Annals of Mathematics, **159** (2004), 407–424

Classification of prime 3-manifolds with σ-invariant greater than $\mathbb{RP}^3$

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Abstract

In this paper we compute the σ-invariants (sometimes also called the smooth Yamabe invariants) of $\mathbb{RP}^3$ and $\mathbb{RP}^2 \times S^1$ (which are equal) and show that the only prime 3-manifolds with larger σ-invariants are S^3, $S^2 \times S^1$, and $S^2 \tilde{\times} S^1$ (the nonorientable S^2 bundle over S^1). More generally, we show that any 3-manifold with σ-invariant greater than $\mathbb{RP}^3$ is either S^3, a connect sum with an S^2 bundle over S^1, or has more than one nonorientable prime component. A corollary is the Poincaré conjecture for 3-manifolds with σ-invariant greater than $\mathbb{RP}^3$.

Surprisingly these results follow from the same inverse mean curvature flow techniques which were used by Huisken and Ilmanen in [7] to prove the Riemannian Penrose Inequality for a black hole in a spacetime. Richard Schoen made the observation [18] that since the constant curvature metric (which is extremal for the Yamabe problem) on $\mathbb{RP}^3$ is in the same conformal class as the Schwarzschild metric (which is extremal for the Penrose inequality) on $\mathbb{RP}^3$ minus a point, there might be a connection between the two problems. The authors found a strong connection via inverse mean curvature flow.

1. Introduction

We begin by reminding the reader of the definition of the σ-invariant of a closed 3-manifold and some of the previously known results. Since our results only apply to 3-manifolds, we restrict our attention to this case.

Given a closed 3-manifold M, the Einstein-Hilbert energy functional on the space of metrics g is defined to be the total integral of the scalar curvature

*The research of the first author was supported in part by NSF grant #DMS-0206483. The research of the second author was kindly supported by FCT-Portugal, grant BD/893/2000.

R_g after the metric has been scaled to have total volume 1. More explicitly,

$$E(g) = \frac{\int_M R_g dV_g}{(\int_M dV_g)^{1/3}}$$

where dV_g is the volume form of g. As will become clear, the most important reference value of this energy function is

$$E(g_0) = 6(2\pi^2)^{2/3} \equiv \sigma_1$$

where g_0 is any constant curvature (or round) metric on S^3. When g_0 has constant sectional curvature 1, $R_{g_0} = 6$ and $\mathrm{Vol}(g_0) = 2\pi^2$.

Since E is unbounded in both the positive and negative directions, it is not interesting to simply maximize or minimize E over the space of all metrics. However, Trudinger, Aubin, and Schoen showed (as conjectured by Yamabe) that a minimum value for E is always realized in each conformal class of metrics by a constant scalar curvature metric, so define the [conformal] Yamabe invariant of the conformal class $[g]$ to be

$$Y(g) = \inf\{E(\bar{g}) \mid \bar{g} = u(x)^4 g,\ u(x) > 0,\ u \in H^1\}$$

where we note that

$$E(\bar{g}) = \frac{\int_M (8|\nabla u|_g^2 + R_g u^2) dV_g}{\left(\int_M u^6 dV_g\right)^{1/3}}. \tag{1}$$

Given any smooth metric g, we can always choose $u(x)$ to be close to zero except near a single point p so that the resulting conformal metric is very close to the round metric on S^3 minus a neighborhood of a point. This construction can be done to make the energy of the resulting conformal metric arbitrarily close to σ_1. Hence,

$$Y(g) \leq \sigma_1$$

for all g and M. Thus, as defined by Schoen in lectures in 1987 and published the following year [17] (see also O. Kobayashi [9] who attended the lectures), let

$$\sigma(M) = \sup\{Y(g) \mid g \text{ any smooth metric on } M\} \quad \leq \sigma_1$$

to get a real-valued smooth invariant of M, called the σ-invariant. We note that the σ-invariant is sometimes called the smooth Yamabe invariant (as opposed to the conformal Yamabe invariant defined above for conformal classes) as well as the Schoen invariant. For clarity, we will adopt the convention of referring to the Yamabe invariant of a conformal class and Schoen's σ-invariant of a smooth manifold.

There are relatively few 3-manifolds for which the σ-invariant is known. Obata [15] showed that for an Einstein metric g we have $Y(g) = E(g)$, which when combined with the above inequality proves that $\sigma(S^3) = \sigma_1$. It is also known that $S^2 \times S^1$ and $S^2 \tilde{\times} S^1$ (the nonorientable S^2 bundle over S^1) have σ-invariant equal to σ_1 [17]. O. Kobayashi showed that as long as at least one of the manifolds has nonnegative σ-invariant, then the σ-invariant of the connect sum of two manifolds is at least the smaller of the two σ-invariants [9]. Hence, any finite number of connect sums of the two S^2 bundles over S^1 has $\sigma = \sigma_1$. In addition, $\sigma(M) > 0$ is equivalent to M admitting a metric with positive scalar curvature. Since T^3 (or T^3 connect sum with any other 3-manifold) does not admit a metric with positive scalar curvature, and since the flat metric on T^3 is easily shown to have $Y = 0$, it follows that $\sigma(T^3) = 0$. From this and O. Kobayashi's result it follows that T^3 connect sum any other 3-manifold with nonnegative σ-invariant has $\sigma = 0$ as well. In addition, any graph manifold which does not admit a metric of positive scalar curvature has $\sigma = 0$. For a more detailed survey of the σ-invariants of 3-manifolds, see the works of Mike Anderson [2], [3] and the works of Claude LeBrun and collaborators [5], [8], [10], [11], [12] for 4-manifolds.

