On planar web geometry through abelian relations and connections

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1. Introduction

Web geometry is devoted to the study of families of foliations which are in general position. We restrict ourselves to the local situation, in the neighborhood of the origin in $\mathbb{C}^2$, with $d \geq 1$ complex analytic foliations of curves in general position. We are interested in the geometry of such configurations, that is, properties of planar $d$-webs which are invariant with respect to analytic local isomorphisms of $\mathbb{C}^2$.

The initiators of the subject are W. Blaschke, G. Thomsen and G. Bol in the 1930’s (cf. [B-B], [B] and for instance [H1]). Methods used here extend some works by S. S. Chern and P. A. Griffiths (cf. for instance [G1], [G2], [C], [C-G]) which bring a resurgence of interest in web geometry closely related to basic results due to N. Abel, S. Lie, H. Poincaré and G. Darboux. For recent results and applications of web geometry in various domains, refer to I. Nakai’s introduction, all papers and references contained in [W].

Let $\mathcal{O} := \mathbb{C}\{x, y\}$ be the ring of convergent power series in two variables. A (germ of a) nonsingular $d$-web $W(d)$ in $(\mathbb{C}^2, 0)$ is defined by a family of leaves which are germs of level sets $\{F_i(x, y) = \text{const.}\}$ where $F_i \in \mathcal{O}$ can be chosen to satisfy $F_i(0) = 0$ such that $dF_i(0) \wedge dF_j(0) \neq 0$ for $1 \leq i < j \leq d$ from the assumption of general position.

From the local inverse theorem, the study of possible configurations for the different $W(d)$ is interesting only for $d \geq 3$. The classification of such $W(d)$ is a widely open problem and the search for invariants of planar webs $W(d)$ motivates the present work.

Let $F(x, y, p) = a_0(x, y) \cdot p^d + a_1(x, y) \cdot p^{d-1} + \cdots + a_d(x, y)$ be an element of $\mathcal{O}[p]$ without multiple factor, not necessarily irreducible and such that $a_0 \neq 0$. We denote by $R = (-1)^{\frac{d(d-1)}{2}} a_0 \cdot \Delta$ the $p$-resultant of $F$ where $\Delta \in \mathcal{O}$ is its $p$-discriminant.

In a neighborhood of $(x_0, y_0) \in \mathbb{C}^2$ such that $R(x_0, y_0) \neq 0$, the Cauchy theorem asserts that the $d$ integral curves of the differential equation of the
first order
\[ F(x, y, y') = 0 \]

are the leaves of a nonsingular web \( \mathcal{W}(d) \) in \( (\mathbb{C}^2, (x_0, y_0)) \).

Every such \( F \in \mathcal{O}[p] \), up to an invertible element in \( \mathcal{O} \), gives rise to an implicit \( d \)-web \( \mathcal{W}(d) \) in \( (\mathbb{C}^2, 0) \) which is generically nonsingular. Inversely, if a nonsingular \( d \)-web in \( (\mathbb{C}^2, 0) \) is given by \( d \) vector fields \( X_i = A_i \partial_x + B_i \partial_y \) in general position, one may assume that \( A_i(0) \neq 0 \) for \( 1 \leq i \leq d \) after a linear change of coordinates. Then "its" differential equation \( F(x, y, y') = 0 \) corresponds to
\[ F(x, y, p) = \prod_{i=1}^{d} (A_i p - B_i). \]

This implicit form of a planar web will be retained throughout the present text. No leaf is preferred and we shall show how this form presents a natural setting for the study of planar webs and their singularities. Moreover, with the help of the web viewpoint, this approach enlarges methods to investigate the geometry of the differential equation \( F(x, y, y') = 0 \).

Basic examples of planar webs come from complex projective algebraic geometry. Let \( C \subset \mathbb{P}^2 \) be a reduced algebraic curve of degree \( d \), not necessarily irreducible and possibly singular. By duality in \( \mathbb{P}^2 \), one can get a special linear \( d \)-web \( L_C(d) \) called the algebraic web associated with \( C \subset \mathbb{P}^2 \) (cf. for instance [H1] for details). This web is singular and its leaves are family of straight lines. It corresponds, in a suitable local coordinate system, to a differential equation of the previous form given by \( F(x, y, p) = P(y - px, p) \) if \( P(s, t) = 0 \) is an affine equation for \( C \). If \( C \) contains no straight lines, the leaves of \( L_C(d) \) are generically the tangents of the dual curve \( \mathcal{C} \subset \mathbb{P}^2 \) of \( C \subset \mathbb{P}^2 \); otherwise, they belong to the corresponding pencils of straight lines.

One of the main invariants of a nonsingular planar web \( \mathcal{W}(d) \) is related to the notion of abelian relation. A \( d \)-uple \( (g_1(F_1), \ldots, g_d(F_d)) \) \in \( \mathcal{O}^d \) satisfying
\[ \sum_{i=1}^{d} g_i(F_i) dF_i = 0 \]
where \( g_i \in \mathbb{C}\{t\} \) is called an abelian relation of \( \mathcal{W}(d) \). By the above component presentation these relations form a \( \mathbb{C} \)-vector space denoted by \( \mathcal{A}(d) \).

For a nonsingular web \( \mathcal{W}(d) \) in \( (\mathbb{C}^2, 0) \), the following optimal inequality holds:
\[ \operatorname{rk} \mathcal{W}(d) := \dim_{\mathbb{C}} \mathcal{A}(d) \leq \frac{1}{2} (d-1)(d-2). \]

This bound is classic and, for example, we will recover it below with new methods coming from basic results in \( \mathcal{D} \)-modules theory (cf. for instance [G-M]). The integer \( \operatorname{rk} \mathcal{W}(d) \) called the rank of \( \mathcal{W}(d) \) defined above is an invariant of \( \mathcal{W}(d) \) which does not depend on the choice of the functions \( F_i \).
From the previous observations and properties, another basic result in planar web geometry is related to linear webs \( L(d) \) (i.e. all leaves of \( L(d) \) are straight lines, not necessarily parallel). For a linear and nonsingular web \( L(d) \) in \((\mathbb{C}^2, 0)\), the following assertions are equivalent:

i) There exists an abelian relation \( \sum_{i=1}^{d} g_i(F_i)dF_i = 0 \) with \( g_i \neq 0 \) for \( 1 \leq i \leq d \);

ii) The linear web \( L(d) \) is algebraic; that is, \( L(d) = L_C(d) \) where \( C \subset \mathbb{P}^2 \) is a reduced algebraic curve of degree \( d \), not necessarily irreducible and possibly singular;

iii) The rank of \( L(d) \) is maximal.

These equivalences play a fundamental role in the foundation of web geometry. Indeed, the implication ii) \( \Rightarrow \) iii) is a special case of Abel’s theorem and asserts that in fact

\[
\text{rk} L_C(d) = \dim_{\mathbb{C}} H^0(C, \omega_C) = \frac{1}{2}(d - 1)(d - 2)
\]

(cf. for instance [H1]). The difficult part i) \( \Rightarrow \) ii) is a kind of converse to Abel’s theorem. In the case \( d = 4 \), it was initiated by Lie’s theorem on surfaces of double translation (cf. for instance [C]) and deeply generalized, for \( d \geq 3 \) and higher codimension questions, by P. A. Griffiths (cf. [G1]). All modern proofs of this implication use the so-called GAGA principle.

Using only the methods introduced here we will get a proof for the above equivalence ii) \( \Leftrightarrow \) iii) and some complements essentially based on partial differential equations and the canonical normalization of \( W(d) \). In particular, these results explain why one condition alone implies all the previous equivalences.

This normalization gives rise to several analytic invariants of \( W(d) \) on \((\mathbb{C}^2, 0)\), where \( d(d - 3) \) of them are functions and the remaining \( d - 2 \) are 2-differential forms. These invariants extend the Blaschke curvature for \( W(3) \) and should be worth studying. A part of their significance will appear below.

