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A new construction of the moonshine vertex operator algebra over the real number field

By Masahiko Miyamoto*

Abstract

We give a new construction of the moonshine module vertex operator algebra $V^\natural$, which was originally constructed in [FLM2]. We construct it as a framed VOA over the real number field $\mathbb{R}$. We also offer ways to transform a structure of framed VOA into another framed VOA. As applications, we study the five framed VOA structures on V_{E_8} and construct many framed VOAs including $V^\natural$ from a small VOA. One of the advantages of our construction is that we are able to construct $V^\natural$ as a framed VOA with a positive definite invariant bilinear form and we can easily prove that $\mathrm{Aut}(V^\natural)$ is the Monster simple group. By similar ways, we also construct an infinite series of holomorphic framed VOAs with finite full automorphism groups. At the end of the paper, we calculate the character of a $3C$ element of the Monster simple group.

1. Introduction

All vertex operator algebras (VOAs) $(V, Y, \mathbf{1}, \omega)$ in this paper are simple VOAs defined over the real number field $\mathbb{R}$ and satisfy $V = \oplus_{i=0}^{\infty} V_i$ and $\dim V_0 = 1$. $\mathbb{C}V$ denotes the complexification $\mathbb{C} \otimes_{\mathbb{R}} V$ of V. Throughout this paper, $v_{(m)}$ denotes a coefficient of vertex operator $Y(v,z) = \sum_{m\in\mathbb{Z}} v_{(m)} z^{-m-1}$ of v at z^{-m-1} and $Y(\omega, z) = \sum_{m\in\mathbb{Z}} L(m) z^{-m-2}$, where ω is the Virasoro element of V. VOAs (conformal field theories) are usually considered over $\mathbb{C}$, but VOAs over $\mathbb{R}$ are extremely important for finite group theory. The most interesting example of VOAs is the moonshine module VOA $V^\natural = \sum_{i=0}^{\infty} V_i^\natural$ over $\mathbb{R}$, constructed in [FLM2], whose second primary space $V_2^\natural$ coincides with the Griess algebra and the full automorphism group is the Monster simple group $\mathbb{M}$. Although it has many interesting properties, the original construction essentially depends on the actions of the centralizer $C_{\mathbb{M}}(\theta) \cong 2^{1+24}Co.1$ of a $2B$-involution θ of $\mathbb{M}$ and it is hard to see the actions of the other elements

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explicitly. The Monster simple group has the other conjugacy class of involutions called $2A$. One of the aims in this paper is to give a new construction of the moonshine module VOA $V^\natural$ from the point of view of an elementary abelian automorphism 2-group generated by $2A$-elements, which gives rise to a framed VOA structure on $V^\natural$. In this paper, we will show several techniques to transform framed VOAs into other framed VOAs. An advantage of our ways is that we can construct many framed VOAs from smaller pieces. As basic pieces, we will use a rational Virasoro VOA $L(\frac{1}{2},0)$ with central charge $\frac{1}{2}$, which is the minimal one of the discrete series of Virasoro VOAs. We note that $L(\frac{1}{2},0)$ over $\mathbb{R}$ satisfies the same fusion rules as the 2-dimensional Ising model $\mathbb{C}L(\frac{1}{2},0)$ does. In particular, we will use a rational conformal vector $e\in V_2$ with central charge $\frac{1}{2}$, that is, a Virasoro element of sub VOA $\langle e\rangle$ which is isomorphic to $L(\frac{1}{2},0)$. In this case, we have an automorphism τ_e of V defined by

$$\tau_e : \begin{cases} 1 & \text{on all } \langle e\rangle\text{-submodules isomorphic to } L(\frac{1}{2},0) \text{ or } L(\frac{1}{2},\frac{1}{2}) \\ -1 & \text{on all } \langle e\rangle\text{-submodules isomorphic to } L(\frac{1}{2},\frac{1}{16})\ , \end{cases} \tag{1.1}$$

whose complexification was given in [Mi1].

In this paper, we will consider a VOA $(V,Y,\mathbf{1},\omega)$ of central charge $\frac{n}{2}$ containing a set $\{e_i \mid i=1,\cdots,n\}$ of mutually orthogonal rational conformal vectors e_i with central charge $\frac{1}{2}$ such that the sum $\sum_{i=1}^n e_i$ is the Virasoro element ω of V. Here, "orthogonal" means $(e_i)_{(1)}e_j = 0$ for $i \neq j$. This is equivalent to the fact that a sub VOA $T = \langle e_1,\cdots,e_n\rangle$ is isomorphic to $L(\frac{1}{2},0)^{\otimes n}$ with Virasoro element ω. Such a VOA V is called "*a framed VOA*" in [DGH] and we will call the set $\{e_1,\ldots,e_n\}$ of conformal vectors "*a coordinate set*." We note that a VOA V of rank $\frac{n}{2}$ is a framed VOA if and only if V is a VOA containing $L(\frac{1}{2},0)^{\otimes n}$ as a sub VOA with the same Virasoro element. It is shown in [DMZ] that $V^\natural$ is a framed VOA of rank 24. Our main purpose in this paper is to reconstruct $V^\natural$ as a framed VOA. Another important example of framed VOAs is a code VOA M_D for an even linear code D, which is introduced by [Mi2]. It is known that every irreducible T-module W is a tensor product $\otimes_{i=1}^n L(\frac{1}{2},h_i)$ of irreducible $L(\frac{1}{2},0)$-modules $L(\frac{1}{2},h_i)$ ($h_i=0,\frac{1}{2},\frac{1}{16}$); see [DMZ]. Define a binary word

$$\tilde{\tau}(W)=(a_1,\cdots,a_n) \tag{1.2}$$

by $a_i=1$ if $h_i=\frac{1}{16}$ and $a_i=0$ if $h_i=0$ or $\frac{1}{2}$. It follows from the fusion rules of $L(\frac{1}{2},0)$-modules that if U is an irreducible M_D-module, then $\tilde{\tau}(W)$ does not depend on the choice of irreducible T-submodules W of U and so we denote it by $\tilde{\tau}(U)$. We call it a (binary) *τ-word* of U since it corresponds to the actions of automorphisms τ_{e_i}. Even if U is not irreducible, we use the same notation $\tilde{\tau}(U)$ if it is well-defined. We note that T is rational and the fusion rules are given by

$$(\otimes_{i=1}^n W^i) \times (\otimes_{i=1}^n U^i) = \otimes_{i=1}^n (W^i \times U^i)$$

for $L(\frac{1}{2},0)$-modules W^i, U^i as proved in [DMZ]. We have to note that their arguments also work for VOAs over $\mathbb{R}$.

As we will show, if V is a framed VOA with a coordinate set $\{e_1,\cdots,e_n\}$, then there are two binary linear codes D and S of length n such that V has the following structure:

(1) $V=\oplus_{\alpha\in S}V^{\alpha}$.

(2) $V^{(0^n)}$ is a code VOA M_D.

(3) V^{α} is an irreducible M_D-module with $\tilde{\tau}(V^{\alpha})=\alpha$ for every $\alpha\in S$.

We will call such a framed VOA a (D,S)-framed VOA.

In order to transform structures of framed VOAs smoothly, the uniqueness of a framed VOA structure is very useful (see Theorem 3.25). Although the uniqueness theorem holds for framed VOAs over $\mathbb{C}$ (see [Mi5]), it is not true for framed VOAs over $\mathbb{R}$. In order to avoid this anomaly, we assume the existence of a positive definite invariant bilinear form (PDIB-form). In this setting, we are able to transform framed VOA structures as in VOAs over $\mathbb{C}$. For example, "tensor product": for a (D,S)-framed VOA $V=\oplus_{\alpha\in S}V^{\alpha}$, $V^{\otimes r}$ is a $(D^{\oplus r},S^{\oplus r})$-framed VOA, and "restriction": for a subcode R of S, $\mathrm{Res}_R(V)=\oplus_{\alpha\in R}V^{\alpha}$ is a (D,R)-framed VOA, are easy transformations. The most important tool is "an induced VOA $\mathrm{Ind}_E^D(V)$." Let us explain it for a while. For $E\subseteq D\subseteq S^{\perp}$, we had constructed "induced $\mathbb{C}M_D$-module" $\mathrm{Ind}_E^D(\mathbb{C}W)$ from an M_E-module W in [Mi3]. We apply it to a VOA and construct a (D,S)-framed VOA $\mathrm{Ind}_E^D(W)$ from an (E,S)-framed VOA W. Fortunately, it preserves the PDIB-form. Moreover, the maximal one $\mathrm{Ind}_E^{S^{\perp}}(W)$ becomes a holomorphic VOA. As an example, we will construct the Leech lattice VOA V_{Λ} from $V^{\natural}$ by restricting and inducing.

We note that it is possible to construct $V^{\natural}$ over the rational number field (even over $\mathbb{Z}[\frac{1}{2}]$) in this way. However, we need several other conditions to get the uniqueness theorem and we will avoid such complications.

Our essential tool is the following theorem, which was proved for VOAs over $\mathbb{C}$ by the author in [Mi5].

Hypotheses I: (1) D and S are both even linear codes of length $8k$.

(2) Let $\{V^{\alpha}\mid\alpha\in S\}$ be a set of irreducible M_D-modules with $\tilde{\tau}(V^{\alpha})=\alpha$.

(3) For any $\alpha,\beta\in S$, there is a fusion rule $V^{\alpha}\times V^{\beta}=V^{\alpha+\beta}$.

(4) For $\alpha,\beta\in S-\{(0^n)\}$ satisfying $\alpha\neq\beta$, it is possible to define a $(D,\langle\alpha,\beta\rangle)$-framed VOA structure with a PDIB-form on

$$V^{\langle\alpha,\beta\rangle}=M_D\oplus V^{\alpha}\oplus V^{\beta}\oplus V^{\alpha+\beta}.$$

(4′) If $S=\langle\alpha\rangle$, $M_D\oplus V^{\alpha}$ is a framed VOA with a PDIB-form.

Theorem 3.25. *Under Hypotheses* I,

$$V=\bigoplus_{\alpha\in S}V^{\alpha}$$

has a structure of (D,S)*-framed* VOA *with a* PDIB*-form. A framed* VOA *structure on* $V=\oplus_{\alpha\in S}V^{\alpha}$ *with a* PDIB*-form is uniquely determined up to* M_D*-isomorphisms.*

Theorem 3.25 states that in order to construct a framed VOA, it is sufficient to check the case $\dim_{\mathbb{Z}_2} S=2$. It is usually difficult to determine the fusion rules $V^{\alpha}\times V^{\beta}$, but an extended $[8,4]$-Hamming code VOA M_{H_8} will solve this problem. For example, the condition (3) may be replaced by the following conditions on codes D and S as we will see.

Theorem 3.20. *Let* W^1 *and* W^2 *be irreducible* M_D*-modules with* $\alpha=\tilde{\tau}(W^1)$, $\beta=\tilde{\tau}(W^2)$. *For a triple* (D,α,β), *assume the following two conditions*:

(3.a) D *contains a self-dual subcode* E *which is a direct sum of* k *extended* $[8,4]$*-Hamming codes such that* $E_{\alpha}=\{\gamma\in E|\mathrm{Supp}(\gamma)\subseteq\mathrm{Supp}(\alpha)\}$ *is a direct factor of* E *or* $\{0\}$.

(3.b) D_{β} *and* $D_{\alpha+\beta}$ *contain maximal self-orthogonal subcodes* H^{β} *and* $H^{\alpha+\beta}$ *containing* E_{β} *and* $E_{\alpha+\beta}$, *respectively, such that they are doubly even and* $H^{\beta}+E=H^{\alpha+\beta}+E$, *where the subscript* S_{α} *denotes a subcode* $\{\beta\in S|\mathrm{Supp}(\beta)\subseteq\mathrm{Supp}(\alpha)\}$ *for any code* S.

Then $W^1\times W^2$ *is irreducible.*

Fortunately, these properties are compatible with induced VOAs.

Theorem 3.21 (Lemma 3.22). *Assume that a triple* (D,α,β) *satisfies the conditions of Theorem* 3.20 *for any* $\alpha,\beta\in\langle\delta,\gamma\rangle$. *Let* $F\subseteq\langle\delta,\gamma\rangle^{\perp}$ *be an even linear code containing* D. *If* $W=M_D\oplus W^{\delta}\oplus W^{\gamma}\oplus W^{\delta+\gamma}$ *is a* $(D,\langle\delta,\gamma\rangle)$*-framed* VOA, *then*

$$\mathrm{Ind}_D^F(W)=M_F\oplus\mathrm{Ind}_D^F(W^{\delta})\oplus\mathrm{Ind}_D^F(W^{\gamma})\oplus\mathrm{Ind}_D^F(W^{\delta+\gamma})$$

has an $(F,\langle\delta,\gamma\rangle)$*-framed* VOA *structure which contains* W *as a sub* VOA.

Corollary 4.2. *Let* $W=M_D\oplus W^{\delta}\oplus W^{\gamma}\oplus W^{\delta+\gamma}$ *be a* $(D,\langle\delta,\gamma\rangle)$*-framed* VOA *with a* PDIB*-form and assume that a triple* (D,α,β) *satisfies the condition of Theorem* 3.20 *for any* $\alpha,\beta\in\langle\delta,\gamma\rangle$. *If* F *is an even linear subcode of* $\langle\alpha,\beta\rangle^{\perp}$ *containing* D, *then* $\mathrm{Ind}_D^F(W)$ *also has a* PDIB*-form.*

Theorems 3.21 and 3.25 state that in order to construct VOAs, it is sufficient to collect M_D-modules satisfying the conditions of Hypotheses I. We will construct such modules from the pieces of the lattice VOA $\tilde{V}_{E_8}$ with a PDIB-form, which is constructed from the root lattice of type E_8. We will show that

$\tilde{V}_{E_8}$ is a (D_{E_8}, S_{E_8})-framed VOA $\oplus_{\alpha\in S_{E_8}}(\tilde{V}_{E_8})^{\alpha}$, where D_{E_8} is isomorphic to the second Reed Müller code RM(2,4) [CS] and

(1.3)
$$S_{E_8} = \big\langle (1^{16}),\ (0^8 1^8),\ (\{0^4 1^4\}^2),\ (\{0^2 1^2\}^4),\ (\{01\}^8)\big\rangle = D_{E_8}^{\perp} \cong \mathrm{RM}(1,4).$$

We will show that a triple (D_{E_8}, α, β) satisfies (3.a) and (3.b) of Theorem 3.20 for any $\alpha, \beta \in S_{E_8}$; see Lemma 5.1. In particular, we have

$$\tilde{V}^{\alpha}_{E_8} \times \tilde{V}^{\beta}_{E_8} = \tilde{V}^{\alpha+\beta}_{E_8} \tag{1.4}$$

for $\alpha, \beta \in S_{E_8}$.

We next explain a new construction of the moonshine module VOA. Set

$$S^{\natural} = \{(\alpha,\alpha,\alpha), (\alpha,\alpha,\alpha^c), (\alpha,\alpha^c,\alpha), (\alpha^c,\alpha,\alpha) \mid \alpha \in S_{E_8}\} \tag{1.5}$$

and $D^{\natural} = (S^{\natural})^{\perp}$, where $\alpha^c = (1^{16}) - \alpha$. $S^{\natural}$ and $D^{\natural}$ are even linear codes of length 48. We note that $D^{\natural}$ is of dimension 41 and contains ${D_{E_8}}^{\oplus 3} := D_{E_8} \oplus D_{E_8} \oplus D_{E_8}$ as a subcode. Clearly, a triple $({D_{E_8}}^{\oplus 3}, \alpha, \beta)$ satisfies the conditions of Theorem 3.20 for any $\alpha, \beta \in S^{\natural}$. Our construction consists of the following three steps. First, $\tilde{V}^{\otimes 3}_{E_8}$ is a $(D^{\oplus 3}_{E_8}, S^{\oplus 3}_{E_8})$-framed VOA with a PDIB-form and

$$V^1 := \bigoplus_{(\alpha,\beta,\gamma)\in S^{\natural}} (\tilde{V}^{\alpha}_{E_8} \otimes \tilde{V}^{\beta}_{E_8} \otimes \tilde{V}^{\gamma}_{E_8}) \tag{1.6}$$

is a sub VOA of $(\tilde{V}_{E_8})^{\otimes 3}$ by the fusion rules (1.4). The second step is to twist it. Set $\xi_1 = (10^{15})$ of length 16 and let R denote a coset module $M_{D_{E_8}+\xi_1}$. To simplify the notation, we denote $R \times \tilde{V}^{\alpha}_{E_8}$ by $R\tilde{V}^{\alpha}_{E_8}$. Set

$$Q = \big\langle (\xi_1\xi_1 0^{16}), (0^{16}\xi_1\xi_1)\big\rangle \subseteq \mathbb{Z}_2^{48}.$$

We induce V^1 from $D^{\oplus 3}_{E_8}$ to $D^{\oplus 3}_{E_8}+Q$:

$$V^2 := \mathrm{Ind}_{D^{\oplus 3}_{E_8}}^{D^{\oplus 3}_{E_8}+Q}(V^1).$$

V^2 is not a VOA, but we are able to find the following $M_{D^{\oplus 3}}$-submodules in V^2:

$$\begin{aligned} W^{(\alpha,\alpha,\alpha)} &:= \tilde{V}^{\alpha}_{E_8} \otimes \tilde{V}^{\alpha}_{E_8} \otimes \tilde{V}^{\alpha}_{E_8},\\ W^{(\alpha,\alpha,\alpha^c)} &:= (R\tilde{V}^{\alpha}_{E_8}) \otimes (R\tilde{V}^{\alpha}_{E_8}) \otimes \tilde{V}^{\alpha^c}_{E_8},\\ W^{(\alpha,\alpha^c,\alpha)} &:= (R\tilde{V}^{\alpha}_{E_8}) \otimes \tilde{V}^{\alpha}_{E_8} \otimes (R\tilde{V}^{\alpha}_{E_8}) \end{aligned}$$

and

$$W^{(\alpha^c,\alpha,\alpha)} := \tilde{V}^{\alpha}_{E_8} \otimes (R\tilde{V}^{\alpha}_{E_8}) \otimes (R\tilde{V}^{\alpha}_{E_8})$$

for $\alpha \in S_{E_8}$. At the end, we extend W^{χ} from $D^{\oplus 3}$ to $D^{\natural}$.

$$(V^{\natural})^{\chi} := \mathrm{Ind}_{D^{\oplus 3}}^{D^{\natural}}(W^{\chi})$$

for $\chi \in S^\natural$. We will show that these $M_{D^\natural}$-modules $(V^\natural)^\chi$ satisfy the conditions in Hypotheses I. Therefore we obtain the desired VOA

$$V^\natural := \bigoplus_{\chi \in S^\natural} (V^\natural)^\chi$$

with a PDIB-form.

Remark. If we construct an induced VOA $\mathrm{Ind}_{D^{\oplus 3}}^{D^\natural}(V^1)$ from V^1 directly, then it is easy to check that it isomorphic to the Leech lattice VOA $\tilde{V}_\Lambda$ (see Section 9). In particular, $\tilde{V}_\Lambda$ has a $(D^\natural, S^\natural)$-framed VOA structure, too.

Since $V^\natural$ is a $(D^\natural, S^\natural)$-framed VOA and $S^\natural = (D^\natural)^\perp$, $V^\natural$ is holomorphic by Theorem 6.1. It comes from the structure of $V^\natural$ and the multiplicity of irreducible $M_{D^\natural}$-submodules that $q^{-1}\sum \dim V_n^\natural q^n = q^{-1} + 196884q + \cdots$ is the J-function $J(q)$. We will also see that the full automorphism group of $V^\natural$ is the Monster simple group (Theorem 9.5). It is also a $\mathbb{Z}_2$-orbifold construction from $\tilde{V}_\Lambda$ (Lemma 9.6). Thus, this is a new construction of the moonshine module VOA and the monster simple group.

In §2.5, we construct a lattice VOA $\tilde{V}_L$ with a PDIB-form. We investigate framed VOA structures on $\tilde{V}_{E_8}$ in §5. In §7, we construct the moonshine VOA $V^\natural$. In Section 8, we will construct a lot of rational conformal vectors of $V^\natural$ explicitly. In Section 9, we prove that $\mathrm{Aut}(V^\natural)$ is the Monster simple group and $V^\natural$ is equal to the one constructed in [FLM2]. In Section 10, we will construct an infinite series of holomorphic VOAs with finite full automorphism groups. In Section 11, we will calculate the characters of some elements of the Monster simple group.

2. Notation and preliminary results

We adopt notation and results from [Mi3] and recall the construction of a lattice VOA from [FLM2]. Codes in this paper are all linear.

2.1. *Notation.*
Throughout this paper, we will use the following notation.

α^c	The complement $(1^n)-\alpha$ of a binary word α of length n.
D_β	$= \{\alpha \in D \mid \mathrm{Supp}(\alpha) \subseteq \mathrm{Supp}(\beta)\}$ for any code D.
$D^\natural, S^\natural$	The moonshine codes. See (1.5).
D_{E_8}, S_{E_8}	See (1.3).
$\widehat{D}$	A group extension $\{\kappa^\alpha \mid \alpha \in D\}$ of D by ± 1.
E_8, $E_8(m)$	An even unimodular lattice of type E_8; also see (5.1).
F_r	The set of all even words of length r.
H_8	The extended $[8,4]$-Hamming code.

$H(\frac{1}{2},\alpha)$, $H(\frac{1}{16},\beta)$	Irreducible M_{H_8}-modules; see Def.13 in [Mi5] or Theorem 3.16.
$\mathrm{Ind}_E^D(U)$	An induced M_D-module from M_E-module U; see Theorem 3.15.
$\iota(x)$	A vector in a lattice VOA $V_L=\bigoplus_{x\in L}S(\bar{H}^-)\iota(x)$; see §2.3.
M	$=M^0\oplus M^1$, $M^0=L(\frac{1}{2},0), M^1=L(\frac{1}{2},\frac{1}{2})$.
$M_{\beta+D}$	A coset module $\bigoplus_{(a_1\cdots a_n)\in\beta+D}\left((\otimes_{i=1}^n M^{a_i})\otimes\kappa^{(a_1\cdots a_n)}\right)$; see §3.
M_D	A code VOA; see §3.
$q^{(1)}$	$=\iota(x)+\iota(-x)\in M^1\cong\mathbf{1}\otimes M^1\subseteq V_{\mathbb{Z}x}$ with $\langle x,x\rangle=1$.
Q	$=\left\langle(1^{15}01^{15}0^{16}),(1^{15}0^{16}01^{15})\right\rangle$.
$RV_{E_8}^{\alpha}$	$M_{(10^7)+D_{E_8}}\times V_{E_8}^{\alpha}$.
$\tilde{\tau}(W)$	A τ-word $(a_1,\cdots,a_n)$; see (1.2).
T	$=\otimes_{i=1}^n L(\frac{1}{2},0)=\langle e_1,\cdots,e_n\rangle=M_{(0^n)}$.
$A(x,z)\sim B(x,z)$	$(x-z)^n(A(x,z)-B(x,z))=0$ for some $n\in\mathbb{N}$.
θ	An automorphism of V_L defined by -1 on L.
ξ_i	A binary word which is 1 in the i-th entry and 0 everywhere else.

2.2. VOA*s over* $\mathbb{R}$ *and* VOA*s over* $\mathbb{C}$. At first, we will quote the following basic results for a VOA over $\mathbb{R}$ from [Mi6]. In this paper, $L(c,0)$ and $L(c,0)_{\mathbb{C}}$ denote simple Virasoro VOAs over $\mathbb{R}$ and $\mathbb{C}$ with central charge c, respectively. Also, Vir denotes the Virasoro algebra over $\mathbb{R}$.

LEMMA 2.1. *Let V be a* VOA *over $\mathbb{R}$ and $U_{\mathbb{C}}$ an irreducible $\mathbb{C}V$-module with real degrees. Then $U_{\mathbb{C}}$ is an irreducible V-module or there is a unique V-module U such that $\mathbb{C}U\cong U_{\mathbb{C}}$ as $\mathbb{C}V$-modules.*

COROLLARY 2.2. *Assume that $L(c,h)_{\mathbb{C}}$ is an irreducible $L(c,0)_{\mathbb{C}}$-module with lowest degree $h\in\mathbb{R}$. Then there exists a unique irreducible $L(c,0)$-module $L(c,h)$ such that $L(c,h)_{\mathbb{C}}\cong\mathbb{C}L(c,h)$. In particular, $\mathbb{C}L(c,0)\cong L(c,0)_{\mathbb{C}}$.*

Proof. First of all, we note that $\mathbb{C}\otimes_{\mathbb{R}}W_{\mathbb{C}}\cong W_{\mathbb{C}}\oplus W_{\mathbb{C}}$ as $L(c,0)_{\mathbb{C}}$-modules for any $L(c,0)_{\mathbb{C}}$-module $W_{\mathbb{C}}$ and $\mathbb{C}\otimes_{\mathbb{R}}U\cong U\oplus U$ as $L(c,0)$-modules for any $L(c,0)$-module U. Therefore, for any proper $L(c,0)$-module W of $L(c,h)_{\mathbb{C}}$, $\mathbb{C}W\cong L(c,h)_{\mathbb{C}}$ or $L(c,h)_{\mathbb{C}}\oplus L(c,h)_{\mathbb{C}}$ as $L(c,0)_{\mathbb{C}}$-modules. Since $\dim_{\mathbb{R}}(L(c,h)_{\mathbb{C}})_h=2$, $L(c,h)_{\mathbb{C}}$ is not irreducible and hence there is an irreducible $L(c,0)$-module $L(c,h)$ such that $L(c,h)_{\mathbb{C}}\cong\mathbb{C}L(c,h)$ by Lemma 2.1. $\square$

In particular, the number of irreducible $L(c,0)$-modules is equal to the number of irreducible $L(c,0)_{\mathbb{C}}$-modules with real degrees.

Corollary 2.3. *The irreducible* $L(\frac{1}{2},0)$*-modules are* $L(\frac{1}{2},0), L(\frac{1}{2},\frac{1}{2})$ *and* $L(\frac{1}{2},\frac{1}{16})$.

Theorem 2.4. *If* $\mathbb{C}V$ *is rational, then so is* V. *In particular,* $L(\frac{1}{2},0)$ *is rational, that is, all modules are completely reducible.*

Proof. We have to show that all V-modules are completely reducible. Suppose this is false and let U be a minimal counterexample; that is, every proper V-submodule of U is a direct sum of irreducible V-modules. By the minimality, we can reduce to the case where U contains a V-submodule W such that U/W and W are irreducible. So, we have a matrix representation of vertex operator

$$Y^U(v,z)=\begin{pmatrix} Y^1(v,z) & Y^2(v,z) \\ 0 & Y^3(v,z)\end{pmatrix}$$

of v on U, where $Y^1(v,z)\in \mathrm{End}(W)[[z,z^{-1}]]$, $Y^2(v,z)\in \mathrm{Hom}(U/W,W)[[z,z^{-1}]]$ and $Y^3(v,z)\in \mathrm{End}(U/W)[[z,z^{-1}]]$. By the assumption, $\mathbb{C}U$ is completely reducible and so $\mathbb{C}U=\mathbb{C}W\oplus X_{\mathbb{C}}$ as $\mathbb{C}V$-modules. Hence there is a matrix $P=\begin{pmatrix} I_U & A \\ 0 & B\end{pmatrix}$ such that $PY(v,z)P^{-1}$ is a diagonal matrix $\begin{pmatrix} Y^1(v,z) & 0 \\ 0 & Y^4(v,z)\end{pmatrix}$ with $Y^4(v,z)\in \mathrm{End}(\mathbb{C}U/\mathbb{C}W)[[z,z^{-1}]]$, where I_U is the identity of $\mathrm{End}(\mathbb{C}W)$, $A\in \mathrm{Hom}(\mathbb{C}U/\mathbb{C}W,\mathbb{C}W)$ and $B\in \mathrm{End}(\mathbb{C}U/\mathbb{C}W)$. Denote A by $A_1+\sqrt{-1}A_2$ with real matrices A_i $(i=1,2)$. By direct calculation,

$$-Y^1(v,z)AB^{-1}+Y^2(v,z)B^{-1}+AY^3(v,z)B^{-1}=0$$

and hence we have

$$-Y^1(v,z)A+Y^2(v,z)+AY^3(v,z)=0$$

and

$$-Y^1(v,z)A_1+Y^2(v,z)+A_1Y^3(v,z)=0.$$

Set $Q=\begin{pmatrix} I_W & A_1 \\ 0 & I_{U/W}\end{pmatrix}$ with an identity map $I_{U/W}$ on U/W; then $QY(v,z)Q^{-1}$ is a diagonal matrix $\begin{pmatrix} Y^1(v,z) & 0 \\ 0 & Y^3(v,z)\end{pmatrix}$, which contradicts the choice of U.

□

About the fusion rules, we have the following:

Lemma 2.5. *Let* W^1, W^2, W^3 *be* V*-modules. Then*

$$\dim I_V\begin{pmatrix} & W^3 & \\ W^1 & & W^2\end{pmatrix} \le \dim I_{\mathbb{C}V}\begin{pmatrix} & \mathbb{C}W^3 & \\ \mathbb{C}W^1 & & \mathbb{C}W^2\end{pmatrix}.$$

Proof. Clearly, if $I \in I_V\begin{pmatrix} & W^3 & \\ W^1 & & W^2\end{pmatrix}$ then we can extend it to an intertwining operator $\tilde{I} \in I_{\mathbb{C}V}\begin{pmatrix} & \mathbb{C}W^3 & \\ \mathbb{C}W^1 & & \mathbb{C}W^2\end{pmatrix}$ by defining $I(\gamma u, z) = \gamma I(u, z)$ for $\gamma \in \mathbb{C}, u \in W^1$. It is easy to see that if $\{I^1, \cdots, I^k\}$ is a basis of $I_V\begin{pmatrix} & W^3 & \\ W^1 & & W^2\end{pmatrix}$ then $\{\tilde{I}^1, \cdots, \tilde{I}^k\}$ is a linearly independent subset of $I_{\mathbb{C}V}\begin{pmatrix} & \mathbb{C}W^3 & \\ \mathbb{C}W^1 & & \mathbb{C}W^2\end{pmatrix}$. For, if $\sum_{i=1}^k (a_i + b_i\sqrt{-1})\tilde{I}^i(v, z)u = 0$ for $v \in W^1, u \in W^2$, then $\sum_{i=1}^k a_i \tilde{I}^i(v, z)u = 0$ and $\sum_{i=1}^k b_i \tilde{I}^i(v, z)u = 0$. □

2.3. *Lattice* VOA*s*. Since we will often use lattice VOAs, we recall the definition from [FLM2].

Let L be a lattice of rank m with a bilinear form $\langle \cdot, \cdot \rangle$. Viewing $H = \mathbb{R} \otimes_{\mathbb{Z}} L$ as a commutative Lie algebra with a bilinear form $\langle , \rangle$, we define the affine Lie algebra

$$\begin{cases} \bar{H} = H[t, t^{-1}] + \mathbb{R}C \\ [C, \bar{H}] = 0, \quad [ht^n, h't^m] = \delta_{m+n,0} n \langle h, h' \rangle C \end{cases}$$

associated with H and the symmetric tensor algebra $S(\bar{H}^-)$ of $\bar{H}^-$, where $\bar{H}^- = H[t^{-1}]t^{-1}$. As in [FLM2], we shall define the Fock space

$$V_L = \oplus_{x \in L} S(\bar{H}^-)\iota(x)$$

with the vacuum $\mathbf{1} = \iota(0)$ and the vertex operators $Y(*, z)$ as follows: The vertex operator of $\iota(a)$ $(a \in L)$ is given by

$$Y(\iota(a), z) = \exp\left(\sum_{n \in \mathbb{Z}_+} \frac{a_{(-n)}}{n} z^n\right) \exp\left(\sum_{n \in \mathbb{Z}_+} \frac{a_{(n)}}{-n} z^{-n}\right) e^a z^a \tag{2.1}$$

and that of $a_{(-1)}\iota(0)$ is

$$Y(a_{(-1)}\iota(0), z) = a(z) = \sum a_{(n)} z^{-n-1}.$$

Here the operator of $a \otimes t^n$ on $M(1)\iota(b)$ is denoted by $a_{(n)}$ and satisfies

$$\begin{aligned} a_{(n)}\iota(b) &= 0 \quad \text{for } n > 0, \\ a_{(0)}\iota(b) &= \langle a, b \rangle \iota(b) \end{aligned}$$

and the operators e^a, z^a are given by

$$\begin{aligned} e^a \iota(b) &= c(a, b)\iota(a+b) \text{ with some } c(a, b) \in \mathbb{R}, \\ z^a \iota(b) &= \iota(b) z^{\langle a, b \rangle}. \end{aligned}$$

If L is an even lattice, then we can take a suitable cocycle $c(a, b)$ such that $e^a e^b = (-1)^{\langle a, b \rangle} e^b e^a$. The vertex operators of the other elements are defined by the normal product:

$$Y(a_{(n)}v,z)=a(z)_nY(v,z)=\mathrm{Res}_x\{(x-z)^na(x)Y(v,z)-(z-x)^nY(v,z)a(x)\}$$

and by extending them linearly. The definition above of vertex operator is very general and so we may think

$$Y(v,z)=\sum_{m\in\mathbb{R}}v_{(m)}z^{-m-1}\in\mathrm{End}(V_{\mathbb{R}\otimes L})\{z\}=\left\{\sum_{j\in\mathbb{C}}s_jz^{-j-1}|s_j\in\mathrm{End}(V_{\mathbb{R}\otimes L})\right\}$$

for $v\in\sum_{a\in\mathbb{R}\otimes_{\mathbb{Z}}L}M(1)\iota(a)$. The Virasoro element ω is given by

$$\tfrac{1}{2}\sum_i(a_i)_{(-1)}(a^i)_{(-1)}\mathbf{1}$$

with $a_i,a^j\in\mathbb{R}L$ satisfying $\langle a_i,a^j\rangle=\delta_{i,j}$. The degree of $(b_1)_{(-i_1)}\cdots(b_k)_{(-i_k)}\iota(d)$ is $i_1+\cdots+i_k+\frac{1}{2}\langle d,d\rangle$ for $b_1,\cdots,b_k,d\in L$. It is shown in [FLM2] that if L is an even positive definite lattice of rank m, then $(V_L,Y,\iota(0),\omega)$ is a VOA of rank m.

