

Metric attractors for smooth unimodal maps

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Abstract

We classify the measure theoretic attractors of general C^3 unimodal maps with quadratic critical points. The main ingredient is the decay of geometry.

1. Introduction

1.1. *Statement of results.* The study of measure theoretical attractors occupied a central position in the theory of smooth dynamical systems in the 1990s. Recall that a forward invariant compact set A is called a (minimal) *metric attractor* for some dynamics if the basin of attraction $B(A) := \{x : \omega(x) \subset A\}$ of A has positive Lebesgue measure and $B(A')$ has Lebesgue measure zero for every forward invariant compact set A' strictly contained in A . Recall that a set is *nowhere dense* if its closure has empty interior, and *meager* if it is a countable union of nowhere dense sets. A forward invariant compact set A is called a (minimal) *topological attractor* if $B(A)$ is not meager while $B(A')$ is meager for every forward invariant compact set A' strictly contained in A . A basic question, known as Milnor's problem, is whether the metric and topological attractors coincide for a given smooth unimodal map.

Milnor's problem has a long and turbulent history; see [16], [11], [5], [2]. In the class of C^3 unimodal maps with negative Schwarzian derivative and a quadratic critical point, an early solution to Milnor's problem was given in [11]. Recently, it was discovered that [11] does not provide a complete proof. The author has told us that his argument can be repaired, [12]. A correct solution using different techniques can be found in [2]. A negative solution when the critical point has high order is given in [1]. The C^3 stability theorem of [8], [10] implies that a generic C^3 unimodal map has finitely many metric attractors which are all attracting cycles, thus solving Milnor's problem in the generic case.

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Our current work solves Milnor's problem for smooth unimodal maps with a quadratic critical point. Historically, the solution is based on two key developments. The first, [2], established decay of geometry for a class of C^3 nonrenormalizable box mappings with finitely many branches and negative Schwarzian derivative everywhere except at the critical point which must be quadratic. The second, [9], recovers negative Schwarzian derivative for smooth unimodal maps with nonflat critical point: the first return map to a neighborhood of the critical value has negative Schwarzian derivative.

Technically, for our study of metric attractors in the smooth category we need a different estimate from that of [9], one which works near the critical point rather than the critical value [3]. We add a new Koebe lemma and exploit the fact that negative Schwarzian derivative is not an invariant of smooth conjugacy to show that the first return map to a neighborhood of the critical point can be real-analytically conjugated to one having negative Schwarzian derivative. This makes it easy to transfer results known for maps with negative Schwarzian to the smooth class. Earlier results in this direction, in particular that high iterates of a smooth critical circle homeomorphism have negative Schwarzian, were obtained in [4].

The classification of metric attractors containing the (nondegenerate) critical point was announced in [3]. Here we give full proofs and explain the structure of metric attractors not containing the critical point (based on the work of Mañé [13]). Consequently, we obtain the classification of all metric attractors for smooth unimodal maps with a nondegenerate critical point.

Classification of metric dynamics. A C^1 map f of a compact interval I is *unimodal* if it has exactly one point ζ where $f'(\zeta) = 0$ (the *critical point*), $\zeta \in \text{int } I$, f' changes sign at ζ , and f maps the boundary of I into itself. The critical point of f is C^n *nonflat of order ℓ* if, near ζ , f can be written as $f(x) = \pm|\phi(x)|^\ell + f(\zeta)$ where ϕ is a C^n diffeomorphism. The critical point is C^n *nonflat* if it is C^n nonflat of some order $\ell > 1$. The set of critical points of f is denoted by Crit .

THEOREM 1. *Let I be a compact interval and $f : I \rightarrow I$ be a C^3 unimodal map with C^3 nonflat critical point of order 2. Then the ω -limit set of Lebesgue almost every point of I is either*

1. *a nonrepelling periodic orbit, or*
2. *a transitive cycle of intervals, or*
3. *a Cantor set of solenoid type.*

A compact interval J is *restrictive* if J contains the critical point of f in its interior and, for some $n > 0$, $f^n(J) \subseteq J$ and $f^n|_J$ is unimodal. In particular, f^n maps the boundary of J into itself. This restriction of f^n to J

is called a *renormalization* of f . We say that f is *infinitely renormalizable* if it has infinitely many restrictive intervals.

A periodic point x of period p is *repelling* if $|Df^p(x)| > 1$, *attracting* if $|Df^p(x)| < 1$, *neutral* if $|Df^p(x)| = 1$ and *super-attracting* if $Df^p(x) = 0$. It is *topologically attracting* if its *basin of attraction* $B(x) := B(\{x, f(x), \dots, f^{p-1}(x)\})$ has nonempty interior.

A *transitive cycle of intervals* is a finite union C of compact intervals such that C is invariant under f , C contains the critical point of f in its interior, and the action of f on C is transitive (has a dense orbit).

We say that f has a *Cantor set of solenoid type* if f is infinitely renormalizable, the solenoid then being the ω -limit set of the critical point.