Web geometry for nonsingular planar webs of maximum rank is, however, larger in extent than the algebraic geometry of plane curves. Indeed, there exist exceptional webs \( E(d) \) in \((\mathbb{C}^2, 0)\). Such a web \( E(d) \) is of maximum rank and cannot be made algebraic, up to an analytic local isomorphism of \( \mathbb{C}^2 \). One knows that necessarily \( d \geq 5 \) and the first known example is Bol’s 5-web \( B(5) \) which is related to the functional relation with five terms satisfied by the dilogarithm (cf. [Bo]). For special models in web geometry and their functional relations as well, a program to study polylogarithm webs is sketched in [H1]. The next exceptional web expected was Kummer’s 9-web \( K(9) \) related to the functional relation with nine terms of the trilogarithm. G. Robert proved in
A refinement of the rank is the finer invariant \((\varrho_3, \ldots, \varrho_d)\) called the weave of a nonsingular planar web \(W(d)\). This sequence of nonnegative integers is defined as follows: in the \(\mathbb{C}\)-vector space \(A(d)\) of abelian relations of \(W(d)\), consider the ascending chain of subspaces

\[A(d)_3 \subseteq A(d)_4 \subseteq \ldots \subseteq A(d)_d = A(d)\]

where \(A(d)_k\) is generated by special abelian relation \((g_1(F_1), \ldots, g_d(F_d))\) of \(W(d)\) containing at most \(k\) nonzero components. Then set

\[\varrho_k := \dim_{\mathbb{C}} A(d)_k / A(d)_{k-1}\]

with \(A(d)_2 = 0\). In particular, we have \(\text{rk } W(d) = \varrho_3 + \cdots + \varrho_d\). For example, the weave of \(B(5)\) is \((5, 0, 1)\) and that of \(K(9)\) is \((17, 3, 3, 3, 0, 0, 2)\). In the algebraic case, the weave of \(\mathcal{L}_C(d)\) is related to the irreducible components of \(C \subset \mathbb{P}^2\).

According to the previous results, methods for determining the rank (resp. the weave) of any nonsingular planar web are of great interest, in particular for the algebraization problem (cf. for instance [H1] through the second order differential equation \(y'' = p_{W(d)}(x, y, y')\) associated to \(W(d)\)) and the study of exceptional webs.

Let \(S\) be the surface defined by \(F(x, y, p) = 0\). The projection \(\pi : S \to (\mathbb{C}^2, 0)\) induced by \((x, y, p) \mapsto (x, y)\) is generically finite with degree \(d\) and gives rise to a trace which is very useful on differential forms.

Coming back to the classical geometric study of differential equations \(F(x, y, y') = 0\), we shall confirm how some basic objects attached to the previous projection govern the geometry of the planar web associated with this equation, from the generic viewpoint as well as the singular one. In fact, even if we restrict our attention to the nonsingular case, most of the objects introduced naturally extend to the singular case.

We suppose from now on that the \(p\)-resultant \(R \in \mathcal{O} \) of \(F\) satisfies \(R(0) \neq 0\). Thus \(\pi\) is a covering map of degree \(d\). The main result in [H2] will be recalled with some details in the next paragraph. Briefly, it is the following: the \(\mathbb{C}\)-vector space of 1-forms

\[a_F := \{ \omega = r \cdot \frac{dy - p dx}{\partial_p(F)} \in \pi_* (\Omega^1_S) ; r \in \mathcal{O}[p] \text{ with } \deg r \leq d - 3 \text{ and } d\omega = 0 \}\]

is identified with the \(\mathbb{C}\)-vector space \(A(d)\) of abelian relations of the web \(W(d)\) generated by \(F\). In this identification an abelian relation is interpreted as the vanishing trace of an element of \(a_F\). By definition the forms in \(a_F\) are closed and moreover appear as solutions of a linear differential operator \(p_0 : J_1(\mathcal{O}^{d-2}) \to \mathcal{O}^{d-1}\) of order 1 induced by the usual differential on 1-forms of the surface \(S\).
Using basic results on overdetermined systems of linear partial differential equations which extend the É. Cartan theory (cf. for instance [S], [B-C-3G]) and in particular the first complex of Spencer of an explicit prolongation $p_k : J_{k+1}(O^{d-2}) \to J_k(O^{d-1})$ of $p_0$, we obtain in the last paragraph one of the main results of this paper:

There exists a $\mathbb{C}$-vector fiber bundle $E$ of rank $\frac{1}{2}(d-1)(d-2)$ on $(\mathbb{C}^2, 0)$ equipped with a connection $\nabla$ such that its $\mathbb{C}$-vector space of horizontal sections is isomorphic to $A(d)$. Moreover, there exists an adapted basis $(e_\ell)$ of $E$ such that the curvature of $(E, \nabla)$ has the following matrix:

$$
\begin{pmatrix}
  k_1 & k_2 & \ldots & k_{\frac{1}{2}(d-1)(d-2)} \\
  0 & 0 & \ldots & 0 \\
  \vdots & \vdots & \ldots & \vdots \\
  0 & 0 & \ldots & 0
\end{pmatrix}
\ dx \wedge dy.
$$

In particular, by the Cauchy-Kowalevski theorem, an explicit way to find maximal rank webs is given, using only the coefficients of $F$. In the case $d = 3$, we find $k_1 \ dx \wedge dy$ as a curvature matrix and it is proved that this 2-form is the usual Blaschke curvature of $W(3)$ (cf. [B-B], [B] and for instance [H1]).

Moreover complete effective results are given for $d = 3$ and $d = 4$. The previous curvature probably depends only on the planar web $W(d)$ and not on the differential equation $F(x, y, y') = 0$ that we use to define it. It is at least true for $d = 3$ and $d = 4$. Thus, the construction of the above $(E, \nabla)$ generalizes the W. Blaschke approach.

For a general linear web some simplifications appear in the description of $(E, \nabla)$ and from the above results some of the previous equivalences for the $L(d)$ are obtained as well as several complements.

Furthermore, it can be noted to close this introduction that in general the previous $(E, \nabla)$ is in fact a meromorphic connection with poles on the discriminant locus of the differential equation $F(x, y, y') = 0$, that is, the analytic germ defined in a neighborhood of $0 \in \mathbb{C}^2$ by $\Delta(x, y) = 0$.

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2. Traces from $S$, abelian relations
and canonical normalisation for $W(d)$

We recall that $R(0) \neq 0$. Thus, the surface $S$ defined by $F$ is nonsingular over $0 \in \mathbb{C}^2$. Locally on $S$, we have the complex $(\Omega^*_{S}, d)$ where

$$
\Omega^*_{S} = \Omega^*_{\mathbb{C}^2}/(dF \wedge \Omega_{\mathbb{C}^2}^{*1}, F \Omega^*_{\mathbb{C}^2}).
$$
Since $\partial_x(F)\,dx + \partial_y(F)\,dy + \partial_p(F)\,dp = 0$ in $\Omega^1_S$, every element $\omega$ in $\Omega^1_S$ gives rise to an expression
\[
\omega := \frac{r_x\,dy - r_y\,dx}{\partial_p(F)}
\]
with $(r_x, r_y, r_p, \theta) \in \mathcal{O}^4_S$ such that the relation $r_x\,\partial_x(F) + r_y\,\partial_y(F) + r_p\,\partial_p(F) = \theta \cdot F$ holds. Inversely the previous expression coupled with this relation corresponds to an element in $\Omega^1_S$ essentially defined through
\[
\omega = \frac{1}{3} \cdot \left( \frac{r_p}{\partial_y(F)} - \frac{r_y}{\partial_p(F)} \right) dx + \left( \frac{r_x}{\partial_p(F)} - \frac{r_p}{\partial_x(F)} \right) dy + \left( \frac{r_y}{\partial_x(F)} - \frac{r_x}{\partial_y(F)} \right) dp
\]
because \[
\frac{r_x\,dy - r_y\,dx}{\partial_p(F)} = \frac{r_y\,dp - r_p\,dy}{\partial_x(F)} = \frac{r_p\,dx - r_x\,dp}{\partial_y(F)}
\]
in $\Omega^1_S$.

Moreover, it can be checked that the exterior differential $d : \Omega^1_S \rightarrow \Omega^2_S$ is defined by
\[
d\omega = d\left( \frac{r_x\,dy - r_y\,dx}{\partial_p(F)} \right) = \left( \partial_x(r_x) + \partial_y(r_y) + \partial_p(r_p) - \theta \right) \frac{dx \wedge dy}{\partial_p(F)}
\]
because \[
\frac{dx \wedge dy}{\partial_p(F)} = \frac{dy \wedge dp}{\partial_x(F)} = \frac{dp \wedge dx}{\partial_y(F)}
\]
in $\Omega^2_S$.