2.4. $L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{1}{2},\frac{1}{16})$. In this subsection, we study a lattice $L=\mathbb{Z}x$ of rank one with $\langle x,x\rangle=1$ and we will not use a cocycle $c(a,b)$ since $\{\iota(mx)\mid m\in\mathbb{Z}\}$ is generated by one element $\iota(x)$. We note that V_L is not a VOA, but a super vertex operator algebra (SVOA); see [Fe]. We also note $\iota(x)\in(V_L)_{\frac{1}{2}}$. As mentioned in [DMZ], there are two mutually orthogonal conformal vectors

$$e^+(2x)=\tfrac{1}{4}(x_{(-1)})^2\iota(0)+\tfrac{1}{4}(\iota(2x)+\iota(-2x))$$

and

$$e^-(2x)=\tfrac{1}{4}(x_{(-1)})^2\iota(0)-\tfrac{1}{4}(\iota(2x)+\iota(-2x))$$

with central charge $\frac{1}{2}$ such that $\omega=e^+(2x)+e^-(2x)=\frac{1}{2}(x_{(-1)})^2\iota(0)$ is the Virasoro element of a VOA $V_{2\mathbb{Z}x}$. Let θ be an automorphism of V_L induced from an automorphism -1 on L, which is given by

$$\theta(x_{(-n_1)}\cdots x_{(-n_i)}\iota(v))=(-1)^ix_{(-n_1)}\cdots x_{(-n_i)}\iota(-v).$$

Note that θ is not an ordinary automorphism defined by

$$\theta(x_{(-n_1)}\cdots x_{(-n_i)}\iota(v))=(-1)^{i+k}x_{(-n_1)}\cdots x_{(-n_i)}\iota(-v)$$

for $\mathrm{wt}(\iota(v))=k$, because we have half integral weights here. Let $(V_{2x\mathbb{Z}})^\theta$ denote the sub VOA of θ-invariants in $V_{2x\mathbb{Z}}$. We note that $V_{2x\mathbb{Z}}$ has a unique invariant bilinear form $\langle\ ,\ \rangle$ with $\langle\mathbf{1},\mathbf{1}\rangle=1$. Then $\langle\ ,\ \rangle$ on $(V_{2\mathbb{Z}x})^\theta$ is positive definite as we will see in the next subsection. Hence $e^\pm(2x)$ generates a vertex operator subalgebra $\langle e^\pm(2x)\rangle$ isomorphic to $L(\frac{1}{2},0)$, since $e^\pm(2x)\in(V_{2x\mathbb{Z}})^\theta$. So V_L contains a sub VOA $T=\langle e^+(2x),e^-(2x)\rangle\cong L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$. Viewing V_L

as a T-module, we see that V_L is a direct sum of irreducible T-modules $L(\frac{1}{2},h_i)\otimes L(\frac{1}{2},k_i)$ with $(h_i,k_i=0,\frac{1}{2},\frac{1}{16})$; see §2.5. There are no $\langle e^{\pm}(2x)\rangle$-submodules isomorphic to $L(\frac{1}{2},\frac{1}{16})$ in V_L since all elements $v\in V_L$ have integral or half integral weights. Since $\dim(V_L)_0=1$, $\dim(V_L)_1=1$ and $\dim(V_L)_{1/2}=2$, V_L is isomorphic to

$$\big(L(\tfrac{1}{2},0)\otimes L(\tfrac{1}{2},0)\big)\oplus\big(L(\tfrac{1}{2},0)\otimes L(\tfrac{1}{2},\tfrac{1}{2})\big)\oplus\big(L(\tfrac{1}{2},\tfrac{1}{2})\otimes L(\tfrac{1}{2},0)\big)\oplus\big(L(\tfrac{1}{2},\tfrac{1}{2})\otimes L(\tfrac{1}{2},\tfrac{1}{2})\big)$$

as T-modules. Since θ fixes $e^{\pm}(2x)$ and $x_{(-1)}(\iota(x)-\iota(-x))$, it keeps the above four irreducible T-submodules invariant. Consequently, we obtain the decomposition:

$$(V_L)^{\theta}\cong\big(L(\tfrac{1}{2},0)\otimes L(\tfrac{1}{2},0)\big)\quad\oplus\quad\big(L(\tfrac{1}{2},\tfrac{1}{2})\otimes L(\tfrac{1}{2},0)\big)$$

as T-modules. Set $M=\{v\in(V_L)^{\theta}\mid(e^-(2x))_{(1)}v=0\}$. It is easy to see that M contains $e^+(2x)$ and has the following decomposition:

$$M=M^0\oplus M^1,\quad M^0=\big\langle e^+(2x)\big\rangle\cong L(\tfrac{1}{2},0)\text{ and }M^1\cong L(\tfrac{1}{2},\tfrac{1}{2})\tag{2.2}$$

as $\langle\!\langle e^+(2x)\rangle\!\rangle$-modules. Since M is closed under the multiplications in V_L, M is an SVOA with the even part M^0 and the odd part M^1. We note that

$$q^{(1)}=\iota(x)+\iota(-x)\tag{2.3}$$

is a lowest degree vector of M^1 and ${q^{(1)}}_{(0)}q^{(1)}=2\iota(0)$. We fix it throughout this paper.

It follows from the definition of vertex operators that $V_{2\mathbb{Z}x+\frac{1}{2}x}$ and $V_{2\mathbb{Z}x-\frac{1}{2}x}$ are irreducible $V_{2\mathbb{Z}x}$-modules. By calculating the eigenvalues of $e^{\pm}(2x)$, we have the following table:

(2.4)

		θ
$e^{\pm}(2x)$	$\in L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$	$+1$
$x(-1)\mathbf{1}$	$\in L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{1}{2},\frac{1}{2})$	-1
$\iota(x)-\iota(-x)$	$\in L(\frac{1}{2},0)\otimes L(\frac{1}{2},\frac{1}{2})$	-1
$\iota(x)+\iota(-x)$	$\in L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{1}{2},0)$	$+1$
$\iota(\pm\frac{x}{2})$	$\in\big(L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{1}{2},\frac{1}{16})\big)\oplus\big(L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{1}{2},\frac{1}{16})\big)$	

Fix lowest weight vectors $\iota(\frac{1}{2}x)$ and $\iota(-\frac{1}{2}x)$ of $V_{2\mathbb{Z}x+x/2}$ and $V_{2\mathbb{Z}x-x/2}$, respectively. Let $W(h)$ denote the eigenspace of $e^-(2x)_{(1)}$ on $V_{L+\frac{1}{2}x}$ with eigenvalue h for $h=0,\frac{1}{2},\frac{1}{16}$. By restricting the actions of the vertex operator $Y(v,z)$ of $v\in M^1$ to $W(h)$, we have the following three intertwining operators:

$$I^{\frac{1}{2},0}(*,z)\in I\begin{pmatrix} & L(\frac{1}{2},\frac{1}{2}) & \\ L(\frac{1}{2},\frac{1}{2}) & & L(\frac{1}{2},0)\end{pmatrix}, \tag{2.5}$$

$$I^{\frac{1}{2},\frac{1}{2}}(*,z)\in I\begin{pmatrix} & L(\frac{1}{2},0) & \\ L(\frac{1}{2},\frac{1}{2}) & & L(\frac{1}{2},\frac{1}{2})\end{pmatrix}$$

and

$$I^{\frac{1}{2},\frac{1}{16}}(*,z)\in I\begin{pmatrix} & L(\frac{1}{2},\frac{1}{16}) & \\ L(\frac{1}{2},\frac{1}{2}) & & L(\frac{1}{2},\frac{1}{16})\end{pmatrix}.$$

Also, for $v\in M^0$ the action of $Y(v,z)$ to $W(h)$ defines the following intertwining operators:

$$I^{0,0}(*,z)\in I\begin{pmatrix} & L(\frac{1}{2},0) & \\ L(\frac{1}{2},0) & & L(\frac{1}{2},0)\end{pmatrix}, \tag{2.6}$$

$$I^{0,\frac{1}{2}}(*,z)\in I\begin{pmatrix} & L(\frac{1}{2},\frac{1}{2}) & \\ L(\frac{1}{2},0) & & L(\frac{1}{2},\frac{1}{2})\end{pmatrix}$$

and

$$I^{0,\frac{1}{16}}(*,z)\in I\begin{pmatrix} & L(\frac{1}{2},\frac{1}{16}) & \\ L(\frac{1}{2},0) & & L(\frac{1}{2},\frac{1}{16})\end{pmatrix},$$

which are actually vertex operators of elements in $\langle\!\langle e^{+}(2x)\rangle\!\rangle$ on $L(\frac{1}{2},h)$ $(h=0,\frac{1}{2},\frac{1}{16})$. We fix these intertwining operators throughout this paper.

We defined the above intertwining operators over $\mathbb{R}$, but they are essentially the same as those of $(V_L)_{\mathbb{C}}$ and so we recall their properties from [Mi3].

Proposition 2.6. (1) *The powers of z in $I^{0,*}(*,z)$, $I^{\frac{1}{2},0}(*,z)$ and $I^{\frac{1}{2},\frac{1}{2}}(*,z)$ are all integers and those of z in $I^{\frac{1}{2},\frac{1}{16}}(*,z)$ are half-integers, that is, in $\frac{1}{2}+\mathbb{Z}$.*

(2) *$I^{*,*}(*,z)$ satisfies the $L(-1)$-derivative property.*

(3) *$I^{*,\frac{1}{16}}(*,z)$ satisfies "supercommutativity":*

$$\begin{aligned} &I^{0,\frac{1}{16}}(v,z_1)I^{0,\frac{1}{16}}(v',z_2)\sim I^{0,\frac{1}{16}}(v',z_2)I^{0,\frac{1}{16}}(v,z_1),\\ &I^{0,\frac{1}{16}}(v,z_1)I^{\frac{1}{2},\frac{1}{16}}(u,z_2)\sim I^{\frac{1}{2},\frac{1}{16}}(u,z_2)I^{0,\frac{1}{16}}(v,z_1)\end{aligned} \tag{2.7}$$

and

$$I^{\frac{1}{2},\frac{1}{16}}(u,z_1)I^{\frac{1}{2},\frac{1}{16}}(u',z_2)\sim -I^{\frac{1}{2},\frac{1}{16}}(u',z_2)I^{\frac{1}{2},\frac{1}{16}}(u,z_1),$$

for $v,v'\in M^0$ and $u,u'\in M^1$.

2.5. *A lattice* VOA *with a* PDIB-*form.* In this subsection, we will construct a lattice VOA $\tilde{V}_L$ over $\mathbb{R}$ with a PDIB-form for an even positive definite lattice L.

Here a bilinear form $\langle\cdot,\cdot\rangle$ on V is said to be invariant if

$$\langle Y(a,z)u,v\rangle = \langle u, Y(e^{zL(1)}(-z^{-2})^{L(0)}a,z^{-1})v\rangle \text{ for } a,u,v\in V.$$

It was proved in [FHL] that any invariant bilinear form on a VOA is automatically symmetric and there is a one-to-one correspondence between invariant bilinear forms and elements of $\mathrm{Hom}(V_0/L(1)V_1,\mathbb{R})$. Since we will only treat VOAs V with $\dim V_0=1$ and $L(1)V_1=0$, there is a unique invariant bilinear form up to scalar multiplication. This bilinear form is given as follows:

$$\text{the coefficient of } Y(e^{zL(1)}(-z^{-2})^{L(0)}u,z^{-1})v \text{ at } z \text{ is } \langle u,v\rangle\mathbf{1}.$$

If we construct a lattice VOA V_L over $\mathbb{R}$ for an even positive definite lattice L as in [FLM2], then $\iota(v)_{(2k-1)}\iota(v)\in S(\bar{H}^-)\iota(2v)\cap(V_L)_0=\{0\}$ for any element $0\neq v\in L$ with $\langle v,v\rangle=2k$ and hence $\langle\iota(v),\iota(v)\rangle=\langle\mathbf{1},(-1)^k\iota(v)_{(2k-1)}\iota(v)\rangle=0$. Namely, V_L does not have a PDIB-form.

Proposition 2.7. *Let L be an even positive definite lattice. Then there is a* VOA $\tilde{V}_L$ *with a* PDIB-*form such that* $\mathbb{C}\otimes\tilde{V}_L\cong(V_L)_{\mathbb{C}}$.

Proof. A lattice VOA $V_L=\bigoplus_{v\in L}S(\mathbb{R}\otimes_{\mathbb{Z}}L^+)\iota(v)$ constructed from a lattice L in [FLM2] has a unique invariant bilinear form $\langle\ ,\ \rangle$ with $\langle\mathbf{1},\mathbf{1}\rangle=1$. That is, it satisfies

$$\langle Y(a,z)u,v\rangle = \langle u,Y(e^{zL(1)}(-z^{-2})^{L(0)}a,z^{-1})v\rangle$$

for $a,u,v\in V_L$; see [FHL]. Here

$$Y^{\dagger}(a,z):=Y(e^{zL(1)}(-z^{-2})^{L(0)}a,z^{-1})=\sum a^{\dagger}_{(m)}z^{-m-1}$$

is the adjoint vertex operator. For $v\in\mathbb{R}\otimes L$, we identify v with $v_{(-1)}\iota(0)\in(V_L)_1$. Since $L(1)v_{(-1)}\iota(0)=0$ and $L(0)v_{(-1)}\iota(0)=v_{(-1)}\iota(0)$, we have $Y^{\dagger}(v,z)=-z^{-2}Y(v,z^{-1})$ and so $v^{\dagger}_{(n)}=-v_{(-n)}$. In [FLM2], the authors used a group extension (a cocycle $c(*,*)$) satisfying $e^{u'}e^{u}=(-1)^{\langle u',u\rangle}e^ue^{u'}$, $e^u\iota(u')=c(u,u')\iota(u+u')$ and $e^v\iota(-v)=\iota(0)$. In particular, for $\iota(v)\in(V_L)_k$,

$$\iota(v)_{(2k-1)}\iota(-v)=\iota(-v)_{(2k-1)}\iota(v)=\iota(0).$$

By definition, $Y^{\dagger}(\iota(v),z)=(-z^{-2})^{\langle v,v\rangle/2}Y(\iota(v),z^{-1})$. We hence have $(\iota(v))^{\dagger}_{(n)}=(-1)^k(\iota(v))_{(2k-n-2)}$ for $\iota(v)\in V_k$ and thus

$$\begin{aligned}
&\langle\iota(v)+\iota(-v),\iota(v)+\iota(-v)\rangle\iota(0)\\
&\qquad=(-1)^k(\iota(v)+\iota(-v))_{(2k-1)}(\iota(v)+\iota(-v))\\
&\qquad=(-1)^k(\iota(v)_{(2k-1)}\iota(-v)+\iota(-v)_{(2k-1)}\iota(v))=(-1)^k2\iota(0).
\end{aligned}$$

Similarly,

$$\langle\iota(v)-\iota(-v),\iota(v)-\iota(-v)\rangle=(-1)^{k+1}2\iota(0).$$

Let $\tilde{\theta}$ be an automorphism of V_L induced from -1 on L, which is given by

$$\tilde{\theta}(v^1_{(-i_1)}\cdots v^m_{(-i_m)}\iota(x)) = (-1)^{k+m} v^1_{(-i_1)}\cdots v^m_{(-i_m)}\iota(-x).$$

Then the space $V^+ = (V_L)^{\tilde{\theta}}$ of $\tilde{\theta}$-invariants is spanned by elements of the forms

$$v^1_{(-n_1)}\cdots v^{2m}_{(-n_{2m})}(\iota(v)+(-1)^k\iota(-v))$$

and

$$v^1_{(-n_1)}\cdots v^{2m+1}_{(-n_{2m+1})}(\iota(v)-(-1)^k\iota(-v))$$

for all $\iota(v)\in V_k, k\in\mathbb{Z}$ and so V^+ has a PDIB-form. Similarly $V^- := \{v\in V_L | \tilde{\theta}(v) = -v\}$ has a negative definite invariant bilinear form. Since $V_L = V^+\oplus V^-$ is a $\mathbb{Z}_2$-graded VOA, $\tilde{V}_L = V^+\oplus\sqrt{-1}V^-$ is also a VOA with a PDIB-form such that $\mathbb{C}\tilde{V}_L = \mathbb{C}V_L \cong (V_L)_{\mathbb{C}}$. □

Clearly, if we define an endomorphism $\bar{\theta}$ of $\tilde{V}_L = V^+\oplus\sqrt{-1}V^-$ by 1 on V^+ and -1 on $\sqrt{-1}V^-$, $\bar{\theta}$ is an automorphism of $\tilde{V}_L$. Since we mainly treat a VOA with a PDIB-form, we sometimes denote the ordinary lattice VOA V_L by $(\tilde{V}_L)^{\bar{\theta}}\oplus\sqrt{-1}\tilde{V}_L^-$, where $\tilde{V}_L^- = \{v\in\tilde{V}_L \mid \bar{\theta}(v) = -v\}$.

In the remainder of this paper, $\tilde{V}_L$ denotes a lattice VOA with a PDIB-form.

2.6. *$L(\frac{1}{2},0)$-modules and framed* VOA*s.* We will show the following result.

LEMMA 2.8. *If V is a framed* VOA *with a coordinate set $\{e_1,\cdots,e_n\}$, then there are two binary linear codes D and S of length n such that V has the following decomposition*:

(1) $V = \oplus_{\alpha\in S}V^\alpha$,

(2) $\mathbb{C}V^{(0^n)}$ *is a code* VOA $(M_D)_{\mathbb{C}}$,

(3) V^α *is an irreducible $V^{(0^n)}$-module with $\tilde{\tau}(V^\alpha) = \alpha$ for $\alpha\in S$.*

Proof. Set $P = \langle\tau_{e_i} \mid i=1,\cdots,n\rangle \subseteq \mathrm{Aut}(V)$, which is an elementary abelian 2-group. Decompose V into a direct sum

$$V = \oplus_{\chi\in\mathrm{Irr}(P)}V^\chi$$

of eigenspaces of P, where $\mathrm{Irr}(P)$ is the set of linear characters of P and V^χ denotes $\{v\in V \mid gv = \chi(g)v \text{ for } g\in P\}$ and $V^{1_P} = V^P$ is the set of P-invariants and 1_P is the trivial character of P. It is known by [DM2] that V^χ is a nonzero irreducible V^P-module for $\chi\in\mathrm{Irr}(P)$. It follows from the definition of τ_{e_i} that $\tilde{\tau}(V^\chi) = (a_i)$ is given by $(-1)^{a_i} = \chi(\tau_{e_i})$. Set $S = \{\tilde{\tau}(V^\chi) \mid \chi\in\mathrm{Irr}(P)\}$ and denote V^χ by $V^{\tilde{\tau}(V^\chi)}$ using a binary word $\tilde{\tau}(V^\chi)$. In particular, $\mathbb{C}V^P$ is a VOA with $\tilde{\tau}(\mathbb{C}V^P) = (0^n)$ and hence it is isomorphic to a code VOA $(M_D)_{\mathbb{C}}$ for some even linear binary code D. Then V has the desired decomposition. □

3. Code VOAs with PDIB-forms

In this section, we review several results from [Mi2]–[Mi5] and prove their $\mathbb{R}$-versions. We will first construct a code VOA M_D with a PDIB-form for an even linear binary code D of length n. Set $M^0 = L(\frac{1}{2},0)$ and $M^1 = L(\frac{1}{2},\frac{1}{2})$. As we showed in §2.4, $M = M^0 \oplus M^1$ has a super VOA structure (M, Y^M). Although an SVOA structure on $\mathbb{C}M$ is uniquely determined, an SVOA structure on M is not unique. For example, if $(M^0 \oplus M^1, Y)$ is an SVOA, then $(M^0 \oplus \sqrt{-1}M^1, Y)$ is the other SVOA. They are isomorphic together as M^0-modules. We already have a VOA structure on $\mathbb{C}M^0 \oplus \mathbb{C}M^1$ and the isomorphism $v^{(0)} + \sqrt{-1}v^{(1)} \to v^{(0)} + v^{(1)}$ defines another VOA structure on $\mathbb{C}M^0 \oplus \mathbb{C}M^1$. So we choose one of them satisfying $q^{(1)}_{(0)}q^{(1)} \in \mathbb{R}^+\mathbf{1}$ and denote it by (M, Y^M), where $q^{(1)}$ is the highest weight vector of M^1 given by (2.3) and $\mathbb{R}^+ = \{r \in \mathbb{R} | r > 0\}$.

An essential property is "super-commutativity":

$$Y^M(v,z_1)Y^M(u,z_2) \sim (-1)^{ij}Y^M(u,z_2)Y^M(v,z_1) \tag{3.1}$$

for $v \in M^i$ and $u \in M^j$ $(i,j = 0,1)$. Here $A(z_1,z_2) \sim B(z_1,z_2)$ means $(z_1-z_2)^N A(z_1,z_2) = (z_1-z_2)^N B(z_1,z_2)$ for some integer N. Take n copies $M^{[i]} = (M^0)^{[i]} \oplus (M^1)^{[i]}$ of $M = M^0 \oplus M^1$ for $i = 1, \cdots, n$ and set $M^{\otimes n} = M^{[1]} \otimes \cdots \otimes M^{[n]}$. For a binary word $\alpha = (a_1, \cdots, a_n) \in \mathbb{Z}_2^n$, set $\tilde{M}_\alpha = \otimes_{i=1}^n (M^{a_i})^{[i]}$, which is a subspace of $M^{\otimes n}$. Define a vertex operator $Y^{\otimes n}(v,z)$ of $v \in M^{\otimes n}$ by setting

$$Y^{\otimes n}(\otimes_{i=1}^n v^i, z)(\otimes_{i=1}^n u^i) = \otimes_{i=1}^n (Y^{M^{[i]}}(v^i,z)u^i) \tag{3.2}$$

for $u^i, v^i \in M^{[i]}$ and extending it to the whole space $M^{\otimes n}$ linearly. It follows from (3.1) that for $v \in \tilde{M}_\alpha, u \in \tilde{M}_\beta$, we have super commutativity:

$$Y^{\otimes n}(v,z_1)Y^{\otimes n}(u,z_2) \sim (-1)^{\langle \alpha,\beta \rangle} Y^{\otimes n}(u,z_2)Y^{\otimes n}(v,z_1), \tag{3.3}$$

where $\langle (a_i),(b_i) \rangle = \sum_{i=1}^n a_i b_i \in \mathbb{Z}_2$. Viewing D as an elementary abelian 2-group with an invariant form, we will show that there is a central extension $\widehat{D} = \{\pm\kappa^\alpha \mid \alpha \in D\}$ of D by ± 1 such that $\kappa^\alpha\kappa^\beta = (-1)^{\langle\alpha,\beta\rangle}\kappa^\beta\kappa^\alpha$ since D is an even linear lattice. Actually, let ξ_i $(i = 1, \cdots, n)$ denote a word $(0^{i-1}10^{n-i})$ and define formal elements κ^{ξ_i} $(i = 1, \cdots, n)$ satisfying $\kappa^{\xi_i}\kappa^{\xi_i} = \kappa^{(0^n)} = 1$ and $\kappa^{\xi_i}\kappa^{\xi_j} = -\kappa^{\xi_j}\kappa^{\xi_i}$ for $i \neq j$. For a word $\alpha = \xi_{j_1} + \cdots + \xi_{j_k}$ with $j_1 < \cdots < j_k$, set

$$\kappa^\alpha = \kappa^{\xi_{j_1}}\kappa^{\xi_{j_2}} \cdots \kappa^{\xi_{j_k}}. \tag{3.4}$$

It is straightforward to check the following:

Lemma 3.1 ([Mi3]). *For α, β,*

$$\begin{aligned} \kappa^\alpha\kappa^\beta &= (-1)^{\langle\alpha,\beta\rangle + |\alpha||\beta|}\kappa^\beta\kappa^\alpha \in \{\pm\kappa^{\alpha+\beta}\} \\ \kappa^\alpha\kappa^\alpha &= (-1)^{\frac{k(k-1)}{2}}\kappa^{(0^n)} \textit{ for } |\alpha| = k. \end{aligned} \tag{3.5}$$

In order to combine (3.3) and (3.5), set

$$M_\delta = \tilde{M}_\delta \otimes \kappa^\delta \tag{3.6}$$

for $\delta \in \mathbb{Z}_2^n$ and

$$M_D = \bigoplus_{\delta \in D} M_\delta. \tag{3.7}$$

Define a new vertex operator $Y(u,z)$ of $u \in M_D$ by setting

$$Y(v \otimes \kappa^\alpha, z)(u \otimes \kappa^\beta) = Y^{\otimes n}(v,z)u \otimes \kappa^\alpha \kappa^\beta \tag{3.8}$$

for $v \otimes \kappa^\alpha \in M_\alpha = \tilde{M}_\alpha \otimes \kappa^\alpha$, $u \otimes \kappa^\beta \in M_\beta$ and extending it linearly. We then obtain the desired commutativity:

$$Y(v,z_1)Y(w,z_2) \sim Y(w,z_2)Y(v,z_1) \tag{3.9}$$

for $v,w \in M_D$. Set $e_i = (1^{[1]} \otimes \cdots \otimes 1^{[i-1]} \otimes \omega^{[i]} \otimes 1^{[i+1]} \otimes \cdots \otimes 1^{[n]}) \otimes \kappa^{(0^n)}$. It is not difficult to see that

$$\omega = e_1 + \cdots + e_n \tag{3.10}$$

is the Virasoro element of M_D and

$$\mathbf{1} = (\mathbf{1}^{[1]} \otimes \cdots \otimes \mathbf{1}^{[n]}) \otimes \kappa^{(0^n)} \tag{3.11}$$

is the vacuum of M_D, where $\omega^{[i]}$ and $\mathbf{1}^{[i]}$ are the Virasoro element and the vacuum of $M^{[i]}$, respectively. To simplify the notation, we will omit superscripts $[i]$ of $M^{[i]}$ from now on. We have proved the following theorem, whose complexification was proved in [Mi2].

THEOREM 3.2. *If D is an even binary linear code, then $(M_D, Y, \omega, \mathbf{1})$ is a* VOA *over $\mathbb{R}$.*

It follows from the construction that $M_{\beta+D} := \oplus_{\alpha \in D} M_{\beta+\alpha}$ is an irreducible M_D-module for any $\beta \in \mathbb{Z}_2^n$ and we will call it a *coset module* of M_D. From the definition of κ^α in (3.4), we have the following lemma.

LEMMA 3.3. *If $g \in \mathrm{Aut}(D)$, there is an automorphism $\tilde{g}$ of a code* VOA *M_D such that $\tilde{g}(e_i) = e_{g(i)}$ and $\tilde{g}(M_\alpha) = M_{g(\alpha)}$.*

Proof. For $g \in \mathrm{Aut}(D)$, we define a permutation g_1 on $\{\tilde{M}_\alpha \mid \alpha \in D\}$ by $g_1(\otimes_{i=1}^n v^{[i]}) = \otimes_{i=1}^n v^{[g(i)]}$ and an automorphism g_2 of $\widehat{D}$ by $g_2(\kappa^{\xi_{i_1}} \cdots \kappa^{\xi_{i_t}}) = \kappa^{\xi_{g(i_1)}} \cdots \kappa^{\xi_{g(i_t)}}$. Combining both actions, we have an automorphism $\tilde{g} = g_1 \otimes g_2$ of $M_D = \oplus_\alpha (\tilde{M}_\alpha \otimes \kappa^\alpha)$. $\square$

Our next aim is to prove that M_D has a PDIB-form $\langle\ ,\ \rangle$ with $\langle \mathbf{1},\mathbf{1}\rangle = 1$. Set $W=\{v\in M_D \mid (e_i)_{(m)}v=0 \text{ for all } m\geq 2, i=1,\cdots,n\}$.

LEMMA 3.4. *$\langle\ ,\ \rangle$ on W is positive definite.*

Proof. Set

$$\tilde{q}^{\alpha} = (q^{(a_1)}\otimes\cdots\otimes q^{(a_n)}) \tag{3.12}$$

for $\alpha=(a_1,\cdots,a_n)\in D$, where $q^{(1)}$ is the highest weight vector of M^1 given by (2.3) and $q^{(0)}$ denotes the vacuum of M^0. It is easy to see that

$$q^{\alpha} = \tilde{q}^{\alpha}\otimes\kappa^{\alpha} \tag{3.13}$$

is a lowest degree element of M_α. Since $M_\alpha \cong \otimes_{i=0}^{n} L(\frac{1}{2},\frac{a_i}{2})$ and $M_D = \oplus_{\alpha\in D}M_\alpha$, $\{q^\alpha : \alpha\in D\}$ spans W. Let k_α denote half of the weight of α. For α,β, we have

$$\begin{aligned}\langle q^\alpha, q^\beta\rangle\mathbf{1} = \langle q^\alpha_{(-1)}\mathbf{1}, q^\beta\rangle\mathbf{1} &= \mathrm{Res}_z\{z^{-1}Y(((-1)^{k_\alpha}z^{-2k_\alpha})q^\alpha, z^{-1})q^\beta\}\\ &= (-1)^{k_\alpha}q^\alpha_{(2k_\alpha-1)}q^\beta = \delta_{\alpha,\beta}2^{2k_\alpha}.\end{aligned}$$

Thus, $\{\frac{1}{2^{k_\alpha}}q^\alpha \mid \alpha\in D\}$ is an orthonormal basis of W. □

Let $V=\oplus_{i=0}^{\infty}V_i$ be a VOA satisfying $\dim V_0=1$ and $L(1)V_1=0$. Set $B=\mathbb{R}L(1)\oplus\mathbb{R}L(0)\oplus\mathbb{R}L(-1)$. Since $B\cong \mathrm{sl}_2(\mathbb{R})$ as Lie algebras and $L(1)V_1=0$, V is a direct sum of irreducible B-modules. If U is an irreducible B-submodule of V and u is a lowest degree vector of U with degree k, then

$$\langle u,v\rangle\mathbf{1} = \langle u_{(-1)}\mathbf{1},v\rangle\mathbf{1} = \mathrm{Res}_z(Y(((-1)^k z^{-2k})u,z^{-1})z^{-1}v = (-1)^k u_{(2k-1)}v \tag{3.14}$$

for any $v\in V_k$. Also we obtain

$$\begin{aligned}\langle L(-1)^i v, L(-1)^j u\rangle &= \langle L(-1)^{i-1}v,\\ &\quad L(1)L(-1)^j u\rangle = (2kj+j^2-j)\langle L(-1)^{i-1}v, L(-1)^{j-1}u\rangle\end{aligned} \tag{3.15}$$

and $(2kj+j^2-j)>0$ for $i,j>0$. Thus $\langle\ ,\ \rangle$ on V is positive definite if and only if

$$u_{(2k-1)}u\in(-1)^k\mathbb{R}^+\mathbf{1} \tag{3.16}$$

for every nonzero homogeneous element $u\in V_k$ satisfying $L(1)u=0$.

We first prove an $\mathbb{R}$-version of Theorem 4.5 in [Mi3].