Note that the critical point ζ of a C^4 unimodal map with $f''(\zeta) \neq 0$ is C^3 nonflat of order 2. The fact that the critical point has order 2 is used in an essential way to exclude the possibility of *wild Cantor attractors*.

COROLLARY 1. *Every metric attractor of f is either*

1. *a topologically attracting periodic orbit, or*
2. *a transitive cycle of intervals, or*
3. *a Cantor set of solenoid type.*

There is at most one metric attractor of type other than 1.

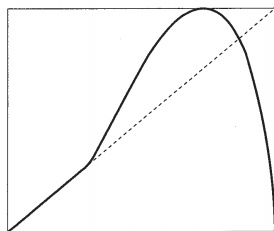


Figure 1: Almost every point is mapped into the interval of fixed points.

Figure 1 shows a unimodal map satisfying our hypotheses for which the ω -limit set of Lebesgue almost-every point is a neutral fixed point. This map has no metric attractors.

COROLLARY 2. *The metric and topological attractors of f coincide.*

Decay of geometry. Following the concept of an adapted interval [13] we call an open set U *regularly returning* for some dynamics f defined in an ambient space containing \bar{U} if $f^n(\partial U) \cap U = \emptyset$ for every $n > 0$.

The *first entry map* E of f into a set U is defined on

$$E_U := \{x : \exists n > 0, f^n(x) \in U\}$$

by the formula $E(x) := f^{n(x)}(x)$ where $n(x) := \min\{n > 0 : f^n(x) \in U\}$. The *first return map* of f into U is the restriction of the first entry map to $E_U \cap U$. The *central domain* of the first return map is the connected component of its domain containing the critical point of f . If U is a regularly returning open interval then the function $n(x)$ is continuous and locally constant on E_U .

Definition 1. Suppose that J is an open interval and $\bar{J} \subset I$. Define $\nu(J, I) := \frac{|J|}{\text{dist}(J, \partial I)}$.

The key property that enables us to exclude wild Cantor attractors is the following result, known as *decay of geometry*.

THEOREM 2. *Let I be a compact interval and $f : I \rightarrow I$ be a C^3 unimodal map with C^3 nonflat critical point ζ of order 2. If ζ is recurrent and nonperiodic and f has only finitely many restrictive intervals then for every $\varepsilon_0 > 0$ there is a regularly returning interval $Y' \ni \zeta$ such that if Y is the central domain of the first return map to Y' , then $\nu(Y, Y') < \varepsilon_0$.*

Decay of geometry occurs when the order of the critical point is 2. Counterexamples exist when the order of the critical point is larger than 2 [1].

A priori bounds. The following important fact known as *a priori bounds* is proved in [9, Lemma 7.4]. An earlier version for nonrenormalizable maps can be found in [18].

FACT 1. *Let f be a C^3 unimodal map with C^3 nonflat nonperiodic critical point ζ . Then there exists a constant K and an infinite sequence of pairs $Y'_i \supset Y_i \ni \zeta$ of open intervals with $|Y_i| \rightarrow 0$ such that, for each i , Y_i is regularly returning, $\nu(Y_i, Y'_i) \leq K$ and for every branch f^n of the first entry map of f into Y_i , f^n extends diffeomorphically onto Y'_i provided the domain of the branch is disjoint from Y_i .*

Negative Schwarzian derivative and conjugation theorem. We say that a C^3 function g has *negative Schwarzian derivative* if

$$S(g)(x) := g'''(x)/g'(x) - \frac{3}{2} (g''(x)/g'(x))^2 < 0$$

whenever $g'(x) \neq 0$. The Schwarzian derivative $S(g)$ satisfies the composition law $S(g \circ h)(x) = S(g)(h(x))h'(x)^2 + S(h)(x)$. Thus iterates of a map with negative Schwarzian derivative also have negative Schwarzian derivative.

In the general smooth case, negative Schwarzian derivative can be recovered [3] in the following sense.

THEOREM 3. *Let I be a compact interval and $f : I \rightarrow I$ be a C^3 unimodal map with C^3 nonflat and nonperiodic critical point. Then there exists a real-analytic diffeomorphism $h : I \rightarrow I$ and an (arbitrarily small) open interval U such that, putting $g = h \circ f \circ h^{-1}$, U is a regularly returning (for g) neighborhood of the critical point of g and the first return map of g to U has uniformly negative Schwarzian derivative.*

1.2. Box mappings.

Definition 2. Consider a finite sequence of compactly nested open intervals around a point $\zeta \in \mathbf{R} : \zeta \in b_0 \subset \bar{b}_0 \subset b_1 \cdots \subset b_k$. Let ϕ be a real-valued C^1 map defined on some open and bounded set $U \subset \mathbf{R}$ containing ζ . Suppose that the derivative of ϕ only vanishes at ζ , which is a local extremum. Assume in addition the following:

- for every $i = 0, \dots, k$, we have $\partial b_k \cap U = \emptyset$,
- b_0 is equal to the connected component of U which contains ζ ,
- for every connected component W of U there exists $0 \leq i \leq k$ so that ϕ maps W into b_i and $\phi : W \rightarrow b_i$ is proper.