The projection $\pi : S \rightarrow (\mathbb{C}^2, 0)$ is a covering map of degree $d$ with local branches $\pi_i(x, y) = (x, y, p_i(x, y))$. Thus, we have
\[
F(x, y, p) = a_0(x, y) \prod_{i=1}^d (p - p_i(x, y)).
\]
Moreover, the vector fields which correspond to the nonsingular $d$-web $\mathcal{W}(d)$ of $(\mathbb{C}^2, 0)$ generated by the differential equation $F(x, y, y') = 0$ have the form
\[
X_i := \partial_x + p_i \partial_y \text{ with } p_i(0) \neq p_j(0) \text{ for } 1 \leq i < j \leq d.
\]
We denote by $\pi_* (\Omega^1_S)$ the fiber in $0 \in \mathbb{C}^2$ of the direct image sheaf of $\Omega^1_S$ with respect to $\pi$. We have the trace morphism $\text{Trace}_\pi : \pi_* (\Omega^1_S) \rightarrow \Omega^1$ defined by $\text{Trace}_\pi(\omega) := \sum_{i=1}^d \pi_i^*(\omega)$ where $\Omega^1$ is the $\mathcal{O}$-module of Pfaff forms on $(\mathbb{C}^2, 0)$. This morphism is $\mathcal{O}$-linear and commutes with the differential $d$. It can be noted that a large part of the previous constructions extends to the singular case by means of the Barlet complex $(\omega^{\bullet}_S, d)$ constructed via special meromorphic forms with poles on the singular set of $S$ (cf. [Ba]).

The following result is proved in [H2]: every $r \in \mathcal{O}[p]$ such that $\deg r \leq d - 2$ gives an element $\omega = r \cdot \frac{dy - p\,dx}{\partial_p(F)}$ which belongs to $\pi_* (\Omega^1_S)$.

More precisely, there exist elements $r_p$ and $t$ in $\mathcal{O}[p]$ with degree less than or equal to $d - 1$ which satisfy the following fundamental relation:
\[
(*) \quad r \cdot (\partial_x(F) + p\,\partial_y(F)) + r_p \cdot \partial_p(F) = (\partial_x(r) + p\,\partial_y(r) + \partial_p(r_p) - t) \cdot F.
\]
Omitting the dependency on \((x, y)\), the proof uses the ubiquitous Lagrange interpolation formula and consists in checking that if
\[
\lambda := \sum_{i=1}^{d} \frac{\rho_i \partial_y(F_i)}{p - p_i}, \quad \mu := \sum_{i=1}^{d} \frac{X_i(p_i) \cdot \rho_i \partial_y(F_i)}{p - p_i}, \quad \text{and} \quad \nu := \sum_{i=1}^{d} \frac{X_i(p_i) \cdot \partial_y(F_i)}{p - p_i}
\]
where \(\rho_i := \frac{r(x, y, p_i)}{\partial_p(F)(x, y, p_i)}\) for \(1 \leq i \leq d\), we have the following equality:
\[
\partial_x(\lambda) + \partial_y(\mu) + \partial_p(\mu) = \nu.
\]
Then it is sufficient to set \(r_p = F \cdot \mu\) and \(t = F \cdot \nu\) since by definition \(r = F \cdot \lambda\).

Moreover if \(\deg r \leq d - 3\), as we shall assume from now on, then \(\deg t \leq d - 2\) by the relation \((\star)\) and from the previous observations, we have the explicit equality
\[
d(r \cdot \frac{dy - p dx}{\partial_p(F)}) = t \cdot \frac{dx \wedge dy}{\partial_p(F)}.
\]

With the notation of the introduction, the main result in [H2] can be stated as the following:

**Theorem a_F.** The map
\[
(g_i(F_i))_i \mapsto \omega := \left(F \cdot \sum_{i=1}^{d} \frac{g_i(F_i) \partial_y(F_i)}{p - p_i}\right) \cdot \frac{dy - p dx}{\partial_p(F)} \in \pi_*(\Omega^1_S)
\]
defines a \(\mathbb{C}\)-isomorphism \(T : A(d) \rightarrow a_F\) such that \(\text{Trace}_x(\omega) = \sum_{i=1}^{d} g_i(F_i) dF_i = 0\). In particular, \(\text{rk} \mathcal{W}(d) = \dim_{\mathbb{C}} a_F\).

It can be noted that the previous map \(T\) is in fact closely related to the application \(E : (\mathbb{C}^2, 0) \times \mathbb{P}^1 \rightarrow \mathbb{P}^{\text{rk} \mathcal{W}(d) - 1}\) which extends a basic construction due to H. Poincaré. This application is very useful in making maximal rank webs algebraic (cf. [H1]).

The relation \((\star)\) implies exactly \(2d - 1\) relations between the coefficients \(a_i, b_j, c_k\) and \(t_l\) where
\[
F = a_0 \cdot p^d + a_1 \cdot p^{d-1} + \cdots + a_d,
\]
\[
r = b_3 \cdot p^{d-3} + b_4 \cdot p^{d-4} + \cdots + b_d,
\]
\[
r_p = c_1 \cdot p^{d-1} + c_2 \cdot p^{d-2} + \cdots + c_d,
\]
\[
t = t_2 \cdot p^{d-2} + t_3 \cdot p^{d-3} + \cdots + t_d
\]
are elements in \(\mathcal{O}[p]\). Moreover, these relations can be viewed in a matrix form.
For $d = 3$, the relation ($\star$) corresponds to the following matrix system:

$$
\begin{pmatrix}
0 & a_0 & -a_0 & 0 & 0 \\
0 & a_0 & a_1 & 0 & -2a_0 & 0 \\
a_0 & a_1 & a_2 & a_2 & -a_1 & -3a_0 \\
a_1 & a_2 & a_2 & a_3 & 0 & -2a_1 \\
a_2 & a_3 & 2a_3 & 0 & a_3 & -a_2
\end{pmatrix}
\begin{pmatrix}
\partial_x(b_3) \\
\partial_y(b_3) \\
c_1 \\
c_2 \\
c_3
\end{pmatrix}
= b_3 \cdot
\begin{pmatrix}
\partial_y(a_0) \\
\partial_x(a_0) + \partial_y(a_1) \\
\partial_x(a_1) + \partial_y(a_2) \\
\partial_x(a_2) + \partial_y(a_3) \\
\partial_x(a_3) + \partial_y(a_4)
\end{pmatrix}
+ t_2 \cdot
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
0
\end{pmatrix}
+ t_3 \cdot
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix}.
$$

It can be verified that the determinant of the $5 \times 5$-matrix above is equal to the $p$-resultant $R$ of $F$. Which is a consequence of the classical formula of Sylvester, namely

$$
R = \begin{vmatrix}
a_0 & a_1 & a_2 & a_3 & 0 \\
0 & a_0 & a_1 & a_2 & a_3 \\
3a_0 & 2a_1 & a_2 & 0 & 0 \\
0 & 3a_0 & 2a_1 & a_2 & 0 \\
0 & 0 & 3a_0 & 2a_1 & a_2
\end{vmatrix}.
$$

Thus, by Cramer formulas, it can be checked since $R(0) \neq 0$ that the previous matrix system is equivalent to the following nonhomogeneous linear differential system:

($\star$)

$$
\begin{align*}
\partial_x(b_3) + A_{1,1} b_3 &= t_3 \\
\partial_y(b_3) + A_{2,1} b_3 &= t_2 
\end{align*}
$$

where, in fact, $A_{i,j} \in \mathcal{O}[1/\Delta]$ which would be interesting in the singular case.

For $d = 4$, the relation ($\star$) corresponds to the following matrix system:

$$
\begin{pmatrix}
0 & 0 & a_0 & -a_0 & 0 & 0 & 0 \\
0 & 0 & a_0 & a_1 & 0 & -2a_0 & 0 & 0 \\
a_0 & a_1 & a_2 & a_2 & -a_1 & -3a_0 & 0 \\
a_1 & a_2 & a_3 & 2a_3 & 0 & -2a_1 & -4a_0 & 0 \\
a_2 & a_3 & a_4 & 3a_4 & a_3 & -a_2 & -3a_1 & -2a_2 \\
a_3 & a_4 & 0 & 0 & 2a_4 & 0 & -2a_2 & 0 \\
a_4 & 0 & 0 & 0 & a_4 & -a_3 & \partial_x(a_4) & 0
\end{pmatrix}
\begin{pmatrix}
\partial_x(b_4) \\
\partial_y(b_4) + \partial_y(b_4) \\
c_1 \\
c_2 \\
c_3 \\
c_4
\end{pmatrix}
= b_3 \cdot
\begin{pmatrix}
\partial_y(a_0) \\
\partial_x(a_0) + \partial_y(a_1) \\
\partial_x(a_1) + \partial_y(a_2) \\
\partial_x(a_2) + \partial_y(a_3) \\
\partial_x(a_3) + \partial_y(a_4)
\end{pmatrix}
+ b_4 \cdot
\begin{pmatrix}
0 \\
\partial_y(a_0) \\
\partial_x(a_0) + \partial_y(a_1) \\
\partial_x(a_1) + \partial_y(a_2) \\
\partial_x(a_2) + \partial_y(a_3) \\
\partial_x(a_3) + \partial_y(a_4)
\end{pmatrix}
+ t_2 \cdot
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix}
+ t_3 \cdot
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix}
+ t_4 \cdot
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix}.
$$
With the same arguments used before, but with a $7 \times 7$-matrix, this system is equivalent to the following:

\[
\begin{align*}
\begin{cases}
\partial_x(b_1) + A_{1,1} b_3 + A_{1,2} b_4 = t_4 \\
\partial_x(b_3) + \partial_y(b_4) + A_{2,1} b_3 + A_{2,2} b_4 = t_3 \\
\partial_y(b_3) + A_{3,1} b_3 + A_{3,2} b_4 = t_2
\end{cases}
\end{align*}
\]  

(*)

with some $A_{i,j} \in O[1/\Delta]$.

