Annals of Mathematics, **159** (2004), 377–405

$(\log t)^{2/3}$ law of the two dimensional asymmetric simple exclusion process

By Horng-Tzer Yau*

Abstract

We prove that the diffusion coefficient for the two dimensional asymmetric simple exclusion process with nearest-neighbor-jumps diverges as $(\log t)^{2/3}$ to the leading order. The method applies to nearest and non-nearest neighbor asymmetric simple exclusion processes.

1. Introduction

The asymmetric simple exclusion process is a Markov process on $\{0,1\}^{\mathbb{Z}^d}$ with asymmetric jump rates. There is at most one particle allowed per site and thus the word exclusion. The particle at a site x waits for an exponential time and then jumps to y with rate $p(x-y)$ provided that the site is not occupied. Otherwise the jump is suppressed and the process starts again. The jump rate is assumed to be asymmetric so that in general there is net drift of the system. The simplicity of the model has made it the default stochastic model for transport phenomena. Furthermore, it is also a basic component for models [5], [12] with incompressible Navier-Stokes equations as the hydrodynamical equation.

The hydrodynamical limit of the asymmetric simple exclusion process was proved by Rezakhanlou [13] to be a viscousless Burgers equation in the Euler scaling limit. If the system is in equilibrium, the Burgers equation is trivial and the system moves with a uniform velocity. This uniform velocity can be removed and the viscosity of the system, or the diffusion coefficient, can be defined via the standard mean square displacement. Although the diffusion coefficient is expected to be finite for dimension $d > 2$, a rigorous proof was obtained only a few years ago [9] by estimating the corresponding resolvent equation. Based on the mode coupling theory, Beijeren, Kutner and Spohn [3]

*Work partially supported by NSF grant DMS-0072098, DMS-0307295 and MacArthur fellowship.

conjectured that $D(t) \sim (\log t)^{2/3}$ in dimension $d = 2$ and $D(t) \sim t^{1/3}$ in $d = 1$. The conjecture at $d = 1$ was also made by Kardar-Parisi-Zhang via the KPZ equation.

This problem has received much attention recently in the context of integrable systems. The main quantity analyzed is fluctuation of the current across the origin in $d = 1$ with the jump restricted to the nearest right site, the totally asymmetric simple exclusion process (TASEP). Consider the special configuration that all sites to the left of the origin were occupied while all sites to the right of the origin were empty. Johansson [6] observed that the current across the origin with this special initial data can be mapped into a last passage percolation problem. By analyzing resulting percolation problem asymptotically in the limit $N \to \infty$, Johansson proved that the variance of the current is of order $t^{2/3}$. In the case of discrete time, Baik and Rains [2] analyzed an extended version of the last passage percolation problem and obtained fluctuations of order t^α, where $\alpha = 1/3$ or $\alpha = 1/2$ depending on the parameters of the model. Both the approaches of [6] and [2] are related to the earlier results of Baik-Deift-Johansson [1] on the distribution of the length of the longest increasing subsequence in random permutations.

In [10] (see also [11]), Prähofer and Spohn succeeded in mapping the current of the TASEP into a last passage percolation problem for a general class of initial data, including the equilibrium case considered in this article. For the discrete time case, the extended problem is closely related to the work [2], but the boundary conditions are different. For continuous time, besides the boundary condition issue, one has to extend the result of [2] from the geometric to the exponential distribution.

To relate these results to our problem, we consider the asymmetric simple exclusion process in equilibrium with a Bernoulli product measure of density ρ as the invariant measure. Define the time dependent correlation function in equilibrium by

$$S(x,t) = \langle \eta_x(t); \eta_0(0) \rangle.$$

We shall choose $\rho = 1/2$ so that there is no net global drift, $\sum_x xS(x,t) = 0$. Otherwise a subtraction of the drift should be performed. The diffusion coefficient we consider is (up to a constant) the second moment of $S(x,t)$:

$$\sum_x x^2 S(x,t) \sim D(t)t$$

for large t. On the other hand the variance of the current across the origin is proportional to

$$\sum_x |x| S(x,t). \tag{1.1}$$

Therefore, Johansson's result on the variance of the current can be interpreted as the spreading of $S(x,t)$ being of order $t^{2/3}$. The result of Johansson is for special initial data and does not directly apply to the equilibrium case. If we combine the work of [10] and [2], neglect various issues discussed above, and extrapolate to the second moment, we obtain growth of the second moment as $t^{4/3}$, consistent with the conjectured $D(t) \sim t^{1/3}$.

We remark that the results based on integrable systems are not just for the variance of the current across the origin, but also for its full limiting distribution. The main restrictions appear to be the rigid requirements of the fine details of the dynamics and the initial data. Furthermore, it is not clear whether the analysis on the current across the origin can be extended to the diffusivity. In particular, the divergence of $D(t)$ as $t \to \infty$ in $d = 1$ has not been proved via this approach even for the TASEP.

Recent work of [8] has taken a completely different approach. It is based on the analysis of the Green function of the dynamics. One first used the duality to map the resolvent equation into a system of infinitely-coupled equations. The hard core condition was proved to be of lower order. Once the hard core condition was removed, the Fourier transform became a very useful tool and the Green function was estimated to degree three. This yielded a lower bound to the full Green function via a monotonicity inequality. Thus one obtained the lower bounds $D(t) \geq t^{1/4}$ in $d = 1$ and $D(t) \geq (\log t)^{1/2}$ in $d = 2$ [8]. In this article, we shall estimate the Green function to degrees high enough to determine the leading order behavior $D(t) \sim (\log t)^{2/3}$ in $d = 2$.

1.1. *Definitions of the models.* Denote the configuration by $\eta = (\eta_x)_{x\in\mathbb{Z}^d}$ where $\eta_x = 1$ if the site x is occupied and $\eta_x = 0$ otherwise. Denote $\eta^{x,y}$ the configuration obtained from η by exchanging the occupation variables at x and y:

$$(\eta^{x,y})_z \;=\; \begin{cases} \eta_z & \text{if } z \neq x,\, y, \\ \eta_x & \text{if } z = y \text{ and} \\ \eta_y & \text{if } z = x. \end{cases}$$

Then the generator of the asymmetric simple exclusion process is given by

$$(\mathcal{L}f)(\eta) \;=\; \sum_{j=1}^{d} \sum_{x,y\in\mathbb{Z}^d} p(x,y)\eta_x[1-\eta_y][f(\eta^{x,y}) - f(\eta)]. \tag{1.2}$$

where $\{e_k,\, 1 \leq k \leq d\}$ stands for the canonical basis of $\mathbb{Z}^d$. For each ρ in $[0,1]$, denote by ν_ρ the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}^d}$ with density ρ and by $< \cdot, \cdot >_\rho$ the inner product in $L^2(\nu_\rho)$. The probability measures ν_ρ are invariant for the asymmetric simple exclusion process.

For two cylinder functions f, g and a density ρ, denote by $\langle f; g\rangle_\rho$ the covariance of f and g with respect to ν_ρ:

$$\langle f; g\rangle_\rho \;=\; \langle fg\rangle_\rho - \langle f\rangle_\rho \langle g\rangle_\rho.$$

Let P_ρ denote the law of the asymmetric simple exclusion process starting from the equilibrium measure ν_ρ. Expectation with respect to P_ρ is denoted by E_ρ. Let

$$S_\rho(x,t) = E_\rho[\{\eta_x(t) - \eta_x(0)\}\eta_0(0)] = \langle \eta_x(t); \eta_0(0)\rangle_\rho$$

denote the time dependent correlation functions in equilibrium with density ρ. The compressibility

$$\chi = \chi(\rho) = \sum_x \langle \eta_x; \eta_0\rangle_\rho = \sum_x S_\rho(x,t)$$

is time independent and $\chi(\rho) = \rho(1-\rho)$ in our setting.

The bulk diffusion coefficient is the variance of the position with respect to the probability measure $S_\rho(x,t)\chi^{-1}$ in $\mathbb{Z}^d$ divided by t; i.e.,

$$D_{i,j}(\rho,t) \;=\; \frac{1}{t}\Big\{ \sum_{x\in\mathbb{Z}^d} x_i x_j S_\rho(x,t)\chi^{-1} - (v_i t)(v_j t)\Big\}, \tag{1.3}$$

where v in $\mathbb{R}^d$ is the velocity defined by

$$vt \;=\; \sum_{x\in\mathbb{Z}^d} x S_\rho(x,t)\chi^{-1}. \tag{1.4}$$

For simplicity, we shall restrict ourselves to the case where the jump is symmetric in the y axis but totally asymmetric in the x axis; i.e., only the jump to the right is allowed on the x axis. Our results hold for other jump rates as well. The generator of this process is given by

(1.5)

$$(\mathcal{L}f)(\eta) \;=\; \sum_{x\in\mathbb{Z}^d}\Big[\eta_x(1-\eta_{x+e_1})(f(\eta^{x,x+e_1}) - f(\eta)) + \frac{1}{2}\big(f(\eta^{x,x+e_2}) - f(\eta)\big)\Big]$$

where we have combined the symmetric jump on the y axis into the last term. We emphasize that the result and method in this paper apply to all asymmetric simple exclusion processes; the special choice is made to simplify the notation. The velocity of the totally asymmetric simple exclusion process is explicitly computed as $v \;=\; 2(1-2\rho)e_1$. We further assume that the density is $1/2$ so that the velocity is zero for simplicity.