Then the map ϕ is called a *box mapping* and the intervals b_i are called *boxes*.

The restriction of a box map to a connected component of its domain will be called a *branch*. Depending on whether the domain of this branch contains the critical point ζ or not, the branch will be called *folding* or *monotone*. The domain b_0 of the folding branch is called the *central domain* and will usually be denoted by b ; b' will denote the box into which the folding branch maps properly. A box map ϕ is said to be *induced* by a map f if each branch of ϕ coincides on its domain with an iterate of f (the iterate may depend on the branch).

Type I and type II box mappings. A box mapping is of type II provided that there are only two boxes $b_0 = b$ and $b_1 = b'$, and every branch is proper into b' . A box mapping is of type I if there are only two boxes, the folding branch is proper into b' and all other branches are diffeomorphisms onto b . A type I box mapping can be canonically obtained from a type II box map ϕ by *filling-in*, in which ϕ outside of b is replaced by the first entry map into b . Note that if f is a unimodal map with critical point ζ and I is a regularly returning open interval containing ζ , then the first return map of f into I is a type II box mapping.

2. Distortion estimates

In this section we prove a strong form of the C^2 Koebe lemma (Proposition 1). In Lemma 2.3 we give a new proof of the required cross-ratio estimates.

Let I be an open interval and $h : I \rightarrow h(I) \subseteq \mathbf{R}$ be a C^1 diffeomorphism. Let a, b, c, d be distinct points of I and define the cross-ratio $\chi(a, b, c, d) := \frac{(c-b)(d-a)}{(c-a)(d-b)}$. By the distortion of χ by h we mean

$$\kappa_h(a, b, c, d) := \chi(h(a), h(b), h(c), h(d)) / \chi(a, b, c, d) .$$

We have the composition rule

$$(1) \quad \log \kappa_{g \circ h}(a, b, c, d) = \log \kappa_g(h(a), h(b), h(c), h(d)) + \log \kappa_h(a, b, c, d) .$$

Define, for $x \neq y$,

$$K_h(x, y) := \frac{\partial}{\partial x} \log \left| \frac{h(x) - h(y)}{x - y} \right| = \frac{h'(x)}{h(x) - h(y)} - \frac{1}{x - y} .$$

An elementary calculation shows that

$$\log \kappa_h(a, b, c, d) = \int_a^b K_h(x, c) - K_h(x, d) dx = \int_{\partial R} K_h(x, y) dx .$$

where R is the rectangle $[a, b] \times [c, d]$ suitably oriented. Note that $K_h(x, y)$ is perhaps integrated across the diagonal $x = y$.

We will also use $\rho_h(a, b, c, d) := \log \kappa_h(a, b, c, d) / (b - a)(d - c)$.

LEMMA 2.1. *Let I be an open interval and let $h : I \rightarrow h(I) \subseteq \mathbf{R}$ be a C^2 diffeomorphism such that $1/\sqrt{|Dh|}$ is convex. Then $\rho_h(a, b, c, d) \geq 0$ for all distinct points a, b, c, d in I .*

Proof. If h is C^3 then the result follows from the formula $\log \kappa_h(a, b, c, d) = \int_a^b \int_c^d 1/(x-y)^2 - h'(x)h'(y)/(h(x)-h(y))^2 dx dy$ since the integrand is nonnegative (equivalent to a standard inequality for maps with nonpositive Schwarzian derivative). The C^2 statement follows by an approximation argument. \square

Definition 3. A continuous increasing function $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ such that $\sigma(0) = 0$ will be called a *gauge function*.

We first consider the case without critical points:

LEMMA 2.2. *Let I be a compact interval and let $h : I \rightarrow h(I) \subseteq \mathbf{R}$ be a C^2 diffeomorphism. Then there exists a gauge function σ , for all distinct points a, b, c, d in I , such that $|\rho_h(a, b, c, d)| \leq |\sigma(d - c)/(d - c)|$.*

Proof. Extend $K_h(x, y)$ to the diagonal of $I \times I$ by defining $K_h(x, x) = \frac{h''(x)}{2h'(x)}$ for $x \in I$. It is easily checked using Taylor expansions that K_h is continuous and thus uniformly continuous. Set $\Delta_h(x, y, z) := K_h(x, y) - K_h(x, z)$ and note that $\Delta_h(x, y, y) = 0$ for all $x, y \in I$. Thus there exists a gauge function σ such that $|\Delta_h(x, y, z)| \leq |\sigma(z - y)|$ for all $x, y, z \in I$. From $\log \kappa_h(a, b, c, d) = \int_a^b \Delta_h(x, c, d) dx$ we see that $|\log \kappa_h(a, b, c, d)| \leq |b - a| |\sigma(d - c)|$. \square

We now allow critical points on the boundary of the interval. The following result generalizes a number of known cross-ratio inequalities; see Theorems 2.1 and 2.2 of [17].