Note that the only two previously computed values of the σ-invariant of 3-manifolds are 0 and σ_1, although it is expected that there are infinitely many different values that the σ-invariant realizes on different manifolds. In fact, if M admits a constant curvature metric g_0 (spherical, hyperbolic, or flat), then Schoen conjectures that $\sigma(M) = E(g_0)$. The flat case is known to be true, but the other two cases appear to be quite challenging.

In particular, if $M = S^3/G_n$ is a smooth manifold and $|G_n| = n$, then it is conjectured that

$$\sigma(M) = \frac{\sigma_1}{n^{2/3}} \equiv \sigma_n. \tag{2}$$

In this paper we prove that this conjecture is true when $n = 2$ and M is $\mathbb{RP}^3$.

2. Main results

Theorems 2.1 and 2.12 (a slight generalization which is more complicated to state but is also very interesting) are the main results of this paper.

THEOREM 2.1. *A closed 3-manifold with $\sigma > \sigma_2$ is either S^3, a connect sum with an S^2 bundle over S^1, or has more than one nonorientable prime component.*

Note that there are two S^2 bundles over S^1, the orientable one $S^2 \times S^1$ and the nonorientable one $S^2 \tilde{\times} S^1$, neither of which is simply-connected. Note also that a simply-connected manifold is always orientable and hence cannot have any nonorientable prime components. Hence, the Poincaré conjecture for 3-manifolds with $\sigma > \sigma_2$ follows.

COROLLARY 2.2. *The only closed, simply-connected* 3-*manifold with* $\sigma > \sigma_2$ *is* S^3.

We are also able to use the above theorem to compute the σ-invariants of some additional 3-manifolds.

COROLLARY 2.3.

$$\sigma(\mathbb{RP}^3) = \sigma_2.$$

The fact that $\sigma(\mathbb{RP}^3) \leq \sigma_2$ follows from Theorem 2.1 since $\mathbb{RP}^3$ is prime and is not S^3 or a connect sum with an S^2 bundle over S^1. $\sigma(\mathbb{RP}^3) \geq \sigma_2$ follows from the fact that $Y(g_0) = \sigma_2$ by Obata's theorem, where g_0 is the constant curvature metric on $\mathbb{RP}^3$.

COROLLARY 2.4.

$$\sigma(\mathbb{RP}^2 \times S^1) = \sigma_2.$$

The fact that $\sigma(\mathbb{RP}^2 \times S^1) \leq \sigma_2$ again follows from Theorem 2.1. Note that $S^2 \times S^1$ is a double cover of $\mathbb{RP}^2 \times S^1$. Furthermore, the standard proof on $S^2 \times S^1$ that there is a sequence of conformal classes $[g_i]$ with $\lim Y(g_i) = \sigma_1$ passes to the quotient to give us a sequence of conformal classes $[\bar{g}_i]$ on $\mathbb{RP}^2 \times S^1$ with $\lim Y(\bar{g}_i) = \sigma_2$, proving that $\sigma(\mathbb{RP}^2 \times S^1) \geq \sigma_2$. We refer the reader to [17] for the details of the $S^2 \times S^1$ result.

COROLLARY 2.5. *Let M be any finite number of connect sums of* $\mathbb{RP}^3$ *and zero or one connect sums of* $\mathbb{RP}^2 \times S^1$. *Then*

$$\sigma(M) = \sigma_2.$$

The upper bound $\sigma(M) \leq \sigma_2$ again comes from Theorem 2.1. The lower bound $\sigma(M) \geq \sigma_2$ comes from the connect sum theorem of O. Kobayashi referred to earlier.

It is possible that the above corollary may be able to be strengthened to allow up to two $\mathbb{RP}^2 \times S^1$ components if these cases can be shown to satisfy Property B (defined below). In any case, it is curious that there is a limit on the number of these factors, and it is certainly interesting to try to understand what happens when you allow for any number of $\mathbb{RP}^2 \times S^1$ components.

Another interesting problem is to compute the σ-invariants of finite connect sums of one or more S^2 bundles over S^1 with one or more of $\mathbb{RP}^3$ and $\mathbb{RP}^2 \times S^1$. At the time of the publication of this paper, Kazuo Akutagawa and the second author found a nice idea to extend the results of this paper to some of those cases [1].

Also, closed 3-manifolds admit a nearly unique prime factorization as the connect sum of prime manifolds [6]. A manifold M is prime if $M = A\#B$ implies that either A or B is S^3. Finite prime factorizations always exist for 3-manifolds and are unique modulo the relation $(S^2 \times S^1)\#(S^2 \tilde{\times} S^1) = (S^2 \tilde{\times} S^1)\#(S^2 \tilde{\times} S^1)$. Consequently classifying closed 3-manifolds reduces to classifying prime 3-manifolds. One natural approach is to try to list prime 3-manifolds in order of their σ-invariants.