Denote the instantaneous currents (i.e., the difference between the rate at which a particle jumps from x to $x+e_i$ and the rate at which a particle jumps from $x+e_i$ to x) by $\tilde{w}_{x,x+e_i}$:

$$\tilde{w}_{x,x+e_1} \;=\; \eta_x[1-\eta_{x+e_1}], \qquad \tilde{w}_{x,x+e_2} = \frac{\eta_x - \eta_{x+e_2}}{2} \tag{1.6}$$

We have the conservation law

$$\mathcal{L}\eta_0 + \sum_{i=1}^{2}\Big\{\tilde{w}_{-e_i,0} - \tilde{w}_{0,e_i}\Big\} = 0.$$

Let $w_i(\eta)$ denote the renormalized current in the i^{th} direction:

$$w_i(\eta) = \tilde{w}_{0,e_i} - \langle \tilde{w}_{0,e_i} \rangle_\rho - \frac{d}{d\theta} \langle \tilde{w}_{0,e_i} \rangle_\theta \Big|_{\theta=\rho} (\eta_0 - \rho).$$

Notice the subtraction of the linear term in this definition. We have

$$w_1(\eta) = -(\eta_0 - \rho)(\eta_{e_1} - \rho) - \rho[\eta_{e_1} - \eta_0], \quad w_2(\eta) = \frac{\eta_0 - \eta_{e_2}}{2}.$$

Define the semi-inner product

$$\langle\!\langle g, h \rangle\!\rangle_\rho \;=\; \sum_{x \in \mathbb{Z}^d} < \tau_x g \,;\, h >_\rho = \sum_{x \in \mathbb{Z}^d} < \tau_x h \,;\, g >_\rho \,, \tag{1.7}$$

where $\tau_x g(\eta) = g(\tau_x \eta)$ and $\tau_x \eta$ is the translation of the configuration to x. Since the subscript ρ is fixed to be $1/2$ in this paper, we shall drop it. All but a finite number of terms in this sum vanish because ν_ρ is a product measure and g, h are mean zero. From this inner product, we define the norm:

$$\|f\|^2 = \langle\!\langle f, f \rangle\!\rangle. \tag{1.8}$$

Notice that all degree one functions vanish in this norm and we shall identify the currents w with their degree two parts. Therefore, for the rest of this paper, we shall put

$$w_1(\eta) = (\eta_0 - \rho)(\eta_{e_1} - \rho), \quad w_2(\eta) = 0. \tag{1.9}$$

Fix a unit vector $\xi \in \mathbb{Z}^d$. From some simple calculation using Ito's formula [7] we can rewrite the diffuseness as

$$\xi \cdot D\xi - \frac{\xi \cdot \xi}{2} \;=\; \frac{1}{\chi} \left\| t^{-1/2} \int_0^t ds\; (\xi \cdot w)(\eta(s)) \right\|^2. \tag{1.10}$$

This is some variant of the Green-Kubo formula. Since $w_2 = 0$, $D - I/2$ is a matrix with all entries zero except

$$D_{11} \;=\; \frac{1}{2} + \frac{1}{\chi} \left\| t^{-1/2} \int_0^t ds\; w_1(\eta(s)) \right\|^2.$$

Recall that $\int_0^\infty e^{-\lambda t} f(t) dt \sim \lambda^{-\alpha}$ as $\lambda \to 0$ means (in some weak sense) that $f(t) \sim t^{\alpha - 1}$. Throughout the following λ will always be a positive real number. The main result of this article is the following theorem. We have restricted ourselves in this theorem to the special process given by (1.5) at $\rho = 1/2$. We believe that the method applies to general cases as well; see the comment at the end of the next section for more details.

Theorem 1.1. *Consider the asymmetric simple exclusion process in $d = 2$ with generator given by* (1.5). *Suppose that the density $\rho = 1/2$. Then there exists a constant $\gamma > 0$ so that for sufficiently small $\lambda > 0$,*

$$\lambda^{-2} |\log \lambda|^{2/3} e^{-\gamma |\log\log\log \lambda|^2} \le \int_0^\infty e^{-\lambda t}\; t D_{11}(t) dt \le \lambda^{-2} |\log \lambda|^{2/3} e^{\gamma |\log\log\log \lambda|^2}.$$

From the definition, we can rewrite the diffusion coefficient as

$$tD_{11}(t) = \frac{t}{2} + \frac{1}{\chi}\int_0^t \int_0^s \langle\!\langle e^{u\mathcal{L}} w_1, w_1 \rangle\!\rangle \, duds.$$

Thus

$$\begin{aligned}(1.11) \qquad & \int_0^\infty e^{-\lambda t} \, tD_{11}(t)dt \\ & = \frac{1}{2\lambda^2} + \frac{1}{\chi}\int_0^\infty dt \int_0^t \int_0^s e^{-\lambda t} \; \langle\!\langle e^{u\mathcal{L}} w_1, w_1 \rangle\!\rangle \, duds \\ & = \frac{1}{2\lambda^2} + \frac{1}{\chi}\int_0^\infty du \Big\{ \int_u^\infty dt \, e^{-\lambda(t-u)} \Big(\int_u^t ds \Big) \Big\} \langle\!\langle e^{-\lambda u} e^{u\mathcal{L}} w_1, \, w_1 \rangle\!\rangle \\ & = \frac{1}{2\lambda^2} + \chi^{-1}\lambda^{-2} \langle\!\langle w_1 \, , \, (\lambda - \mathcal{L})^{-1} w_1 \rangle\!\rangle.\end{aligned}$$

Therefore, Theorem 1.1 follows from the next estimate on the resolvent.

THEOREM 1.2. *There exists a constant $\gamma > 0$ such that for sufficiently small $\lambda > 0$,*

$$|\log \lambda|^{2/3} e^{-\gamma |\log\log\log\lambda|^2} \leq \langle\!\langle w_1, (\lambda - \mathcal{L})^{-1} w_1 \rangle\!\rangle \leq |\log\lambda|^{2/3} e^{\gamma|\log\log\log\lambda|^2}.$$

From the following well-known lemma, the upper bound holds without the time integration. For a proof, see [9].

LEMMA 1.1. *Suppose μ is an invariant measure of a process with generator $\mathcal{L}$. Then*

$$(1.12) \qquad E^\mu\Big[\Big(t^{-1/2}\int_0^t w(\eta(s))\, ds\Big)^2\Big] \leq \; \langle\!\langle w_1, (t^{-1} - \mathcal{L})^{-1} w_1 \rangle\!\rangle.$$

Since w_1 is the only non-vanishing current, we shall drop the subscript 1.

2. Duality and removal of the hard core condition

Denote by $\mathcal{C} = \mathcal{C}(\rho)$ the space of ν_ρ-mean-zero-cylinder functions. For a finite subset Λ of $\mathbb{Z}^d$, denote by ξ_Λ the mean zero cylinder function defined by

$$\xi_\Lambda \; = \prod_{x\in\Lambda} \xi_x, \qquad \xi_x = \frac{\eta_x - \rho}{\sqrt{\rho(1-\rho)}}.$$

Denote by $\mathcal{M}_n$ the space of cylinder homogeneous functions of degree n, i.e., the space generated by all homogeneous monomials of degree n :

$$\mathcal{M}_n \; = \; \Big\{ h \in \mathcal{C} \, ; \; h = \sum_{|\Lambda| = n} h_\Lambda \xi_\Lambda \, , \; h_\Lambda \in \mathbb{R} \Big\}.$$

Notice that in this definition all but a finite number of coefficients h_Λ vanish because h is assumed to be a cylinder function. Denote by $\mathcal{C}_n = \cup_{1\leq j\leq n} \mathcal{M}_j$

the space of cylinder functions of degree less than or equal to n. All mean zero cylinder functions h can be decomposed as a finite linear combination of cylinder functions of finite degree : $\mathcal{C} = \cup_{n\geq 1}\mathcal{M}_n$. Let $\mathcal{L} = \mathcal{S} + \mathcal{A}$ where $\mathcal{S}$ is the symmetric part and $\mathcal{A}$ is the asymmetric part. Fix a function g in $\mathcal{M}_n$: $g = \sum_{\Lambda,|\Lambda|=n} g_\Lambda \xi_\Lambda$. A simple computation shows that the symmetric part is given by

$$(\mathcal{S}g)(\eta) = -\frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{j=1}^{d}\sum_{\substack{\Omega,\,|\Omega|=n-1\\ \Omega\cap\{x,x+e_j\}=\phi}} \Big\{g_{\Omega\cup\{x+e_j\}} - g_{\Omega\cup\{x\}}\Big\}\Big[\xi_{\Omega\cup\{x+e_j\}} - \xi_{\Omega\cup\{x\}}\Big].$$