In the general case, using again the Sylvester formula for the resultant, the relation (*) gives rise to the following nonhomogeneous linear differential system:

\[
\begin{align*}
\begin{cases}
\partial_x(b_d) + A_{1,1} b_3 + \cdots + A_{1,d-2} b_d = t_d \\
\partial_x(b_{d-1}) + \partial_y(b_d) + A_{2,1} b_3 + \cdots + A_{2,d-2} b_d = t_{d-1} \\
\vdots \\
\partial_x(b_3) + \partial_y(b_4) + A_{d-2,1} b_3 + \cdots + A_{d-2,d-2} b_d = t_3 \\
\partial_y(b_3) + A_{d-1,1} b_3 + \cdots + A_{d-1,d-2} b_d = t_2
\end{cases}
\end{align*}
\]  

(*d)

with explicit $A_{i,j} \in O[1/\Delta]$ obtained only from the coefficients of $F$ by Cramer formulas.

Let $\mathcal{M}(d)$ be the homogeneous linear differential system associated with (*d). Then, using the previous theorem and the fact that $a_F$ is uniquely determined by the analytic solutions of $\mathcal{M}(d)$, we have the following identifications:

\[\mathcal{A}(d) = a_F = \text{Sol} \mathcal{M}(d)\]

where $\text{Sol} \mathcal{M}(d)$ denotes the $\mathbb{C}$-vector space of analytic solutions of $\mathcal{M}(d)$.

In particular, using only the symbol of the linear differential system $\mathcal{M}(d)$, we recover the classical optimal bound $\frac{1}{2}(d-1)(d-2)$ for the rank $rk \mathcal{W}(d)$.

Indeed, let $\mathcal{D}$ be the ring of linear differential operators with coefficients in $O$ (cf. for instance [G-M] for basic results and terminology). We denote by $\mathcal{M}(d)$ the left $\mathcal{D}$-module associated with $\mathcal{M}(d)$ and $gr \mathcal{M}(d)$ its natural associated graded $O[\xi, \eta]$-module. The special form of the system $\mathcal{M}(d)$, namely its symbol, implies that

\[(\xi, \eta)^{d-2} \subseteq \text{Fitt}_0(gr \mathcal{M}(4)) \subseteq \text{Ann}(gr \mathcal{M}(d))\]

where $\text{Fitt}_0(gr \mathcal{M}(4))$ is the 0-th Fitting ideal of $gr \mathcal{M}(d)$ and $\text{Ann}(gr \mathcal{M}(d))$ its annihilator. This proves that we have the following identification:

\[\mathcal{M}(d) = O^{rk \mathcal{W}(d)}\text{ as left } \mathcal{D}-\text{modules.}\]

In other words, we obtain either $\mathcal{M}(d) = 0$, which is the generic case for webs $\mathcal{W}(d)$ or $\mathcal{M}(d)$ is an integrable connection. Moreover, the previous inclusions give the optimal bound for $rk \mathcal{W}(d)$ since

\[rk \mathcal{W}(d) = \text{mult} \mathcal{M}(d) := \text{mult} gr \mathcal{M}(d) \leq \text{mult} O[\xi, \eta]/(\xi, \eta)^{d-2} = \frac{1}{2}(d-1)(d-2).\]
The previous identification with the $\mathcal{D}$-modules viewpoint comes from the Hilbert-Rückert Nullstellensatz and does not give rise, in a simple way, to general methods for determining the rank of the nonsingular planar web $W(d)$. Another method to study the system $\mathcal{M}(d)$ will be presented in the next section with more geometric objects.

However, in the case $d = 3$, a formula for the rank depending only on the coefficients of $F$ has been given by G. Mignard (cf. [M]) using the $\mathcal{D}$-modules approach (cf. [H1]). A different proof of it is given in [H2]. It uses some normalization for $W(d)$ that we introduce for $d \geq 3$ to end this section.

Let $\omega := \frac{dy - p \, dx}{\partial_p(F)}$ be the particular 1-form in $\Omega^1_S$. For $1 \leq i \leq d$, the forms $\omega_i := \pi^*_i(\omega)$ define the $d$-web $W(d)$ associated to $F(x, y, y') = 0$. For these forms, we have

$$\sum_{i=1}^{d} \omega_i = 0, \quad \sum_{i=1}^{d} p_i \cdot \omega_i = 0, \ldots, \quad \sum_{i=1}^{d} p_i^{d-3} \cdot \omega_i = 0;$$

that is, $\text{Trace}_x(p^k \cdot \omega) = 0$ for $0 \leq k \leq d - 3$ since $\sum_{i=1}^{d} \frac{p^j_i(x, y)}{\partial_p(F)(x, y, p_i(x, y))} = 0$ for $0 \leq j \leq d - 2$ by the Lagrange interpolation formula.

Any family $(\tilde{\omega}_i)$ of 1-forms which defines $W(d)$ and such that the following $d - 2$ relations are satisfied:

$$\sum_{i=1}^{d} p^k_i \cdot \tilde{\omega}_i = 0 \text{ for } 0 \leq k \leq d - 3$$

will be called a normalization of the nonsingular planar web $W(d)$. From the general position hypothesis, it may be remarked that the $d - 2$ previous relations which are satisfied by the $(\tilde{\omega}_i)$ are necessarily independent.

Such a normalization exists and the previous one $(\omega_i)$ constructed from the particular 1-form $\omega = \frac{dy - p \, dx}{\partial_p(F)} \in \Omega^1_S$ will be called the canonical normalization of $W(d)$. This terminology is justified by some properties.

The first one gives a useful means to compare two normalisations of $W(d)$.

**Proposition 1.** Let $(\tilde{\omega}_i)$ be a normalization of $W(d)$. Now, $\tilde{\omega}_i = g \cdot \omega_i$ for $1 \leq i \leq d$ where $g$ is an invertible element of $\mathcal{O}$ and $(\omega_i)$ is the canonical normalization of this web.

**Proof.** For $d = 3$, this proposition is a basic result to obtain the property of the Blaschke curvature for any $W(3)$ (cf. for instance [B] and below). This proof naturally extends to $d \geq 4$ and we give the method for $d = 4$. For any normalization and naturally for $(\omega_i)$ the canonical one, we have the
following form:
\[
\begin{aligned}
\omega_1 + \omega_2 &= -\omega_3 - \omega_4 \\
p_1 \cdot \omega_1 + p_2 \cdot \omega_2 &= -p_3 \cdot \omega_3 - p_4 \cdot \omega_4
\end{aligned}
\]
which implies \((p_4 - p_1) \cdot \omega_1 \wedge \omega_2 = (p_4 - p_3) \cdot \omega_2 \wedge \omega_3\). By circular permutation we get, with classical notation, a nonsingular 2-form \(\Omega\) on \((\mathbb{C}^2, 0)\) such that
\[
\Omega := (p_1 - p_2)(p_2 - p_3)(p_3 - p_4)(p_4 - p_1) \cdot \omega_1 \wedge \omega_2
\]
\[
= -(p_1 - p_2)(p_2 - p_3)(p_3 - p_4)(p_4 - p_1) \cdot \omega_2 \wedge \omega_3
\]
\[
= (p_1 - p_2)(p_2 - p_3)(p_3 - p_4)(p_4 - p_1) \cdot \omega_3 \wedge \omega_4
\]
\[
= -(p_1 - p_2)(p_2 - p_3)(p_3 - p_4)(p_4 - p_1) \cdot \omega_4 \wedge \omega_1.
\]
For \((\tilde{\omega}_i)\), we have \(\tilde{\omega}_i = g_i \cdot \omega_i\) with \(g_i \in \mathcal{O}^*\). From the previous observations, we get a nonsingular 2-form \(\tilde{\Omega}\) which satisfies the following equalities:
\[
\tilde{\Omega} = g_1g_2 \cdot \Omega = g_2g_3 \cdot \Omega = g_3g_4 \cdot \Omega = g_4g_1 \cdot \Omega.
\]
This proves \(g_1 = g_2\) and \(g_2 = g_4\). But also \((p_2 - p_1) \cdot \omega_1 \wedge \omega_2 = (p_4 - p_3) \cdot \omega_1 \wedge \omega_3\); thus we obtain \(g_2 = g_3\) and \(\tilde{\Omega} = g^2 \cdot \Omega\) with \(g := g_1 = g_2 = g_3 = g_4\) which ends the proof of the proposition. \(\square\)