Corollary 2.6. *The first five prime* 3*-manifolds ordered by their* σ*-invariants are* S^3, $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, $\mathbb{RP}^3$, *and* $\mathbb{RP}^2 \times S^1$. *The first three manifolds have* $\sigma = \sigma_1$ *and the last two have* $\sigma = \sigma_2$. *All other prime* 3*-manifolds have* $\sigma \leq \sigma_2$.

We conjecture that in fact all other prime 3-manifolds have $\sigma < \sigma_2$.

Theorem 2.1 has the advantage of being concise but is actually a special case of Theorem 2.12. However, to properly state Theorem 2.12 it is convenient to make the following topological definitions.

Definition 2.7. A 3-manifold M^3 has *Property* A if M^3 is not S^3 or a connect sum with an S^2 bundle over S^1 *and* M^3 can be expressed as $P^3\#Q^3$ where P^3 is prime and Q^3 is orientable.

Definition 2.8. A 3-manifold M^3 has *Property* B if M^3 is not S^3 or a connect sum with an S^2 bundle over S^1 *and* M^3 can be expressed as $P^3\#Q^3$ where P^3 is prime and $\alpha(Q^3) = 2$.

Definition 2.9. Define $\alpha(Q^3)$ to be the supremum of the Euler characteristic of the boundary (not necessarily connected) of all smooth connected regions (with two-sided boundaries) whose complements are also connected.

Note that by smooth and two-sided we mean that at every boundary point of the region, the region in the manifold locally looks like a neighborhood around the origin of the upper half space in $\mathbb{R}^3$. Also, considering a small ball in Q^3 proves that $\alpha(Q^3) \geq 2$ always. We also make a nonessential comment that Property B is equivalent to saying that M^3 is not S^3 and M^3 can be expressed as $I^3\#Q^3$ where I^3 is irreducible and $\alpha(Q^3) = 2$.

Lemma 2.10. *Property* A *implies Property* B.

Proof. Assume M^3 has Property A. Then the first part of Property B is immediate. For the last part, by Property A we know that M^3 can be expressed as $P\#Q$ where P is prime and Q is orientable. We will show that $\alpha(Q) = 2$.

Let U be a smooth, regular, connected region in Q^3, and let Σ be the boundary of U. Since Q^3 is orientable, it follows that Σ (which has a globally defined normal vector pointing in the direction of U for example) is also orientable. Hence, the connected components of Σ are spheres and surfaces of higher genus with nonpositive Euler characteristic.

Lemma 3.8 on page 27 of [6] states that if Q^3 minus an embedded 2-sphere is connected, then Q^3 is a connect sum of an S^2 bundle over S^1 with some other 3-manifold. Hence, since Property A assumes that M^3 and hence Q^3 are not connect sums with S^2 bundles over S^1, any sphere component of Σ must already split Q^3 into two regions. In this case, Σ must be exactly a single sphere, since any other components of Σ would split Q^3 into more than two connected regions. Hence, the two possibilities are that either Σ is a single sphere, or Σ is the disjoint union of any number of connected surfaces with nonpositive Euler characteristic. In both cases the Euler characteristic of Σ is less than or equal to 2, so $\alpha(Q^3) = 2$, proving Property B.

The topological invariant α is new to the authors. We make a couple of nonessential comments about it here. Besides always having to be at least two, consideration of the connect sum operation implies that $\alpha(A\#B) \geq \alpha(A) + \alpha(B) - 2$. This inequality is an equality when both A and B are orientable due to the following lemma.

Lemma 2.11. *If M^3 is orientable and has exactly k $S^2 \times S^1$ components in its prime factorization, then $\alpha(M^3) = 2(k+1)$.*

Sketch of Proof. The fact that $\alpha(S^2 \times S^1) \geq 4$ implies (by the connect sum observation just stated) that $\alpha(M^3) \geq 2(k+1)$. Conversely, $\alpha(M^3) \geq 2(k+1)$ implies that there must be at least $(k+1)$ spheres in Σ since the boundary surface Σ is orientable (since M^3 is orientable). Referring the reader to the argument used by Hempel in [6] in Lemma 3.8 on page 27 implies that there must be at least k $S^2 \times S^1$ bundles in M^3, proving the lemma.

However, if M^3 is not orientable, then it is harder to understand $\alpha(M^3)$. This is because the boundary surface Σ does not have to be orientable and therefore can have $\mathbb{RP}^2$'s contributing positive Euler characteristic. We leave this case as an interesting problem to investigate.

Theorem 2.12. *A closed 3-manifold M^3 with Property* A *or* B *has $\sigma(M^3) \leq \sigma_2$.*

The above theorem could be thought of as the main theorem of this paper and implies Theorem 2.1 by considering the negation of Property A. In the next section we will see how the above theorem follows from Theorem 3.2.

3. The basic approach and some definitions

The purpose of the remainder of this paper is to prove Theorem 2.12. In this section we will show that Theorem 2.12, a statement about closed 3-manifolds, follows from Theorem 3.2, a statement about the Sobolev constants of asymptotically flat 3-manifolds with nonnegative scalar curvature.

