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On planar web geometry through abelian relations and connections

By Alain Hénaut

1. Introduction

Web geometry is devoted to the study of families of foliations which are in general position. We restrict ourselves to the local situation, in the neighborhood of the origin in $\mathbb{C}^2$, with $d \geq 1$ complex analytic foliations of curves in general position. We are interested in the *geometry* of such configurations, that is, properties of planar d-webs which are invariant with respect to analytic local isomorphisms of $\mathbb{C}^2$.

The initiators of the subject are W. Blaschke, G. Thomsen and G. Bol in the 1930's (cf. [B-B], [B] and for instance [H1]). Methods used here extend some works by S. S. Chern and P. A. Griffiths (cf. for instance [G1], [G2], [C], [C-G]) which bring a resurgence of interest in web geometry closely related to basic results due to N. Abel, S. Lie, H. Poincaré and G. Darboux. For recent results and applications of web geometry in various domains, refer to I. Nakai's introduction, all papers and references contained in [W].

Let $\mathcal{O} := \mathbb{C}\{x, y\}$ be the ring of convergent power series in two variables. A (germ of a) nonsingular *d-web* $\mathcal{W}(d)$ in $(\mathbb{C}^2, 0)$ is defined by a family of *leaves* which are germs of level sets $\{F_i(x, y) = const.\}$ where $F_i \in \mathcal{O}$ can be chosen to satisfy $F_i(0) = 0$ such that $dF_i(0) \wedge dF_j(0) \neq 0$ for $1 \leq i < j \leq d$ from the assumption of general position.

From the local inverse theorem, the study of possible configurations for the different $\mathcal{W}(d)$ is interesting only for $d \geq 3$. The classification of such $\mathcal{W}(d)$ is a widely open problem and the search for invariants of planar webs $\mathcal{W}(d)$ motivates the present work.

Let $F(x, y, p) = a_0(x, y) \,.\, p^d + a_1(x, y) \,.\, p^{d-1} + \cdots + a_d(x, y)$ be an element of $\mathcal{O}[p]$ without multiple factor, not necessarily irreducible and such that $a_0 \neq 0$. We denote by $R = (-1)^{\frac{d(d-1)}{2}} a_0 \,.\, \Delta$ the p-resultant of F where $\Delta \in \mathcal{O}$ is its p-discriminant.

In a neighborhood of $(x_0, y_0) \in \mathbb{C}^2$ such that $R(x_0, y_0) \neq 0$, the Cauchy theorem asserts that the d integral curves of the differential equation of the first order

$$F(x, y, y') = 0$$

are the leaves of a nonsingular web $\mathcal{W}(d)$ in $(\mathbb{C}^2, (x_0, y_0))$.

Every such $F \in \mathcal{O}[p]$, up to an invertible element in $\mathcal{O}$, gives rise to an *implicit* d-web $\mathcal{W}(d)$ in $(\mathbb{C}^2, 0)$ which is generically nonsingular. Inversely, if a nonsingular d-web in $(\mathbb{C}^2, 0)$ is given by d vector fields $X_i = A_i \partial_x + B_i \partial_y$ in general position, one may assume that $A_i(0) \neq 0$ for $1 \leq i \leq d$ after a linear change of coordinates. Then "its" differential equation $F(x, y, y') = 0$ corresponds to $F(x, y, p) = \prod_{i=1}^{d} (A_i p - B_i)$.

This implicit form of a planar web will be retained throughout the present text. No leaf is preferred and we shall show how this form presents a natural setting for the study of planar webs and their singularities. Moreover, with the help of the web viewpoint, this approach enlarges methods to investigate the geometry of the differential equation $F(x, y, y') = 0$.

Basic examples of planar webs come from complex projective algebraic geometry. Let $C \subset \mathbb{P}^2$ be a reduced algebraic curve of degree d, not necessarily irreducible and possibly singular. By duality in $\check{\mathbb{P}}^2$, one can get a special linear d-web $\mathcal{L}_C(d)$ called the *algebraic web* associated with $C \subset \mathbb{P}^2$ (cf. for instance [H1] for details). This web is singular and its leaves are family of straight lines. It corresponds, in a suitable local coordinate system, to a differential equation of the previous form given by $F(x, y, p) = P(y - px, p)$ if $P(s, t) = 0$ is an affine equation for C. If C contains no straight lines, the leaves of $\mathcal{L}_C(d)$ are *generically* the tangents of the dual curve $\check{C} \subset \check{\mathbb{P}}^2$ of $C \subset \mathbb{P}^2$; otherwise, they belong to the corresponding pencils of straight lines.

One of the main invariants of a nonsingular planar web $\mathcal{W}(d)$ is related to the notion of abelian relation. A d-uple $\big(g_1(F_1), \ldots, g_d(F_d)\big) \in \mathcal{O}^d$ satisfying

$$\sum_{i=1}^{d} g_i(F_i) dF_i = 0$$

where $g_i \in \mathbb{C}\{t\}$ is called an *abelian relation* of $\mathcal{W}(d)$. By the above component presentation these relations form a $\mathbb{C}$-vector space denoted by $\mathcal{A}(d)$.

For a nonsingular web $\mathcal{W}(d)$ in $(\mathbb{C}^2, 0)$, the following *optimal inequality* holds:

$$\operatorname{rk} \mathcal{W}(d) := \dim_{\mathbb{C}} \mathcal{A}(d) \leq \frac{1}{2}(d-1)(d-2).$$

This bound is classic and, for example, we will recover it below with new methods coming from basic results in $\mathcal{D}$-modules theory (cf. for instance [G-M]). The integer $\operatorname{rk} \mathcal{W}(d)$ called the *rank* of $\mathcal{W}(d)$ defined above is an invariant of $\mathcal{W}(d)$ which does not depend on the choice of the functions F_i.

From the previous observations and properties, another basic result in planar web geometry is related to *linear* webs $\mathcal{L}(d)$ (i.e. all leaves of $\mathcal{L}(d)$ are straight lines, not necessarily parallel). For a linear and nonsingular web $\mathcal{L}(d)$ in $(\mathbb{C}^2, 0)$, the following assertions are *equivalent*:

i) There exists an abelian relation $\sum_{i=1}^{d} g_i(F_i)dF_i = 0$ with $g_i \neq 0$ for $1 \leq i \leq d$;

ii) The linear web $\mathcal{L}(d)$ is algebraic; that is, $\mathcal{L}(d) = \mathcal{L}_C(d)$ where $C \subset \mathbb{P}^2$ is a reduced algebraic curve of degree d, not necessarily irreducible and possibly singular;

iii) The rank of $\mathcal{L}(d)$ is maximal.

These equivalences play a fundamental role in the foundation of web geometry. Indeed, the implication ii) $\Rightarrow$ iii) is a special case of Abel's theorem and asserts that in fact

$$\operatorname{rk} \mathcal{L}_C(d) = \dim_{\mathbb{C}} H^0(C, \omega_C) = \frac{1}{2}(d-1)(d-2)$$

(cf. for instance [H1]). The difficult part i) $\Rightarrow$ ii) is a kind of converse to Abel's theorem. In the case $d = 4$, it was initiated by Lie's theorem on surfaces of double translation (cf. for instance [C]) and deeply generalized, for $d \geq 3$ and higher codimension questions, by P. A. Griffiths (cf. [G1]). All modern proofs of this implication use the so-called GAGA principle.

Using only the methods introduced here we will get a proof for the above equivalence ii) $\Leftrightarrow$ iii) and some complements essentially based on partial differential equations and the canonical normalization of $\mathcal{W}(d)$. In particular, these results explain why one condition alone implies all the previous equivalences.

This normalization gives rise to several analytic *invariants* of $\mathcal{W}(d)$ on $(\mathbb{C}^2, 0)$, where $d(d-3)$ of them are functions and the remaining $d-2$ are 2-differential forms. These invariants extend the Blaschke curvature for $\mathcal{W}(3)$ and should be worth studying. A part of their significance will appear below.

Web geometry for nonsingular planar webs of maximum rank is, however, larger in extent than the algebraic geometry of plane curves. Indeed, there exist *exceptional* webs $\mathcal{E}(d)$ in $(\mathbb{C}^2, 0)$. Such a web $\mathcal{E}(d)$ is of maximum rank and cannot be made algebraic, up to an analytic local isomorphism of $\mathbb{C}^2$. One knows that necessarily $d \geq 5$ and the first known example is Bol's 5-web $\mathcal{B}(5)$ which is related to the functional relation with five terms satisfied by the dilogarithm (cf. [Bo]). For special models in web geometry and their functional relations as well, a program to study polylogarithm webs is sketched in [H1]. The next exceptional web expected was Kummer's 9-web $\mathcal{K}(9)$ related to the functional relation with nine terms of the trilogarithm. G. Robert proved in

[R] that this 9-web is indeed exceptional and he found “on the road” some others $\mathcal{E}(d)$ (cf. also L. Pirio’s paper [P]).