The asymmetric part $\mathcal{A}$ is decomposed into two pieces $\mathcal{A} = M + J$ so that M maps $\mathcal{M}_n$ into itself and $J = J_+ + J_-$ maps $\mathcal{M}_n$ into $\mathcal{M}_{n-1}\cup\mathcal{M}_{n+1}$:

(2.1)
$$(Mg)(\eta) = \frac{1-2\rho}{2}\sum_{x\in\mathbb{Z}^d}\sum_{\substack{\Omega,|\Omega|=n-1\\ \Omega\cap\{x,x+e_1\}=\phi}} \Big\{g_{\Omega\cup\{x+e_1\}} - g_{\Omega\cup\{x\}}\Big\}\Big[\xi_{\Omega\cup\{x+e_1\}} + \xi_{\Omega\cup\{x\}}\Big],$$

(2.2)
$$(J_+g)(\eta) = -\sqrt{\rho(1-\rho)}\sum_{x\in\mathbb{Z}^d}\sum_{\substack{\Omega,\,|\Omega|=n-1\\ \Omega\cap\{x,x+e_1\}=\phi}} \Big\{g_{\Omega\cup\{x+e_1\}} - g_{\Omega\cup\{x\}}\Big\}\xi_{\Omega\cup\{x,x+e_1\}},$$

(2.3)
$$(J_-g)(\eta) = -\sqrt{\rho(1-\rho)}\sum_{x\in\mathbb{Z}^d}\sum_{\substack{\Omega,\,|\Omega|=n-1\\ \Omega\cap\{x,x+e_1\}=\phi}} \Big\{g_{\Omega\cup\{x+e_1\}} - g_{\Omega\cup\{x\}}\Big\}\xi_{\Omega}\ .$$

Clearly, $J_+^* = -J_-$. Restricting to the case $\rho = 1/2$, we have $M = 0$ and thus $J = \mathcal{A}$. We shall now identify monomials of degree n with symmetric functions of n variables. Let $\mathcal{E}_1$ denote the set with no double sites, i.e.,

$$\mathcal{E}_1 = \{\mathbf{x}_n := (x_1,\cdots,x_n) : x_i \neq x_j,\ \text{for } i\neq j\}$$

Define

$$f(x_1,\cdots,x_n) = f_{\{x_1,\cdots,x_n\}}, \qquad \text{if } \mathbf{x}_n\in\mathcal{E}_1, \tag{2.4}$$
$$= 0, \qquad \text{if } \mathbf{x}_n\notin\mathcal{E}_1.$$

Notice that

$$E\big\{\big[\sum_{|A|=n} f_A\xi_A\big]^2\big\} = \frac{1}{n!}\sum_{x_1,\cdots,x_n\in\mathbb{Z}^d}|f(x_1,\cdots,x_n)|^2.$$

From now on, we shall refer to $f(x_1,\cdots,x_n)$ as a homogeneous function of degree n vanishing on the complement of $\mathcal{E}_1$.

With this identification, the coefficients of the current are given by

$$w_1(0,e_1) = w_1(e_1,0) := (w_1)_{\{0,e_1\}} = -1/4$$

and zero otherwise. Since we only have one nonvanishing current, we shall drop the subscript 1 for the rest of this paper.

If g is a symmetric homogeneous function of degree n, we can check that

$$
\begin{aligned}
(2.5)\quad A_+g(x_1,\cdots,x_{n+1}) = -\frac{1}{2}\sum_{i=1}^{n+1}\sum_{j\neq i}&[g(x_1,\cdots,x_i+e_1,\cdots,\widehat{x_j},\cdots x_{n+1})\\
&- g(x_1,\cdots,x_i,\cdots,\widehat{x_j},\cdots,x_{n+1})]\\
&\times\ \delta(x_j-x_i-e_1)\prod_{k\neq j}\big(1-\delta(x_j-x_k)\big)
\end{aligned}
$$

where $\delta(0)=1$ and zero otherwise. We can check that

(2.6)

$$
\begin{aligned}
\mathcal{S}g(x_1,\cdots,x_n) = \alpha\sum_{i=1}^{n}\sum_{\sigma=\pm}\sum_{\beta=1,2}\prod_{k\neq i}&\big(1-\delta(x_i+\sigma e_\beta-x_k)\big)\\
&\times[g(x_1,\cdots x_i+\sigma e_\beta,\cdots,x_n)-g(x_1,\cdots,x_i,,\cdots,x_n)]
\end{aligned}
$$

where α is some constant and $\delta(0)=1$ and zero otherwise. The constant α is not important in this paper and we shall fix it so that $\mathcal{S}$ is the same as the discrete Laplacian with Neumann boundary condition on $\mathcal{E}_1$.

The hard core condition makes various computations very complicated. In particular, the Fourier transform is difficult to apply. However, if we are interested only in the orders of magnitude, this condition was removed in [8]. We now summarize the main result in [8].

For a function F, we shall use the same symbol $\langle F\rangle$ to denote the expectation

$$
\frac{1}{n!}\sum_{x_1,\cdots,x_n\in\mathbb{Z}^2}F(x_1,\cdots,x_n).
$$

We now define A_+F using the same formula except we drop the last delta function, i.e,

(2.7)

$$
\begin{aligned}
A_+F(x_1,\cdots,x_{n+1}) = -\frac{1}{2}\sum_{i=1}^{n+1}\sum_{j\neq i}&\Big[\,F(x_1,\cdots,x_i+e_1,\cdots,\widehat{x_j},\cdots x_{n+1})\\
&-F(x_1,\cdots,x_i,\cdots,\widehat{x_j},\cdots,x_{n+1})\,\Big]\,\delta(x_j-x_i-e_1).
\end{aligned}
$$

Notice that $\langle A_+F\rangle = 0$. Thus the counting measure is invariant and we define $A_- = -A_+^*$; i.e.,

$$
(2.8)\qquad \langle A_-G,F\rangle = -\langle G,A_+F\rangle.
$$

Finally, we define

$$
L=\Delta+A,\qquad A=A_++A_-,
$$

where the discrete Laplacian is given by

$$\Delta F(x_1, \dots , x_n) = \sum_{i=1}^{n} \sum_{\sigma=\pm} \sum_{\alpha=1,2} [F(x_1, \dots x_i + \sigma e_\alpha, \dots , x_n) - F(x_1, \dots , x_i, , \dots , x_n)].$$

For the rest of this paper, we shall only work with F and L. So all functions are defined everywhere and L has no hard core condition.

Denote by π_n the projection onto functions with degrees less than or equal to n. Let L_n be the projection of L onto the image of π_n, i.e., $L = \pi_n L \pi_n$. The key result of [8] is the following lemma.

LEMMA 2.1. *For any $\lambda > 0$ fixed, for $k \geq 1$,*

$$C^{-1} k^{-6} \langle\!\langle w, (\lambda - L_{2k+1})^{-1} w \rangle\!\rangle \leq \langle\!\langle w, (\lambda - \mathcal{L})^{-1} w \rangle\!\rangle \leq C k^4 \langle\!\langle w, (\lambda - L_{2k})^{-1} w \rangle\!\rangle. \tag{2.9}$$

The expression $\langle\!\langle w, L_n^{-1} w \rangle\!\rangle$ was also calculated in [8]. The resolvent equation $(\lambda - L_n)u = w$ can be written as

$$\begin{aligned} (\lambda - S)u_n - A_+ u_{n-1} &= 0, \\ A_+^* u_{k+1} + (\lambda - S)u_k - A_+ u_{k-1} &= 0, \quad n-1 \geq k \geq 3, \\ A_+^* u_3 + (\lambda - S)u_2 &= w. \end{aligned} \tag{2.10}$$

We can solve the first equation of (2.10) by

$$u_n = (\lambda - S)^{-1} A_+ u_{n-1}.$$

Substituting this into the equation of degree $n-1$, we have

$$u_{n-1} = \Big[(\lambda - S) + A_+^* (\lambda - S)^{-1} A_+\Big]^{-1} u_{n-2}.$$

Solving iteratively we arrive at

$$u_2 = \Big[(\lambda - S) + A_+^* \Big\{(\lambda - S) + \cdots \cdots + A_+^* \Big((\lambda - S) + A_+^* (\lambda - S)^{-1} A_+\Big)^{-1} A_+ \Big\}^{-1} A_+ \Big]^{-1} w.$$

This gives an explicit expression for $\langle\!\langle w, (\lambda - L_n)^{-1} w \rangle\!\rangle$, for example,

$$\begin{aligned} &\langle\!\langle w, (\lambda - L_3)^{-1} w \rangle\!\rangle = \langle\!\langle w, \Big[\lambda - S + A_+^* (\lambda - S)^{-1} A_+\Big]^{-1} w \rangle\!\rangle. \\ &\langle\!\langle w, (\lambda - L_4)^{-1} w \rangle\!\rangle = \langle\!\langle w, \Big[\lambda - S + A_+^* \Big\{\lambda - S + A_+^* (\lambda - S)^{-1} A_+\Big\}^{-1} A_+\Big]^{-1} w \rangle\!\rangle. \\ &\langle\!\langle w, (\lambda - L_5)^{-1} w \rangle\!\rangle \\ &= \langle\!\langle w, \Big[\lambda - S + A_+^* \Big\{\lambda - S + A_+^* [\lambda - S + A_+^* (\lambda - S)^{-1} A_+]^{-1} A_+\Big\}^{-1} A_+\Big]^{-1} w \rangle\!\rangle. \end{aligned} \tag{2.11}$$