Suppose that M has Property A or B. Then we want to prove that $\sigma(M) \leq \sigma_2$. This would follow if we could show that

$$Y(g) \leq \sigma_2$$

for all conformal classes of metrics $[g]$ on M.

If $Y(g) \leq 0$, then we are done. Otherwise, $Y(g) > 0$ implies that the metric g_0 which minimizes E in $[g]$ has constant positive scalar curvature R_0. Working inside of (M, g_0) now, define

$$L_0 \equiv \Delta_0 - \frac{1}{8} R_0$$

to be the "conformal Laplacian" with respect to g_0. Now choose any point $p \in M$ and define $G_p(x)$ to be the Green's function of L_0 at p scaled so that

$$L_0 G_p = 0$$

on $M - \{p\}$ and

$$\lim_{q \to p} d(p,q) G_p(q) = 1.$$

This Green's function exists and is positive since $R_0 > 0$ and by the maximum principle.

Definition 3.1. A Riemannian 3-manifold (M, g) is said to be *asymptotically flat* if there's a compact set $K \subseteq M$ such that $M - K$ is diffeomorphic to $\mathbb{R}^3 - \{|x| \leq 1\}$ and in the coordinate chart defined by this diffeomorphism we have

$$g = \sum_{i,j} g_{ij}(x) dx^i dx^j,$$

where

$$g_{ij} = \delta_{ij} + O(|x|^{-1}), \quad g_{ij,k} = O(|x|^{-2}), \quad g_{ij,kl} = O(|x|^{-3}).$$

Let $g_{AF} = G_p(x)^4 g_0$ on $M - \{p\}$. Then $(M - \{p\}, g_{AF})$ is an asymptotically flat Riemannian manifold with zero scalar curvature where the point p has been sent to infinity. Note that the formula for the scalar curvature of a conformal metric is

$$R_{AF} = -8 G_p^{-5} L_0(G_p) = 0.$$

Also note that the metrics g, g_0, and g_{AF} are all in the same conformal class, so

$$Y(g) = Y(g_0) = Y(g_{AF})$$

as long as the conformal factors on g_{AF} are required to go to zero at infinity sufficiently rapidly. Then since g_{AF} has zero scalar curvature, it follows from equation 1 that

$$\begin{aligned} C(g_{AF}) &= \inf\left\{ \frac{\int_M 8|\nabla u|^2 dV}{\left(\int_M u^6 dV\right)^{1/3}} \mid u \in H^1(M - \{p\}, g_{AF}) \text{ and has compact support} \right\} \\ &\equiv 8\, S(g_{AF}) \end{aligned}$$

where $S(g_{AF})$ is the Sobolev constant of $(M - \{p\}, g_{AF})$. Note that requiring conformal factors on $(M - \{p\}, g_{AF})$ to have compact support is equivalent to requiring conformal factors on (M, g) and (M, g_0) to be zero in an arbitrarily small open neighborhood around p which does not affect that values of $Y(g)$ or $Y(g_0)$. It is also okay to use $u(x)$ in the above Sobolev expression which do not have compact support but instead are in $H^1_{\text{loc}} \cap L^6$ and satisfy

$$\lim_{x\to\infty} u(x)|x|^{1/2} = 0 \tag{3}$$

where $|x|$ is defined as the distance from some base point in $(M - \{p\}, g_{AF})$. The reason is that this decay condition guarantees that it is possible to cut off u at infinity to yield a compactly supported function with energy arbitrarily close to the energy of u.

By the discussion in this section, Theorem 2.12 follows from the following result on asymptotically flat 3-manifolds with nonnegative scalar curvature which we will prove in the remainder of this paper using inverse mean curvature flow techniques.

Theorem 3.2. *Let (M, g) be an asymptotically flat 3-manifold with nonnegative scalar curvature satisfying Property* A *or* B*. Then*

$$S(g) \le \sigma_2/8.$$

4. Some intuition

The (Riemannian) Schwarzschild metric on $\mathbb{RP}^3$ minus a point p is the only case which gives equality in Theorem 3.2, so this case deserves discussion. We begin by working on the covering space of $(\mathbb{RP}^3, g_0)$ which is of course (S^3, g_0), where g_0 is again the constant curvature round metric. Removing a point on

$\mathbb{RP}^3$ is equivalent to removing two antipodal points n and s on S^3. Note that $(S^3 - n - s, g_0)$ still has an $O(3)$ symmetry as well as a $\mathbb{Z}_2$ symmetry. Next, let $G(x)$ be the Green's function of the conformal Laplacian at p as in the previous section and lift $G(x)$ to S^3. Then $(S^3 - n - s, g_{AF})$, where $g_{AF} = G(x)^4 g_0$, is a zero scalar curvature metric with *two* asymptotically flat ends. Note that since $G(x)$ satisfies $LG = 0$ on $S^3 - n - s$ with identical asymptotics on n and s, G has the $O(3)$ and $\mathbb{Z}_2$ symmetries as well. Hence, $(S^3 - n - s, g_{AF})$ has these same symmetries. Said another way, $(S^3 - n - s, g_{AF})$ is a spherically symmetric, zero scalar curvature, asymptotically flat manifold with two ends.