A refinement of the rank is the finer invariant $(\varrho_3, \dots, \varrho_d)$ called the *weave* of a nonsingular planar web $\mathcal{W}(d)$. This sequence of nonnegative integers is defined as follows: in the $\mathbb{C}$-vector space $\mathcal{A}(d)$ of abelian relations of $\mathcal{W}(d)$, consider the ascending chain of subspaces

$$\mathcal{A}(d)_3 \subseteq \mathcal{A}(d)_4 \subseteq \dots \subseteq \mathcal{A}(d)_d = \mathcal{A}(d)$$

where $\mathcal{A}(d)_k$ is generated by special abelian relation $\big(g_1(F_1), \dots, g_d(F_d)\big)$ of $\mathcal{W}(d)$ containing at most k nonzero components. Then set

$$\varrho_k := \dim_{\mathbb{C}} \mathcal{A}(d)_k / \mathcal{A}(d)_{k-1}$$

with $\mathcal{A}(d)_2 = 0$. In particular, we have $\operatorname{rk} \mathcal{W}(d) = \varrho_3 + \dots + \varrho_d$. For example, the weave of $\mathcal{B}(5)$ is $(5, 0, 1)$ and that of $\mathcal{K}(9)$ is $(17, 3, 3, 3, 0, 0, 2)$. In the algebraic case, the weave of $\mathcal{L}_C(d)$ is related to the irreducible components of $C \subset \mathbb{P}^2$.

According to the previous results, methods for determining the rank (resp. the weave) of any nonsingular planar web are of great interest, in particular for the algebraization problem (cf. for instance [H1] through the second order differential equation $y'' = P_{\mathcal{W}(d)}(x, y, y')$ associated to $\mathcal{W}(d)$) and the study of exceptional webs.

Let S be the surface defined by $F(x, y, p) = 0$. The projection $\pi : S \longrightarrow (\mathbb{C}^2, 0)$ induced by $(x, y, p) \longmapsto (x, y)$ is generically finite with degree d and gives rise to a *trace* which is very useful on differential forms.

Coming back to the classical geometric study of differential equations $F(x, y, y') = 0$, we shall confirm how some basic objects attached to the previous projection govern the geometry of the planar web associated with this equation, from the generic viewpoint as well as the singular one. In fact, even if we restrict our attention to the nonsingular case, most of the objects introduced naturally extend to the singular case.

We suppose from now on that the p-resultant $R \in \mathcal{O}$ of F satisfies $R(0) \neq 0$. Thus π is a covering map of degree d. The main result in [H2] will be recalled with some details in the next paragraph. Briefly, it is the following: the $\mathbb{C}$-vector space of 1-forms

$$\mathfrak{a}_F := \Big\{ \omega = r \cdot \frac{dy - p\,dx}{\partial_p(F)} \in \pi_*(\Omega^1_S) \; ; \; r \in \mathcal{O}[p] \text{ with } \deg r \leq d-3 \text{ and } d\omega = 0 \Big\}$$

is identified with the $\mathbb{C}$-vector space $\mathcal{A}(d)$ of abelian relations of the web $\mathcal{W}(d)$ generated by F. In this identification an abelian relation is interpreted as the vanishing trace of an element of $\mathfrak{a}_F$. By definition the forms in $\mathfrak{a}_F$ are closed and moreover appear as solutions of a linear differential operator $p_0 : J_1(\mathcal{O}^{d-2}) \longrightarrow \mathcal{O}^{d-1}$ of order 1 induced by the usual differential on 1-forms of the surface S.

Using basic results on overdetermined systems of linear partial differential equations which extend the É. Cartan theory (cf. for instance [S], [B-C-3G]) and in particular the first complex of Spencer of an explicit prolongation $p_k : J_{k+1}(\mathcal{O}^{d-2}) \longrightarrow J_k(\mathcal{O}^{d-1})$ of p_0, we obtain in the last paragraph one of the main results of this paper:

There exists a $\mathbb{C}$*-vector fiber bundle* $\mathcal{E}$ *of rank* $\frac{1}{2}(d-1)(d-2)$ *on* $(\mathbb{C}^2, 0)$ *equipped with a connection* ∇ *such that its* $\mathbb{C}$*-vector space of horizontal sections is isomorphic to* $\mathcal{A}(d)$*. Moreover, there exists an adapted basis* (e_ℓ) *of* $\mathcal{E}$ *such that the curvature of* $(\mathcal{E}, \nabla)$ *has the following matrix*:

$$\begin{pmatrix} k_1 & k_2 & \dots & k_{\frac{1}{2}(d-1)(d-2)} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} dx \wedge dy.$$

In particular, by the Cauchy-Kowalevski theorem, an explicit way to find maximal rank webs is given, using only the coefficients of F. In the case $d = 3$, we find $k_1\, dx \wedge dy$ as a curvature matrix and it is proved that this 2-form is the usual Blaschke curvature of $\mathcal{W}(3)$ (cf. [B-B], [B] and for instance [H1]).

Moreover complete effective results are given for $d = 3$ and $d = 4$. The previous curvature probably depends *only* on the planar web $\mathcal{W}(d)$ and not on the differential equation $F(x, y, y') = 0$ that we use to define it. It is at least true for $d = 3$ and $d = 4$. Thus, the construction of the above $(\mathcal{E}, \nabla)$ generalizes the W. Blaschke approach.

For a general linear web some simplifications appear in the description of $(\mathcal{E}, \nabla)$ and from the above results some of the previous equivalences for the $\mathcal{L}(d)$ are obtained as well as several complements.

Furthermore, it can be noted to close this introduction that in general the previous $(\mathcal{E}, \nabla)$ is in fact a meromorphic connection with poles on the *discriminant locus* of the differential equation $F(x, y, y') = 0$, that is, the analytic germ defined in a neighborhood of $0 \in \mathbb{C}^2$ by $\Delta(x, y) = 0$.

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2. Traces from S, abelian relations and canonical normalisation for $\mathcal{W}(d)$

We recall that $R(0) \neq 0$. Thus, the surface S defined by F is nonsingular over $0 \in \mathbb{C}^2$. Locally on S, we have the complex $(\Omega_S^\bullet, d)$ where

$$\Omega_S^\bullet = \Omega_{\mathbb{C}^3}^\bullet / (dF \wedge \Omega_{\mathbb{C}^3}^{\bullet - 1}, F\Omega_{\mathbb{C}^3}^\bullet).$$

Since $\partial_x(F)\,dx + \partial_y(F)\,dy + \partial_p(F)\,dp = 0$ in Ω^1_S, every element ω in Ω^1_S gives rise to an expression

$$\omega := \frac{r_x\,dy - r_y\,dx}{\partial_p(F)}$$

with $(r_x, r_y, r_p, \theta) \in \mathcal{O}^4_S$ such that the relation $r_x\,\partial_x(F) + r_y\,\partial_y(F) + r_p\,\partial_p(F) = \theta.F$ holds. Inversely the previous expression coupled with this relation corresponds to an element in Ω^1_S essentially defined through

$$\omega = \frac{1}{3}\cdot\big\{\big(\frac{r_p}{\partial_y(F)} - \frac{r_y}{\partial_p(F)}\big)dx + \big(\frac{r_x}{\partial_p(F)} - \frac{r_p}{\partial_x(F)}\big)dy + \big(\frac{r_y}{\partial_x(F)} - \frac{r_x}{\partial_y(F)}\big)dp\big\}$$

because $\dfrac{r_x\,dy - r_y\,dx}{\partial_p(F)} = \dfrac{r_y\,dp - r_p\,dy}{\partial_x(F)} = \dfrac{r_p\,dx - r_x\,dp}{\partial_y(F)}$ in Ω^1_S.

Moreover, it can be checked that the exterior differential $d : \Omega^1_S \longrightarrow \Omega^2_S$ is defined by

$$d\omega = d\big(\frac{r_x\,dy - r_y\,dx}{\partial_p(F)}\big) = \big(\partial_x(r_x) + \partial_y(r_y) + \partial_p(r_p) - \theta\big)\frac{dx\wedge dy}{\partial_p(F)}$$

because $\dfrac{dx\wedge dy}{\partial_p(F)} = \dfrac{dy\wedge dp}{\partial_x(F)} = \dfrac{dp\wedge dx}{\partial_y(F)}$ in Ω^2_S.

The projection $\pi : S \longrightarrow (\mathbb{C}^2, 0)$ is a covering map of degree d with local branches $\pi_i(x, y) = (x, y, p_i(x, y))$. Thus, we have

$$F(x, y, p) = a_0(x, y)\,.\prod_{i=1}^{d}\big(p - p_i(x, y)\big).$$

Moreover, the vector fields which correspond to the nonsingular d-web $\mathcal{W}(d)$ of $(\mathbb{C}^2, 0)$ generated by the differential equation $F(x, y, y') = 0$ have the form

$$X_i := \partial_x + p_i\partial_y \text{ with } p_i(0) \neq p_j(0) \text{ for } 1 \leq i < j \leq d.$$

We denote by $\pi_*(\Omega^1_S)$ the fiber in $0 \in \mathbb{C}^2$ of the direct image sheaf of Ω^1_S with respect to π. We have the trace morphism $\mathrm{Trace}_\pi : \pi_*(\Omega^1_S) \longrightarrow \Omega^1$ defined by $\mathrm{Trace}_\pi(\omega) := \displaystyle\sum_{i=1}^{d} \pi_i^*(\omega)$ where Ω^1 is the $\mathcal{O}$-module of Pfaff forms on $(\mathbb{C}^2, 0)$. This morphism is $\mathcal{O}$-linear and commutes with the differential d. It can be noted that a large part of the previous constructions extends to the singular case by means of the Barlet complex $(\omega^\bullet_S, d)$ constructed *via* special meromorphic forms with poles on the singular set of S (cf. [Ba]).

