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# Index theorems for holomorphic self-maps

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## Introduction

The usual index theorems for holomorphic self-maps, like for instance the classical holomorphic Lefschetz theorem (see, e.g., [GH]), assume that the fixed-points set contains only isolated points. The aim of this paper, on the contrary, is to prove index theorems for holomorphic self-maps having a positive dimensional fixed-points set.

The origin of our interest in this problem lies in holomorphic dynamics. A main tool for the complete generalization to two complex variables of the classical Leau-Fatou flower theorem for maps tangent to the identity achieved in [A2] was an index theorem for holomorphic self-maps of a complex surface fixing pointwise a smooth complex curve $S$. This theorem (later generalized in [BT] to the case of a singular $S$) presented uncanny similarities with the Camacho-Sad index theorem for invariant leaves of a holomorphic foliation on a complex surface (see [CS]). So we started to investigate the reasons for these similarities; and this paper contains what we have found.

The main idea is that the simple fact of being pointwise fixed by a holomorphic self-map $f$ induces a lot of structure on a (possibly singular) subvariety $S$ of a complex manifold $M$. First of all, we shall introduce (in §3) a canonically defined holomorphic section $X_f$ of the bundle $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$, where $N_S$ is the normal bundle of $S$ in $M$ (here we are assuming $S$ smooth; however, we can also define $X_f$ as a section of a suitable sheaf even when $S$ is singular — see Remark 3.3 — but it turns out that only the behavior on the regular part of $S$ is relevant for our index theorems), and $\nu_f$ is a positive integer, the *order of contact* of $f$ with $S$, measuring how close $f$ is to being the identity in a neighborhood $S$ (see §1). Roughly speaking, the section $X_f$ describes the directions in which $S$ is pushed by $f$; see Proposition 8.1 for a more precise description of this phenomenon when $S$ is a hypersurface.

The canonical section $X_f$ can also be seen as a morphism from $N_S^{\otimes \nu_f}$ to $TM|_S$; its image $\Xi_f$ is the *canonical distribution.* When $\Xi_f$ is contained in $TS$ (we shall say that $f$ is *tangential*) and integrable (this happens for instance if $S$ is a hypersurface), then on $S$ we get a singular holomorphic foliation induced by $f$ — and this is a first concrete connection between our discrete dynamical theory and the continuous dynamics studied in foliation theory. We stress, however, that we get a well-defined foliation on $S$ *only,* while in the continuous setting one usually assumes that $S$ is invariant under a foliation defined in a *whole neighborhood* of $S$. Thus even in the tangential codimension-one case our results will not be a direct consequence of foliation theory.

As we shall momentarily discuss, to get index theorems it is important to have a section of $TS \otimes (N_S^*)^{\otimes \nu_f}$ (as in the case when $f$ is tangential) instead of merely a section of $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$. In Section 3, when $f$ is not tangential (which is a situation akin to dicriticality for foliations; see Propositions 1.4 and 8.1) we shall define other holomorphic sections $H_{\sigma,f}$ and $H^1_{\sigma,f}$ of $TS \otimes (N_S^*)^{\otimes \nu_f}$ which are as good as $X_f$ when $S$ satisfies a geometric condition which we call *comfortably embedded* in $M$, meaning, roughly speaking, that $S$ is a first-order approximation of the zero section of a vector bundle (see §2 for the precise definition, amounting to the vanishing of two sheaf cohomology classes — or, in other terms, to the triviality of two canonical extensions of $N_S$).

The canonical section is not the only object we are able to associate to $S$. Having a section $X$ of $TS \otimes F^*$, where $F$ is any vector bundle on $S$, is equivalent to having an $F^*$-valued derivation $X^\#$ of the sheaf of holomorphic functions $\mathcal{O}_S$ (see §5). If $E$ is another vector bundle on $S$, a *holomorphic action of $F$ on $E$ along $X$* is a $\mathbb{C}$-linear map $\tilde{X} \colon \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$ (where $\mathcal{E}$ and $\mathcal{F}$ are the sheafs of germs of holomorphic sections of $E$ and $F$) satisfying $\tilde{X}(gs) = X^\#(g) \otimes s + g\tilde{X}(s)$ for any $g \in \mathcal{O}_S$ and $s \in E$; this is a generalization of the notion of $(1,0)$-connection on $E$ (see Example 5.1).

In Section 5 we shall show that when $S$ is a hypersurface and $f$ is tangential (or $S$ is comfortably embedded in $M$) there is a natural way to define a holomorphic action of $N_S^{\otimes \nu_f}$ on $N_S$ along $X_f$ (or along $H_{\sigma,f}$ or $H^1_{\sigma,f}$). And this will allow us to bring into play the general theory developed by Lehmann and Suwa (see, e.g., [Su]) on a cohomological approach to index theorems. So, exactly as Lehmann and Suwa generalized, to any dimension, the Camacho-Sad index theorem, we are able to generalize the index theorems of [A2] and [BT] in the following form (see Theorem 6.2):

THEOREM 0.1. *Let $S$ be a compact, globally irreducible, possibly singular hypersurface in an $n$-dimensional complex manifold $M$. Let $f \colon M \to M$, $f \not\equiv \mathrm{id}_M$, be a holomorphic self-map of $M$ fixing pointwise $S$, and denote by* $\mathrm{Sing}(f)$ *the zero set of $X_f$. Assume that*

(a) *$f$ is tangential to $S$, and then set $X = X_f$, or that*

(b) *$S^0 = S \setminus \big(\mathrm{Sing}(S) \cup \mathrm{Sing}(f)\big)$ is comfortably embedded into $M$, and then set $X = H_{\sigma,f}$ if $\nu_f > 1$, or $X = H^1_{\sigma,f}$ if $\nu_f = 1$.*

*Assume moreover* $X \not\equiv O$ (*a condition always satisfied when* $f$ *is tangential*), *and denote by* $\mathrm{Sing}(X)$ *the zero set of* $X$. *Let* $\mathrm{Sing}(S) \cup \mathrm{Sing}(X) = \bigcup_\lambda \Sigma_\lambda$ *be the decomposition of* $\mathrm{Sing}(S) \cup \mathrm{Sing}(X)$ *in connected components. Finally, let* $[S]$ *be the line bundle on* $M$ *associated to the divisor* $S$. *Then there exist complex numbers* $\mathrm{Res}(X, S, \Sigma_\lambda) \in \mathbb{C}$ *depending only on the local behavior of* $X$ *and* $[S]$ *near* $\Sigma_\lambda$ *such that*

$$\sum_\lambda \mathrm{Res}(X, S, \Sigma_\lambda) = \int_S c_1^{n-1}([S]),$$

*where* $c_1([S])$ *is the first Chern class of* $[S]$.

Furthermore, when $\Sigma_\lambda$ is an isolated point $\{x_\lambda\}$, we have explicit formulas for the computation of the residues $\mathrm{Res}(X, S, \{x_\lambda\})$; see Theorem 6.5.

Since $X$ is a global section of $TS \otimes (N_S^*)^{\otimes \nu_f}$, if $S$ is smooth and $X$ has only isolated zeroes it is well-known that the top Chern class $c_{n-1}\big(TS \otimes (N_S^*)^{\otimes \nu_f}\big)$ counts the zeroes of $X$. Our result shows that $c_1^{n-1}(N_S)$ is related in a similar (but deeper) way to the zero set of $X$. See also Section 8 for examples of results one can obtain using both Chern classes together.

If the codimension of $S$ is greater than one, and $S$ is smooth, we can blow-up $M$ along $S$; then the exceptional divisor $E_S$ is a hypersurface, and we can apply to it the previous theorem. In this way we get (see Theorem 7.2):

THEOREM 0.2. *Let* $S$ *be a compact complex submanifold of codimension* $1 < m < n$ *in an* $n$*-dimensional complex manifold* $M$. *Let* $f\colon M \to M$, $f \not\equiv \mathrm{id}_M$, *be a holomorphic self-map of* $M$ *fixing pointwise* $S$, *and assume that* $f$ *is tangential, or that* $\nu_f > 1$ (*or both*). *Let* $\bigcup_\lambda \Sigma_\lambda$ *be the decomposition in connected components of the set of singular directions* (*see* §7 *for the definition*) *for* $f$ *in* $E_S$. *Then there exist complex numbers* $\mathrm{Res}(f, S, \Sigma_\lambda) \in \mathbb{C}$, *depending only on the local behavior of* $f$ *and* $S$ *near* $\Sigma_\lambda$, *such that*

$$\sum_\lambda \mathrm{Res}(f, S, \Sigma_\lambda) = \int_S \pi_* c_1^{n-1}([E_S]),$$

*where* $\pi_*$ *denotes integration along the fibers of the bundle* $E_S \to S$.

Theorems 0.1 and 0.2 are only two of the index theorems we can derive using this approach. Indeed, we are also able to obtain versions for holomorphic self-maps of two other main index theorems of foliation theory, the Baum-Bott index theorem and the Lehmann-Suwa-Khanedani (or variation) index theorem: see Theorems 6.3, 6.4, 6.6, 7.3 and 7.4. In other words, it turns out that the existence of holomorphic actions of suitable complex vector bundles defined only on $S$ is an efficient tool to get index theorems, both in our setting and (under slightly different assumptions) in foliation theory; and this is another reason for the similarities noticed in [A2].

Finally, in Section 8 we shall present a couple of applications of our results to the discrete dynamics of holomorphic self-maps of complex surfaces, thus closing the circle and coming back to the arguments that originally inspired our work.

## 1. The order of contact

Let $M$ be an $n$-dimensional complex manifold, and $S \subset M$ an irreducible subvariety of codimension $m$. We shall denote by $\mathcal{O}_M$ the sheaf of holomorphic functions on $M$, and by $\mathcal{I}_S$ the subsheaf of functions vanishing on $S$. With a slight abuse of notations, we shall use the same symbol to denote both a germ at $p$ and any representative defined in a neighborhood of $p$. We shall denote by $TM$ the holomorphic tangent bundle of $M$, and by $\mathcal{T}_M$ the sheaf of germs of local holomorphic sections of $TM$. Finally, we shall denote by $\mathrm{End}(M,S)$ the set of (germs about $S$ of) holomorphic self-maps of $M$ fixing $S$ pointwise.

Let $f \in \mathrm{End}(M,S)$ be given, $f \not\equiv \mathrm{id}_M$, and take $p \in S$. For every $h \in \mathcal{O}_{M,p}$ the germ $h \circ f$ is well-defined, and we have $h \circ f - h \in \mathcal{I}_{S,p}$.

*Definition* 1.1. The *$f$-order of vanishing at $p$ of $h \in \mathcal{O}_{M,p}$* is given by

$$\nu_f(h;p) = \max\{\mu \in \mathbb{N} \mid h \circ f - h \in \mathcal{I}_{S,p}^{\mu}\},$$

and the *order of contact $\nu_f(p)$ of $f$ at $p$* with $S$ by

$$\nu_f(p) = \min\{\nu_f(h;p) \mid h \in \mathcal{O}_{M,p}\}.$$

We shall momentarily prove that $\nu_f(p)$ does not depend on $p$.

Let $(z^1,\ldots,z^n)$ be local coordinates in a neighborhood of $p$. If $h$ is any holomorphic function defined in a neighborhood of $p$, the definition of order of contact yields the important relation

$$h \circ f - h = \sum_{j=1}^{n} (f^j - z^j) \frac{\partial h}{\partial z^j} \qquad (\mathrm{mod}\ \mathcal{I}_{S,p}^{2\nu_f(p)}), \tag{1.1}$$

where $f^j = z^j \circ f$.

As a consequence we have

Lemma 1.1. (i) *Let $(z^1,\ldots,z^n)$ be any set of local coordinates at $p \in S$. Then*

$$\nu_f(p) = \min_{j=1,\ldots,n}\{\nu_f(z^j;p)\}.$$

(ii) *For any $h \in \mathcal{O}_{M,p}$ the function $p \mapsto \nu_f(h;p)$ is constant in a neighborhood of $p$.*

(iii) *The function $p \mapsto \nu_f(p)$ is constant.*

*Proof.* (i) Clearly, $\nu_f(p) \le \min_{j=1,\dots,n}\{\nu_f(z^j;p)\}$. The opposite inequality follows from (1.1).

(ii) Let $h \in \mathcal{O}_{M,p}$, and choose a set $\{\ell^1,\dots,\ell^k\}$ of generators of $\mathcal{I}_{S,p}$. Then we can write

$$h\circ f - h = \sum_{|I|=\nu_f(h;p)} \ell^I g_I, \tag{1.2}$$

where $I=(i_1,\dots,i_k)\in\mathbb{N}^k$ is a $k$-multi-index, $|I| = i_1+\cdots+i_k$, $\ell^I = (\ell^1)^{i_1}\cdots(\ell^k)^{i_k}$ and $g_I \in \mathcal{O}_{M,p}$. Furthermore, there is a multi-index $I_0$ such that $g_{I_0}\notin\mathcal{I}_{S,p}$. By the coherence of the sheaf of ideals of $S$, the relation (1.2) holds for the corresponding germs at all points $q\in S$ in a neighborhood of $p$. Furthermore, $g_{I_0}\notin\mathcal{I}_{S,p}$ means that $g_{I_0}|_S\not\equiv 0$ in a neighborhood of $p$, and thus $g_{I_0}\notin\mathcal{I}_{S,q}$ for all $q\in S$ close enough to $p$. Putting these two observations together we get the assertion.

(iii) By (i) and (ii) we see that the function $p\mapsto\nu_f(p)$ is locally constant and since $S$ is connected, it is constant everywhere. □

We shall then denote by $\nu_f$ the *order of contact* of $f$ with $S$, computed at any point $p\in S$.

As we shall see, it is important to compare the order of contact of $f$ with the $f$-order of vanishing of germs in $\mathcal{I}_{S,p}$.

*Definition* 1.2. We say that $f$ is *tangential* at $p$ if

$$\min\bigl\{\nu_f(h;p) \mid h\in\mathcal{I}_{S,p}\bigr\} > \nu_f.$$

LEMMA 1.2. *Let $\{\ell^1,\dots,\ell^k\}$ be a set of generators of $\mathcal{I}_{S,p}$. Then*

$$\nu_f(h;p)\ge\min\{\nu_f(\ell^1;p),\dots,\nu_f(\ell^k;p),\nu_f+1\}$$

*for all $h\in\mathcal{I}_{S,p}$. In particular, $f$ is tangential at $p$ if and only if*

$$\min\{\nu_f(\ell^1;p),\dots,\nu_f(\ell^k;p)\}>\nu_f.$$

*Proof.* Let us write $h = g_1\ell^1+\cdots+g_k\ell^k$ for suitable $g_1,\dots,g_k\in\mathcal{O}_{M,p}$. Then

$$h\circ f - h = \sum_{j=1}^k \bigl[(g_j\circ f)(\ell^j\circ f-\ell^j)+(g_j\circ f-g_j)\ell^j\bigr],$$

and the assertion follows. □

COROLLARY 1.3. *If $f$ is tangential at one point $p\in S$, then it is tangential at all points of $S$.*

*Proof.* The coherence of the sheaf of ideals of $S$ implies that if $\{\ell^1,\dots,\ell^k\}$ are generators of $\mathcal{I}_{S,p}$ then the corresponding germs are generators of $\mathcal{I}_{S,q}$ for all $q \in S$ close enough to $p$. Then Lemmas 1.1.(ii) and 1.2 imply that both the set of points where $f$ is tangential and the set of points where $f$ is not tangential are open; hence the assertion follows because $S$ is connected. $\square$

Of course, we shall then say that $f$ is *tangential* along $S$ if it is tangential at any point of $S$.

*Example* 1.1. Let $p$ be a smooth point of $S$, and choose local coordinates $z = (z^1, \ldots, z^n)$ defined in a neighborhood $U$ of $p$, centered at $p$ and such that $S \cap U = \{z^1 = \cdots = z^m = 0\}$. We shall write $z' = (z^1, \ldots, z^m)$ and $z'' = (z^{m+1}, \ldots, z^n)$, so that $z''$ yields local coordinates on $S$. Take $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$; then in local coordinates the map $f$ can be written as $(f^1, \ldots, f^n)$ with

$$f^j(z) = z^j + \sum_{h \geq 1} P_h^j(z', z''),$$

where each $P_h^j$ is a homogeneous polynomial of degree $h$ in the variables $z'$, with coefficients depending holomorphically on $z''$. Then Lemma 1.1 yields

$$\nu_f = \min\{h \geq 1 \mid \exists\, 1 \leq j \leq n : P_h^j \not\equiv 0\}.$$

Furthermore, $\{z^1, \ldots, z^m\}$ is a set of generators of $\mathcal{I}_{S,p}$; therefore by Lemma 1.2 the map $f$ is tangential if and only if

$$\min\{h \geq 1 \mid \exists\, 1 \leq j \leq m : P_h^j \not\equiv 0\} > \min\{h \geq 1 \mid \exists\, m+1 \leq j \leq n : P_h^j \not\equiv 0\}.$$

*Remark* 1.1. When $S$ is smooth, the differential of $f$ acts linearly on the normal bundle $N_S$ of $S$ in $M$. If $S$ is a hypersurface, $N_S$ is a line bundle, and the action is multiplication by a holomorphic function $b$; if $S$ is compact, this function is a constant. It is easy to check that in local coordinates chosen as in the previous example the expression of the function $b$ is exactly $1 + P_1^1(z)/z^1$; therefore we must have $P_1^1(z) = (b_f - 1)z^1$ for a suitable constant $b_f \in \mathbb{C}$. In particular, if $b_f \neq 1$ then necessarily $\nu_f = 1$ and $f$ is not tangential along $S$.

*Remark* 1.2. The number $\mu$ introduced in [BT, (2)] is, by Lemma 1.1, our order of contact; therefore our notion of tangential is equivalent to the notion of nondegeneracy defined in [BT] when $n = 2$ and $m = 1$. On the other hand, as already remarked in [BT], a nondegenerate map in the sense defined in [A2] when $n = 2$, $m = 1$ and $S$ is smooth is tangential if and only if $b_f = 1$ (which was the case mainly considered in that paper).

*Example* 1.2. A particularly interesting example (actually, the one inspiring this paper) of map $f \in \mathrm{End}(M, S)$ is obtained by blowing up a map tangent to the identity. Let $f_o$ be a (germ of) holomorphic self-map of $\mathbb{C}^n$ (or of any complex $n$-manifold) fixing the origin (or any other point) and *tangent to the* *identity,* that is, such that $d(f_o)_O = \mathrm{id}$. If $\pi\colon M \to \mathbb{C}^n$ denotes the blow-up of the origin, let $S = \pi^{-1}(O) \cong \mathbb{P}^{n-1}(\mathbb{C})$ be the exceptional divisor, and $f \in \mathrm{End}(M, S)$ the lifting of $f_o$, that is, the unique holomorphic self-map of $M$ such that $f_o \circ \pi = \pi \circ f$ (see, e.g., [A1] for details). If

$$f_o^j(w) = w^j + \sum_{h \geq 2} Q_h^j(w)$$

is the expansion of $f_o^j$ in a series of homogeneous polynomials (for $j = 1, \ldots, n$), then in the canonical coordinates centered in $p = [1 : 0 : \cdots : 0]$ the map $f$ is given by

$$f^j(z) = \begin{cases} z^1 + \sum\limits_{h \geq 2} Q_h^1(1, z'')(z^1)^h & \text{for } j = 1, \\ z^j + \dfrac{\sum_{h \geq 2} \left[Q_h^j(1, z'') - z^j Q_h^1(1, z'')\right](z^1)^{h-1}}{1 + \sum_{h \geq 2} Q_h^1(1, z'')(z^1)^{h-1}} & \text{for } j = 2, \ldots, n, \end{cases}$$

where $z'' = (z^2, \ldots, z^n)$. Therefore $b_f = 1$,

$$\nu_f(z^1; p) = \min\{h \geq 2 \mid Q_h^1(1, z'') \not\equiv 0\},$$

and

$$\nu_f = \min\big\{\nu_f(z^1; p), \\ \min\{h \geq 1 \mid \exists\, 2 \leq j \leq n : Q_{h+1}^j(1, z'') - z^j Q_{h+1}^1(1, z'') \not\equiv 0\}\big\}.$$

Let $\nu(f_o) \geq 2$ be the order of $f_o$, that is, the minimum $h$ such that $Q_h^j \not\equiv 0$ for some $1 \leq j \leq n$. Clearly, $\nu_f(z^1; p) \geq \nu(f_o)$ and $\nu_f \geq \nu(f_o) - 1$. More precisely, if there is $2 \leq j \leq n$ such that $Q_{\nu(f_o)}^j(1, z'') \not\equiv z^j Q_{\nu(f_o)}^1(1, z'')$, then $\nu_f = \nu(f_o) - 1$ and $f$ is tangential. If on the other hand we have $Q_{\nu(f_o)}^j(1, z'') \equiv z^j Q_{\nu(f_o)}^1(1, z'')$ for all $2 \leq j \leq n$, then necessarily $Q_{\nu(f_o)}^1(1, z'') \not\equiv 0$, $\nu_f(z^1; p) = \nu(f_o) = \nu_f$, and $f$ is not tangential.