PROPOSITION 3.5. *Let V be a framed* VOA *with a coordinate set $\{e_1,\cdots,e_n\}$. If $\tilde{\tau}(V)=(0^n)$ and V has a* PDIB*-form, then there is an even linear code D of length n such that V is isomorphic to a code* VOA *M_D.*

Proof. Since $\tilde{\tau}(V)=(0^n)$, $\tau_{e_i}=1$ and so we can define automorphisms σ_{e_i} for $i=1,\cdots,n$, where σ_{e_i} is defined by $\exp(2\pi\sqrt{-1}(e_i)_{(1)})$ on V; see [Mi1]. We note that the eigenvalues of $(e_i)_{(1)}$ on V are in $\mathbb{Z}/2$. Set $Q=\langle\sigma_{e_i}\mid i=1,\cdots,n\rangle$, which is an elementary abelian 2-group. Let

$$V=\oplus_{\chi\in\mathrm{Irr}(Q)}V^{\chi}$$

be the decomposition of V into the direct sum of eigenspaces of Q, where $\mathrm{Irr}(Q)$ is the set of linear characters of Q. Since $\dim V_0=1$ and V^{χ} is an irreducible V^Q-module by [DM2], we have $V^Q=T$ and $V^{\chi}\cong\otimes_{i=1}^{n}L(\frac{1}{2},\frac{h_i}{2})$ as T-modules, where $h_i\in\{0,1\}$ is defined by $\chi(\sigma_{e_i})=(-1)^{h_i}$. Identifying χ and a binary word (h_i), $V^{\chi}\cong M_{\chi}=\tilde{M}_{\chi}\otimes\kappa^{\chi}$ as T-modules. Since all weights of V^{χ} are integers, the weight of χ is even, say $2k_{\chi}$. Let $p^{\chi}\in V^{\chi}$ be a lowest degree vector with $\langle p^{\chi},p^{\chi}\rangle=2^{2k_{\chi}}$. We identify p^{χ} with $\tilde{q}^{\chi}\otimes\tilde{\kappa}^{\chi}$, see $\tilde{q}^{\chi}$ at (3.12). Since $\tilde{q}^{\chi}_{(2k_{\chi}-1)}\tilde{q}^{\chi}=2^{k_{\chi}}\mathbf{1}$, we have

$$\begin{aligned}2^{k_{\chi}}\mathbf{1}&=\langle\tilde{q}^{\chi}\otimes\tilde{\kappa}^{\chi},\tilde{q}^{\chi}\otimes\tilde{\kappa}^{\chi}\rangle\mathbf{1}\\&=\langle\mathbf{1},(-1)^{k}(\tilde{q}^{\chi}\otimes\tilde{\kappa}^{\chi})_{(2k_{\chi}-1)}\tilde{q}^{\chi}\otimes\tilde{\kappa}^{\chi}\rangle\mathbf{1}\\&=2^{2k_{\chi}}\langle\mathbf{1},(-1)^{k}\tilde{\kappa}^{\chi}\tilde{\kappa}^{\chi}\rangle\mathbf{1}.\end{aligned}\tag{3.17}$$

Hence $\tilde{\kappa}^{\chi}\tilde{\kappa}^{\chi}=(-1)^{k_{\chi}}\tilde{\kappa}^{0}$ for any χ, which determines a cocycle uniquely and it coincides with (3.5). This completes the proof of Proposition 3.5. □

As a corollary, we have:

Corollary 3.6. *For an even linear code D, M_D has a* PDIB-*form. In particular, if α is even, then a coset module $M_{D+\alpha}$ also has a* PDIB-*form.*

Proof. It is sufficient to show that there is a VOA V with a PDIB-form such that V contains M_D. Since M_D is a sub VOA of M_S if $D\subseteq S$ and we can also embed $M_D\cong M_D\otimes\mathbf{1}\subseteq M_D\otimes M_D$, we may assume that D is the set of all even words of length 2n. Let $\{x_1,\cdots,x_n\}$ be an orthonormal basis of a Euclidian space of dimension n and set

$$L=\left\{\sum_{i=1}^{n}a_ix_i\mid a_i\in\mathbb{Z},\sum_{i=1}^{n}a_i\equiv 0\pmod 2\right\}.\tag{3.18}$$

Clearly, L is an even lattice and $\tilde{V}_L$ denotes a lattice VOA with a PDIB-form. Since $\tilde{V}_L$ contains $2n$ mutually orthogonal rational conformal vectors

$$e(2x_i)^{\pm}=\tfrac{1}{4}((x_i)_{(-1)})^2\mathbf{1}\pm\tfrac{1}{4}(\iota(2x_i)+\iota(-2x_i))\quad(i=1,\cdots,n)\tag{3.19}$$

with central charge $\frac{1}{2}$, $\tilde{V}_L$ is a framed VOA. Since $\langle v,2x_j\rangle\in 2\mathbb{Z}$ for $v\in L$ and $j=1,\cdots,n$, (2.4) implies $\tilde{\tau}(\tilde{V}_L)=(0^{2n})$ and hence $\tilde{V}_L$ is isomorphic to a code VOA M_S for some even linear code S of length $2n$ by Proposition 3.5. It is easy to see $\dim(M_S)_1=n(2n-1)$ and so S is the set of all even words of length $2n$. Hence M_D has a PDIB-form. □

LEMMA 3.7. *If a* VOA V *contains a code* VOA M_D *and* D *contains a codeword* δ *of weight* 2, *then* $\mathbb{C}V$ *contains an automorphism* g *satisfying*

$$g=(-1)^{\langle\beta,\delta\rangle} \text{ on } M_\beta \text{ for } \beta\in D.$$

In particular, g *coincides with* $\sigma_{e_i}\sigma_{e_j}$ *on* M_D *if* $\mathrm{Supp}(\delta)=\{i,j\}$.

Proof. Let $\alpha\in D$ be a codeword of weight 2, say $\alpha=(110^{n-2})$, then $(M_\alpha)_1\neq 0$. Set $E=\{(00),(11)\}$, then $M_{\langle\alpha\rangle}=M_E\otimes\left(L(\frac{1}{2},0)^{\otimes n-2}\right)$ and M_E is isomorphic to $V_{2\mathbb{Z}x}$ with $\langle x,x\rangle=1$ as given in §2.4. Let v be an element of V_1 corresponding to $x_{(-1)}\mathbf{1}$. Define $g=\exp(2\pi\sqrt{-1}v_{(0)})$. Since $v\in V_1$ and M_E is rational, $v_{(0)}$ acts on V semisimply and g is an automorphism of V satisfying the desired conditions. □

We propose one conjecture.

CONJECTURE 1. *If* V *is a* (D,S)*-framed* VOA *and* $\beta\in D$, *then there is an automorphism* g *of* V *such that* $g=\prod_{i\in\mathrm{Supp}(\beta)}\sigma_{e_i}$ *on* M_D.

3.1. *M_D-modules.* We recall the structures of irreducible $\mathbb{C}M_D$-modules from [Mi3]. Let W be an irreducible M_D-module with $\tilde{\tau}(W)=\mu$. Then $\mathbb{C}W$ is a $\mathbb{C}M_D$-module and $\mathbb{C}W=W\oplus W$ as M_D-modules. Since we have defined nonzero intertwining operators $I^{0,*}(v,z)$ and $I^{\frac{1}{2},*}(u,z)$ over $\mathbb{R}$ in §2.4, we have an $\mathbb{R}$-version of Theorem 5.1 in [Mi3]:

THEOREM 3.8. *Let* (W,Y^W) *be an irreducible* M_D*-module with* $\tilde{\tau}(W)=\mu$ *and* $\{X^i\mid i=1,\cdots,m\}$ *the set of all nonisomorphic irreducible* T*-submodules of* W. *Set* $D_\mu=\{\alpha\in D|\mathrm{Supp}(\alpha)\subseteq\mathrm{Supp}(\mu)\}$ *and let* $\hat{D}_\mu$ *denote a group extension* $\{\pm\kappa^\alpha|\alpha\in D_\mu\}$ *given by* (3.4). *Then there are irreducible* $\mathbb{R}\widehat{D}_\mu$*-modules* Q^i *and representations* $\phi^i:\widehat{D}_\mu\to\mathrm{End}(Q^i)$ *satisfying* $\phi^i(-\kappa^{(0^n)})=-I_{Q^i}$ *for* $i=1,\cdots,m$ *such that* $W\cong\oplus_{i=1}^m(X^i\otimes Q^i)$ *as* M_{D_μ}*-modules.*

Here the vertex operator $Y^W(q^\alpha,z)$ of $q^\alpha=(\otimes_{i=1}^n q^{(a_i)})\otimes\kappa^\alpha\in M_\alpha$ on $\oplus_{j=1}^m(X^j\otimes Q^j)$ is given by

$$\oplus_{j=1}^m\left(\otimes_{i=1}^n I^{a_i/2,*}(q^{(a_i)},z)\otimes\phi^j(\kappa^\alpha)\right)$$

for $\alpha=(a_1,\cdots,a_n)$. See (3.13), §2.2 and §2.3 for q^α and $\otimes_{i=1}^n I^{a_i/2,*}(q^{(a_i)},z)$.

Before we study M_D-modules, we explain the structure of a 2-group $\widehat{D}$. An important property of our cocycle is that if a maximal self-orthogonal subcode H of D_μ is doubly even (for example, an extended $[8,4]$-Hamming code), then $\widehat{H}=\{\pm\kappa^\alpha\mid\alpha\in H\}$ is an elementary abelian 2-group and hence every irreducible $\mathbb{R}\widehat{H}$-representation is linear. If $\chi:\widehat{D_\mu}\to\mathrm{End}(Q)$ is an irreducible $\mathbb{R}\widehat{D_\mu}$-module with $\chi(-\kappa^{(0^n)})=-I_Q$, then $K:=\mathrm{Ker}(\chi)$ is in the center of $\widehat{D_\mu}$. Since $\widehat{H}$ is a maximal normal abelian subgroup of $\widehat{D_\mu}$, $\widehat{H}/K$ is a maximal normal abelian subgroup of $\widehat{D_\mu}/K$. Since $\widehat{D_\mu}/K$ has a faithful irreducible representation, the center $Z(\widehat{D_\mu}/K)$ is cyclic and so is of order 2. Hence $\widehat{D_\mu}/K$ is an extra-special 2-group and $Q_{|H}$ is a direct sum of distinct $\widehat{H}$-irreducible modules.

In the remainder of this section, we use the following notation:

$$B(D) := \left\{ \beta \in \mathbb{Z}_2^n \mid \begin{array}{l} \text{one of the maximal self-orthogonal} \\ \text{subcodes of } D_\beta \text{ is doubly even} \end{array} \right\}.$$

Corollary 3.9. *If H is a doubly even code and W is an irreducible M_H-module with $\tilde{\tau}(W) = (1^n)$, then W is also irreducible as a T-module.*

Lemma 3.10. *Let W be an irreducible M_D-module with $\tilde{\tau}(W) \in B(D)$. Then $\mathbb{C}W$ is an irreducible $\mathbb{C}M_D$-module.*

Proof. Let H be a maximal self-orthogonal doubly even subcode of $D_{\tilde{\tau}(W)}$. Since $\mathbb{C}W \cong W \oplus W$ as M_D-modules and W is a direct sum of distinct M_{D_μ}-modules, we may assume $D_\mu = D$ and $W \cong X \otimes Q$, where X is an irreducible M_H-module and Q is an irreducible $\mathbb{R}\widehat{D}$-module by Theorem 3.8. As mentioned above, $Q_{|\widehat{H}}$ is a direct sum of distinct linear $\widehat{H}$-modules and $\mathbb{C}Q_{|\widehat{H}}$ is a direct sum of distinct irreducible $\mathbb{C}\widehat{H}$-modules. Hence $\mathbb{C}Q$ is an irreducible $\mathbb{C}\widehat{D}$-module and so $\mathbb{C}W$ is an irreducible $\mathbb{C}M_D$-module. □

Corollary 3.11. *If $I_{\mathbb{C}M_D}\begin{pmatrix} \mathbb{C}W^3 \\ \mathbb{C}W^1\ \mathbb{C}W^2 \end{pmatrix} \neq 0$, then $I_{M_D}\begin{pmatrix} W^3 \\ W^1\ W^2 \end{pmatrix} \neq 0$ for M_D-modules W^1, W^2 and W^3.*

Proof. Choose $0 \neq I(*,z) \in I_{\mathbb{C}M_D}\begin{pmatrix} \mathbb{C}W^3 \\ \mathbb{C}W^1\ \mathbb{C}W^2 \end{pmatrix}$. By restricting $I(*,z)$ on W^1 and W^2, we have a nonzero intertwining operator $\tilde{I}(*,z) \in I_{M_D}\begin{pmatrix} W^3 \oplus W^3 \\ W^1\ \ W^2 \end{pmatrix}$. Taking the first entry and the second entry of $\mathbb{C}W^3 = W^3 \oplus \sqrt{-1}W^3$, we have two intertwining operators $\tilde{I}^1(*,z)$ and $\tilde{I}^2(*,z)$ in $I_{M_D}\begin{pmatrix} W^3 \\ W^1\ W^2 \end{pmatrix}$ and one of them at least is nonzero. □

One of the attributes of lattice VOAs and their modules is that we can find all M_D-modules inside of them in some sense. This fact is very useful in studying the fusion rules among M_D-modules. For example, one obtains:

Lemma 3.12. *If W^1, W^2 are M_D-modules, then $W^1 \times W^2$ is nonzero.*

Proof. By Corollary 3.11, we may assume that all VOAs are considered over $\mathbb{C}$, and so we omit the subscript $\mathbb{C}$. If $W^1 \times W^2 = 0$, then $(W^1)^{\otimes 2} \times$

$(W^2)^{\otimes 2} = 0$ as $(M_D)^{\otimes 2}$-modules. We may hence assume that $\tilde{\tau}(W^1) = (1^{2h+2k}0^{2s+2t})$ and $\tilde{\tau}(W^2) = (0^{2h}1^{2k}1^{2s}0^{2t})$ by rearranging the order. Set $\alpha = \tilde{\tau}(W^1)$, $\beta = \tilde{\tau}(W^2)$ and $n = 2(h+k+s+t)$. Let F_r denote the set of all even words of length r. We may also assume that $D = \langle \alpha, \beta \rangle^{\perp}$. Set $D^1 = \langle \alpha \rangle^{\perp}$ and $D^2 = \langle \beta \rangle^{\perp}$. Clearly, $D^1 = F_{2h+2k} \oplus F_{2s+2t}$. Generally, $M_{F_{2r}}$ is isomorphic to a lattice VOA $V_{N(r)}$, where $N(r) = \{\sum_{i=1}^r a_i x_i \mid a_i \in \mathbb{Z}, \sum a_i \equiv 0 \pmod 2\}$ with an orthonormal basis $\{x_1, \cdots, x_r\}$ as we showed in the proof of Corollary 3.6. An irreducible V_L-module $V_{L+\frac{x_1+\cdots+x_r}{2}}$ is isomorphic to $L(\frac{1}{2}, \frac{1}{16})^{\otimes 2r} \otimes Q$ as $L(\frac{1}{2}, 0)^{\otimes 2r}$-modules and Q is an irreducible $\widehat{F_n}$-module. Since $\widehat{F_n}$ is a direct sum of an extra-special 2-group and a group of order 2, $Q_{|\hat{H}}$ contains all irreducible $\widehat{H}$-modules on which $-\kappa^{(0^n)}$ acts as –1. It is easy to see that $M_D \subseteq M_{D^1}$ and $M_{D^1} \cong V_{N(h+k)} \otimes V_{N(s+t)}$ and $W^1 \subseteq V_{\{N(h+k)+\frac{1}{2}(x_1+\cdots x_{h+k})\}} \otimes V_{N(s+t)}$. Similarly, we can find W^2 in $V_{\mathbb{R}L}$. It follows from the definition of vertex operators that there are $v \in W^1$ and $u \in W^2$ such that $Y(v, z)u \neq 0$. Since commutativity holds for $Y(v, z)$ and $Y(u, z)$ for $u \in M_D$ and $v \in W^1$, we have an intertwining operator $Y(*, z) \in I_{M_D}\begin{pmatrix} V_{\mathbb{R}L} \\ W^1 \; W^2 \end{pmatrix}$ by restriction. Namely, $W^1 \times W^2$ is nonzero. $\square$

An irreducible V-module X is called a "simple current" if $W \times X$ is irreducible for any irreducible V-module W.

Corollary 3.13. *If X is an irreducible M_D-module with $\tilde{\tau}(X) \in B(D)$, then the fusion product*

$$M_{\alpha+D} \times X$$

is an irreducible M_D-module for any α.

Proof. Since $\mathbb{C}X$ is an irreducible $\mathbb{C}M_D$-module by Lemma 3.10 and $\mathbb{C}M_{\alpha+D}$ is a simple current, $\mathbb{C}M_{\alpha+D} \times \mathbb{C}X$ is also irreducible. If $I\begin{pmatrix} U \\ M_{\alpha+D} \quad X \end{pmatrix} \neq 0$, then $\tilde{\tau}(U) = \tilde{\tau}(X) \in B(X)$ and so $\mathbb{C}U$ is irreducible and $\mathbb{C}U = \mathbb{C}M_{\alpha+D} \times \mathbb{C}X$. Hence $\dim I\begin{pmatrix} U \\ M_{\alpha+D} \quad X \end{pmatrix} \leq \dim I\begin{pmatrix} \mathbb{C}U \\ \mathbb{C}M_{\alpha+D} \quad \mathbb{C}X \end{pmatrix} = 1$ and so $M_{\alpha+D} \times X = U$. $\square$

Lemma 3.14. *Let (W, Y^W) be an irreducible M_D-module with $\tilde{\tau}(W) = \mu$ and let $W = \oplus_{i=0}^r U^i$ be the decomposition of W into the direct sum of distinct homogeneous M_{D_μ}-submodules U^i. Then U^i is irreducible and Y^W is uniquely determined by U^i for any i.*

Proof. Let X be an irreducible T-submodule of U^0 and set $X \cong \otimes_{i=1}^n L(\frac{1}{2}, h_i)$ ($h_i = 0, \frac{1}{2}, \frac{1}{16}$). By the fusion rule of $L(\frac{1}{2}, 0)$-modules, U^0 is homogeneous as a T-module; that is, every irreducible T-submodule of U^0 is isomorphic to X. By Proposition 4.1 in [DM2], $\{v_{(m)}u \mid u\in \mathbb{C}X, v\in \mathbb{C}M_\alpha, \alpha\in D\}$ spans $\mathbb{C}W$. On the other hand, if $\alpha=(a_i)\notin D_\mu$, then the irreducible $\mathbb{C}T$-submodule generated by $v_{(m)}u$ is isomorphic to $\otimes_{i=1}^n \mathbb{C}L(\frac{1}{2}, h_i+\frac{a_i}{2})$ and hence $\langle v_{(m)}u \mid u\in X, v\in \mathbb{C}M_\alpha, \alpha\in D\rangle \cap \mathbb{C}U^0=\mathbb{C}X$, which proves $\mathbb{C}U^0=\mathbb{C}X$ and $U^0=X$. We also have that $\langle v_{(m)}u \mid u\in U^0, v\in M_{\alpha+D_\mu}\rangle$ is an irreducible M_{D_μ}-module U^j for some j by the same arguments, which we denote by U^α. Corollary 3.13 implies that $M_{\alpha+D_\mu}\times U^0$ is irreducible. Considering the image of $Y(v,z)$ from U^0, we have a nonzero intertwining operator $Y(v,z): U^0\to U^\alpha[[z,z^{-1}]]$ for $v\in M_{\alpha+D_\mu}$. We hence conclude $M_{\alpha+D_\mu}\times U^\beta=U^{\alpha+\beta}$. That is, if one of the $\{U^i \mid i=1,\cdots,r\}$ is given, then the other U^j's are uniquely determined as M_{D_μ}-modules. Assume that there is another M_D-module S such that $S_{|M_{D_\mu}}\cong \oplus_{\beta\in D/D_\mu}U^\beta$ as M_{D_μ}-modules. Denote the restriction of $Y^W(*,z)$ on U^β by $I^{\alpha,\beta}(*,z): U^\beta\to U^{\alpha+\beta}$ and that of $Y^S(*,z)$ on U^β by $J^{\alpha,\beta}(*,z): U^\beta\to U^{\alpha+\beta}$ for $v\in M_{\alpha+D_\beta}$. Since $\dim I\begin{pmatrix} & U^{\alpha+\beta} \\ M_{D_\mu+\alpha} & U^\beta\end{pmatrix}=1$, there are scalars $\lambda_{\beta,\beta+\alpha}$ such that $J^{\alpha,\beta}(v,z)=\lambda_{\beta,\beta+\alpha}I^{\alpha,\beta}(v,z)$ for any $v\in M_{\alpha+D_\mu}$. For each α, let $A(\alpha)$ be a $|D/D_\mu|\times|D/D_\mu|$-matrix whose $(\beta,\beta+\alpha)$-entry is $\lambda_{\beta,\beta+\alpha}$ for any $\beta\in D/D_\mu$ and 0 otherwise. Since $\{Y^W(v,z)|v\in M_D\}$ and $\{Y^S(v,z)|v\in M_D\}$ satisfy mutual commutativity and associativity, respectively, $A: D/D_\mu\to M(|D/D_\mu|\times|D/D_\mu|,\mathbb{R})$ is a regular representation. We are hence able to reform $A(\alpha)$ into a permutation matrix by changing the basis. Therefore we may assume $J^{\alpha,\beta}=I^{\alpha,\beta}$ and so W is isomorphic to S as an M_D-module. □

Combining the arguments above, we have the following theorem:

THEOREM 3.15. *Let W be an irreducible M_E-module with $\tilde{\tau}(W)=\mu\in B(E)$. Let D be an even code containing E such that $\langle D,\mu\rangle=0$. Assume that there is a maximal self-orthogonal (doubly even) subcode H of E_μ such that H is also a maximal self-orthogonal subcode of D_μ. Then there is a unique irreducible M_D-module X containing W as an M_E-submodule. Here the subscript S_μ denotes $\{\alpha\in S|\mathrm{Supp}(\alpha)\subseteq \mathrm{Supp}(\mu)\}$ for any code S.*

We will call X in Theorem 3.15 an *induced M_D-module* and denote it by $\mathrm{Ind}_E^D(W)$.

We next quote the results about an extended $[8,4]$-Hamming code VOA $\mathbb{C}V_{H_8}$ from [Mi2]. Here an extended $[8,4]$-Hamming code H_8 is a subspace of $\mathbb{Z}_2^8$ spanned by $\{(1^8),(1^40^4),(1^20^21^20^2),(\{10\}^4)\}$, which is isomorphic to the Reed Müller code RM(1,3). Let $\{e_1,\cdots,e_8\}$ be a coordinate set of an extended $[8,4]$-Hamming code VOA M_{H_8}. Let W be an irreducible M_{H_8}-module. If $\tilde{\tau}(W)=(0^8)$, then $\mathbb{C}W$ is isomorphic to a coset module $\mathbb{C}M_{H_8+\alpha}$ for some

$\alpha\in\mathbb{Z}_2^8$ and hence W is isomorphic to $M_{H_8+\alpha}$. We denote it by $H(\frac{1}{2},\alpha)$. If $\tilde{\tau}(W)=(1^8)$, then there is a linear representation $\chi:\widehat{H_8}\to\{\pm1\}$ such that $\mathbb{C}W$ is isomorphic to $(L(\frac{1}{2},\frac{1}{16})^{\otimes 8})\otimes\ \mathbb{C}_\chi$. If we fix a basis $\{\alpha^1,\alpha^2,\alpha^3,\alpha^4\}$ of H_8, then there is a word β such that $\chi(\kappa^{\alpha^i})=(-1)^{\langle\beta,\alpha^i\rangle}$. In particular, χ is realizable over $\mathbb{R}$ and so W is isomorphic to $(L(\frac{1}{2},\frac{1}{16})^{\otimes 8})\otimes\mathbb{R}_\chi$, which we denote by $H(\frac{1}{16},\beta)$. We should also note that $H(\frac{1}{16},\beta)$ depends on the choice of the basis of H_8. So, we fix a basis $\{(1^8),(1^40^4),(1^20^21^20^2),((10)^4)\}$ of H_8 throughout this paper. We should also note that $\mathbb{C}H(h,\alpha)$ is denoted by $H(h,\alpha)$ in [Mi5]. Reforming the results in [Mi5] into those for VOAs over $\mathbb{R}$ by a similar argument as in §2.2, we have the following result.

Theorem 3.16. *Let W be an irreducible M_{H_8}-module. If $\tilde{\tau}(W)=(0^8)$, then W is isomorphic to one of*

$$\{H(\tfrac{1}{2},\alpha)\mid\alpha\in\mathbb{Z}_2^8\}.$$

If $\tilde{\tau}(W)=(1^8)$, then W is isomorphic to one of

$$\{H(\tfrac{1}{16},\alpha)\mid\alpha\in\mathbb{Z}_2^8\}.$$

$H(\frac{1}{2},\alpha)\cong H(\frac{1}{2},\beta)$ if and only if $\alpha+\beta\in H_8$ and $H(\frac{1}{16},\alpha)\cong H(\frac{1}{16},\beta)$ if and only if $\alpha+\beta\in H_8$. $H(\frac{1}{2},\alpha)$ is a coset module $M_{H_8+\alpha}$ and $H(\frac{1}{16},\beta)$ is isomorphic to $L(\frac{1}{2},\frac{1}{16})^{\otimes 8}$ as an $L(\frac{1}{2},0)^{\otimes 8}$-module.

In [Mi5], the author obtained the fusion rules among

$$\{\mathbb{C}H(r,\alpha)\mid r=\tfrac{1}{2},\tfrac{1}{16},\alpha\in\mathbb{Z}_2^8\}.$$

Since H_8 is doubly even, we have the following by Lemma 2.5 and Lemma 3.12.

Lemma 3.17.

$$H(\tfrac{1}{2},\alpha)\times H(\tfrac{1}{2},\beta)=H(\tfrac{1}{2},\alpha+\beta),$$
$$H(\tfrac{1}{16},\alpha)\times H(\tfrac{1}{2},\beta)=H(\tfrac{1}{16},\alpha+\beta)$$

and

$$H(\tfrac{1}{16},\alpha)\times H(\tfrac{1}{16},\beta)=H(\tfrac{1}{2},\alpha+\beta).$$

We next show that M_{H_8} contains the other two coordinate sets. To simplify the notation, we will choose another cocycle of $\widehat{H_8}$ for a while. We have already fixed a basis $\{\alpha_1,\cdots,\alpha_4\}$ of H_8. Set $\bar{\kappa}^\alpha=\kappa^{a_1\alpha_1}\cdots\kappa^{a_4\alpha_4}$ for $\alpha=\sum_{i=1}^4 a_i\alpha_i\in H_8$. Note that H_8 contains 14 words of weight 4. For such a codeword (or a 4 points set) $\beta=(b_1\cdots b_8)$, let

$$\bar{q}^\beta=\tfrac{1}{4}(\otimes_{i=1}^8 q^{(b_i)})\otimes\bar{\kappa}^\alpha\in(M_{H_8})_2.$$

It follows from a direct calculation that

$$s^\alpha=\tfrac{1}{8}(e_1+\cdots+e_8)+\tfrac{1}{8}\sum_{\beta\in H_8,\ |\beta|=4}(-1)^{(\alpha,\beta)}\bar{q}^\beta$$

is a conformal vector with central charge $\frac{1}{2}$ for every word $\alpha\in\mathbb{Z}_2^8$ as we showed in [Mi2]. Clearly, $s^\alpha=s^\beta$ if and only if $\alpha+\beta\in H_8$. It is also straightforward to check that $\langle s^\alpha,s^\beta\rangle=0$ if and only if $\alpha+\beta$ is an even word. Therefore we have two new coordinate sets $\{d_1,\cdots,d_8\}$ and $\{f_1,\cdots,f_8\}$ in M_{H_8}. Set $T_d=\langle d_1,\cdots,d_8\rangle$ and $T_f=\langle f_1,\cdots,f_8\rangle$. With M_{H_8} a T_d-module and a T_f module, $\bigoplus_{(a_1,\cdots,a_8)\in H_8}\left(\otimes_{i=1}^8 L(\frac{1}{2},\frac{a_i}{2})\right)\cong M_{H_8}$. Therefore there is an automorphism σ of M_{H_8} such that $\sigma(e_i)=d_i$ and $\sigma(d_i)=f_i$ for every i, which is obtained by rearrangment of the orders of $\{d_i\}$ and $\{f_i\}$. Viewing an M_{H_8}-module as a T_d-module and a T_f-module, we have the following correspondence (see Proposition 2.2 and Lemma 2.7 in [Mi5]):

Lemma 3.18. *There is an automorphism σ of M_{H_8} such that*

$$\sigma(H(\tfrac{1}{2},(0^8)))\cong H(\tfrac{1}{2},(0^8)),$$
$$\sigma(H(\tfrac{1}{2},\xi_1))\cong H(\tfrac{1}{16},(0^8)),$$
$$\sigma(H(\tfrac{1}{16},(0^8)))\cong H(\tfrac{1}{16},\xi_1)$$

and

$$\sigma(H(\tfrac{1}{16},\xi_1))\cong H(\tfrac{1}{2},\xi_1),$$

where ξ_1 denotes (10^7). In particular, $\sigma(q^{(1^8)})_{(3)}$ acts on $H(\frac{1}{16},(0^8))$ as $-q^{(1^8)}_{(3)}$, where $q^{(1^8)}=\left((\otimes_{i=1}^8 q^{(1)})\otimes\kappa^{(1^8)}\right)$.

Since all codewords of H_8 are in $B(H_8)$, we have the following as a corollary.

Corollary 3.19. *$H(\frac{1}{2},\alpha)$ and $H(\frac{1}{16},\alpha)$ are all simple currents.*

We will next prove the following important theorem.

Theorem 3.20. *Let W^1 and W^2 be irreducible M_D-modules with $\alpha=\tilde\tau(W^1)$, $\beta=\tilde\tau(W^2)$. For a triple (D,α,β), the following two conditions are assumed:*

(3.a) *D contains a self-dual subcode E which is a direct sum of k extended $[8,4]$-Hamming codes such that $E_\alpha=\{\gamma\in E|\mathrm{Supp}(\gamma)\subseteq\mathrm{Supp}(\alpha)\}$ is a direct factor of E or $\{0\}$.*

(3.b) *There are maximal self-orthogonal subcodes H^β and $H^{\alpha+\beta}$ of D_β and $D_{\alpha+\beta}$ containing E_β and $E_{\alpha+\beta}$, respectively, such that they are doubly even and*

$$H^\beta+E=H^{\alpha+\beta}+E,$$

where the subscript S_α denotes a subcode $\{\beta\in S|\mathrm{Supp}(\beta)\subseteq\mathrm{Supp}(\alpha)\}$ for any code S.

Then $W^1\times W^2$ is irreducible.

Proof. Suppose the conclusion is false and choose D as a minimal counterexample. If $\alpha=0$ or $\beta=0$, then W^1 or W^2 is a coset module and the assertion follows from Corollary 3.13, since $\tilde{\tau}(W^i)\in B(D)$. By the assumption (3.a), the weight of α is a multiple of eight. We may assume $\alpha=(1^{8r}0^{8s})$ and $\beta\neq 0$. Let Q be an irreducible M_D-module so that $0\neq I\begin{pmatrix} & Q & \\ W^1 & & W^2\end{pmatrix}$. Clearly $\tilde{\tau}(Q)=\alpha+\beta$. By the assumption (3.a), there is a self-dual subcode $E=E_\alpha\oplus E_{\alpha^c}$ of D such that E is a direct sum of extended $[8,4]$-Hamming codes.

(1) Assume first that $E_\beta=\{\gamma\in E|\mathrm{Supp}(\gamma)\subseteq \mathrm{Supp}(\beta)\}$ is a direct factor of E; that is, $E=E_\beta\oplus E_{\beta^c}$. Let U^i be an irreducible M_E-submodule of W^i for each $i=1,2$. By Theorem 3.16, $U^1\cong(\otimes_{i=1}^r H(\frac{1}{16},\alpha^i))\otimes(\otimes_{j=1}^s H(\frac{1}{2},\beta^j))$ as M_E-modules and hence $U^1\times U^2$ is an irreducible M_E-module. Since Q contains $U^1\times U^2$ as an M_E-module, Q is uniquely determined as an M_D-module. Since Q is a direct sum of distinct irreducible M_E-submodules and the restrictions $I\begin{pmatrix} & Q & \\ W^1 & & W^2\end{pmatrix}\to I\begin{pmatrix} & Q & \\ U^1 & & U^2\end{pmatrix}\to I\begin{pmatrix} U^1\times U^2 \\ U^1 \quad U^2\end{pmatrix}$ are injective, we have $W^1\times W^2=Q$.