The following result is proved in [H2]: *every* $r \in \mathcal{O}[p]$ *such that* $\deg r \leq d-2$ *gives an element* $\omega = r\cdot\dfrac{dy - p\,dx}{\partial_p(F)}$ *which belongs to* $\pi_*(\Omega^1_S)$.

More precisely, there exist elements r_p and t in $\mathcal{O}[p]$ with degree less than or equal to $d-1$ which satisfy the following fundamental relation:

$$(\star)\qquad r\,.\big(\partial_x(F) + p\,\partial_y(F)\big) + r_p\,.\,\partial_p(F) = \big(\partial_x(r) + p\,\partial_y(r) + \partial_p(r_p) - t\big).\,F.$$

Omitting the dependency on (x, y), the proof uses the ubiquitous Lagrange interpolation formula and consists in checking that if

$$\lambda := \sum_{i=1}^{d} \frac{\rho_i \, \partial_y(F_i)}{p - p_i}\,, \quad \mu := \sum_{i=1}^{d} \frac{X_i(p_i) \,.\, \rho_i \, \partial_y(F_i)}{p - p_i} \text{ and } \nu := \sum_{i=1}^{d} \frac{X_i(\rho_i) \,.\, \partial_y(F_i)}{p - p_i}$$

where $\rho_i := \dfrac{r(x, y, p_i)}{\partial_p(F)(x, y, p_i)\partial_y(F_i)(x, y)}$ for $1 \leq i \leq d$, we have the following equality: $\partial_x(\lambda) + p\,\partial_y(\lambda) + \partial_p(\mu) = \nu$. Then it is sufficient to set $r_p = F \,.\, \mu$ and $t = F \,.\, \nu$ since by definition $r = F \,.\, \lambda$.

Moreover if $\deg r \leq d - 3$, as we shall assume from now on, then $\deg t \leq d - 2$ by the relation $(\star)$ and from the previous observations, we have the explicit equality

$$d\big(r \cdot \frac{dy - p\,dx}{\partial_p(F)}\big) = t \cdot \frac{dx \wedge dy}{\partial_p(F)}\cdot$$

With the notation of the introduction, the main result in [H2] can be stated as the following:

Theorem $\mathfrak{a}_F$. *The map*

$$\big(g_i(F_i)\big)_i \longmapsto \omega := \big(F \cdot \sum_{i=1}^{d} \frac{g_i(F_i)\partial_y(F_i)}{p - p_i}\big) \cdot \frac{dy - p\,dx}{\partial_p(F)} \in \pi_*(\Omega^1_S)$$

defines a $\mathbb{C}$*-isomorphism* $T : \mathcal{A}(d) \longrightarrow \mathfrak{a}_F$ *such that* $\operatorname{Trace}_\pi(\omega) = \sum_{i=1}^{d} g_i(F_i) dF_i = 0$. *In particular*, $\operatorname{rk} \mathcal{W}(d) = \dim_{\mathbb{C}} \mathfrak{a}_F$.

It can be noted that the previous map T is in fact closely related to the application $E : (\mathbb{C}^2, 0) \times \mathbb{P}^1 \longrightarrow \mathbb{P}^{\operatorname{rk} \mathcal{W}(d) - 1}$ which extends a basic construction due to H. Poincaré. This application is very useful in making maximal rank webs algebraic (cf. [H1]).

The relation $(\star)$ implies exactly $2d - 1$ relations between the coefficients a_i, b_j, c_k and t_l where

$$\begin{aligned} F &= a_0 \,.\, p^d + a_1 \,.\, p^{d-1} + \cdots + a_d, \\ r &= b_3 \,.\, p^{d-3} + b_4 \,.\, p^{d-4} + \cdots + b_d, \\ r_p &= c_1 \,.\, p^{d-1} + c_2 \,.\, p^{d-2} + \cdots + c_d, \\ t &= t_2 \,.\, p^{d-2} + t_3 \,.\, p^{d-3} + \cdots + t_d \end{aligned}$$

are elements in $\mathcal{O}[p]$. Moreover, these relations can be viewed in a matrix form.

For $d=3$, the relation $(\star)$ corresponds to the following matrix system:

$$\begin{pmatrix} 0 & a_0 & -a_0 & 0 & 0 \\ a_0 & a_1 & 0 & -2a_0 & 0 \\ a_1 & a_2 & a_2 & -a_1 & -3a_0 \\ a_2 & a_3 & 2a_3 & 0 & -2a_1 \\ a_3 & 0 & 0 & a_3 & -a_2 \end{pmatrix} \begin{pmatrix} \partial_x(b_3) \\ \partial_y(b_3) \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= b_3 \cdot \begin{pmatrix} \partial_y(a_0) \\ \partial_x(a_0)+\partial_y(a_1) \\ \partial_x(a_1)+\partial_y(a_2) \\ \partial_x(a_2)+\partial_y(a_3) \\ \partial_x(a_3) \end{pmatrix} + t_2 \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ 0 \end{pmatrix} + t_3 \cdot \begin{pmatrix} 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

It can be verified that the determinant of the 5×5-matrix above is equal to the p-resultant R of F. Which is a consequence of the classical formula of Sylvester, namely

$$R = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 3a_0 & 2a_1 & a_2 & 0 & 0 \\ 0 & 3a_0 & 2a_1 & a_2 & 0 \\ 0 & 0 & 3a_0 & 2a_1 & a_2 \end{vmatrix}.$$

Thus, by Cramer formulas, it can be checked since $R(0)\neq 0$ that the previous matrix system is equivalent to the following nonhomogeneous linear differential system:

$$(\star_3) \qquad \begin{cases} \partial_x(b_3) + A_{1,1}\, b_3 = t_3 \\ \partial_y(b_3) + A_{2,1}\, b_3 = t_2 \end{cases}$$

where, in fact, $A_{i,j} \in \mathcal{O}[1/\Delta]$ which would be interesting in the singular case.

For $d=4$, the relation $(\star)$ corresponds to the following matrix system:

$$\begin{pmatrix} 0 & 0 & a_0 & -a_0 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & 0 & -2a_0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_2 & -a_1 & -3a_0 & 0 \\ a_1 & a_2 & a_3 & 2a_3 & 0 & -2a_1 & -4a_0 \\ a_2 & a_3 & a_4 & 3a_4 & a_3 & -a_2 & -3a_1 \\ a_3 & a_4 & 0 & 0 & 2a_4 & 0 & -2a_2 \\ a_4 & 0 & 0 & 0 & 0 & a_4 & -a_3 \end{pmatrix} \begin{pmatrix} \partial_x(b_4) \\ \partial_x(b_3)+\partial_y(b_4) \\ \partial_y(b_3) \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

$$= b_3 \cdot \begin{pmatrix} \partial_y(a_0) \\ \partial_x(a_0)+\partial_y(a_1) \\ \partial_x(a_1)+\partial_y(a_2) \\ \partial_x(a_2)+\partial_y(a_3) \\ \partial_x(a_3)+\partial_y(a_4) \\ \partial_x(a_4) \\ 0 \end{pmatrix} + b_4 \cdot \begin{pmatrix} 0 \\ \partial_y(a_0) \\ \partial_x(a_0)+\partial_y(a_1) \\ \partial_x(a_1)+\partial_y(a_2) \\ \partial_x(a_2)+\partial_y(a_3) \\ \partial_x(a_3)+\partial_y(a_4) \\ \partial_x(a_4) \end{pmatrix} + t_2 \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ 0 \\ 0 \end{pmatrix} + t_3 \cdot \begin{pmatrix} 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ 0 \end{pmatrix} + t_4 \cdot \begin{pmatrix} 0 \\ 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}.$$

With the same arguments used before, but with a 7×7-matrix, this system is equivalent to the following:

$$(\star_4)\qquad \begin{cases} \partial_x(b_4) & + \quad A_{1,1}\, b_3 + A_{1,2}\, b_4 = t_4 \\ \partial_x(b_3) + \partial_y(b_4) & + \quad A_{2,1}\, b_3 + A_{2,2}\, b_4 = t_3 \\ \partial_y(b_3) & + \quad A_{3,1}\, b_3 + A_{3,2}\, b_4 = t_2 \end{cases}$$

with some $A_{i,j} \in \mathcal{O}[1/\Delta]$.

In the general case, using again the Sylvester formula for the resultant, the relation $(\star)$ gives rise to the following nonhomogeneous linear differential system:

$$(\star_d)\quad \begin{cases} \partial_x(b_d) & + & A_{1,1}\, b_3 & + & \cdots & + & A_{1,d-2}\, b_d & = & t_d \\ \partial_x(b_{d-1}) + \partial_y(b_d) & + & A_{2,1}\, b_3 & + & \cdots & + & A_{2,d-2}\, b_d & = & t_{d-1} \\ & & & & & & & \vdots & \\ \partial_x(b_3) + \partial_y(b_4) & + & A_{d-2,1}\, b_3 & + & \cdots & + & A_{d-2,d-2}\, b_d & = & t_3 \\ \partial_y(b_3) & + & A_{d-1,1}\, b_3 & + & \cdots & + & A_{d-1,d-2}\, b_d & = & t_2 \end{cases}$$

with explicit $A_{i,j} \in \mathcal{O}[1/\Delta]$ obtained only from the coefficients of F by Cramer formulas.