LEMMA 2.3. *Let I be a compact interval and $f : I \rightarrow \mathbf{R}$ be a C^2 map with all critical points C^2 nonflat. Then there exists a gauge function σ such that for all distinct points a, b, c, d in I contained in the closure of a subinterval J on which f is a diffeomorphism,*

$$\rho_f(a, b, c, d) \geq -\min \left(\frac{\sigma(b-a)}{b-a}, \frac{\sigma(d-c)}{d-c} \right).$$

Proof. It suffices to prove $\rho_f(a, b, c, d) \geq -\sigma(d-c)/d-c$ since $\rho_f(a, b, c, d) = \rho_f(c, d, a, b)$. By C^2 nonflatness of the critical points, for every $c \in \text{Crit}$ there exist ε_c and a C^2 diffeomorphism ϕ_c such that $f(x) = f(c) \pm |\phi_c(x)|^{\ell_c}$, $\ell_c > 1$, in $U_c = [c - \varepsilon_c, c + \varepsilon_c] \cap I$. Let $\varepsilon := \inf_{c \in \text{Crit}} \varepsilon_c/2$. Since f has at most finitely many critical points, ε is positive.

Suppose that $[a, d]$ is contained in an interval J whose endpoints are either in Crit or in ∂I and $f' \neq 0$ inside J . Set $\Omega_\eta = \{(x, y) \in J^2 : |x - y| < \eta\}$ and note that $K_f(x, y)$ is continuous for (x, y) in the compact set $J^2 \setminus \Omega_\eta$. If $[a, b] \times [c, d] \cap \Omega_\eta = \emptyset$ then

$$(2) \quad |\log \kappa_f(a, b, c, d)| = \left| \int_a^b K_f(x, c) - K_f(x, d) dx \right| \leq |(b-a)\tilde{\sigma}(d-c)|$$

for some gauge function $\tilde{\sigma}$ and consequently, $|\rho_f(a, b, c, d)| \leq \tilde{\sigma}(d-c)/d-c$.

Now subdivide the rectangle $R = [a, b] \times [c, d]$ into N equal rectangles $R_i = [a_i, b_i] \times [c_i, d_i]$ with the sides smaller than $\eta := \varepsilon/3$ and the orientation induced by R . In particular, the sign of $(b_i - a_i)(d_i - c_i)$ does not depend on i . We will use the fact that

$$\rho_f(a, b, c, d) = \frac{1}{(b-a)(d-c)} \sum_i \int_{\partial R_i} K_f(x, y) dx = \frac{1}{N} \sum_i \rho_f(a_i, b_i, c_i, d_i).$$

If $R_i \cap \Omega_{\varepsilon/3} = \emptyset$ then the estimate (2) works. If $R_i \cap \Omega_{\varepsilon/3} \neq \emptyset$ then R_i is contained in Δ_ε . In particular, a_i, b_i, c_i, d_i are contained in the interval J_i of length $\leq \varepsilon$. We consider two cases.

- (i) If J_i is not contained in $\bigcup_{c \in \text{Crit}} U_c(\varepsilon_c)$ then the distance of J_i to Crit is bigger than ε . To estimate $\int_{\partial R_i} K_f(x, y) dx$ we apply Lemma 2.2 for f restricted $J \setminus \bigcup_{c \in \text{Crit}} U_c(\varepsilon)$.
- (ii) If J_i is contained in $\bigcup_{c \in \text{Crit}} U_c(\varepsilon)$ then we write $f(x) = f(c) \pm |\phi_c(x)|^{\ell_c}$ for $x \in U_c(\varepsilon_c)$. If $g = |\cdot|^\ell$ then $\rho_g(a_i, b_i, c_i, d_i) \geq 0$ and it is enough, by the composition rule (1), to consider the effect of ϕ . Lemma 2.2 gives us the desired estimate.

We finish the proof by summing up the contributions from all rectangles R_i . \square

PROPOSITION 1 (the Koebe principle). *Let I be a compact interval and $f : I \rightarrow I$ be a C^2 map with all critical points C^2 nonflat. Then there exists a gauge function σ with the following property. If $J \subset T$ are open intervals and $n \in \mathbb{N}$ is such that f^n is a diffeomorphism on T then, for every $x, y \in J$, we have*

$$\frac{(f^n)'(x)}{(f^n)'(y)} \geq \frac{e^{-\sigma(\max_{i=0}^{n-1} |f^i(T)|) \sum_{i=0}^{n-1} |f^i(J)|}}{(1 + \nu(f^n(J), f^n(T)))^2}.$$

