Proof of Sub-Lemma* 3.1. Clearly

$$|\mathbb{A}^s_\delta \varphi|_\infty \leq |\varphi|_\infty; \quad |\mathbb{A}^s_\delta \varphi - \varphi|_\infty \leq \delta^\beta H_{s,\beta}(\varphi). \tag{C.1}$$

The estimate of the smoothness of $\mathbb{A}^s_\delta \varphi$ is a bit more subtle; to investigate it, it is convenient to introduce an appropriate coordinate system.

Since all the quantities are related to the same stable manifold, from now on we will consider the Riemannian metric restricted to the stable manifold.

Given $x, y$ belonging to the same stable manifold, we first identify the tangent spaces at $x$ and $y$ by parallel transport; then we consider normal coordinates at $x$ and at $y$. Clearly in such coordinates the balls $W^s_\delta(x)$ and $W^s_\delta(y)$ are actual balls of radius $\delta$; of course this is not the case for $W^s_\delta(y)$ in the normal coordinates at $x$. We call $I_{xy} : \mathcal{T}_x\mathcal{M} \to \mathcal{T}_y\mathcal{M}$ the isometry that identifies the tangent spaces and we define the map $\Upsilon_{xy} : \mathcal{M} \to \mathcal{M}$ as

$$\Upsilon_{xy}(z) = \exp_y(I_{xy}\exp_x^{-1}(z)).$$

where exp is the exponential map defined by the metric on the stable manifold.

First of all notice that, by construction

$$\Upsilon_{xy}(W^s_\delta(x)) = W^s_\delta(y). \tag{C.2}$$

Next, to study $\Upsilon_{xy}$ we describe it in the normal coordinates of the point $x$. We will then identify all the tangent spaces by the Cartesian structure of such a chart. When, as usual, the $\Gamma^k_{ij}$ are called the Christoffel symbols, the equation of parallel transport for a vector $v$ along the curve $\gamma$ reads

$$\frac{dv^k}{dt} = -\sum_{ij} \Gamma_{ij}^k v^i \frac{d\gamma^j}{dt}.$$

Moreover, in the normal coordinates of the point $x$,[29]

$$|\Gamma_{ij}^k(\xi)| \le C|\xi|.$$

Assuming $d(x,y) \le \delta$, we are interested only in a region contained in the ball $W_{2\delta}^s(x)$; thus $|\Gamma_{ij}^k|_\infty \le C\delta$. Hence, by a standard use of the Gronwald inequality,

$$|I_{x,y}v - v| \le C_1 d(x,y)^2 |v|. \tag{C.3}$$

Arguing in the same manner on the equations defining the geodesics, and taking into account (C.3), we see that

$$d(\Upsilon_{xy}(z), z) \le (1 + C_2\delta) d(x,y). \tag{C.4}$$

This implies that the symmetric difference $W_\delta^s(x) \Delta W_\delta^s(y)$ is contained in the spherical shell $W_{\delta+C_2 d(x,y)}^s(x) \backslash W_{\delta - C_2 d(x,y)}^s(x)$ whose measure is proportional to $\delta^{d-1} d(x,y)$.

To see that the Jacobian is close to one, a bit more work is needed. Namely, we must linearize the geodesic equations along the geodesic. This is a standard procedure and it is best done via the Jacobi fields [6]. By using Gronwald again, and the fact that the manifolds are uniformly $\mathcal{C}^4$, we see that

$$|J\Upsilon_{xy} - 1|_\infty \le C_3 d(x,y). \tag{C.5}$$

From this it follows immediately

$$H_{s,1}(\mathbb{A}_\delta^s \varphi) \le C\delta^{-1} |\varphi|_\infty. \tag{C.6}$$

We can then conclude by using (C.2), (C.4) and (C.5),

$$\begin{aligned} &\left| \int_{W_\delta^s(x)} \varphi - \int_{W_\delta^s(y)} \varphi \right| \\ &\quad \le \int_{B_\delta(0)} |\varphi(\xi)\rho(x,\xi) - \varphi \circ \Upsilon_{xy}(\xi) \rho(y, \Upsilon_{xy}(\xi)) J\Upsilon_{xy}(\xi)| \, d\xi \\ &\quad \le \left[ (1 + c_2\delta)^\beta H_{s,\beta}(\varphi) d(x,y)^\beta + C_4 |\varphi|_\infty d(x,y) \right] m^s(W_\delta^s(x)). \end{aligned} \tag{C.7}$$

□

Next we need an estimate of how much two nearby manifolds can drift apart.

[29] Clearly the smoothness of the metric will depend on the smoothness of the tangent planes (that in our case are uniformly $\mathcal{C}^3$); see [17]. Accordingly $\Gamma$ will be uniformly $\mathcal{C}^2$.