Besides $\mathbb{R}^3$, the only other spherically symmetric, zero scalar curvature, geodesically complete 3-manifolds are scalings of the Schwarzschild metric (with mass set to 2 here) which is most conveniently written as

$$(\mathbb{R}^3 - \{0\}, (1 + 1/|x|)^4 \delta_{ij}).$$

Note that since the conformal factor blows up at 0, the above metric has two asymptotically flat ends, one at ∞ and one at 0. The $O(3)$ symmetry of the Schwarzschild metric in the above picture is clear, but the $\mathbb{Z}_2$ symmetry (which sends x to $x/|x|^2$) is harder to see. Another good picture of the Schwarzschild metric with mass 2 is as the submanifold of the Euclidean space $\mathbb{R}^4$ which satisfies

$$|(x, y, z)| = \frac{w^2}{16} + 4,$$

which is a parabola rotated about an S^2. Here both the $O(3)$ and $\mathbb{Z}_2$ symmetries are clear as well as the fact that there is a minimal sphere which is fixed by the $\mathbb{Z}_2$ symmetry.

Thus, in the first model for the Schwarzschild metric, when we mod out by the $\mathbb{Z}_2$ symmetry we get

$$(\mathbb{R}^3 - B_1(0), (1 + 1/|x|)^4 \delta_{ij}) \quad \equiv \quad (L, s)$$

where the antipodal points of the minimal sphere $|x| = 1$ are identified. By the uniqueness of this construction, $(\mathbb{RP}^3 - \{p\}, g_{AF})$ must be isometric to some constant scaling of (L, s).

By the previous section, we know that $S(g_{AF}) = Y(g_{AF})/8 = Y(g_0)/8$. But Obata's theorem tells us that $Y(g_0) = \sigma_2$. Hence, we see that the Sobolev constants of $(\mathbb{RP}^3 - \{p\}, g_{AF})$ and therefore (L, s) are both $\sigma_2/8$.

Define $u_0(x)$ on (L, s) such that $(L, u_0(x)^4 s)$ is isometric to $(\mathbb{RP}^3 - \{p\}, g_0)$. For convenience, scale $u_0(x)$ so that its maximum value is 1. By the previous section we know that it is this function $u_0(x)$ which has Sobolev ratio $\sigma_2/8$ which is the minimum. The key point here is that $u_0(x)$ also has the $O(3)$ symmetry.

The main idea for proving Theorem 3.2 will be to show that on M which satisfy Property B (or Property A since A implies B), we can always construct a function $u(x)$ with Sobolev ratio $\leq \sigma_2/8$. Then it follows that the Sobolev constant, which is the minimum of the Sobolev ratios, must be $\leq \sigma_2/8$. We now perform this construction on (L, s) since we already know what the answer has to be here.

Let $\Sigma(0)$ be the minimal sphere in (L, s) which is the coordinate sphere $|x| = 1$. Note that Σ_0 weakly bounds a compact region. Flow this sphere in the outward normal direction to define a family of surfaces $\Sigma(t)$ where at each point the speed of the flow equals $1/H$, where H is the mean curvature of $\Sigma(t)$. In our case there is a slight issue at the beginning of the flow since $\Sigma(0)$ has zero mean curvature, but this flow can still be defined for all $t > 0$ since all of the other spherically symmetric spheres $\Sigma(t)$ have positive H. For now we just require that $\Sigma(t)$ converges to $\Sigma(0)$ as t goes to zero. We can also think of this flow of surfaces using a level set formulation where we define $\phi(x)$ such that

$$\Sigma(t) = \{x \mid \phi(x) = t\}.$$

Define $f(t)$ for $t \geq 0$ such that

$$f(t) = u_0(\Sigma(t)) \tag{4}$$

which makes sense since $u_0(x)$ is constant on each $\Sigma(t)$. Note that we have used our knowledge of $u_0(x)$, which has the minimal Sobolev ratio on (L, s), to define f. Equivalently, given f we can recover u_0 since $u_0(x) = f(\phi(x))$.

In the general case on (M, g_{AF}), the results of Huisken and Ilmanen [7] say that if we have an outermost minimal surface with a connected component Σ_0 which weakly bounds a compact region, then it is always possible to define a weak version of inverse mean curvature flow using a level set formulation to get a locally Lipschitz function $\phi(x)$. Then we will show that $u(x) = f(\phi(x))$ has Sobolev ratio $\leq \sigma_2/8$. We see that this construction does indeed work in the model case (L, s). The fact that this case works in general follows from some very special properties of inverse mean curvature flow. The most central of these amazing facts is that the Hawking mass of $\Sigma(t)$, which is defined entirely in terms of the geometry of $\Sigma(t)$, is nondecreasing in t.

When the weak inverse mean curvature flow is smooth, it agrees with the classical flow. The classical $1/H$ flow though clearly has problems since it is not clear how to define the flow if the mean curvature is zero anywhere on $\Sigma(t)$. The trick to defining the weak flow is to have the family of surfaces $\Sigma(t)$ occasionally "jump" outward. This corresponds to flat regions in the level set function $\phi(x)$. The fact that these occasional jumps allow the flow to be defined such that the Hawking mass is still nondecreasing is really quite remarkable.

5. Inverse mean curvature flow

We will use the weak formulation of inverse mean curvature flow as developed by Huisken and Ilmanen in [7]. Before stating the results in [7] which we need, we introduce some of their terminology.