(2) We assume that E_β is not a direct factor of E. By the assumption (3.b), there are maximal self-orthogonal (doubly even) subcodes H^β and $H^{\alpha+\beta}$ of D_β and $D_{\alpha+\beta}$ containing E_β and $E_{\alpha+\beta}$, respectively, such that $H^\beta+E=H^{\alpha+\beta}+E$. Set $D'=H^\beta+D$. It is easy to check that (D',α,β) satisfies (3.a) and (3.b).

Assume that $D\neq D'$. Let X^1 and X^2 be irreducible $M_{D'}$-submodules of W^1 and W^2, respectively. By the minimality of D, $X^1\times X^2$ is irreducible. Since Q contains a submodule isomorphic to $X^1\times X^2$ as an $M_{D'}$-module and $D'_{\alpha+\beta}$ contains $H^{\alpha+\beta}$, Q is uniquely determined. Since Q contains only one irreducible submodule isomorphic to $X^1\times X^2$, we have $W^1\times W^2=Q$ and $D=H^\beta+E$.

(2.1) We claim that W^2 and Q are irreducible as M_E-modules.

First, note that $\tilde{\tau}(Q)=\alpha+\beta$ and $D=H^\beta+E=H^{\alpha+\beta}+E$. Since the proofs are almost the same, we will prove the assertion only for W^2. Since H^β contains E_β and $D=H^\beta+E$, we obtain $D_\beta=H^\beta$. If P is an irreducible M_{H^β}-submodule of W^2, then $W^2=\mathrm{Ind}_{H^\beta}^D(P)$ and P is irreducible as a T-module. In particular, P is irreducible as an M_{E_β}-module. Since $\tilde{\tau}(P)=\beta$, $\mathrm{Ind}_{E_\beta}^E(P)$ is an irreducible M_E-submodule of W^2. On the other hand, since $D/H^\beta\cong E/E_\beta$, we have $\dim(\mathrm{Ind}_{E_\beta}^E(P))=\dim(\mathrm{Ind}_{H^\beta}^D(P))=\dim W^2$ so that W^2 is an irreducible M_E-module, which proves the claim.

(2.2) Let U^1 be an irreducible M_E-submodule of W^1. Since $\tilde{\tau}(U^1)=\alpha$ and E_α is a direct sum of E, U^1 is a simple current. Since W^2 and U are both irreducible M_E-modules by the claim above, $U^1\times W^2$ is irreducible.

Furthermore, since

$$0 \neq \dim I_{M_D}\begin{pmatrix} & Q & \\ W^1 & & W^2\end{pmatrix} \leq \dim I_{M_E}\begin{pmatrix} & Q & \\ U^1 & & W^2\end{pmatrix} \leq 1, \tag{3.20}$$

we have $U^1 \times W^2 \cong Q$ as M_E-modules. Fix a nonzero intertwining operator

$$I^1(*, z) \in I_{M_E}\begin{pmatrix} & Q & \\ U^1 & & W^2\end{pmatrix}.$$

For $I(*, z) \in I_{M_D}\begin{pmatrix} & Q & \\ U^1 & & W^2\end{pmatrix}$, there is a scalar $\lambda \in \mathbb{R}$ such that $I(v, z) = \lambda I^1(v, z)$ for $v \in U^1$. Since $Y^Q(u, z)I(v, z) \sim I(v, z)Y^2(u, z)$, we have

$$Y^Q(u, z)I^1(v, z) = I^1(v, z)Y^2(u, z)$$

for $u \in M_D$ and $v \in U^1$. Since the coefficients of $\{I^1(v, z)w \mid v \in U^1, w \in W^2\}$ spans Q, $Y^Q(u, z)$ is uniquely determined by $Y^2(u, z)$ and hence the action of M_D on Q is uniquely determined. Thus $W^1 \times W^2 = Q$ by (3.20). □

We now arrive at the main result of this section, which is an $\mathbb{R}$-version of Theorem 6.5 in [Mi5]:

Theorem 3.21. *Let $W = M_D \oplus W^\delta \oplus W^\gamma \oplus W^{\delta+\gamma}$ be a $(D, \langle\delta, \gamma\rangle)$-framed* VOA *and let F be an even linear subcode of $\langle\delta, \gamma\rangle^\perp$ containing D. Assume that $\langle\delta, \gamma\rangle \subseteq B(D)$, D_μ contains a maximal self-orthogonal (doubly even) subcode of F_μ for any $\mu \in \langle\delta, \gamma\rangle$ and*

$$\mathrm{Ind}_D^F(W^\alpha) \times \mathrm{Ind}_D^F(W^\beta) = \mathrm{Ind}_D^F(W^{\alpha+\beta}) \tag{3.21}$$

for $\alpha, \beta \in \langle\delta, \gamma\rangle$. Then

$$\mathrm{Ind}_D^F(W) := M_F \oplus \mathrm{Ind}_D^F(W^\delta) \oplus \mathrm{Ind}_D^F(W^\gamma) \oplus \mathrm{Ind}_D^F(W^{\delta+\gamma})$$

has an $(F, \langle\delta, \gamma\rangle)$-framed VOA *structure, which contains W as a sub* VOA.

We will also prove that if $M_D \oplus W^\delta$ is a $(D, \langle\delta\rangle)$-framed VOA and $\mathrm{Ind}_D^F(W^\delta) \times \mathrm{Ind}_D^F(W^\delta) = M_F$, then $M_F \oplus \mathrm{Ind}_D^F(W^\delta)$ is an $(F, \langle\delta\rangle)$-framed VOA. Before we prove Theorem 3.21, we note that the conditions of Theorem 3.21 including the fusion rule (3.21) follow from the conditions of Theorem 3.20.

Proposition 3.22. *Assume the triple (D, α, β) satisfies the conditions of Theorem 3.20 for any $\alpha, \beta \in \langle\delta, \gamma\rangle$. Then $\mathrm{Ind}_D^F(W^\alpha) \times \mathrm{Ind}_D^F(W^\beta) = \mathrm{Ind}_D^F(W^{\alpha+\beta})$ for $\alpha, \beta \in \langle\delta, \gamma\rangle$.*

Proof of Theorem 3.21. Set $V = \mathrm{Ind}_D^F(W)$. For simplicity, we denote Ind_D^F by Ind. Let $Y^W(v, z) \in \mathrm{End}(W)[[z, z^{-1}]]$ be the given vertex operator of $v \in W$. For $\alpha', \beta' \in S = \langle\delta, \gamma\rangle$, let

$$J^{\alpha', \beta'}(v, z) \in I_{M_D}\begin{pmatrix} & W^{\alpha'+\beta'} & \\ W^{\alpha'} & & W^{\beta'}\end{pmatrix}$$

be the restriction of $Y^W(v,z)$ on $W^{\beta'}$ for $v\in W^{\alpha'}$ and $\alpha',\beta'\in S=\langle\delta,\gamma\rangle$. Since Theorem 11.9 in [DL] implies that a natural restriction

$$\phi: I_{M_F}\begin{pmatrix} & \mathrm{Ind}(W^{\gamma'}) & \\ \mathrm{Ind}(W^{\alpha'}) & & \mathrm{Ind}(W^{\beta'})\end{pmatrix} \to I_{M_D}\begin{pmatrix} & W^{\gamma'} & \\ W^{\alpha'} & & W^{\beta'}\end{pmatrix}$$

is injective and the multiplicity of $W^{\alpha'}\times W^{\beta'}$ in $\mathrm{Ind}(W^{\alpha'+\beta'})$ is one, we can choose

$$I^{\alpha',\beta'}(*,z)\in I_{M_F}\begin{pmatrix} & \mathrm{Ind}(W^{\alpha'+\beta'}) & \\ \mathrm{Ind}(W^{\alpha'}) & & \mathrm{Ind}(W^{\beta'})\end{pmatrix}$$

such that $I^{\alpha',\beta'}(v,z)u=J^{\alpha',\beta'}(v,z)u$ for any $v\in W^{\alpha'}$ and $u\in W^{\beta'}$. Define $Y(v,z)\in\mathrm{End}(V)[[z,z^{-1}]]$ by $Y(v,z)u=I^{\alpha',\beta'}(v,z)u$ for $v\in\mathrm{Ind}(W^{\alpha'})$ and $u\in\mathrm{Ind}(W^{\beta'})$. Note that $Y(v,z)u=Y^W(v,z)u$ for $u,v\in W$. Moreover, the powers of z in $Y(v,z)$ are all integers since $\langle\tilde\tau(\mathrm{Ind}(W)),F\rangle=0$ by Proposition 2.6. For $u,v\in W$, we have $Y(u,z_1)Y(v,z_2)\sim Y(v,z_2)Y(u,z_1)$. We also have that $Y(v,z)|_{\mathrm{Ind}(W^\beta)}$ is at least an intertwining operator for $v\in V$ and so $Y(v,z_1)Y(u,z_2)\sim Y(u,z_2)Y(v,z_1)$ for $u\in M_F$ and $v\in\mathrm{Ind}(W^\alpha)$. Hence

$$T^{u,u'}:=\{w\in\mathrm{Ind}(W)\mid Y(u',z)Y(u,x)w\sim Y(u,x)Y(u',z)w\} \tag{3.22}$$

is an M_F-module for $u,u'\in W$. Since $T^{u,u'}$ contains W, it coincides with V. Namely, $\{Y(u,z)\mid u\in W\cup M_D\}$ satisfies mutual commutativity on V. Clearly, $\{Y(v,z)\mid v\in M_D\cup W\}$ generates vertex operators for all elements of V by the normal products and hence $\{Y(v,z)|v\in V\}$ satisfies mutual commutativity by Dong's lemma. The other required conditions are also easy to check and so we have a desired VOA structure on $V=\mathrm{Ind}(W)$. □

LEMMA 3.23. *Let $V=\oplus_{\alpha\in S}V^\alpha$ be a (D,S)-framed* VOA *satisfying the conditions of Theorem* 3.20 *and assume that W is an irreducible V-module. Let $W=\oplus_{\beta\in S'}W^\beta$ be the decomposition into the direct sum of nonzero M_D-modules W^β with $\tilde\tau(W^\beta)=\beta$ for all $\beta\in S'$. Then W^β are all irreducible M_D-modules and there is a word γ such that $S'=S+\gamma$.*

Proof. We note that M_D is rational. By arguments similar to those in the proof of Theorem 3.8, we have that W^β is irreducible. We note $\tilde\tau(V^\alpha\times W^\beta)=\alpha+\beta$. Since $Y(v,z)u\neq 0$ for $0\neq v\in V^\alpha$ and $0\neq u\in W^\beta$ by [DL], S' contains $\gamma+S$ for any $\gamma\in S'$. Since W is irreducible, S' is a coset. □

Hypotheses I.

(1) D and S are both even linear codes of length $8k$.

(2) V is a direct sum $\oplus_{\alpha\in S}V^\alpha$ of irreducible M_D-modules V^α satisfying $\tilde\tau(V^\alpha)=\alpha$.

(3) For any $\alpha,\beta\in S$, there is a fusion rule $V^\alpha\times V^\beta=V^{\alpha+\beta}$.

(4) For $\alpha, \beta \in S - \{(0^n)\}$ satisfying $\alpha \neq \beta$, it is possible to define a framed VOA structure with a PDIB-form on

$$V^{\langle \alpha,\beta\rangle} = M_D \oplus V^{\alpha} \oplus V^{\beta} \oplus V^{\alpha+\beta}.$$

As a special case, if $S = \langle \alpha \rangle$, then we assume that $V^{\langle \alpha \rangle} = M_D \oplus V^{\alpha}$ has a framed VOA structure with a PDIB-form.

LEMMA 3.24. *Let $V = \oplus_{\alpha \in S} V^{\alpha}$ be a* VOA *satisfying the conditions of Hypotheses* I *and $W = \oplus_{\beta \in S+\gamma} W^{\beta}$ an irreducible V-module. Assume that (D, α, β) satisfies the conditions of Theorem* 3.20 *for any $\alpha, \beta \in S + \mathbb{Z}_2\gamma$. Then W is uniquely determined by W^{β} for any $\beta \in S + \gamma$.*

Proof. Since $V^{\alpha} \times W^{\beta} = W^{\alpha+\beta}$ by Theorem 3.20, an M_D-module structure on W is uniquely determined by W^{β}. By arguments similar to those in the proof of Theorem 3.15, we have the desired conclusion. □

Since the intertwining operators among $L(\frac{1}{2}, 0)$-modules are all well-defined over $\mathbb{R}$ (even over $\mathbb{Q}$), we can rewrite Theorem 4.1 of [Mi5] into the following theorem.

THEOREM 3.25. *Under Hypotheses* I,

$$V = \bigoplus_{\alpha \in S} V^{\alpha}$$

has a structure of (D, S)-framed VOA *with a* PDIB-*form. A framed* VOA *structure on $V = \oplus_{\alpha \in S} V^{\alpha}$ with a* PDIB-*form is uniquely determined up to M_D-isomorphisms.*

Proof. First, we fix vertex operators $Y^{V^{\alpha}}(v, z)$ of $v \in M_D$ on M_D-modules V^{α}. Set $I^{0,\alpha}(v, z) = Y^{V^{\alpha}}(v, z)$. Let $Y^{\langle \alpha,\beta\rangle}$ denote a vertex operator of the VOA $V^{\langle \alpha,\beta\rangle} = M_D \oplus V^{\alpha} \oplus V^{\beta} \oplus V^{\alpha+\beta}$. We may assume that $Y^{\langle \alpha,\beta\rangle}(v, z)u = Y^{V^{\delta}}(v, z)u$ for $v \in M_D$ and $u \in V^{\delta}$ for $\delta \in \langle \alpha, \beta\rangle$. Define $I^{\alpha,0}(*, z) \in I\begin{pmatrix} & V^{\alpha} & \\ V^{\alpha} & & M_D \end{pmatrix}$ by the skew-symmetry property: $I^{\alpha,0}(u, z)v = e^{zL(-1)} Y^{V^{\alpha}}(v, -z)u$ for $v \in M_D$ and $u \in V^{\alpha}$, which is equal to $Y^{\langle \alpha,\beta\rangle}(u, z)_{|M_D} v$ for any β. We also define $I(*, z) \in I\begin{pmatrix} & M_D & \\ V^{\alpha} & & V^{\alpha} \end{pmatrix}$ by $I(u', z)u = Y^{\langle \alpha,\beta\rangle}(u', z)u$ for $u, u' \in V^{\alpha}$ for some β. We will show that this does not depend on β. Since $V^{\alpha} \times V^{\alpha} = M_D$ and our VOAs are over $\mathbb{R}$, there are two possibilities of VOA structures on $M_D \oplus V^{\alpha}$ given by $Y^{\pm}(v, z) = \begin{pmatrix} 0 & \pm I(v, z) \\ I^{\alpha,0}(v, z) & 0 \end{pmatrix}$ for $v \in V^{\alpha}$. Since we also assumed that $M_D \oplus V^{\alpha}$ has a PDIB-form, there is a unique VOA structure on $M_D \oplus V^{\alpha}$ up to M_D-isomorphism. That is, if we fix an orthonormal basis $\{u_i^{\alpha} \mid i \in I_{\alpha}\}$

of V^α, then $Y^{\langle\alpha,\beta\rangle}(u,z)v$ for $u,v\in V^\alpha$ does not depend on the choice of β. So set $I^{\alpha,\alpha}(v,z)=I(v,z)$. Define a nonzero intertwining operator

$$I^{\alpha,\beta}(*,z)\in I\begin{pmatrix} & V^{\alpha+\beta} & \\ V^\alpha & & V^\beta\end{pmatrix}$$

for $\alpha,\beta\in S$ satisfying $\dim\langle\alpha,\beta\rangle=2$ by $I^{\alpha,\beta}(v,z)u=Y^{\langle\alpha,\beta\rangle}(v,z)u$ for $v\in V^\alpha$ and $u\in V^\beta$. Now we have $I^{\alpha,\beta}(*,z)$ for all $\alpha,\beta\in S$.

Our next step is to choose suitable scalars $\lambda^{\alpha,\beta}$ and define a new vertex operator $Y(v,z)\in\mathrm{End}(V)[[z,z^{-1}]]$ by

$$Y(v,z)u:=\lambda^{\alpha,\beta}I^{\alpha,\beta}(v,z)u \tag{3.23}$$

for $v\in V^\alpha$ and $u\in V^\beta$ so that $\{Y(v,z)\mid v\in V\}$ satisfies mutual commutativity. We note that intertwining operators already satisfy the $L(-1)$-derivative property and the other conditions except mutual commutativity and so "mutual commutativity" is the only thing we have to prove. Let $\{\alpha_1,\cdots,\alpha_t\}$ be a basis of S and set $S_i=\langle\alpha_1,\cdots,\alpha_i\rangle$ for $i=0,1,\cdots,t$ and $V^{[i]}=\oplus_{\alpha\in S_i}V^\alpha$. We will choose $\lambda^{\alpha,\beta}$ inductively so that (3.23) becomes a vertex operator of VOA $V^{[i]}$ by restriction to $V^{[i]}$ and also is a vertex operator on $V^{[i]}$-module V. Since the V^α are all M_D-modules, the vertex operators $Y^V(v,z)$ of $v\in V^{[0]}$ ($\cong M_D$) on V satisfy mutual commutativity and so set $\lambda^{0,\alpha}=1$. We next assume that there are an integer r and scalars $\lambda^{\alpha,\beta}$ for $\alpha\in S_r$ and $\beta\in S$ such that $Y(v,z)$ given by (3.23) is a vertex operator of $V^{[r]}$ by restricting on $V^{[r]}$ and is also a vertex operator of VOA $V^{[r]}$ on $V^{[r]}$-module V. It is clear that $V^{S_r+\delta}=\oplus_{\gamma\in S_r}V^{\delta+\gamma}$ is an irreducible $V^{[r]}$-module for each $\delta\in S$ by the fusion rules and hence V decomposes into the direct sum of irreducible $V^{[r]}$-modules. It follows from the fusion rule of M_D-modules V^β and Lemma 3.24, that

$$V^{\delta+S_r}\times V^{\gamma+S_r}=V^{\delta+\gamma+S_r}$$

as $V^{[r]}$-modules. Decompose $V^{[r+1]}=V^{[r]}\oplus V^{\alpha_{r+1}+S_r}$ as $V^{[r]}$-modules. To simplify the notation, we denote α_{r+1} by α. Let $\{\gamma_i\in S\mid i\in J\}$ be a set of representatives of cosets S/S_{r+1}. Since the natural restriction

$$\pi: I_{V^{[r]}}\begin{pmatrix} & V^{S_r+\alpha+\gamma_i} & \\ V^{S_r+\alpha} & & V^{S_r+\gamma_i}\end{pmatrix}\to I_{M_D}\begin{pmatrix} & V^{S_r+\alpha+\gamma_i} & \\ V^{\alpha} & & V^{\gamma_i}\end{pmatrix}$$

is injective and $\dim I_{M_D}\begin{pmatrix} & V^{S_r+\alpha+\gamma_i} & \\ V^{\alpha} & & V^{\gamma_i}\end{pmatrix}=1$, we can choose a nonzero intertwining operator $I^{\alpha+S_r,\gamma_i+S_r}(*,z)\in I\begin{pmatrix} & V^{S_r+\alpha+\gamma_i} & \\ V^{S_r+\alpha} & & V^{S_r+\gamma_i}\end{pmatrix}$ such that

$$I^{\alpha+S_r,\gamma_i+S_r}(v,z)u=I^{\alpha,\gamma_i}(v,z)u$$

for $v\in V^\alpha,u\in V^{\gamma_i}$. Restricting $I^{\alpha+S_r,\gamma_i+S_r}(*,z)$ to $V^{\alpha+\beta,\gamma_i+\delta}$ for $\beta,\delta\in S_r$, we have a scalar $\lambda_{\alpha+\beta,\gamma_i+\delta}$ such that

$$I^{\alpha+S_r,\gamma_i+S_r}(v,z)u=\lambda_{\alpha+\beta,\gamma_i+\delta}I^{\alpha+\beta,\gamma_i+\delta}(v,z)u$$

for $v\in V^{\alpha+\beta}$ and $u\in V^{\gamma_i+\delta}$. We will show that $V^{[r+1]}$ is a VOA and V is a module with a vertex operator $Y(v,z)=I^{\alpha+S_r,r_i+S_r}(v,z)$ for $v\in V^{S_r+\alpha}$, which proves the assertion. Set

$$Q=\{w\in V|Y(u,z)Y(u',x)w\sim Y(u',x)Y(u,z)w \text{ for } u,u'\in V^{\alpha}\}.$$

Since $Y(*,z)$ is an intertwining operator of $V^{[r]}$-modules, Q is a $V^{[r]}$-module. On the other hand, by the definition of Y, Q contains V^{γ^i} for all i. Hence Q coincides with V. In particular, $\{Y(u,z)\mid u\in V^{[r]}\cup V^{\alpha}\}$ satisfies mutual commutativity. Since $V^{[r+1]}$ is generated by $V^{[r]}$ and V^{α}, we have the desired result. This completes the construction of our VOA.

We next show that a framed VOA structure on $V=\oplus_{\alpha\in S}V^{\alpha}$ is unique. Assume that there are two VOA structures (V,Y) and (V,Y') on V. Clearly, the $V^{\langle\alpha,\beta\rangle}$ are sub VOAs of both (V,Y) and (V,Y'). Since $\dim I_{M_D}\begin{pmatrix} & V^{\alpha+\beta} & \\ V^{\alpha} & & V^{\beta}\end{pmatrix}=1$, there are real numbers $\lambda_{\alpha,\beta}$ such that $Y'(v,z)u=\lambda_{\alpha,\beta}Y(v,z)u$ for $v\in V^{\alpha}$, $u\in V^{\beta}$. Clearly $\lambda_{*,*}$ is a cocycle of an elementary abelian 2-group S. We will show that it is a coboundary so that we have the desired result. Let $\widehat{S}$ be a group extension of S by a cocycle $\lambda_{*,*}$. Since both $\{Y(v,z)|v\in V\}$ and $\{Y'(v,z)|v\in V\}$ satisfy mutual commutativity, respectively, $\widehat{S}$ is an abelian 2-group. By the assumption, $\lambda_{(0^n),\beta}=1$ and so $\lambda_{\beta,(0^n)}=1$ by the skew symmetry. Since both have a PDIB-form, we may assume $\lambda_{\alpha,\alpha}=1$ for all $\alpha\in S$ by changing the basis of (V,Y'), which implies that $\widehat{S}$ is an elementary abelian 2-group and $\lambda_{*,*}$ is a coboundary of S over $\mathbb{R}$. $\square$

For a word α, we can define an automorphism σ_α of $M_D=\oplus_{\beta\in D}M_\beta$ by

$$\sigma_\alpha:(-1)^{\langle\beta,\alpha\rangle} \text{ on } M_\beta$$

and extend it by linearity. We will next show a relation between σ_α and a fusion product $M_{\alpha+D}\times W$.

Lemma 3.26. *Let W be an irreducible M_D-module with $\beta:=\tilde{\tau}(W)\in B(D)$. Let H be a maximal self orthogonal (doubly even) subcode H of D_β and α a binary word in $H^{\perp}$. Then $\sigma_\alpha W$ is isomorphic to W as an M_D-module.*

Proof. Decompose M_D into $M_D^+\oplus M_D^-$, where

$$M_D^{\pm}=\{v\in M_D\mid \sigma_\alpha(v)=\pm v\}.$$

Set $E=\{\gamma\in D\mid\langle\gamma,\alpha\rangle=0\}$. Clearly, $M_D^+=M_E$. Since E contains H, there is an M_E-module U such that $\mathrm{Ind}_E^D(U)=W$ by Theorem 3.15. It follows from the definition of the induced modules that $\mathrm{Ind}_E^D(U)\cong U\oplus(M_D^-\times U)$ as M_E-modules. The actions of M_D^- switch U and $M_D^-\times U$; that is, $u_{(m)}(U)\subseteq M_D^-\times U$ and $u_{(m)}(M_D^-\times U)\subseteq U$ for any $m\in\mathbb{Z}$ and $u\in M_D^-$. Moreover, $u_{(m)}\sigma_\alpha v=-u_{(m)}v$ for $u\in M_D^-$ and $v\in\mathrm{Ind}_E^D(U)$. It is easy to check that $(1_U,-1_{M_D^-\times U})$ on $U\oplus M_D^-\times U$ is an isomorphism from $\sigma_\alpha(\mathrm{Ind}_E^D(U))$ to $\mathrm{Ind}_E^D(U)$. $\square$

For an irreducible M_D-module W, $\sigma_\alpha W$ is also an irreducible M_D-module. Clearly, W and $\sigma_\alpha W$ are isomorphic as T-modules and $\sigma_\alpha=\sigma_\beta$ if and only if $\alpha+\beta\in D^\perp$. Let α be a word satisfying $\mathrm{Supp}(\alpha)\subseteq\mathrm{Supp}(\tilde\tau(W))$. In this case $M_{D+\alpha}\times W$ is isomorphic to W as a T-module. The following lemma is important.

LEMMA 3.27. *Let W be an irreducible M_D-module with $\tilde\tau(W)\in B(D)$ and assume* $\mathrm{Supp}(\alpha)\subseteq\mathrm{Supp}(\tilde\tau(W))$. *Then $M_{\alpha+D}\times W$ is isomorphic to $\sigma_\alpha W$ as an M_D-module.*

Proof. Set $\beta=\tilde\tau(W)$. Clearly, $\tilde\tau(M_{D+\alpha}\times W)=\tilde\tau(\sigma_\alpha W)=\beta$. By Corollary 3.13, $W'=M_{\alpha+D}\times W$ is irreducible. Let H be a maximal self-orthogonal (doubly even) subcode of D_β. Since an M_D-module W with $\tilde\tau(W)=\beta$ is uniquely determined by an M_H-submodule, we may assume that D is a self-orthogonal doubly even code and $\mathrm{Supp}(D)\subseteq\mathrm{Supp}(\beta)$. In particular, we may also assume that W and W' are both isomorphic to $L(\frac12,\frac1{16})^{\otimes n}$ as T-modules. Since $1\le\dim I_{M_D}\begin{pmatrix} & W' & \\ U & & W\end{pmatrix}\le\dim I_T\begin{pmatrix} & L(\frac12,\frac1{16})^{\otimes n} & \\ M_\gamma & & L(\frac12,\frac1{16})^{\otimes n}\end{pmatrix}=1$, an intertwining operator of type $\begin{pmatrix} & W' & \\ M_{\gamma+D}, & & W\end{pmatrix}$ is uniquely determined up to scalar multiple for $\gamma\in D+\alpha$. As shown in §2.4 or in [Mi5], we can choose a nonzero intertwining operator $I(*,z)\in I_T\begin{pmatrix} & L(\frac12,\frac1{16})^{\otimes n} & \\ M_\gamma & & L(\frac12,\frac1{16})^{\otimes n}\end{pmatrix}$ by

$$I(q^\gamma,z)=I(\hat q^\gamma\otimes\kappa^\gamma,z)=\otimes_{i=1}^n I^{g_i,\frac1{16}}(q^{g_i},z)\otimes\kappa^\gamma,$$

where $\gamma=(g_1,\cdots,g_n)$ and $I^{g_i,\frac1{16}}(*,z)$ are the fixed intertwining operators of type $\begin{pmatrix} & L(\frac12,\frac1{16}) & \\ L(\frac12,\frac{g_i}2) & & L(\frac12,\frac1{16})\end{pmatrix}$ given by (2.5) and (2.6). By Theorem 3.8, there are linear modules $\mathbb{R}_\chi$ and $\mathbb{R}_\phi$ of $\widehat D=\{\kappa^\alpha|\alpha\in D\}$ such that $W\cong L(\frac12,\frac1{16})^{\otimes n}\otimes\mathbb{R}_\chi$ and $W'\cong L(\frac12,\frac1{16})^{\otimes n}\otimes\mathbb{R}_\phi$, respectively. By associativity of intertwining operators, we have

$$\begin{aligned}&I(q^\beta_{(m)}q^\alpha,z)\\&=\mathrm{Res}_x\{(x-z)^mY^{W'}(q^\beta,x)I(q^\alpha,z)-(-z+x)^mI(q^\alpha,z)Y^W(q^\beta,x)\}\\&=\mathrm{Res}_x\{(x-z)^mI^{\otimes n}(\hat q^\beta,x)\phi(\kappa^\beta)I(q^\alpha,z)-(-z+x)^mI(q^\alpha,z)I^{\otimes n}(\hat q^\beta,x)\chi(\kappa^\beta)\}\end{aligned}$$

for $q^\beta\in M_\beta\subseteq M_D$ and $q^\alpha\in M_\alpha$. In particular, for a sufficiently large N, we obtain

$$0=\mathrm{Res}_x\{(x-z)^NI^{\otimes n}(\hat q^\beta,x)\phi(\kappa^\beta)I(q^\alpha,z)-(-z+x)^NI(q^\alpha,z)I^{\otimes n}(\hat q^\beta,x)\chi(\kappa^\beta)\}.$$

On the other hand, as we showed in Proposition 2.6, $I(*,z)$ satisfies super-commutativity:

$$(x-z)^NI^{\otimes n}(\hat q^\beta,x)I^{\otimes n}(\hat q^\alpha,z)-(-1)^{\langle\alpha,\beta\rangle}(-z+x)^NI^{\otimes n}(\hat q^\alpha,z)I^{\otimes n}(\hat q^\gamma,x)=0.$$

Therefore

$$\mathrm{Res}_x\{(x-z)^N\phi(\kappa^\beta)-(-1)^{\langle\alpha,\beta\rangle}(-z+x)^N\chi(\kappa^\beta)\}=0$$

and so $\phi(\kappa^\beta)=(-1)^{\langle\alpha,\beta\rangle}\chi(\kappa^\beta)$ for $\beta\in D$. Hence W' is isomorphic to $\sigma_\alpha W$ as an M_D-module. □

Remark 1. The above lemma may look a little strange since we usually obtain relations $\sigma(W^1)\times\sigma(W^2)=\sigma(W^1\times W^2)$ and $(M_{\alpha+D}\times W^1)\times(M_{\alpha+D}\times W^2)=(W^1\times W^2)$ for an automorphism σ and a coset module $M_{\alpha+D}$, respectively. However, if $\sigma(W^i)\cong M_{D+\alpha}\times W^i$ for $i=1,2$, then $W^1\times W^2$ does not satisfy the condition of the above lemma by the fusion rules of $L(\frac{1}{2},0)$-modules and so $\sigma(W^1\times W^2)=W^1\times W^2$.

4. Positive definite invariant bilinear form

In our construction, "induced VOAs" play important roles. We will show that they inherit PDIB-forms.

THEOREM 4.1. *Assume that W^α is an irreducible M_D-module with $\tilde\tau(W^\alpha)=\alpha$ and that (D,α,α) satisfies the conditions of Theorem* 3.20. *Let F be an even code containing D such that $\langle F,\alpha\rangle=0$. If a* VOA $U=M_D\oplus W^\alpha$ *has a* PDIB-*form, then so does the induced* VOA $\mathrm{Ind}_D^F(U)$ $(=M_F\oplus\mathrm{Ind}_D^F(W^\alpha))$.

Proof. Clearly, it is sufficient to prove the assertion for $F=\langle\alpha,(1^n)\rangle^\perp$. Since $\langle\alpha,(1^n)\rangle^\perp$ is generated by words of weight 2, it is also sufficient to prove the assertion for $F=D+\mathbb{Z}_2\beta$ where the weight of $\beta\in\langle\alpha\rangle^\perp$ is 2. We may assume $\beta=(110^{n-2})$. Since $\langle\beta,\alpha\rangle=0$, we have $\mathrm{Supp}(\beta)\subseteq\mathrm{Supp}(\alpha)$ or $\mathrm{Supp}(\beta)\cap\mathrm{Supp}(\alpha)=\emptyset$.