We have assumed that the process is given by (1.5); i.e., the jump is symmetric in the y axis and totally asymmetric in the x axis. But the setup in this section clearly applies to general jump rates as well. The only difference is the analysis of the equation (2.10). Since our main tool in the next few sections is the Fourier transform, we expect it to be applicable to general translationally invariant jump rates and to yield similar results. The more important assumption for Theorem 1.1 is the density $\rho = 1/2$. For the current across the origin in one dimension [2], [10], $\rho = 1/2$ is the only equilibrium density for which the variance of the current across the origin is not the standard Gaussian. For the diffuseness the density $\rho = 1/2$ may not play such a critical role. The reason is that the operator M in (2.1) behaves like a drift operator. In Fourier space, it becomes a multiplication operator $p_1 + \cdots + p_n$. Due to the average over the translation, the relevant inner product (3.2) restricts the Fourier modes to the hyperplane $p_1 + \cdots + p_n = 0$. Therefore, M essentially vanishes for all densities with respect to the norm defined by the inner product (3.2). More careful analysis is still needed to determine if this heuristic argument is correct. For the current across the origin, on the other hand, there is no average over translation and p_1 behaves like an elliptic operator in $d = 1$. This explains its Gaussian behavior for $\rho \neq 1/2$.

3. Main estimate

We now introduce the following conventions: Denote the component of p by (r, s). Denote $\mathbf{p}_n = (p_1, \cdots, p_n), \mathbf{r}_n = (r_1, \cdots, r_n)$ and $\mathbf{s}_n = (s_1, \cdots, s_n)$. The Fourier transform of

$$[F(x_1 + e_1, \cdots, x_n) - F(x_1, \cdots, x_n)]\delta(x_{n+1} = x_1 + e_1)$$

is given by

$$\begin{aligned}
\sum_x [F(x_1 + e_1, \cdots, x_n) - F(x_1, \cdots, x_n)]e^{-i[x_1p_1 + \cdots + x_np_n + (x_1+e_1)p_{n+1}]} \\
= \left[e^{ir_1} - e^{-ir_{n+1}}\right] \hat{F}(p_1 + p_{n+1}, \cdots, p_n) \\
\sim [i(r_1 + r_{n+1})] \ \hat{F}(p_1 + p_{n+1}, \cdots, p_n).
\end{aligned}$$

All functions considered for the rest of this paper are symmetric periodic functions of period 2π.

Since F is symmetric,

$$\widehat{A_+F}(\mathbf{p}_{n+1}) = -\sum_{j<m}^{n+1} (e^{ir_j} - e^{-ir_m})\hat{F}(p_1, \cdots, p_j + p_m, \cdots, \widehat{p_m}, \cdots, p_{n+1}). \tag{3.1}$$

We can also compute the discrete Laplacian acting on F,

$$\widehat{-\Delta F}(\mathbf{p}_n) = -\sum_{j=1}^{n} \sum_{k=1,2} \left[e^{ie_k p_j} - 2 + e^{-ie_k p_j} \right] \ \hat{F}(p_1, \cdots, p_n) = \boldsymbol{\omega}(\mathbf{p}_n)\hat{F}(\mathbf{p}_n)$$

where $\boldsymbol{\omega}(\mathbf{p}_n) = \sum_{j=1}^{n} \omega(p_j)$ and

$$\omega(p_j) = -\left[e^{ir_j} - 2 + e^{-ir_j} \right] - \left[e^{is_j} - 2 + e^{-is_j} \right]. \tag{3.2}$$

We shall abuse the notation a bit by denoting also

$$\omega(r_j) = -\left[e^{ir_j} - 2 + e^{-ir_j} \right] = 4\big(\sin \frac{r_j}{2} \big)^2.$$

Notice that $\sqrt{\omega(x)} = 0$ if and only if $x \equiv 0 \bmod \pi$. When $x \sim 0 \bmod \pi$,

$$\sqrt{\omega(x)} \sim |\sin x|. \tag{3.3}$$

By definition,

$$\begin{aligned} \langle\!\langle F, G \rangle\!\rangle &= \sum_z \frac{1}{n!} \sum_{x_1, \cdots, x_n} \bar{F}(x_1, \cdots, x_n) G(x_1 + z, \cdots, x_n + z) \\ &= \sum_z \frac{1}{n!} \int dp_1 \cdots dp_n \overline{\hat{F}(p_1, \cdots, p_n)} \hat{G}(p_1, \cdots, p_n) e^{i(p_1 + \cdots + p_n)z} \\ &= \frac{1}{n!} \int dp_1 \cdots dp_n \delta\big(p_1 + \cdots + p_n\big) \overline{\hat{F}(p_1, \cdots, p_n)} \hat{G}(p_1, \cdots, p_n). \end{aligned} \tag{3.4}$$

In other words, when considering the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, we can consider the class of $\hat{F}(p_1, \cdots, p_n)$ defined only on the subspace $\sum_j p_j \equiv 0 \bmod 2\pi$. We shall simply use the notation $\sum_j p_j = 0$ to denote the last condition.

From now on, we work only on the moment space and all functions are defined in terms of the momentum variables. Let $d\mu_n(\mathbf{p}_n)$ denote the measure

$$d\mu_n(\mathbf{p}_n) = \frac{1}{n!} \, \delta\Big(\sum_{j=1}^{n} p_j \Big) \prod_{j=1}^{n} dp_j. \tag{3.5}$$

3.1. *Statement of the main estimate.* Let τ be a positive constant and define

$$\mathcal{G}^\tau(\mathbf{p}_n) = \{ \, \mathbf{p}_n : \boldsymbol{\omega}(\mathbf{p}_n) \le |\log \lambda|^{-2\tau} \, \}. \tag{3.6}$$

Denote the complement of $\mathcal{G}^\tau$ by $\mathcal{B}^\tau$. Define for $\kappa \ge 0$ the two operators

$$\begin{aligned} \mathbf{U}^n_{\kappa,\tau}(\mathbf{p}_n) &= \boldsymbol{\omega}(\mathbf{r}_n) |\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n))|^\kappa, && \mathbf{p}_n \in \mathcal{G}^\tau; \\ &= \boldsymbol{\omega}(\mathbf{r}_n) |\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n))|, && \mathbf{p}_n \in \mathcal{B}^\tau \end{aligned} \tag{3.7}$$

$$\begin{aligned} \mathbf{V}^n_{\kappa,\tau}(\mathbf{p}_n) &= \boldsymbol{\omega}(\mathbf{r}_n) |\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n))|^\kappa, && \mathbf{p}_n \in \mathcal{G}^\tau; \\ &= -|\log \log \lambda|^2 \boldsymbol{\omega}(\mathbf{r}_n), && \mathbf{p}_n \in \mathcal{B}^\tau. \end{aligned} \tag{3.8}$$

The main estimates of this paper are contained in the following theorem.

Theorem 3.1. *Let κ and τ be nonnegative numbers satisfying*

$$0 \le \kappa \le 1 < \tau. \tag{3.9}$$

Let n be any positive integers such that

$$n^{10} \le |\log\log\lambda|^{1/2}. \tag{3.10}$$

Suppose that for some $\gamma \le |\log\log\lambda|^{-3}$

$$\Omega_{n+1} \ge \gamma \mathbf{V}^{n+1}_{\kappa,2\tau} \tag{3.11}$$

as an operator. Let

$$\tilde\kappa = 1 - \kappa/2. \tag{3.12}$$

Then

$$A_+^*(\lambda - S_{n+1} + \Omega_{n+1})^{-1} A_+ \le \gamma^{-1} |\log\log\lambda|^2 \mathbf{U}^n_{\tilde\kappa,\tau} \tag{3.13}$$

as an operator.

On the other hand, if

$$\Omega_{n+1} \le \gamma^{-1} \mathbf{U}^{n+1}_{\kappa,\tau}, \tag{3.14}$$

then,

$$A_+^*(\lambda - S_{n+1} + \Omega_{n+1})^{-1} A_+ \ge C\gamma \mathbf{V}^n_{\tilde\kappa,2\tau} \tag{3.15}$$

as an operator.

4. Upper bound

We first recall that for any two positive operators A, B,

$$0 < A \le B \quad \text{if and only if} \quad 0 < B^{-1} \le A^{-1}.$$

Furthermore, the map $B \to C^* B C$ is monotonic. For $\gamma \le |\log\log\lambda|^{-3}$, we have

$$\boldsymbol{\omega}(\mathbf{p}_{n+1}) + \gamma \mathbf{V}^{n+1}_{\kappa,2\tau}(\mathbf{p}_{n+1}) \ge 0.$$

Thus we can replace Ω in Theorem 3.1 by either $\mathbf{V}$ or $\mathbf{U}$ in the proof. For the rest of this paper, we shall follow the convention to denote the characteristic function of a set A by A itself (instead of χ_A).