The second property of the canonical normalization \((\omega_i)\) of \(W(d)\) is to be related to the columns of the \((d - 1) \times (d - 2)\)-matrix \((A_{i,j})\) which appears in the previous differential system \((*_d)\). In fact, from the explicit expression of \(d : \Omega^1_S \longrightarrow \Omega^2_S\), we obtain for \(0 \leq k \leq d - 3\) and for the particular 1-form \(\omega = \frac{dy - p \, dx}{\partial_p(F)} \in \Omega^1_S\) the following equalities:
\[
d(p^k \cdot \omega) = (A_{d-1,d-2-k} \cdot p^{d-2} + A_{d-2,d-2-k} \cdot p^{d-3} + \cdots + A_{1,d-2-k}) \cdot \frac{dx \wedge dy}{\partial_p(F)(x, y, p)}.
\]
In particular for \(0 \leq k \leq d - 3\) and \(1 \leq i \leq d\), we have the main equalities
\[(k_i) \quad d(p^k \cdot \omega_i) = (A_{1,d-2-k} \, dx + \sum_{j=0}^{d-3} p^j_i \cdot A_{j+2,d-2-k} \, dy) \wedge \omega_i
\]
which gives by differentiation for \(d \geq 4\) the following \(d - 3\) relations:
\[
A_{1,d-3} + \sum_{j=1}^{d-2} (A_{j+1,d-3} - A_{j,d-2}) \cdot p^j_i - A_{d-1,d-2} \cdot p^{d-1}_i = X_i(p_i)
\]
\[
\vdots
\]
\[
\sum_{j=0}^{d-4} A_{j+1,1} \cdot p^j_i + (A_{d-2,1} - A_{1,d-2}) \cdot p^{d-3}_i + (A_{d-1,1} - A_{2,d-2}) \cdot p^{d-2}_i - \sum_{j=1}^{d-3} A_{j+2,d-2} \cdot p^{j+d-2}_i = (d - 3) \cdot p^{d-4}_i \cdot X_i(p_i).
\]
These relations prove that the matrix \((A_{i,j})\) has the following particular form:

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & A_1 \\
0 & 0 & \cdots & A_1 & A_2 \\
0 & 0 & \cdots & A_2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & A_1 & \cdots & 0 & 0 \\
A_1 & A_2 & \cdots & 0 & 0 \\
A_2 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

if and only if the web \(W(d)\) is linear; that is, \(W(d) = \mathcal{L}(d)\).

Since we have the equivalence: \(W(d)\) is linear if and only if \(X_i(p_i) = 0\) for \(1 \leq i \leq d\), the matrix \((A_{i,j})\) above is obtained step by step from the previous relations by using the general position hypothesis and a Vandermonde determinant. Moreover, in the linear case we have \(r_p = 0\) in the relation \((\star)\) from the previous equivalence. Thus, by identification with the \(t = 0\) and the relation \((\star)\) above is obtained step by step from the
determinant. Moreover, in the linear case we have \(r_p = 0\) in the relation \((\star)\) from the previous equivalence. Thus, by identification with the \(t\) corresponding to \(r = 1\), we obtain \(A_2 = -\frac{\partial_y(a_0)}{a_0}\) and \(A_1 = -\frac{\partial_x(a_0)}{a_0} - \frac{\partial_y(a_1)}{a_0}\) from the coefficients of \(F\).

Furthermore in the algebraic case, that is \(W(d) = \mathcal{L}_C(d)\) with \(C \subset \mathbb{P}^2\), the matrix \((A_{i,j})\) obtained from \(F(x, y, p) = P(y - px, p)\) where \(P(s, t) = 0\) is an affine equation for \(C\) is the null matrix. Indeed, here we have \(\partial_x(F) + p\partial_y(F) = 0\) and the relation \((\star)\) in this linear case implies that \(r = 1\) corresponds to \(t = 0\).

For any normalization \((\tilde{\omega}_i)\) of \(W(d)\), the general position hypothesis gives a matrix \((\tilde{A}_{i,j})\) which satisfies the analogue to the relations \((k_i)\) before, but with \(\tilde{\omega}_i\) and \(\tilde{A}_{i,j}\). Moreover, the similar \(d - 3\) relations obtained above are satisfied. This normalization gives rise to several invariants of \(W(d)\) as follows:

**Theorem 1.** With the previous notation, the \((d - 1) \times (d - 2)\)-matrix \((A_{i,j})\) coming from \(F(x, y, y') = 0\) gives analytic invariants on \((\mathbb{C}^2, 0)\) of the nonsingular planar web \(W(d)\) generated by this differential equation: on the one hand, \(d(d - 3)\) functions

\[A_{m,n}\] for \(2 \leq m + n \leq d - 2\) and \(d + 1 \leq m + n \leq 2d - 3\),

\[A_{u,d-1-u} - A_{1,d-2}\] for \(2 \leq u \leq d - 2\) and \(A_{v,d-v} - A_{2,d-2}\) for \(3 \leq v \leq d - 1\) and on the other hand, \(d - 2\) differential forms of degree 2

\[d\Gamma_q\] for \(1 \leq q \leq d - 2\) where \(\Gamma_q := A_{d-q-1,q}dx + A_{d-q,q}dy\).

In particular in the linear case the \(2\)-differential form \(d\Gamma = kdx \wedge dy\) is an invariant of \(\mathcal{L}(d)\) where \(\Gamma := A_1dx + A_2dy\), and, explicitly, \(k := \partial_x(A_2) - \partial_y(A_1) = \partial_y^2(a_1)/a_0\).
Proof. For the functions, using the general position hypothesis, we have \( \tilde{A}_{m,n} = A_{m,n} \) for suitable index and the other equalities. It is a direct consequence of the relations induced by \((k_i)\) and the analogue for any normalization \((\tilde{\omega}_i)\) of \(W(d)\). Using Proposition 1 and the general position hypothesis, we have for \(1 \le q \le d-2\) and from the relation \((k_i)\), the equalities \(\tilde{A}_{d-q-1,q} - A_{d-q-1,q} = \frac{\partial_x(g)}{g}\) and \(\tilde{A}_{d-q,q} - A_{d-q,q} = \frac{\partial_y(g)}{g}\) which prove the result for the 2-forms. In the linear case, the result comes from the previous calculations.

For \(d = 3\), we set \(\gamma = A_{1,1}dx + A_{2,1}dy\). From the previous observations and with the canonical normalization \((\omega_i)\) of \(W(3)\), we have \(d\omega_i = \gamma \wedge \omega_i\) for \(1 \le i \le 3\), which proves the following result of [H2]:

*The Blaschke curvature of the nonsingular planar web \(W(3)\) is equal to \(d\gamma\).*

Indeed, we know (cf. for instance [B] or the previous theorem) that the Blaschke curvature \(d\gamma\) of \(W(3)\) does not depend on the normalization used to define it, contrary to the 1-form \(\gamma\) on \((\mathbb{C}^2,0)\) such that \(d\omega_i = \gamma \wedge \omega_i\) for \(1 \le i \le 3\).

Moreover, according to the integrability condition of the homogeneous differential system \(M(3)\) associated with \((\star_3)\), we get the nonsurprising result:

\[
\text{rk } W(3) = \begin{cases} 0 & \text{if } d\gamma \neq 0 \\ 1 & \text{if } d\gamma = 0 \end{cases}.
\]

Using integrability conditions for \(M(d)\), this concrete approach will be generalized for \(d \geq 3\) in the next section.