Let $\mathcal{M}(d)$ be the homogeneous linear differential system associated with $(\star_d)$. Then, using the previous theorem and the fact that $\mathfrak{a}_F$ is uniquely determined by the analytic solutions of $\mathcal{M}(d)$, *we have the following identifications*:

$$\mathcal{A}(d) = \mathfrak{a}_F = \operatorname{Sol}\mathcal{M}(d)$$

where $\operatorname{Sol}\mathcal{M}(d)$ *denotes the* $\mathbb{C}$*-vector space of analytic solutions of* $\mathcal{M}(d)$.

In particular, using only the symbol of the linear differential system $\mathcal{M}(d)$, *we recover the classical optimal bound* $\frac{1}{2}(d-1)(d-2)$ *for the rank* $\operatorname{rk}\mathcal{W}(d)$. Indeed, let $\mathcal{D}$ be the ring of linear differential operators with coefficients in $\mathcal{O}$ (cf. for instance [G-M] for basic results and terminology). We denote by $\mathcal{M}(d)$ the left $\mathcal{D}$-module associated with $\mathcal{M}(d)$ and $\operatorname{gr}\mathcal{M}(d)$ its natural associated graded $\mathcal{O}[\xi,\eta]$-module. The special form of the system $\mathcal{M}(d)$, namely its symbol, implies that

$$(\xi,\eta)^{d-2} \subseteq \operatorname{Fitt}_0\big(\operatorname{gr}\mathcal{M}(4)\big) \subseteq \operatorname{Ann}\big(\operatorname{gr}\mathcal{M}(d)\big)$$

where $\operatorname{Fitt}_0\big(\operatorname{gr}\mathcal{M}(4)\big)$ is the 0-th Fitting ideal of $\operatorname{gr}\mathcal{M}(d)$ and $\operatorname{Ann}\big(\operatorname{gr}\mathcal{M}(d)\big)$ its annihilator. This proves that we have the following identification:

$$\mathcal{M}(d) = \mathcal{O}^{\operatorname{rk}\mathcal{W}(d)} \text{ as left } \mathcal{D}\text{–modules.}$$

In other words, we obtain either $\mathcal{M}(d) = 0$, which is the *generic case* for webs $\mathcal{W}(d)$ or $\mathcal{M}(d)$ is an *integrable connection.* Moreover, the previous inclusions give the optimal bound for $\operatorname{rk}\mathcal{W}(d)$ since

$$\begin{aligned} \operatorname{rk}\mathcal{W}(d) &= \operatorname{mult}\mathcal{M}(d) := \operatorname{mult}\operatorname{gr}\mathcal{M}(d) \\ &\le \operatorname{mult}\mathcal{O}[\xi,\eta]/(\xi,\eta)^{d-2} = \frac{1}{2}(d-1)(d-2). \end{aligned}$$

The previous identification with the $\mathcal{D}$-modules viewpoint comes from the Hilbert-Rückert Nullstellensatz and does not give rise, in a simple way, to general methods for determining the rank of the nonsingular planar web $\mathcal{W}(d)$. Another method to study the system $\mathcal{M}(d)$ will be presented in the next section with more geometric objects.

However, in the case $d = 3$, a formula for the rank depending only on the coefficients of F has been given by G. Mignard (cf. [M]) using the $\mathcal{D}$-modules approach (cf. [H1]). A different proof of it is given in [H2]. It uses some normalization for $\mathcal{W}(d)$ that we introduce for $d \geq 3$ to end this section.

Let $\omega := \dfrac{dy - p\,dx}{\partial_p(F)}$ be the *particular* 1-form in Ω^1_S. For $1 \leq i \leq d$, the forms $\omega_i := \pi_i^*(\omega)$ define the d-web $\mathcal{W}(d)$ associated to $F(x, y, y') = 0$. For these forms, we have

$$\sum_{i=1}^{d} \omega_i = 0\,,\ \sum_{i=1}^{d} p_i\,.\,\omega_i = 0\,, \ldots\,,\ \sum_{i=1}^{d} p_i^{d-3}\,.\,\omega_i = 0;$$

that is, $\text{Trace}_\pi(p^k.\,\omega) = 0$ for $0 \leq k \leq d-3$ since $\displaystyle\sum_{i=1}^{d} \frac{p_i^j(x,y)}{\partial_p(F)(x,y,p_i(x,y))} = 0$ for $0 \leq j \leq d-2$ by the Lagrange interpolation formula.

Any family $(\widetilde{\omega}_i)$ of 1-forms which defines $\mathcal{W}(d)$ and such that the following $d-2$ relations are satisfied:

$$\sum_{i=1}^{d} p_i^k\,.\,\widetilde{\omega}_i = 0 \text{ for } 0 \leq k \leq d-3$$

will be called a *normalization* of the nonsingular planar web $\mathcal{W}(d)$. From the general position hypothesis, it may be remarked that the $d-2$ previous relations which are satisfied by the $(\widetilde{\omega}_i)$ are necessarily independent.

Such a normalization exists and the previous one (ω_i) constructed from the particular 1-form $\omega = \dfrac{dy - p\,dx}{\partial_p(F)} \in \Omega^1_S$ will be called the *canonical normalization* of $\mathcal{W}(d)$. This terminology is justified by some properties.

The first one gives a useful means to compare two normalisations of $\mathcal{W}(d)$.

Proposition 1. *Let $(\widetilde{\omega}_i)$ be a normalization of $\mathcal{W}(d)$. Now, $\widetilde{\omega}_i = g\,.\,\omega_i$ for $1 \leq i \leq d$ where g is an invertible element of $\mathcal{O}$ and (ω_i) is the canonical normalization of this web.*

Proof. For $d = 3$, this proposition is a basic result to obtain the property of the Blaschke curvature for any $\mathcal{W}(3)$ (cf. for instance [B] and below). This proof naturally extends to $d \geq 4$ and we give the method for $d = 4$. For any normalization and naturally for (ω_i) the canonical one, we have the following form:

$$\begin{cases} \omega_1 + \omega_2 = -\omega_3 - \omega_4 \\ p_1 \,.\, \omega_1 + p_2 \,.\, \omega_2 = -p_3 \,.\, \omega_3 - p_4 \,.\, \omega_4 \end{cases}$$

which implies $(p_4 - p_1) \,.\, \omega_1 \wedge \omega_2 = (p_4 - p_3) \,.\, \omega_2 \wedge \omega_3$. By circular permutation we get, with classical notation, a nonsingular 2-form Ω on $(\mathbb{C}^2, 0)$ such that

$$\begin{aligned} \Omega :&= (p_1 - p_2)(p_2 - p_3)(\widehat{p_3 - p_4})(p_4 - p_1) \,.\, \omega_1 \wedge \omega_2 \\ &= -(p_1 - p_2)(p_2 - p_3)(p_3 - p_4)(\widehat{p_4 - p_1}) \,.\, \omega_2 \wedge \omega_3 \\ &= (\widehat{p_1 - p_2})(p_2 - p_3)(p_3 - p_4)(p_4 - p_1) \,.\, \omega_3 \wedge \omega_4 \\ &= -(p_1 - p_2)(\widehat{p_2 - p_3})(p_3 - p_4)(p_4 - p_1) \,.\, \omega_4 \wedge \omega_1. \end{aligned}$$

For $(\widetilde{\omega_i})$, we have $\widetilde{\omega_i} = g_i \,.\, \omega_i$ with $g_i \in \mathcal{O}^*$. From the previous observations, we get a nonsingular 2-form $\widetilde{\Omega}$ which satisfies the following equalities:

$$\widetilde{\Omega} = g_1 g_2 \,.\, \Omega = g_2 g_3 \,.\, \Omega = g_3 g_4 \,.\, \Omega = g_4 g_1 \,.\, \Omega.$$

This proves $g_1 = g_3$ and $g_2 = g_4$. But also $(p_2 - p_4) \,.\, \omega_1 \wedge \omega_2 = (p_4 - p_3) \,.\, \omega_1 \wedge \omega_3$; thus we obtain $g_2 = g_3$ and $\widetilde{\Omega} = g^2 \,.\, \Omega$ with $g := g_1 = g_2 = g_3 = g_4$ which ends the proof of the proposition. $\square$

The second property of the canonical normalization (ω_i) of $\mathcal{W}(d)$ is to be related to the columns of the $(d-1) \times (d-2)$-matrix $(A_{i,j})$ which appears in the previous differential system $(\star_d)$. In fact, from the explicit expression of $d : \Omega^1_S \longrightarrow \Omega^2_S$, we obtain for $0 \leq k \leq d-3$ and for the particular 1-form $\omega = \dfrac{dy - p\,dx}{\partial_p(F)} \in \Omega^1_S$ the following equalities:

$$d(p^k . \,\omega) = \big(A_{d-1,d-2-k} \,.\, p^{d-2} + A_{d-2,d-2-k} \,.\, p^{d-3} + \cdots + A_{1,d-2-k}\big) \cdot \frac{dx \wedge dy}{\partial_p(F)(x,y,p)} \cdot$$