Proof. Without loss of generality $T = (\alpha, \beta)$, $J = (x, y)$ and $\alpha < x < y < \beta$. Write $F = f^n$ and let σ be as in Proposition 2.3. Set $P = \sum_{i=0}^{n-1} \sigma(|f^i(T)|) |f^i(J)|$. By Proposition 2.3 and (1),

$$\begin{aligned} \log \kappa_F(\alpha, x, x + \varepsilon, y) &\geq \sum_{i=0}^{n-1} \log \kappa_f(f^i(\alpha), f^i(x), f^i(x + \varepsilon), f^i(y)) \\ &\geq - \sum_{i=0}^{n-1} \sigma(|f^i(\alpha, x)|) |f^i(x + \varepsilon, y)| \\ &\geq - \sum_{i=0}^{n-1} \sigma(|f^i(T)|) |f^i(J)| = -P. \end{aligned}$$

Taking $\varepsilon \downarrow 0$ yields

$$\frac{F(y) - F(\alpha)}{y - \alpha} F'(x) \geq e^{-P} \frac{F(x) - F(\alpha)}{x - \alpha} \frac{|F(J)|}{|J|}$$

which after rearranging becomes

$$(3) \quad F'(x) \geq e^{-P} \frac{|\alpha - y|}{|\alpha - x|} \frac{|F(\alpha) - F(x)|}{|F(\alpha) - F(y)|} \frac{|F(J)|}{|J|} \geq \frac{e^{-P}}{1 + \nu(F(J), F(T))} \frac{|F(J)|}{|J|}.$$

Likewise, considering $\kappa_F(x, y - \varepsilon, x + \varepsilon, y)$ and taking $\varepsilon \downarrow 0$ yields

$$\frac{|F(J)|^2}{|J|^2} \geq e^{-P} F'(x) F'(y).$$

Equation (3) now gives $F'(x)/F'(y) \geq e^{-3P}/(1 + \nu(F(J), F(T)))^2$. \square

3. Proof of the conjugation theorem

In the following easy lemma we consider diffeomorphisms with constant negative Schwarzian derivative. These will be useful in defining the conjugacy in Theorem 3.

LEMMA 3.1. *For $s > 0$ consider the function*

$$\phi_s(x) := \frac{\tanh(\sqrt{\frac{s}{2}}x)}{\tanh(\sqrt{\frac{s}{2}})},$$

which is a real-analytic diffeomorphism of the real line into itself, fixing -1 , 0 and 1 . The Schwarzian derivative of ϕ_s is everywhere equal to $-s$.

The following lemma is included for completeness. An interval J is *symmetric* for a unimodal map f if $J = f^{-1}(f(J))$.

LEMMA 3.2. *Let I be a compact interval and $f : I \rightarrow I$ be a unimodal map. If f does not have arbitrarily small regularly returning symmetric open intervals containing the critical point ζ then ζ is periodic.*

Proof. Let J be the interior of the intersection of all regularly returning symmetric open intervals containing ζ . We must show that if $J \neq \emptyset$ then ζ is periodic. Indeed, if $J \neq \emptyset$ then J is clearly a regularly returning symmetric open interval containing ζ . By the minimality of J , ζ is mapped inside J by some iterate of f . Let ϕ be the first return map to J , which by minimality has only one branch. Again by minimality ϕ cannot have fixed points inside J other than ζ . Moreover ζ is indeed a fixed point of ϕ since otherwise we could easily construct an appropriate regularly returning interval inside J containing ζ . \square

The next lemma is a standard consequence of the nonexistence of wandering intervals [6].

LEMMA 3.3. *Let f be a C^2 unimodal map with C^2 nonflat, nonperiodic critical point ζ . For every interval $Y \ni \zeta$ there exists $\varepsilon_0(Y) > 0$ such that if I is an interval mapped diffeomorphically onto Y by some iterate f^n then $|f^i(I)| \leq \varepsilon_0(Y)$ for every $i = 0, \dots, n$, and $\varepsilon_0(Y) \rightarrow 0$ as $|Y| \rightarrow 0$.*

Proof. Otherwise there exists $\delta > 0$, a sequence $Y_i \downarrow \{\zeta\}$ of open intervals, intervals I_i with $|I_i| > \delta$ and $n_i \rightarrow \infty$ such that f^{n_i} maps I_i diffeomorphically onto Y_i . Passing to a subsequence, we may suppose the I_i converge to some limit interval I_∞ with $|I_\infty| \geq \delta$. Let I be an interval compactly contained in the interior of I_∞ . By definition $f^{n_i}|_I$ is diffeomorphic for arbitrarily large n_i . Thus $f^n|_I$ is diffeomorphic for all $n > 0$, which shows that I is a homterval. Since f has no wandering intervals [6], this means that $\omega(x)$ is a periodic orbit for some $x \in I$. However $\zeta \in \omega(x)$ by definition; thus ζ is periodic, a contradiction. \square