**3. On the connection \((\mathcal{E}, \nabla)\) associated with \(W(d)\) and some applications**

From the previous section, the exterior differential \(d : \pi_* (\Omega^1_S) \longrightarrow \pi_* (\Omega^2_S)\) coming from the surface \(S\), gives by restriction a linear differential operator \(\rho : \mathcal{O}^{d-2} \longrightarrow \mathcal{O}^{d-1}\) on \((\mathbb{C}^2,0)\). It is defined by \(\rho(b_3, \ldots, b_d) = (t_2, \ldots, t_d)\) where \(d(r \cdot \frac{dy - p dx}{\partial_p(F)}) = t \cdot \frac{dx \wedge dy}{\partial_p(F)}\) with \(r = b_3 \cdot p^{d-3} + \cdots + b_d\) and \(t = t_2 \cdot p^{d-2} + \cdots + t_d\).

This operator of order 1 is induced by the homogeneous linear differential system \(M(d)\) associated with \((\star_d)\). Its corresponding morphism of \(\mathcal{O}\)-modules \(p_0 : J_1(\mathcal{O}^{d-2}) \longrightarrow \mathcal{O}^{d-1}\) satisfies \(p_0 \circ j_1 = \rho\) where in matrix form \(j_1(b) = (b, \partial_x(b), \partial_y(b))\) as jets. From the nature of the system \((\star_d)\), the kernel \(R_0\) of \(p_0\) is a free \(\mathcal{O}\)-module of finite type.
We use, with minor modifications, the now classical notation from the works of D. Spencer and H. Goldschmidt (cf. [S] and for instance [B-C-3G]). In particular, \( p_k : J_{k+1}(O^{d-2}) \to J_k(O^{d-1}) \) denotes the \( k \)-th prolongation of \( p_0 \) for \( k \geq 0 \) obtained by successive derivations and \( R_k \) is the kernel of \( p_k \). For symbols, we have natural exact sequences:

\[
0 \to g_k \to S_{k+1}(O^{d-2}) \xrightarrow{\sigma_k} S_k(O^{d-1}) \xrightarrow{\tau_k} \text{Coker } \sigma_k \to 0.
\]

With this notation and among others the “snake” lemma, we have for \( k \geq 0 \) the following exact commutative diagram of \( O \)-modules:

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & & & & & & & \\
0 & g_{k+1} & \to & S_{k+2}(O^{d-2}) & \xrightarrow{\sigma_{k+1}} & S_{k+1}(O^{d-1}) & \xrightarrow{\tau_k} & \mathfrak{R}_k & \to & 0 \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & R_{k+1} & \to & J_{k+2}(O^{d-2}) & \xrightarrow{p_{k+1}} & J_{k+1}(O^{d-1}) & & & & & \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & R_k & \to & J_{k+1}(O^{d-2}) & \xrightarrow{p_k} & J_k(O^{d-1}) & & & & & \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{R}_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

where \( \pi_k \) (resp. \( \hat{\pi}_k \)) is the natural projection, \( \pi_k \) is the morphism induced on kernels and \( \mathfrak{R}_k := \text{Coker } \sigma_{k+1} \) is the obstruction space of formal integrability of \( p_k \).

Moreover, we have the following equivalence: \( \pi_k \) is a surjective morphism if and only if \( \beta_k = 0 \).

For \( p \geq 1 \) and \( k \geq 0 \) we have \( \text{rk } J_k(O^p) = \frac{p}{2} (k+1)(k+2) \); thus \( \text{rk } S_k(O^p) = p(k+1) \) and we set \( J_l(O^p) = 0 \) if \( l < 0 \).

For \( k \geq 1 \) the \( O \)-modules \( R_k \) and \( g_{k+1} \) are of finite type but not necessarily free. However, using the special nature of the symbols of the prolongations of the system \( M(d) \), one can determine the constant rank of each \( \sigma_k \).

In particular, it can be checked that \( g_k = 0 \) for \( k \geq d-3 \) and \( g_{d-4} \) is a free \( O \)-module of rank 1. Moreover, one obtains that \( \mathfrak{R}_k = 0 \) for \( k \leq d-4 \) and that it is a free \( O \)-module of rank \( k-d+4 \) for \( k \geq d-3 \).

Since \( \sigma_{d-3} \) is an isomorphism, we get among others that \( \pi_{d-4} : R_{d-3} \to R_{d-4} \) is an isomorphism of free \( O \)-modules with rank \( \frac{1}{2} (d-1)(d-2) \). Furthermore, we have (cf. [S]) a natural exact sequence of \( \mathbb{C} \)-vector spaces

\[
0 \to O^p \xrightarrow{j_l} J_l(O^p) \xrightarrow{D} \Omega^1 \otimes_O J_{l-1}(O^p) \xrightarrow{D} \Omega^2 \otimes_O J_{l-2}(O^p) \to 0.
\]
For example with \( p = 1, \) \( l = 2 \) and the natural projection \( \pi : J_1(\mathcal{O}) \to J_0(\mathcal{O}) \), we have explicitly the following three applications:

\[
\begin{align*}
\mathcal{J}^2(f) &= (f, \partial_x(f), \partial_y(f), \partial_x^2(f), \partial_x \partial_y(f), \partial_y^2(f)), \\
D(z, p, q, r, s, t) &= dx \otimes (\partial_x(z) - p, \partial_x(p) - r, \partial_x(q) - s), \\
&\quad + dy \otimes (\partial_y(z) - q, \partial_y(p) - s, \partial_y(q) - t)
\end{align*}
\]

and

\[
D(\omega \otimes (z, p, q)) = d\omega \otimes \pi(z, p, q) - \omega \wedge D(z, p, q)
\]

\[
= d\omega \otimes z - \omega \wedge (dx \otimes (\partial_x(z) - p) + dy \otimes (\partial_y(z) - q)).
\]

With the notation used before, the previous exact sequence induces on kernels the first Spencer complex associated with the prolongation \( p_k \) of \( p_0 \), that is, the following complex (of families) of \( \mathbb{C} \)-vector spaces exact at \( \mathbb{R}^k \) with injective \( j^k_{k+1} \):

\[
0 \to \text{Sol} \mathcal{M}(d) \xrightarrow{j^k_{k+1}} R_k \xrightarrow{D} \Omega^1 \otimes \mathcal{O} R_{k-1} \xrightarrow{D} \Omega^2 \otimes \mathcal{O} R_{k-2} \to 0.
\]

Moreover, from the preceding constructions, we have the following commutative diagram of \( \mathbb{C} \)-vector spaces with exact rows and such that the columns are complex exact at \( \mathbb{R}^k \) (resp. \( \mathbb{R}^k + 1 \)) with injective \( j^k_{k+1} \) (resp. \( j^k_{k+2} \)):

\[
\begin{array}{ccccccc}
0 & \to & \text{Sol} \mathcal{M}(d) & \xrightarrow{\text{id}} & \text{Sol} \mathcal{M}(d) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & g_{k+1} & \xrightarrow{j^k_{k+2}} & R_{k+1} & \xrightarrow{D} & R_k \\
\downarrow & & \downarrow & & \downarrow & & \\
\Omega^1 \otimes \mathcal{O} g_k & \xrightarrow{D} & \Omega^1 \otimes \mathcal{O} R_k & \xrightarrow{\pi_{k-1}} & \Omega^1 \otimes \mathcal{O} R_{k-1} & \xrightarrow{\beta_{k-1}} & \mathcal{K}_{k-1} \\
\downarrow & & \downarrow & & \downarrow & & \\
\Omega^2 \otimes \mathcal{O} g_{k-1} & \xrightarrow{D} & \Omega^2 \otimes \mathcal{O} R_{k-1} & \xrightarrow{\pi_{k-2}} & \Omega^2 \otimes \mathcal{O} R_{k-2} & \xrightarrow{\beta_{k-2}} & \mathcal{K}_{k-2} \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0.
\end{array}
\]

Using the previous results one gets a long sequence of \( \mathcal{O} \)-modules of finite type

\[
\cdots \xrightarrow{\pi_{d-2}} R_{d-2} \xrightarrow{\pi_{d-3}} \mathcal{E} := R_{d-3} \xrightarrow{\pi_{d-4}} R_{d-4}
\]

with injective morphisms which begin with an isomorphism \( \pi_{d-4} \) of free \( \mathcal{O} \)-modules with rank \( \frac{1}{2}(d - 1)(d - 2) \).
The operator $D : R_k \rightarrow \Omega^1 \otimes \mathcal{O} R_{k-1}$ of the first Spencer complex associated with $p_k$ satisfies $D(fe) = df \otimes \pi_{k-1}(e) + fD(e)$ for $e \in R_k$ and $f \in \mathcal{O}$. Thus, it allows us to make for $k \geq d - 3$ the following construction related to the previous long sequence: if $\beta_{k-1} = 0$, that is, $\pi_{k-1}$ is an isomorphism, then $R_k = \ker p_k \subset \mathcal{J}_{k+1}((\mathcal{O}^{d-2})$ is equipped with a connection defined by

$$\nabla_k := (\pi_{k-1})^{-1} \circ D : R_k \rightarrow \Omega^1 \otimes \mathcal{O} R_k$$

such that its $\mathbb{C}$-vector space $\ker \nabla_k$ of horizontal sections is isomorphic to $\text{Sol}\mathcal{M}(d)$. Moreover, its curvature

$$\nabla_k^{(1)} \circ \nabla_k : R_k \rightarrow \Omega^2 \otimes \mathcal{O} R_k$$

takes its values in $\Omega^2 \otimes \mathcal{O} g_{k-1}$ with the identification $\Omega^2 \otimes \mathcal{O} R_k \xrightarrow{\pi_{k-1}} \Omega^2 \otimes \mathcal{O} R_{k-1}$.