In particular for $0 \leq k \leq d-3$ and $1 \leq i \leq d$, we have the main equalities

$$(k_i) \qquad d(p_i^k . \,\omega_i) = \Big(A_{1,d-2-k}\, dx + \big(\sum_{j=0}^{d-3} p_i^j \,.\, A_{j+2,d-2-k}\big)\, dy\Big) \wedge \omega_i$$

which gives by differentiation for $d \geq 4$ the following $d-3$ relations:

$$A_{1,d-3} + \sum_{j=1}^{d-2} (A_{j+1,d-3} - A_{j,d-2}) \,.\, p_i^j - A_{d-1,d-2} \,.\, p_i^{d-1} = X_i(p_i)$$

$$\vdots$$

$$\begin{aligned} &\sum_{j=0}^{d-4} A_{j+1,1} \,.\, p_i^j + (A_{d-2,1} - A_{1,d-2}) \,.\, p_i^{d-3} \\ &\qquad + (A_{d-1,1} - A_{2,d-2}) \,.\, p_i^{d-2} - \sum_{j=1}^{d-3} A_{j+2,d-2} \,.\, p_i^{j+d-2} = (d-3)\, p_i^{d-4} \,.\, X_i(p_i). \end{aligned}$$

These relations prove that the matrix $(A_{i,j})$ has the following *particular form*:

$$\begin{pmatrix} 0 & 0 & & \cdots & & 0 & A_1 \\ 0 & 0 & & \cdots & & A_1 & A_2 \\ 0 & 0 & & \cdots & & A_2 & 0 \\ \vdots & \vdots & \iddots & & \iddots & \vdots & \vdots \\ 0 & A_1 & & \cdots & & 0 & 0 \\ A_1 & A_2 & & \cdots & & 0 & 0 \\ A_2 & 0 & & \cdots & & 0 & 0 \end{pmatrix}$$

if and only if the web $\mathcal{W}(d)$ *is linear*; *that is*, $\mathcal{W}(d) = \mathcal{L}(d)$.

Since we have the equivalence: $\mathcal{W}(d)$ *is linear if and only if* $X_i(p_i) = 0$ *for* $1 \le i \le d$, the matrix $(A_{i,j})$ above is obtained step by step from the previous relations by using the general position hypothesis and a Vandermonde determinant. Moreover, in the linear case we have $r_p = 0$ in the relation $(\star)$ from the previous equivalence. Thus, by identification with the t corresponding to $r = 1$, we obtain $A_2 = -\dfrac{\partial_y(a_0)}{a_0}$ and $A_1 = -\dfrac{\partial_x(a_0)}{a_0} - \partial_y\big(\dfrac{a_1}{a_0}\big)$ from the coefficients of F.

Furthermore in the algebraic case, that is $\mathcal{W}(d) = \mathcal{L}_C(d)$ with $C \subset \mathbb{P}^2$, the matrix $(A_{i,j})$ obtained from $F(x,y,p) = P(y - px, p)$ where $P(s,t) = 0$ is an affine equation for C is the *null matrix*. Indeed, here we have $\partial_x(F) + p\,\partial_y(F) = 0$ and the relation $(\star)$ in this linear case implies that $r = 1$ corresponds to $t = 0$.

For any normalization $(\widetilde{\omega_i})$ of $\mathcal{W}(d)$, the general position hypothesis gives a matrix $(\widetilde{A_{i,j}})$ which satisfies the analogue to the relations (k_i) before, but with $\widetilde{\omega_i}$ and $\widetilde{A_{i,j}}$. Moreover, the similar $d-3$ relations obtained above are satisfied. This normalization gives rise to several invariants of $\mathcal{W}(d)$ as follows:

THEOREM 1. *With the previous notation, the* $(d-1)\times(d-2)$*-matrix* $(A_{i,j})$ *coming from* $F(x,y,y') = 0$ *gives analytic invariants on* $(\mathbb{C}^2, 0)$ *of the nonsingular planar web* $\mathcal{W}(d)$ *generated by this differential equation : on the one hand* $d(d-3)$ *functions*

$$A_{m,n} \text{ for } 2 \le m+n \le d-2 \text{ and } d+1 \le m+n \le 2d-3,$$

$$A_{u,d-1-u} - A_{1,d-2} \text{ for } 2 \le u \le d-2 \text{ and } A_{v,d-v} - A_{2,d-2} \text{ for } 3 \le v \le d-1$$

and on the other hand $d-2$ *differential forms of degree* 2

$$d\Gamma_q \text{ for } 1 \le q \le d-2 \text{ where } \Gamma_q := A_{d-q-1,q}\,dx + A_{d-q,q}\,dy.$$

In particular in the linear case the 2*-differential form* $d\Gamma = k\,dx \wedge dy$ *is an invariant of* $\mathcal{L}(d)$ *where* $\Gamma := A_1\,dx + A_2\,dy$, *and, explicitly,* $k := \partial_x(A_2) - \partial_y(A_1) = \partial_y^2\big(\dfrac{a_1}{a_0}\big)\cdot$

Proof. For the functions, using the general position hypothesis, we have $\widetilde{A_{m,n}} = A_{m,n}$ for suitable index and the other equalities. It is a direct consequence of the relations induced by (k_i) and the analogue for any normalization $(\widetilde{\omega_i})$ of $\mathcal{W}(d)$. Using Proposition 1 and the general position hypothesis, we have for $1 \leq q \leq d-2$ and from the relation (k_i), the equalities $\widetilde{A_{d-q-1,q}} - A_{d-q-1,q} = \dfrac{\partial_x(g)}{g}$ and $\widetilde{A_{d-q,q}} - A_{d-q,q} = \dfrac{\partial_y(g)}{g}$ which prove the result for the 2-forms. In the linear case, the result comes from the previous calculations. □

For $d = 3$, we set $\gamma = A_{1,1}dx + A_{2,1}dy$. From the previous observations and with the canonical normalization (ω_i) of $\mathcal{W}(3)$, we have $d\omega_i = \gamma \wedge \omega_i$ for $1 \leq i \leq 3$, which proves the following result of [H2]:

The Blaschke curvature of the nonsingular planar web $\mathcal{W}(3)$ is equal to $d\gamma$.

Indeed, we know (cf. for instance [B] or the previous theorem) that the Blaschke curvature $d\gamma$ of $\mathcal{W}(3)$ does not depend on the normalization used to define it, contrary to the 1-form γ on $(\mathbb{C}^2, 0)$ such that $d\omega_i = \gamma \wedge \omega_i$ for $1 \leq i \leq 3$.

Moreover, according to the *integrability condition* of the homogeneous differential system $\mathcal{M}(3)$ associated with $(\star_3)$, we get the nonsurprising result:

$$\operatorname{rk} \mathcal{W}(3) = \begin{cases} 0 & \text{if} \quad d\gamma \neq 0 \\ 1 & \text{if} \quad d\gamma = 0 \end{cases} .$$

Using integrability conditions for $\mathcal{M}(d)$, this concrete approach will be generalized for $d \geq 3$ in the next section.

3. On the connection $(\mathcal{E}, \nabla)$ associated with $\mathcal{W}(d)$ and some applications

From the previous section, the exterior differential $d : \pi_*(\Omega^1_S) \longrightarrow \pi_*(\Omega^2_S)$ coming from the surface S, gives by restriction a linear differential operator

$$\rho : \mathcal{O}^{d-2} \longrightarrow \mathcal{O}^{d-1}$$

on $(\mathbb{C}^2, 0)$. It is defined by $\rho\,(b_3, \dots, b_d) = (t_2, \dots, t_d)$ where $d\big(r \cdot \dfrac{dy - p\,dx}{\partial_p(F)}\big) = t \cdot \dfrac{dx \wedge dy}{\partial_p(F)}$ with $r = b_3 \,.\, p^{d-3} + \cdots + b_d$ and $t = t_2 \,.\, p^{d-2} + \cdots + t_d$.

This operator of order 1 is induced by the homogeneous linear differential system $\mathcal{M}(d)$ associated with $(\star_d)$. Its corresponding morphism of $\mathcal{O}$-modules

$$p_0 : J_1(\mathcal{O}^{d-2}) \longrightarrow \mathcal{O}^{d-1}$$

satisfies $p_0 \circ j_1 = \rho$ where in matrix form $j_1(b) = \big(b, \partial_x(b), \partial_y(b)\big)$ as jets. From the nature of the system $(\star_d)$, the kernel R_0 of p_0 is a free $\mathcal{O}$-module of finite type.