By definition,

$$\begin{aligned}&\langle\!\langle F,\ A_+^*(\lambda - S_{n+1} + \gamma \mathbf{V}^{n+1}_{\kappa,2\tau})^{-1} A_+ F \rangle\!\rangle \\ &\qquad = \int d\mu_{n+1}(\mathbf{p}_{n+1})\, \frac{|A_+ F(p_1, \cdots, p_{n+1})|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}) + \gamma \mathbf{V}^{n+1}_{\kappa,2\tau}(\mathbf{p}_{n+1})}.\end{aligned}$$

Let $\mathbf{V}^{\pm,n+1}_{\kappa,2\tau}$ denote the positive and negative parts of $\mathbf{V}^{n+1}_{\kappa,2\tau}$. Then

$$\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}) + \gamma \mathbf{V}^{n+1}_{\kappa,2\tau}(\mathbf{p}_{n+1}) \geq (1-\gamma)\boldsymbol{\omega}(\mathbf{p}_{n+1}) + \gamma \mathbf{V}^{-,n+1}_{\kappa,2\tau}(\mathbf{p}_{n+1}) \\ + \gamma\big[\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}) + \mathbf{V}^{n+1}_{\kappa,2\tau}(\mathbf{p}_{n+1})\big].$$

Since $\gamma \leq |\log\log\lambda|^{-3}$,

$$(1-\gamma)\boldsymbol{\omega}(\mathbf{p}_{n+1}) + \gamma \mathbf{V}^{-,n+1}_{\kappa,2\tau}(\mathbf{p}_{n+1}) \geq 0.$$

Thus,

$$\begin{aligned} &\langle\!\langle F,\ A^*(\lambda - S_{n+1} + \gamma \mathbf{V}^{n+1}_{\kappa,2\tau})^{-1} AF \rangle\!\rangle \\ &\qquad \leq \gamma^{-1} \int d\mu_{n+1}(\mathbf{p}_{n+1})\ \frac{|A_+ F(p_1,\cdots,p_{n+1})|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}) + \mathbf{V}^{+,n+1}_{\kappa,2\tau}(\mathbf{p}_{n+1})}. \end{aligned} \tag{4.1}$$

We now divide the integration into the good region $\mathcal{G}^{2\tau}(\mathbf{p}_{n+1})$ and the bad region $\mathcal{B}^{2\tau}(\mathbf{p}_{n+1})$. In the good region,

$$\mathbf{V}^{+,n+1}_{\kappa,2\tau}(\mathbf{p}_{n+1}) = \boldsymbol{\omega}(\mathbf{r}_{n+1})|\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}))|^{\kappa}.$$

Thus the contribution is

$$\int d\mu_{n+1}(\mathbf{p}_{n+1})\ \mathcal{G}^{2\tau}(\mathbf{p}_{n+1}) \frac{|A_+ F(p_1,\cdots,p_{n+1})|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}) + \boldsymbol{\omega}(\mathbf{r}_{n+1})|\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}))|^{\kappa}}. \tag{4.2}$$

Since $\mathbf{V}^{+,n+1}_{\kappa,2\tau} = 0$ in the bad region, the contribution from this region is

$$\int d\mu_{n+1}(\mathbf{p}_{n+1})\ \mathcal{B}^{2\tau}(\mathbf{p}_{n+1}) \frac{|A_+ F(p_1,\cdots,p_{n+1})|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1})}. \tag{4.3}$$

4.1. *Decomposition into diagonal and off-diagonal terms.* Denote by Θ_κ the function

$$\Theta_\kappa(\mathbf{p}_{n+1}) = [\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}) + \boldsymbol{\omega}(\mathbf{r}_{n+1})|\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}))|^{\kappa}]^{-1}. \tag{4.4}$$

The contribution from the good region can be decomposed into diagonal and off-diagonal terms:

$$\begin{aligned} &\int d\mu_{n+1}(\mathbf{p}_{n+1})\ \mathcal{G}^{2\tau}(\mathbf{p}_{n+1})\Theta_\kappa(\mathbf{p}_{n+1})|A_+ F(p_1,\cdots,p_{n+1})|^2 \\ &\qquad = \frac{n(n+1)}{2}\langle\!\langle\, F,\ \mathcal{K}^{\kappa,\mathcal{G}^{2\tau}}_n F \,\rangle\!\rangle + n(n-1)(n+1)\langle\!\langle\, F,\ \Phi^{\kappa,\mathcal{G}^{2\tau}}_n F \,\rangle\!\rangle \\ &\qquad\quad + \frac{n(n-1)(n-2)(n+1)}{4}\langle\!\langle\, F,\ \Psi^{\kappa,\mathcal{G}^{2\tau}}_n F \,\rangle\!\rangle \end{aligned} \tag{4.5}$$

where

(4.6)

$$\langle\!\langle F,\ \mathcal{K}_n^{\kappa,\mathcal{G}^{2\tau}} F \rangle\!\rangle = \int d\mu_{n+1}(\mathbf{p}_{n+1})\ \Theta_\kappa(\mathbf{p}_{n+1}) |e^{ir_n} - e^{-ir_{n+1}}|^2 \times\ \mathcal{G}^{2\tau}(\mathbf{p}_{n+1})\,|\,F(p_1, \cdots, p_{n-1}, p_n + p_{n+1})\,|^2 ,$$

(4.7)

$$\langle\!\langle F,\ \Phi_n^{\kappa,\mathcal{G}^{2\tau}} F \rangle\!\rangle = \frac{1}{2} \int d\mu_{n+1}(\mathbf{p}_{n+1})\ \mathcal{G}^{2\tau}(\mathbf{p}_{n+1}) \Theta_\kappa(\mathbf{p}_{n+1}) \times (e^{ir_1} - e^{-ir_{n+1}})(e^{ir_2} - e^{-ir_{n+1}}) \times\ \left[\overline{F}(p_1 + p_{n+1}, p_2 \cdots, p_n) F(p_1, p_2 + p_{n+1}, \cdots, p_n) + \text{c.c.}\right] ,$$

$$(4.8) \qquad \langle\!\langle F,\ \Psi_n^{\kappa,\mathcal{G}^{2\tau}} F \rangle\!\rangle = \frac{1}{2} \int d\mu_{n+1}(\mathbf{p}_{n+1})\ \mathcal{G}^{2\tau}(\mathbf{p}_{n+1}) \Theta_\kappa(\mathbf{p}_{n+1}) \times (e^{ir_1} - e^{-ir_2})(e^{ir_3} - e^{-ir_4}) \times\ \left[\overline{F}(p_1 + p_2, p_3 \cdots, p_{n+1}) F(p_1, p_2, p_3 + p_4, \cdots, p_{n+1}) + \text{c.c.}\right] ,$$

where "c.c." denotes the complex conjugate.

To check the combinatorics, we notice that the total number of terms is

$$\Big(\frac{n(n+1)}{2}\Big)\Big[1 + 2(n-1) + \Big(\frac{(n-1)(n-2)}{2}\Big)\Big] = \Big(\frac{n(n+1)}{2}\Big)^2,$$

the same as the total number of terms in $(AF)^2$. The factors are obtained in the following way. Notice that in the formula of $(AF)^2$ we have to choose two indices. We first fix the special two indices in one F to be, say, $(1, 2)$. This gives a factor $n(n+1)/2$. There is only one choice for the second index to be $(1, 2)$ and this gives the first factor for the diagonal term. There are $2(n-1)$ choices to have either 1 or 2 and $(n-1)(n-2)/2$ choices to have neither 1 nor 2. These give the last two factors.

Notice that by the Schwarz inequality, the off-diagonal term is bounded by the diagonal term. For the purposes of upper bound we only have to estimate the diagonal term. Since the number of the off-diagonal terms is bigger than the diagonal terms by a factor of order n^2, we have the upper bound

$$(4.9) \qquad \int d\mu_{n+1}(\mathbf{p}_{n+1})\ \mathcal{G}^{2\tau}(\mathbf{p}_{n+1}) \Theta_\kappa(\mathbf{p}_{n+1}) |A_+ F(p_1, \cdots, p_{n+1})|^2 \le C n^4 \langle\!\langle F,\ \mathcal{K}_n^{\kappa,\mathcal{G}^{2\tau}} F \rangle\!\rangle .$$

4.2. *Preliminary remarks.* Notice in the expression for $\mathcal{K}_n^{\kappa,\mathcal{G}^{2\tau}}$ that we can integrate the variables $p_n - p_{n+1}$. So we make the change of variables and define some notation:

(4.10)

$$u_+ = p_n + p_{n+1}, \quad u_- = p_n - p_{n+1}, \qquad \sqrt{2}x = r_n - r_{n+1}, \quad \sqrt{2}y = s_n - s_{n+1}.$$