By the assumption, D_α contains a direct sum E_α of extended $[8,4]$-Hamming codes such that $\mathrm{Supp}(E_\alpha)=\mathrm{Supp}(\alpha)$. Since $E_\alpha\subseteq D_\alpha$, $\mathrm{Ind}_D^F(W^\alpha)$ is irreducible. Set

$$V=M_F\oplus\mathrm{Ind}_D^F(W^\alpha).$$

By an argument similar to that in the proof of Theorem 3.25, we are able to prove that V has a framed VOA structure. Since $\mathrm{Ind}_D^F(W^\alpha)\times\mathrm{Ind}_D^F(W^\alpha)=M_F$ by Lemma 3.22, there are two possibilities of VOA structures on V. Namely, if one is $(M_F\oplus\mathrm{Ind}_D^F(W^\alpha),Y)$, then the other is $(M_F\oplus\sqrt{-1}\mathrm{Ind}_D^F(W^\alpha),Y)$. Since $W^\alpha\times W^\alpha=M_D$, we may assume $(M_F\oplus\mathrm{Ind}_D^F(W^\alpha),Y)$ contains $U=M_D\oplus W^\alpha$ as a sub VOA. As an M_{E_α}-module, W^α is a direct sum $\bigoplus_{i\in I}W^i$ of distinct irreducible M_{E_α}-modules W^i and $V^i=M_{E_\alpha}\oplus M_{E_\alpha+\beta}\oplus W^i\oplus(M_{E_\alpha+\beta}\times W^i)$ is a sub VOA of V for each i. Since $(M_{E_\alpha}\oplus W^i,Y_{|M_{E_\alpha}\oplus W^i})$ is a sub VOA of $M_D\oplus W^\alpha$, $(M_{E_\alpha}\oplus W^i,Y_{|M_{E_\alpha}\oplus W^i})$ has a PDIB-form.

If we once prove that a VOA structure (V^i,Y) on V^i has a PDIB-form, then $W^i\oplus(M_{E+\beta}\times W^i)$ has an orthonormal basis with respect to Y and so we have the desired result, since $M_{D+\beta}\times W^{\alpha}$ coincides with $\oplus_{i\in I}(M_{E+\beta}\times W^{i})$. Therefore we may assume that $\mathrm{Supp}(D)=\mathrm{Supp}(\alpha)$ and D is a direct sum $E^1\oplus\cdots\oplus E^s$ of extended $[8,4]$-Hamming codes E^i. In particular, W^{α} is irreducible as a T-module, where $T=M_{(0^n)}$. Since a VOA structure (V,Y) on V containing U is uniquely determined, we have to show that there exists a VOA structure on (V,Y) with a PDIB-form. For if (V,Y') is the other VOA structure on V, then (W^{α},Y') has a negative definite invariant bilinear form and it is impossible for (V,Y') to contain U. We will divide the proof into two cases:

(1) If $\mathrm{Supp}(\beta)\cap\mathrm{Supp}(\alpha)=\emptyset$, then there is a code D^0 of length $n-2$ such that $D=\{(00\alpha)|\alpha\in D^0\}$, $M_D=L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)\otimes M_{D^0}$ and $M_{D+\beta}=L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{1}{2},\frac{1}{2})\otimes M_{D^0}$. By the decompositions above, we are able to write

$$W^{\alpha}\cong L(\tfrac{1}{2},h_1)\otimes L(\tfrac{1}{2},h_2)\otimes W'$$

and

$$M_{D+\beta}\times W^{\alpha}\cong L(\tfrac{1}{2},h_1+\tfrac{1}{2})\otimes L(\tfrac{1}{2},h_2+\tfrac{1}{2})\otimes W'.$$

for some irreducible M_{D^0}-module W' and $h_1,h_2=0,\frac{1}{2}$, where $h_i+\frac{1}{2}$ denotes 0 if $h_i=\frac{1}{2}$ and $\frac{1}{2}$ if $h_i=0$. Since $L(\frac{1}{2},0)^{\otimes 2}\oplus L(\frac{1}{2},\frac{1}{2})^{\otimes 2}\cong\tilde{V}_{2\mathbb{Z}x}=(V_{2\mathbb{Z}x})^{\theta}\oplus\sqrt{-1}(V_{2\mathbb{Z}x})^{-}$ for $\langle x,x\rangle=1$, $\sqrt{-1}x_{(0)}$ is an isomorphism from $L(\frac{1}{2},h_1)\otimes L(\frac{1}{2},h_2)$ to $L(\frac{1}{2},h_1+\frac{1}{2})\otimes L(\frac{1}{2},h_2+\frac{1}{2})$ and $(x_{(0)})^2$ acts diagonally on $L(\frac{1}{2},h_1)\otimes L(\frac{1}{2},h_2)$ with positive eigenvalues. Let $\{v^i\mid i\in I\}$ be an orthogonal basis such that each v^i is in an eigenspace of $(x_{(0)})^2$. Then $\{\sqrt{-1}x_{(0)}v^i\mid i\in I\}$ is a basis of $L(\frac{1}{2},h_1+\frac{1}{2})\otimes L(\frac{1}{2},h_2+\frac{1}{2})$ and

$$\langle\sqrt{-1}x_{(0)}v^i,\sqrt{-1}x_{(0)}v^j\rangle=\langle v^i,(x_{(0)})^2v^j\rangle=\delta_{ij}\langle v^i,(x_{(0)})^2v^j\rangle\geq 0.$$

Hence $\mathrm{Ind}_D^F(U)$ has a PDIB-form.

(2) We next assume $\mathrm{Supp}(\beta)\subseteq\mathrm{Supp}(\alpha)$. Since D is a direct sum of extended $[8,4]$-Hamming codes and the weight of β is 2, we have to treat the following two cases:

(2.1) $\mathrm{Supp}(\beta)\subseteq\mathrm{Supp}(E^1)$.

(2.2) $D=E_8\oplus\cdots\oplus E_8$ and $\beta=(10^710^70^{n-16})$.

Case (2.1). By Lemma 3.18, there is an automorphism σ of M_D such that $\sigma(W^{\alpha})$ is isomorphic to a coset module $M_{D+\gamma}$. Since $\mathrm{Supp}(\beta)\subseteq\mathrm{Supp}(E^1)$ and β has an even weight, $\sigma(M_{\beta+D})$ is also isomorphic to a coset module $M_{\delta+D}$ for some δ. Namely, $\sigma(\mathrm{Ind}_D^F(U))$ is isomorphic to a code VOA $M_{\langle D,\delta,\gamma\rangle}$. Therefore it has a PDIB-form.

Case (2.2). We may assume that $\alpha=(1^n)$ and $\beta=(10^710^70^{n-16})$. Since $L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{1}{2},\frac{1}{2})$ has a PDIB-form and the lowest weight is an integer, we may also assume that $n=16$ and $\alpha=(1^{16})$. We will find such a VOA as a sub VOA of $\tilde{V}_{E_8}$ in the next section. This will complete the proof of Theorem 4.1.

COROLLARY 4.2. *Let* $W = M_D \oplus W^{\delta} \oplus W^{\gamma} \oplus W^{\delta+\gamma}$ *be a* $(D, \langle \delta, \gamma \rangle)$*-framed* VOA *with a* PDIB*-form and assume that a triple* (D, α, β) *satisfies the condition of Theorem* 3.20 *for any* $\alpha, \beta \in \langle \delta, \gamma \rangle$. *If* F *is an even linear subcode of* $\langle \alpha, \beta \rangle^{\perp}$ *containing* D, *then* $\mathrm{Ind}_D^F(W)$ *also has a* PDIB*-form.*

Proof. By Theorem 3.21, $V = \mathrm{Ind}_D^F(W)$ is an $(F, \langle \alpha, \beta \rangle)$-framed VOA $\bigoplus_{\gamma \in \langle \alpha, \beta \rangle} V^{\gamma}$ containing W, where $V^{\gamma} = \mathrm{Ind}_D^F(W^{\gamma})$. It follows from Theorem 4.1 and from the fact that $V^{\gamma} \times V^{\gamma} = M_F$ by Lemma 3.22 that V^{γ} has a PDIB-form or a negative definite invariant bilinear form. However, since W^{γ} has a PDIB-form, V has a PDIB-form. $\square$

5. E_8-lattice VOA

As mentioned in the introduction, we will construct the parts of $V^{\natural}$ by using the decomposition of $\tilde{V}_{E_8}$, where $\tilde{V}_{E_8}$ is a lattice VOA constructed from the root lattice of type E_8 with a PDIB-form; (see §2.5). The main purpose of this section is to study five framed VOA structures of V_{E_8} and $\tilde{V}_{E_8}$. In particular, we will show that there are codes D_{E_8} and S_{E_8} of length 16 such that $\tilde{V}_{E_8}$ is a (D_{E_8}, S_{E_8})-framed VOA satisfying the conditions (1)–(4) of Hypotheses I and triple sets (D_{E_8}, α, β) satisfy the conditions of Theorem 3.20 for any $\alpha, \beta \in S_{E_8}$. Incidentally, we will see that an orbifold construction from VOA $\mathbb{C}V_{E_8}$ coincides with the changing of coordinate sets of extended $[8,4]$-Hamming code sub VOAs of $\mathbb{C}V_{E_8}$.

Let E_8 denote the root lattice of type E_8. It is known that E_8 is the unique even unimodular positive definite lattice of rank 8. We first define four expressions of E_8, that is, lattices $E_8(m) : m = 1,2,3,4,5$. Let $\{x_1, \cdots, x_8\}$ be an orthonormal basis and set

$$E_8(1) = \left\langle \tfrac{1}{2}\left(\sum_{i=1}^{8} x_i\right), x_i \pm x_j \mid i, j = 1, \cdots, 8 \right\rangle \tag{5.1}$$

and $\tilde{N}(1) = \langle x_i \mid i = 1, \cdots, 8 \rangle$, where $\langle u_i \mid i \in I \rangle$ denotes a lattice generated by $\{u_i \mid i \in I\}$. It is easy to check that $E_8(1)$ is isomorphic to E_8. We can define other expressions of lattice E_8 as follows:

(5.2)

$$\begin{aligned}
E_8(2) = \ & \left\langle \tfrac{1}{2}(x_1 - x_2 - x_3 - x_4) + x_5, \tfrac{1}{2}(x_5 + x_6 + x_7 + x_8) + x_1, \right. \\
& \left. x_i \pm x_j \mid i, j \in \{1,2,3,4\}, \text{ or } i, j \in \{5,6,7,8\} \right\rangle. \\
E_8(3) = \ & \left\langle \tfrac{1}{2}(x_1 - x_2 - x_5 - x_6) + x_3, \tfrac{1}{2}(x_1 + x_2 - x_3 - x_4) - x_7, \right. \\
& \left. \tfrac{1}{2}(-x_5 - x_6 + x_7 + x_8) + x_1, x_1 + x_3 + x_5 + x_7, x_{2i-1} \pm x_{2i}, \ (i = 1,2,3,4) \right\rangle \\
E_8(4) = \ & \left\langle \tfrac{1}{2}(x_1 - x_3 - x_5 - x_7) + x_2, \tfrac{1}{2}(x_1 - x_2 + x_5 - x_6) - x_3, \right. \\
& \left. \tfrac{1}{2}(-x_1 + x_2 - x_3 - x_4) - x_7, \tfrac{1}{2}(x_1 + x_3 - x_6 + x_8) + x_5, 2x_1, \cdots, 2x_8 \right\rangle.
\end{aligned}$$

Fix $m=1,2,3,4$ and denote $E_8(m)$ by L. Let V_L be a lattice VOA constructed from L as in [FLM2] and θ an automorphism of V_L induced from -1 on L. We note that all V_L are isomorphic to V_{E_8}. Since $E_8(m)$ contains an orthogonal basis $\{2x_1,\cdots,2x_8\}$ of square length 4, V_L is a framed VOA with a coordinate set $I=\{e_i \mid i=1,\cdots,16\}$ given by

$$(5.3)\quad e_{2i-j} = \tfrac{1}{4}(x_i)^2_{(-1)}\mathbf{1}-(-1)^j\tfrac{1}{4}(\iota(2x_i)+\iota(-2x_i)) \quad (i=1,2\ldots,8 \text{ and } j=0,1)$$

by [DMZ]. Since they are all in the set $V_L^{\tilde\theta}$ of $\tilde\theta$-invariants, we can also take this set as a coordinate set of $\tilde V_L$.

Let $P(m)=\langle \tau_{e_i} \mid i=1,\cdots,16\rangle \subseteq \mathrm{Aut}(\tilde V_L)$ and denote $E_8(m)\cap \tilde N(1)$ by $N(m)$. It is straightforward to verify that $\tilde V_{N(m)}$ contains $\langle e_1,\cdots,e_{16}\rangle$ and $(\tilde V_L)^{P(m)}$ coincides with $\tilde V_{N(m)}$ by (2.4). Since $(\tilde V_L)^{P(m)}$ has a PDIB-form and $\tilde\tau((\tilde V_L)^{P(m)})=(0^{16})$, there is a code $D(m)$ of length 16 such that $(\tilde V_L)^{P(m)}$ is isomorphic to a code VOA $M_{D(m)}$. It is also not difficult to check that $(D(m),\alpha,\beta)$ satisfies the conditions of Theorem 3.20 for $\alpha,\beta\in S^m:=D(m)^\perp$ and $(\tilde V_L)$ is a $(D(m),D(m)^\perp)$-framed VOA satisfying Hypotheses I. However, these are not the pieces we will use to construct $V^\natural$ since $D(m)$ has a root and $(M_{D(m)})_1\neq 0$ for $m=1,2,3,4$. In order to construct the moonshine VOA $V^\natural$, we need a code D without roots. To find the desired decomposition, we will change coordinate sets. Incidentally, this process coincides with a $\mathbb{Z}_2$-orbifold construction of $\tilde V_{E_8}$ from itself as we will see.

Let us explain the relation between a $\mathbb{Z}_2$-orbifold construction and changing the coordinate sets. It is known that a $\mathbb{Z}_2$-orbifold model from $\mathbb{C}V_{E_8}$ is isomorphic to itself. Let θ be an automorphism of V_L induced from -1 on L. Also, θ fixes $\iota(x_i)+\iota(-x_i)$ and acts as -1 on $\mathbb{C}(x_i)_{(-1)}\mathbf{1}$ and $\mathbb{C}(\iota(x_i)-\iota(-x_i))$. Hence θ acts on M_α as $(-1)^{\langle\alpha,(\{01\}^8)\rangle}$ and hence the fixed point space $M^\theta_{D(m)}$ is equal to the direct sum $\bigoplus_{\alpha\in D(m,+)} M_\alpha$, where $D(m,+)=\{\alpha\in D(m) \mid \langle\alpha,(\{01\}^8)\rangle=0\}$. Suppose that $V=\oplus_{\alpha\in S}V^\alpha$ is a (D,S)-framed VOA satisfying Hypotheses I, where D is a code of length $2n$ containing $(0^{2i}110^{2n-2i-2})$ for all $i=1,\cdots,m$. Set $\beta=(\{01\}^n)$. Assume that the twisted part of the $\mathbb{Z}_2$-orbifold model does not contain any coset modules. Then the $\mathbb{Z}_2$-orbifold construction is corresponding to the following three steps as we will see in the next example.

(1) Take a half $M_{D(+)}$ of M_D, where $D(+)=\{\alpha\in D \mid \langle\alpha,\beta\rangle=0\}$.

(2) Take an $M_{D(+)}$-module V^β with $\tilde\tau(V^\beta)=\beta$ and generate $M_{D(+)}$-modules $V^{\beta+\gamma}$ with $\tilde\tau(V^{\beta+\gamma})=\beta+\gamma$ by $V^{\beta+\gamma}=V^\beta\times V^\gamma$ for $\gamma\in S$.

(3) Define a VOA structure on $\tilde V=\oplus_{\alpha\in\langle S,\beta\rangle}V^\alpha$.

If we start from $E_8(1)$, $\tilde\tau(V_{N(1)+v})=(1^{16})$ for $v=\frac{1}{2}(\sum_{i=1}^8 x_i)$ and so $S^1=\big\langle(1^{16})\big\rangle$ and $D(1)$ is the set of all even words of length 16. $D(1)$ contains a self dual subcode $H=H_8^1\oplus H_8^2$, where H_8^i are extended $[8,4]$-Hamming codes and

$\mathrm{Supp}(H_8^1)=\{1,2,\cdots,8\}$ and $\mathrm{Supp}(H_8^2)=\{9,\cdots,16\}$. Since $\langle((10)^8),\beta\rangle=0$ for any $\beta\in H$, we have $M_H\subseteq V_L^\theta$. Therefore the decompositions of V_L and $\tilde{V}_L$ as M_H-modules are exactly the same. Since $D(1)$ consists of all even words, the center $Z(\widehat{D(1)})$ is $\langle\pm\kappa^{(0^{16})},\pm\kappa^{(1^{16})}\rangle$ and hence there are exactly two irreducible $M_{D(1)}$-modules $\mathrm{Ind}_H^{D(1)}(H(\frac{1}{16},(0^8))\otimes H(\frac{1}{16},(0^8)))$ and $\mathrm{Ind}_H^{D(1)}(H(\frac{1}{16},(0^8))\otimes H(\frac{1}{16},\xi^1))$ by Theorem 3.8. The difference between them is possibly to be judged by the action of $q^{(1^{16})}:=((q^{(1)})^{\otimes 16})\otimes\kappa^{(1^{16})}$. By Table (2.4) and the proof of Proposition 2.7, we have $q^{(1^{16})}=(x_1)_{(-1)}\cdots(x_8)_{(-1)}\mathbf{1}$ and $(x_i)_{(-1)}\mathbf{1}=\sqrt{-1}((q^{(1)})^{\otimes 2})\otimes\kappa^{\xi_{2i-1}}\kappa^{\xi_{2i}}$. Since the eigenvalue of $q^{(1^{16})}$ on $\mathbb{R}\iota(\frac{1}{2}\sum x_i)$ is positive,

$$\tilde{V}_{E_8}\cong M_{D(1)}\oplus\mathrm{Ind}_H^{D(1)}(H(\tfrac{1}{16},(0^8))\otimes H(\tfrac{1}{16},(0^8))) \tag{5.4}$$

by the choice of $E(1)$. By Lemma 3.18, there is an automorphism $\sigma\in\mathrm{Aut}(M_{H_8})$ such that $\{\sigma(e_1),\cdots,\sigma(e_8)\}$ is another coordinate set of M_{H_8} satisfying $\sigma(H(\frac{1}{16},(\xi_1)))\cong H(\frac{1}{2},\xi_1)\cong M_{H_8+\xi_1}$ and $\sigma(H(\frac{1}{2},\xi_1))\cong H(\frac{1}{16},(0^8))$. Take a new coordinate set

$$J=\{\sigma(e_1),\cdots,\sigma(e_8),e_9,\cdots,e_{16}\}$$

of V_{E_8}. Then for $\beta\in D(1)$ with $\langle\beta,(1^80^8)\rangle=1$, $\tilde{\tau}(\sigma(M_{H+\beta}))=(1^80^8)$ and $\sigma(M_{H+\alpha})$ is also a coset module for $\langle\alpha,(1^80^8)\rangle=0$. We also have $\tilde{\tau}(\sigma(H(\frac{1}{16},\xi_1))\otimes H(\frac{1}{16},\xi_1))=(0^81^8)$. Hence the set $\tilde{\tau}(V_L)$ with respect to J is

$$S^2=\{(0^{16}),(1^80^8),(0^81^8),(1^{16})\}.$$

Set $P^2=\langle\tau_{\sigma(e_i)},\tau_{e_j}\mid i=1,\cdots,8,j=9,\cdots,16\rangle$ and define a linear code D_2 by $(V_L)^{P^2}\cong M_{D_2}$ with respect to J; then D_2 splits into a direct sum $D_2^1\oplus D_2^2$ such that D_2^1 and D_2^2 are the sets of all even words whose supports are in $\{1,2,\cdots,8\}$ and $\{9,\cdots,16\}$, respectively. Note that this process corresponds to an orthogonal transformation

$$\frac{1}{2}\begin{pmatrix}1&-1&-1&-1\\1&-1&1&1\\1&1&-1&1\\1&1&1&-1\end{pmatrix} \tag{5.5}$$

by (2.4). Therefore this decomposition coincides with the decomposition given by $E_8(2)$ and $D(2)$. Note that $(1^{16})\in D_2$ and $\sigma(M_{(1^{16})})\cong M_{(1^{16})}$.

We next consider the case of $E_8(2)$ and $S^2=\langle(1^80^8),(0^81^8)\rangle$. We use the decomposition above again by renaming $J=\{\sigma(e_1),\cdots,\sigma(e_8),e_9,\cdots,e_{16}\}$ and D_2 by $I=\{e_1,\cdots,e_{16}\}$ and $D(2)$, respectively. Set

$$I_1=\{\alpha\in D(2)\mid\mathrm{Supp}(\alpha)\subseteq\{1,2,3,4,9,10,11,12\}\},$$
$$I_2=\{\alpha\in D(2)\mid\mathrm{Supp}(\alpha)\subseteq\{5,6,7,8,13,14,15,16\}\}.$$

It is clear that I_i contains an extended $[8,4]$-Hamming code H_i for $i=1,2$. Take a new coordinate set $\{f_1,\cdots,f_4,f_9,\cdots,f_{12}\}$ of H_1 and define a new coordinate set

$$J=\{f_1,\cdots,f_4,e_5,\cdots,e_8,f_9,\cdots,f_{12},e_{13},\cdots,e_{16}\}$$

of V_L. Then if an $M_{H_1}\otimes M_{H_2}$-module U has a τ-word

$$(\alpha,\beta)\in\{1,\cdots,4,9,\cdots,12\}\oplus\{5,\cdots,8,13,\cdots,16\}$$

with respect to I, then the τ-word with respect to J is either (α,β) or (α^c,β). Moreover, there is a submodule with a τ-word $(1^40^41^40^4)$ with respect to J. An example is $M_{H_1\oplus H_2+\alpha}$, where α is a word with $\langle\alpha,(1^40^41^40^4)\rangle=1$. Therefore we have

$$D_3=\left\langle D_3^1\oplus D_3^2\oplus D_3^3\oplus D_3^4,\{1,5,9,13\}\right\rangle \tag{5.6}$$

where D_3^i is the set of all even words in $\{4i-3,4i-2,4i-1,4i\}$ for $i=1,\cdots,4$. We also obtain

$$S^3=\left\langle(1^{16}),(1^80^8),(1^40^41^40^4)\right\rangle. \tag{5.7}$$

This corresponds to the decomposition with respect to $E_8(3)$ and $D_3=D(3)$. $D(3)$ also contains two orthogonal extended $[8,4]$-Hamming codes $H_1(3)$ and $H_2(3)$ whose supports are

$$\{1,2,5,6,9,10,13,14\}\quad\text{and}\quad\{3,4,7,8,11,12,15,16\}.$$

Repeating the arguments above, we have

$$S^4=\left\langle(1^{16}),(1^80^8),(1^40^41^40^4),(\{1^20^2\}^4)\right\rangle \tag{5.8}$$

and $D_4=(S^4)^\perp$. We have $D_4=D(4)$ and $D(4)$ still contains a direct sum of 2 extended $[8,4]$-Hamming codes whose supports are $(\{10\}^8)$ and $(\{01\}^8)$. Repeating the same arguments again, we finally obtain new codes

$$S^5=\left\langle(1^{16}),(1^80^8),(1^40^41^40^4),(\{1100\}^4),(\{10\}^8)\right\rangle \tag{5.9}$$

and $D(5)=(S^5)^\perp$, which are not codes we can get from lattice constructions.

Let us finish the proof of Theorem 4.1. Set $\xi_1=(10^7)$ so that $\beta=(\xi_1\xi_1)$. Consider a framed VOA structure

$$\tilde{V}_{E_8}\cong M_{D(1)}\oplus\mathrm{Ind}_H^{D(1)}\left(H\left(\tfrac{1}{16},(0^8)\right)\otimes H\left(\tfrac{1}{16},(0^8)\right)\right).$$

Set $H=H_8\oplus H_8$ and $M_H\subseteq M_{D(1)}$. Since $\tilde{V}_{E_8}$ is an M_H-module, it is a direct sum of distinct irreducible M_H-modules. Since $D(1)$ is the set of all even words, $M_{D(1)}$ contains $H(\frac12,\xi_1)\otimes H(\frac12,\xi_1)$ and so $\tilde{V}_{E_8}$ has a sub VOA isomorphic to

$$\begin{aligned}(5.10)\qquad &\left(H\left(\tfrac{1}{2},(0^8)\right)\otimes H\left(\tfrac{1}{2},(0^8)\right)\right)\oplus\left(H\left(\tfrac{1}{2},\xi_1\right)\otimes H\left(\tfrac{1}{2},\xi_1\right)\right)\\ &\oplus\left(H\left(\tfrac{1}{16},(0^8)\right)\otimes H\left(\tfrac{1}{16},(0^8)\right)\right)\oplus\left(H\left(\tfrac{1}{16},\xi_1\right)\otimes H\left(\tfrac{1}{16},\xi_1\right)\right).\end{aligned}$$

This is the desired VOA in Theorem 4.1.

Set $D_{E_8}=D(5)$ and $S_{E_8}=S^5$. We note that D_{E_8} is a Reed Müller code RM(2,4) and S_{E_8} is a Reed Müller code RM(1,4).

Lemma 5.1. *Triples* (RM(2,4), α,β) *satisfy the conditions* (3.a) *and* (3.b) *of Theorem* 3.20 *for any* $\alpha,\beta\in\mathrm{RM}(1,4)$.

Proof. To simplify the notation, set $D=\mathrm{RM}(2,4)$ and $S=\mathrm{RM}(1,4)$. The weight enumerator of RM(1,4) is $x^{16}+30x^8y^8+y^{16}$. If $\alpha=(0^{16})$ or (1^{16}), then for any maximal self-orthogonal (doubly even) subcodes H^β and H^{β^c} of D_β and D_{β^c} which are direct sums of extended [8,4]-Hamming codes or zero, $E=H^\beta\oplus H^{\beta^c}$ satisfies the desired conditions. So we may assume that the weight of α is eight. We note that D_α and D_{α^c} are isomorphic to the extended [8,4]-Hamming code. Set $E=D_\alpha\oplus D_{\alpha^c}$ and $H^\alpha=D_\alpha$. If β is $(0^{16}),(1^{16})$, α or α^c, then E and $H^{(1^{16})}=E$ satisfy the desired conditions.

The remaining case is that all of $\alpha,\beta,\alpha+\beta$ have weight eight. Say $\alpha=(1^80^8)$ and $\beta=(1^40^41^40^4)$. We use an expression

$$\mathbb{Z}_2^{16}=\{(\delta_1,\delta_2,\delta_3,\delta_4)\mid\delta\in\mathbb{Z}_2^4\}.$$

Clearly, since $E_\gamma=H^\gamma=D_\gamma$ is an extended [8,4]-Hamming code for $\gamma\in S$ with $|\gamma|=8$, we have

$$\begin{aligned}E_\alpha&=\{(\delta\delta0^40^4),\ (\delta\delta^c0^40^4)\mid\delta\in\mathbb{Z}_2^4\text{ even}\},\\ E_{\alpha^c}&=\{(0^40^4\delta\delta),\ (0^40^4\delta\delta^c)\mid\delta\in\mathbb{Z}_2^4\text{ even}\},\\ H^\beta&=\{(\delta0^4\delta0^4),\ (\delta0^4\delta^c0^4)\mid\delta\in\mathbb{Z}_2^4\text{ even}\}\end{aligned}$$

and

$$H^{\alpha+\beta}=\{(0^4\delta\delta0^4),(0^4\delta\delta^c0^4)\mid\delta\in\mathbb{Z}_2^4\text{ even}\}.$$

Since $(0^4\delta\delta0^4)-(\delta0^4\delta0^4)=(\delta\delta0^40^4)$ and $(0^4\delta\delta^c0^4)=(\delta0^4\delta^c0^4)+(\delta\delta0^40^4)$, we obtain $H^{\alpha+\beta}+E=H^\beta+E$. □

Proposition 5.2. $\tilde{V}_{E_8}$ *is a* (D_{E_8},S_{E_8})*-framed* VOA *with a* PDIB*-form.*

We found a (D_{E_8},S_{E_8})-framed VOA structure on $\tilde{V}_{E_8}$ from the $(D(m),S^m)$-framed structure on $\tilde{V}_{E_8}$. Although it is easy to reverse the process, there is another important step. Namely, let

$$\tilde{V}_{E_8}=\oplus_{\alpha\in S^m}V^\alpha$$

be the decomposition such that $V^{(0^{16})} \cong M_{D(m)}$. Let β be an even word so that $\langle\beta\rangle^{\perp} \cap S^m = S^{m-1}$. Then $V^+ = \oplus_{\alpha\in S^m} V^{\alpha}$ is a sub VOA and we can define the induced VOA

$$\tilde{V}^{m-1} = \mathrm{Ind}_{D(m)}^{D(m-1)}(V^+),$$

which is also a VOA containing $M_{D(m-1)}$. Thus, the above process to get an induced VOA is a reverse step of $\mathbb{Z}_2$-orbifold construction. As an application, we will explain properties of automorphisms of a lattice VOA V_L for an even lattice L in the remainder of this section. Let L_2 denote the set of all elements of L with squared length 4. As we showed, for any $a\in L_2$, we can define two conformal vectors

$$\begin{aligned} e^+(a) &= \tfrac{1}{16}(a_{(-1)})^2\mathbf{1} + \tfrac{1}{4}(\iota(a)+\iota(-a)),\\ e^-(a) &= \tfrac{1}{16}(a_{(-1)})^2\mathbf{1} + \tfrac{1}{4}(\iota(a)+\iota(-a)). \end{aligned}$$

Then we have:

Lemma 5.3. *Let $\tau_{e^+(a)} = \tau_{e^-(a)}$ on V_L. Then $\tau_a = \tau_{e^+(a)}$, $[\tau_a, y_{(m)}] = 0$ for $y\in L$ and*

$$\tau_a : \iota(x) \to (-1)^{\langle x,a\rangle}\iota(x)$$

for $x \in L$. In particular, $\langle \tau_a \mid a\in L_2\rangle$ is an elementary abelian 2-subgroup of $\mathrm{Aut}(V_L)$. If $\langle a, b\rangle$ is odd for $a, b\in L_2$, then $\tau_b(e^{\pm}(a)) = e^{\mp}(a)$.