We use, with minor modifications, the now classical notation from the works of D. Spencer and H. Goldschmidt (cf. [S] and for instance [B-C-3G]). In particular, $p_k : J_{k+1}(\mathcal{O}^{d-2}) \longrightarrow J_k(\mathcal{O}^{d-1})$ denotes the k-th *prolongation* of p_0 for $k \geq 0$ obtained by successive derivations and R_k is the kernel of p_k. For symbols, we have natural exact sequences:

$$0 \longrightarrow g_k \longrightarrow S_{k+1}(\mathcal{O}^{d-2}) \xrightarrow{\sigma_k} S_k(\mathcal{O}^{d-1}) \xrightarrow{\tau_{k-1}} \operatorname{Coker} \sigma_k \longrightarrow 0.$$

With this notation and among others the "snake" lemma, we have for $k \geq 0$ the following exact commutative diagram of $\mathcal{O}$-modules:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & g_{k+1} \arrow[r] \arrow[d] & S_{k+2}(\mathcal{O}^{d-2}) \arrow[r, "\sigma_{k+1}"] \arrow[d] & S_{k+1}(\mathcal{O}^{d-1}) \arrow[r, "\tau_k"] \arrow[d] & \mathfrak{K}_k \arrow[r] & 0 \\
0 \arrow[r] & R_{k+1} \arrow[r] \arrow[d, "\overline{\pi}_k"'] & J_{k+2}(\mathcal{O}^{d-2}) \arrow[r, "p_{k+1}"] \arrow[d, "\pi_k"'] & J_{k+1}(\mathcal{O}^{d-1}) \arrow[d, "\widehat{\pi}_k"'] & & \\
0 \arrow[r] & R_k \arrow[r] \arrow[d, "\beta_k"'] & J_{k+1}(\mathcal{O}^{d-2}) \arrow[r, "p_k"] \arrow[d] & J_k(\mathcal{O}^{d-1}) \arrow[d] & & \\
 & \mathfrak{K}_k & 0 & 0 & &
\end{tikzcd}$$

where π_k (resp. $\widehat{\pi}_k$) is the natural projection, $\overline{\pi}_k$ is the morphism induced on kernels and $\mathfrak{K}_k := \operatorname{Coker} \sigma_{k+1}$ is the obstruction space of formal integrability of p_k .

Moreover, we have the following equivalence: $\overline{\pi}_k$ *is a surjective morphism if and only if* $\beta_k = 0$.

For $p \geq 1$ and $k \geq 0$ we have $\operatorname{rk} J_k(\mathcal{O}^p) = \frac{p}{2}(k+1)(k+2)$; thus $\operatorname{rk} S_k(\mathcal{O}^p) = p(k+1)$ and we set $J_l(\mathcal{O}^p) = 0$ if $l < 0$.

For $k \geq 1$ the $\mathcal{O}$-modules R_k and g_{k+1} are of finite type but not necessarily free. However, using the special nature of the symbols of the prolongations of the system $\mathcal{M}(d)$, one can determine the constant rank of each σ_k.

In particular, it can be checked that $g_k = 0$ for $k \geq d-3$ and g_{d-4} is a free $\mathcal{O}$-module of rank 1. Moreover, one obtains that $\mathfrak{K}_k = 0$ for $k \leq d-4$ and that it is a free $\mathcal{O}$-module of rank $k-d+4$ for $k \geq d-3$.

Since σ_{d-3} is an isomorphism, we get among others that $\overline{\pi}_{d-4} : R_{d-3} \longrightarrow R_{d-4}$ is an isomorphism of free $\mathcal{O}$-modules with rank $\frac{1}{2}(d-1)(d-2)$.

Furthermore, we have (cf. [S]) a natural exact sequence of $\mathbb{C}$-vector spaces

$$0 \longrightarrow \mathcal{O}^p \xrightarrow{j_l} J_l(\mathcal{O}^p) \xrightarrow{D} \Omega^1 \otimes_{\mathcal{O}} J_{l-1}(\mathcal{O}^p) \xrightarrow{D} \Omega^2 \otimes_{\mathcal{O}} J_{l-2}(\mathcal{O}^p) \longrightarrow 0.$$

For example with $p=1$, $l=2$ and the natural projection $\pi : J_1(\mathcal{O}) \longrightarrow J_0(\mathcal{O})$, we have explicitly the following three applications:

$$\begin{aligned} j_2(f) &= \big(f, \partial_x(f), \partial_y(f), \partial_x^2(f), \partial_x\partial_y(f), \partial_y^2(f)\big),\\ D(z,p,q,r,s,t) &= dx \otimes \big(\partial_x(z)-p, \partial_x(p)-r, \partial_x(q)-s\big)\\ &\quad + dy \otimes \big(\partial_y(z)-q, \partial_y(p)-s, \partial_y(q)-t\big)\\ \text{and}\quad D\big(\omega\otimes(z,p,q)\big) &= d\omega\otimes\pi(z,p,q) - \omega\wedge D(z,p,q)\\ &= d\omega\otimes z - \omega\wedge\big(dx\otimes(\partial_x(z)-p) + dy\otimes(\partial_y(z)-q)\big). \end{aligned}$$

With the notation used before, the previous exact sequence induces on kernels the *first Spencer complex* associated with the prolongation p_k of p_0, that is, the following complex (of families) of $\mathbb{C}$-vector spaces exact at R_k with injective j_{k+1}:

$$0 \longrightarrow \operatorname{Sol}\mathcal{M}(d) \xrightarrow{j_{k+1}} R_k \xrightarrow{D} \Omega^1 \otimes_{\mathcal{O}} R_{k-1} \xrightarrow{D} \Omega^2 \otimes_{\mathcal{O}} R_{k-2} \longrightarrow 0.$$

Moreover, from the preceding constructions, we have the following commutative diagram of $\mathbb{C}$-vector spaces with exact rows and such that the columns are complex exact at R_k (resp. R_{k+1}) with injective j_{k+1} (resp. j_{k+2}):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Sol}\mathcal{M}(d) \arrow[r, "\mathrm{id}"] \arrow[d, "j_{k+2}"'] & \operatorname{Sol}\mathcal{M}(d) \arrow[r] \arrow[d, "j_{k+1}"'] & 0 \\
g_{k+1} \arrow[r] & R_{k+1} \arrow[r, "\overline{\pi}_k"] \arrow[d, "D"'] & R_k \arrow[r, "\beta_k"] \arrow[d, "D"'] & \mathfrak{K}_k \\
\Omega^1 \otimes_{\mathcal{O}} g_k \arrow[r] & \Omega^1 \otimes_{\mathcal{O}} R_k \arrow[r, "\overline{\pi}_{k-1}"] \arrow[d, "D"'] & \Omega^1 \otimes_{\mathcal{O}} R_{k-1} \arrow[r, "\beta_{k-1}"] \arrow[d, "D"'] & \Omega^1 \otimes_{\mathcal{O}} \mathfrak{K}_{k-1} \\
\Omega^2 \otimes_{\mathcal{O}} g_{k-1} \arrow[r] & \Omega^2 \otimes_{\mathcal{O}} R_{k-1} \arrow[r, "\overline{\pi}_{k-2}"] \arrow[d] & \Omega^2 \otimes_{\mathcal{O}} R_{k-2} \arrow[r, "\beta_{k-2}"] \arrow[d] & \Omega^2 \otimes_{\mathcal{O}} \mathfrak{K}_{k-2} \\
 & 0 & 0. &
\end{tikzcd}$$

Using the previous results one gets a long sequence of $\mathcal{O}$-modules of finite type

$$\cdots \xrightarrow{\overline{\pi}_{d-2}} R_{d-2} \xrightarrow{\overline{\pi}_{d-3}} \mathcal{E} := R_{d-3} \xrightarrow{\overline{\pi}_{d-4}} R_{d-4}$$

with injective morphisms which begin with an isomorphism $\overline{\pi}_{d-4}$ of free $\mathcal{O}$-modules with rank $\frac{1}{2}(d-1)(d-2)$.

The operator $D : R_k \longrightarrow \Omega^1 \otimes_{\mathcal{O}} R_{k-1}$ of the first Spencer complex associated with p_k satisfies $D(fe) = df \otimes \overline{\pi}_{k-1}(e) + fD(e)$ for $e \in R_k$ and $f \in \mathcal{O}$. Thus, it allows us to make for $k \geq d-3$ the following *construction* related to the previous long sequence: if $\beta_{k-1} = 0$, that is, $\overline{\pi}_{k-1}$ is an isomorphism, then $R_k = \operatorname{Ker} p_k \subset J_{k+1}(\mathcal{O}^{d-2})$ is equipped with a connection defined by

$$\nabla_k := (\overline{\pi}_{k-1})^{-1} \circ D : R_k \longrightarrow \Omega^1 \otimes_{\mathcal{O}} R_k$$

such that its $\mathbb{C}$-vector space $\operatorname{Ker} \nabla_k$ of horizontal sections is isomorphic to $\operatorname{Sol} \mathcal{M}(d)$. Moreover its curvature

$$\nabla_k^{(1)} \circ \nabla_k : R_k \longrightarrow \Omega^2 \otimes_{\mathcal{O}} R_k$$

takes its values in $\Omega^2 \otimes_{\mathcal{O}} g_{k-1}$ with the identification $\Omega^2 \otimes_{\mathcal{O}} R_k \xrightarrow{\overline{\pi}_{k-1}} \Omega^2 \otimes_{\mathcal{O}} R_{k-1}$. In particular, if this connection (R_k, ∇_k) exists it is always integrable for $k \geq d-2$. Indeed, it can be checked that, with some abuse of notation, we have $D = \overline{\pi}_{k-1} \circ \nabla_k^{(1)}$. Thus, it follows that $\overline{\pi}_{k-2} \circ \overline{\pi}_{k-1} \circ \nabla_k^{(1)} \circ \nabla_k = \overline{\pi}_{k-2} \circ D \circ (\overline{\pi}_{k-1})^{-1} \circ D = D^2 = 0$. From a basic result on connections, if $\beta_{k-1} = 0$ then R_k is a free $\mathcal{O}$-module of finite type, even if the connection (R_k, ∇_k) is not integrable. Moreover, by the previous commutative diagram, it can be checked that the inclusion $\operatorname{Ker} \beta_k \subseteq \operatorname{Ker}(\nabla_k^{(1)} \circ \nabla_k)$ holds.