Suppose at least one of $|r_n|, |r_{n+1}|, |s_n|, |s_{n+1}|$ is not near 0 or π, say

$$\pi/100 \leq |r_n| < 99\pi/100.$$

Then,

$$\boldsymbol{\omega}(\mathbf{p}_{n+1}) \geq \omega(r_n) \geq C$$

for some constant. Therefore, we can bound the kernel $\Theta_\kappa(\mathbf{p}_{n+1}) \leq C^{-1}$ and

$$|e^{ir_n} - e^{-ir_{n+1}}| = |e^{i(r_n+r_{n+1})} - 1| \leq C\sqrt{\omega(r_n + r_{n+1})}. \tag{4.11}$$

After integrating $p_n - p_{n+1}$, we change the variable $u_+ = p_n + p_{n+1}$ to p_n. Recall the normalization difference $((n+1)!)^{-1}$ and $(n!)^{-1}$ for $d\mu_{n+1}$ and $d\mu_n$. Thus,

$$\begin{aligned}\int d\mu_{n+1}(\mathbf{p}_{n+1})\ &\{\pi/100 \leq |r_n| < 99\pi/100\} \\ &\Theta_\kappa(\mathbf{p}_{n+1})|e^{ir_n} - e^{-ir_{n+1}}|^2 \,|\, F(p_1, \cdots, p_{n-1}, p_n + p_{n+1}) \,|^2 \\ &\quad \leq Cn^{-1} \int d\mu_n(\mathbf{p}_n)\, \omega(r_n) \,|\, F(\mathbf{p}_n) \,|^2 .\end{aligned} \tag{4.12}$$

Since we are interested only in terms that diverge as $\lambda \to 0$, this term is negligible. Therefore, we shall assume that

$$|r_n|, |r_{n+1}|, |s_n|, |s_{n+1}| \in [0, \pi/100] \cup [99\pi/100, \pi]. \tag{4.13}$$

We now divide the integration region according to $|r_n|, |r_{n+1}|, |s_n|, |s_{n+1}|$ in $[0, \pi/100]$ or $[99\pi/100, \pi]$. There are sixteen disjoint regions and the final results are obtained by adding together the estimates from these sixteen disjoint regions. For simplicity, we shall consider only the region that all these variables are in the interval $[0, \pi/100]$. The estimates in all other regions are the same. For example, suppose that $r_{n+1} \in [99\pi/100, \pi]$ and the other three variables belong to $[0, \pi/100]$. Let $p_{n+1} = (\pi, 0) + \tilde{p}_{n+1}$ and define

$$G(\mathbf{p}_n, \tilde{p}_{n+1}) = F(\mathbf{p}_{n+1}).$$

Now we have $|\tilde{r}_{n+1}|, |\tilde{s}_{n+1}| \in [0, \pi/100]$ and we can estimate on G instead of on F.

Therefore, we now assume the following generality:

$$G_I : |r_n|, |r_{n+1}|, |s_n|, |s_{n+1}| \in [0, \pi/100]. \tag{4.14}$$

This argument applies to all terms for the rest of this paper and we shall from now on consider only this case. The indices $n, n+1$ are the two indices appearing in $F(p_1, \cdots, p_{n-1}, p_n + p_{n+1})$; they may change depending on the variables we use in the future. Notice that in this region,

$$\omega(p_j) \sim p_j^2, \quad j = n, n+1, \quad \omega(p_n \pm p_{n+1}) \sim (p_n \pm p_{n+1})^2. \tag{4.15}$$

Since we are concerned only with the order of magnitude, *for the rest of the proof for Theorem* 3.1 *in Sections* 4–6, *we shall replace* $\omega(p)$ *by* p^2 *whenever it is more convenient.*

4.3. *The upper bound of the diagonal term: The good region.* The following lemma is the main estimate on the diagonal term in the good region.

LEMMA 4.1.

(4.16)
$$\langle\!\langle\, F,\ \mathcal{K}_n^{\kappa,\mathcal{G}^{2\tau}} F \,\rangle\!\rangle \le \frac{C}{(n+1)} \int d\mu_n(\mathbf{p}_n)\, \omega(r_n) |\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n))|^{1-\kappa/2}\, |F(\mathbf{p}_n)|^2 \,.$$

Recall the change of variables (4.10). We can bound the diagonal term from above as

$$\begin{aligned}\langle\!\langle\, F,\ \mathcal{K}_n^{\kappa,\mathcal{G}^{2\tau}} F \,\rangle\!\rangle \le \frac{C}{(n+1)!} &\int_{\sum_{j=1}^{n-1} p_j + u_+ = 0} \prod_{j=1}^{n-1} dp_j \\ &\times \int du_+\, \omega(e_1 \cdot u_+) \mathcal{G}^{2\tau}(\mathbf{p}_{n+1})\, |F(p_1, \cdots, p_{n-1}, u_+)|^2 \\ &\times \int_{-\pi/10}^{\pi/10} \int_{-\pi/10}^{\pi/10} dxdy\, \Big[\lambda + a^2 + b^2 + x^2 + y^2 \\ &\qquad + \big(a^2 + x^2\big) \big| \log\big(\lambda + a^2 + b^2 + x^2 + y^2\big)\big|^\kappa \Big]^{-1}\end{aligned}$$

where
$$b^2 = \boldsymbol{\omega}(\mathbf{s}_{n-1}) + \omega(e_2 \cdot u_+), \quad a^2 = \boldsymbol{\omega}(\mathbf{r}_{n-1}) + \omega(e_1 \cdot u_+).$$

Clearly, we have
$$\mathcal{G}^{2\tau}(\mathbf{p}_{n+1}) \subset \big\{x^2 + y^2 \le C|\log\lambda|^{-4\tau}\big\}\big\{a^2 + b^2 \le C|\log\lambda|^{-4\tau}\big\}.$$

We now replace u_+ by p_n. Recall the normalization difference $((n+1)!)^{-1}$ and $(n!)^{-1}$ for $d\mu_{n+1}$ and $d\mu_n$. Thus we have the upper bound

$$\begin{aligned}\langle\!\langle\, F,\ \mathcal{K}_n^{\kappa,\mathcal{G}^{2\tau}} F \,\rangle\!\rangle \le &\frac{C}{(n+1)} \int d\mu_n(\mathbf{p}_n)\, \omega(r_n)\, |F(\mathbf{p}_n)|^2 \,\big\{a^2 + b^2 \le C|\log\lambda|^{-4\tau}\big\} \\ &\times \int\!\!\int dxdy\, \big\{x^2 + y^2 \le C|\log\lambda|^{-4\tau}\big\} \\ &\times \Big[\lambda + b^2 + y^2 + (a^2 + x^2) \\ &\quad \times \big\{1 + |\log(\lambda + a^2 + b^2 + x^2 + y^2)|^\kappa\big\}\Big]^{-1},\end{aligned}$$

where
$$b^2 = \boldsymbol{\omega}(\mathbf{s}_n), \quad a^2 = \boldsymbol{\omega}(\mathbf{r}_n).$$

We need the following lemma which will be used in several places later on.

LEMMA 4.2. *Let* $\tau > 1$ *and*

$$K_\kappa^\tau(a,b) = \int\int dxdy \left\{x^2+y^2 \le |\log\lambda|^{-2\tau}\right\} \cdot \Big[\lambda + b^2 + y^2 + (a^2+x^2)\big\{1 + |\log(\lambda + a^2+b^2+x^2+y^2)|^\kappa\big\}\Big]^{-1}.$$

Suppose that

$$a^2+b^2 \le |\log\lambda|^{-2\tau}. \tag{4.17}$$

Then for $0 \le \kappa \le 1$,

$$K_\kappa^\tau(a,b) \le C\Big|\log(\lambda + a^2+b^2)\Big|^{1-\kappa/2}. \tag{4.18}$$

On the other hand, if

$$a^2+b^2 \le |\log\lambda|^{-4\tau}, \tag{4.19}$$

then the lower bound

$$K_\kappa^\tau(a,b) \ge C^{-1}\Big|\log(\lambda + a^2+b^2)\Big|^{1-\kappa/2}. \tag{4.20}$$

Also there exists the trivial bound

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} dxdy\Big[\lambda + a^2+b^2+y^2+x^2\Big]^{-1} \le C\Big|\log(\lambda + a^2+b^2)\Big|. \tag{4.21}$$

Proof. Clearly the trivial bound can be checked easily. We now prove the rest. Fix a constant m, $1 < m < \tau$. Let

$$G(x,y) = \Big\{(x,y) : |x| \le |\log\lambda|^m|y| \le |\log\lambda|^{2m}|x|\Big\}$$

and B be its complement.