In particular, if this connection $(R_k, \nabla_k)$ exists it is always integrable for $k \geq d - 2$. Indeed, it can be checked that, with some abuse of notation, we have $D = \pi_{k-1} \circ \nabla_k^{(1)}$. Thus, it follows that $\pi_{k-2} \circ \pi_{k-1} \circ \nabla_k^{(1)} \circ \nabla_k = \pi_{k-2} \circ D \circ (\pi_{k-1})^{-1} \circ D = D^2 = 0$. From a basic result on connections, if $\beta_{k-1} = 0$ then $R_k$ is a free $\mathcal{O}$-module of finite type, even if the connection $(R_k, \nabla_k)$ is not integrable. Moreover, by the previous commutative diagram, it can be checked that the inclusion $\ker \beta_k \subseteq \ker (\nabla_k^{(1)} \circ \nabla_k)$ holds.

Since $\beta_{d-4} = 0$, this construction starts with $(\mathcal{E}, \nabla) := (R_{d-3}, \nabla_{d-3})$ which is called the connection associated with the planar web $\mathcal{W}(d)$ generated by the differential equation $F(x, y, y') = 0$. The curvature $K := \nabla_{d-3}^{(1)} \circ \nabla_{d-3}$ of $(\mathcal{E}, \nabla)$ is $\mathcal{O}$-linear and since $g_{d-4}$ is a free $\mathcal{O}$-module of rank 1, there exists an adapted basis $(e_\ell)$ of the free $\mathcal{O}$-module $\mathcal{E}$ which verifies

$$K(e_\ell) = k_\ell \, dx \wedge dy \otimes e_1 \quad \text{for} \quad 1 \leq \ell \leq \frac{1}{2}(d-1)(d-2) \quad \text{with} \quad k_\ell \in \mathcal{O}$$

where $e_1 \in \mathcal{E}$ corresponds to a generator of $g_{d-4}$. Moreover from results above, $\beta_{d-3} = 0$ implies $K = 0$.

The successive morphisms $\beta_k$ vanish or not and the construction mentioned above jointed with the Cauchy-Kowalevski theorem as below can be used to give a theoretical approach to the determination of the exact rank of $\mathcal{W}(d)$. However, the level $d - 3$ at least gives the following effective method to characterize maximal rank webs:

**Theorem 2.** With the previous notation, the following conditions are equivalent:

i) The connection $(\mathcal{E}, \nabla)$ is integrable, that is $k_\ell = 0$ for $1 \leq \ell \leq \frac{1}{2}(d-1)(d-2)$;

ii) The planar web $\mathcal{W}(d)$ associated with $F(x, y, y') = 0$ is of maximal rank.
Proof. With natural identification, the Cauchy-Kowalevski theorem asserts that the evaluation map \( \text{Ker} \nabla = \text{Sol} M(d) \to \mathbb{C} \frac{1}{2}(d-1)(d-2) \) close to \( 0 \in \mathbb{C}^2 \) is an isomorphism if \( K = 0 \); hence i) \( \Rightarrow \) ii). Conversely, since \( \Omega^1 \otimes \mathcal{O} \) \((\mathcal{O} \otimes \mathbb{C} \text{Sol} M(d))\) is identified with \( \Omega^1 \otimes \mathbb{C} \text{Sol} M(d)\), the local system \( \text{Sol} M(d)\) of dimension \( \frac{1}{2}(d-1)(d-2) \) induces an integrable connection \((\mathcal{O} \otimes \mathbb{C} \text{Sol} M(d), \nabla_S)\) defined by \( \nabla_S(f \otimes s) = df \otimes s \). Then we have a morphism of connections \((\mathcal{O} \otimes \mathbb{C} \text{Sol} M(d), \nabla_S) \to (E, \nabla)\) given by \( f \otimes s \mapsto fs \), which is an isomorphism. This proves that the connection \((E, \nabla)\) is integrable. \( \square \)

Examples. 1. For \( d = 3 \), that is for \( \mathcal{W}(3) \), the matrix \((A_{i,j})\) introduced before has the following simplified form:

\[
(A_{i,j}) = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.
\]

It can be checked that the row matrix \( e_1 = (1 \ -A_1 \ -A_2) \in J_1(\mathcal{O}) \) is an adapted basis of \( E := R_0 \). In this basis, the connection \((E, \nabla)\) associated with \( \mathcal{W}(3) \) has the following matrix:

\[
\gamma = (A_1 \ dx + A_2 \ dy)
\]

with the corresponding curvature matrix

\[
d\gamma + \gamma \wedge \gamma = (\partial_x(A_2) - \partial_y(A_1)) \ dx \wedge dy
\]

which is the Blaschke curvature of \( \mathcal{W}(3) \).

2. For \( d = 4 \) in the linear case, that is \( \mathcal{W}(4) = \mathcal{L}(4) \), the matrix \((A_{i,j})\) has the following particular form:

\[
(A_{i,j}) = \begin{pmatrix} 0 & A_1 \\ A_1 & A_2 \\ A_2 & 0 \end{pmatrix}.
\]

In this case, one constructs an adapted basis \((e_1, e_2, e_3)\) of \( E := R_1 \) with explicit elements of \( J_2(\mathcal{O}^2) \) in matrix forms:

\[
e_1 = \begin{pmatrix} 0 & -1 & 0 & 2A_1 & A_2 & 0 \\ 0 & 0 & 1 & 0 & -A_1 & -2A_2 \end{pmatrix},
\]

\[
e_2 = \begin{pmatrix} 1 & -A_1 & -A_2 & A_1^2 - \partial_x(A_1) & A_1A_2 - \partial_x(A_2) & A_2^2 - \partial_y(A_2) \\ 0 & 0 & 0 & 0 & 0 & k \end{pmatrix},
\]

\[
e_3 = \begin{pmatrix} 0 & 0 & 0 & -k & 0 & 0 \\ 1 & -A_1 & -A_2 & A_1^2 - \partial_x(A_1) & A_1A_2 - \partial_y(A_1) & A_2^2 - \partial_y(A_2) \end{pmatrix}.
\]
where \( k := \partial_x(A_2) - \partial_y(A_1) \). In this basis, the connection \((\mathcal{E}, \nabla)\) associated with \(W(4)\) has the following matrix:

\[
\gamma = \begin{pmatrix}
\Gamma & -k \ dy & -k \ dx \\
(dx & \Gamma & 0 \\
-dy & 0 & \Gamma
\end{pmatrix}
\]

with the corresponding curvature matrix

\[
d\gamma + \gamma \wedge \gamma = \begin{pmatrix}
3k & -\partial_x(k) & \partial_y(k) \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \ dx \wedge \ dy
\]

where \( \Gamma := A_1 \ dx + A_2 \ dy \).