Since $\beta_{d-4} = 0$, this construction starts with $(\mathcal{E}, \nabla) := (R_{d-3}, \nabla_{d-3})$ which is called the *connection* associated with the planar web $\mathcal{W}(d)$ generated by the differential equation $F(x, y, y') = 0$. The curvature $K := \nabla_{d-3}^{(1)} \circ \nabla_{d-3}$ of $(\mathcal{E}, \nabla)$ is $\mathcal{O}$-linear and since g_{d-4} is a free $\mathcal{O}$-module of rank 1, there exists an *adapted basis* (e_ℓ) of the free $\mathcal{O}$-module $\mathcal{E}$ which verifies

$$K(e_\ell) = k_\ell \, dx \wedge dy \otimes e_1 \quad \text{for} \quad 1 \leq \ell \leq \frac{1}{2}(d-1)(d-2) \quad \text{with} \quad k_\ell \in \mathcal{O}$$

where $e_1 \in \mathcal{E}$ corresponds to a generator of g_{d-4}. Moreover from results above, $\beta_{d-3} = 0$ implies $K = 0$.

The successive morphisms β_k vanish or not and the construction mentioned above jointed with the Cauchy-Kowalevski theorem as below can be used to give a theoretical approach to the determination of the exact rank of $\mathcal{W}(d)$. However, the level $d-3$ at least gives the following effective method to characterize maximal rank webs:

THEOREM 2. *With the previous notation, the following conditions are equivalent*:

i) *The connection* $(\mathcal{E}, \nabla)$ *is integrable, that is* $k_\ell = 0$ *for* $1 \leq \ell \leq \frac{1}{2}(d-1)(d-2)$;

ii) *The planar web* $\mathcal{W}(d)$ *associated with* $F(x, y, y') = 0$ *is of maximal rank.*

Proof. With natural identification, the Cauchy-Kowalevski theorem asserts that the evaluation map $\operatorname{Ker}\nabla = \operatorname{Sol}\mathcal{M}(d) \longrightarrow \mathbb{C}^{\frac{1}{2}(d-1)(d-2)}$ close to $0 \in \mathbb{C}^2$ is an isomorphism if $K = 0$; hence i) $\Rightarrow$ ii). Conversely, since $\Omega^1 \otimes_{\mathcal{O}} \big(\mathcal{O}\otimes_{\mathbb{C}}\operatorname{Sol}\mathcal{M}(d)\big)$ is identified with $\Omega^1\otimes_{\mathbb{C}}\operatorname{Sol}\mathcal{M}(d)$, the local system $\operatorname{Sol}\mathcal{M}(d)$ of dimension $\frac{1}{2}(d-1)(d-2)$ induces an integrable connection $(\mathcal{O}\otimes_{\mathbb{C}}\operatorname{Sol}\mathcal{M}(d), \nabla_S)$ defined by $\nabla_S(f \otimes s) = df \otimes s$. Then we have a morphism of connections $(\mathcal{O} \otimes_{\mathbb{C}} \operatorname{Sol}\mathcal{M}(d), \nabla_S) \longrightarrow (\mathcal{E}, \nabla)$ given by $f \otimes s \longmapsto fs$, which is an isomorphism. This proves that the connection $(\mathcal{E}, \nabla)$ is integrable. □

Examples. 1. For $d = 3$, that is for $\mathcal{W}(3)$, the matrix $(A_{i,j})$ introduced before has the following simplified form:

$$(A_{i,j}) = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

It can be checked that the row matrix $e_1 = \begin{pmatrix} 1 & -A_1 & -A_2 \end{pmatrix} \in J_1(\mathcal{O})$ is an adapted basis of $\mathcal{E} := R_0$. In this basis, the connection $(\mathcal{E}, \nabla)$ associated with $\mathcal{W}(3)$ has the following matrix:

$$\gamma = \big(A_1\, dx + A_2\, dy\big)$$

with the corresponding curvature matrix

$$d\gamma + \gamma \wedge \gamma = \big(\partial_x(A_2) - \partial_y(A_1)\big)\, dx \wedge dy$$

which is the Blaschke curvature of $\mathcal{W}(3)$.

2. For $d = 4$ in the linear case, that is $\mathcal{W}(4) = \mathcal{L}(4)$, the matrix $(A_{i,j})$ has the following particular form:

$$(A_{i,j}) = \begin{pmatrix} 0 & A_1 \\ A_1 & A_2 \\ A_2 & 0 \end{pmatrix}.$$

In this case, one constructs an adapted basis (e_1, e_2, e_3) of $\mathcal{E} := R_1$ with explicit elements of $J_2(\mathcal{O}^2)$ in matrix forms:

$$e_1 = \begin{pmatrix} 0 & -1 & 0 & 2A_1 & A_2 & 0 \\ 0 & 0 & 1 & 0 & -A_1 & -2A_2 \end{pmatrix},$$

$$e_2 = \begin{pmatrix} 1 & -A_1 & -A_2 & A_1^2 - \partial_x(A_1) & A_1A_2 - \partial_x(A_2) & A_2^2 - \partial_y(A_2) \\ 0 & 0 & 0 & 0 & 0 & k \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & -k & 0 & 0 \\ 1 & -A_1 & -A_2 & A_1^2 - \partial_x(A_1) & A_1A_2 - \partial_y(A_1) & A_2^2 - \partial_y(A_2) \end{pmatrix},$$

where $k := \partial_x(A_2) - \partial_y(A_1)$. In this basis, the connection $(\mathcal{E}, \nabla)$ associated with $\mathcal{W}(4)$ has the following matrix:

$$\gamma = \begin{pmatrix} \Gamma & -k\,dy & -k\,dx \\ dx & \Gamma & 0 \\ -dy & 0 & \Gamma \end{pmatrix}$$

with the corresponding curvature matrix

$$d\gamma + \gamma \wedge \gamma = \begin{pmatrix} 3\,k & -\partial_x(k) & \partial_y(k) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} dx \wedge dy$$

where $\Gamma := A_1\,dx + A_2\,dy$.

Remarks. 1. In the general case $\mathcal{W}(4)$ with the previous method and from the matrix $(A_{i,j}) = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ A_{3,1} & A_{3,2} \end{pmatrix}$, one gets an adapted basis of $(\mathcal{E}, \nabla)$ with curvature matrix

$$d\gamma + \gamma \wedge \gamma = \begin{pmatrix} k_1 & k_2 & k_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} dx \wedge dy,$$

where we have explicitly

$$\begin{aligned} k_1 &= 2\kappa_1 + \kappa_2 + \partial_y(\lambda_1), \\ k_2 &= -\partial_x(\kappa_1) - \partial_y^2(A_{1,1}) + \lambda_1\kappa_1 + \partial_y(\lambda_2 A_{1,1}) + \partial_x(A_{1,1}A_{3,2}) + A_{1,1}\partial_x(A_{3,2}), \\ k_3 &= \partial_y(\kappa_2) - \partial_x^2(A_{3,2}) + \lambda_2\kappa_2 - \partial_x(\lambda_1 A_{3,2}) + \partial_y(A_{1,1}A_{3,2}) + A_{3,2}\partial_y(A_{1,1}), \end{aligned}$$

with $\kappa_1 = \partial_x(A_{3,1}) - \partial_y(A_{2,1})$ and $\kappa_2 = \partial_x(A_{2,2}) - \partial_y(A_{1,2})$; that is, $d\Gamma_q = \kappa_q\,dx \wedge dy$ for $1 \leq q \leq 2$, $\lambda_1 = A_{2,1} - A_{1,2}$ and $\lambda_2 = A_{3,1} - A_{2,2}$. Moreover, as with the Blaschke curvature for any $\mathcal{W}(3)$, the previous relations prove that (k_ℓ) does not depend on a normalization of $\mathcal{W}(4)$. In other words, the collection (k_ℓ) is an *invariant* of the planar web $\mathcal{W}(4)$; that is, the curvature of its associated connection $(\mathcal{E}, \nabla)$ is "canonical".

2. For the weave of a general planar web $\mathcal{W}(4)$ introduced in the first section, it can be noted from the theorem called $\mathfrak{a}_F$ that $(b_3, b_4) \in \mathcal{A}(4)_3 \subseteq \mathcal{A}(4)_4 = \mathcal{A}(4)$ if and only if $b_3^4\,.\,F(x, y, \dfrac{-b_4}{b_3}) = 0$. For $d \geq 4$, one can get the same kind of description of elements in $\mathcal{A}(d)_k$ by adding suitable new equations on $(b_3, \ldots, b_d) \in \mathcal{A}(d)$.