If Σ is a C^1 surface of a Riemannian 3-manifold (N, h), we say that $H \in L^1_{\mathrm{loc}}(\Sigma)$ is the *weak mean curvature* of Σ provided

$$\int_\Sigma \operatorname{div}_N(\vec{X})dA_h = \int_\Sigma H\langle \vec{X}, \vec{\nu}\rangle\, dA_h$$

for all compactly supported vector fields on N, where $\vec{\nu}$ is the exterior unit normal.

In the case that Σ is smooth, $-H\vec{\nu}$ coincides with the usual mean curvature vector of Σ.

Definition 5.1. Given a compact C^1 hypersurface Σ with weak mean curvature H in $L^2(\Sigma)$, the *Hawking mass* is defined to be

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|}{(16\pi)^3}}\left(16\pi - \int_\Sigma H^2 dA_h\right),$$

where $\left|\Sigma\right|$ is the two dimensional Hausdorff measure of Σ.

Finally, a compact set E is said to be a *minimizing hull* in N if it minimizes area on the outside, i.e., if

$$\left|\partial^* E\right| \le \left|\partial^* F\right|$$

for all F containing E such that $F - E \subset\subset N$, where $\partial^* F$ is the reduced boundary of F.

THEOREM 5.2 (Huisken-Ilmanen). *Let (N, h) be an asymptotically flat 3-manifold with nonnegative scalar curvature. If E_0 is an open precompact minimizing hull in N having a smooth boundary ∂E_0, then there is a precompact locally Lipschitz ϕ satisfying*:

(i) *For $t \ge 0$, $\Sigma_t = \partial\{\phi < t\}$ defines an increasing family of $C^{1,\alpha}$ surfaces such that $\Sigma_0 = \partial E_0$;*

(ii) *For almost all $t \ge 0$, the weak mean curvature of Σ_t is $|\nabla\phi|_{\Sigma_t}$;*

(iii) *For almost all $t \ge 0$,*

$$|\nabla\phi|_{\Sigma_t} \neq 0 \quad \textit{for almost all } x \in \Sigma_t$$

(*with respect to the surface measure*) *and*

$$\left|\Sigma_t\right| = \left|\Sigma_0\right| e^t \quad \textit{for all } t \ge 0;$$

(iv) *The Hawking mass* $m_H(\Sigma_t)$ *is a nondecreasing function of* $t \geq 0$ *provided the Euler characteristic* $\chi(\Sigma_t) \leq 2$ *for all* $t \geq 0$.

Note that when ϕ is smooth with nonvanishing gradient then (ii) is just saying that the surfaces $\{\phi = t\}_{t\geq 0}$ solve the inverse mean curvature flow with initial condition ∂E_0 because in this situation the speed of this flow is given by

$$|\nabla\phi|^{-1}\vec{\nu},$$

where $\vec{\nu}$ is the exterior unit normal to $\{\phi = t\}$.

6. Application of inverse mean curvature flow

Let (M, g) be an asymptotically flat 3-manifold with nonnegative scalar curvature which has property B (or A since A implies B).

Proof of Theorem 3.2. Given an asymptotically flat 3-manifold (M, g), there is a very useful concept of its "outermost minimal surface." An outermost minimal surface is a minimal surface which encloses all other compact minimal surfaces. A result due to Meeks, Simon, and Yau [14] is that as long as (M, g) is not topologically $\mathbb{R}^3$ (that is, S^3 before one point was removed), then the outermost minimal surface always exists and is the disjoint union of a finite number of [weakly] embedded 2-spheres (limits of uniformly smooth embedded 2-spheres). Even more remarkably, their result also states that the region exterior to the outermost minimal surface is topologically $\mathbb{R}^3$ minus a finite number of balls.

Hence, Property B implies that an outermost minimal surface must exist in our case since our original manifold is not allowed to be S^3. Furthermore, since our manifold is not a connect sum with an S^2 bundle over S^1, then (as referred to earlier) each sphere in the outermost minimal surface splits the manifold into two parts and therefore bounds a connected, compact region. In fact, since the exterior region is topologically trivial, we may view M as a connect sum along the spheres of the outermost minimal surface of the regions inside the minimal spheres.

Now recall that Property B states that $M = P\#Q$, where P is prime and $\alpha(Q) \leq 2$. By the uniqueness of prime decompositions of 3-manifolds which do not have $S^2\tilde{\times}S^1$ components, the prime manifold P must be entirely inside one of the spheres of the outermost minimal surface. Let us call this particular sphere Σ.

Σ will be the starting point for our inverse mean curvature flow. Note that since inverse mean curvature flow always flows outward, we may equivalently view the flow as always being inside a subset of Q.

Because Σ is the boundary of a region which is a minimizing hull, Theorem 5.2 may be applied. What is also crucial is that the surface Σ_t in the inverse mean curvature flow always has Euler characteristic less than or equal to 2. The reason for this is that there is a critical step in the Huisken-Ilmanen argument [7] which uses

$$\int_{\Sigma_t} K \leq 4\pi, \tag{5}$$

where K is the Gauss curvature of Σ_t, to prove that the Hawking mass is nondecreasing. By the Gauss-Bonnet theorem, this condition will be satisfied if and only if Σ_t always has Euler characteristic less than or equal to 2.