Remarks. 1. In the general case \(W(4)\) with the previous method and from the matrix \( (A_{i,j}) = \begin{pmatrix}
A_{1,1} & A_{1,2} \\
A_{2,1} & A_{2,2} \\
A_{3,1} & A_{3,2}
\end{pmatrix} \), one gets an adapted basis of \((\mathcal{E}, \nabla)\) with curvature matrix

\[
d\gamma + \gamma \wedge \gamma = \begin{pmatrix}
k_1 & k_2 & k_3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \ dx \wedge \ dy,
\]

where we have explicitly

\[
\begin{align*}
k_1 &= 2\kappa_1 + \kappa_2 + \partial_y(\lambda_1), \\
k_2 &= -\partial_x(\kappa_1) - \partial_y^2(\lambda_{1,1}) + \lambda_1 \kappa_1 + \partial_y(\lambda_2 A_{1,1}) + \partial_x(\lambda_1 A_{3,2}) + A_{1,1} \partial_x(A_{3,2}), \\
k_3 &= \partial_y(\kappa_2) - \partial_y^2(\lambda_{3,2}) + \lambda_2 \kappa_2 - \partial_x(\lambda_1 A_{3,2}) + \partial_y(\lambda_1 A_{3,2}) + A_{3,2} \partial_y(A_{1,1}),
\end{align*}
\]

with \( \kappa_1 = \partial_x(A_{3,1}) - \partial_y(A_{2,1}) \) and \( \kappa_2 = \partial_x(A_{2,2}) - \partial_y(A_{1,2}) \); that is, \( d\Gamma_{q} = \kappa_q \ dx \wedge \ dy \) for \( 1 \leq q \leq 2 \), \( \lambda_1 = A_{2,1} - A_{1,2} \) and \( \lambda_2 = A_{3,1} - A_{2,2} \). Moreover, as with the Blaschke curvature for any \(W(3)\), the previous relations prove that \( (k_\ell) \) does not depend on a normalization of \(W(4)\). In other words, the collection \( (k_\ell) \) is an invariant of the planar web \(W(4)\); that is, the curvature of its associated connection \((\mathcal{E}, \nabla)\) is “canonical”.

2. For the weave of a general planar web \(W(4)\) introduced in the first section, it can be noted from the theorem called \( a_F \) that \((b_3, b_4) \in \mathcal{A}(4)_3 \subseteq \mathcal{A}(4) = \mathcal{A}(4) \) if and only if \( b_3^4 \cdot F(x,y, \frac{-b_4}{b_3}) = 0 \). For \( d \geq 4 \), one can get the same kind of description of elements in \(\mathcal{A}(d)_k\) by adding suitable new equations on \((b_3, \ldots, b_d) \in \mathcal{A}(d)\).

To end this section, we give some applications of the previous methods and results. Let \(\mathcal{L}(d)\) be a linear and nonsingular web in \((\mathbb{C}^2, 0)\). The previous relation \((\ast)\) is reduced to the following:

\[
(\ast \mathcal{L}) \quad r \cdot (\partial_x(F) + p \partial_y(F)) = (\partial_x(r) + p \partial_y(r) - t) \cdot F.
\]
Since $p_i = \xi_i(F_i)$ with $\xi_i \in \mathbb{C}\{z\}$ for $1 \leq i \leq d$ in the linear case, $r_p = 0$. Moreover for $\mathcal{L}(d)$, the homogeneous linear differential system $\mathcal{M}(d)$ associated to $(\ast_d)$ has the following particular form:

\[
\begin{align*}
\partial_x(b_d) + A_1 b_d &= 0 \\
\partial_x(b_{d-1}) + \partial_y(b_d) + A_1 b_{d-1} + A_2 b_d &= 0 \\
&\quad \vdots \\
\partial_x(b_3) + \partial_y(b_4) + A_1 b_3 + A_2 b_4 &= 0 \\
\partial_y(b_3) + A_2 b_3 &= 0.
\end{align*}
\]

With the theorem $a_{F^*}$, it can be checked, with the notation used before, that if $d\Gamma = k dx \wedge dy = 0$ then the rank of $\mathcal{L}(d)$ is maximally equal to $\frac{1}{2}(d-1)(d-2)$. Indeed, we have the following explicit description, through the $\mathbb{C}$-vector space $a_{F^*}$, of its $\mathbb{C}$-vector space of abelian relations:

\[
\{ e^{-\varphi}(y - px)^{\ell_1} p^{\ell_2} \cdot \frac{dy - p dx}{\partial_p(F)} \} \in \pi_*(\Omega^1_S) \text{ with } 0 \leq \ell_1 + \ell_2 \leq d - 3,
\]

with $\Gamma = A_1 dx + A_2 dy = d\varphi$ where $\varphi \in \mathcal{O}$ is given by the Poincaré lemma.

In particular we recover, without using Abel’s theorem and traces of elements in $H^0(C, \omega_C^1)$, that if $\mathcal{L}(d) = \mathcal{L}_c(d)$ is an algebraic web with $C \subset \mathbb{P}^2$ then it necessarily has maximal rank since in this case we have seen that $\Gamma = 0$. Moreover its space $a_{F^*}$ has the previous description with $\varphi = 0$.

In the linear case, an adapted basis $(e_\ell)$ for the connection $(\mathcal{E}, \nabla)$ associated with $\mathcal{L}(d)$ can be constructed, step by step on $d$, following the examples given before. At each step, $e_1$ is chosen and the other vectors $e_\ell$ are constructed from the steps before, installed on different rows with suitable zeros. Moreover it can be checked that in this special case, the $\ell$-component of each $\nabla(e_\ell)$ is $(A_1 dx + A_2 dy) \otimes e_\ell$ which proves the following result:

**Proposition 2.** Let $\mathcal{L}(d)$ be a linear and nonsingular planar web. Then, the trace of the curvature $K$ of the connection $(\mathcal{E}, \nabla)$ associated with $\mathcal{L}(d)$ satisfies the following equalities:

\[
tr(K) = k_1 dx \wedge dy = \frac{1}{2}(d-1)(d-2). d\Gamma.
\]

As announced in the introduction and using only the previous methods, the following result and its proof give several complements of a basic result in planar web geometry:

**Theorem 3.** Let $\mathcal{L}(d)$ be a linear and nonsingular planar web associated with a differential equation $F(x, y, y') = 0$ with canonical normalization $(\omega_i)$. 

Then, the following conditions are equivalent:

1) \( \mathcal{L}(d) \) is of maximal rank;
2) \( \omega_i = \rho \, du_i \) for \( 1 \leq i \leq d \) with elements \( u_i \) in \( \mathcal{O} \) and \( \rho \) in \( \mathcal{O}^* \);
3) The connection \( (\mathcal{E}, \nabla) \) associated with \( \mathcal{L}(d) \) is integrable;
4) With the previous notation \( k = \partial_2 \left( \frac{a_1}{a_0} \right) = 0 \); that is, \( d\Gamma = 0 \);
5) \( \mathcal{L}(d) \) is algebraic.

Proof. From the previous results, there are the following implications:

5) \( \Rightarrow \) 1) \( \iff \) 3) \( \Rightarrow \) 4). 2) \( \iff \) 4): For \( 1 \leq i \leq d \), we have \( d\omega_i = (A_1 \, d\lambda + A_2 \, dy) \wedge \omega_i = \frac{d\rho}{\rho} \wedge \omega_i \) if \( \omega_i = \rho \, du_i \); hence \( d\Gamma = 0 \) from the general position hypothesis. Conversely, if \( d\Gamma = 0 \) then \( \Gamma = d\varphi = \frac{d\rho}{\rho} \) if \( \rho = e^{\varphi} \). But \( d\left( \frac{\omega_i}{\rho} \right) = -\frac{d\rho}{\rho^2} \wedge \omega_i + \frac{d\omega_i}{\rho} = 0 \) since \( d\omega_i = \Gamma \wedge \omega_i \), hence \( \omega_i = \rho \, du_i \) for \( 1 \leq i \leq d \) by the Poincaré lemma. 4) \( \Rightarrow \) 5): We have \( \Gamma = d\varphi \) and from the previous description of \( a_F \) in this case, the element \( r := e^{-\varphi} \) verifies the relation \( (\ast \mathcal{L}) \) with \( t = 0 \). Thus, \( F \) is a solution of the partial differential equation

\[
\partial_x(f) + p \partial_y(f) = -\left( \partial_x(\varphi) + p \partial_y(\varphi) \right) \cdot f
\]

with unknown \( f \in \mathbb{C}\{x, y, p\} \). By a classical argument (cf. for instance [Ca]), the general solution of the previous equation is \( f = e^{-\varphi} \cdot \Phi(y - px, p) \) with \( \Phi \in \mathbb{C}\{s, t\} \). This implies the equality \( F = e^{-\varphi} \cdot P(y - px, p) \) where \( P \) is a reduced element in \( \mathbb{C}\{s, t\} \) with degree \( d \) from the hypothesis on \( F \in \mathcal{O}[p] \). Thus \( \mathcal{L}(d) \) is algebraic with \( \mathcal{L}(d) = \mathcal{L}_C(d) \), where \( P(s, t) = 0 \) is an affine equation of the reduced algebraic curve \( C \subset \mathbb{P}^2 \).

References


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