Suppose now that f^n is a branch of the first entry map into an interval $Y := Y_i$ given by fact 1, and that the domain J of the branch is disjoint from Y . There is a number $\varepsilon(Y) > 0$ independent of the branch such that for all $x \in J$ we have $S(f^n)(x) \leq \frac{\varepsilon(Y)}{|J|^2}$ and $\varepsilon(Y) \rightarrow 0$ as $|Y| \rightarrow 0$. Indeed, letting

$L = \max(0, \sup S(f)) < \infty$, the composition law for the Schwarzian derivative, the Koebe principle and the disjointness of $J, \dots, f^{n-1}(J)$ yield

$$\begin{aligned} S(f^n)(x) &= \sum_{i=0}^{n-1} S(f)(f^i(x))(f^i)'(x)^2 \leq \frac{LK^4}{|J|^2} \sum_{i=0}^{n-1} |f^i(J)|^2 \\ &\leq \frac{\varepsilon_0(Y)LK^4}{|J|^2} \sum_{i=0}^{n-1} |f^i(J)| \leq \frac{\varepsilon(Y)}{|J|^2}, \end{aligned}$$

where $K > 0$ comes from the Koebe lemma.

We will also need the fact that if f has a C^3 nonflat critical point ζ then there is some $\eta > 0$ such that $S(f)(x) < -\eta/(x - \zeta)^2$ for all $x \neq \zeta$ sufficiently close to ζ .

Proof of Theorem 3. Fix some $0 < s < \eta/4$ and consider an interval $Y := Y_i$ given by fact 1. Let F denote the first return map to Y and let A be the increasing affine map taking Y to $(-1, 1)$. Observe that $h := \phi_s \circ A$ has constant Schwarzian derivative $S(h)(x) = -4s/|Y|^2$. The composition law for the Schwarzian derivative gives

$$(4) \quad S(h \circ F \circ h^{-1})(h(x))h'(x)^2 = \frac{4s}{|Y|^2}(1 - F'(x)^2) + SF(x)$$

for all x in the domain of F . Let I be the domain of a branch f^{n+1} of F . Then $f(I)$ is contained in a domain J of a branch f^n of the first entry map into Y , and J is disjoint from Y (if $\zeta \notin I$ then $J = f(I)$). Let $G = h \circ F \circ h^{-1}$. Equation 4 and the results noted above yield, for $x \in I$,

$$\begin{aligned} S(G)(h(x))h'(x)^2 &= \frac{4s}{|Y|^2}(1 - f'(x)^2(f^n)'(f(x))^2) + S(f^n) \circ f f'(x)^2 + S(f)(x) \\ &\leq \frac{4s}{|Y|^2} \left(1 - f'(x)^2 \frac{|Y|^2}{K^2|J|^2} \right) + \varepsilon(Y) \frac{f'(x)^2}{|J|^2} + S(f)(x) \\ &= \frac{f'(x)^2}{|J|^2} \left(\varepsilon(Y) - \frac{4s}{K^2} \right) + S(f)(x) + \frac{4s}{|Y|^2}. \end{aligned}$$

Now $\varepsilon(Y) - 4s/K^2$ will be negative as long as Y is small enough. Since $x \in Y$ we have $S(f)(x) + 4s/|Y|^2 < (4s - \eta)/|Y|^2 < 0$ if $|Y|$ is small enough. Thus $S(G)(y) < 0$ for all y in the domain of G if Y is small enough. \square

We immediately obtain a weak form of the finiteness of attractors theorem [6]:

COROLLARY 3. *Let I be a compact interval and $f : I \rightarrow I$ be a C^3 unimodal map with C^3 nonflat critical point. Then there exists $N \in \mathbf{Z}^+$ such that any periodic orbit with period greater than N is repelling.*

Proof. It is well known [13, Th. C] that if nonrepelling periodic orbits of arbitrarily high period exist, then they must accumulate the critical point. This is impossible if the critical point ζ is periodic. If ζ is not periodic then, by the conjugation theorem, after a real-analytic coordinate change, the first return map ϕ of f to a regularly returning interval U containing ζ has negative Schwarzian derivative. Because of the negative Schwarzian derivative, all nonrepelling periodic orbits of ϕ must attract ζ , so there can be at most one of these. Since any periodic orbit of f passing through U is periodic for ϕ , this proves the result. \square

4. Decay of geometry

PROPOSITION 2. *Let F denote a C^3 type II box mapping with C^3 nonflat critical point ζ of order 2 and negative Schwarzian derivative. Assume that the orbit of ζ is infinite, recurrent and that F has no restrictive interval. Suppose that F has the following expansivity property: for every $\eta > 0$ there is some $\delta > 0$ such that if I is an interval mapped by a nonnegative iterate of F diffeomorphically onto an interval of length less than δ containing ζ , then $|I| < \eta$. Now, for every $\varepsilon_0 > 0$ there is a regularly returning interval Y' which contains ζ such that, if Y denotes the central domain of the first return map into Y' , then $\nu(Y, Y') < \varepsilon_0$.*

The proof will be split into two cases depending on whether $\omega(\zeta)$ intersects the domains of finitely many branches of F , or infinitely many.