Borrowing a term from continuous dynamics, we say that a map $f_o$ tangent to the identity at the origin is *dicritical* if $w^h Q_{\nu(f_o)}^k(w) \equiv w^k Q_{\nu(f_o)}^h(w)$ for all $1 \leq h, k \leq n$. Then we have proved that:

Proposition 1.4. *Let $f_o \in \mathrm{End}(\mathbb{C}^n, O)$ be a (germ of) holomorphic self-map of $\mathbb{C}^n$ tangent to the identity at the origin, and let $f \in \mathrm{End}(M, S)$ be its blow-up. Then $f$ is not tangential if and only if $f_o$ is dicritical. Furthermore, $\nu_f = \nu(f_o) - 1$ if $f_o$ is not dicritical, and $\nu_f = \nu(f_o)$ if $f_o$ is dicritical.*

In particular, most maps obtained with this procedure are tangential.

## 2. Comfortably embedded submanifolds

Up to now $S$ was any complex subvariety of the manifold $M$. However, some of the proofs in the following sections do not work in this generality; so this section is devoted to describe the kind of properties we shall (sometimes) need on $S$.

Let $E'$ and $E''$ be two vector bundles on the same manifold $S$. We recall (see, e.g., [Ati, §1]) that an *extension* of $E''$ by $E'$ is an exact sequence of vector bundles

$$O \longrightarrow E' \xrightarrow{\iota} E \xrightarrow{\pi} E'' \longrightarrow O.$$

Two extensions are *equivalent* if there is an isomorphism of exact sequences which is the identity on $E'$ and $E''$.

A *splitting* of an extension $O \longrightarrow E' \xrightarrow{\iota} E \xrightarrow{\pi} E'' \longrightarrow O$ is a morphism $\sigma\colon E'' \to E$ such that $\pi \circ \sigma = \mathrm{id}_{E''}$. In particular, $E = \iota(E') \oplus \sigma(E'')$, and we shall say that the extension *splits.* We explicitly remark that an extension splits if and only if it is equivalent to the trivial extension $O \to E' \to E' \oplus E'' \to E'' \to O$.

Let $S$ now be a complex submanifold of a complex manifold $M$. We shall denote by $TM|_S$ the restriction to $S$ of the tangent bundle of $M$, and by $N_S = TM|_S/TS$ the normal bundle of $S$ into $M$. Furthermore, $\mathcal{T}_{M,S}$ will be the sheaf of germs of holomorphic sections of $TM|_S$ (which is different from the restriction $\mathcal{T}_M|_S$ to $S$ of the sheaf of holomorphic sections of $TM$), and $\mathcal{N}_S$ the sheaf of germs of holomorphic sections of $N_S$.

*Definition* 2.1. Let $S$ be a complex submanifold of codimension $m$ in an $n$-dimensional complex manifold $M$. A chart $(U_\alpha, z_\alpha)$ of $M$ is *adapted* to $S$ if either $S \cap U_\alpha = \emptyset$ or $S \cap U_\alpha = \{z^1_\alpha = \cdots = z^m_\alpha = 0\}$, where $z_\alpha = (z^1_\alpha, \ldots, z^n_\alpha)$. In particular, $\{z^1_\alpha, \ldots, z^m_\alpha\}$ is a set of generators of $\mathcal{I}_{S,p}$ for all $p \in S \cap U_\alpha$. An atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ of $M$ is *adapted* to $S$ if all charts in $\mathfrak{U}$ are. If $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ is adapted to $S$ we shall denote by $\mathfrak{U}_S = \{(U''_\alpha, z''_\alpha)\}$ the atlas of $S$ given by $U''_\alpha = U_\alpha \cap S$ and $z''_\alpha = (z^{m+1}_\alpha, \ldots, z^n_\alpha)$, where we are clearly considering only the indices such that $U_\alpha \cap S \neq \emptyset$. If $(U_\alpha, z_\alpha)$ is a chart adapted to $S$, we shall denote by $\partial_{\alpha,r}$ the projection of $\partial/\partial z^r_\alpha|_{S \cap U_\alpha}$ in $N_S$, and by $\omega^r_\alpha$ the local section of $N^*_S$ induced by $dz^r_\alpha|_{S\cap U_\alpha}$; thus $\{\partial_{\alpha,1}, \ldots, \partial_{\alpha,m}\}$ and $\{\omega^1_\alpha, \ldots, \omega^m_\alpha\}$ are local frames for $N_S$ and $N^*_S$ respectively over $U_\alpha \cap S$, dual to each other.

From now on, every chart and atlas we consider on $M$ will be adapted to $S$.

*Remark* 2.1. We shall use the Einstein convention on the sum over repeated indices. Furthermore, indices like $j$, $h$, $k$ will run from 1 to $n$; indices like $r$, $s$, $t$, $u$, $v$ will run from 1 to $m$; and indices like $p$, $q$ will run from $m+1$ to $n$.

*Definition* 2.2. We shall say that $S$ *splits into* $M$ if the extension $O \to TS \to TM|_S \to N_S \to O$ splits.

*Example* 2.1. It is well-known that if $S$ is a rational smooth curve with negative self-intersection in a surface $M$, then $S$ splits into $M$.

PROPOSITION 2.1. *Let $S$ be a complex submanifold of codimension $m$ in an $n$-dimensional complex manifold $M$. Then $S$ splits into $M$ if and only if there is an atlas $\hat{\mathfrak{U}} = \{(\hat{U}_\alpha, \hat{z}_\alpha)\}$ adapted to $S$ such that*

$$\left.\frac{\partial \hat{z}^p_\beta}{\partial \hat{z}^r_\alpha}\right|_S \equiv 0, \tag{2.1}$$

*for all $r = 1, \ldots, m$, $p = m+1, \ldots, n$ and indices $\alpha$ and $\beta$.*

*Proof.* It is well known (see, e.g., [Ati, Prop. 2]) that there is a one-to-one correspondence between equivalence classes of extensions of $N_S$ by $TS$ and the cohomology group $H^1\big(S, \mathrm{Hom}(\mathcal{N}_S, \mathcal{T}_S)\big)$, and an extension splits if and only if it corresponds to the zero cohomology class.

The class corresponding to the extension $O \to TS \to TM|_S \to N_S \to O$ is the class $\delta(\mathrm{id}_{N_S})$, where $\delta\colon H^0\big(S, \mathrm{Hom}(\mathcal{N}_S, \mathcal{N}_S)\big) \to H^1\big(S, \mathrm{Hom}(\mathcal{N}_S, \mathcal{T}_S)\big)$ is the connecting homomorphism in the long exact sequence of cohomology associated to the short exact sequence obtained by applying the functor $\mathrm{Hom}(\mathcal{N}_S, \cdot)$ to the extension sequence. More precisely, if $\mathfrak{U}$ is an atlas adapted to $S$, we get a local splitting morphism $\sigma_\alpha\colon N_{U''_\alpha} \to TM|_{U''_\alpha}$ by setting $\sigma_\alpha(\partial_{r,\alpha}) = \partial/\partial z^r_\alpha$, and then the element of $H^1\big(\mathfrak{U}_S, \mathrm{Hom}(\mathcal{N}_S, \mathcal{T}_S)\big)$ associated to the extension is $\{\sigma_\beta - \sigma_\alpha\}$. Now,

$$(\sigma_\beta - \sigma_\alpha)(\partial_{r,\alpha}) = \left.\frac{\partial z^s_\beta}{\partial z^r_\alpha}\right|_S \frac{\partial}{\partial z^s_\beta} - \frac{\partial}{\partial z^r_\alpha} = \left.\frac{\partial z^s_\beta}{\partial z^r_\alpha}\,\frac{\partial z^p_\alpha}{\partial z^s_\beta}\right|_S \frac{\partial}{\partial z^p_\alpha}.$$

So, if (2.1) holds, then $S$ splits into $M$. Conversely, assume that $S$ splits into $M$; then we can find an atlas $\mathfrak{U}$ adapted to $S$ and a 0-cochain $\{c_\alpha\} \in H^0(\mathfrak{U}_S, \mathcal{T}_S \otimes \mathcal{N}^*_S)$ such that

$$\left.\frac{\partial z^s_\beta}{\partial z^r_\alpha}\,\frac{\partial z^p_\alpha}{\partial z^s_\beta}\right|_S = (c_\beta)^q_s \left.\frac{\partial z^s_\beta}{\partial z^r_\alpha}\,\frac{\partial z^p_\alpha}{\partial z^q_\beta}\right|_S - (c_\alpha)^p_r \tag{2.2}$$

on $U_\alpha \cap U_\beta \cap S$. We claim that the coordinates

$$\begin{cases} \hat{z}^r_\alpha = z^r_\alpha, \\ \hat{z}^p_\alpha = z^p_\alpha + (c_\alpha)^p_s(z''_\alpha) z^s_\alpha \end{cases} \tag{2.3}$$

satisfy (2.1) when restricted to suitable open sets $\hat{U}_\alpha \subseteq U_\alpha$. Indeed, (2.2) yields

$$\begin{aligned}\frac{\partial \hat{z}^p_\beta}{\partial \hat{z}^r_\alpha} &= \frac{\partial \hat{z}^p_\beta}{\partial z^s_\alpha}\frac{\partial z^s_\alpha}{\partial \hat{z}^r_\alpha} + \frac{\partial \hat{z}^p_\beta}{\partial z^q_\alpha}\frac{\partial z^q_\alpha}{\partial \hat{z}^r_\alpha} = \frac{\partial \hat{z}^p_\beta}{\partial z^r_\alpha} - (c_\alpha)^q_r \frac{\partial \hat{z}^p_\beta}{\partial z^q_\alpha} + R_1 \\ &= \frac{\partial z^p_\beta}{\partial z^r_\alpha} + (c_\beta)^p_s \frac{\partial z^s_\beta}{\partial z^r_\alpha} - (c_\alpha)^q_r \frac{\partial z^p_\beta}{\partial z^q_\alpha} + R_1 = R_1,\end{aligned}$$

where $R_1$ denotes terms vanishing on $S$, and we are done. □

*Definition* 2.3. Assume that $S$ splits into $M$. An atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ adapted to $S$ and satisfying (2.1) will be called a *splitting atlas* for $S$. It is easy to see that for any splitting morphism $\sigma\colon N_S \to TM|_S$ there exists a splitting atlas $\mathfrak{U}$ such that $\sigma(\partial_{r,\alpha}) = \partial/\partial z^r_\alpha$ for all $r = 1, \ldots m$ and indices $\alpha$; we shall say that $\mathfrak{U}$ is *adapted* to $\sigma$.

*Example* 2.2. A *local holomorphic retraction* of $M$ onto $S$ is a holomorphic retraction $\rho\colon W \to S$, where $W$ is a neighborhood of $S$ in $M$. It is clear that the existence of such a local holomorphic retraction implies that $S$ splits into $M$.

*Example* 2.3. Let $\pi\colon M \to S$ be a rank $m$ holomorphic vector bundle on $S$. If we identify $S$ with the zero section of the vector bundle, $\pi$ becomes a (global) holomorphic retraction of $M$ on $S$. The charts given by the trivialization of the bundle clearly give a splitting atlas. Furthermore, if $(U_\alpha, z_\alpha)$ and $(U_\beta, z_\beta)$ are two such charts, we have $z''_\beta = \varphi_{\beta\alpha}(z''_\alpha)$ and $z'_\beta = a_{\beta\alpha}(z''_\alpha) z'_\alpha$, where $a_{\beta\alpha}$ is an invertible matrix depending only on $z''_\alpha$. In particular, we have

$$\frac{\partial z^p_\beta}{\partial z^r_\alpha} \equiv 0 \qquad \text{and} \qquad \frac{\partial^2 z^r_\beta}{\partial z^s_\alpha \partial z^t_\alpha} \equiv 0$$

for all $r$, $s$, $t = 1, \ldots, m$, $p = m+1, \ldots, n$ and indices $\alpha$ and $\beta$.

The previous example, compared with (2.1), suggests the following

*Definition* 2.4. Let $S$ be a codimension $m$ complex submanifold of an $n$-dimensional complex manifold $M$. We say that $S$ is *comfortably embedded* in $M$ if $S$ splits into $M$ and there exists a splitting atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ such that

$$\left.\frac{\partial^2 z^r_\beta}{\partial z^s_\alpha \partial z^t_\alpha}\right|_S \equiv 0 \tag{2.4}$$

for all $r$, $s$, $t = 1, \ldots, m$ and indices $\alpha$ and $\beta$.

An atlas satisfying the previous condition is said to be *comfortable* for $S$. Roughly speaking, then, a comfortably embedded submanifold is like a first-order approximation of the zero section of a vector bundle.

Let us express condition (2.4) in a different way. If $(U_\alpha, z_\alpha)$ and $(U_\beta, z_\beta)$ are two charts about $p \in S$ adapted to $S$, we can write

$$z_\beta^r = (a_{\beta\alpha})_s^r \, z_\alpha^s \tag{2.5}$$

for suitable $(a_{\beta\alpha})_s^r \in \mathcal{O}_{M,p}$. The germs $(a_{\beta\alpha})_s^r$ (unless $m = 1$) are not uniquely determined by (2.5); indeed, all the other solutions of (2.5) are of the form $(a_{\beta\alpha})_s^r + e_s^r$, where the $e_s^r$'s are holomorphic and satisfy

$$e_s^r z_\alpha^s \equiv 0. \tag{2.6}$$

Differentiating with respect to $z_\alpha^t$ we get

$$e_t^r + \frac{\partial e_s^r}{\partial z_\alpha^t} \, z_\alpha^s \equiv 0; \tag{2.7}$$

in particular, $e_t^r|_S \equiv 0$, and so the restriction of $(a_{\beta\alpha})_s^r$ to $S$ is uniquely determined — and it indeed gives the 1-cocycle of the normal bundle $N_S$ with respect to the atlas $\mathfrak{U}_S$.

Differentiating (2.7) we obtain

$$\frac{\partial e_t^r}{\partial z_\alpha^s} + \frac{\partial e_s^r}{\partial z_\alpha^t} + \frac{\partial^2 e_u^r}{\partial z_\alpha^s \partial z_\alpha^t} \, z_\alpha^u \equiv 0; \tag{2.8}$$

in particular,

$$\left[ \frac{\partial e_t^r}{\partial z_\alpha^s} + \frac{\partial e_s^r}{\partial z_\alpha^t} \right] \Bigg|_S \equiv 0,$$

and so the restriction of

$$\frac{\partial (a_{\beta\alpha})_t^r}{\partial z_\alpha^s} + \frac{\partial (a_{\beta\alpha})_s^r}{\partial z_\alpha^t}$$

to $S$ is uniquely determined for all $r$, $s$, $t = 1, \ldots, m$.

With this notation, we have

$$\frac{\partial^2 z_\beta^r}{\partial z_\alpha^s \partial z_\alpha^t} = \frac{\partial (a_{\beta\alpha})_s^r}{\partial z_\alpha^t} + \frac{\partial (a_{\beta\alpha})_t^r}{\partial z_\alpha^s} + \frac{\partial^2 (a_{\beta\alpha})_u^r}{\partial z_\alpha^s \partial z_\alpha^t} z_\alpha^u;$$

therefore (2.4) is equivalent to requiring

$$\left( \frac{\partial (a_{\beta\alpha})_t^r}{\partial z_\alpha^s} + \frac{\partial (a_{\beta\alpha})_s^r}{\partial z_\alpha^t} \right) \Bigg|_S \equiv 0 \tag{2.9}$$

for all $r$, $s$, $t = 1, \ldots, m$, and indices $\alpha$ and $\beta$.

*Example* 2.4. It is easy to check that the exceptional divisor $S$ in Example 1.2 is comfortably embedded into the blow-up $M$.

Then the main result of this section is

THEOREM 2.2. *Let $S$ be a codimension $m$ complex submanifold of an $n$-dimensional complex manifold $M$. Assume that $S$ splits into $M$, and let $\mathfrak{U}=\{(U_\alpha,z_\alpha)\}$ be a splitting atlas. Define a* 1*-cochain $\{h_{\beta\alpha}\}$ of $\mathcal{N}_S\otimes\mathcal{N}_S^*\otimes\mathcal{N}_S^*$ by setting*

$$h_{\beta\alpha}=\frac{1}{2}\,\frac{\partial z_\alpha^r}{\partial z_\beta^u}\frac{\partial^2 z_\beta^u}{\partial z_\alpha^s\partial z_\alpha^t}\bigg|_S\,\partial_{\alpha,r}\otimes\omega_\alpha^s\otimes\omega_\alpha^t \tag{2.10}$$
$$=\frac{1}{2}(a_{\alpha\beta})_u^r\left(\frac{\partial(a_{\beta\alpha})_s^u}{\partial z_\alpha^t}+\frac{\partial(a_{\beta\alpha})_t^u}{\partial z_\alpha^s}\right)\bigg|_S\,\partial_{\alpha,r}\otimes\omega_\alpha^s\otimes\omega_\alpha^t.$$

*Then*:

(i) $\{h_{\beta\alpha}\}$ *defines an element $[h]\in H^1(S,\mathcal{N}_S\otimes\mathcal{N}_S^*\otimes\mathcal{N}_S^*)$ independent of $\mathfrak{U}$;*

(ii) *$S$ is comfortably embedded in $M$ if and only if $[h]=0$.*

*Proof.* (i) Let us first prove that $\{h_{\beta\alpha}\}$ is a 1-cocycle with values in $\mathcal{N}_S\otimes\mathcal{N}_S^*\otimes\mathcal{N}_S^*$. We know that

$$(a_{\alpha\beta})_u^r(a_{\beta\alpha})_s^u=\delta_s^r+e_s^r,$$

where $\delta_s^r$ is Kronecker's delta, and the $e_s^r$'s satisfy (2.6) . Differentiating we get

$$\frac{\partial(a_{\alpha\beta})_u^r}{\partial z_\alpha^t}(a_{\beta\alpha})_s^u+(a_{\alpha\beta})_u^r\frac{\partial(a_{\beta\alpha})_s^u}{\partial z_\alpha^t}=\frac{\partial e_s^r}{\partial z_\alpha^t};$$

therefore (2.8) yields

$$(a_{\beta\alpha})_s^u\,\frac{\partial(a_{\alpha\beta})_u^r}{\partial z_\alpha^t}\bigg|_S+(a_{\beta\alpha})_t^u\,\frac{\partial(a_{\alpha\beta})_u^r}{\partial z_\alpha^s}\bigg|_S=-(a_{\alpha\beta})_u^r\left(\frac{\partial(a_{\beta\alpha})_s^u}{\partial z_\alpha^t}+\frac{\partial(a_{\beta\alpha})_t^u}{\partial z_\alpha^s}\right)\bigg|_S.$$

Hence

$$\begin{aligned}
h_{\alpha\beta}&=\frac{1}{2}(a_{\beta\alpha})_u^r\left(\frac{\partial(a_{\alpha\beta})_s^u}{\partial z_\beta^t}+\frac{\partial(a_{\alpha\beta})_t^u}{\partial z_\beta^s}\right)\bigg|_S\,\partial_{\beta,r}\otimes\omega_\beta^s\otimes\omega_\beta^t\\
&=\frac{1}{2}(a_{\beta\alpha})_u^r(a_{\alpha\beta})_r^{r_1}(a_{\beta\alpha})_{s_1}^s(a_{\beta\alpha})_{t_1}^t\\
&\quad\times\left((a_{\alpha\beta})_t^{t_2}\frac{\partial(a_{\alpha\beta})_s^u}{\partial z_\alpha^{t_2}}+(a_{\alpha\beta})_s^{s_2}\frac{\partial(a_{\alpha\beta})_t^u}{\partial z_\alpha^{s_2}}\right)\bigg|_S\,\partial_{\alpha,r_1}\otimes\omega_\alpha^{s_1}\otimes\omega_\alpha^{t_1}\\
&=\frac{1}{2}\left((a_{\beta\alpha})_{s_1}^s\frac{\partial(a_{\alpha\beta})_s^{r_1}}{\partial z_\alpha^{t_1}}+(a_{\beta\alpha})_{t_1}^t\frac{\partial(a_{\alpha\beta})_t^{r_1}}{\partial z_\alpha^{s_1}}\right)\bigg|_S\,\partial_{\alpha,r_1}\otimes\omega_\alpha^{s_1}\otimes\omega_\alpha^{t_1}\\
&=-h_{\beta\alpha},
\end{aligned}$$

where in the second equality we used (2.1). Analogously one proves that $h_{\alpha\beta}+h_{\beta\gamma}+h_{\gamma\alpha}=0$, and thus $\{h_{\beta\alpha}\}$ is a 1-cocycle as claimed.