Proof. Since $\langle a, L\rangle \in \mathbb{Z}$ and $\langle a, a\rangle = 4$, $L \subseteq \frac{1}{4}\mathbb{Z}a \oplus \frac{1}{4}\langle a\rangle^{\perp}$. In particular, we may view $V_L \subseteq V_{\frac{1}{4}\mathbb{Z}a} \oplus V_{\frac{1}{4}\langle a\rangle^{\perp}}$. From Table (2.4), we have

$$\tau_{e^{\pm}(a)} : \begin{cases} 1 & \text{on} \quad \mathbb{R}a(-1)\mathbf{1},\ \mathbb{R}\iota((\frac{1}{2}+\mathbb{Z})a),\ \mathbb{R}\iota(\mathbb{Z}a) \\ -1 & \text{on} \quad \mathbb{R}\iota((\frac{1}{4}+\frac{1}{2}\mathbb{Z})a). \end{cases}$$

Hence $[\tau_{e^{\pm}(a)}, y_{(m)}] = 1$ for $y\in L$ and

$$\tau_{e^{\pm}(a)} : \iota(x) \to (-1)^{\langle x,a\rangle}\iota(x)$$

for $x\in L$. Therefore we obtain the desired results. □

Theorem 5.4. *For $g\in \mathrm{Aut}(S_{E_8})$, there is an automorphism $\tilde{g}$ of $\tilde{V}_{E_8}$ such that $\tilde{g}(e_i) = e_{g(i)}$ for all $i = 1, \cdots, 16$.*

Proof. Recalling the definition of a Reed Müller code RM$(1, 4)$, letting $F = \mathbb{Z}_2^4$ be a vector space over $\mathbb{Z}_2$ of dimension 4 and denote (1000), (0100), (0010), (0001) by v^1, v^2, v^3, v^4, respectively. Define $\langle (a_i), (b_i)\rangle = \sum_{i=1}^4 a_i b_i$. The coordinate set of a Reed Müller code RM$(1, 4)$ is the set of all 16 vectors of F and $\mathrm{RM}(1,4)$ consists of $(0^{16}),(1^{16})$ and the codewords of length eight given by hyperplanes. It is easy to see that

$$\begin{aligned}\mathrm{Aut}(\mathrm{RM}(1,4)) &= \mathrm{Aut}(\mathrm{RM}(2,4)) \cong \mathrm{GL}(5,2)_1 \\ &= \{\sigma\in \mathrm{GL}(5,2) | \sigma^t(10000) = {}^t(10000)\}\end{aligned}$$

and it is generated by

$$\begin{aligned}\alpha(i) &: v\in F \to v+v^i \qquad \text{and} \\ \alpha(i,j) &: v\in F \to v+\langle v,v^j\rangle v^i \quad \text{ for } i\neq j.\end{aligned}$$

Choose $g\in \mathrm{Aut}(S_{E_8})$. By Lemma 3.8, we may assume $g\in\mathrm{Aut}(M_{D_{E_8}})$ and $g(M_{(1^{16})}) = M_{(1^{16})}$. Set $q=q^{(1^{16})}$. Since g is an even permutation, we may assume $g(\kappa^{(1^{16})})=\kappa^{(1^{16})}$ and $g(q)=q$. For an $M_{D_{E_8}}$-module W, $g(W)$ denotes an $M_{D_{E_8}}$-module defined by $v_{(n)}(g(u)) = g(v^g_{(n)}u)$ for $v\in M_{D_{E_8}}$ and $u\in W$. Clearly,

$$g(\tilde{V}_{E_8}) := \oplus_{\alpha\in S_{E_8}} g(\tilde{V}^{\alpha}_{E_8})$$

is a VOA with a PDIB-form. Note that $g(\tilde{V}_{E_8})$ contains $g(M_{D_{E_8}}) \cong M_{D_{E_8}}$. Using the backward processes according to the sequences

$$\begin{array}{rlllll} S^5 = g(S^5) & \supseteq g(S^4) & \supseteq g(S^3) & \supseteq g(S^2) & \supseteq g(S^1) = S^1, \\ D(5)=g(D(5)) & \subseteq g(D(4)) & \subseteq g(D(3)) & \subseteq g(D(2)) & \subseteq g(D(1)) = D(1), \\ M_{D(5)} = g(M_{D(5)}) & \subseteq g(M_{D(4)}) & \subseteq g(M_{D(3)}) & \subseteq g(M_{D(2)}) & \subseteq g(M_{D(1)}) \cong M_{D(1)}, \end{array}$$

we obtain a coordinate set $\{\tilde{e}_1,\cdots,\tilde{e}_{16}\}$ of $\tilde{V}_{E_8}$ such that $g(\tilde{V}_{E_8})$ has the decomposition

$$g(\tilde{V}_{E_8}) \cong M_{D(1)} \oplus W.$$

Here we note that $D(1)$ coincides with the set of all even words of length 16 and W is an irreducible $M_{D(1)}$-module with $\tilde{\tau}(W)=(1^{16})$. So W is isomorphic to $\mathrm{Ind}_E^{D(1)}(H(\frac{1}{16},(0^8))\otimes H(\frac{1}{16},(0^8))$ or $\mathrm{Ind}_E^{D(1)}(H(\frac{1}{16},\xi_1)\otimes H(\frac{1}{16},(0^8))$. The action $q_{(7)}$ on $(\tilde{V}_{E_8})^{(1^{16})}$ is equal to $q_{(7)} = g(q)_{(7)}$ on $g((\tilde{V}_{E_8})^{(1^{16})})$ by the definition. Since the coordinates sets are changing parallel, the expression of q by $\{\tilde{e}^1,\cdots,\tilde{e}^{16}\}$ is equal to $\{e_1,\cdots,e_{16}\}$. We note that $\kappa^{(1^{16})}$ is in the center of $\widehat{D(1)}$. Therefore we conclude that $W\cong \mathrm{Ind}_E^{D(1)}(H(\frac{1}{16},(0^8))\otimes H(\frac{1}{16},(0^8)))$, which coincides with (5.4). Therefore there is a VOA isomorphism

$$\phi: \tilde{V}_{E_8} \to g(\tilde{V}_{E_8})$$

such that $\phi(e_i)=\tilde{e}^i$ for $i=1,\cdots,16$. By changing the coordinate sets according to

$$\begin{array}{lllll} S^1 & \subseteq S^2 & \subseteq S^3 & \subseteq S^4 & \subseteq S^5 \\ g(S^1) & \subseteq g(S^2) & \subseteq g(S^3) & \subseteq g(S^4) & \subseteq g(S^5), \end{array}$$

respectively, we have an isomorphism ϕ of $\tilde{V}_{E_8}$ to $g(\tilde{V}_{E_8})$ with $\phi(e_i)=e_i$ for all i. Hence we have the desired automorphism $\phi^{-1}g$ of $\tilde{V}_{E_8}$. □

6. Holomorphic VOA

Let V be a (D,S)-framed VOA with a coordinate set $\{e_i \mid i=1,\cdots,n\}$. As we showed in [Mi5], S is orthogonal to D.

THEOREM 6.1. *If $S=D^{\perp}$, then V is the only irreducible V-module. That is, V is holomorphic.*

Proof. Let (U,Y^U) be an irreducible V-module. Since M_D is rational, U is a direct sum of irreducible M_D-modules. Decompose U into the direct sum $\oplus_\beta U^\beta$ of M_D-modules such that $\tilde{\tau}(U^\beta)=\beta$. Choose β so that $U^\beta \neq 0$. Since U^β is an M_D-module, $\beta\in D^{\perp}=S$ and so $V^\beta \neq 0$. Since

$$U=\langle v_{(n)}u \mid v\in V^\alpha, n\in\mathbb{Z}, \alpha\in S\rangle$$

for any $0\neq u\in U^\beta$ by [DM2],

$$U^\beta=\langle v_{(n)}u \mid v\in M_D, n\in\mathbb{Z}\rangle$$

for any $0\neq u\in U^\beta$ and hence U^β is an irreducible M_D-module. Since the restrictions

$$I\begin{pmatrix} & U & \\ V & & U\end{pmatrix}\to I\begin{pmatrix} & U & \\ V^\beta & & U^\beta\end{pmatrix}\to I\begin{pmatrix} & U^{(0^n)} & \\ V^\beta & & U^\beta\end{pmatrix}$$

are injective, we have $U^{(0^n)}\neq 0$ and $U^{(0^n)}$ is isomorphic to a coset module $M_{D+\alpha}$ for some word $\alpha\in\mathbb{Z}_2^n$. Using the skew symmetry, we can define a nonzero intertwining operator $I(*,z)\in I_{M_D}\begin{pmatrix} & U & \\ U & & V\end{pmatrix}$ with integral powers of z by $I(u,z)v=e^{zL(-1)}Y^U(v,-z)u$ for $v\in V$ and $u\in U$. By restriction, we have a nonzero intertwining operator $I^\gamma(v,z)\in I_{M_D}\begin{pmatrix} & U^\gamma & \\ M_{\alpha+D} & & V^\gamma\end{pmatrix}$ for $\gamma\in S$. Since $I^\gamma(v,z)$ has integral powers of z, α is orthogonal to S and so $\alpha\in S^{\perp}=D$. Hence $U^{(0^n)}$ is isomorphic to M_D. Let q be a lowest degree vector of $U^{(0^n)}$ corresponding to the vacuum of M_D. Since $L(-1)q=0$, $I(q,z)\in \mathrm{Hom}(V,U[[z,z^{-1}]])$ is a scalar and gives an M_D-isomorphism of V to U. This completes the proof of Theorem 6.1. □

7. Construction of the moonshine VOA

In this section, we will construct a framed VOA $V^\natural$, which is equal to the moonshine module VOA constructed in [FLM2], as we will see in Section 9. In Section 5, we found that $\tilde{V}_{E_8}$ is a (D_{E_8},S_{E_8})-framed VOA with a coordinate set $\{e_i \mid i=1,\cdots,16\}$ and $S_{E_8}=D_{E_8}^{\perp}$ is spanned by

(7.1) $$\{(1^{16}),(0^8 1^8),(\{0^4 1^4\}^2),(\{0^2 1^2\}^4),(\{01\}^8)\}.$$

To simplify the notation, we denote D_{E_8} and S_{E_8} by D and S in this section, respectively. In Lemma 5.1 and Proposition 5.2, we showed that (D,S) satisfies the conditions in Theorem 3.20 and that $\tilde{V}_{E_8}$ is a (D,S)-framed VOA

$$\tilde{V}_{E_8}=\bigoplus_{\alpha\in S}{V_{E_8}}^{\alpha} \tag{7.2}$$

satisfying the conditions of Hypotheses I.

We note that all codewords of S except (0^{16}) and (1^{16}) are of weight eight. We define a new code $S^{\natural}$ of length 48 by

$$S^{\natural}=\left\langle(1^{16}0^{16}0^{16}),(0^{16}1^{16}0^{16}),(0^{16}0^{16}1^{16}),(\alpha,\alpha,\alpha)\mid\alpha\in S\right\rangle. \tag{7.3}$$

The weight enumerator of $S^{\natural}$ is $X^{48}+3X^{32}+120X^{24}+3X^{16}+1$ which has another expression:

$$S^{\natural}=\{(\alpha,\alpha,\alpha),\ (\alpha,\alpha,\alpha^c),\ (\alpha,\alpha^c,\alpha),\ (\alpha^c,\alpha,\alpha)\mid\alpha\in S\}. \tag{7.4}$$

Set $D^{\natural}=(S^{\natural})^{\perp}$ and call it "*the moonshine code.*" Now $D^{\natural}$ contains $D^{\oplus 3}=\{(\alpha,\beta,\gamma)\mid\alpha,\beta,\gamma\in D\}$ and it is easy to see that

$$D^{\natural}=\{(\alpha,\beta,\gamma)\mid\alpha+\beta+\gamma\in D,\alpha,\beta,\gamma\text{ is even}\}. \tag{7.5}$$

Hence $D^{\natural}$ is of dimension 41 and has no codewords of weight 2. We note that a triple $(D^{\oplus 3},\alpha,\beta)$ satisfies the conditions (3.a) and (3.b) of Theorem 3.20 for any $\alpha,\beta\in S^{\natural}$, since $S^{\natural}\subseteq S^{\oplus 3}$. Denote (10^{15}) by ξ_1 and set

$$Q=\left\langle(\xi_1\xi_1 0^{16}),(0^{16}\xi_1\xi_1)\right\rangle. \tag{7.6}$$

To simplify the notation, we let R denote a coset module M_{ξ_1+D} and RW denote a fusion product (tensor product) $R\times W$ for an M_D-module W. As explained in the introduction, our construction consists of the following steps. First, $V_{E_8}\otimes V_{E_8}\otimes V_{E_8}$ is a $(D^{\oplus 3},S^{\oplus 3})$-framed VOA with a coordinate set

$$\{e_i\otimes\mathbf{1}\otimes\mathbf{1},\quad \mathbf{1}\otimes e_j\otimes\mathbf{1},\quad \mathbf{1}\otimes\mathbf{1}\otimes e_k\quad\mid i,j,k=1,\cdots,16\},$$

where $\mathbf{1}$ is the vacuum of V_{E_8}. Decompose it into

$$V_{E_8}\otimes V_{E_8}\otimes V_{E_8}=\bigoplus_{\alpha,\beta,\gamma\in S}({V_{E_8}}^{\alpha}\otimes {V_{E_8}}^{\beta}\otimes {V_{E_8}}^{\gamma}). \tag{7.7}$$

By the fusion rules,

$$V^1=\bigoplus_{(\alpha,\beta,\gamma)\in S^{\natural}}({V_{E_8}}^{\alpha}\otimes {V_{E_8}}^{\beta}\otimes {V_{E_8}}^{\gamma}) \tag{7.8}$$

is a sub $(D^{\oplus 3},S^{\natural})$-framed VOA. Using induction we obtain

$$V^2=\mathrm{Ind}_{D^{\oplus 3}}^{D^{\oplus 3}+Q}(V^1). \tag{7.9}$$

Note that since $\langle Q, S^\natural\rangle \neq 0$, a vertex operator of some element in V^2 does not have integral powers of z. In particular, V^2 is not a VOA. However, as $M_{D^{\oplus 3}}$-modules, we have

$$\begin{aligned}\mathrm{Ind}_{D^{\oplus 3}}^{D^{\oplus 3}+Q}&({V_{E_8}}^{\alpha}\otimes {V_{E_8}}^{\beta}\otimes {V_{E_8}}^{\gamma})\\ &= ({V_{E_8}}^{\alpha}\otimes {V_{E_8}}^{\beta}\otimes {V_{E_8}}^{\gamma}) \oplus (R{V_{E_8}}^{\alpha}\otimes R{V_{E_8}}^{\beta}\otimes {V_{E_8}}^{\gamma})\\ &\quad \oplus({V_{E_8}}^{\alpha}\otimes R{V_{E_8}}^{\beta}\otimes R{V_{E_8}}^{\gamma})\oplus(R{V_{E_8}}^{\alpha}\otimes {V_{E_8}}^{\beta}\otimes R{V_{E_8}}^{\gamma}).\end{aligned}$$

Using (7.4), define $W^{(\alpha,\beta,\gamma)}$ for $(\alpha,\beta,\gamma)\in S^\natural$ as follows:

$$\begin{aligned} W^{(\alpha,\alpha,\alpha)} &= {V_{E_8}}^{\alpha} \otimes {V_{E_8}}^{\alpha} \otimes {V_{E_8}}^{\alpha},\\ W^{(\alpha,\alpha,\alpha^c)} &= (R{V_{E_8}}^{\alpha}) \otimes (R{V_{E_8}}^{\alpha}) \otimes {V_{E_8}}^{\alpha^c},\\ W^{(\alpha,\alpha^c,\alpha)} &= (R{V_{E_8}}^{\alpha}) \otimes {V_{E_8}}^{\alpha^c} \otimes (R{V_{E_8}}^{\alpha}),\\ W^{(\alpha^c,\alpha,\alpha)} &= {V_{E_8}}^{\alpha^c} \otimes (R{V_{E_8}}^{\alpha}) \otimes (R{V_{E_8}}^{\alpha}). \end{aligned} \tag{7.10}$$

Since all $R{V_{E_8}}^{\alpha}$ are irreducible M_D-modules by Corollary 3.13, all $W^{(\alpha,\beta,\gamma)}$ are irreducible $M_{D^{\oplus 3}}$-modules. Induce them into

$$V^\chi = \mathrm{Ind}_{D^{\oplus 3}}^{D^\natural}(W^\chi) \tag{7.11}$$

for $\chi\in S^\natural$. Finally, set

$$V^\natural = \bigoplus_{\chi\in S^\natural} V^\chi. \tag{7.12}$$

This is the desired Fock space. We will show that $V^\natural$ has a $(D^\natural, S^\natural)$-framed VOA structure.

Since $(D^\natural,\alpha,\beta)$ satisfies the conditions of Theorem 3.20 for $\alpha,\beta\in S^\natural$, it only remains to prove that

$$V^{\langle\chi,\mu\rangle} = M_{D^\natural} \oplus V^\chi \oplus V^\mu \oplus V^{\chi+\mu}$$

has a VOA structure with a PDIB-form for any $\mu,\chi\in S^\natural$ with $\dim\langle\mu,\chi\rangle = 2$. We note that since $M_{D^{\oplus 3}} \oplus W^{(\alpha,\alpha,\alpha)}$ and $M_{D^\natural} \oplus W^{(\alpha,\alpha,\alpha^c)}$ are sub VOAs of $\mathrm{Ind}_{D^{\oplus 3}}^{\langle D^{\oplus 3},(\xi_1\xi_1 0^{16})\rangle}(M_{D^{\oplus 3}} \oplus W^{(\alpha,\alpha,\alpha)})$, they have VOA structures with PDIB-forms. Take a sub VOA

$$(V^1)^{\langle\chi,\mu\rangle} = M_{D^{\oplus 3}} \oplus (V^1)^\chi \oplus (V^1)^\mu \oplus (V^1)^{\chi+\mu}$$

of V^1 in (7.8) and set

$$W^{\langle\chi,\mu\rangle} = M_{D^{\oplus 3}} \oplus W^\chi \oplus W^\mu \oplus W^{\chi+\mu},$$

for $\chi,\mu\in S^\natural$. If $\langle\chi,\mu\rangle$ is orthogonal to $(\xi_1\xi_1 0^{16})$, then $\mathrm{Ind}_{D^{\oplus 3}}^{\langle D^{\oplus 3},(\xi_1\xi_1 0^{16})\rangle}((V^1)^{\chi,\mu})$ is a VOA with the desired properties and it contains $W^{\langle\chi,\mu\rangle}$ as a sub VOA. Similarly, if $\langle\chi,\mu\rangle$ is orthogonal to $(0^{16}\xi_1\xi_1)$ or $(\xi_1 0^{16}\xi_1)$, then we have the desired properties. Therefore we may assume that $\chi = (\alpha,\alpha,\alpha^c)$ and $\mu = (\beta,\beta^c,\beta)$. Set $\gamma = \alpha^c+\beta$. We divide the proof into two cases.

Case (1): Assume that $\mathrm{Supp}(\alpha)\cap\mathrm{Supp}(\beta)\neq\emptyset$. Choose $t\in\mathrm{Supp}(\alpha)\cap\mathrm{Supp}(\beta)$. Set $\xi_t=(0^{t-1}10^{15-t})$ and $R^t=M_{D+\xi_t}$. Since

$$(\xi_t\xi_t0^{16})+(\xi_1\xi_10^{16})\in D^\natural,\quad (\xi_t0^{16}\xi_t)+(\xi_10^{16}\xi_1)\in D^\natural,\quad (0^{16}\xi_t\xi_t)+(0^{16}\xi_1\xi_1)\in D^\natural,$$

we have

$$\begin{aligned}
\mathrm{Ind}_{D^{\oplus 3}}^{D^\natural}(R^tV_{E_8}{}^{\alpha}\otimes R^tV_{E_8}{}^{\alpha}\otimes V_{E_8}{}^{\alpha^c}) &= \mathrm{Ind}_{D^{\oplus 3}}^{D^\natural}(RV_{E_8}{}^{\alpha}\otimes RV_{E_8}{}^{\alpha}\otimes V_{E_8}{}^{\alpha^c}),\\
\mathrm{Ind}_{D^{\oplus 3}}^{D^\natural}(R^tV_{E_8}{}^{\beta}\otimes V_{E_8}{}^{\beta^c}\otimes R^tV_{E_8}{}^{\beta}) &= \mathrm{Ind}_{D^{\oplus 3}}^{D^\natural}(RV_{E_8}{}^{\beta}\otimes V_{E_8}{}^{\beta^c}\otimes RV_{E_8}{}^{\beta}),\\
\mathrm{Ind}_{D^{\oplus 3}}^{D^\natural}(V_{E_8}{}^{\gamma^c}\otimes R^tV_{E_8}{}^{\gamma}\otimes R^tV_{E_8}{}^{\gamma}) &= \mathrm{Ind}_{D^{\oplus 3}}^{D^\natural}(V_{E_8}{}^{\gamma^c}\otimes RV_{E_8}{}^{\gamma}\otimes RV_{E_8}{}^{\gamma}).
\end{aligned}$$

Set

$$\rho_1=(\xi_t\xi_t0^{16}),\quad \rho_2=(\xi_t0^{16}\xi_t),\quad \rho_3=(0^{16}\xi_t\xi_t).$$

Since $\mathrm{Supp}(\rho_1)\subseteq\mathrm{Supp}(\chi)$, $\mathrm{Supp}(\rho_2)\subseteq\mathrm{Supp}(\mu)$ and $\mathrm{Supp}(\rho_3)\subseteq\mathrm{Supp}(\chi+\mu)$, it follows from Lemma 3.27 that

$$\begin{aligned}
R^t(V_{E_8})^{\alpha}\otimes R^t(V_{E_8})^{\alpha}\otimes(V_{E_8})^{\alpha^c} &\cong \sigma_{\rho_1}((V^1)^{(\alpha,\alpha,\alpha^c)}),\\
R^t(V_{E_8})^{\beta}\otimes(V_{E_8})^{\beta^c}\otimes R^t(V_{E_8})^{\beta} &\cong \sigma_{\rho_2}(V^1)^{(\beta,\beta^c,\beta)},\\
(V_{E_8})^{\gamma^c}\otimes R^1(V_{E_8})^{\gamma}\otimes(V_{E_8})^{\gamma} &\cong \sigma_{\rho_3}(V^1)^{(\gamma^c,\gamma,\gamma)}.
\end{aligned}$$

Since $M_{D^{\oplus 3}}\oplus(V^1)^{(\alpha,\alpha,\alpha^c)}\oplus(V^1)^{(\beta,\beta^c,\beta)}\oplus(V^1)^{(\gamma^c,\gamma,\gamma)}$ has a VOA structure with a PDIB-form, so does $\sigma_{\mu^1+\mu^2}(M_{D^{\oplus 3}})\oplus\sigma_{\rho_1+\rho_2}((V^1)^{(\alpha,\alpha,\alpha^c)})\oplus\sigma_{\rho_1+\rho_2}((V^1)^{(\beta,\beta^c,\beta)})\oplus\sigma_{\rho_1+\rho_2}((V\ 1)^{(\gamma^c,\gamma,\gamma)})$. Clearly, we have

$$\begin{aligned}
\sigma_{\rho_1+\rho_2}(M_{D^{\oplus 3}}) &\cong M_{D^{\oplus 3}},\\
\sigma_{\rho_1+\rho_2}((V^1)^{(\alpha,\alpha,\alpha^c)}) &\cong \sigma_{\rho_1}((V^1)^{(\alpha,\alpha,\alpha^c)})
\end{aligned}$$

and

$$\sigma_{\rho_1+\mu^2}((V^1)^{(\beta,\beta^c,\beta)})\cong\sigma_{\rho_2}((V^1)^{(\beta,\beta^c,\beta)})$$

by Lemma 3.26. Since $\rho_1+\rho_2+\rho_3=0$, $\sigma_{\rho_1+\rho_2}(V^1)^{(\gamma^c,\gamma,\gamma)}\cong\sigma_{\rho_3}(V^1)^{(\gamma^c,\gamma,\gamma)}$. Hence $W^{\langle\chi,\mu\rangle}=M_{D^{\oplus 3}}\oplus W^{(\alpha,\alpha,\alpha^c)}\oplus W^{(\beta,\beta^c,\beta)}\oplus W^{(\gamma^c,\gamma,\gamma)}$ has the desired VOA structure and so does $(V^\natural)^{\langle\chi,\mu\rangle}$.

Case (2): Assume $\mathrm{Supp}(\alpha)\cap\mathrm{Supp}(\beta)=\emptyset$. Then one of $\{\alpha,\beta,\alpha+\beta^c\}$ is at least (0^{16}) since $\alpha,\beta\in S$. We may assume $\alpha=(0^{16})$. Note that $\chi=(0^{32}1^{16})$. It follows from the structure of D that there is a self dual subcode E of $D^{\oplus 3}$ which is a direct sum $\bigoplus_{i=1}^{6}E^i$ of 6 extended $[8,4]$-Hamming codes E^i such that $E_\delta=\{\mu\in E|\mathrm{Supp}(\mu)\subseteq\mathrm{Supp}(\delta)\}$ is a direct factor of E for any $\delta\in\langle\beta,\beta^c\rangle$. In particular, there are M_E-modules U^χ, U^μ, $U^{\chi+\mu}$ such that

$$\begin{aligned}
\mathrm{Ind}_E^{D^\natural}(U^\chi) &= (V^\natural)^{(0^{16}0^{16}1^{16})},\\
\mathrm{Ind}_E^{D^\natural}(U^\mu) &= (V^\natural)^{(\beta,\beta^c,\beta)}
\end{aligned}$$

and

$$\mathrm{Ind}_E^{D^\natural}(U^{\chi+\mu})=(V^\natural)^{(\beta,\beta^c,\beta^c)}.$$

In the following, we will only prove the case $|\beta|=8$, but we are able to prove the assertions for $\beta=(0^{16})$ or $\beta=(1^{16})$ by similar arguments. We may assume $\beta=(1^8 0^8)$ and $\delta=(0^8 1^8)$. As shown in Section 5, we have a VOA $\tilde{V}_{E_8}^{\langle\beta,\delta\rangle}=\tilde{V}_{E_8}^{(0^{16})}\oplus\tilde{V}_{E_8}^{(1^8 0^8)}\oplus\tilde{V}_{E_8}^{(0^8 1^8)}\oplus\tilde{V}_{E_8}^{(1^{16})}$ with a PDIB-form such that

$$\begin{aligned}\tilde{V}_{E_8}^{(0^{16})}&\cong \mathrm{Ind}_F^D\left(H\left(\tfrac{1}{2},(0^8)\right)\otimes H\left(\tfrac{1}{2},(0^8)\right)\right),\\ \tilde{V}_{E_8}^{(1^8 0^8)}&\cong \mathrm{Ind}_F^D\left(H\left(\tfrac{1}{16},\xi_1\right)\otimes H\left(\tfrac{1}{2},\xi_1\right)\right),\\ \tilde{V}_{E_8}^{(0^8 1^8)}&\cong \mathrm{Ind}_F^D\left(H\left(\tfrac{1}{2},\xi_1\right)\otimes H\left(\tfrac{1}{16},\xi_1\right)\right)\end{aligned}$$

and

$$\tilde{V}_{E_8}^{(1^{16})}\cong \mathrm{Ind}_F^D\left(H\left(\tfrac{1}{16},(0^8)\right)\otimes H\left(\tfrac{1}{16},(0^8)\right)\right),$$

where $F=D_{(1^8 0^8)}\oplus D_{(0^8 1^8)}$ is a direct sum of two extended $[8,4]$-Hamming codes. In order to simplify the notation, we omit "$\otimes$" between $H(*,*)$ and $H(*,*)$. As a sub VOA,

$$\begin{aligned}H\left(\tfrac{1}{2},(0^8)\right)H\left(\tfrac{1}{2},(0^8)\right)\oplus H(\tfrac{1}{16},\xi_1)H\left(\tfrac{1}{2},\xi_1\right)\oplus H\left(\tfrac{1}{2},\xi_1\right)H\left(\tfrac{1}{16},\xi_1\right)\\ \oplus H\left(\tfrac{1}{16},(0^8)\right)H\left(\tfrac{1}{16},(0^8)\right)\end{aligned}$$

has a VOA structure with a PDIB-form. Since $W^{(0^{16}0^{16}1^{16})}$ is given by $R\tilde{V}_{E_8}^{(0^{16})}\otimes R\tilde{V}_{E_8}^{(0^{16})}\otimes\tilde{V}_{E_8}^{(1^{16})}=M_{D+\xi}\otimes M_{D+\xi}\otimes {V_{E_8}}^{(1^{16})}$, we have

$$U^{\chi}=H(\tfrac{1}{2},\xi_1)H(\tfrac{1}{2},(0^8))H(\tfrac{1}{2},\xi_1)H(\tfrac{1}{2},(0^8))H(\tfrac{1}{16},(0^8))H(\tfrac{1}{16},(0^8)).$$

We similarly obtain

$$U^{\mu}=H(\tfrac{1}{16},(0^8))H(\tfrac{1}{2},\xi_1)H(\tfrac{1}{2},\xi_1)H(\tfrac{1}{16},\xi_1)H(\tfrac{1}{16},(0^8))H(\tfrac{1}{2},\xi_1)$$

and

$$U^{\chi+\mu}=H(\tfrac{1}{16},\xi_1)H(\tfrac{1}{2},\xi_1)H(\tfrac{1}{2},0)H(\tfrac{1}{16},\xi_1)H(\tfrac{1}{2},0)H(\tfrac{1}{16},\xi_1).$$

By changing the order of the components, $(123456)\to(243516)$, we have

$$\begin{aligned}M_E&\cong H(\tfrac{1}{2},(0^8))H(\tfrac{1}{2},(0^8))H(\tfrac{1}{2},(0^8))H(\tfrac{1}{2},(0^8))H(\tfrac{1}{2},(0^8))H(\tfrac{1}{2},(0^8)),\\ U^{\chi}&\cong H(\tfrac{1}{2},(0^8))H(\tfrac{1}{2},(0^8))H(\tfrac{1}{2},(\xi_1))H(\tfrac{1}{16},(0^8))H(\tfrac{1}{2},(\xi_1))H(\tfrac{1}{16},(0^8)),\\ U^{\mu}&\cong H(\tfrac{1}{2},(\xi_1))H(\tfrac{1}{16},(\xi_1))H(\tfrac{1}{2},(\xi_1))H(\tfrac{1}{16},(0^8))H(\tfrac{1}{16},(0^8))H(\tfrac{1}{2},(\xi_1))\end{aligned}$$

and

$$U^{\chi+\mu}=H(\tfrac{1}{2},(\xi_1))H(\tfrac{1}{16},(\xi_1))H(\tfrac{1}{2},(0^8))H(\tfrac{1}{2},(\xi_1))H(\tfrac{1}{16},(\xi_1))H(\tfrac{1}{16},(\xi_1)).$$

By Lemma 3.17, there is an automorphism σ of M_{H_8} such that

$$\begin{aligned}\sigma(H(\tfrac{1}{2},\xi_1))&\cong H(\tfrac{1}{16},(0^8)),\\ \sigma(H(\tfrac{1}{16},\xi_1))&\cong H(\tfrac{1}{2},(\xi_1))\end{aligned}$$

and

$$\sigma(H(\tfrac{1}{16},(0^8)))\cong H(\tfrac{1}{16},(\xi_1)).$$

Changing the coordinate set by $\{\sigma(e_1),\cdots,\sigma(e^8)\}$, we have

$$\begin{aligned}\tilde{V}_{E_8}^{(0^{16})} &\cong \mathrm{Ind}_F^D(H(\tfrac{1}{2},(0^8))\otimes H(\tfrac{1}{2},(0^8))),\\ \tilde{V}_{E_8}^{(1^8 0^8)} &\cong \mathrm{Ind}_F^D(H(\tfrac{1}{2},(\xi_1))\otimes H(\tfrac{1}{16},(0^8))),\\ \tilde{V}_{E_8}^{(0^8 1^8)} &\cong \mathrm{Ind}_F^D(H(\tfrac{1}{16},(0^8))\otimes H(\tfrac{1}{2},(\xi_1)))\end{aligned}$$

and

$$\tilde{V}_{E_8}^{(1^{16})} \cong \mathrm{Ind}_F^D(H(\tfrac{1}{16},(\xi_1))\otimes H(\tfrac{1}{16},(\xi_1))).$$

Therefore $U^{\langle\chi,\mu\rangle}=M_E\oplus U^\chi\oplus U^\mu\oplus U^{\chi+\mu}$ is a subset of $\tilde{V}_{E_8}\otimes\tilde{V}_{E_8}\otimes\tilde{V}_{E_8}$. It is also easy to check that $U^{\langle\chi,\mu\rangle}$ is closed under the products given by vertex operators. Consequently, $U^{\langle\chi,\mu\rangle}$ is a VOA with a PDIB-form and so is $(V^\natural)^{\langle\chi,\mu\rangle}=\mathrm{Ind}_E^{D^\natural}(U^{\langle\chi,\mu\rangle})$. This completes the construction of $V^\natural$. □

Corollary 7.1. *$V^\natural$ has a* PDIB-*form.*

Remark 2. Because of our construction, a VOA satisfying Hypotheses I is a direct sum of the tensor product of $L(\frac{1}{2},0), L(\frac{1}{2},\frac{1}{2}), L(\frac{1}{2},\frac{1}{16})$ and we know the multiplicities of irreducible $L(\frac{1}{2},0)^{\otimes n}$-modules by Theorem 3.8 (cf. Corollary 5.2 in [Mi3]). Hence it is not difficult to calculate its character

$$\mathrm{ch}_V(z)=e^{2\pi iz(\mathrm{rank}(V))/24}\left(\sum_{n=0}^{\infty}\dim V_n\ e^{2\pi iz}\right).$$

For example, let us show that $(V^\natural)_1=0$. We first have $(M_{D^\natural})_1=0$ since $D^\natural$ has no codewords of weight 2. Also, if $(V^\natural)_1^\chi\neq 0$ for some χ, then the weight of χ is equal to 16 and hence χ is one of $(1^{16}0^{16}0^{16})$, $(0^{16}1^{16}0^{16})$ or $(0^{16}0^{16}1^{16})$. Say $\chi=(1^{16}0^{16}0^{16})$. Since $(V^\natural)^\chi=\mathrm{Ind}_{D^3_{E_8}}^{D^\natural}(V_{E_8}^{(1^{16})}\otimes M_{D_{E_8}+\xi_1}\otimes M_{D_{E_8}+\xi_1})$ and $D^\natural$ does not contains any words of the form (α,ξ_1,ξ_1), the minimal weight of $(V^\natural)^\chi$ is greater than 1, which contradicts the choice of χ. Therefore we obtain $V_1^\natural=0$.

8. Conformal vectors

Since each rational conformal vector $e\in V$ with central charge $\frac{1}{2}$ gives rise to an automorphism τ_e, it is very important to find such conformal vectors for studying the automorphism group $\mathrm{Aut}(V)$. Therefore we will construct several conformal vectors of $V^\natural$ explicitly.