The case with infinitely many branches. For each domain Δ of a branch which intersects $\omega(\zeta)$, we define $n(\Delta)$, the first entry time of the orbit of ζ into Δ . We choose a sequence Δ_i with $|\Delta_i| \rightarrow 0$. For each i the interval Δ_i can be pulled back by $F^{1-n(\Delta_i)}$ as a diffeomorphism to a neighborhood of the critical value. The lengths of these neighborhoods tend to 0 by the expansivity hypothesis. Pulling back by one more iterate of F we get a sequence of regularly returning neighborhoods Y'_i of ζ , each of which is mapped into Δ_i by $F^{n(\Delta_i)}$ as a proper unimodal map.

Let us construct Y_i , the corresponding central domain. It is the preimage by $F^{-n(\Delta_i)}$ of some domain δ_i of the first entry map into Y'_i . We begin by considering the first entry map into the central domain of F . Let δ'_i be the domain of the first entry map into the central domain of F which contains δ_i . Since the nesting of the central domain inside the range of F is preserved under pull-back by negative Schwarzian diffeomorphisms, we get $\nu(\delta'_i, \Delta_i) \geq \eta > 0$ where η is independent of i . Likewise, by the classical Koebe lemma for maps with negative Schwarzian derivative, the distortion of the first entry map on δ'_i is bounded independently of i . By the expansivity hypothesis again,

lengths of domains of branches of the first entry map into Y'_i tend to 0 with i . Hence $\nu(\delta_i, \Delta_i) \rightarrow 0$. Again by negative Schwarzian derivative, this implies $\nu(Y_i, Y'_i) \rightarrow 0$ which implies the result.

Observe that this argument did not make use of the hypothesis about the nondegeneracy of the critical point.

The case with finitely many branches. In this case will use the following theorem.

THEOREM 4. *Let F denote a C^3 type II box mapping with finitely many branches, negative Schwarzian derivative, with the central branch which factors as $F(\zeta)(1 - (h(x))^2)$ where h is a C^3 increasing diffeomorphism of the closure of the central domain onto its image and $h(\zeta) = 0$. Assume that the orbit of ζ is infinite, recurrent and that F has no restrictive interval. Now, for every $\varepsilon_0 > 0$ there is a regularly returning interval Y' which contains ζ such that, if Y denotes the central domain of the first return map into Y' , then $\nu(Y, Y') < \varepsilon_0$.*

Theorem 4 is a weaker restatement of Theorem 1 from [2].

The map F from Proposition 2 can be restricted to only those branches whose domains intersect $\omega(\zeta)$ and the remaining hypotheses will still apply. Now, we see that Theorem 4 is directly applicable under the hypotheses of Proposition 2 except for the special form of the central branch. This problem is taken care of by an elementary calculation.

Proof of Theorem 2. We apply Proposition 2 to the first return map F of f to a regularly returning open interval U containing the critical point. We may assume that F has negative Schwarzian derivative by Theorem 3. If U is small enough then F will have no restrictive intervals. Moreover, we take U small enough that its closure contains no nonrepelling periodic points (Corollary 3). The expansivity hypothesis is then satisfied. Indeed, if not, then arguing by contradiction as in Lemma 3.3, we see that F (and thus f) would have a homterval I . Since f has no wandering intervals, this means that some point of I is attracted to a nonrepelling periodic orbit. This orbit must intersect the closure of U , a contradiction. \square

Induced expansion. We will say that a unimodal map f induces expansion if there is a regularly returning open interval J containing the critical point of f , an open subset U of J , a map $F : U \rightarrow J$ and $\rho > 1$ with the following properties: for each connected component V of U there is a positive integer $n(V)$ such that F coincides with $f^{n(V)}$ on V , F maps V diffeomorphically onto J with derivative (in absolute value) at least ρ , and if A is the set of points in J which return to J infinitely often under iteration by f , then U contains Lebesgue almost every point of A .

PROPOSITION 3. *Let I be a compact interval and $f : I \rightarrow I$ be a C^3 unimodal map with C^3 nonflat critical point ζ of order 2. If ζ is nonperiodic and f has only finitely many restrictive intervals then f induces expansion.*

Proof. As in the proof of Theorem 2, we consider the first return map F of f to a regularly returning interval U containing the critical point. We may assume that F has negative Schwarzian by Theorem 3. By Theorem 2, we may take the central domain of F to be as small as we like, proportionally to U , by taking U small enough. Adapting Proposition 5 from [7] to F , we see that F induces an expanding Markov map on some perhaps smaller regularly returning open interval U' containing ζ , in other words f induces expansion on U' . \square

5. Attractors

Dynamics away from critical points. Our study of the dynamics away from critical points is based on the following result of Mañé [13, Th. D]:

FACT 2. *Let I be a compact interval, $f : I \rightarrow I$ be a C^2 map and $K \subseteq I$ be a compact invariant set not containing critical points. Then either K has Lebesgue measure zero or there exist an interval $J \subseteq I$ and $n \geq 1$ such that $f^n(J) \subseteq J$, $f^n|_J$ has no critical points and $J \cap K$ has positive Lebesgue measure.*

An interval map f is *nonsingular* if $f^{-1}(A)$ has zero Lebesgue measure for every Borel set A with zero Lebesgue measure. A map with a finite number of critical points is nonsingular.