Now we have to prove that the cohomology class $[h]$ is independent of the atlas $\mathfrak{U}$. Let $\hat{\mathfrak{U}}=\{(\hat{U}_\alpha,\hat{z}_\alpha)\}$ be another splitting atlas; up to taking a common refinement we can assume that $U_\alpha=\hat U_\alpha$ for all $\alpha$. Choose $(A_\alpha)^r_s\in\mathcal{O}(U_\alpha)$ so that $\hat z^r_\alpha=(A_\alpha)^r_s z^s_\alpha$; as usual, the restrictions to $S$ of $(A_\alpha)^r_s$ and of

$$\frac{\partial(A_\alpha)^r_s}{\partial z^t_\alpha}+\frac{\partial(A_\alpha)^r_t}{\partial z^s_\alpha}$$

are uniquely defined. Set, now,

$$C_\alpha=\frac{1}{2}(A_\alpha^{-1})^r_u\left[\frac{\partial(A_\alpha)^u_s}{\partial z^t_\alpha}+\frac{\partial(A_\alpha)^u_t}{\partial z^s_\alpha}\right]\Bigg|_S\partial_{\alpha,r}\otimes\omega^s_\alpha\otimes\omega^t_\alpha;$$

then it is not difficult to check that

$$h_{\beta\alpha}-\hat h_{\beta\alpha}=C_\beta-C_\alpha,$$

where $\{\hat h_{\beta\alpha}\}$ is the 1-cocycle built using $\hat{\mathfrak{U}}$, and this means exactly that both $\{h_{\beta\alpha}\}$ and $\{\hat h_{\beta\alpha}\}$ determine the same cohomology class.

(ii) If $S$ is comfortably embedded, using a comfortable atlas we immediately see that $[h]=0$. Conversely, assume that $[h]=0$; therefore we can find a splitting atlas $\mathfrak{U}$ and a 0-cochain $\{c_\alpha\}$ of $\mathcal{N}_S\otimes\mathcal{N}_S^*\otimes\mathcal{N}_S^*$ such that $h_{\beta\alpha}=c_\alpha-c_\beta$. Writing

$$c_\alpha=(c_\alpha)^r_{st}\,\partial_{\alpha,r}\otimes\omega^s_\alpha\otimes\omega^t_\alpha,$$

with $(c_\alpha)^r_{ts}$ symmetric in the lower indices, we define $\hat z_\alpha$ by setting

$$\begin{cases}\hat z^r_\alpha=z^r_\alpha+(c_\alpha)^r_{st}(z''_\alpha)\,z^s_\alpha z^t_\alpha & \text{for } r=1,\dots,m,\\ \hat z^p_\alpha=z^p_\alpha & \text{for } p=m+1,\dots,n,\end{cases}$$

on a suitable $\hat U_\alpha\subseteq U_\alpha$. Then $\hat{\mathfrak{U}}=\{(\hat U_\alpha,\hat z_\alpha)\}$ clearly is a splitting atlas; we claim that it is comfortable too. Indeed, by definition the functions

$$(\hat a_{\beta\alpha})^r_s=[\delta^r_u+(c_\beta)^r_{uv}(a_{\beta\alpha})^v_t z^t_\alpha](a_{\beta\alpha})^u_{u_1}d^{u_1}_s$$

satisfy (2.5) for $\hat{\mathfrak{U}}$, where the $d^{u_1}_s$'s are such that $z^{u_1}_\alpha=d^{u_1}_s\hat z^s_\alpha$. Hence

$$\left(\frac{\partial(\hat a_{\beta\alpha})^r_s}{\partial\hat z^t_\alpha}+\frac{\partial(\hat a_{\beta\alpha})^r_t}{\partial\hat z^s_\alpha}\right)\Bigg|_S=2(c_\beta)^r_{uv}(a_{\beta\alpha})^u_s(a_{\beta\alpha})^v_t|_S+\left(\frac{\partial(a_{\beta\alpha})^r_s}{\partial z^t_\alpha}+\frac{\partial(a_{\beta\alpha})^r_t}{\partial z^s_\alpha}\right)\Bigg|_S$$
$$+(a_{\beta\alpha})^r_u\left(\frac{\partial d^u_s}{\partial z^t_\alpha}+\frac{\partial d^u_t}{\partial z^s_\alpha}\right)\Bigg|_S.$$

Now, differentiating

$$z^u_\alpha=d^u_v\big(z^v_\alpha+(c_\alpha)^v_{rs}z^r_\alpha z^s_\alpha\big)$$

we get

$$\delta^u_t=\frac{\partial d^u_v}{\partial z^t_\alpha}\big(z^v_\alpha+(c_\alpha)^v_{rs}z^r_\alpha z^s_\alpha\big)+d^u_v\big(\delta^v_t+2(c_\alpha)^v_{rt}z^r_\alpha\big)$$

and

$$0=\left(\frac{\partial d^u_s}{\partial z^t_\alpha}+\frac{\partial d^u_t}{\partial z^s_\alpha}\right)\Bigg|_S+2(c_\alpha)^u_{st}.$$

Recalling that $h_{\beta\alpha}=c_\alpha-c_\beta$ we then see that $\hat{\mathfrak{U}}$ satisfies (2.9), and we are done. □

*Remark* 2.2. Since $N_S \otimes N_S^* \otimes N_S^* \cong \mathrm{Hom}\big(N_S, \mathrm{Hom}(N_S, N_S)\big)$, the previous theorem asserts that to any submanifold $S$ splitting into $M$ we can canonically associate an extension

$$O \to \mathrm{Hom}(N_S, N_S) \to E \to N_S \to O$$

of $N_S$ by $\mathrm{Hom}(N_S, N_S)$, and $S$ is comfortably embedded in $M$ if and only if this extension splits. See also [ABT] for more details on comfortably embedded submanifolds.

## 3. The canonical sections

Our next aim is to associate to any $f \in \mathrm{End}(M, S)$ different from the identity a section of a suitable vector bundle, indicating (very roughly speaking) how $f$ would move $S$ if it did not keep it fixed. To do so, in this section we still assume that $S$ is a smooth complex submanifold of a complex manifold $M$; however, in Remark 3.3 we shall describe the changes needed to avoid this assumption.

Given $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, it is clear that $df|_{TS} = \mathrm{id}$; therefore $df - \mathrm{id}$ induces a map from $N_S$ to $TM|_S$, and thus a holomorphic section over $S$ of the bundle $TM|_S \otimes N_S^*$. If $(U, z)$ is a chart adapted to $S$, we can define germs $g_r^h$ for $h = 1, \ldots, n$ and $r = 1, \ldots, m$ by writing

$$z^h \circ f - z^h = z^1 g_1^h + \cdots + z^m g_m^h.$$

It is easy to check that the germ of the section of $TM|_S \otimes N_S^*$ defined by $df - \mathrm{id}$ is locally expressed by

$$g_r^h|_{U\cap S}\, \frac{\partial}{\partial z_h} \otimes \omega^r,$$

where we are again indicating by $\omega^r$ the germ of section of the conormal bundle induced by the 1-form $dz^r$ restricted to $S$.

A problem with this section is that it vanishes identically if (and only if) $\nu_f > 1$. The solution consists in expanding $f$ at a higher order.

*Definition* 3.1. Given a chart $(U, z)$ adapted to $S$, set $f^j = z^j \circ f$, and write

$$f^j - z^j = z^{r_1} \cdots z^{r_{\nu_f}}\, g^j_{r_1 \ldots r_{\nu_f}}, \tag{3.1}$$

where the $g^j_{r_1\ldots r_{\nu_f}}$'s are symmetric in $r_1, \ldots, r_{\nu_f}$ and do not all vanish restricted to $S$. Let us then define

$$\mathcal{X}_f = g^h_{r_1 \ldots r_{\nu_f}}\, \frac{\partial}{\partial z^h} \otimes dz^{r_1} \otimes \cdots \otimes dz^{r_{\nu_f}}. \tag{3.2}$$

This is a local section of $TM \otimes (T^*M)^{\otimes \nu_f}$, defined in a neighborhood of a point of $S$; furthermore, when restricted to $S$, it induces a local section of $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$.

*Remark* 3.1. When $m>1$ the $g^j_{r_1\dots r_{\nu_f}}$'s are *not* uniquely determined by (3.1). Indeed, if $e^j_{r_1\dots r_{\nu_f}}$ are such that

$$e^j_{r_1\dots r_{\nu_f}} z^1\cdots z^{r_{\nu_f}} \equiv 0 \tag{3.3}$$

then $g^j_{r_1\dots r_{\nu_f}}+e^j_{r_1\dots r_{\nu_f}}$ still satisfies (3.1). This means that the section (3.2) is not uniquely determined too; but, as we shall see, this will not be a problem. For instance, (3.3) implies that $e^j_{r_1\dots r_{\nu_f}}\in\mathcal{I}_S$; therefore $\mathcal{X}_f|_{U\cap S}$ is always uniquely determined — though *a priori* it might depend on the chosen chart. On the other hand, when $m=1$ both the $g^j_{r_1\dots r_{\nu_f}}$'s and $\mathcal{X}_f$ *are* uniquely determined; this is one of the reasons making the codimension-one case simpler than the general case.

We have already remarked that when $\nu_f=1$ the section $\mathcal{X}_f$ restricted to $U\cap S$ coincides with the restriction of $df-\mathrm{id}$ to $S$. Therefore when $\nu_f=1$ the restriction of $\mathcal{X}_f$ to $S$ gives a globally well-defined section. Actually, this holds for any $\nu_f\ge 1$:

PROPOSITION 3.1. *Let $f\in\mathrm{End}(M,S)$, $f\not\equiv\mathrm{id}_M$. Then the restriction of $\mathcal{X}_f$ to $S$ induces a global holomorphic section $X_f$ of the bundle $TM|_S\otimes(N_S^*)^{\otimes\nu_f}$.*

*Proof.* Let $(U,z)$ and $(\hat U,\hat z)$ be two charts about $p\in S$ adapted to $S$. Then we can find holomorphic functions $a^r_s$ such that

$$\hat z^r = a^r_s z^s; \tag{3.4}$$

in particular,

$$\frac{\partial\hat z^r}{\partial z^s}=a^r_s \pmod{\mathcal{I}_S}\qquad\text{and}\qquad \frac{\partial\hat z^r}{\partial z^p}=0 \pmod{\mathcal{I}_S}. \tag{3.5}$$

Now set $f^j=z^j\circ f$, $\hat f^j=\hat z^j\circ f$, and define $g^j_{r_1\cdots r_{\nu_f}}$ and $\hat g^j_{r_1\cdots r_{\nu_f}}$ using (3.1) with $(U,z)$ and $(\hat U,\hat z)$ respectively. Then (3.4) and (1.1) yield

$$\begin{aligned}
a^{r_1}_{s_1}\cdots a^{r_{\nu_f}}_{s_{\nu_f}}\hat g^j_{r_1\dots r_{\nu_f}} z^{s_1}\cdots z^{s_{\nu_f}} &= \hat g^j_{r_1\dots r_{\nu_f}}\hat z^{r_1}\cdots \hat z^{r_{\nu_f}}\\
&= \hat f^j-\hat z^j = (f^h-z^h)\frac{\partial\hat z^j}{\partial z^h}+R_{2\nu_f}\\
&= g^h_{s_1\dots s_{\nu_f}}\frac{\partial\hat z^j}{\partial z^h} z^{s_1}\cdots z^{s_{\nu_f}}+R_{2\nu_f},
\end{aligned}$$

where the remainder terms $R_{2\nu_f}$ belong to $\mathcal{I}_S^{2\nu_f}$. Therefore we find

$$a^{r_1}_{s_1}\cdots a^{r_{\nu_f}}_{s_{\nu_f}}\hat g^j_{r_1\dots r_{\nu_f}} = \frac{\partial\hat z^j}{\partial z^h} g^h_{s_1\dots s_{\nu_f}} \pmod{\mathcal{I}_S}. \tag{3.6}$$

Recalling (3.5) we then get

$$\begin{aligned}\hat g^j_{r_1\dots r_{\nu_f}}\frac{\partial}{\partial\hat z^j}\otimes d\hat z^{r_1}\otimes\cdots\otimes d\hat z^{r_{\nu_f}}\\ &=\frac{\partial z^h}{\partial\hat z^j}\frac{\partial\hat z^{r_1}}{\partial z^{k_1}}\cdots\frac{\partial\hat z^{r_{\nu_f}}}{\partial z^{k_{\nu_f}}}\hat g^j_{r_1\dots r_{\nu_f}}\frac{\partial}{\partial z^h}\otimes dz^{k_1}\otimes\cdots\otimes dz^{k_{\nu_f}}\\ &=a^{r_1}_{s_1}\cdots a^{r_{\nu_f}}_{s_{\nu_f}}\hat g^j_{r_1\dots r_{\nu_f}}\frac{\partial z^h}{\partial\hat z^j}\frac{\partial}{\partial z^h}\otimes dz^{s_1}\otimes\cdots\otimes dz^{s_{\nu_f}}\pmod{\mathcal{I}_S}\\ &=g^h_{s_1\dots s_{\nu_f}}\frac{\partial}{\partial z^h}\otimes dz^{s_1}\otimes\cdots\otimes dz^{s_{\nu_f}}\pmod{\mathcal{I}_S},\end{aligned}$$

and we are done. □

*Remark* 3.2. For later use, we explicitly notice that when $m=1$ the germs $a^r_s$ are uniquely determined, and (3.6) becomes

$$(a^1_1)^{\nu_f}\hat g^j_{1\dots1}=\frac{\partial\hat z^j}{\partial z^h}\,g^h_{1\dots1}\pmod{\mathcal{I}_S^{\nu_f}}.\tag{3.7}$$

*Definition* 3.2. Let $f\in\mathrm{End}(M,S)$, $f\not\equiv\mathrm{id}_M$. The *canonical section* $X_f\in H^0\big(S,\,\mathcal{T}_{M,S}\otimes(\mathcal{N}_S^*)^{\otimes\nu_f}\big)$ *associated to* $f$ is defined by setting

$$X_f=g^h_{s_1\dots s_{\nu_f}}|_S\,\frac{\partial}{\partial z^h}\otimes\omega^{s_1}\otimes\cdots\otimes\omega^{s_{\nu_f}}\tag{3.8}$$

in any chart adapted to $S$. Since $(N_S^*)^{\otimes\nu_f}=(N_S^{\otimes\nu_f})^*$, we can also think of $X_f$ as a holomorphic section of $\mathrm{Hom}(N_S^{\otimes\nu_f},TM|_S)$, and introduce the *canonical distribution* $\Xi_f=X_f(N_S^{\otimes\nu_f})\subseteq TM|_S$.

In particular we can now justify the term "tangential" previously introduced:

Corollary 3.2. *Let* $f\in\mathrm{End}(M,S)$, $f\not\equiv\mathrm{id}_M$. *Then* $f$ *is tangential if and only if the canonical distribution is tangent to* $S$, *that is if and only if* $\Xi_f\subseteq TS$.

*Proof.* This follows from Lemma 1.2. □

*Example* 3.1. By the notation introduced in Example 1.2, if $f$ is obtained by blowing up a map $f_o$ tangent to the identity, then the canonical coordinates centered in $p=[1:0:\cdots:0]$ are adapted to $S$. In particular, if $f_o$ is non-dicritical (that is, if $f$ is tangential) then in a neighborhood of $p$,

$$X_f=\big[Q^q_{\nu(f_o)}(1,z'')-z^qQ^1_{\nu(f_o)}(1,z'')\big]\frac{\partial}{\partial z^q}\otimes(\omega^1)^{\otimes(\nu(f_o)-1)}.$$

*Remark* 3.3. To be more precise, $X_f$ is a section of the subsheaf $\mathcal{T}_{M,S}\otimes\mathrm{Sym}^{\nu_f}(\mathcal{N}_S^*)$, where $\mathrm{Sym}^{\nu_f}(\mathcal{N}_S^*)$ is the symmetric $\nu_f$-fold tensor product of $\mathcal{N}_S^*$.

Now, the sheaf $\mathcal{N}_S^*$ is isomorphic to $\mathcal{I}_S/\mathcal{I}_S^2$, and it is known that $\mathrm{Sym}^{\nu_f}\mathcal{I}_S/\mathcal{I}_S^2$ is isomorphic to $\mathcal{I}_S^{\nu_f}/\mathcal{I}_S^{\nu_f+1}$. This allows us to define $X_f$ as a global section of the coherent sheaf $\mathcal{T}_{M,S}\otimes\mathrm{Sym}^{\nu_f}(\mathcal{I}_S/\mathcal{I}_S^2)$ even when $S$ is singular. Indeed, if $(U,z)$ is a local chart adapted to $S$, for $j=1,\dots,n$ the functions $f^j-z^j$ determine local sections $[f^j-z^j]$ of $\mathcal{I}_S^{\nu_f}/\mathcal{I}_S^{\nu_f+1}$. But, since for any other chart $(\hat{U},\hat{z})$,

$$\hat{f}^j-\hat{z}^j=(f^h-z^h)\frac{\partial\hat{z}^j}{\partial z^h}+R_{2\nu_f},$$

then $(\partial/\partial z^j)\otimes[f^j-z^j]$ is a well-defined global section of $\mathcal{T}_{M,S}\otimes\mathrm{Sym}^{\nu_f}(\mathcal{I}_S/\mathcal{I}_S^2)$ which coincides with $X_f$ when $S$ is smooth.

*Remark* 3.4. When $f$ is tangential and $\Xi_f$ is involutive as a sub-distribution of $TS$ — for instance when $m=1$ — we thus get a holomorphic singular foliation on $S$ canonically associated to $f$. As already remarked in [Br], this possibly is the reason explaining the similarities discovered in [A2] between the local dynamics of holomorphic maps tangent to the identity and the dynamics of singular holomorphic foliations.

*Definition* 3.3. A point $p\in S$ is *singular* for $f$ if there exists $v\in(N_S)_p$, $v\neq O$, such that $X_f(v\otimes\cdots\otimes v)=O$. We shall denote by $\mathrm{Sing}(f)$ the set of singular points of $f$.

In Section 7 it will become clear why we choose this definition for singular points. In Section 8 we shall describe a dynamical interpretation of $X_f$ at nonsingular points in the codimension-one case; see Proposition 8.1.

*Remark* 3.5. If $S$ is a hypersurface, the normal bundle is a line bundle. Therefore $\Xi_f$ is a 1-dimensional distribution, and the singular points of $f$ are the points where $\Xi_f$ vanishes. Recalling (3.8), we then see that $p\in\mathrm{Sing}(f)$ if and only if $g^1_{1\dots1}(p)=\cdots=g^n_{1\dots1}(p)=0$ for any adapted chart, and thus both the strictly fixed points of [A2] and the singular points of [BT], [Br] are singular in our case as well.

As we shall see later on, our index theorems will need a section of $TS\otimes(N_S^*)^{\otimes\nu_f}$; so it will be natural to assume $f$ tangential. When $f$ is not tangential but $S$ splits in $M$ we can work too.