8.1. *Case* I. Set $D_1=\langle H_8\oplus H_8,(\xi_1\xi_1)\rangle$ and $S=\langle(1^{16})\rangle$, where $\xi_1=(10^7)$. Then the pair (α,β,S) satisfies the conditions (3.a) and (3.b) of Theorem 3.20 for any $\alpha,\beta\in S_1$. Set

$$U=H(\tfrac{1}{2},0)H(\tfrac{1}{2},0)\oplus H(\tfrac{1}{2},\xi_1)H(\tfrac{1}{2},\xi_1)\oplus H(\tfrac{1}{16},\xi_1)H(\tfrac{1}{16},0)\oplus H(\tfrac{1}{16},0)H(\tfrac{1}{16},\xi_1).$$

U is isomorphic to a sub VOA of V_{E_8}. It is easy to see that

$$\dim(H(\tfrac{1}{2},0)H(\tfrac{1}{2},0))_1 = 0$$

and

$$\begin{aligned}\dim(H(\tfrac{1}{2},\xi_1)H(\tfrac{1}{2},\xi_1))_1 &= \dim(H(\tfrac{1}{16},\xi_1)H(\tfrac{1}{16},0))_1\\ &= \dim(H(\tfrac{1}{16},0)H(\tfrac{1}{16},\xi_1))_1 = 1.\end{aligned}$$

Hence the weight-one space U_1 of U is isomorphic to sl(2) as a Lie algebra. If we view $(H(\frac{1}{2},\xi_1)H(\frac{1}{2},\xi_1))_1$ as a Cartan subalgebra of sl(2), $H(\frac{1}{16},\xi_1)H(\frac{1}{16},0) \oplus H(\frac{1}{16},0)H(\frac{1}{16},\xi_1)$ contains two roots α and β. A sub VOA generated by U_1 is isomorphic to a lattice $V_{\mathbb{Z}x}$ of type A_1 with $\langle x,x\rangle = 2$. Identifying α and β with $\iota(x)$ and $\iota(-x)$, respectively, we obtain the following elements:

$$\begin{aligned}x_{(-1)}\mathbf{1} &\in (H(\tfrac{1}{2},\xi_1)\otimes H(\tfrac{1}{2},\xi_1))_1,\\ \iota(x)+\iota(-x) &\in (H(\tfrac{1}{16},\xi_1)\otimes H(\tfrac{1}{16},0))_1,\end{aligned}$$

and

$$\iota(x)-\iota(-x) \in (H(\tfrac{1}{16},\xi_1)\otimes H(\tfrac{1}{16},0))_1.$$

Take another copy of these and set

$$\begin{aligned}y_{(-1)}\mathbf{1} &\in (H(\tfrac{1}{2},\xi_1)\otimes H(\tfrac{1}{2},\xi_1))_1,\\ \iota(y)+\iota(-y) &\in (H(\tfrac{1}{16},\xi_1)\otimes H(\tfrac{1}{16},0))_1,\end{aligned}$$

and

$$\iota(y)-\iota(-y) \in (H(\tfrac{1}{16},\xi_1)\otimes H(\tfrac{1}{16},0))_1.$$

Then we have

$$\begin{aligned}\iota(\pm x)\otimes\iota(\pm y)+\iota(\mp x)\otimes\iota(\mp y) &\in H(\tfrac{1}{16},0)H(\tfrac{1}{16},\xi_1)H(\tfrac{1}{16},0)H(\tfrac{1}{16},\xi_1)\\ &\quad\oplus H(\tfrac{1}{16},\xi_1)H(\tfrac{1}{16},0)H(\tfrac{1}{16},\xi_1)H(\tfrac{1}{16},0)\\ x_{(-1)}y_{(-1)}\mathbf{1} &\in H(\tfrac{1}{2},\xi_1)H(\tfrac{1}{2},\xi_1)H(\tfrac{1}{2},\xi_1)H(\tfrac{1}{2},\xi_1)\end{aligned}$$

and

$$(x_{(-1)})^2\mathbf{1},\ (y_{(-1)})^2\mathbf{1} \in H(\tfrac{1}{2},0)H(\tfrac{1}{2},0)H(\tfrac{1}{2},0)H(\tfrac{1}{2},0).$$

It follows from $\langle x\pm y, x\pm y\rangle = 2$ that

$$e^{+}(x\pm y) = \tfrac{1}{16}((x\pm y)_{(-1)})^2\mathbf{1}+\tfrac{1}{4}(\iota(x\pm y)+\iota(-x\mp y))$$

and

$$e^{-}(x\pm y) = \tfrac{1}{16}((x\pm y)_{(-1)})^2\mathbf{1}-\tfrac{1}{4}(\iota(x\pm y)+\iota(-x\mp y))$$

are rational conformal vectors with central charge $\frac{1}{2}$. Therefore we obtain four rational conformal vectors $e^{\pm}(x\pm y)$ in

$$\begin{aligned}&H(\tfrac{1}{2},0)H(\tfrac{1}{2},0)H(\tfrac{1}{2},0)H(\tfrac{1}{2},0)\oplus H(\tfrac{1}{2},\xi_1)H(\tfrac{1}{2},\xi_1)H(\tfrac{1}{2},\xi_1)H(\tfrac{1}{2},\xi_1)\\ &\oplus H(\tfrac{1}{16},0)H(\tfrac{1}{16},\xi_1)H(\tfrac{1}{16},0)H(\tfrac{1}{16},\xi_1)\oplus H(\tfrac{1}{16},\xi_1)H(\tfrac{1}{16},0)H(\tfrac{1}{16},\xi_1)H(\tfrac{1}{16},0).\end{aligned}$$

8.2. *Case* II. We treat the first component $V_{E_8}\otimes\mathbf{1}\otimes\mathbf{1}$ of $V_{E_8}\otimes V_{E_8}\otimes V_{E_8}$. For simplicity, we denote D_{E_8}, S_{E_8} and V_{E_8} by D, S, V, respectively. For $\alpha,\beta\in S$ with $|\alpha|=|\beta|=|\alpha+\beta|=8$, V contains a sub VOA

$$V^{\langle\alpha^c,\beta^c\rangle}=M_D\oplus V^{\alpha^c}\oplus V^{\beta^c}\oplus V^{\alpha+\beta}.$$

Since D_{α^c}, D_{β^c} and $D_{\alpha+\beta}$ are all isomorphic to H_8, the multiplicities of the irreducible $L(\frac{1}{2},0)^{\otimes 8}$-modules in $V^{\alpha^c}\oplus V^{\beta^c}\oplus V^{\alpha+\beta}$ are all one by Theorem 3.8. Hence $\dim(V^{\alpha^c})_1=\dim(V^{\beta^c})_1=\dim(V^{\alpha+\beta})_1=8$. Since D does not contain any words of weight 2, $(M_D)_1=0$ and so $(M_D\oplus V^{\alpha^c})_1$, $(M_D\oplus V^{\beta^c})_1$ and $(M_D\oplus V^{\alpha+\beta})_1$ are all commutative Lie algebras. Since $V^{\langle\alpha^c,\beta^c\rangle}$ is a sub VOA of a lattice VOA V of rank 8 and hence $(V^{\langle\alpha^c,\beta^c\rangle})_1$ is isomorphic to $\mathrm{sl}(2)^{\oplus 8}$. Let $\{x_1,\cdots,x_8\}$ be the set of positive roots of $A_1^{\oplus 8}$. Viewing $(V^{\alpha+\beta})_1$ as a Cartan subalgebra of $sl(2)^{\oplus 8}$ and embedding it into a lattice VOA $V_{A_1^{\otimes 8}}$ of root lattice $A_1^{\otimes 8}$, we are able to denote the positive roots by $\iota(x_1),\cdots,\iota(x_8)$ and the negative roots by $\iota(-x_1),\cdots,\iota(-x_8)$. In addition, we may assume

$$\begin{aligned}(x_i)_{(-1)}\mathbf{1}&\in V_1^{\alpha+\beta},\\ \iota(x_i)+\iota(-x_i)&\in V_1^{\alpha^c},\\ \iota(x_i)-\iota(-x_i)&\in V_1^{\beta^c}\end{aligned}$$

for $i=1,\cdots,8$.

We next treat the second and third components of $V_{E_8}\otimes V_{E_8}\otimes V_{E_8}$. Set $V^{(\gamma,\gamma)}=V^{\gamma}\otimes V^{\gamma}$ and $V^{(\bar{\gamma},\bar{\gamma})}=RV^{\gamma}\otimes RV^{\gamma}$ for $\gamma\in S$. We also set $F=\{(\alpha',\beta')\mid \alpha'+\beta'\in D,\alpha',\beta' \text{ even }\}$ and $W^{(\delta,\delta)}=\mathrm{Ind}_{D^{\oplus 2}}^{F}(V^{(\delta,\delta)})$ for δ. We note that F does not contain any roots and $D\oplus F\subseteq D^{\natural}$. By a similar argument as in the construction of the moonshine VOA,

$$W^{\langle(\bar{\alpha},\bar{\alpha}),(\bar{\beta},\bar{\beta})\rangle}=M_{D^{\oplus 2}}\oplus W^{(\bar{\alpha},\bar{\alpha})}\otimes W^{(\bar{\beta},\bar{\beta})}\oplus W^{(\alpha+\beta,\alpha+\beta)}$$

has a VOA structure. By Theorem 3.25, we have a VOA

$$W^{\langle(\bar{\alpha},\bar{\alpha}),(\bar{\beta},\bar{\beta})\rangle}=M_F\oplus W^{(\bar{\alpha},\bar{\alpha})}\oplus W^{(\bar{\beta},\bar{\beta})}\oplus W^{(\alpha+\beta,\alpha+\beta)}.$$

Since the numbers of codewords in $F_{(\alpha,\alpha)}$, $F_{(\beta,\beta)}$ and $F_{((\alpha+\beta),(\alpha+\beta))}$ are all 2^{11}, the multiplicities of irreducible $L(\frac{1}{2},0)^{\otimes 16}$-submodules are all $2^{11-8}=8$, where $F_{(\gamma,\gamma)}=\{\delta\in F|\mathrm{Supp}(\delta)\subseteq\mathrm{Supp}((\gamma,\gamma))\}$. Hence $\dim(W_1^{(\gamma,\gamma)})=8$ for $\gamma\in\{\bar{\alpha},\bar{\beta},\alpha+\beta\}$. We also have that

$$\begin{aligned}X=M_F&\oplus W^{(1^{\bar{1}6},1^{\bar{1}6})}\oplus W^{(\bar{al},\bar{\alpha})}\oplus W^{(\bar{\beta},\bar{\beta})}\oplus W^{(\alpha+\beta,\alpha+\beta)}\\ &\oplus W^{(\alpha^c,\alpha^c)}\oplus W^{(\beta^c,\beta^c)}\oplus W^{(\overline{\alpha+be^c},\overline{\alpha+\beta^c})}\end{aligned}$$

has a VOA structure. If $|\delta|=16$, only irreducible T-submodules of W^{δ} isomorphic to $\otimes_{i=1}^{32}L(\frac{1}{2},\frac{d_i}{16})$ contribute the weight-one space for $\delta=(d_1,\cdots,d_{32})$. Since $|\alpha+\beta^c|=|\alpha+\beta|=8$, $(M_F\oplus W^{(\bar{1^{16}}),(\bar{1^{16}})})_1=0$ and $(W^{\overline{\alpha+\beta^c},\overline{\alpha+\beta^c}}\oplus W^{\alpha+\beta,\alpha+\beta})_1$

is of dimension 16. Since X is a sub VOA of a lattice VOA of rank 16, X_1 is isomorphic to $\mathrm{sl}(2)^{\oplus 16}$ and $W_1^{\langle(\bar{\alpha},\bar{\alpha}),(\bar{\beta},\bar{\beta})\rangle}$ is isomorphic to $\mathrm{sl}(2)^{\oplus 8}$. Viewing $(W^{(\alpha+\beta,\alpha+\beta)})_1$ as a Cartan subalgebra and embedding it in a lattice VOA $V_{A_1^{\oplus 8}}$ of the root lattice $A_1^{\oplus 8}$, we are able to denote the positive roots by $\iota(y_1),\cdots,\iota(y_8)$ and the negative roots by $\iota(-y_1),\cdots,\iota(-y_8)$. Then we may assume that

$$\begin{aligned}(y_i)_{(-1)}\mathbf{1} &\in (W^{(\alpha+\beta,\alpha+\beta)})_1,\\ \iota(y_i)+\iota(-y_i) &\in (W^{(\bar{\alpha},\bar{\alpha})})_1\end{aligned}$$

and

$$\iota(y_i)-\iota(-y_i)\in (W^{(\bar{\beta},\bar{\beta})})_1$$

for $i=1,\cdots,8$.

Set

$$\begin{aligned}U^{(\alpha^c,\alpha,\alpha)} &= V_{E_8}{}^{\alpha^c}\otimes W^{(\bar{\alpha},\bar{\alpha})},\\ U^{(\beta^c,\beta,\beta)} &= V_{E_8}{}^{\beta^c}\otimes U^{(\bar{\beta},\bar{\beta})}\end{aligned}$$

and

$$U^{(\alpha+\beta,\alpha+\beta,\alpha+\beta)}=V_{E_8}{}^{\alpha+\beta}\otimes W^{(\alpha+\beta,\alpha+\beta)}.$$

Then

$$U=M_{D\oplus F}\oplus U^{(\alpha^c,\alpha,\alpha)}\oplus U^{(\beta^c,\beta,\beta)}\oplus U^{(\alpha+\beta,\alpha+\beta,\alpha+\beta)}$$

is a sub VOA of $V^\natural$. We have

$$\begin{aligned}((x_i)_{(-1)})^2\mathbf{1} &\in M_D,\\ ((y_i)_{(-1)})^2\mathbf{1} &\in M_F,\\ (x_i)_{(-1)}(y_i)_{(-1)}\mathbf{1} &\in U^{(\alpha+\beta,\alpha+\beta,\alpha+\beta)},\\ (\iota(x_i)+\iota(-x_i))\otimes(\iota(y_i)+\iota(-y_i)) &\in U^{(\alpha^c,\alpha,\alpha)}\end{aligned}$$

and,

$$(\iota(x_i)-\iota(-x_i))\otimes(\iota(y_i)-\iota(-y_i))\in W^{(\beta^c,\beta,\beta)}.$$

By the same arguments as in the case I, we have 32 mutually orthogonal conformal vectors

$$\begin{aligned}d_{4i-3} &= \tfrac{1}{16}((x_i+y_i)_{(-1)})^2\mathbf{1}+\tfrac{1}{4}(\iota(x_i+y_i)+\iota(-x_i-y_i))\\ d_{4i-2} &= \tfrac{1}{16}((x_i+y_i)_{(-1)})^2\mathbf{1}-\tfrac{1}{4}(\iota(x_i+y_i)+\iota(-x_i-y_i))\\ d_{4i-1} &= \tfrac{1}{16}((x_i-y_i)_{(-1)})^2\mathbf{1}+\tfrac{1}{4}(\iota(x_i-y_i)+\iota(-x_i+y_i))\\ d_{4i} &= \tfrac{1}{16}((x_i-y_i)_{(-1)})^2\mathbf{1}-\tfrac{1}{4}(\iota(x_i-y_i)+\iota(-x_i+y_i))\end{aligned}$$

in V^1, where $\iota(x_i+y_i)$ denotes $\iota(x_i)\otimes\iota(y_i)$.

9. The automorphism group

In this section, we will prove that the full automorphism group of $V^\natural$ is the Monster simple group.

Hypotheses II.
(1) $V=\sum_{i=0}^{\infty} V_i$ is a framed VOA over $\mathbb{R}$ with a PDIB-form $\langle\ ,\ \rangle$.
(2) $V_1=0$.

We recall the following results from [Mi4].

THEOREM 9.1. *Under Hypotheses* II, *if* e, f *are two distinct conformal vectors with central charge* $\frac{1}{2}$, *then*

$$\langle e,f\rangle \le \tfrac{1}{12} \quad \textit{and} \quad \langle e-f,e-f\rangle \ge \tfrac{1}{3}.$$

In particular, there are only finitely many conformal vectors with central charge $\frac{1}{2}$.

Proof. By a product $ab=a_{(1)}b$ and an inner product $\langle a,b\rangle\mathbf{1}=a_{(3)}b$ for $a,b\in V_2$, V_2 becomes a commutative algebra called a Griess algebra. Decompose V_2 as $\mathbb{R}e\oplus\mathbb{R}e^{\perp}$ with $\mathbb{R}e^{\perp}=\{v\in V_2 | \langle v,e\rangle=0\}$. For a conformal vector f, there are $r\in\mathbb{R}$ and $u\in\mathbb{R}e^{\perp}$ such that

$$f=re+u.$$

Since $\langle eu,e\rangle=\langle u,e^2\rangle=\langle u,2e\rangle=0$, we have $eu\in\mathbb{R}e^{\perp}$ and hence

$$2re+2u=2f=ff=\{2r^2e+(uu)_e\}+\{(uu-(uu)_e)+2reu\},$$

where $(uu)_e$ denotes the first entry of uu in the decomposition $\mathbb{R}e\oplus\mathbb{R}e^{\perp}$. Hence

$$r^2/2+\langle e,(uu)_e\rangle=\langle e,2r^2e+(uu)_e\rangle=\langle e,ff\rangle=\langle e,2f\rangle=\langle e,2re\rangle=r/2$$

and so $\langle e,(uu)_e\rangle=r(1-r)/2$. On the other hand, we have

$$\tfrac{1}{4}=\langle f,f\rangle=r^2\tfrac{1}{4}+\langle u,u\rangle,$$

and hence $\langle u,u\rangle=\frac{1}{4}(1-r^2)$. Since $\langle e\rangle\cong L(\frac{1}{2},0)$ as VOAs and every irreducible $L(\frac{1}{2},0)$-module is isomorphic to one of $L(\frac{1}{2},0), L(\frac{1}{2},\frac{1}{2}), L(\frac{1}{2},\frac{1}{16})$, the eigenvalues of $e_{(1)}$ on V are $0, 1+\mathbb{Z}^+, \frac{1}{2}, \frac{1}{2}+\mathbb{Z}^+, \frac{1}{16}, \frac{1}{16}+\mathbb{Z}^+$. Let v be an element in $\mathbb{R}e^{\perp}\subseteq V_2$. Since $e_{(m)}v\in V_{3-m}$ for $m\in\mathbb{Z}$, we have $e_{(m)}v=0$ for $m=2,4,5,\cdots$. Also since $\langle e,v\rangle=0$, we have $e_{(3)}v=0$. Therefore v is a sum of highest weight vectors of $\langle e\rangle$-modules. Hence the eigenvalues of $e_{(1)}$ on $\mathbb{R}e^{\perp}$ are 0, $\frac{1}{2}$, or $\frac{1}{16}$. Consequently, we obtain

$$r/2-r^2/2=\langle e,(uu)_e\rangle=\langle e,uu\rangle=\langle ue,u\rangle\le\tfrac{1}{2}\langle u,u\rangle=\tfrac{1}{8}(1-r^2)$$

and thus $3r^2-4r+1\ge 0$. This implies $r\ge 1$ or $r\le\frac{1}{3}$. If $r\ge 1$, then it contradicts $\langle u,u\rangle>0$. We now have $r\le\frac{1}{3}$ and so $\langle e,f\rangle\le\frac{1}{12}$, which implies $\langle e-f,e-f\rangle\ge\frac{1}{3}$. Therefore there are only finitely many conformal vectors with central charge $\frac{1}{2}$ since $\{v\in V_2 | \langle v,v\rangle=4\}$ is a compact space. $\square$

THEOREM 9.2. *If V satisfies Hypothesis* II, *then* $\mathrm{Aut}(V)$ *is finite.*

Proof. Suppose the theorem is false and let G be an automorphism group of V of infinite order. Since G acts on the set J of all conformal vectors with central charge $\frac{1}{2}$ and J is a finite set by Theorem 9.1, we may assume that G fixes all conformal vectors with central charge $\frac{1}{2}$. In particular, G fixes every conformal vector e_i in a coordinate set $\{e_i|i=1,\cdots,n\}$. Set $P=\langle\tau_{e_i}\mid i=1,\cdots,n\rangle$. By the definition of τ_{e_i}, P is an elementary abelian 2-group. Let $V=\oplus_{\chi\in\mathrm{Irr}(P)}V^\chi$ be the decomposition of V into the direct sum of eigenspaces of P, where $\mathrm{Irr}(P)$ is the set of all linear characters of P and $V^\chi=\{v\in V\mid gv=\chi(g)v\ \forall g\in P\}$. As we mentioned in the introduction, $\tilde{\tau}(V^\chi)=(a_1,\cdots,a_n)\in\mathbb{Z}_2^n$ is given by $(-1)^{a_i}=\chi(e_i)$. Since G fixes all e_i and $g^{-1}\tau_{e_i}g=\tau_{g(e_i)}$ for $g\in\mathrm{Aut}(V)$ by the definition of τ_{e_i}, $[G,P]=1$ and hence G leaves all V^χ invariant. In particular, G acts on V^{1_G}. We think over the action of G on V^{1_G} $(=V^P)$ for a while. Set $T=\langle e_1,\cdots,e_n\rangle$, which is isomorphic to $L(\frac{1}{2},0)^{\otimes n}$. Since $\dim V_0=1$, T is the only irreducible T-submodule of V isomorphic to $L(\frac{1}{2},0)^{\otimes n}$ as a T-module. By the hypotheses, V has a PDIB-form and so V^P is simple. Hence V^P is isomorphic to a code VOA $M_D=\oplus_{\alpha\in D}M_\alpha$ for some even linear code D. Since T is generated by $\{e_i\mid i=1,\cdots,n\}$ and G fixes all e_i, G fixes all elements of T and so $g\in G$ acts on M_α as a scalar $\lambda_\alpha(g)$. Since V has a PDIB-form, we have $0\neq\langle v,v\rangle=\langle g(v),g(v)\rangle=\lambda_\alpha^2(g)\langle v,v\rangle$ and hence $\lambda_\alpha(g)=\pm1$. Since the order of D is finite, we may assume that G fixes all elements in V^P. Since V^χ is an irreducible V^P-module by [DM2], $g\in G$ acts on V^χ as a scalar $\mu_\chi(g)$. By the same arguments as above, we have a contradiction. □

In Lemma 3.3, we showed that we are able to induce every automorphism of D into an automorphism of M_D. We will show that we can induce every automorphism of $S^\natural$ into an automorphism of $V^\natural$.

LEMMA 9.3. *For any $g\in\mathrm{Aut}(S^\natural)$, there is an automorphism $\tilde{g}$ of $V^\natural$ such that $\tilde{g}(e_i)=e_{g(i)}$.*

Proof. By Lemma 3.3, we may assume that g is an automorphism of $M_{D^\natural}$. Let $g((V^\natural)^\chi)$ be an $M_{D^\natural}$-module defined by $v_{(m)}(g\cdot u))=g\cdot(g^{-1}(v)_{(m)}u)$ for $v\in M_{D^\natural}$, $u\in(V^\natural)^\chi$ and $m\in\mathbb{Z}$. Clearly, $\tilde{\tau}(g((V^\natural)^\chi)=g^{-1}(\chi)$ and

$$g(V^\natural)=\oplus_{\chi\in S^\natural}g((V^\natural)^\chi)$$

has a $(D^\natural,S^\natural)$-framed VOA structure by Theorem 3.25. We will prove that there is an $M_{D^\natural}$-isomorphism

$$\pi_\chi: g((V^\natural)^\chi)\to(V^\natural)^{g(\chi)}$$

for $\chi \in S^\natural$. In this case, by the uniqueness theorem (Theorem 3.25), there are scalars λ_χ such that an endomorphism

$$\phi : g(V^\natural) \to V^\natural$$

given by $\phi = \oplus_\chi \lambda_\chi \pi\chi$ on $\oplus_\chi g((V^\natural)^\chi)$ is a VOA-isomorphism. Hence $\tilde{g}(v) = \phi(g \cdot v)$ for $v \in V^\natural$ becomes one of the desired automorphisms of $V^\natural$.

Since $S^\natural = \{(\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in S_{E_8}, \beta, \gamma = \alpha \text{ or } \alpha^c\}$, $\mathrm{Aut}(S^\natural) = \Sigma_3 \times \mathrm{Aut}(S_{E_8})$, where Σ_3 is the symmetric group on three letters. As we showed in the proof of Theorem 5.4,

$$\mathrm{Aut}(S_{E_8}) \cong \mathrm{GL}(5,2)_1 = \{g \in \mathrm{GL}(5,2) \mid g^t(10000) = {}^t(10000)\}.$$

In particular, g leaves $D^{\oplus 3} = D_{E_8} \oplus D_{E_8} \oplus D_{E_8}$ and $D^\natural$ invariant. Set $\chi = (\alpha, \beta, \gamma)$. We first assume that $g \in \Sigma_3$. Since $(V^\natural)^\chi = \mathrm{Ind}_{D^{\oplus 3}}^{D^\natural}(W^{(\alpha,\beta,\gamma)})$ and $W^{(\alpha,\beta,\gamma)}$ is given by (7.10), we have $g(W^{(\alpha,\beta,\gamma)}) \cong W^{g(\alpha,\beta,\gamma)}$ as $M_{D^{\oplus 3}}$-modules and so we have the desired isomorphism for $g \in \Sigma_3$. Assume $g = (h, h, h)$ with $h \in \mathrm{Aut}(S_{E_8})$. By Theorem 5.4, $h(\tilde{V}_{E_8}^{\alpha}) \cong \tilde{V}_{E_8}^{h(\alpha)}$ and hence $g(W^{(\alpha,\alpha,\alpha)}) \cong W^{(h(\alpha),h(\alpha),h(\alpha))}$. For $\chi = (\alpha, \alpha, \alpha^c)$,

$$\begin{aligned} g(W^{(\alpha,\alpha,\alpha^c)}) = {} & h(R\tilde{V}_{E_8}^{\alpha}) \otimes h(R\tilde{V}_{E_8}^{\alpha}) \otimes h(\tilde{V}_{E_8}^{\alpha^c}) \\ \cong {} & (h(R))\tilde{V}_{E_8}^{h(\alpha)} \otimes (h(R))\tilde{V}_{E_8}^{h(\alpha)} \otimes \tilde{V}_{E_8}^{h(\alpha^c)} \end{aligned}$$

as $M_{D_{E_8}} \otimes M_{D_{E_8}} \otimes M_{D_{E_8}}$-modules. Since $R \cong M_{D_{E_8}+\xi_1}$, $h(R) \cong M_{D_{E_8}+\xi_j}$, where $j = h(1)$ and $\xi_j = (0^{j-1}10^{16-j})$. Since $(\xi_1 + \xi_j, \xi_1 + \xi_j, 0^{16}) \in D^\natural$, $(R \times h(R)) \otimes (R \times h(R)) \otimes M_{D_{E_8}}$ is a submodule $M_{D^{\oplus 3}+(\xi_1+\xi_j,\xi_1+\xi_j,0^{16})}$ of $M_{D^\natural}$ and so we have the desired conclusion:

$$\begin{aligned} g(V^\natural)^\chi &= g(\mathrm{Ind}_{D_{E_8}^3}^{D^\natural} W^{(\alpha,\alpha,\alpha^c)}) \\ &= \mathrm{Ind}_{D_{E_8}^3}^{D^\natural} (h(R))\tilde{V}_{E_8}^{h(\alpha)}) \otimes (h(R))\tilde{V}_{E_8}^{h(\alpha)}) \otimes (\tilde{V}_{E_8}^{h(\alpha^c)}) \\ &\cong \mathrm{Ind}_{D_{E_8}^3}^{D^\natural} R\tilde{V}_{E_8}^{h(\alpha)} \otimes R\tilde{V}_{E_8}^{h(\alpha)} \otimes \tilde{V}_{E_8}^{h(\alpha)^c} \\ &\cong (V^\natural)^{g(\chi)}. \end{aligned}$$

□

Let Λ be the Leech lattice and let V_Λ be a lattice VOA constructed from Λ. The following result easily comes from the construction of V_Λ in [FLM2].

LEMMA 9.4. $\mathrm{Aut}(V_\Lambda) \cong ((\mathbb{R}^\times)^{\oplus 24})Co.0$, *where* $\mathbb{R}^\times = \mathbb{R} - \{0\}$ *is the multiplicative group of* $\mathbb{R}$. ($Co.0$ *does not mean a subgroup.*)

Proof. Since $(V_\Lambda)_1$ is a commutative Lie algebra $\mathbb{R}\Lambda$ of rank 24 and $\exp(\alpha_{(0)}) = \sum_{i=0}^{\infty} \frac{1}{i!}(\alpha_{(0)})^i$ is an automorphism acting on $\mathbb{R}\iota(x)$ as a scalar $\exp(\langle \alpha, x \rangle)$ for $\alpha \in (V_\Lambda)_1$ and $x \in \Lambda$, we have an automorphism group $\mathbb{R}^{\times \oplus 24}$, which is a normal subgroup of $\mathrm{Aut}(V_\Lambda)$. On the other hand, Frenkel, Lepowsky and Meurman [FLM2] induced $g \in \mathrm{Aut}(\Lambda)$ into an automorphism of the group extension $\hat{\Lambda} = \{\pm\iota(x) \mid x \in \Lambda\}$ and also into an automorphism of V_Λ using cocycles. Hence V_Λ has an automorphism group $(\mathbb{R}^{\times\oplus 24})Co.0$. Conversely, suppose $\mathrm{Aut}(V_\Lambda) \neq (\mathbb{R}^{\times\oplus 24})Co.0$ and $g\in \mathrm{Aut}(V_\Lambda)-(\mathbb{R}^{\times\oplus 24})Co.0$; then g leaves $(V_\Lambda)_1$ invariant and hence it leaves a sub VOA $\langle (V_\Lambda)_1\rangle$ of free bosons invariant. Then g acts on the lattice of highest weights of V_Λ as a $\langle (V_\Lambda)_1\rangle$-module, which is isomorphic to the Leech lattice. Multiplying an element of $Co.0 := \mathrm{Aut}(\Lambda)$, we may assume that g fixes all highest weight vectors $\{\iota(x) \mid x\in\Lambda\}$ of V_Λ as a $\langle (V_\Lambda)_1\rangle$-module up to scalar multiple and so g commutes with $x(0)$ for $x\in\Lambda$. Consequently, g fixes all elements of $(V_\Lambda)_1$ and acts on $\mathbb{R}\iota(x)$ as a scalar and so $g\in(\mathbb{R}^{\times\oplus 24})$, which contradicts the choice of g. □

THEOREM 9.5. $\mathrm{Aut}(V^\natural)$ *is the Monster simple group.*

Proof. As we proved, the full automorphism group of $V^\natural$ is finite. Set $\delta=\tau_{e_1}\tau_{e_2}$ and decompose $V^\natural$ into the direct sum

$$V^\natural = V^+ \oplus V^-$$

of the eigenspaces of δ, where $V^\pm = \{v \in V^\natural \mid \delta(v) = \pm v\}$. By the definition of τ_{e_i},

$$V^+ = \sum_{\alpha\in S^\natural,\ \langle \alpha,(110^{46})\rangle=0} (V^\natural)^\alpha.$$

Set $S_\Lambda = \left\langle (110^{46})\right\rangle^\perp \cap S^\natural$ and $D_\Lambda = S_\Lambda^\perp$. Since

$$S^\natural = \{(\alpha,\beta,\gamma) \mid \alpha,\beta,\gamma\in S_{E_8}, \beta,\gamma\in\{\alpha,\alpha^c\}\}$$

and

$$S_{E_8} = \left\langle (1^{16}),(1^8 0^8),(1^4 0^4)^2,(1^2 0^2)^4,(10)^8\right\rangle,$$

we have an expression:

$$S_\Lambda = \left\{(a_1,\cdots,a_{24})\in S^\natural \mid a_i\in\{(00),(11)\}\right\}.$$

In particular, δ is equal to $\tau_{e_{2m-1}}\tau_{e_{2m}}$ for any $m=1,\cdots,24$. We note that V^+ is a $(D^\natural, S_\Lambda)$-framed VOA. Since $S_\Lambda^\perp$ is larger than $D^\natural$, we can construct an induced VOA

$$\tilde{V} = \mathrm{Ind}_{D^\natural}^{D_\Lambda}(V^+).$$

Since $(S_\Lambda)^\perp = D_\Lambda$, $\tilde{V}$ is a holomorphic VOA of rank 24 by Theorem 6.1. It follows from the direct calculation that the codewords of D_Λ of weight 2 are

$$\{(110^{46}),(00110^{44}),\cdots,(0^{46}11)\}.$$

We assert that $(\mathrm{Ind}_{D^\natural}^{D_\Lambda}(V^\natural)^\alpha)_1 = 0$ for $\alpha \neq 0$. Suppose false and assume $(\mathrm{Ind}_{D^\natural}^{D_\Lambda}(V^\natural)^\alpha)_1 \neq 0$ for some α. Then the weight of α is 16 and so α is one of $(1^{16}0^{32}),(0^{16}1^{16}0^{16}),(0^{32}1^{16})$, say $\alpha = (1^{16}0^{32})$. Since $(V^\natural)^\alpha$ is given by $\mathrm{Ind}_{D_{E_8}^{\oplus 3}}^{D^\natural}(V_{E_8}^{(1^{16})}\otimes M_{D_{E_8}+\xi_1}\otimes M_{D_{E_8}+\xi_1})$ and D_Λ does not contain any word of the form $(*\xi_1\xi_1)$, we have a contradiction. Consequently,

$$\mathcal{G}=(\tilde{V})_1=(M_{D_\Lambda})_1=\oplus_{\alpha\in D_\Lambda,|\alpha|=2}(M_\alpha)_1$$

is a commutative Lie algebra of rank 24 and $\tilde{\mathcal{G}}:=\left\langle(\tilde{V})_1\right\rangle$ is a VOA of free bosons of rank 24. We note that $\mathcal{G}$ has a PDIB-form $\langle\cdot,\cdot\rangle$ given by $v_{(1)}u=\langle v,u\rangle\mathbf{1}$ since $\tilde{V}$ has a PDIB-form. Hence $\mathbb{C}\tilde{V}$ is isomorphic to a lattice VOA $\mathbb{C}V_\Lambda$ of the Leech lattice Λ by [Mo]. More precisely, we will show the following lemmas in order to continue the proof of the theorem.