LEMMA C.1. *There exists a constant $C > 0$ such that for each $x \in \mathcal{M}$ and $y \in W^s_\delta(x)$,*[30]

$$\operatorname{dist}(W^u_\delta(x), W^u_\delta(y)) \le C d(x, y)^\tau.$$

*Proof.* Clearly $d(W^u_\delta(x), W^u_\delta(y))$ is bounded by the distance computed along the stable manifold. For each $\xi \in W^u_\delta(x)$ consider the unstable holonomy between $W^{sc}(x)$ and $W^{sc}(\xi)$. Let $\{\eta\} := W^{sc}(\xi) \cap W^u_\delta(y)$. By A.3 it follows that $d_s(\xi, \eta) \le C d_s(x, y)^\tau$ and from this the lemma is proved. □

The other needed results concerning averages are all based on a sort of change of order of integration formula. Although such a result may already exist in some form in the literature (after all it is a sort of Fubini with respect to a foliation with Hölder smoothness), I find it more convenient to derive it in the following.

To proceed it is helpful to choose special coordinates in which the unstable, or the stable manifolds, are straight. Let us do the construction for the unstable manifold, the one for the stable being similar.

First notice that such a straightening can be only local, we can then choose an appropriate covering $\{U_i\}$ of $\mathcal{M}$ (appropriate means that the open sets must be sufficiently small) and perform the wanted construction in each open set $U_i$.

Let $U$ be a sufficiently small open ball. Let us choose a coordinate system in $U$, since the Euclidean norm in the coordinate is equivalent to the Riemannian length we will use it instead and we will, from now on, confuse $U$ with its coordinate representation.

It is particularly convenient to choose the chart in such a way that, given a preferred point $\bar{x} \in U$, $\{(u, 0)\}_{u \in \mathbb{R}^{d_u}} = W^u(\bar{x})$ and $\{(0, s)\}_{s \in \mathbb{R}^{d_s+1}} = W^{sc}(\bar{x})$.

At this point we can define the function $H : \mathbb{R}^{d_u+d_s+1} \to \mathbb{R}^{d_s+1}$ by the requirement

$$\{(u, H(u, s))\}_{u \in \mathbb{R}^{d_u}} = W^u((0, s)).$$

Clearly this implies $H(0, s) = s; \quad H(u, 0) = 0$. We define then the change of coordinates

$$\Psi(\bar{u}, \bar{s}) = (\bar{u}, H(\bar{u}, \bar{s})).$$

In the coordinates $(\bar{u}, \bar{s})$ the unstable manifolds are just all the vector spaces of the type $\{(\bar{u}, a)\}$ for some $a \in \mathbb{R}^{d_s+1}$.

In addition, a trivial computation shows that, calling $J\Psi$ the Jacobian of the change of coordinates $\Psi$, we have that $J\Psi(\bar{u}, \bar{s})$ is nothing else than the Jacobian of the unstable holonomy between $\{(0, \xi)\}_{\xi \in \mathbb{R}^{d_s+1}}$ and $\{(\bar{u}, \xi)\}_{\xi \in \mathbb{R}^{d_s+1}}$.

[30] Here by "dist" we mean the Hausdorff distance.

LEMMA C.2. *There exists $\bar{c} > 0$ such that the kernel $\tilde{Z}_\varepsilon$, defined in* (4.5), *satisfies*

$$|\tilde{Z}_\varepsilon|_{\mathcal{C}^\tau} \leq \bar{c}|Z_\varepsilon|_\infty;$$

*moreover $\tilde{Z}(x,\xi)$ is Lipschitz with respect to the second variable, limited to the flow direction, with Lipschitz constant $\bar{c}|Z_\varepsilon|_\infty$.*

*Proof.* Since all the relevant quantities are local, we can compute in a chart $\Psi$ as above.

Let $U$ be an open set in the chart and consider $f : \mathcal{M}^2 \to \mathbb{C}$ supported in $U^2$. Then

$$\int_{\mathcal{M}} m(dx) \int_{W^u_\delta(x)} f(x,\xi) m^u(d\xi) = \int_{\{(x,\xi)\in U^2 \mid d^u(x,\xi)\leq\delta\}} f(x,\xi) m^u(d\xi) m(dx).$$

Now we set $\Xi_\delta := \{(x,\xi) \in U^2 \mid d^u(x,\xi) \leq \delta\}$ and we change variables: $x = \Psi(u,s)$ and $\xi = \Psi(u',s)$.

$$\int_{\mathcal{M}} m(dx) \int_{W^u_\delta(x)} f(x,\xi) m^u(d\xi) = \int_{\Xi_\delta} f(\Psi(u,s),\Psi(u',s))\rho(u_1,s) J\Psi(u,s) du'\, du\, ds,$$

where $\rho \circ \Psi^{-1}$ is a uniformly $\tau$-Hölder function. Accordingly,

$$\tilde{Z}_\varepsilon(x,\xi) = \frac{J\Psi(x)\rho(\Psi^{-1}(\xi))}{J\Psi(\xi)\rho(\Psi^{-1}(x))} Z_\varepsilon(x).$$

The smoothness of $\tilde{Z}_\varepsilon(x,\xi)$ follows then from previous results on holonomy smoothness and the smoothness of $Z_\varepsilon$. In turn, the latter is proved exactly as in equation (C.7) where we exchanged the rôle of the stable and unstable manifolds and set $\varphi = 1$. □

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