Let $O\longrightarrow TS\overset{\iota}{\longrightarrow}TM|_S\overset{\pi}{\longrightarrow}N_S\longrightarrow O$ be the usual extension. Then we can associate to any splitting morphism $\sigma\colon N_S\to TM|_S$ a morphism $\sigma'\colon TM|_S\to TS$ such that $\sigma'\circ\iota=\mathrm{id}_{TS}$, by $\sigma'=\iota^{-1}\circ(\sigma\circ\pi-\mathrm{id}_{TM|_S})$. Conversely, if there is a morphism $\sigma'\colon TM|_S\to TS$ such that $\sigma'\circ\iota=\mathrm{id}_{TS}$, we get a splitting morphism by setting $\sigma=(\pi|_{\mathrm{Ker}\,\sigma'})^{-1}$. Then

*Definition* 3.4. Let $f\in\mathrm{End}(M,S)$, $f\not\equiv\mathrm{id}_M$, and assume that $S$ splits in $M$. Choose a splitting morphism $\sigma\colon N_S\to TM|_S$ and let $\sigma'\colon TM|_S\to TS$

be the induced morphism. We set

$$H_{\sigma,f} = (\sigma' \otimes \mathrm{id}) \circ X_f \in H^0\big(S, \mathcal{T}_S \otimes (N_S^*)^{\otimes \nu_f}\big).$$

Since the differential of $f$ induces a morphism from $N_S$ into itself, we have a dual morphism $(df)^*\colon N_S^* \to N_S^*$. Then if $\nu_f = 1$ we also set

$$H^1_{\sigma,f} = \big(\mathrm{id} \otimes (df)^*\big) \circ H_{\sigma,f} \in H^0\big(S, \mathcal{T}_S \otimes N_S^*\big).$$

*Remark* 3.6. We defined $H^1_{\sigma,f}$ only for $\nu_f = 1$ because when $\nu_f > 1$ one has $(df)^* = \mathrm{id}$. On the other hand, when $\nu_f = 1$ one has $(df)^* = \mathrm{id}$ if and only if $f$ is tangential. Finally, we have $X_f \equiv H_{\sigma,f}$ if and only if $f$ is tangential, and $H_{\sigma,f} \equiv O$ if and only if $\Xi_f \subseteq \mathrm{Im}\,\sigma = \mathrm{Ker}\,\sigma'$.

Finally, if $(U, z)$ is a chart in an atlas adapted to the splitting $\sigma$, locally we have

$$H_{\sigma,f} = g^p_{s_1 \ldots s_{\nu_f}}|_S \, \frac{\partial}{\partial z^p} \otimes \omega^{s_1} \otimes \cdots \otimes \omega^{s_{\nu_f}},$$

and, if $\nu_f = 1$,

$$H^1_{\sigma,f} = (\delta^s_r + g^s_r) g^p_s|_S \, \frac{\partial}{\partial z^p} \otimes \omega^r.$$

## 4. Local extensions

As we have already remarked, while $X_f$ is well-defined, its extension $\mathcal{X}_f$ in general is not. However, we shall now derive formulas showing how to control the ambiguities in the definition of $\mathcal{X}_f$, at least in the cases that interest us most.

In this section we assume $m = 1$, i.e., that $S$ has codimension one in $M$. To simplify notation we shall write $g^j$ for $g^j_{1\ldots 1}$ and $a$ for $a^1_1$. We shall also use the following notation:

- $T_1$ will denote any sum of terms of the form $g \frac{\partial}{\partial z^p} \otimes dz^{h_1} \otimes \cdots \otimes dz^{h_{\nu_f}}$ with $g \in \mathcal{I}_S$;
- $R_k$ will denote any local section with coefficients in $\mathcal{I}_S^k$.

For instance, if $(U, z)$ and $(\hat{U}, \hat{z})$ are two charts adapted to $S$,

$$\begin{aligned}(4.1)\qquad \frac{\partial}{\partial \hat{z}^h} \otimes (d\hat{z}^1)^{\otimes \nu_f} = {}& a^{\nu_f} \frac{\partial z^k}{\partial \hat{z}^h} \frac{\partial}{\partial z^k} \otimes (dz^1)^{\otimes \nu_f} \\ & + \frac{\partial z^1}{\partial \hat{z}^h} a^{\nu_f - 1} z^1 \sum_{\ell=1}^{\nu_f} \frac{\partial a}{\partial z^{j_\ell}} \frac{\partial}{\partial z^1} \otimes dz^1 \otimes \cdots \\ & \cdots \otimes dz^{j_\ell} \otimes \cdots \otimes dz^1 + T_1 + R_2,\end{aligned}$$

where

$$T_1 = \frac{\partial z^p}{\partial \hat{z}^h} a^{\nu_f - 1} z^1 \sum_{\ell=1}^{\nu_f} \frac{\partial a}{\partial z^{j_\ell}} \frac{\partial}{\partial z^p} \otimes dz^1 \otimes \cdots \otimes dz^{j_\ell} \otimes \cdots \otimes dz^1.$$

Assume now that $f$ is tangential, and let $(U, z)$ be a chart adapted to $S$. We know that $f^1 - z^1 \in \mathcal{I}_S^{\nu_f+1}$, and thus we can write

$$f^1 - z^1 = h^1 (z^1)^{\nu_f+1},$$

where $h^1$ is uniquely determined. Now, if $(\hat{U}, \hat{z})$ is another chart adapted to $S$ then

$$\begin{aligned} a^{\nu_f+1} \hat{h}^1 (z^1)^{\nu_f+1} &= \hat{f}^1 - \hat{z}^1 = (a \circ f) f^1 - a z^1 \\ &= a(f^1 - z^1) + (a \circ f - a) z^1 + (a \circ f - a)(f^1 - z^1) \\ &= a(f^1 - z^1) + \frac{\partial a}{\partial z^p} (f^p - z^p) z^1 + R_{\nu_f+2} \\ &= \left[ a h^1 + \frac{\partial a}{\partial z^p} g^p \right] (z^1)^{\nu_f+1} + R_{\nu_f+2}. \end{aligned}$$

Therefore

$$a^{\nu_f+1} \hat{h}^1 = a h^1 + \frac{\partial a}{\partial z^p} g^p + R_1. \tag{4.2}$$

Since $g^1 = h^1 z^1$ we then get

$$a^{\nu_f} \hat{g}^1 = a g^1 + \frac{\partial a}{\partial z^p} g^p z^1 + R_2, \tag{4.3}$$

which generalizes (3.6) when $f$ is tangential and $m = 1$.

Putting (4.3), (3.6) and (4.1) into (3.2) we then get

Lemma 4.1. *Let $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$. Assume that $f$ is tangential, and that $S$ has codimension* 1. *Let $(\hat{U}, \hat{z})$ and $(U, z)$ be two charts about $p \in S$ adapted to $S$, and let $\hat{\mathcal{X}}_f$, $\mathcal{X}_f$ be given by* (3.2) *in the respective coordinates. Then*

$$\hat{\mathcal{X}}_f = \mathcal{X}_f + T_1 + R_2.$$

When $S$ is comfortably embedded in $M$ and of codimension one we shall also need nice local extensions of $H_{\sigma,f}$ and $H^1_{\sigma,f}$, and to know how they behave under change of (comfortable) coordinates.

*Definition* 4.1. Let $S$ be comfortably embedded in $M$ and of codimension 1, and take $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$. Let $(U, z)$ be a chart in a comfortable atlas, and set $b^1(z) = g^1(O, z'')$; notice that $f$ is tangential if and only if $b^1 \equiv O$. Write $g^1 = b^1 + h^1 z^1$ for a well-defined holomorphic function $h^1$; then set

$$\mathcal{H}_{\sigma,f} = h^1 z^1 \frac{\partial}{\partial z^1} \otimes (dz^1)^{\otimes \nu_f} + g^p \frac{\partial}{\partial z^p} \otimes (dz^1)^{\otimes \nu_f}, \tag{4.4}$$

and if $\nu_f = 1$ set

$$\mathcal{H}^1_{\sigma,f} = h^1 z^1 \frac{\partial}{\partial z^1} \otimes dz^1 + g^p(1+b^1)\frac{\partial}{\partial z^p} \otimes dz^1. \tag{4.5}$$

Notice that $\mathcal{H}_{\sigma,f}$ (respectively, $\mathcal{H}^1_{\sigma,f}$) restricted to $S$ yields $H_{\sigma,f}$ (respectively, $H^1_{\sigma,f}$).

Proposition 4.2. *Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$. Assume that $S$ is comfortably embedded in $M$, and of codimension one. Fix a comfortable atlas $\mathfrak{U}$, and let $(U,z)$, $(\hat{U},\hat{z})$ be two charts in $\mathfrak{U}$ about $p \in S$. Then if $\nu_f = 1$,*

$$\hat{\mathcal{H}}^1_{\sigma,f} = \mathcal{H}^1_{\sigma,f} + T_1 + R_2, \tag{4.6}$$

*while if $\nu_f > 1$,*

$$\hat{\mathcal{H}}_{\sigma,f} = \mathcal{H}_{\sigma,f} + T_1 + R_2, \tag{4.7}$$

*where $T_1 = T_1^o + T_1^1$ with*

$$T_1^o = \frac{1}{a} g^q z^1 \sum_{\ell=1}^{\nu_f} \frac{\partial a}{\partial z^{p_\ell}} \frac{\partial}{\partial z^q} \otimes dz^1 \otimes \cdots \otimes dz^{p_\ell} \otimes \cdots \otimes dz^1,$$
$$T_1^1 = -a g^1 \frac{\partial z^q}{\partial \hat{z}^1} \frac{\partial}{\partial z^q} \otimes (dz^1)^{\otimes \nu_f}.$$

*Proof.* First of all, from (3.7), $a^{\nu_f}\hat{b}^1 = ab^1 (\mathrm{mod}\, \mathcal{I}_S)$. But since we are using a comfortable atlas we get

$$\frac{\partial(a^{\nu_f}\hat{b}^1 - ab^1)}{\partial z^1} = (\nu_f a^{\nu_f - 1}\hat{b}^1 - b^1)\frac{\partial a}{\partial z^1} + R_1 = R_1,$$

and thus

$$a^{\nu_f}\hat{b}^1 = ab^1 \pmod{\mathcal{I}_S^2}. \tag{4.8}$$

If $\nu_f > 1$ then by (3.7) and (4.8),

$$a^{\nu_f}\hat{h}^1\hat{z}^1 = (ah^1 + \frac{\partial a}{\partial z^p} g^p) z^1 \pmod{\mathcal{I}_S^2},$$

which implies

$$a^{\nu_f+1}\hat{h}^1 = ah^1 + \frac{\partial a}{\partial z^p} g^p \pmod{\mathcal{I}_S}. \tag{4.9}$$

If $\nu_f = 1$, using (2.4) we can write

$$\begin{aligned}\hat{b}^1\hat{z}^1 + \hat{h}^1(\hat{z}^1)^2 &= \hat{f}^1 - \hat{z}^1 \\ &= \frac{\partial \hat{z}^1}{\partial z^j}(f^j - z^j) + \frac{1}{2}\frac{\partial^2 \hat{z}^1}{\partial z^h \partial z^k}(f^h - z^h)(f^k - z^k) + R_3 \\ &= ab^1 z^1 + \left[ah^1 + \frac{\partial a}{\partial z^p} g^p (1+b^1)\right](z^1)^2 + R_3,\end{aligned}$$

and by (4.8),

$$a^2\hat{h}^1 = ah^1 + \frac{\partial a}{\partial z^p} g^p(1+b^1) \pmod{\mathcal{I}_S}. \tag{4.10}$$

So if we compute $\hat{\mathcal{H}}_{\sigma,f}$ for $\nu_f > 1$ (respectively, $\hat{\mathcal{H}}^1_{\sigma,f}$ for $\nu_f = 1$) using (3.7), (4.1) and (4.9) (respectively, (3.7), (4.1), (4.8) and (4.10)), we get the assertions. □

## 5. Holomorphic actions

The index theorems to be discussed depend on actions of vector bundles. This concept was introduced by Baum and Bott in [BB], and later generalized in [CL], [LS], [LS2] and [Su]. Let us recall here the relevant definitions.

Let $S$ again be a submanifold of codimension $m$ in an $n$-dimensional complex manifold $M$, and let $\pi_F\colon F \to S$ be a holomorphic vector bundle on $S$. We shall denote by $\mathcal{F}$ the sheaf of germs of holomorphic sections of $F$, by $\mathcal{T}_S$ the sheaf of germs of holomorphic sections of $TS$, and by $\Omega^1_S$ (respectively, $\Omega^1_M$) the sheaf of holomorphic 1-forms on $S$ (respectively, on $M$).

A section $X$ of $\mathcal{T}_S \otimes \mathcal{F}^*$ (or, equivalently, a holomorphic section of $TS \otimes F^*$) can be interpreted as a morphism $X\colon \mathcal{F} \to \mathcal{T}_S$. Therefore it induces a derivation $X^\#\colon \mathcal{O}_S \to \mathcal{F}^*$ by setting

$$X^\#(g)(u) = X(u)(g) \tag{5.1}$$

for any $p \in S$, $g \in \mathcal{O}_{S,p}$ and $u \in \mathcal{F}_p$. If $\{f_1^*, \dots, f_k^*\}$ is a local frame for $F^*$ about $p$, and $X$ is locally given by $X = \sum_j v_j \otimes f_j^*$, then

$$X^\#(g) = \sum_j v_j(g) f_j^*. \tag{5.2}$$

Notice that if $X^*\colon \Omega^1_S \to \mathcal{F}^*$ denotes the dual morphism of $X\colon \mathcal{F} \to \mathcal{T}_S$, by definition we have

$$X^*(\omega)(u) = \omega\big(X(u)\big)$$

for any $p \in S$, $\omega \in (\Omega^1_S)_p$ and $u \in \mathcal{F}_p$, and so

$$X^\#(g) = X^*(dg).$$

*Definition* 5.1. Let $\pi_E\colon E \to S$ be another holomorphic vector bundle on $S$, and denote by $\mathcal{E}$ the sheaf of germs of holomorphic sections of $E$. Let $X$ be a section of $\mathcal{T}_S \otimes \mathcal{F}^*$. A *holomorphic action of $F$ on $E$ along $X$* (or an *$X$-connection* on $E$) is a $\mathbb{C}$-linear map $\tilde{X}\colon \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$ such that

$$\tilde{X}(gs) = X^\#(g) \otimes s + g\tilde{X}(s) \tag{5.3}$$

for any $g \in \mathcal{O}_S$ and $s \in \mathcal{E}$.

*Example* 5.1. If $F = TS$, and the section $X$ is the identity $\mathrm{id}\colon TS \to TS$, then $X^{\#}(g) = dg$, and a holomorphic action of $TS$ on $E$ along $X$ is just a (1,0)-connection on $E$.

*Definition* 5.2. A point $p \in S$ is a *singularity* of a holomorphic section $X$ of $\mathcal{T}_S \otimes \mathcal{F}^*$ if the induced map $X_p\colon F_p \to T_pS$ is not injective. The set of singular points of $X$ will be denoted by $\mathrm{Sing}(X)$, and we shall set $S^0 = S \setminus \mathrm{Sing}(X)$ and $\Xi_X = X(F|_{S^0}) \subseteq TS^0$. Notice that $\Xi_X$ is a holomorphic subbundle of $TS^0$.

The canonical section previously introduced suggests the following definition:

*Definition* 5.3. A *Camacho-Sad action* on $S$ is a holomorphic action of $N_S^{\otimes \nu}$ on $N_S$ along a section $X$ of $\mathcal{T}_S \otimes (N_S^{\otimes \nu})^*$, for a suitable $\nu \geq 1$.

*Remark* 5.1. The rationale behind the name is the following: as we shall see, the index theorem in [A2] is induced by a holomorphic action of $N_S^{\otimes \nu_f}$ on $N_S$ along $X_f$ when $f$ is tangential, and this index theorem was inspired by the Camacho-Sad index theorem [CS].

Let us describe a way to get Camacho-Sad actions. Let $\pi\colon TM|_S \to N_S$ be the canonical projection; we shall use the same symbol for all other projections naturally induced by it. Let $X$ be any global section of $TS \otimes (N_S^{\otimes \nu})^*$. Then we might try to define an action $\tilde{X}\colon \mathcal{N}_S \to (\mathcal{N}_S^{\otimes \nu})^* \otimes \mathcal{N}_S = \mathrm{Hom}(\mathcal{N}_S^{\otimes \nu}, \mathcal{N}_S)$ by setting

$$\tilde{X}(s)(u) = \pi([\mathcal{X}(\tilde{u}), \tilde{s}]|_S) \tag{5.4}$$

for any $s \in \mathcal{N}_S$ and $u \in \mathcal{N}_S^{\otimes \nu}$, where: $\tilde{s}$ is any element in $\mathcal{T}_M|_S$ such that $\pi(\tilde{s}|_S) = s$; $\tilde{u}$ is any element in $\mathcal{T}_M|_S^{\otimes \nu_f}$ such that $\pi(\tilde{u}|_S) = u$; and $\mathcal{X}$ is a *suitably chosen* local section of $\mathcal{T}_M \otimes (\Omega^1_M)^{\otimes \nu}$ that restricted to $S$ induces $X$.

Surprisingly enough, we can make this definition work in the cases interesting to us:

THEOREM 5.1. *Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$, be given. Assume that $S$ has codimension one in $M$ and that*

(a) *$f$ is tangential to $S$, or that*

(b) *$S$ is comfortably embedded into $M$.*

*Then we can use* (5.4) *to define a Camacho-Sad action on $S$ along $X_f$ in case* (a), *along $H_{\sigma,f}$ in case* (b) *when $\nu_f > 1$, and along $H^1_{\sigma,f}$ in case* (b) *when $\nu_f = 1$.*

*Proof.* We shall denote by $X$ the section $X_f$, $H_{\sigma,f}$ or $H^1_{\sigma,f}$ depending on the case we are considering. Let $\mathfrak{U}$ be an atlas adapted to $S$, comfortable and adapted to the splitting morphism $\sigma$ in case (b), and let $\mathcal{X}$ be the local extension of $X$ defined in a chart belonging to $\mathfrak{U}$ by Definition 3.1 (respectively, Definition 4.1). We first prove that the right-hand side of (5.4) does not depend on the chart chosen. Take $(U, z)$, $(\hat{U}, \hat{z}) \in \mathfrak{U}$ to be local charts about $p \in S$. Using Lemma 4.1 and Proposition 4.2 we get

$$[\hat{\mathcal{X}}(\tilde{u}), \tilde{s}] = [(\mathcal{X} + T_1 + R_2)(\tilde{u}), \tilde{s}] = [\mathcal{X}(\tilde{u}) + T_1 + R_2, \tilde{s}] = [\mathcal{X}(\tilde{u}), \tilde{s}] + T_0 + R_1,$$

where $T_0$ represents a local section of $TM$ that restricted to $S$ is tangent to it. Thus

$$\pi\big([\hat{\mathcal{X}}(\tilde{u}), \tilde{s}]|_S\big) = \pi\big([\mathcal{X}(\tilde{u}), \tilde{s}]|_S\big),$$

as desired.

We must now show that the right-hand side of (5.4) does not depend on the extensions of $s$ and $u$ chosen. If $\tilde{s}'$ and $\tilde{u}'$ are other extensions of $s$ and $u$ respectively, we have $(\tilde{s}' - \tilde{s})|_S = T_0$, while $(\tilde{u}' - \tilde{u})|_S$ is a sum of terms of the form $V_1 \otimes \cdots \otimes V_{\nu_f}$ with at least one $V_\ell$ tangent to $S$. Therefore $\mathcal{X}(\tilde{u}' - \tilde{u})|_S = O$ and

$$\begin{aligned}[\mathcal{X}(\tilde{u}'), \tilde{s}']|_S = {} & [\mathcal{X}(\tilde{u}), \tilde{s}]|_S + [\mathcal{X}(\tilde{u}), \tilde{s}' - \tilde{s}]|_S + [\mathcal{X}(\tilde{u}' - \tilde{u}), \tilde{s}]|_S \\ & + [\mathcal{X}(\tilde{u}' - \tilde{u}), \tilde{s}' - \tilde{s}]|_S = [\mathcal{X}(\tilde{u}), \tilde{s}]|_S + T_0,\end{aligned}$$

so that $\pi\big([\mathcal{X}(\tilde{u}'), \tilde{s}']|_S\big) = \pi\big([\mathcal{X}_f(\tilde{u}), \tilde{s}]|_S\big)$, as wanted.