LEMMA 9.6. *$\tilde{V}$ is isomorphic to the lattice* VOA $\tilde{V}_\Lambda$ *of the Leech lattice* Λ *given in Proposition* 2.7. *In particular, one can choose a set of mutually orthogonal vectors* $\{x_1,\cdots,x_{24}\}$ *in* Λ *of squared length* 4 *such that every conformal vector* e_k *in a coordinate set of* $\tilde{V}$ *is written as*

$$e_{2j-i}=\tfrac{1}{16}((x_j)_{(-1)})^2\mathbf{1}+(-1)^i\tfrac{1}{4}(\iota(x_j)+\iota(-x_j))$$

for $j=1,\cdots,24$ *and* $i=0,1$ *by identifying* $\tilde{V}$ *and* $\tilde{V}_\Lambda$. *Moreover,*

$$(b_1b_1b_2b_2\cdots b_{24}b_{24})\in S_\Lambda$$

if and only if there is $(a_i)\in\mathbb{Z}^{24}$ *such that*

$$x=\tfrac{1}{2}\sum_{i=1}^{24}a_ix_i+\tfrac{1}{4}\sum_{i=1}^{24}b_ix_i\in\Lambda,$$

where $b_i\in\{0,1\}$ *denotes integers and binary words, by an abuse of notation.*

Proof. Set

$$W=\{v\in\tilde{V}\mid x_{(n)}v=0\text{ for all }x\in\mathcal{G}\text{ and }n>0\}.$$

Then the action of $\{x_{(0)}\mid x\in\mathcal{G}\}$ on $\mathbb{C}W$ is diagonalizable since $\mathcal{G}$ is commutative. Let L be the set of highest weights of $\tilde{\mathcal{G}}$-submodules of $\mathbb{C}W$ as a $\tilde{\mathcal{G}}$-module. It is easy to see that L is an even unimodular positive definite lattice without roots since $W_1=0$. Hence L is the Leech lattice Λ and $\mathbb{C}\tilde{V}\cong\mathbb{C}V_\Lambda$.

On the other hand, $\tilde{V}$ has a PDIB-form and it also has a $\mathbb{Z}_2$-grading

$$\tilde{V}=(V^\natural)^{\langle\delta\rangle}\oplus\tilde{V}^-$$

by the definition of induced VOAs, where $\tilde{V}^-=M_{(110^{46})+D^\natural}\times(V^\natural)^{\langle\delta\rangle}$. Let θ be an automorphism of $\mathbb{C}\tilde{V}$ defined by 1 on $\mathbb{C}(V^\natural)^{\langle\delta\rangle}$ and -1 on $\mathbb{C}\tilde{V}^-$. Now θ is acting on $\mathbb{C}(\tilde{V})_1$ as -1 and so we may assume that it is equal to an automorphism of $\mathbb{C}V_\Lambda$ induced from -1 on Λ by taking a conjugate. When $V=(V^\natural)^{\langle\delta\rangle}\oplus\sqrt{-1}\tilde{V}^-$, it is also a sub VOA of $\mathbb{C}\tilde{V}$. Let $\iota(x)$ denote a highest weight vector for $\tilde{\mathcal{G}}$ which lies in $\mathbb{C}\tilde{V}$ with highest weight $x\in\Lambda$. Namely, $u_{(0)}\iota(x)=\langle u,x\rangle\iota(x)$ for $u\in\mathcal{G}$. We note that $\theta(\iota(x))=(-1)^k\iota(x)$ for $\langle x,x\rangle=2k$. The space W spanned by highest weight vectors for $\tilde{\mathcal{G}}$ is a direct sum of irreducible $\mathcal{G}$-modules W^i whose dimensions are less than or equal to 2. If $\dim W^i=1$, then $\mathbb{C}W^i=\mathbb{C}\iota(x)$ for some $x\in\Lambda$. On the other hand, if $\dim W^i=2$, then $\mathbb{C}W^i=\mathbb{C}\iota(x)+\mathbb{C}\iota(y)$. Since W^i is irreducible, $\iota(x)$ and $\iota(y)$ are in the same homogeneous space $\mathbb{C}(\tilde{V})_k$ for some k. Since $\mathbb{C}\mathcal{G}=\mathbb{C}\tilde{V}_1\cong\mathbb{C}\Lambda$, we have $\mathbb{Z}x=\mathbb{Z}y$ and so $y=-x$. Hence W^i has a basis $\{a\iota(x)+b\iota(-x),c\iota(x)+d\iota(-x)\}$ for some $a,b,c,d\in\mathbb{C}$. We may assume that $a\in\mathbb{R}$. Since $\tilde{V}$ has a PDIB-form, we may also assume that $\{\frac{1}{\sqrt{2}}(a\iota(x)+b\iota(-x)),\frac{1}{\sqrt{2}}(c\iota(x)+d\iota(-x))\}$ is an orthonormal basis of W^i. Therefore $b=(-1)^k a^{-1}$, $d=(-1)^k c^{-1}$ and $ad+bc=(-1)^k(ac^{-1}+a^{-1}c)=0$. Hence $a^2=-c^2>0$ and we hence have $c=\sqrt{-1}a$ and $d=-\sqrt{-1}b$. Since $\mathbb{C}W^i=\mathbb{C}\iota(x)+\mathbb{C}\iota(-x)$ and $W^i=\mathbb{C}W^i\cap\tilde{V}$, θ keeps W^i invariant. Therefore $\theta(a\iota(x)+(-1)^k a^{-1}\iota(-x))=a^{-1}\iota(x)+(-1)^k a\iota(-x)\in W^i$, which implies $a=\pm1$. Hence $\iota(x)+(-1)^k\iota(-x)$, $\sqrt{-1}(\iota(x)-(-1)^k\iota(-x))\in W$ and $\sqrt{-1}x_{(0)}\mathbf{1}\in\mathcal{G}$ for $x\in\Lambda$. Consequently, $\tilde{V}$ coincides with the lattice VOA $\tilde{V}_\Lambda$ defined in Proposition 2.7 and V coincides with V_Λ.

We recall the structure $V_{\mathbb{Z}x}\cong L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)\oplus L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{1}{2},\frac{1}{2})$ and $(L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{1}{2},\frac{1}{2}))_1=\mathbb{R}\sqrt{-1}x_{(-1)}\mathbf{1}$ for a VOA $V_{\mathbb{Z}x}$ with $\langle x,x\rangle=4$. Since $(\tilde{V})_1=(M_{D_\Lambda})_1=\oplus_{i=1}^{24}(M_{\xi_{2i-1}+\xi_{2i}})_1$, we have

$$e_{2j}-e_{2j-1}\in W=\{v\in\tilde{V}|x_{(n)}v=\ 0\text{ for all }x\in(\tilde{V})_1\text{ and }n>0\}$$

and $\mathbb{R}(e_{2j}-e_{2j-1})+\sqrt{-1}\mathbb{R}(x_j)_{(0)}(e_{2j}-e_{2j-1})$ is an irreducible $\mathcal{G}$-submodule of L. Hence, by the arguments above, we have

$$e_{2j-i}=\tfrac{1}{16}((x_j)_{(-1)})^2\mathbf{1}+(-1)^i\tfrac{1}{4}(\iota(x_j)+\iota(-x_j))$$

for some $x_j\in\Lambda$. Since

$$0=(e_{2j-1}+e_{2j})_{(1)}(e_{2k}-e_{2k-1})=\tfrac{1}{64}\langle x_j,x_k\rangle^2(\iota(x_k)+\iota(-x_k))$$

for $k\neq j$, we have $\langle x_j,x_k\rangle=0$. Namely, $\{x_1,\cdots,x_{24}\}$ is a set of mutually orthogonal vectors of Λ with squared length 4. If $y=\sum_{i=1}^{24}c_ix_i\in\Lambda$, then $c_i\in\frac{1}{4}\mathbb{Z}$ since $\langle y,x_i\rangle\in\mathbb{Z}$. Assume that $y=\frac{1}{4}\sum b_ix_i$ is in Λ and set $U=V_{\langle x_1,\cdots,x_{24}\rangle+y}$ and $T^j=\langle e_{2j-1},e_{2j}\rangle$. As we showed in Section 2,

(1) $b_j\in1+2\mathbb{Z}$ if and only if an irreducible T^j-submodule of U is isomorphic to $L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{1}{2},\frac{1}{16})$. In particular, $(b_1b_1b_2b_2\cdots b_{24}b_{24})\in S_\Lambda$.

(2) $b_j\in2+4\mathbb{Z}$ if and only if an irreducible T^j-submodule of U is isomorphic to $L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{1}{2},0)$ or $L(\frac{1}{2},0)\otimes L(\frac{1}{2},\frac{1}{2})$.

(3) $b_i\in4\mathbb{Z}$ if and only if an irreducible T^j-submodule of U is isomorphic to $L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$ or $L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{1}{2},\frac{1}{2})$.

Conversely, if $\gamma=(b_1b_1b_2b_2\cdots b_{24}b_{24})\in S_\Lambda$, then $\tilde{\mathcal{G}}$ acts on $(\tilde{V}_\Lambda)^\gamma$ and so $(\tilde{V}_\Lambda)^\gamma\cap W\neq 0$. By the arguments above, there is an element $x\in\Lambda$ such that $\iota(x)\in\tilde{V}_\Lambda$ or $\iota(x)+(-1)^{|x|/2}\iota(-x)\in(\tilde{V}_\Lambda)^\gamma$. Hence there is a codeword $(a_1\cdots a_{24})\in\mathbb{Z}_2^{24}$ such that $x=\frac{1}{2}\sum a_ix_i+\frac{1}{4}\sum b_ix_i\in\Lambda$. $\square$

Lemma 9.7. *For any $y\in\Lambda$ with squared length 4, $\tau_{e(y)^+}=\tau_{e(y)^-}$ in $\mathrm{Aut}(V_\Lambda)$ and $\tau_{e(y)^+}\in\langle\pm1\rangle^{\oplus 24}\subseteq(\mathbb{R}^\times)^{\oplus 24}$.*

Proof. Since $Co.0$ acts on the set of all vectors in Λ with squared length 4 transitively, we may assume that $y=x_1$ and $e(y)^+=e_1$ and $e(y)^-=e_2$, where $\{x_1,\cdots,x_{24}\}$ is the set defined in the above lemma. By the arguments in the proof of the above lemma, it is clear that $\tau_{e(y)^+}=\tau_{e(y)^-}$. Since $\tau_{e_1}\iota(x)=(-1)^{\langle x_1,x\rangle}\iota(x)$ and $[\tau_{e_1},x_{(-n)}]=0$, we have $\tau_{e_1}\in\langle\pm1\rangle^{\oplus 24}$. $\square$

Returning to the proof of Theorem 9.5, we have $V_\Lambda\cong(V^\natural)^{\langle\delta\rangle}\oplus\sqrt{-1}\tilde{V}^-$. Let θ be an automorphism of V_Λ defined by 1 on $(V^\natural)^{\langle\delta\rangle}$ and -1 on $\sqrt{-1}\tilde{V}^-$. We identify $(V^\natural)^{\langle\delta\rangle}$ with V_Λ^θ. Let J be the set of all rational conformal vectors in $(V^\natural)^{\langle\delta\rangle}$ with central charge $\frac{1}{2}$. Set $G=\mathrm{Aut}(V^\natural)$, $K^\natural=\langle\tau_e\mid e\in J\rangle\subseteq\mathrm{Aut}(V^\natural)$, $K=\langle\tau_e\mid e\in J\rangle\subseteq\mathrm{Aut}((V^\natural)^{\langle\delta\rangle})$, $H=\mathrm{Aut}(V_\Lambda)$ and $K_\Lambda=\langle\tau_e\mid e\in J\rangle\subseteq\mathrm{Aut}(V_\Lambda)$. By Lemma 9.4, $H\cong(\mathbb{R}^{\times\oplus 24})Co.0$ and $C_H(\langle\theta\rangle)\cong 2^{24}Co.0$. ($Co.0$ does not imply a subgroup.) Clearly, $K^\natural\subseteq C_G(\langle\delta\rangle)$ and $K_\Lambda\subseteq C_H(\langle\theta\rangle)$.

By restricting automorphisms of $V^\natural$ and V_Λ to $(V^\natural)^{\langle\delta\rangle}$ and $V_\Lambda^{\langle\theta\rangle}$, respectively, we have epimorphisms $\pi^\natural: K^\natural\to K$ and $\pi_\Lambda: K_\Lambda\to K$. By [DM2], $\mathrm{Ker}(\pi^\natural)=\langle\delta\rangle$ and $\mathrm{Ker}(\pi_\Lambda)=\langle\theta\rangle\cap K_\Lambda$. So we have the following diagram.

$$
\begin{tikzcd}[arrows=dash]
G=\mathrm{Aut}(V^\natural) \arrow[d] & & & \mathrm{Aut}((V^\natural)^\delta) \arrow[dl] \arrow[dr] \arrow[ddd] & & & H=\mathrm{Aut}(V_\Lambda) \arrow[d] \\
C_G(\delta) \arrow[dd] \arrow[r] & & \overline{C_G(\delta)} \arrow[ddr] & & \overline{C_H(\theta)} \arrow[ddl] & & C_H(\theta) \arrow[dd] \arrow[ll] \\
 & & & & & & \\
K^\natural \arrow[d] \arrow[rrr] & & & K \arrow[d] & & & K_\Lambda \arrow[d] \arrow[lll] \\
\langle\delta\rangle & & & 1 & & & \langle\theta\rangle\cap K_\Lambda
\end{tikzcd}
$$

First, we will show that $K_\Lambda\not\subseteq 2^{24}\langle\theta\rangle$, where 2^{24} denotes the elementary normal abelian 2-subgroup $\langle\pm1\rangle^{\oplus 24}$ of $(\mathbb{R}^\times)^{\otimes 24}Co.0$. Let $g=(2,4)(6,8)(10,12)\cdots(46,48)\in S_{48}$. It is straightforward to check that g is an automorphism of $S^\natural$. By Lemma 9.3, there is an automorphism $\tilde{g}\in\mathrm{Aut}(V^\natural)$ such that $\tilde{g}(e_i)=e_{g(i)}$. Set $\delta'=\tau_{e_1}\tau_{e_4}$ $(=\tilde{g}(\delta))$ and $\tilde{L}'_\Lambda=g(\tilde{L}_\Lambda)$. By Lemma 9.7, there is a set of mutually orthogonal vectors $\{x_1, \cdots, x_{24}\}$ in Λ of squared length 4 such that

$$e_{2j-i} = \frac{1}{16}((x_j)_{(-1)})^2\iota(0) + (-1)^i\frac{1}{4}(\iota(x_j) + \iota(-x_j)).$$

It is easy to see that $\gamma = (0^8 1^8 0^8 1^8 0^8 1^8) \in S_\Lambda$. Since $((V^\natural)^\gamma)_2 \neq 0$, there is $y \in \Lambda$ of squared length 4 such that $\langle y, x_i\rangle \equiv 1 \pmod 2$ if and only if $i \in \mathrm{Supp}(\gamma)$. For each $y \in \Lambda$, $e^+(y) = \frac{1}{16}(y_{(-1)})^2\iota(0) + \frac{1}{4}(\iota(y) + \iota(-y))$ is a rational conformal vector in $(V_\Lambda)^{\langle\theta, \tau_{e_1}, \tau_{e_2}, \cdots, \tau_{e_8}\rangle}$. In particular, $g(e^+(y)) \in (V^\natural)^{\langle\delta\rangle}$. Since $\langle y, x_5\rangle \equiv 1$ (mod 2), we have $\tau_{e(y)}(\iota(\pm x_5)) = -\iota(\pm x_5)$ and so $\tau_{e(y)}$ exchanges e_9 and e_{10}. On the other hand, $\tilde{g}$ fixes e_9 and exchanges e_{10} and e_{12}. Hence $\tau_{\tilde{g}}(e(y))$ exchanges e_9 and e_{12} and hence $\tau_{\tilde{g}(e(y))}$ does not belong to $2^{24}\langle\theta\rangle$. Hence $K_\Lambda \not\subseteq 2^{24}\langle\theta\rangle$.

Since K_Λ is generated by all automorphisms given by conformal vectors in $(V_\Lambda)^{\langle\theta\rangle}$, K_Λ is a normal subgroup of $C_H(\langle\theta\rangle) \cong 2^{24}Co.0$ and so we have $K_\Lambda = C_H(\langle\theta\rangle)$. Consequently, $K \cong 2^{24}Co.1$, $K^\natural = O_2(K^\natural)Co.1$ and $O_2(K^\natural)$ is of order 2^{25}, where $O_2(G)$ denotes the maximal normal 2-subgroup of G. If $O_2(K^\natural)$ is an abelian 2-group, then $O_2(K^\natural)$ is an elementary 2-group of order 2^{25} and decomposes into $\langle\delta\rangle \oplus N$ as a $Co.1$-module. Let y be a vector of Λ of squared length 4 satisfying $\langle y, x_{24}\rangle = 1$. Then $e^\pm(y) \in (V^\natural)^{\langle\delta\rangle}$ and $\tau_{e^+(y)}$ fixes $\delta = \tau_{e_1}\tau_{e_2} = \tau_{e_{47}}\tau_{e_{48}}$ and exchanges e_{47} and e_{48}. By Lemma 9.7, $\tau_{e_{47}}, \tau_{e_{48}} \in O_2(K^\natural)$. Since $\delta = \tau_{e_{47}}\tau_{e_{48}}$, we may assume $e_{47} \in N$ and $e_{48} \not\in N$, which contradicts that $\tau_{e(y)}$ exchanges e_{47} and e_{48}. Hence $O_2(K^\natural)$ is not abelian and hence $O_2(K^\natural)$ is isomorphic to a central extension of $\Lambda/2\Lambda$ given by the inner product of $\Lambda/2\Lambda$, since $Co.1$ acts on $O_2(K^\natural)/\langle\delta\rangle$ faithfully. That is, $O_2(K^\natural)$ is an extra-special 2-group of order 2^{25}, which is denoted by 2^{1+24}. By Lemma 9.3, $\mathrm{Aut}(V^\natural)$ contains a subgroup whose restriction on $\{e_1, \cdots, e_{48}\}$ is isomorphic to $\mathrm{GL}(5,2)_1 \times \Sigma_3$, where Σ_3 is the symmetric group on three letters and permutes three components of $V_{E_8}^{\otimes 3}$, and $\mathrm{GL}(5,2)_1$ denotes

$$\{A \in \mathrm{GL}(5,2) \mid A\,{}^t(10000) = {}^t(10000)\}.$$

Set $\delta_1 = \tau_{e_1}\tau_{e_3}$ and $B^2 = \langle\delta, \delta_1\rangle$. Denote δ and $\delta\delta_1$ by δ_0 and δ_2, respectively. Since a subgroup of $\mathrm{GL}(5,2)_1$ acts on $\{\delta_0, \delta_1, \delta_2\}$ transitively and e_3 is given by a vector of Λ of squared length 4, we have $N_{\mathrm{Aut}(V^\natural)}(B^2) \cong 2^{2+12+22}(\Sigma_3 \times M_{24})$ from the structure of $C_{\mathrm{Aut}(V^\natural)}(\delta) \cong 2^{1+24}Co.1$. Similarly, all nontrivial elements of $B^3 = \langle\tau_{e_1}\tau_{e_2}, \tau_{e_1}\tau_{e_3}, \tau_{e_1}\tau_{e_5}\rangle$ are conjugate by the actions of $\mathrm{GL}(5,2)_1 \subseteq \mathrm{Aut}(V^\natural)$ and so $N_{\mathrm{Aut}(V^\natural)}(B^3) \cong 2^{3+6+12+18}(3\Sigma_6 \times \mathrm{PSL}(3,2))$. By the same arguments, we can calculate the normalizer of $B^4 = \langle\tau_{e_1}\tau_{e_2}, \tau_{e_1}\tau_{e_3}, \tau_{e_1}\tau_{e_5}, \tau_{e_1}\tau_{e_9}\rangle$. We leave these calculation to the reader.

We will next prove that $\mathrm{Aut}(V^\natural)$ is a simple group. If H is a nontrivial minimal normal subgroup of $\mathrm{Aut}(V^\natural)$, then $C_H(\delta_i)$ is a normal subgroup of $C(\delta_i) = 2^{1+24}Co.1$ for $i = 0, 1, 2$. Hence $C_H(\delta_i) = 2^{1+24}Co.1$ or $C_H(\delta_i) = 2^{1+24}$ or $C_H(\delta_i) = \langle\delta_i\rangle$. We note that $\delta_i (i = 0, 1, 2)$ are conjugate to each other in $\mathrm{Aut}(V^\natural)$ and hence $C_H(\delta_i) \cong C_H(\delta_0)$ for $i = 1, 2$. In any case, $\delta_i \in H$ and so $C_H(\delta_i) \neq \langle\delta_i\rangle$ since $\delta_j \in \langle C_H(\delta_i) \mid i = 1, 2, 3\rangle = H$. If $C_H(\delta_1) = 2^{1+24}$ then $P := C_H(\delta_1)$ is a

Sylow 2-subgroup of H. Since $|P : C_P(\delta_2)| = 2$ and $C_P(\delta_2)$ is not abelian, we have $[C_P(\delta_2), C_P(\delta_2)] = \langle\delta_1\rangle$, which contradicts $[C_H(\delta_2), C_H(\delta_2)] = \langle\delta_2\rangle$. Therefore we have $C_H(\delta_i) = 2^{1+24}Co.1$. Since $\langle\delta_i\rangle$ is a characteristic subgroup of a Sylow 2-subgroup of H, we have $H = \mathrm{Aut}(V^\natural)$ and hence $\mathrm{Aut}(V^\natural)$ is a simple group. By the characterization of the Monster simple group and the above facts, we know that $\mathrm{Aut}(V^\natural)$ is the Monster simple group; see [I], [S], [T]. $\square$

As shown above, $V^\natural$ is a holomorphic VOA with rank 24 with $(V^\natural)_1 = 0$ and the Monster simple group $\mathbb{M}$ acts on $B = V_2^\natural$ faithfully. Since the $\mathbb{M}$-invariant commutative algebraic structure on a vector space of dimension 196884 B is unique, B is isomorphic to the Griess algebra constructed in [Gr]. We have also proved that $(V^\natural)^\delta$ is isomorphic to $(\tilde{V}_\Lambda)^\theta$, which means that $V^\natural$ is a VOA given by a $\mathbb{Z}_2$-orbifold construction from the Leech lattice VOA $\tilde{V}_\Lambda$. Hence $V^\natural$ is equal to the moonshine module VOA constructed in [FLM2].

10. Holomorphic VOAs

In this section, we will construct an infinite series of holomorphic VOAs whose full automorphism groups are finite. We will adopt the notation from Section 7 and repeat the similar constructions as in Section 7.

For $n = 1, 2, \cdots$, set

$$S^\natural(n) = \left\langle (\{0^{16}\}^i 1^{16} \{0^{16}\}^{2n-i}), (\{\alpha\}^{2n+1}) \mid \alpha \in S_{E_8}, i = 1, \cdots, 2n \right\rangle.$$

$S^\natural(n)$ is an even linear code of length $16+32n$ and $(S^\natural(n))^\perp$ contains a direct sum $(D_{E_8})^{\oplus 2n+1}$ of $2n+1$ copies of D_{E_8} for each n. When γ is an element of $S^\natural(n)$, then there is $\alpha \in S_{E_8}$ such that

$$\gamma = (\beta_1, \cdots, \beta_{2n+1}),$$

where $\beta_i \in \{\alpha, \alpha^c\}$. We may assume that the number of β_i satisfying $\beta_i = \alpha$ is odd. Set

$$W^\gamma = \otimes_{i=1}^{2n+1} \tilde{W}^{\beta_i},$$

where

$$\tilde{W}^{\beta_i} = {V_{E_8}}^\alpha \quad \text{if } \beta_i = \alpha$$

and

$$\tilde{W}^{\beta_i} = R{V_{E_8}}^{\alpha^c} \quad \text{if } \beta_i = \alpha^c.$$

Set

$$V^3(n) = \bigoplus_{\gamma \in S^\natural(n)} W^\gamma$$

and

$$V^\natural(n) = \mathrm{Ind}_{(D_{E_8})^{\oplus 2n+1}}^{(S^\natural(n))^\perp} (V^3(n)).$$

Then we can show that $V^\natural(n)$ has a framed VOA structure by exactly the same proof as in the construction of $V^\natural$. It also satisfies $(V^\natural(n))_1=0$. Moreover, it is a holomorphic VOA by Theorem 6.1 and its full automorphism group is finite by Theorem 9.2.

11. Characters

In this section, we will calculate the characters of the $3C$ element and the $2B$ element of the Monster simple group. By Lemma 9.3, we are able to induce an automorphism of $D^\natural$ into an automorphism of $V^\natural$.

11.1. $3C$. Clearly, $g=(1,17,33)(2,18,34)\cdots(16,32,48)$ is an automorphism of $D^\natural$. Let $\tilde{g}$ be an automorphism of $V^\natural$ induced from g. By the definition, $\tilde{g}$ acts on $\{e_i \mid i=1,\cdots,48\}$ as $(1,17,33)(2,18,34)\cdots(16,32,48)$.

In this subsection, we denote D_{E_8} by D. $V^\natural$ contains $M_{D^{\oplus 3}}=M_D\otimes M_D\otimes M_D$. We view $V^\natural$ as an $M_D\otimes M_D\otimes M_D$-module. Since $\tilde{g}$ permutes $\{V^\chi \mid \chi\in S^\natural\}$, we obtain

$$\begin{aligned}\mathrm{ch}_{V^\natural}(g,z) &= \mathrm{tr}_{g,z}(V^\natural)\\ &= \mathrm{tr}_{g,z}\left(\bigoplus_{\chi^g=\chi\in S^\natural} V^\chi\right)\\ &= \mathrm{tr}_{g,z}\left(\bigoplus_{\alpha\in D_{E_8}} V^{(\alpha,\alpha,\alpha)}\right),\end{aligned}$$

where $\mathrm{tr}_{g,z}(V)=\sum_{m\in\mathbb{Z}}\mathrm{tr}(\tilde{g})_{|V_m}e^{2\pi imz}$ for $V=\oplus_{m\in\mathbb{Z}}V_m$.

By the definition of $V^{(\alpha,\alpha,\alpha)}$,

$$V^{(\alpha,\alpha,\alpha)}=\mathrm{Ind}_{D^{\oplus 3}}^{D^\natural}({V_{E_8}}^\alpha\otimes {V_{E_8}}^\alpha\otimes {V_{E_8}}^\alpha).$$

It follows from the definition of induced modules that

$$\mathrm{Ind}_{D^{\oplus 3}}^{D^\natural}(U)\cong\bigoplus_{\mu\in D^\natural/D^{\oplus 3}} M_{D^{\oplus 3}+\mu}\times U$$

as $M_{D^{\oplus 3}}$-modules. Since $D^\natural=\{(\alpha,\beta,\gamma)\mid \alpha+\beta+\gamma\in D, \alpha,\beta,\delta \text{ even }\}$, we obtain $\tilde{g}(D^{\oplus 3}+\mu)=D^{\oplus 3}+\mu$ if and only if $\mu\in D^{\oplus 3}$. Hence

$$\begin{aligned}\mathrm{tr}_{\tilde{g},z}(V^{(\alpha,\alpha,\alpha)}) &= \mathrm{tr}_{\tilde{g},z}({V_{E_8}}^\alpha\otimes {V_{E_8}}^\alpha\otimes {V_{E_8}}^\alpha)\\ &= \mathrm{tr}_{1,3z}({V_{E_8}}^\alpha).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{ch}_{V^\natural}(\tilde{g},z) &= \sum_{\alpha\in D_{E_8}}\mathrm{tr}_{1,3z}({V_{E_8}}^\alpha)\\ &= \mathrm{tr}_{1,3z}(V_{E_8})=\mathrm{ch}_{V_{E_8}}(1,3z).\end{aligned}$$

11.2. 1 *and* 2*B*. Let $\delta = \tau_{e_1}\tau_{e_2}$. We proved that $(V^\natural)^{\langle\delta\rangle}$ is isomorphic to $(V_\Lambda)^{\langle\theta\rangle}$. Hence

$$\mathrm{ch}\ ((V^\natural)^{\langle\delta\rangle}) = 1 + 98580q^2 + \cdots .$$

So we will calculate the character of $(V^\natural)^- = \{v \in V^\natural \mid \delta(v) = -v\}$. It follows from the definition of τ_{e_i} that

$$\mathrm{ch}\ ((V^\natural)^-) = \sum_{\langle\chi,(110^{46})\rangle=1} \mathrm{ch}\ ((V^\natural)^\chi).$$

Set $\chi = (\alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma \in \mathbb{Z}_2^{16}$. Assume $\langle\chi, (110^{46})\rangle = 1$. Then the weight of α is 8 and so the weight of χ is 24 since $\chi \in S^\natural$. Consequently, $\dim D^\natural_\chi = 7+7+4$ and hence the multiplicity of every irreducible T-submodule of $(V^\natural)^\chi$ is 2^6. Let U be an irreducible T-submodule of $(V^\natural)^\chi$. It follows from the total degree that the number of $L(\frac{1}{2}, \frac{1}{2})$ in $U = \otimes_{i=1}^{48} L(\frac{1}{2}, h_i)$ is odd. On the other hand, let γ be an odd word with $\mathrm{Supp}(\gamma) \cap \mathrm{Supp}(\chi) = \emptyset$. By the action of $M_{D^\natural}$, there exists an irreducible T-submodule isomorphic to $\otimes_{i=1}^{48} L(\frac{1}{2}, h_i)$ with $h_i = \frac{1}{2}$ for $i \in \mathrm{Supp}(\gamma)$, $h_i = \frac{1}{16}$ for $i \in \mathrm{Supp}(\chi)$ and $h_i = 0$ for $i \notin \mathrm{Supp}(\chi+\gamma)$. Hence

$$\begin{aligned}
\mathrm{ch}((V^\natural)^\chi) &= 2^6 \mathrm{ch}\ \{L(\tfrac{1}{2}, \tfrac{1}{16})^{\otimes 24} \tfrac{1}{2}((L(\tfrac{1}{2}, 0) + L(\tfrac{1}{2}, \tfrac{1}{2}))^{\otimes 24} - (L(\tfrac{1}{2}, 0) - L(\tfrac{1}{2}, \tfrac{1}{2}))^{\otimes 24})\} \\
&= 32q^{3/2} \prod_{n\in\mathbb{N}} (1+q^n)^{24} \left(\prod_{n\in\mathbb{N}+\frac{1}{2}} (1+q^n)^{24} - \prod_{n\in\mathbb{N}+\frac{1}{2}} (1-q^n)^{24} \right).
\end{aligned}$$

Since there are 64 codewords χ such that $\langle\chi, (110^{46})\rangle = 1$, we have

$$\begin{aligned}
\mathrm{ch}((V^\natural)^-) &= 2^{11} q^{3/2} \prod_{n\in\mathbb{N}} (1+q^n)^{24} \left(\prod_{n\in\mathbb{N}+\frac{1}{2}} (1+q^n)^{24} - \prod_{n\in\mathbb{N}+\frac{1}{2}} (1-q^n)^{24} \right) \\
&= 2^{11} q^{3/2} (1 + 24q + \cdots)(48q^{1/2} + \cdots) \\
&= 2^{12}(24q^2 + \cdots).
\end{aligned}$$

In particular, we obtain $(V^\natural)_1 = 0$ and $\dim (V^\natural)_2 = 196884$.

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Institute of Mathematics, University of Tsukuba, Tsukuba, Japan
E-mail address: miyamoto@math.tsukuba.ac.jp

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