To end this section, we give some applications of the previous methods and results. Let $\mathcal{L}(d)$ be a linear and nonsingular web in $(\mathbb{C}^2, 0)$. The previous relation $(\star)$ is reduced to the following:

$$(\star\mathcal{L}) \qquad r\,.\big(\partial_x(F) + p\,\partial_y(F)\big) = \big(\partial_x(r) + p\,\partial_y(r) - t\big).\,F.$$

Since $p_i = \xi_i(F_i)$ with $\xi_i \in \mathbb{C}\{z\}$ for $1 \leq i \leq d$ in the linear case, $r_p = 0$. Moreover for $\mathcal{L}(d)$, the homogeneous linear differential system $\mathcal{M}(d)$ associated to $(\star_d)$ has the following particular form:

$$\begin{cases} \partial_x(b_d) & + & & & A_1\, b_d & = & 0 \\ \partial_x(b_{d-1}) + \partial_y(b_d) & + & A_1\, b_{d-1} & + & A_2\, b_d & = & 0 \\ & & & & & \vdots & \\ \partial_x(b_3) + \partial_y(b_4) & + & A_1\, b_3 & + & A_2\, b_4 & = & 0 \\ \partial_y(b_3) & + & A_2\, b_3 & & & = & 0. \end{cases}$$

With the theorem $\mathfrak{a}_F$, it can be checked, with the notation used before, that if $d\Gamma = k\, dx \wedge dy = 0$ then the rank of $\mathcal{L}(d)$ is maximally equal to $\frac{1}{2}(d-1)(d-2)$. Indeed, we have the following explicit description, through the $\mathbb{C}$-vector space $\mathfrak{a}_F$, of its $\mathbb{C}$-vector space of abelian relations:

$$\big\{\, e^{-\varphi}(y-px)^{\ell_1} p^{\ell_2} \cdot \frac{dy - p\, dx}{\partial_p(F)} \big\} \in \pi_*(\Omega^1_S) \;\text{ with }\; 0 \leq \ell_1 + \ell_2 \leq d-3,$$

with $\Gamma = A_1\, dx + A_2\, dy = d\varphi$ where $\varphi \in \mathcal{O}$ is given by the Poincaré lemma.

In particular we recover, without using Abel's theorem and traces of elements in $H^0(C, \omega^1_C)$, that if $\mathcal{L}(d) = \mathcal{L}_C(d)$ is an algebraic web with $C \subset \mathbb{P}^2$ then it necessarily has maximal rank since in this case we have seen that $\Gamma = 0$. Moreover its space $\mathfrak{a}_F$ has the previous description with $\varphi = 0$.

In the linear case, an adapted basis (e_ℓ) for the connection $(\mathcal{E}, \nabla)$ associated with $\mathcal{L}(d)$ can be constructed, step by step on d, following the examples given before. At each step, e_1 is chosen and the other vectors e_ℓ are constructed from the steps before, installed on different rows with suitable zeros. Moreover it can be checked that in this special case, the ℓ-component of each $\nabla(e_\ell)$ is $(A_1\, dx + A_2\, dy) \otimes e_\ell$ which proves the following result:

Proposition 2. *Let $\mathcal{L}(d)$ be a linear and nonsingular planar web. Then, the trace of the curvature K of the connection $(\mathcal{E}, \nabla)$ associated with $\mathcal{L}(d)$ satisfies the following equalities*:

$$tr(K) = k_1\, dx \wedge dy = \frac{1}{2}(d-1)(d-2).\, d\Gamma.$$

As announced in the introduction and using only the previous methods, the following result and its proof give several complements of a basic result in planar web geometry:

Theorem 3. *Let $\mathcal{L}(d)$ be a linear and nonsingular planar web associated with a differential equation $F(x, y, y') = 0$ with canonical normalization (ω_i).*

Then, the following conditions are equivalent:

1) $\mathcal{L}(d)$ *is of maximal rank*;
2) $\omega_i = \rho\, du_i$ *for* $1 \le i \le d$ *with elements* u_i *in* $\mathcal{O}$ *and* ρ *in* $\mathcal{O}^*$;
3) *The connection* $(\mathcal{E}, \nabla)$ *associated with* $\mathcal{L}(d)$ *is integrable*;
4) *With the previous notation* $k = \partial_y^2(\dfrac{a_1}{a_0}) = 0$; *that is,* $d\Gamma = 0$;
5) $\mathcal{L}(d)$ *is algebraic.*

Proof. From the previous results, there are the following implications: $5) \Rightarrow 1) \Leftrightarrow 3) \Rightarrow 4)$. $2) \Leftrightarrow 4)$: For $1 \le i \le d$, we have $d\omega_i = (A_1\, dx + A_2\, dy) \wedge \omega_i = \dfrac{d\rho}{\rho} \wedge \omega_i$ if $\omega_i = \rho\, du_i$; hence $d\Gamma = 0$ from the general position hypothesis. Conversely, if $d\Gamma = 0$ then $\Gamma = d\varphi = \dfrac{d\rho}{\rho}$ if $\rho = e^{\varphi}$. But $d(\dfrac{\omega_i}{\rho}) = -\dfrac{d\rho}{\rho^2} \wedge \omega_i + \dfrac{d\omega_i}{\rho} = 0$ since $d\omega_i = \Gamma \wedge \omega_i$, hence $\omega_i = \rho\, du_i$ for $1 \le i \le d$ by the Poincaré lemma. $4) \Rightarrow 5)$: We have $\Gamma = d\varphi$ and from the previous description of $\mathfrak{a}_F$ in this case, the element $r := e^{-\varphi}$ verifies the relation $(\star\mathcal{L})$ with $t = 0$. Thus, F is a solution of the partial differential equation

$$\partial_x(f) + p\, \partial_y(f) = -\big(\partial_x(\varphi) + p\, \partial_y(\varphi)\big).\, f$$

with unknown $f \in \mathbb{C}\{x, y, p\}$. By a classical argument (cf. for instance [Ca]), the general solution of the previous equation is $f = e^{-\varphi}.\, \Phi(y - px, p)$ with $\Phi \in \mathbb{C}\{s, t\}$. This implies the equality $F = e^{-\varphi}.\, P(y - px, p)$ where P is a reduced element in $\mathbb{C}[s, t]$ with degree d from the hypothesis on $F \in \mathcal{O}[p]$. Thus $\mathcal{L}(d)$ is algebraic with $\mathcal{L}(d) = \mathcal{L}_C(d)$, where $P(s, t) = 0$ is an affine equation of the reduced algebraic curve $C \subset \mathbb{P}^2$. □

Bordeaux I University, Talence, France
E-mail address: Alain.Henaut@math.u-bordeaux.fr

References

[Ba] D. Barlet, Le faisceau $\omega_X^{\bullet}$ sur un espace analytique X de dimension pure, in *Fonctions de Plusieurs Variables Complexes* III (Séminaire F. Norguet, 1975–1977), *Lecture Notes in Math.* **670**, Springer-Verlag, Berlin, 1978, 187–204.

[B] W. Blaschke, *Einführung in die Geometrie der Waben*, Birkhäuser Verlag, Basel, 1955.

[B-B] W. Blaschke und G. Bol, *Geometrie der Gewebe*, Springer-Verlag, Berlin, 1938.

[Bo] G. Bol, Über ein bemerkenswertes Fünfgewebe in der Ebene, *Abh. Hamburg* **11** (1936), 387–393.

[B-C-3G] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt and P. A. Griffiths, *Exterior Differential Systems*, *Math. Sci. Res. Inst. Publ.* **18**, Springer-Verlag, New York, 1991.

[Ca] H. Cartan, *Calcul Différentiel*, Hermann, Paris, 1967.

[C] S. S. Chern, Web geometry, *Bull. Amer. Math. Soc.* **6** (1982), 1–8.

[C-G] S. S. Chern and P. A. Griffiths, Abel's theorem and webs, *Jahresber. Deutsch. Math.-Verein.* **80** (1978), 13–110, and Corrections and addenda to our paper: "Abel's theorem and webs", *Jahresber. Deutsch. Math.-Verein.* **83** (1981), 78–83.

[G1] P. A. Griffiths, Variations on a theorem of Abel, *Invent. Math.* **35** (1976), 321–390.

[G2] ———, On Abel's differential equations, in *Algebraic Geometry* (J. J. Sylvester Sympos., Johns Hopkins Univ., Baltimore, MD, 1976) (J.-I. Igusa, ed.), Johns Hopkins Univ. Press, Baltimore, Md., 1977, 26–51.

[G-M] M. Granger and P. Maisonobe, A basic course on differential modules, in $\mathcal{D}$-modules *Cohérents et Holonomes* (Nice, 1990), *Travaux en Cours* **45**, Hermann, Paris, 1993, 103–168.

[H1] A. Hénaut, Analytic web geometry, in *Web Theory and Related Topics* (Toulouse, 1996) (J. Grifone and É. Salem, eds.), World Sci. Publishing Co., River Edge, NJ, 2001, 6–47.

[H2] ———, Sur la courbure de Blaschke et le rang des tissus de $\mathbb{C}^2$, *Natur. Sci. Rep. Ochanomizu Univ.* **51** (2000), 11–25.

[M] G. Mignard, Rang et courbure des 3-tissus de $\mathbb{C}^2$, *C. R. Acad. Sci. Paris* **329** (1999), 629–632.

[P] L. Pirio, Study of a functional equation associated to Kummer's equation of the trilogarithm. Applications, Prépub. Univ. Paris 6, 2002.

[R] G. F. Robert, Relations fonctionnelles polylogarithmiques et tissus plans, Prépub. Univ. Bordeaux 1, 2002.

[S] D. C. Spencer, *Selecta*, Vol. 3, World Sci. Publishing Co., Philadelphia, 1985.

[W] J. Grifone and É. Salem (Eds.), *Web Theory and Related Topics*, World Sci. Publishing Co., River Edge, NJ, 2001.

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