In the region B we drop $(a^2+x^2)\big\{1 + |\log(\lambda+a^2+b^2+y^2)|^\kappa\big\}$ to have an upper bound. The angle integration of x, y gives a factor $|\log\lambda|^{-m}$. Thus the contribution from this region is bounded by

$$C|\log\lambda|^{-m+1} \le C.$$

In the region G we have

$$\begin{aligned}\log(\lambda+a^2+b^2+y^2)^{-1} - \log(1+|\log\lambda|^{2m}) &\le \log(\lambda+a^2+b^2+x^2+y^2)^{-1}\\ &\le \log(\lambda+a^2+b^2+y^2)^{-1}.\end{aligned}$$

By assumption (4.17) or (4.19), $a^2+b^2+x^2+y^2 \le 2|\log\lambda|^{-2\tau}$. Thus for $\tau > m$,

$$\log(\lambda+a^2+b^2+y^2)^{-1} - \log(1+|\log\lambda|^{2m}) \ge C\log(\lambda+a^2+b^2+y^2)^{-1}$$

for some constant depending on τ, m. Therefore,

$$\begin{aligned}C\log(\lambda+a^2+b^2+y^2)^{-1} &\le \log(\lambda+a^2+b^2+x^2+y^2)^{-1}\\ &\le \log(\lambda+a^2+b^2+y^2)^{-1}.\end{aligned} \tag{4.22}$$

Upper bound. We now replace $\log(\lambda + a^2 + b^2 + x^2 + y^2)^{-1}$ by $C\log(\lambda + a^2 + b^2 + y^2)^{-1}$ and drop $a^2\{1 + |\log(\lambda + a^2 + b^2 + x^2 + y^2)|^\kappa\}$ to have an upper bound for K_κ^τ. Thus we can bound $K_\kappa^\tau(a,b)$ by

$$K_\kappa^\tau(a,b) \le C \int\int dxdy \left\{x^2 + y^2 \le |\log\lambda|^{-2\tau}\right\}$$
$$\times \left[\lambda + b^2 + y^2 + a^2 + x^2|\log(\lambda + a^2 + b^2 + y^2)|^\kappa\}.\right]^{-1}$$

Change the variable and let

$$z = x|\log(\lambda + a^2 + b^2 + y^2)|^{\kappa/2}.$$

Then

$$z^2 \le x^2|\log(\lambda + a^2 + b^2)|^\kappa.$$

Thus for x, y in the integration region we have

$$z^2 + y^2 \le |\log\lambda|^{-2\tau}|\log(\lambda + a^2 + b^2)|^\kappa \le C.$$

We can bound $K_\kappa^\tau(a,b)$ by

$$K_\kappa^\tau(a,b) \le C \int\int dzdy \left\{z^2 + y^2 \le C\right\}$$
$$\times \left[\lambda + a^2 + b^2 + y^2 + z^2\right]^{-1}|\log(\lambda + a^2 + b^2 + y^2)|^{-\kappa/2}.$$

Denote

$$\rho^2 = z^2 + y^2. \tag{4.23}$$

Since

$$\log(\lambda + a^2 + b^2 + y^2)^{-1} \ge \log(\lambda + a^2 + b^2 + \rho^2)^{-1},$$

we can bound the integration by

$$C\int_0^C d\rho^2\big(\lambda+a^2+b^2+\rho^2\big)^{-1}|\log(\lambda+a^2+b^2+\rho^2)|^{-\kappa/2} \le C|\log(\lambda+a^2+b^2)|^{1-\kappa/2}.$$

This proves the upper bound.

Lower bound. We now replace $\log(\lambda + a^2 + b^2 + x^2 + y^2)^{-1}$ by $\log(\lambda + a^2 + b^2 + y^2)^{-1}$ to have a lower bound for K_κ^τ. We change the variable to the same z and ρ as in the upper bound. We now restrict the angle $\theta(z,y)$ of the two dimensional vector (z,y) to be between $\pi/3$ and $2\pi/3$, i.e.,

$$\pi/3 \le \theta(z,y) \le 2\pi/3. \tag{4.24}$$

In this region, $|z| \sim |y| \sim \rho$. Denote by $q = \rho^2 + a^2 + b^2$. We further restrict the integration to

$$\mathcal{W} = \left\{\, 2(a^2|\log(\lambda + a^2 + b^2)|^\kappa + b^2) \le q \le |\log\lambda|^{-2\tau}/2 \,\right\}. \tag{4.25}$$

From the last restriction, we also have $\rho^2 \le |\log\lambda|^{-2\tau}/2$. Since a^2+b^2 satisfies (4.19) and $|x| \le |z|$, the condition $x^2+y^2 \le |\log\lambda|^{-2\tau}$ is satisfied.

The integral is thus bounded below by

$$\int dzdy\, \{\pi/3 \le \theta(z,y) \le 2\pi/3\} \mathcal{W}(q) |\log(\lambda + a^2+b^2+\rho^2)|^{-\kappa/2} \\ \times \Big[\lambda + b^2 + \rho^2 + a^2 |\log(\lambda+a^2+b^2+\rho^2)|^{\kappa}\Big]^{-1}.$$

From the restriction on q, we have

$$\rho^2 \ge a^2|\log(\lambda+a^2+b^2)|^\kappa \ge a^2|\log(\lambda+a^2+b^2+\rho^2)|^\kappa.$$

Thus

$$\Big[\lambda + b^2 + \rho^2 + a^2|\log(\lambda+a^2+b^2+\rho^2)|^\kappa\Big]^{-1} \ge (1/2)(\lambda+a^2+b^2+\rho^2)^{-1}.$$

The angle integration produces just some constant factor. Thus the integral is bigger than

$$\begin{aligned} (4.26)\quad & C\int_{2(a^2|\log a^2|^\kappa + b^2)}^{|\log\lambda|^{-2\tau}/2} dq\, (\lambda+q)^{-1} |\log(\lambda+q)|^{-\kappa/2} \\ & \ge C\Big[\big|\log(\lambda + 2a^2|\log a^2|^\kappa + 2b^2)\big|^{1-\kappa/2} - \big[2\tau\log|\log\lambda| + \log 2\big]^{1-\kappa/2}\Big]. \end{aligned}$$

Since $a^2+b^2 \le |\log\lambda|^{-4\tau}$ we have

$$\log(\lambda + 2a^2|\log a^2|^\kappa + 2b^2)^{-1} \ge (7\tau/2)|\log\log\lambda|. \tag{4.27}$$

Therefore, we have the bound

$$\begin{aligned} &\big|\log(\lambda + 2a^2|\log a^2|^\kappa + 2b^2)\big|^{1-\kappa/2} - \big[2\tau\log|\log\lambda| + \log 2\big]^{1-\kappa/2} \\ &\qquad\qquad \ge (1/20)\big|\log(\lambda+a^2+b^2)\big|^{1-\kappa/2}. \end{aligned}$$

We have thus proved the lower bound. □

From this lemma, we have proved Lemma 4.1 concerning the estimate in the good region. Observe that the main contribution of the $p_n - p_{n+1}$ integration comes from the region $|p_n - p_{n+1}| \gg |p_n + p_{n+1}| + \boldsymbol{\omega}(\mathbf{p}_{n-1})$. In fact, we have the following lemma.

Lemma 4.3. *For any $m > 0$ there is a constant C_m such that*

$$\begin{aligned} (4.28)\quad & \int d\mu_{n+1}(\mathbf{p}_{n+1}) \Big\{ |p_n - p_{n+1}|^2 \le |\log\lambda|^{2m} \big[\, |p_n+p_{n+1}|^2 + \boldsymbol{\omega}(\mathbf{p}_{n-1}) \,\big] \Big\} \\ & \qquad \times \frac{|e^{ir_n} - e^{-ir_{n+1}}|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1})} \, |\, F(p_1, \cdots, p_{n-1}, p_n + p_{n+1})\,|^2 \\ & \qquad \le C_m n^{-1}\, |\log\log\lambda| \int d\mu_n(\mathbf{p}_n)\, \omega(r_n)\, |\, F(\mathbf{p}_n)\,|^2 . \end{aligned}$$

Proof. We have the bound

$$\begin{aligned}
&\int d(p_n - p_{n+1}) \, \frac{|e^{ir_n} - e^{-ir_{n+1}}|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1})} \\
&\qquad \times \big\{ |p_n - p_{n+1}|^2 \le |\log \lambda|^{2m} \big[\, |p_n + p_{n+1}|^2 + \boldsymbol{\omega}(\mathbf{p}_{n-1}) \,\big] \big\} \\
&\quad \le \log \Big(\frac{\lambda + (1 + |\log \lambda|^{2m}) \big[|p_n + p_{n+1}|^2 + \boldsymbol{\omega}(\mathbf{p}_{n-1}) \big]}{\lambda + |p_n + p_{n+1}|^2 + \boldsymbol{\omega}(\mathbf{p}_{n-1})} \Big) \le C_m |\log \log \lambda|.
\end{aligned}$$

Having changed the variable $p_n + p_{n+1} \to p_n$, we have proved the lemma. □

Therefore, at a price of the term on the right side of (4.28) we can assume the following (II):

$$G_{\mathrm{II}}: \qquad |p_n - p_{n+1}|^2 \ge |\log \lambda|^{2m} \big[\, |p_n + p_{n+1}|^2 + \boldsymbol{\omega}(\mathbf{p}_{n-1}) \,\big]. \tag{4.29}$$

Under the assumptions (3.9) (3.10), the term on the right side of (4.28) is much smaller than the accuracy we need for Theorem 3.1. Therefore this condition will be imposed for the rest of the paper.