We are left to show that $\tilde{X}$ is actually an action. Take $g \in \mathcal{O}_S$, and let $\tilde{g} \in \mathcal{O}_M|_S$ be any extension. First of all,

$$\tilde{X}(s)(gu) = \pi\big([\mathcal{X}(\tilde{g}\tilde{u}), \tilde{s}]|_S\big) = g\tilde{X}(s)(u) - \tilde{s}(\tilde{g})|_S \pi\big(X(u)\big) = g\tilde{X}(s)(u),$$

and so $\tilde{X}(s)$ is a morphism. Finally, (5.1) yields

$$\mathcal{X}(\tilde{u})(\tilde{g})|_S = X^{\#}(g)(u),$$

and so

$$\tilde{X}(gs)(u) = \pi\big([\mathcal{X}(\tilde{u}), \tilde{g}\tilde{s}]|_S\big) = g\tilde{X}(s)(u) + \mathcal{X}(\tilde{u})(\tilde{g})|_S\, s = g\tilde{X}(s)(u) + X^{\#}(g)(u)s,$$

and we are done. □

*Remark* 5.2. If $\nu_f = 1$ and $f$ is not tangential then (5.4) with $\mathcal{X} = \mathcal{H}_{\sigma,f}$ does not define an action. This is the reason why we introduced the new section $H^1_{\sigma,f}$ and its extension $\mathcal{H}^1_{\sigma,f}$.

Later it will be useful to have an expression of $\tilde{X}_f, \tilde{H}_{\sigma,f}$ and $\tilde{H}^1_{\sigma,f}$ in local coordinates. Let then $(U, z)$ be a local chart belonging to a (comfortable, if necessary) atlas adapted to $S$, so that $\{\partial_1\}$ is a local frame for $N_S$, and $\{(\omega^1)^{\otimes \nu_f} \otimes \partial_1\}$ is a local frame for $(N_S^{\otimes \nu_f})^* \otimes N_S$. There is a holomorphic function $M_f$ such that

$$\tilde{X}_f(\partial_1)(\partial_1^{\otimes \nu_f}) = M_f \partial_1.$$

Now, recalling (3.2), we obtain

$$\tilde{X}_f(\partial_1)(\partial_1^{\otimes\nu_f}) = \pi\left(\left[\mathcal{X}_f\left((\frac{\partial}{\partial z^1})^{\otimes\nu_f}\right), \frac{\partial}{\partial z_1}\right]\Big|_S\right)$$
$$= \pi\left(\left[g^j\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^1}\right]\Big|_S\right) = -\left.\frac{\partial g^1}{\partial z^1}\right|_S \partial_1,$$

and so

$$M_f = -\left.\frac{\partial g^1}{\partial z^1}\right|_S. \tag{5.5}$$

In particular, recalling that $f$ is tangential we can write $g^1 = z^1 h^1$, and hence (5.5) yields

$$M_f = -h^1|_S. \tag{5.6}$$

Similarly, if we write $\tilde{H}_{\sigma,f}(\partial_1)(\partial_1^{\otimes\nu_f}) = M_{\sigma,f}\partial_1$ and $\tilde{H}^1_{\sigma,f}(\partial_1)(\partial_1) = M^1_{\sigma,f}\partial_1$, we obtain

$$M_{\sigma,f} = M^1_{\sigma,f} = -h^1|_S, \tag{5.7}$$

where $h^1$ is defined by $f^1 - z^1 = b^1(z^1)^{\nu_f} + h^1(z^1)^{\nu_f+1}$.

Following ideas originally due to Baum and Bott (see [BB]), we can also introduce a holomorphic action on the virtual bundle $TS - N_S^{\otimes\nu_f}$. But let us first define what we mean by a holomorphic action on such a bundle.

*Definition* 5.4. Let $S^0$ be an open dense subset of a complex manifold $S$, $F$ a vector bundle on $S$, $X \in H^0(S, \mathcal{T}_S \otimes \mathcal{F}^*)$, $W$ a vector bundle over $S^0$ and $\tilde{W}$ any extension of $W$ over $S$ in $K$-theory. Then we say that $F$ *acts holomorphically on* $\tilde{W}$ along $X$ if $F|_{S^0}$ acts holomorphically on $W$ along $X|_{S^0}$.

Let $S$ be a codimension-one submanifold of $M$ and take $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, as usual. If $f$ is tangential set $X = X_f$. If not, assume that $S$ is comfortably embedded in $M$ and set $X = H_{\sigma,f}$ or $X = H^1_{\sigma,f}$ according to the value of $\nu_f$; in this case, we shall also assume that $X \not\equiv O$. Set $S^0 = S \backslash \mathrm{Sing}(X)$, and let $\mathcal{Q}_f = \mathcal{T}_S / X(\mathcal{N}_S^{\otimes\nu_f})$. The sheaf $\mathcal{Q}_f$ is a coherent analytic sheaf which is locally free over $S^0$. The associated vector bundle (over $S^0$) is denoted by $Q_f$ and it is called the *normal bundle of* $f$. Then the virtual bundle $TS - N_S^{\otimes\nu_f}$, represented by the sheaf $\mathcal{Q}_f$, is an extension (in the sense of $K$-theory) of $Q_f$.

*Definition* 5.5. A *Baum-Bott action* on $S$ is a holomorphic action of $N_S^{\otimes\nu}$ on the virtual bundle $TS - N_S^{\otimes\nu}$ along a section $X$ of $\mathcal{T}_S \otimes N_S^{\otimes\nu}$, for a suitable $\nu \geq 1$.

THEOREM 5.2. *Let* $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, *be given. Assume that* $S$ *has codimension one in* $M$, *and that either* $f$ *is tangential to* $S$ (*and then set* $X = X_f$) *or* $S$ *is comfortably embedded into* $M$ (*and then set* $X = H_{\sigma,f}$ *or* $X = H^1_{\sigma,f}$ *according to the value of* $\nu_f$). *Assume moreover that* $X \not\equiv 0$. *Then there exists a Baum-Bott action* $\tilde{B}\colon \mathcal{Q}_f \to (\mathcal{N}_S^{\otimes\nu_f})^* \otimes \mathcal{Q}_f$ *of* $N_S^{\otimes\nu_f}$ *on* $TS - N_S^{\otimes\nu_f}$ *along* $X$ *defined by*

$$\tilde{B}(s)(u) = \pi_f([X(u), \tilde{s}]) \tag{5.8}$$

*where* $\pi_f\colon \mathcal{T}_S \to \mathcal{Q}_f$ *is the natural projection, and* $\tilde{s} \in \mathcal{T}_S$ *is any section such that* $\pi_f(\tilde{s}) = s$.

*Proof.* If $\hat{s} \in \mathcal{T}_S$ is another section such that $\pi_f(\hat{s}) = s$ we have $\hat{s} - \tilde{s} \in X(\mathcal{N}_S^{\otimes\nu_f})$; hence $\pi_f([X(u), \hat{s}-\tilde{s}]) = O$, and (5.8) does not depend on the choice of $\tilde{s}$. Finally, one can easily check that $\tilde{B}$ is a holomorphic action on $S^0$. $\square$

*Remark* 5.3. Since $S$ has codimension one, $X\colon N_S^{\otimes\nu_f} \to TS$ yields a (possibly singular) holomorphic foliation on $S$, and the previous action coincides with the one used in [BB] for the case of foliations.

We can also define a third holomorphic action, on the virtual bundle $TM|_S - N_S^{\otimes\nu_f}$. Assume that $f$ is tangential, and let $S^0 = S \setminus \mathrm{Sing}(X_f)$, as before. Then the sheaf $\mathcal{W}_f = \mathcal{T}_{M,S}/X_f(\mathcal{N}_S^{\otimes\nu_f})$ is a coherent analytic sheaf, locally free over $S^0$; let $W_f = TM|_{S^0}/X_f(N_S^{\otimes\nu_f}|_{S^0})$ be the associated vector bundle over $S^0$. Then the virtual bundle $TM|_S - N_S^{\otimes\nu_f}$, represented by the sheaf $\mathcal{W}_f$, is an extension (in the sense of $K$-theory) of $W_f$.

*Definition* 5.6. A *Lehmann-Suwa action* on $S$ is a holomorphic action of $N_S^{\otimes\nu}$ on $TM|_S - N_S^{\otimes\nu}$ along a section $X$ of $\mathcal{T}_S \otimes N_S^{\otimes\nu}$, for a suitable $\nu \geq 1$.

Again, the name is chosen to honor the ones who first discovered the analogous action for holomorphic foliations in any dimension; see [LS], [LS2] (and [KS] for dimension two).

To present an example of such an action we first need a definition.

*Definition* 5.7. Let $S$ be a codimension-one, comfortably embedded submanifold of $M$, and choose a comfortable atlas $\mathfrak{U}$ adapted to a splitting morphism $\sigma\colon N_S \to TM|_S$. If $v \in (\mathcal{N}_S^{\otimes\nu})_p$ and $(U, \varphi) \in \mathfrak{U}$ is a chart about $p \in S$, we can write $v = \lambda(z'')\partial_1^{\otimes\nu}$ for a suitable $\lambda \in \mathcal{O}(U \cap S)$. Then the *local extension of* $v$ *along the fibers of* $\sigma$ is the local section $\tilde{v} = \lambda(z'')(\partial/\partial z^1)^{\otimes\nu} \in (\mathcal{T}_M|_S^{\otimes\nu})_p$.

If $(\hat{U}, \hat{z})$ is another chart in $\mathfrak{U}$ about $p$, and $v \in (\mathcal{N}_S^{\otimes\nu})_p$, we can also write $v = \hat{\lambda}\hat{\partial}_1^{\otimes\nu}$, and we clearly have $\hat{\lambda} = (a|_S)^\nu\lambda$. But since $S$ is comfortably embedded in $M$ we also have

$$\left.\frac{\partial(\hat{\lambda} - a^\nu \lambda)}{\partial z^1}\right|_S \equiv 0,$$

and thus

$$a^\nu \lambda = \hat{\lambda} + R_2.$$

Therefore if $\hat{v}$ denotes the local extension of $v$ along the fibers of $\sigma$ in the chart $(\hat{U}, \hat{\varphi})$ we have

$$\hat{v} = \hat{\lambda} \left(\frac{\partial}{\partial \hat{z}^1}\right)^{\otimes \nu} = a^\nu \lambda \frac{\partial z^{h_1}}{\partial \hat{z}^1} \cdots \frac{\partial z^{h_\nu}}{\partial \hat{z}^1} \frac{\partial}{\partial z^{h_1}} \otimes \cdots \otimes \frac{\partial}{\partial z^{h_\nu}} + R_2 = \tilde{v} + T_1 + R_2, \tag{5.9}$$

where

$$T_1 = a\lambda \sum_{\ell=1}^{\nu} \frac{\partial z^{p_\ell}}{\partial \hat{z}^1} \frac{\partial}{\partial z^1} \otimes \cdots \otimes \frac{\partial}{\partial z^{p_\ell}} \otimes \cdots \otimes \frac{\partial}{\partial z^1}.$$

Hence:

THEOREM 5.3. *Let $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, be given. Assume that $S$ is of codimension one and comfortably embedded in $M$, and that $f$ is tangential with $\nu_f > 1$. Let $\rho_f \colon \mathcal{T}_{M,S} \to \mathcal{W}_f$ be the natural projection. Then a Lehmann-Suwa action $\tilde{V} \colon \mathcal{W}_f \to (\mathcal{N}_S^{\otimes \nu_f})^* \otimes \mathcal{W}_f$ of $N_S^{\otimes \nu_f}$ on $TM|_S - N_S^{\otimes \nu_f}$ may be defined along $X_f$ by setting*

$$\tilde{V}(s)(v) = \rho_f([\mathcal{X}_f(\tilde{v}), \tilde{s}]|_S), \tag{5.10}$$

*for $s \in \mathcal{W}_f$ and $v \in \mathcal{N}_S^{\otimes \nu}$, where $\tilde{s}$ is any element in $\mathcal{T}_M|_S$ such that $\rho_f(\tilde{s}|_S) = s$, and $\tilde{v} \in \mathcal{T}_M|_S^{\otimes \nu_f}$ is an extension of $v$ constant along the fibers of a splitting morphism $\sigma$.*

*Proof.* Since $\mathcal{X}_f(\tilde{v})|_S \in \mathcal{T}_S$ then clearly (5.10) does not depend on the extension $\tilde{s}$ chosen. Using (5.9) and (4.7), since $f$ tangential implies $\mathcal{X}_f = \mathcal{H}_{\sigma,f}$ and $T_1^1 = R_2$, we have

$$[\hat{\mathcal{X}}_f(\hat{v}), \tilde{s}] = [(\mathcal{X}_f + T_1^o + R_2)(\tilde{v} + T_1 + R_2), \tilde{s}] = [\mathcal{X}_f(\tilde{v}), \tilde{s}] + R_1,$$

and therefore (5.10) does not depend on the comfortable coordinates chosen to define it. Finally, arguing as in Theorem 5.1 we can show that $\tilde{V}$ actually is a holomorphic action, and we are done. □

## 6. Index theorems for hypersurfaces

Let $S$ be a compact, globally irreducible, possibly singular hypersurface in a complex manifold $M$, and set $S' = S \setminus \mathrm{Sing}(S)$. Given the following data:

(a) a line bundle $F$ over $S'$;

(b) a holomorphic section $X$ of $TS' \otimes F^*$;

(c) a vector bundle $E$ defined on $M$;

(d) a holomorphic action $\tilde{X}$ of $F|_{S'}$ on $E|_{S'}$ along $X$;

we can recover a partial connection (in the sense of Bott) on $E$ restricted to $S^0 = S' \setminus \mathrm{Sing}(X)$ as follows: since, by definition of $S^0$, the dual map $X^*\colon \Xi_X^* \to F^*|_{S^0}$ is an isomorphism, we can define a partial connection (in the sense of Bott [Bo]) $D\colon \Xi_X \times H^0(S^0, E|_{S^0}) \to H^0(S^0, E|_{S^0})$ by setting

$$D_v(s) = (X^* \otimes \mathrm{id})^{-1}\big(\tilde{X}(s)\big)(v)$$

for $p \in S^0$, $v \in (\Xi_X)_p$ and $s \in H^0(S^0, E|_{S^0})$. Furthermore, we can always extend this partial connection $D$ to a $(1,0)$-connection on $E|_{S^0}$, for instance by using a partition of unity (see, e.g., [BB]). Any such connection (which is a $\Xi_X$-*connection* in the terminology of [Bo], [Su]) will be said to be *induced* by the holomorphic action $\tilde{X}$.

We can then apply the general theory developed by Lehmann and Suwa for foliations (see in particular Theorem $1'$ and Proposition 4 of [LS], as well as [Su, Th. VI.4.8]) to get the following:

THEOREM 6.1. *Let $S$ be a compact, globally irreducible, possibly singular hypersurface in an $n$-dimensional complex manifold $M$, and set $S' = S \setminus \mathrm{Sing}(S)$. Let $F$ be a line bundle over $S'$ admitting an extension to $M$, and $X$ a holomorphic section of $TS' \otimes F^*$. Set $S^0 = S' \setminus \mathrm{Sing}(X)$, and let $\mathrm{Sing}(S) \cup \mathrm{Sing}(X) = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of $\mathrm{Sing}(S) \cup \mathrm{Sing}(X)$ in connected components. Finally, let $E$ be a vector bundle defined on $M$. Then for any holomorphic action $\tilde{X}$ of $F|_{S'}$ on $E|_{S'}$ along $X$ and any homogeneous symmetric polynomial $\varphi$ of degree $n-1$, there are complex numbers $\mathrm{Res}_\varphi(\tilde{X}, E, \Sigma_\lambda) \in \mathbb{C}$, depending only on the local behavior of $\tilde{X}$ and $E$ near $\Sigma_\lambda$, such that*

$$\sum_\lambda \mathrm{Res}_\varphi(\tilde{X}, E, \Sigma_\lambda) = \int_S \varphi(E),$$

*where $\varphi(E)$ is the evaluation of $\varphi$ on the Chern classes of $E$.*

Recalling the results of the previous section, we then get the following index theorem for holomorphic self-maps:

THEOREM 6.2. *Let $S$ be a compact, globally irreducible, possibly singular hypersurface in an $n$-dimensional complex manifold $M$. Let $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, be given. Assume that*

(a) *$f$ is tangential to $S$, and $X = X_f$, or that*

(b) *$S^0 = S \setminus \big(\mathrm{Sing}(S) \cup \mathrm{Sing}(f)\big)$ is comfortably embedded into $M$, and $X = H_{\sigma,f}$ if $\nu_f > 1$, or $X = H^1_{\sigma,f}$ if $\nu_f = 1$.*

*Assume moreover* $X \not\equiv O$. *Let* $\mathrm{Sing}(S) \cup \mathrm{Sing}(X) = \bigcup_\lambda \Sigma_\lambda$ *be the decomposition of* $\mathrm{Sing}(S) \cup \mathrm{Sing}(X)$ *in connected components. Finally, let* $[S]$ *be the line bundle on* $M$ *associated to the divisor* $S$. *Then there exist complex numbers* $\mathrm{Res}(X, S, \Sigma_\lambda) \in \mathbb{C}$, *depending only on the local behavior of* $X$ *and* $[S]$ *near* $\Sigma_\lambda$, *such that*

$$\sum_\lambda \mathrm{Res}(X, S, \Sigma_\lambda) = \int_S c_1^{n-1}([S]).$$

*Proof.* By Theorem 5.1 we have a Camacho-Sad action on $S$ along $X$ on $N_{S^0}$. Since $[S]$ is an extension to $M$ of $N_{S^0}$, we can apply Theorem 6.1. □

*Remark* 6.1. If $M$ has dimension two, and $S$ has at least one singularity or $X_f$ has at least one zero, then $S' \setminus \mathrm{Sing}(f)$ is *always* comfortably embedded in $M$. Indeed, it is an open Riemann surface; so $H^1(S' \setminus \mathrm{Sing}(f), \mathcal{F}) = O$ for any coherent analytic sheaf $\mathcal{F}$, and the result follows from Proposition 2.1 and Theorem 2.2.

In a similar way, applying [Su, Th. IV.5.6], Theorem 5.3, and recalling that $\varphi(H - L) = \varphi(H \otimes L^*)$ for any vector bundle $H$, line bundle $L$ and homogeneous symmetric polynomial $\varphi$, we get

Theorem 6.3. *Let* $S$ *be a compact, globally irreducible, possibly singular hypersurface in an* $n$*-dimensional complex manifold* $M$. *Let* $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, *be given. Assume that* $S' = S \setminus \mathrm{Sing}(S)$ *is comfortably embedded into* $M$, *and that* $f$ *is tangential to* $S$ *with* $\nu_f > 1$. *Let* $\mathrm{Sing}(S) \cup \mathrm{Sing}(X_f) = \bigcup_\lambda \Sigma_\lambda$ *be the decomposition of* $\mathrm{Sing}(S) \cup \mathrm{Sing}(X_f)$ *in connected components. Finally, let* $[S]$ *be the line bundle on* $M$ *associated to the divisor* $S$. *Then for any homogeneous symmetric polynomial* $\varphi$ *of degree* $n-1$ *there exist complex numbers* $\mathrm{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \Sigma_\lambda) \in \mathbb{C}$, *depending only on the local behavior of* $X_f$ *and* $TM|_S - [S]^{\otimes \nu_f}$ *near* $\Sigma_\lambda$, *such that*

$$\sum_\lambda \mathrm{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \Sigma_\lambda) = \int_S \varphi\big(TM|_S \otimes ([S]^*)^{\otimes \nu_f}\big).$$

Finally, applying the Baum-Bott index theorem (see [Su, Th. III.7.6]) and Theorem 5.2 we get

Theorem 6.4. *Let* $S$ *be a compact, globally irreducible, smooth complex hypersurface in an* $n$*-dimensional complex manifold* $M$. *Let* $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, *be given. Assume that*

(a) $f$ *is tangential to* $S$, *and* $X = X_f$, *or that*

(b) $S^0 = S \setminus \mathrm{Sing}(f)$ *is comfortably embedded into* $M$, *and* $X = H_{\sigma,f}$ *if* $\nu_f > 1$, *or* $X = H^1_{\sigma,f}$ *if* $\nu_f = 1$.