Fortunately, one property of the Huisken-Ilmanen inverse mean curvature flow is that both the interior and exterior regions of Σ_t stay connected for all t. The interior region stays connected by Huisken and Ilmanen's energy minimization argument: if the interior region ever became disconnected, the energy could be decreased by simply removing the new component. The exterior region stays connected because if it ever developed more than one component, filling in the components not connected to infinity would decrease the area of Σ, which would violate the condition that Σ is not enclosed by surfaces with less area.

Thus, since our flow takes place entirely in a subset of Q, and since Σ_t splits M (and therefore Q) into connected interior and exterior regions, Property B guarantees us that the Euler characteristic of Σ_t is always ≤ 2 since $\alpha(Q) \leq 2$ by assumption.

Let ϕ be the precompact function given by Theorem 5.2 from an inverse mean curvature flow starting at Σ. Define a locally Lipschitz function u on M by

$$u(x) := \begin{cases} f(0) & \text{if } \phi(x) \leq 0 \\ f(\phi(x)) & \text{if } \phi(x) > 0 \end{cases}$$

where $f(t)$ is defined in equation 4. The goal is to show that this particular test function for the Sobolev quotient has

$$\frac{\displaystyle\int_M |\nabla u|^2 dV_g}{\left(\displaystyle\int_M u^6 dV_g\right)^{1/3}} \leq \frac{\sigma_2}{8}$$

and that $u(x)$ decays at infinity sufficiently rapidly. Then since the Sobolev constant is the infimum of the above ratio over all test functions, it will follow that $S(g) \leq \frac{\sigma_2}{8}$, proving Theorem 3.2.

Let $\Sigma_t := \partial\{\phi < t\}$. To estimate the numerator the coarea formula gives

$$\begin{aligned}\int_M |\nabla u|^2 dV_g &= \int_{\{\phi\geq 0\}} f'(\phi(x))^2\, |\nabla\phi\,|^2 dV_g\\ &= \int_0^\infty f'(t)^2 \int_{\Sigma_t} |\nabla\phi\,| dA_g\, dt\\ &= \int_0^\infty f'(t)^2 \int_{\Sigma_t} H dA_g\, dt,\end{aligned}$$

where the last equality comes from Theorem 5.2 (ii).

To estimate the denominator we apply again the coarea formula. In order to do so set

$$C_\varepsilon := \{x \in M|\ |\nabla\phi\,| > \varepsilon\,\} \cap \{\phi \geq 0\}.$$

Then

$$\begin{aligned}\int_M u^6\, dV_g &\geq \int_{C_\varepsilon} u^6\, dV_g\\ &= \int_0^\infty f(t)^6 \int_{\Sigma_t\cap C_\varepsilon} |\nabla\phi\,|^{-1}\, dA_g\, dt.\end{aligned}$$

Note that by Theorem 5.2 (iii), for almost all $t \geq 0$, Σ_t differs from $\Sigma_t \cap C_0$ by a null measure set with respect to the surface measure. So, making ε going to zero in the previous inequality, the Lebesgue Monotone Convergence Theorem implies:

$$\begin{aligned}\int_M u^6\, dV_g &\geq \int_0^\infty f(t)^6 \int_{\Sigma_t} |\nabla\phi\,|^{-1}\, dA_g\, dt\\ &= \int_0^\infty f(t)^6 \int_{\Sigma_t} H^{-1}\, dA_g\, dt\\ &\geq \int_0^\infty f(t)^6 \big|\Sigma_t\big|^2 \left(\int_{\Sigma_t} H\, dA_g\right)^{-1} dt\\ &= \int_0^\infty f(t)^6 e^{2t}\, \big|\Sigma_0\big|^2 \left(\int_{\Sigma_t} H\, dA_g\right)^{-1} dt\end{aligned}$$

where in the last two steps we used Hölder's inequality and $\big|\Sigma_t\big| = e^t\big|\Sigma_0\big|$ respectively.

Next, using the monotonicity of the Hawking mass (Theorem 5.2 (iv)), we find an upper bound for $\int_{\Sigma_t} H\, dA_g$.

LEMMA 6.1. *For all* $t \geq 0$,

$$\int_{\Sigma_t} H dA_g \leq \sqrt{16\pi\big|\Sigma_0\big|(e^t - e^{t/2})}.$$

Proof. Using the monotonicity of the Hawking mass (Theorem 5.2 (iv)) we have

$$\begin{aligned}\sqrt{\frac{|\Sigma_t|}{(16\pi)^3}}\left(16\pi - \int_{\Sigma_t} H^2\, dA_g\right) &= m_H(\Sigma_t)\\ &\geq m_H(\Sigma_0)\\ &= \sqrt{\frac{|\Sigma_0|}{16\pi}}\end{aligned}$$

and so, by Theorem 5.2 (iii),

$$\begin{aligned}\int_{\Sigma_t} H^2\, dA_g &\leq 16\pi\left(1 - \sqrt{|\Sigma_0||\Sigma_t|^{-1}}\right)\\ &= 16\pi\left(1 - e^{-t/2}\right).\end{aligned}$$