COROLLARY 4. *Let I be a compact interval and $f : I \rightarrow I$ be C^2 and nonsingular. Then, for Lebesgue almost every $x \in I$, either $\omega(x)$ contains a critical point or $\omega(x)$ coincides with a nonrepelling periodic orbit.*

Proof. Let A be the set of all $x \in I$ such that $\omega(x) \cap \text{Crit} = \emptyset$ and $\omega(x)$ is not a nonrepelling periodic orbit. If A is of positive Lebesgue measure then there exists an open neighborhood U of Crit and a forward invariant $B \subset A$ of positive Lebesgue measure so that the forward orbit of every point of B is disjoint from U . The support K of Lebesgue measure restricted to B is forward invariant also. By fact 2, there exist an interval $J \subseteq I$ and $n \geq 1$ such that $f^n(J) \subseteq J$, $f^n|_J$ has no critical points and $J \cap K$ has positive measure. It follows that $J \cap A$ has positive measure also. But $\omega(x)$ is a nonrepelling periodic orbit for almost every $x \in J$, a contradiction. \square

COROLLARY 5. *Let I be a compact interval and $f : I \rightarrow I$ be C^2 and nonsingular. Then every metric attractor of f that does not contain critical points coincides with a topologically attracting periodic orbit.*

Proof. Let K be a metric attractor that contains no critical points. Since $\omega(x) = K$ for almost every $x \in \rho(K)$, it follows from the preceding corollary that K is a nonrepelling periodic orbit. If this orbit is not topologically attracting then $\rho(K)$ coincides with $\bigcup_{n \geq 0} f^{-n}(K)$ and has measure zero, a contradiction. \square

It may be instructive to examine Figure 1 in the light of these results.

Attractors containing the critical point. In the light of Corollary 4, Theorem 1 is reduced to the following assertion.

PROPOSITION 4. *Let I be a compact interval and $f : I \rightarrow I$ be a C^3 unimodal map with C^3 nonflat critical point of order 2. Then, for Lebesgue almost every $x \in I$, either $\omega(x)$ does not contain a critical point or $\omega(x)$ coincides with either*

- *a super-attracting periodic orbit, or*
- *a transitive cycle of intervals, or*
- *a Cantor set of solenoid type.*

The proof of the proposition uses the following lemma.

LEMMA 5.1. *Let I be a compact interval and $f : I \rightarrow I$ be a unimodal map with critical point ζ . If there exists a subinterval Y containing ζ in its interior such that, for Lebesgue almost every $x \in Y$, the orbit of x intersects Y in a set of full Lebesgue measure, then f has a metric attractor which is a transitive cycle of intervals.*

Proof. Note that some iterate of Y intersects Y since almost every point in Y returns to Y . Thus C , the closure of the union of the iterates of Y , is a finite union of intervals. Clearly C is forward invariant and contains ζ in its interior. By the definition of Y , the orbit of almost every $x \in C$ is dense in C which implies that C is a transitive metric attractor. \square

Proof of Proposition 4. If the critical point is periodic then it is a super-attracting periodic point and the result is obvious. If f is infinitely renormalizable and $\zeta \in \omega(x)$, then $\omega(x)$ is obviously contained in the intersection of the orbits of all restrictive intervals. It is easily shown that this is a Cantor set of solenoid type coinciding with $\omega(c)$. In fact, one can even use the classical

S-unimodal theory since, by Theorem 3, after a real-analytic coordinate change, all renormalizations of f of sufficiently high period have negative Schwarzian derivative.

We may therefore suppose that the critical point is not recurrent and that f has only finitely many restrictive intervals. From Corollary 3 we know that f induces expansion on an open interval J containing the critical point, the induced map F being defined on an open subset U of J . If U has full (Lebesgue) measure in J then the orbit by F of almost every point in J is dense in J ; thus, by Lemma 5.1, f has a metric attractor which is a transitive cycle of intervals. If U does not have full measure in J then almost every point in U leaves U under iteration by F . By the definition of induced expansion, almost every point $x \in J$ with $c \in \omega(x)$ never leaves U under iteration by F . Thus in this case we see that the set of points x with $c \in \omega(x)$ has measure zero, and so the result is trivial. \square

Proof of Corollary 1. Suppose that A is a metric attractor of f . If A contains the critical point then, by Proposition 4, A is either a super-attracting periodic orbit or a cycle of intervals or a solenoid. It follows from their definitions that these three possibilities are mutually exclusive. If A does not contain the critical point then, by Corollary 5, A is a topologically attracting periodic orbit. \square

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