*Assume moreover* $X \not\equiv O$. *Let* $\mathrm{Sing}(X) = \bigcup_\lambda \Sigma_\lambda$ *be the decomposition of* $\mathrm{Sing}(X)$ *in connected components. Finally, let* $[S]$ *be the line bundle on* $M$ *associated to the divisor* $S$. *Then for any homogeneous symmetric polynomial* $\varphi$ *of degree* $n-1$ *there exist complex numbers* $\mathrm{Res}_\varphi(X, TS - [S]^{\otimes \nu_f}, \Sigma_\lambda) \in \mathbb{C}$, *depending only on the local behavior of* $X$ *and* $TS - [S]^{\otimes \nu_f}$ *near* $\Sigma_\lambda$, *such that*

$$\sum_\lambda \mathrm{Res}_\varphi(X, TS - [S]^{\otimes \nu_f}, \Sigma_\lambda) = \int_S \varphi\big(TS \otimes ([S]^*)^{\otimes \nu_f}\big).$$

Thus, we have recovered three main index theorems of foliation theory in the setting of holomorphic self-maps fixing pointwise a hypersurface.

Clearly, these index theorems are as useful as the formulas for the computation of the residues are explicit; the rest of this section is devoted to deriving such formulas in many important cases.

Let us first describe the general way these residues are defined in Lehmann-Suwa theory. Assume the hypotheses of Theorem 6.1. Let $\tilde{U}_0$ be a tubular neighborhood of $S^0$ in $M$, and denote by $\rho\colon \tilde{U}_0 \to S^0$ the associated retraction. Given any connection $D$ on $E|_{S^0}$ induced by the holomorphic action $\tilde{X}$ of $F$ along $X$, set $D^0 = \rho^*(D)$. Next, choose an open set $\tilde{U}_\lambda \subset M$ such that $\tilde{U}_\lambda \cap \big(\mathrm{Sing}(S) \cup \mathrm{Sing}(X)\big) = \Sigma_\lambda$, and a compact real $2n$-dimensional manifold $\tilde{R}_\lambda \subset \tilde{U}_\lambda$ with $C^\infty$ boundary containing $\Sigma_\lambda$ in its interior and such that $\partial\tilde{R}_\lambda$ intersects $S$ transversally. Let $D^\lambda$ be any connection on $E|_{\tilde{U}_\lambda}$, and denote by $B\big(\varphi(D^0), \varphi(D^\lambda)\big)$ the Bott difference form of $\varphi(D^0)$ and $\varphi(D^\lambda)$ on $\tilde{U}_0 \cap \tilde{U}_\lambda$. Then (see [LS] and [Su, Chap. IV])

$$\mathrm{Res}_\varphi(\tilde{X}, E, \Sigma_\lambda) = \int_{R_\lambda} \varphi(D^\lambda) - \int_{\partial R_\lambda} B\big(\varphi(D^0), \varphi(D^\lambda)\big), \tag{6.1}$$

where $R_\lambda = \tilde{R}_\lambda \cap S$. A similar formula holds for virtual vector bundles too; see again [Su, Chap. IV].

*Remark* 6.2. When $\Sigma_\lambda = \{x_\lambda\}$ is an isolated singularity of $S$, the second integral in (6.1) is taken on the link of $x_\lambda$ in $S$. In particular if $S$ is not irreducible at $x_k$ then the residue is the sum of several terms, one for each irreducible component of $S$ at $x_k$.

We now specialize (6.1) to our situation. Let us begin with the Camacho-Sad action: we shall compute the residues for connected components $\Sigma_\lambda$ reduced to an isolated point $x_\lambda$. Let again $[S]$ be the line bundle associated to the divisor $S$, and choose an open set $\tilde{U}_\lambda \subset M$ containing $x_\lambda$ so that $\tilde{U}_\lambda \cap \big(\mathrm{Sing}(S) \cup \mathrm{Sing}(X)\big) = \{x_\lambda\}$ and $[S]$ is trivial on $\tilde{U}_\lambda$; take as $D^\lambda$ the trivial connection for $[S]$ on $W$ with respect to some frame. In particular, then, $\varphi(D^\lambda) = O$ on $R_\lambda$. By (6.1) the residue is then obtained simply by integrating

$B\big(\varphi(D^0),\varphi(D^\lambda)\big)$ over $\partial R_\lambda$. Notice furthermore that since $[S]$ is a line bundle there is only one nontrivial $\varphi$ to consider: the $(n-1)^{\text{th}}$ power of the linear symmetric function, so that $\varphi(D)=c_1^{n-1}([S])$.

Let $\eta^j$ be a connection one-form of $D^j$, for $j=0,\ \lambda$; with respect to a suitable frame for $[S]$ we can assume that $\eta^\lambda\equiv O$. Let

$$\tilde\eta := t\eta^0+(1-t)\eta^\lambda = t\eta^0,$$

and let $\tilde K := d\tilde\eta+\tilde\eta\wedge\tilde\eta = d\tilde\eta$. From the very definition of the Bott difference form, it follows that

$$B\big(\varphi(D^0),\varphi(D^\lambda)\big)=\left(\frac{1}{2\pi i}\right)^{n-1}\int_0^1 \tilde K^{n-1}.$$

A straightforward computation shows that

$$\tilde K^{n-1}=(n-1)t^{n-2}dt\wedge\eta^0\wedge\overbrace{d\eta^0\wedge\cdots\wedge d\eta^0}^{n-2}+N,$$

where $N$ is a term not containing $dt$. Therefore

$$B\big(\varphi(D^0),\varphi(D^\lambda)\big)=\left(\frac{1}{2\pi i}\right)^{n-1}\eta^0\wedge\overbrace{d\eta^0\wedge\cdots\wedge d\eta^0}^{n-2}. \tag{6.2}$$

Assume now that $x_\lambda\in\operatorname{Sing}(X)$ and $S$ is smooth at $x_\lambda$. Up to shrinking $\tilde U_\lambda$ we may assume that $\tilde U_\lambda$ is the domain of a chart $z$ adapted to $S$ (and belonging to a comfortable atlas if necessary), so that $\{\partial_1\}$ is a local frame for $N_{S^0}$, and $\{dz^2,\dots,dz^n\}$ is a local frame for $T^*S^0$. Then any connection $D$ induced by the Camacho-Sad action is locally represented by the (1,0)-form $\eta^0$ such that $D(\partial_1)=\eta^0\otimes\partial_1$. To compute $\eta^0$, we first of all notice that $X=g^p\frac{\partial}{\partial z^p}\otimes(\omega^1)^{\otimes\nu_f}$, if $X=X_f$ or $X=H_{\sigma,f}$, and $X=(1+b^1)g^p\frac{\partial}{\partial z^p}\otimes\omega^1$ if $X=H^1_{\sigma,f}$. Then, when $X$ is $X_f$ or $H_{\sigma,f}$,

$$(X^*)^{-1}((\omega^1)^{\otimes\nu_f})=\frac{1}{g^p}\,dz^p,$$

while when $X=H^1_{\sigma,f}$,

$$(X^*)^{-1}((\omega^1)^{\otimes\nu_f})=\frac{1}{(1+b^1)g^p}\,dz^p.$$

Therefore, recalling formulas (5.6) and (5.7), we can choose $D$ so that when $X$ is $X_f$ or $H_{\sigma,f}$,

$$\eta^0=(X^*\otimes\mathrm{id})^{-1}\big(\tilde X(\partial_1)\big)=-\left.\frac{h^1}{g^p}\right|_S dz^p, \tag{6.3}$$

while when $X=H^1_{\sigma,f}$,

$$\eta^0=(X^*\otimes\mathrm{id})^{-1}\big(\tilde H^1_{\sigma,f}(\partial_1)\big)=-\left.\frac{h^1}{(1+b^1)g^p}\right|_S dz^p. \tag{6.4}$$

*Remark* 6.3. When $n = 2$ and $X = X_f$ we recover the connection form obtained in [Br]. The form $\eta$ introduced in [A2], which is the opposite of $\eta^0$, is the connection form of the dual connection on $N^*_{S^0}$, by [A2, (1.7)]. Since the definition of Chern class implicitly used in [A2] is the opposite of the one used in [Br] everything is coherent. Finally, when $n = 2$ and $X = H^1_{\sigma,f}$ we have obtained the correct multiple of the form $\eta$ introduced in [A2] when $S$ was the smooth zero section of a line bundle (notice that $1 + b^1$ is constant because $S$ is compact, and that the form $\eta$ of [A2] must be divided by $b = 1 + b^1$ to get a connection form).

Now we can take $R_1 = \{|g^p(x)| \le \varepsilon \mid p = 2, \dots, n\}$ for a suitable $\varepsilon > 0$ small enough. In particular, if we set $\Gamma = \{|g^p(x)| = \varepsilon \mid p = 2, \dots, n\} \cap S$, oriented so that $d\theta^2 \wedge \cdots \wedge d\theta^n > 0$ where $\theta^p = \arg(g^p)$, then arguing as in [L, §5] or [LS, §4] (see also [Su, pp.105–107]) we obtain

$$\text{(6.5)} \qquad \operatorname{Res}(X, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_\Gamma \frac{(h^1)^{n-1}}{g^2 \cdots g^n}\, dz^2 \wedge \cdots \wedge dz^n,$$

when $X = X_f$ or $H_{\sigma,f}$, while when $X = H^1_{\sigma,f}$ we have

$$\text{(6.6)} \quad \operatorname{Res}(H^1_{\sigma,f}, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_\Gamma \frac{(h^1)^{n-1}}{(1+b^1)^{n-1} g^2 \cdots g^n}\, dz^2 \wedge \cdots \wedge dz^n.$$

*Remark* 6.4. For $n = 2$, formulas (6.5) and (6.6) give the indices defined in [A2]. Thus, if $S$ is smooth, Theorem 6.2 implies the index theorem of [A2], because $c_1([S]) = c_1(N_S)$. In an analogous way, Lehmann and Suwa (see [L], [LS], [LS2]) proved that the Camacho-Sad index theorem also is a consequence of Theorem 6.1.

When $x_\lambda$ is an isolated singular point of $S$ the computation of the residue is more complicated, because one cannot apply directly the results in [LS] as before, for in general there is no natural extension of $\Xi_X$ and the Camacho-Sad action to $\operatorname{Sing}(S)$. However we are able to compute explicitly the index in this case too when $n = 2$, and when $n > 2$ and $f$ is tangential with $\nu_f > 1$.

If $n = 2$ we can choose local coordinates $\{(w^1, w^2)\}$ in $\tilde{U}_\lambda$ so that $S \cap \tilde{U}_\lambda = \{l(w^1, w^2) = 0\}$ for some holomorphic function $l$, and $dl \wedge dw^2 \neq 0$ on $S \cap \tilde{U}_\lambda \setminus \{x_\lambda\}$. In particular $(l, w^2)$ are local coordinates adapted to $S^0$ near $S \cap \tilde{U}_\lambda \setminus \{x_\lambda\}$ and $\frac{\partial}{\partial l}$ can be chosen as a local frame for $N_{S^0}$ on $\partial R_1$.

*Remark* 6.5. When $S^0$ is comfortably embedded in $M$ the chart $(l, w^2)$ should belong to a comfortable atlas. Studying the proofs of Propositions 2.1 and of Theorem 2.2 one sees that this is possible up to replacing $l$ by a function of the form $\hat{l} = \big(1 + c(w^2)l\big)l$, where $c$ is a holomorphic function defined on $S \cap \tilde{U}_\lambda \setminus \{x_\lambda\}$. Since to compute the residues we only need the behavior of $l$ and

$w^2$ near $\partial R_1$, it is easy to check that using $\hat{l}$ or $l$ in the following computations yields the same results. So for the sake of simplicity we shall not distinguish between $l$ and $\hat{l}$ in the sequel.

Up to shrinking $\tilde{U}_\lambda$, we can again assume that $[S]$ is trivial on $\tilde{U}_\lambda$. The function $l$ is a local generator of $\mathcal{I}_S$ on $\tilde{U}_\lambda$. Then the dual of $[l] \in \mathcal{I}_S/\mathcal{I}_S^2$, denoted by $s$, is a holomorphic frame of $[S]$ on $\tilde{U}_\lambda$ which extends the holomorphic frame $\frac{\partial}{\partial l}$ of $N_{S'}$ (see [Su, p.86]). In particular $s|_{\partial R_1} = \frac{\partial}{\partial l}$. We then choose on $[S]|_{\tilde{U}_\lambda}$ the trivial connection with respect to $s$, so that $\eta^\lambda = O$. We are left with the computation of the form $\eta^0$ near $\partial R^1$. But if $X = X_f$ or $X = H_{\sigma,f}$ we can apply (6.3) to get

$$\eta^0|_{\partial R_1} = -\left.\frac{(l\circ f - l) - b^1 l^{\nu_f}}{l\cdot(w^2\circ f - w^2)}\right|_{\partial R_1} dw^2,$$

where

$$b^1 = \left.\frac{l\circ f - l}{l^{\nu_f}}\right|_S$$

is identically zero when $f$ is tangential. On the other hand, when $X = H^1_{\sigma,f}$, applying (6.4) we get

$$\eta^0|_{\partial R_1} = -\left.\frac{(l\circ f - l) - b^1 l}{(l + (l\circ f - l))(w^2\circ f - w^2)}\right|_{\partial R_1} dw^2.$$

Hence the residue is

$$\mathrm{Res}(X, S, \{x_\lambda\}) = \frac{1}{2\pi i}\int_{\partial R_1} \left.\frac{(l\circ f - l) - b^1 l^{\nu_f}}{l\cdot(w^2\circ f - w^2)}\right|_S dw^2, \tag{6.7}$$

when $X = X_f$ or $X = H_{\sigma,f}$, while when $X = H^1_{\sigma,f}$,

$$\mathrm{Res}(H^1_{\sigma,f}, S, \{x_\lambda\}) = \frac{1}{2\pi i}\int_{\partial R_1} \left.\frac{(l\circ f - l) - b^1 l}{(l + (l\circ f - l))(w^2\circ f - w^2)}\right|_S dw^2. \tag{6.8}$$

*Remark* 6.6. When $f$ is tangential we have $b^1 \equiv 0$; therefore the formula (6.7) gives the index defined in [BT], and Theorem 6.2 implies the index theorem of [BT].

When $n > 2$, $f$ is tangential and $\nu_f > 1$, we can define a local vector field $\tilde{v}_f$ which generates the Camacho-Sad action $\tilde{X}_f$ and compute explicitly the residue even at a singular point $x_\lambda$ of $S$. To see this, assume $(w^1, \dots, w^n)$ are local coordinates in $\tilde{U}_\lambda$ so that $S \cap \tilde{U}_\lambda = \{l(w^1, \dots, w^n) = 0\}$ for some holomorphic function $l$. Define the vector field $\tilde{v}_f$ on $\tilde{U}_\lambda$ by

$$\tilde{v}_f = \frac{w^1\circ f - w^1}{l^{\nu_f}}\frac{\partial}{\partial w^1} + \dots + \frac{w^n\circ f - w^n}{l^{\nu_f}}\frac{\partial}{\partial w^n}. \tag{6.9}$$

We claim that the "holomorphic action" $\theta_{\tilde{v}_f}$ in the sense of Bott [Bo] of $\tilde{v}_f$ on $N_{S'}$ as defined in [LS, p.177] coincides with our Camacho-Sad action, and thus we can apply [LS, Th. 1] to compute the residue. To prove this we consider $W_1 = \{x \in \tilde{U}_\lambda | \frac{\partial l}{\partial w^1}(x) \neq 0\}$. On this open set we make the following change of coordinates:

$$\begin{cases} z^1 = l(w^1, \dots, w^n), & \\ z^p = w^p & \text{for } p = 2, \dots, n. \end{cases}$$

The new coordinates $(z^1, \dots, z^n)$ are adapted to $S$ on $W_1$. If $f^j = z^j + g^j (z^1)^{\nu_f}$ as usual, we have

$$w^p \circ f - w^p = g^p (z^1)^{\nu_f}, \tag{6.10}$$

and

(6.11)

$$w^1 \circ f - w^1 = \frac{\partial w^1}{\partial z^j} g^j (z^1)^{\nu_f} + R_{2\nu_f} = \left(\frac{\partial l}{\partial w^1}\right)^{-1} \left[g^1 - \frac{\partial l}{\partial w^p} g^p\right] (z^1)^{\nu_f} + R_{2\nu_f}.$$

Therefore, from (6.10) and (6.11), taking into account that $\nu_f > 1$, we get

$$\begin{aligned} \tilde{v}_f = &\left(\frac{w^1 \circ f - w^1}{(z^1)^{\nu_f}} \frac{\partial l}{\partial w^1} + \frac{w^p \circ f - w^p}{(z^1)^{\nu_f}} \frac{\partial l}{\partial w^p}\right) \frac{\partial}{\partial z^1} \\ &+ \frac{w^q \circ f - w^q}{(z^1)^{\nu_f}} \frac{\partial}{\partial z^q} = \mathcal{X}_f(\partial_1^{\otimes \nu_f}) + R_2, \end{aligned} \tag{6.12}$$

which gives the claim on $W_1$. Since the same holds on each $W_j = \{x \in \tilde{U}_\lambda | \frac{\partial l}{\partial w^j}(x) \neq 0\}$, $j = 1, \dots, n$, and $(\tilde{U}_\lambda \cap S) \setminus \{x_\lambda\} = \bigcup_j W_j$, it follows that the Bott holomorphic action induced by $\tilde{v}_f$ is the same as the Camacho-Sad action given by $\tilde{X}_f$. Thus, if we choose — as we can — the coordinates $(w^1, \dots, w^n)$ as in [LS, Th. 2], that is so that $\{l, (w^p \circ f - w^p)/l^{\nu_f}\}$ form a regular sequence at $x_\lambda$, the residue is expressed by the formula after [LS, Th. 2]. Taking into account that, since $f$ is tangential and by (6.13), the function $l$ divides $dl(\tilde{v}_f)$, we get

(6.13)

$$\mathrm{Res}(X_f, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi i}\right)^{n-1} \int_\Gamma \frac{\left[\sum_{j=1}^n \frac{\partial l}{\partial w^j}(w^j \circ f - w^j)\right]^{n-1}}{l^{n-1} \prod_{p=2}^n (w^p \circ f - w^p)} \, dw^2 \wedge \cdots \wedge dw^n,$$

where this time

$$\Gamma = \left\{ w \in \tilde{U}_\lambda \;\middle|\; \left|\frac{w^p \circ f - w^p}{l^{\nu_f}}(w)\right| = \epsilon, \; l(w) = 0 \right\},$$

for a suitable $0 < \epsilon << 1$, and $\Gamma$ is oriented as usual (in particular $\Gamma = (-1)^{[\frac{n-1}{2}]} R_{u_0}$ where $R_{u_0}$ is the set defined in [LS, Th. 2]).

Note that for $n = 2$ we recover, when $\nu_f > 1$, formula (6.7). On the other hand, if $x_\lambda$ is nonsingular for $S$, then the previous argument with $l = w^1$ works for $\nu_f = 1$ as well, and we get formula (6.5).