Thus, it follows from Hölder's inequality that

$$\begin{aligned}\int_{\Sigma_t} H dA_g &\leq \sqrt{16\pi|\Sigma_t|(1 - e^{-t/2})}\\ &= \sqrt{16\pi|\Sigma_0|(e^t - e^{t/2})}.\end{aligned}$$

□

Combining Lemma 6.1 with the previous estimates for the numerator and denominator we obtain

$$\begin{aligned}\frac{\displaystyle\int_M |\nabla u|^2 dV_g}{\left(\displaystyle\int_M u^6 dV_g\right)^{1/3}} &\leq \frac{(16\pi)^{2/3}\displaystyle\int_0^\infty f'(t)^2\sqrt{e^t - e^{t/2}}\, dt}{\left(\displaystyle\int_0^\infty f(t)^6 e^{2t}\,(e^t - e^{t/2})^{-1/2} dt\right)^{1/3}}\\ &= \frac{\sigma_2}{8}.\end{aligned}$$

Observe that a dramatic simplification occurs at the last step. The point is that we are not actually computing the two fairly complicated integrals. Instead, we observe that we have equality in all of our inequalities in the model case where we take a Schwarzschild metric on $\mathbb{RP}^3 - \{p\}$. Then since $f(t)$ is chosen to give the optimal test function in this case and we know that the Sobolev constant of the Schwarzschild metric on $\mathbb{RP}^3 - \{p\}$ is $\sigma_2/8$, we get the simplification. Note that there is no need to even compute $f(t)$ explicitly much less to actually evaluate the above integrals.

However, for those who are interested,

$$f(t) = \frac{1}{\sqrt{2e^t - e^{t/2}}}.$$

We leave it to the brave reader to actually plug this expression into the above integrals. We congratulate and thank Kevin Iga of Pepperdine University for being the first person to actually perform this check successfully.

As a final note, we need to verify that the test function $u(x)$ which we defined using inverse mean curvature flow actually decays sufficiently rapidly at infinity to be a valid test function. This is straightforward and follows from the fact that in the asymptotically flat coordinate chart, Σ_t lies in the annular region $\{x \mid c_1 < |x|e^{-t/2} < c_2\}$ by the maximum principle for some positive constants c_1 and c_2 for sufficiently large t. From this it can be shown that

$$k_1 \leq u(x)|x| \leq k_2$$

for some positive k_1 and k_2 and sufficiently large t. Hence, $u(x)$ satisfies equation 3, so Theorem 3.2 follows. $\square$

7. Open problems and acknowledgments

Since this paper was motivated by the computation of $\sigma(\mathbb{RP}^3)$, it makes since to next try to verify equation 2 for all n. One could also hope that solving these problems might have the corresponding spin-offs that this paper did and could be used to classify prime manifolds which admit metrics with nonnegative scalar curvature. The expected result is that the only such manifolds are quotients of S^3 and $S^2 \times S^1$. Naturally these same questions are interesting in higher dimensions as well.

Another great conjecture comes from considering the negative σ-invariant case. Suppose M^n is compact and admits a hyperbolic metric g_0. Then Schoen conjectures that $\sigma(M) = E(g_0)$ (as well as the corresponding positive constant curvature statement). This conjecture about hyperbolic 3-manifolds is equivalent to the following volume comparison conjecture: If g is any other metric on M^n with scalar curvature larger than that of g_0, then $\mathrm{Vol}(g) \geq \mathrm{Vol}(g_0)$. It is possible that these conjectures will follow from Perelman's work on the Ricci flow (see below).

More generally, it would be extremely interesting to have a procedure under which the σ-invariant could be related to a natural topological decomposition or geometrization of the 3-manifold. In the negative case, Perelman's recent work on the Ricci flow appears on the verge of answering this request. In this case, it would appear that the σ-invariant is likely given by the long time limit of $\bar{\lambda}$ (defined at the end of Perelman's first paper [16]) under Ricci flow, which in turn can be understood in terms of the geometrization of the original manifold into graph manifold pieces and hyperbolic pieces. Other connections between Ricci flow and the σ-invariant would also be very interesting, since both can be broadly interpreted as attempts at some kind of geometrization.

However, so far Ricci flow techniques have not been useful for finding the σ-invariants when these invariants are positive. This is related to the fact that Perelman's $\bar{\lambda}$ quantity is only necessarily nondecreasing when $\bar{\lambda}$ is negative. As mentioned earlier, at the time of this paper Kazuo Akutagawa and the second author found a way to extend the results of this paper to compute $\sigma(\mathbb{RP}^3\#(S^2\times S^1))$ as well as the σ-invariants of some other 3-manifold. Since their paper is still in progress, we refer the reader to [1] for their full results. We also believe that the 3-manifold topological invariant α defined in this paper deserves further consideration. It would also be very nice to show that except for the five manifolds listed in Corollary 2.6, all other prime 3-manifolds have $\sigma < \sigma_2$. This last conjecture may follow from a strengthening of some of the techniques in this paper.

This paper began after a talk given by Richard Schoen at the Gilbarg Memorial Conference at Stanford University in April 2002. We would also like to thank John Hempel and Steven Kerckhoff for helping us formulate the topological hypotheses of this paper as cleanly as possible and Jeff Viaclovsky for helpful discussions.

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(Received July 27, 2002)