Summing up, we have proved the following:

THEOREM 6.5. *Let $S$ be a compact, globally irreducible, possibly singular hypersurface in an $n$-dimensional complex manifold $M$. Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$, be given. Assume that*

(a) *$f$ is tangential to $S$, and $X = X_f$, or that*

(b) *$S^0 = S \setminus \big(\mathrm{Sing}(S) \cup \mathrm{Sing}(f)\big)$ is comfortably embedded into $M$, and $X = H_{\sigma,f}$ if $\nu_f > 1$, or $X = H^1_{\sigma,f}$ if $\nu_f = 1$.*

*Assume $X \not\equiv O$. Let $x_\lambda \in S$ be an isolated point of $\mathrm{Sing}(S) \cup \mathrm{Sing}(X)$. Then the number $\mathrm{Res}(X,S,\{x_\lambda\}) \in \mathbb{C}$ introduced in Theorem 6.2 is given*

(i) *if $x_\lambda \in \mathrm{Sing}(X) \cap (S \setminus \mathrm{Sing}(S))$, and $f$ is tangential or $S^0$ is comfortably embedded in $M$ and $\nu_f > 1$, by*

$$\mathrm{Res}(X,S,\{x_\lambda\}) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_\Gamma \frac{(h^1)^{n-1}}{g^2 \cdots g^n}\, dz^2 \wedge \cdots \wedge dz^n;$$

(ii) *if $x_\lambda \in \mathrm{Sing}(X) \cap (S \setminus \mathrm{Sing}(S))$, $S^0$ is comfortably embedded in $M$ and $\nu_f = 1$, by*

$$\mathrm{Res}(H^1_{\sigma,f},S,\{x_\lambda\}) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_\Gamma \frac{(h^1)^{n-1}}{(1+b^1)^{n-1} g^2 \cdots g^n}\, dz^2 \wedge \cdots \wedge dz^n;$$

(iii) *if $n = 2$, $x_\lambda \in \mathrm{Sing}(S)$, and $f$ is tangential or $S^0$ is comfortably embedded in $M$ and $\nu_f > 1$, by*

$$\mathrm{Res}(X,S,\{x_\lambda\}) = \frac{1}{2\pi i} \int_{\partial R_1} \left.\frac{(l \circ f - l) - b^1 l^{\nu_f}}{l \cdot (w^2 \circ f - w^2)}\right|_S dw^2;$$

(iv) *if $n = 2$, $x_\lambda \in \mathrm{Sing}(S)$, $S^0$ is comfortably embedded in $M$ and $\nu_f = 1$, by*

$$\mathrm{Res}(H^1_{\sigma,f},S,\{x_\lambda\}) = \frac{1}{2\pi i} \int_{\partial R_1} \left.\frac{(l \circ f - l) - b^1 l}{(l + (l \circ f - l))(w^2 \circ f - w^2)}\right|_S dw^2;$$

(v) *if $n > 2$, $x_\lambda \in \mathrm{Sing}(S)$, $f$ is tangential and $\nu_f > 1$, by*

$$\mathrm{Res}(X_f,S,\{x_\lambda\}) = \left(\frac{-i}{2\pi i}\right)^{n-1} \int_\Gamma \frac{\left[\sum_{j=1}^n \frac{\partial l}{\partial w^j}(w^j \circ f - w^j)\right]^{n-1}}{l^{n-1} \prod_{p=2}^n (w^p \circ f - w^p)}\, dw^2 \wedge \cdots \wedge dw^n.$$

Our next aim is to compute the residue for the Lehmann-Suwa action, at least for an isolated smooth point $x_\lambda \in \mathrm{Sing}(X_f)$. Let $(W,w)$ be a local chart about $x_\lambda$ belonging to a comfortable atlas. Set $l = w^1$ and define $\tilde{v}_f$ as in (6.9). By (6.13) the Lehmann-Suwa action $\tilde{V}$ is given by the holomorphic action (in the sense of Bott) of $\tilde{v}_f$ on $TM|_S - [S]^{\otimes \nu_f}$. Therefore we can apply [L], [LS] (see also [Su, Ths. IV.5.3, IV.5.6], and [Su, Remark IV.5.7]) to obtain

$$\mathrm{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \{x_\lambda\}) = \mathrm{Res}_\varphi(X_f, TM|_S, \{x_\lambda\}),$$

where $\mathrm{Res}_\varphi(X_f, TM|_S, \{x_\lambda\})$ is the residue for the local Lie derivative action of $\tilde{v}_f$ on $TM|_S$ given by

$$\tilde{V}_l(s)(\tilde{v}_f) = [\tilde{v}_f, \tilde{s}]|_S,$$

where $s$ is a section of $TM|_S$ and $\tilde{s}$ is a local extension of $s$ constant along the fibers of $\sigma$.

We can write an expression of $\tilde{V}_l$ in local coordinates. Let $(U, z)$ be a local chart belonging to a comfortable atlas. Then $\{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}\}$ is a local frame for $TM$, and $\{(\omega^1)^{\otimes \nu_f} \otimes \frac{\partial}{\partial z^1}\big|_S, \ldots, (\omega^1)^{\otimes \nu_f} \otimes \frac{\partial}{\partial z^n}\big|_S\}$ is a local frame for $(N_S^{\otimes \nu_f})^* \otimes TM|_S$. Thus there exist holomorphic functions $V_j^k$ (for $j$, $k = 1, \ldots, n$) so that

$$\tilde{V}_l(\frac{\partial}{\partial z^j})(\partial_1^{\otimes \nu_f}) = V_j^k \frac{\partial}{\partial z^k}.$$

Now, from (4.4) we get

$$\begin{aligned}\tilde{V}_l(\frac{\partial}{\partial z^j})(\partial_1^{\otimes \nu_f}) &= \left[\mathcal{X}_f\left((\frac{\partial}{\partial z^1})^{\otimes \nu_f}\right), \frac{\partial}{\partial z^j}\right]\Bigg|_S \\ &= \left[h^1 z^1 \frac{\partial}{\partial z^1} + g^p \frac{\partial}{\partial z^p}, \frac{\partial}{\partial z^j}\right]\Bigg|_S = -h^1|_S \delta_j^1 \frac{\partial}{\partial z^1} - \frac{\partial g^p}{\partial z^j}\bigg|_S \frac{\partial}{\partial z^p},\end{aligned}$$

and hence

$$V_1^1 = -h^1|_S, \qquad V_p^1 \equiv 0, \qquad V_j^p = -\frac{\partial g^p}{\partial z^j}\bigg|_S. \tag{6.14}$$

Therefore [Su, Th. IV.5.3] yields

THEOREM 6.6. *Let $S$ be a compact, globally irreducible, possibly singular hypersurface in an $n$-dimensional complex manifold $M$. Let $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, be given. Assume that $S' = S \setminus \mathrm{Sing}(S)$ is comfortably embedded into $M$, and that $f$ is tangential to $S$ with $\nu_f > 1$. Let $x_\lambda \in \mathrm{Sing}(X_f)$ be an isolated smooth point of $\mathrm{Sing}(S) \cup \mathrm{Sing}(X_f)$. Then for any homogeneous symmetric polynomial $\varphi$ of degree $n-1$ the complex number*

$$\mathrm{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \{x_\lambda\})$$

*introduced by Theorem 6.3 is given by*

$$\mathrm{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \{x_\lambda\}) = \int_\Gamma \frac{\varphi(V)\, dz^2 \wedge \cdots \wedge dz^n}{g^2 \cdots g^n}, \tag{6.15}$$

*where $V = (V_j^k)$ is the matrix given by (6.14) and $\Gamma$ is as in (6.5).*

*Remark* 6.7. We adopt here the convention that if $V$ is an $n \times n$ matrix then $c_j(V)$ is the $j^{\text{th}}$ symmetric function of the eigenvalues $V$ multiplied by $(i/2\pi)^j$, and $\varphi(V) = \varphi\big(c_1(V), \dots, c_{n-1}(V)\big)$.

Finally, if $x_\lambda$ is an isolated point in $\operatorname{Sing}(X)$, the complex numbers $\operatorname{Res}_\varphi(X, TS - [S]^{\otimes \nu_f}, \{x_\lambda\})$ appearing in Theorem 6.4 can be computed exactly as in the foliation case using a Grothendieck residue with a formula very similar to (6.15); see [BB], [Su, Th. III.5.5].

## 7. Index theorems in higher codimension

Let $S \subset M$ be a complex submanifold of codimension $1 < m < n$ in a complex $n$-manifold $M$. A way to get index theorems for holomorphic self-maps of $M$ fixing pointwise $S$ is to blow-up $S$ and then apply the index theorems for hypersurfaces; this is what we plan to do in this section.

We shall denote by $\pi \colon M_S \to M$ the blow-up of $M$ along $S$, and by $E_S = \pi^{-1}(S)$ the exceptional divisor, which is a hypersurface in $M_S$ isomorphic to the projectivized normal bundle $\mathbb{P}(N_S)$.

*Remark* 7.1. If $S$ is singular, the blow-up $M_S$ is in general singular too. So this approach works only for smooth submanifolds.

If $(U, z)$ is a chart adapted to $S$ centered in $p \in S$, in $M_S$ we have $m$ charts $(\tilde{U}_r, w_r)$ centered in $[\partial_1], \dots, [\partial_m]$ respectively, where if $v \in N_{S,p}$, $v \neq O$, we denote by $[v]$ its projection in $\mathbb{P}(N_S)$. The coordinates $z^j$ and $w_r^h$ are related by

$$z^j(w_r) = \begin{cases} w_r^j & \text{if } j = r, m+1, \dots, n, \\ w_r^r w_r^j & \text{if } j = 1, \dots, r-1, r+1, \dots, m. \end{cases}$$

*Remark* 7.2. We have $\tilde{U}_r \cap E_S = \{w_r^r = 0\}$, and thus $(\tilde{U}_r, w_r)$ is adapted to $E_S$ up to a permutation of the coordinates.

Now take $f \in \operatorname{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, and assume that $df$ acts as the identity on $N_S$ (this is automatic if $\nu_f > 1$, while if $\nu_f = 1$ it happens if and only if $f$ is tangential). Then we can lift $f$ to a unique map $\tilde{f} \in \operatorname{End}(M_S, E_S)$, $\tilde{f} \not\equiv \mathrm{id}_{M_S}$, such that $f \circ \pi = \pi \circ \tilde{f}$ (see, e.g., [A1] for details). If $(U, z)$ is a chart adapted to $S$ and we set $f^j = z^j \circ f$ and $\tilde{f}_r^j = w_r^j \circ \tilde{f}$,

$$\tilde{f}_r^j(w_r) = \begin{cases} f^j\big(z(w_r)\big) & \text{if } j = r, m+1, \dots, n, \\ \dfrac{f^j\big(z(w_r)\big)}{f^r\big(z(w_r)\big)} & \text{if } j = 1, \dots, r-1, r+1, \dots, m. \end{cases} \tag{7.1}$$

If $f$ is tangential we can find holomorphic functions $h^r_{r_1\ldots r_{\nu_f+1}}$ symmetric in the lower indices such that

$$f^r - z^r = h^r_{r_1\ldots r_{\nu_f+1}} z^{r_1}\cdots z^{r_{\nu_f+1}} + R_{\nu_f+2}; \tag{7.2}$$

as usual, only the restriction to $S$ of each $h^r_{r_1\ldots r_{\nu_f+1}}$ is uniquely defined. Set then

$$Y = h^r_{r_1\ldots r_{\nu_f+1}}|_S\, \partial_r \otimes \omega^{r_1}\otimes\cdots\otimes\omega^{r_{\nu_f+1}};$$

it is a local section of $N_S\otimes (N_S^*)^{\otimes(\nu_f+1)}$.

On the other hand, if $f$ is not tangential set $B = (\pi\otimes \mathrm{id})_* \circ X_f$, where $\pi\colon TM|_S \to N_S$ is the canonical projection. In this way we get a global section of $N_S\otimes (N_S^*)^{\otimes\nu_f}$, not identically zero if and only if $f$ is not tangential, and given in local adapted coordinates by

$$B = g^r_{r_1\ldots r_{\nu_f}}|_S\, \partial_r\otimes\omega^{r_1}\otimes\cdots\otimes\omega^{r_{\nu_f}}.$$

*Definition* 7.1. Take $p\in S$. If $f$ is tangential, a non-zero vector $v\in (N_S)_p$ is a *singular direction* for $f$ at $p$ if $X_f(v\otimes\cdots\otimes v) = O$ and $Y(v\otimes\cdots\otimes v)\wedge v = O$. If $f$ is not tangential, $v$ is a *singular direction* if $B(v\otimes\cdots\otimes v)\wedge v = O$.

*Remark* 7.3. The condition $Y(v\otimes\cdots\otimes v)\wedge v = O$ is equivalent to requiring $Y(v\otimes\cdots\otimes v) = \lambda v$ for some $\lambda\in\mathbb{C}$.

Of course, in the tangential case we must check that this definition is well-posed, because the morphism $Y$ depends on the local coordinates chosen to define it. First of all, if $(U,z)$ is a chart adapted to $S$ and centered at $p$ then $X_f(v\otimes\cdots\otimes v) = O$ when $f$ is tangential means

$$g^p_{r_1\ldots r_{\nu_f}}(O)\, v^{r_1}\cdots v^{r_{\nu_f}}\,\frac{\partial}{\partial z^p} = O, \tag{7.3}$$

where $v = v^r\partial_r$. Now let $(\hat U,\hat z)$ be another chart adapted to $S$ centered in $p$. Then we can find holomorphic functions $a^r_s$ and $\hat a^r_s$ such that $\hat z^r = a^r_s z^s$ and $z^r = \hat a^r_s \hat z^s$. Arguing as in the proof of (4.2) we get

$$a^{r_1}_{s_1}\cdots a^{r_{\nu_f+1}}_{s_{\nu_f+1}}\hat h^r_{r_1\ldots r_{\nu_f+1}} = a^r_s h^s_{s_1\ldots s_{\nu_f+1}} + \sum_{\ell=1}^{\nu_f+1}\frac{\partial a^r_{s_\ell}}{\partial z^p} g^p_{s_1\ldots \hat{s_\ell}\ldots s_{\nu_f+1}} + R_1,$$

where the index with the hat is missing from the list. Therefore

$$\hat Y = Y + \hat a^s_r \sum_{\ell=1}^{\nu_f+1}\frac{\partial a^r_{s_\ell}}{\partial z^p} g^p_{s_1\ldots\hat{s_\ell}\ldots s_{\nu_f+1}}\bigg|_S\, \partial_s\otimes\omega^{s_1}\otimes\cdots\otimes\omega^{s_{\nu_f+1}};$$

in particular if $X_f(v\otimes\cdots\otimes v) = O$ equation (7.3) yields

$$\hat Y(v\otimes\cdots\otimes v) = Y(v\otimes\cdots\otimes v),$$

and the notion of singular direction when $f$ is tangential is well-defined.

Proposition 7.1. *Let $S \subset M$ be a complex submanifold of codimension $1 < m < n$ of a complex $n$-manifold $M$, and take $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, such that $df$ acts as the identity on $N_S$ (that is $f$ is tangential, or $\nu_f > 1$, or both). Denote by $\pi\colon M_S \to M$ the blow-up of $M$ along $S$ with exceptional divisor $E_S$, and let $\tilde{f} \in \mathrm{End}(M_S, E_S)$ be the lifted map. Then:*

(i) *if $S$ is comfortably embedded in $M$ then $E_S$ is comfortably embedded in $M_S$, and the choice of a splitting morphism for $S$ induces a splitting morphism for $E_S$;*

(ii) *$d\tilde{f}$ acts as the identity on $N_{E_S}$;*

(iii) *$\tilde{f}$ is always tangential; furthermore $\nu_{\tilde{f}} = \nu_f$ if $f$ is tangential, $\nu_{\tilde{f}} = \nu_f - 1$ otherwise;*

(iv) *a direction $[v] \in E_S$ is a singular point for $\tilde{f}$ if and only if it is a singular direction for $f$.*

*Proof.* (i) Let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be a comfortable atlas adapted to $S$; we claim that $\tilde{\mathfrak{U}} = \{(\tilde{U}_{\alpha,r}, w_{\alpha,r})\}$ is a comfortable atlas adapted to $E_S$ (and in particular determines a splitting morphism for $E_S$). Let us first prove that it is a splitting atlas, that is that

$$\left.\frac{\partial w^j_{\beta,s}}{\partial w^r_{\alpha,r}}\right|_{E_S} \equiv 0$$

for every $r$, $s$, $j \neq s$ and indices $\alpha$ and $\beta$. We have

$$z^j_\beta = z^j_\beta|_S + \left.\frac{\partial z^j_\beta}{\partial z^s_\alpha}\right|_S z^s_\alpha + \frac{1}{2}\left.\frac{\partial^2 z^j_\beta}{\partial z^u_\alpha \partial z^v_\alpha}\right|_S z^u_\alpha z^v_\alpha + R_3.$$

Since $w^r_{\alpha,r} = z^r_\alpha$, if $j = p > m$ we immediately get

$$\left.\frac{\partial w^p_{\beta,s}}{\partial w^r_{\alpha,r}}\right|_{E_S} = \left.\frac{\partial z^p_\beta}{\partial z^r_\alpha}\right|_S \equiv 0,$$

because $\mathfrak{U}$ is a splitting atlas. If $j = t \leq m$,

$$\begin{aligned} z^t_\beta &= \left.\frac{\partial z^t_\beta}{\partial z^s_\alpha}\right|_S z^s_\alpha + \frac{1}{2}\left.\frac{\partial^2 z^t_\beta}{\partial z^u_\alpha \partial z^v_\alpha}\right|_S z^u_\alpha z^v_\alpha + R_3 \\ &= \left[\left.\frac{\partial z^t_\beta}{\partial z^r_\alpha}\right|_S + \sum_{u\neq r}\left.\frac{\partial z^t_\beta}{\partial z^u_\alpha}\right|_S w^u_{\alpha,r}\right] w^r_{\alpha,r} + O\big((w^r_{\alpha,r})^3\big), \end{aligned} \tag{7.4}$$

because $\mathfrak{U}$ is a comfortable atlas. Therefore if $t \neq s$,

$$w^t_{\beta,s} = \frac{z^t_\beta}{z^s_\beta} = \frac{\left.\frac{\partial z^t_\beta}{\partial z^r_\alpha}\right|_S + \sum_{u\neq r}\left.\frac{\partial z^t_\beta}{\partial z^u_\alpha}\right|_S w^u_{\alpha,r} + O\big((w^r_{\alpha,r})^2\big)}{\left.\frac{\partial z^s_\beta}{\partial z^r_\alpha}\right|_S + \sum_{u\neq r}\left.\frac{\partial z^s_\beta}{\partial z^u_\alpha}\right|_S w^u_{\alpha,r} + O\big((w^r_{\alpha,r})^2\big)},$$

and so

$$\frac{\partial w^t_{\beta,s}}{\partial w^r_{\alpha,r}} = O(w^r_{\alpha,r}),$$

as required.

Finally, since $w^s_{\beta,s} = z^s_\beta$, equation (7.4) yields

$$\frac{\partial^2 w^s_{\beta,s}}{\partial (w^r_{\alpha,r})^2} = O(w^r_{\alpha,r}),$$

and $\tilde{\mathfrak{U}}$ is a comfortable atlas, as claimed.

(ii) Since *df* acts as the identity on $N_S$, in local coordinates we can write

$$f^j(z) = z^j + g^j_{r_1\ldots r_{\nu_f+1}} z^{r_1}\cdots z^{r_{\nu_f}} + R_{\nu_f+1},$$

with $g^s_{r_1}|_S \equiv 0$ if $\nu_f = 1$. Then (7.1) yields

$$\tilde f^j_r(w_r) = w^j_r + (w^r_r)^{\nu_f} g^j_{r_1\ldots r_{\nu_f}}\big(z(w_r)\big) w_r^{\hat r_1}\cdots w_r^{\hat r_{\nu_f}} + O\big((w^r_r)^{\nu_f+1}\big) \tag{7.5}$$

if $j = r, m+1, \ldots, n$, and

$$\tilde f^j_r(w_r) = w^j_r + (w^r_r)^{\nu_f-1}\big[g^j_{r_1\ldots r_{\nu_f}}\big(z(w_r)\big) - w^j_r g^r_{r_1\ldots r_{\nu_f}}\big(z(w_r)\big)\big] w_r^{\hat r_1}\cdots w_r^{\hat r_{\nu_f}} + O\big((w^r_r)^{\nu_f}\big) \tag{7.6}$$

if $j = 1, \ldots, r-1, r+1, \ldots, m$, where $w_r^{\hat s} = w^s_r$ if $s \neq r$, and $w_r^{\hat r} = 1$. In particular, $d\tilde f$ acts as the identity on $N_{E_S}$.

(iii) We have

$$g^j_{r_1\ldots r_{\nu_f}}|_{E_S}\big(z(w_r)\big) = g^j_{r_1\ldots r_{\nu_f}}|_S(O, w''_r);$$

therefore if $f$ is tangential then $w^r_r$ divides all $g^s_{r_1\ldots r_{\nu_f}}\big(z(w_r)\big)$, while it does not divide some $g^p_{r_1\ldots r_{\nu_f}}\big(z(w_r)\big)$. In particular, then, $\tilde f$ is tangential and $\nu_{\tilde f} = \nu_f$, by (7.5) and (7.6). On the other hand, if $f$ is not tangential $w^r_r$ does not divide some $g^s_{r_1\ldots r_{\nu_f}}\big(z(w_r)\big)$; therefore

$$\begin{aligned}\big[g^s_{r_1\ldots r_{\nu_f}}\big(z(w_r)\big) &- w^s_r g^r_{r_1\ldots r_{\nu_f}}\big(z(w_r)\big)\big]\big|_{E_S}\\ &= g^s_{r_1\ldots r_{\nu_f}}(O, w''_r) - w^s_r g^r_{r_1\ldots r_{\nu_f}}(O, w''_r) \not\equiv 0,\end{aligned}$$

and thus $\nu_{\tilde f} = \nu_f - 1$ and $\tilde f$ is again tangential.

(iv) Take $v \in (N_S)_p$, $v \neq O$, and a chart $(U, z)$ adapted to $S$ centered in $p$. Then $v = v^s \partial_s$, with $v^r \neq 0$ for some $r$. Hence $[v] \in \tilde U_r$ has coordinates

$$w^j_r([v]) = \begin{cases} 0 & \text{if } j = r, m+1, \ldots, n,\\ v^j/v^r & \text{if } j = 1, \ldots, r-1, r+1, \ldots, m.\end{cases}$$

If $f$ is not tangential, then $[v]$ is a singular point for $\tilde f$ if and only if

$$[v^r g^s_{r_1\ldots r_{\nu_f}}(O) - v^s g^r_{r_1\ldots r_{\nu_f}}(O)] v^{r_1}\cdots v^{r_{\nu_f}} = 0$$

for all $s$, and thus if and only if $B(v\otimes\cdots\otimes v)\wedge v = O$, as claimed.

If $f$ is tangential, writing $f^s - z^s$ as in (7.2) we get

$$\tilde{f}^s_r(w_r) = w^s_r + (w^r_r)^{\nu_f}\left[h^s_{r_1\ldots r_{\nu_f+1}}\big(z(w_r)\big) - w^s_r h^r_{r_1\ldots r_{\nu_f+1}}\big(z(w_r)\big)\right] w_r^{\hat{r}_1}\cdots w_r^{\hat{r}_{\nu_f+1}} + O\big((w^r_r)^{\nu_f+1}\big)$$

for $s \neq r$, and then it is clear that $[v]$ is a singular point for $\tilde{f}$ if and only if $v$ is a singular direction for $f$. □

We therefore get index theorems in any codimension:

Theorem 7.2. *Let $S$ be a compact complex submanifold of codimension $1 < m < n$ in an $n$-dimensional complex manifold $M$. Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$, be given, and assume that $df$ acts as the identity on $N_S$. Let $\bigcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the set of singular directions for $f$ in $\mathbb{P}(N_S)$. Then there exist complex numbers $\mathrm{Res}(f,S,\Sigma_\lambda) \in \mathbb{C}$, depending only on the local behavior of $f$ and $S$ near $\Sigma_\lambda$, such that*

$$\sum_\lambda \mathrm{Res}(f,S,\Sigma_\lambda) = \int_{E_S} c_1^{n-1}([E_S]) = \int_S \pi_* c_1^{n-1}([E_S]),$$

*where $E_S$ is the exceptional divisor in the blow-up $\pi\colon M_S \to M$ of $M$ along $S$, and $\pi_*$ denotes the integration along the fibers of the bundle $\pi|_{E_S}\colon E_S \to S$.*

*Proof.* This follows immediately from Theorem 6.2, Proposition 7.1, and the projection formula (see, e.g., [Su, Prop. II.4.5]). □

*Remark* 7.4. The restriction to $E_S$ of the cohomology class $c_1([E_S])$ is the Chern class $\zeta = c_1(T)$ of the tautological bundle $T$ on the bundle $\pi|_{E_S}\colon E_S \to S$ and it satisfies the relation

$$\zeta^{n-m} - \pi|^*_{E_S} c_1(N_S)\zeta^{n-m-1} + \pi|^*_{E_S} c_2(N_S)\zeta^{n-m-2} + \cdots + (-1)^{n-m}\pi|^*_{E_S} c_{n-m}(N_S) = 0$$

in the cohomology ring of the bundle (see, e.g., [GH, pp. 606–608]). This formula can sometimes be used to compute $\zeta$ in terms of the Chern classes of $N_S$ and $TM$ in specific examples.

Theorem 7.3. *Let $S$ be a compact complex submanifold of codimension $1 < m < n$ in an $n$-dimensional complex manifold $M$. Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$, be given, and set $\nu = \nu_f$ if $f$ is tangential, and $\nu = \nu_f - 1$ otherwise. Assume that $S$ is comfortably embedded into $M$, and that $\nu > 1$. Let $\bigcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the set of singular directions for $f$ in $\mathbb{P}(N_S)$. Finally, let $\pi\colon M_S \to M$ be the blow-up of $M$ along $S$, with exceptional divisor $E_S$. Then for any homogeneous symmetric polynomial $\varphi$ of degree $n-1$ there exist complex numbers $\mathrm{Res}_\varphi(f, TM_S|_{E_S} - N_{E_S}^{\otimes\nu}, \Sigma_\lambda) \in \mathbb{C}$,* *depending only on the local behavior of* $f$ *and* $TM_S|_{E_S} - N_{E_S}^{\otimes\nu}$ *near* $\Sigma_\lambda$, *such that*

$$\sum_\lambda \mathrm{Res}_\varphi(f, TM_S|_{E_S} - N_{E_S}^{\otimes\nu}, \Sigma_\lambda) = \int_S \pi_*\varphi\big(TM_S|_{E_S} \otimes (N_{E_S}^*)^{\otimes\nu}\big),$$

*where* $\pi_*$ *denotes the integration along the fibers of the bundle* $E_S \to S$.

*Proof.* This follows immediately from Theorem 6.3, Proposition 7.1, and the projection formula. □

THEOREM 7.4. *Let* $S$ *be a compact complex submanifold of codimension* $1 < m < n$ *in an* $n$*-dimensional complex manifold* $M$. *Let* $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, *be given, and assume that* $df$ *acts as the identity on* $N_S$. *Set* $\nu = \nu_f$ *if* $f$ *is tangential, and* $\nu = \nu_f - 1$ *otherwise. Let* $\bigcup_\lambda \Sigma_\lambda$ *be the decomposition in connected components of the set of singular directions for* $f$ *in* $\mathbb{P}(N_S)$. *Finally, let* $\pi\colon M_S \to M$ *be the blow-up of* $M$ *along* $S$, *with exceptional divisor* $E_S$. *Then for any homogeneous symmetric polynomial* $\varphi$ *of degree* $n-1$ *there exist complex numbers* $\mathrm{Res}_\varphi(f, TE_S - N_{E_S}^{\otimes\nu}, \Sigma_\lambda) \in \mathbb{C}$, *depending only on the local behavior of* $f$ *and* $TE_S - N_{E_S}^{\otimes\nu}$ *near* $\Sigma_\lambda$, *such that*

$$\sum_\lambda \mathrm{Res}_\varphi(f, TE_S - N_{E_S}^{\otimes\nu}, \Sigma_\lambda) = \int_S \pi_*\varphi\big(TE_S \otimes (N_{E_S}^*)^{\otimes\nu}\big),$$

*where* $\pi_*$ *denotes the integration along the fibers of the bundle* $E_S \to S$.

*Proof.* This follows immediately from Theorem 6.4, Proposition 7.1, and the projection formula. □

## 8. Applications to dynamics

We conclude this paper with applications to the study of the dynamics of endomorphisms of complex manifolds, first recalling a definition from [A2]:

*Definition* 8.1. Let $f \in \mathrm{End}(M, p)$ be a germ at $p \in M$ of a holomorphic self-map of a complex manifold $M$ fixing $p$. A *parabolic curve* for $f$ at $p$ is a injective holomorphic map $\varphi\colon \Delta \to M$ satisfying the following properties:

(i) $\Delta$ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$;

(ii) $\varphi$ is continuous at the origin, and $\varphi(0) = p$;

(iii) $\varphi(\Delta)$ is invariant under $f$, and $(f|_{\varphi(\Delta)})^n \to p$ as $n \to \infty$.

Furthermore, we say that $\varphi$ is *tangent to a direction* $v \in T_pM$ at $p$ if for one (and hence any) chart $(U, z)$ centered at $p$ the direction of $z\big(\varphi(\zeta)\big)$ converges to the direction $dz_p(v)$ as $\zeta \to 0$.

Now we have the promised dynamical interpretation of $X_f$ at nonsingular points:

Proposition 8.1. *Assume that $S$ has codimension one in $M$, and take $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$. Let $p \in S$ be a regular point of $X_f$, that is $X_f(p) \neq O$. Then*

(i) *If $f$ is tangential then no infinite orbit of $f$ can stay arbitrarily close to $p$. More precisely, there is a neighborhood $U$ of $p$ such that for every $q \in U$ there is $n_0 \in \mathbb{N}$ such that $f^{n_0}(q) \notin U$ or $f^{n_0}(q) \in S$.*

(ii) *If $\Xi_f(p)$ is transversal to $T_pS$ (so in particular $f$ is non-tangential) and $\nu_f > 1$ then there exists at least one parabolic curve for $f$ at $p$ tangent to $\Xi_f(p)$.*

(iii) *If $\Xi_f(p)$ is transversal to $T_pS$, $\nu_f = 1$, and $|b(p)| \neq 0, 1$ or $b(p) = \exp(2\pi i\theta)$ where $\theta$ satisfies the Bryuno condition (and $b$ is the function defined in Remark 1.1) then there exists an $f$-invariant one-dimensional holomorphic disk $\Delta$ passing through $p$ tangent to $\Xi_f(p)$ such that $f|_\Delta$ is holomorphically conjugated to the multiplication by $b(p)$.*

*Proof.* In local adapted coordinates centered at $p \in S$ we can write

$$f^j(z) = z^j + (z^1)^{\nu_f} g^j(z),$$

so that

$$\Xi_f(p) = \mathrm{Span}\left( g^1(O)\,\frac{\partial}{\partial z^1}\bigg|_p + \cdots + g^n(O)\,\frac{\partial}{\partial z^n}\bigg|_p \right).$$

In case (i), we have $g^1 = z^1 h^1$ for a suitable holomorphic function $h^1$, and $g^{p_0}(O) \neq 0$ for some $2 \leq p_0 \leq n$, because $p$ is not singular. Therefore we can apply [AT, Prop. 2.1] (see also [A2, Prop. 2.1]), and the assertion follows.

Now, $\Xi_f(p)$ is transversal to $T_pS$ if and only if $g^1(O) \neq 0$. In case (ii) we can then write

$$f^j(z) = z^j + g^j(O)(z^1)^{\nu_f} + O(\|z\|^{\nu_f+1})$$

with $g^1(O) \neq 0$. Then $\Xi_f(p)$ is a non-degenerate characteristic direction of $f$ at $p$ in the sense of Hakim, and thus by [H1, 2] there exist at least $\nu_f - 1$ parabolic curves for $f$ at $p$ tangent to $\Xi_f(p)$.

If $\nu_f = 1$, it is easy to see that $b^1(p) = 1 + g^1(O)$, and $b^1(p) \neq 1$ because $\Xi_f(p)$ is transversal to $T_pS$. Therefore we can write

$$f^j(z) = \begin{cases} b^1(p)z^1 + O(\|z\|^2) & \text{if } j = 1, \\ z^j + g^j(O)z^1 + O(\|z\|^2) & \text{if } 2 \leq j \leq n, \end{cases}$$

and the assertion in case (iii) follows immediately from [Pö] (see also [N]). □

In other words, $X_f$ essentially dictates the dynamical behavior of $f$ away from the singularities — or, from another point of view, we can say that the interesting dynamics is concentrated near the singularities of $X_f$.

*Remark* 8.1. If $\Xi_f(p)$ is transversal to $T_pS$, $\nu_f = 1$ and $b(p) = 0$ or $b(p) = \exp(2\pi i\theta)$ with $\theta$ irrational not satisfying the Bryuno condition, there might still be an $f$-invariant one-dimensional holomorphic disk passing through $p$ and tangent to $\Xi_f(p)$. On the other hand, if $b(p) = \exp(2\pi i\theta)$ is a $k^{\text{th}}$ root of unity, necessarily different from one, several things might happen. For instance, if $b(p) = -1$, up to a linear change of coordinates we can write

$$f^j(z) = \begin{cases} z^1 + z^1\bigl(-2 + (z^1)^{\mu_1}\hat{g}^1(z)\bigr) & \text{if } j = 1, \\ z^j + (z^1)^{\mu_j+1}\hat{g}^j(z) & \text{if } j = 2, \dots, n, \end{cases}$$

for suitable $\mu_1, \dots, \mu_n \in \mathbb{N}$ and holomorphic functions $\hat{g}^j$ not divisible by $z^1$ and such that $\hat{g}^j(O) = 0$ if $\mu_j = 0$. Then if $\mu_1 = 0$,

$$\begin{aligned}(f \circ f)^j(z)& \\ &= \begin{cases} z^1 - z^1\bigl[\hat{g}^1(z) + \hat{g}^1\bigl(f(z)\bigr) - \hat{g}^1(z)\hat{g}\bigl(f(z)\bigr)\bigr] & \text{if } j = 1, \\ z^j + (z^1)^{\mu_j+1}\bigl[\hat{g}^j(z) - \bigl(-1 + \hat{g}^1(z)\bigr)^{\mu_j+1}\hat{g}^j\bigl(f(z)\bigr)\bigr] & \text{if } j = 2, \dots, n. \end{cases}\end{aligned}$$

So $\nu_{f\circ f} = 1$, $f \circ f$ is non-tangential but $p$ is singular for $f \circ f$. On the other hand, if $\mu_1 = 1$,

$$\begin{aligned}(f \circ f)^j(z)& \\ &= \begin{cases} z^1 - (z^1)^2\bigl[\hat{g}^1(z) - \hat{g}^1\bigl(f(z)\bigr) + O(z^1)\bigr] & \text{if } j = 1, \\ z^j + (z^1)^{\mu_j+1}\bigl[\hat{g}^j(z) + (-1)^{\mu_j}\hat{g}^j\bigl(f(z)\bigr) + O(z^1)\bigr] & \text{if } j = 2, \dots, n. \end{cases}\end{aligned}$$

Now if, for instance, $\mu_2 = 0$ we get $\nu_{f\circ f} = 1$, but $f \circ f$ is tangential and $p$ is singular for $f \circ f$. But if $\mu_2 = 2$ and $\mu_j \geq 2$ for $j \geq 3$ we get $\nu_{f\circ f} = 3$ and $p$ can be either singular or nonsingular for $f \circ f$.

*Remark* 8.2. If $\nu_f = 1$, $\Xi_f(p) \subset T_pS$ and $S$ is compact, necessarily $f$ is tangential, because $b \equiv 1$ and then $g^1(0, z'') \equiv 0$. If $S$ is not compact we might have an isolated point of tangency, and in that case we might have parabolic curves at $p$ not tangent to $\Xi_f(p)$. For instance, the methods of [A1] show that this happens for the map

$$f^j(z) = \begin{cases} z^1 + z^1\bigl(az^2 + bz^3 + h_1(z'') + z^1h_2(z)\bigr) & \text{if } j = 1, \\ z^2 + z^1\bigl(c + h_3(z)\bigr) & \text{if } j = 2, \\ z^3 + z^1g^3(z) & \text{if } j = 3, \end{cases}$$

when $a$, $c \neq 0$.

Finally, we describe a couple of applications to endomorphisms of complex surfaces:

COROLLARY 8.2. *Let $S$ be a smooth compact one-dimensional submanifold of a complex surface $M$, and take $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$. Assume that $f$ is tangential, or that $S \setminus \mathrm{Sing}(f)$ is comfortably embedded in $M$, and let $X$ denote $X_f$, $H_{\sigma,f}$ or $H^1_{\sigma,f}$ as usual; assume moreover that $X \not\equiv O$. Then*

(i) *if $c_1(N_S) \neq 0$ then $\chi(S) - \nu_f c_1(N_S) > 0$;*

(ii) *if $c_1(N_S) > 0$ then $S$ is rational, $\nu_f = 1$ and $c_1(N_S) = 1$.*

*Proof.* The well-known theorem about the localization of the top Chern class at the zeroes of a global section (see, e.g., [Su, Th. III.3.5]) yields

$$\sum_{x \in \mathrm{Sing}(X)} N(X;x) = \chi(S) - \nu_f\, c_1(N_S), \tag{8.1}$$

where $N(X;x)$ is the multiplicity of $x$ as a zero of $X$. Now, If $c_1(N_S) \neq 0$ then by Theorem 6.2 the set $\mathrm{Sing}(X)$ is not empty. Therefore the sum in (8.1) must be strictly positive, and the assertions follow. □

*Definition* 8.2. Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$. We say that a point $p \in S$ is *weakly attractive* if there are infinite orbits arbitrarily close to $p$, that is, if for every neighborhood $U$ of $p$ there is $q \in U$ such that $f^n(q) \in U \setminus S$ for all $n \in \mathbb{N}$. In particular, this happens if there is an infinite orbit converging to $p$.

Then we can prove the following

PROPOSITION 8.3. *Let $S$ be a smooth compact one-dimensional submanifold of a complex surface $M$, and take $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$. If $f$ is tangential then there are at most $\chi(S) - \nu_f c_1(N_S)$ weakly attractive points for $f$ on $S$.*

*Proof.* By (8.1) the sum of zeros of the section $X_f$ (counting multiplicity) is equal to $\chi(S) - \nu_f c_1(N_S)$. Thus the number of zeros (not counting multiplicity) is at most $\chi(S) - \nu_f c_1(N_S)$. The assertion then follows from Proposition 8.1. □

Finally, the previous index theorems allow a classification of the smooth curves which are fixed by a holomorphic map and are dynamically trivial.

THEOREM 8.4. *Let $S$ be a smooth compact one-dimensional submanifold of a complex surface $M$, and take $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$. Moreover assume that $\mathrm{sp}(df_p) = \{1\}$ for some $p \in S$. If there are no weakly attractive points for $f$ on $S$ then only one of the following cases occurs:*

(i) $\chi(S) = 2$, $c_1(N_S) = 0$, *or*

(ii) $\chi(S) = 2$, $c_1(N_S) = 1$, $\nu_f = 1$, *or*

(iii) $\chi(S) = 0$, $c_1(N_S) = 0$.

*Proof.* Since $N_S$ is a line bundle over a compact curve $S$, the action of $df$ on $N_S$ is given by multiplication by a constant, and since $df_p$ has only the eigenvalue 1 then this constant must be 1. If $f$ were nontangential then by Proposition 8.1.(ii) all but a finite number of points of $S$ would be weakly attractive. Therefore $f$ is tangential. By [A2, Cor. 3.1] (or [Br, Prop. 7.7]) if there is a point $q \in S$ so that $\mathrm{Res}(X_f, N_S, p) \notin \mathbb{Q}^+$ then $q$ is weakly attractive. Thus the sum of the residues is nonnegative and by Theorem 6.2 it follows that $c_1(N_S) \geq 0$. Thus (8.1) yields

$$\chi(S) \geq \nu_f c_1(N_S) \geq 0. \tag{8.2}$$

Therefore the only possible cases are $\chi(S) = 0, 2$. If $\chi(S) = 0$ then (8.2) implies $c_1(N_S) = 0$. Assume that $\chi(S) = 2$. Thus $c_1(N_S) = 0$, 1, 2. However if $c_1(N_S) = 1$ and $\nu_f = 2$ or if $c_1(N_S) = 2$ (and necessarily $\nu_f = 1$) then (8.1) would imply that $X_f$ has no zeroes, and thus $c_1(N_S) = 0$ by Theorem 6.2. □

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