4.4. *Upper bound of the diagonal term*: *The bad region.* The contribution from the bad region can be decomposed into diagonal and off-diagonal terms. Again, we shall use the Schwarz inequality to bound the off-diagonal terms by the diagonal terms. Therefore, we have the bound

$$\begin{aligned}
&\int d\mu_{n+1}(\mathbf{p}_{n+1}) \, \mathcal{B}^{2\tau}(\mathbf{p}_{n+1}) \frac{|A_+ F(p_1, \cdots, p_{n+1})|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1})} \\
&\quad \le 2n^4 \int d\mu_{n+1}(\mathbf{p}_{n+1}) \, \mathcal{B}^{2\tau}(\mathbf{p}_{n+1}) \frac{|e^{ir_n} - e^{-ir_{n+1}}|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1})} \\
&\qquad \times \big| F(p_1, \cdots, p_{n-1}, p_n + p_{n+1}) \big|^2 .
\end{aligned}$$

Again the variable $p_n - p_{n+1}$ does not appear in F and we can perform the integration.

We subdivide $\mathcal{B}^{2\tau}(\mathbf{p}_{n+1})$ into

$$\mathcal{B}^{2\tau}(\mathbf{p}_{n+1}) \mathcal{B}_n^{4\tau}(\mathbf{p}_{n-1}, p_n + p_{n+1}) \,\cup\, \mathcal{B}^{2\tau}(\mathbf{p}_{n+1}) \mathcal{G}^{4\tau}(\mathbf{p}_{n-1}, p_n + p_{n+1}).$$

In the first case, we drop the characteristic function $\mathcal{B}^{2\tau}(\mathbf{p}_{n+1})$ to have an upper bound. We now use the trivial bound (4.21) to estimate the integration of the variable $p_n - p_{n+1}$ by

(4.30)

$$
\begin{aligned}
2n^4 \int d\mu_{n+1}(\mathbf{p}_{n+1})\, &\mathcal{B}^{4\tau}(\mathbf{p}_{n-1}, p_n + p_{n+1}) \\
&\times \frac{|e^{ir_n} - e^{-ir_{n+1}}|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1})} \,|\, F(p_1, \cdots, p_{n-1}, p_n + p_{n+1})\,|^2 \\
&\leq Cn^3 \int d\mu_n(\mathbf{p}_n)\, \mathcal{B}^{4\tau}(\mathbf{p}_n)\omega(r_n)|\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n))|\,|\, F(p_1, \cdots, p_n)\,|^2 \\
&\leq Cn^3 |\log\log\lambda| \int d\mu_n(\mathbf{p}_n)\, \omega(r_n)\,|\, F(p_1, \cdots, p_n)\,|^2 .
\end{aligned}
$$

Here we have used the change of the normalization between

We now estimate the region $\mathcal{B}^{2\tau}(\mathbf{p}_{n+1})\mathcal{G}^{4\tau}(\mathbf{p}_{n-1}, p_n + p_{n+1})$ which is the transition from the bad set to the good set. In this region,

$$
|p_n - p_{n+1}|^2 \geq C|\log\lambda|^{-4\tau}.
$$

The contribution is bounded by

$$
\begin{aligned}
2n^4 \int d\mu_{n+1}(\mathbf{p}_{n+1})\, &\left\{|p_n - p_{n+1}|^2 \geq C|\log\lambda|^{-4\tau}\right\} \\
&\times \frac{|e^{ir_n} - e^{-ir_{n+1}}|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1})} \,|\, F(p_1, \cdots, p_{n-1}, p_n + p_{n+1})\,|^2 \\
&\leq Cn^3 |\log\log\lambda| \int d\mu_n(\mathbf{p}_n)\, \omega(r_n)\,|\, F(\mathbf{p}_n)\,|^2 .
\end{aligned} \tag{4.31}
$$

Combining the estimates (4.30) and (4.31), we can bound the contribution from the bad region by

$$
\begin{aligned}
\int d\mu_{n+1}(\mathbf{p}_{n+1})\, &\mathcal{B}^{2\tau}(\mathbf{p}_{n+1}) \frac{|A_+ F(p_1, \cdots, p_{n+1})|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1})} \\
&\leq Cn^3 |\log\log\lambda| \int d\mu_n(\mathbf{p}_n)\, \omega(r_n)\,|\, F(\mathbf{p}_n)\,|^2 .
\end{aligned}
$$

With the estimate on the good region, Lemma 4.1, we can bound the right side of (4.1) by

$$
\begin{aligned}
\int d\mu_{n+1}(\mathbf{p}_{n+1})\, &\frac{|A_+ F(p_1, \cdots, p_{n+1})|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}) + \mathbf{V}^{+,n+1}_{\kappa,2\tau}(\mathbf{p}_{n+1})} \\
&\leq Cn^3 \int d\mu_n(\mathbf{p}_n)\, \omega(r_n)|\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n))|^{1-\kappa/2}\, |F(\mathbf{p}_n)|^2 \\
&\quad + Cn^3 |\log\log\lambda| \int d\mu_n(\mathbf{p}_n)\, \omega(r_n)\,|\, F(\mathbf{p}_n)\,|^2 .
\end{aligned}
$$

Under the condition (3.10), it is easy to check that for the symmetric function F the right side of the last equation is bounded above by $|\log\log\lambda|^2 \mathbf{U}^n_{\tilde{\kappa},\tau}(\mathbf{p}_n)$. This proves the upper bound for Theorem 3.1.

5. Lower bound: The diagonal terms

By definition, we have

$$
\langle\!\langle FA^*(\lambda - S_{n+1} + \gamma^{-1}\mathbf{U}^{n+1}_{\kappa,\tau})^{-1}AF\rangle\!\rangle = \int d\mu_{n+1}(\mathbf{p}_{n+1})\, \frac{|A_+F(p_1,\cdots,p_{n+1})|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}) + \gamma^{-1}\mathbf{U}^{n+1}_{\kappa,\tau}(\mathbf{p}_{n+1})}. \tag{5.1}
$$

Since $\gamma \le 1$ and $\lambda + \boldsymbol{\omega} \ge 0$, the integral is bigger than

$$
\gamma \int d\mu_{n+1}(\mathbf{p}_{n+1})\, \frac{|A_+F(p_1,\cdots,p_{n+1})|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}) + \mathbf{U}^{n+1}_{\kappa,\tau}(\mathbf{p}_{n+1})}.
$$

Divide the integral into $\mathbf{p}_{n+1} \in \mathcal{G}^\tau$ and $\mathbf{p}_{n+1} \in \mathcal{B}^\tau$. In the bad set $\mathcal{B}^\tau$, we bound the integral in this region from below by zero. In the good set, we have

$$
\mathbf{U}^{n+1}_{\kappa,\tau}(\mathbf{p}_{n+1}) = \boldsymbol{\omega}(\mathbf{r}_{n+1})|\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}))|^\kappa, \qquad \mathbf{p}_{n+1} \in \mathcal{G}^\tau.
$$

Thus

$$
\begin{aligned}
\langle\!\langle FA^*(\lambda - S_{n+1} + \mathbf{U}^{n+1}_{\kappa,\tau})^{-1}AF\rangle\!\rangle
&\ge \int d\mu_{n+1}(\mathbf{p}_{n+1})\, \mathcal{G}^\tau(\mathbf{p}_{n+1}) \frac{|A_+F(p_1,\cdots,p_{n+1})|^2}{\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}) + \mathbf{U}^{n+1}_{\kappa,\tau}(\mathbf{p}_{n+1})}\\
&\ge \int d\mu_{n+1}(\mathbf{p}_{n+1})\, \mathcal{G}^\tau(\mathbf{p}_{n+1})\Theta_\kappa(\mathbf{p}_{n+1})|A_+F(p_1,\cdots,p_{n+1})|^2,
\end{aligned} \tag{5.2}
$$

where $\Theta_\kappa(\mathbf{p}_{n+1})$ is as defined in (4.4). We now decompose the last term into diagonal and off-diagonal terms:

$$
\frac{n(n+1)}{2}\langle\!\langle F,\ \mathcal{K}^{\kappa,\mathcal{G}^\tau}_n F\rangle\!\rangle + n(n-1)(n+1)\langle\!\langle F,\ \Phi^{\kappa,\mathcal{G}^\tau}_n F\rangle\!\rangle + \frac{n(n-1)(n-2)(n+1)}{4}\langle\!\langle F,\ \Psi^{\kappa,\mathcal{G}^\tau}_n F\rangle\!\rangle \tag{5.3}
$$

where these operators are as defined in (4.6)–(4.8).

5.1. *Lower bound on the diagonal terms.* The main estimate on the lower bound of the diagonal term (4.6) is the following lemma. Define

$$
F^{2\tau}_{\mathcal{G}}(\mathbf{p}_n) = F(\mathbf{p}_n)\mathcal{G}^{2\tau}(\mathbf{p}_n), \qquad F^{2\tau}_{\mathcal{B}}(\mathbf{p}_n) = F(\mathbf{p}_n)\mathcal{B}^{2\tau}(\mathbf{p}_n). \tag{5.4}
$$

LEMMA 5.1. *Recall κ, τ and n satisfy the assumptions* (3.9) *and* (3.10). *Then the diagonal term is bounded below by*

$$
\langle\!\langle F,\ \mathcal{K}^{\kappa,\mathcal{G}^\tau}_n F\rangle\!\rangle \ge\ Cn^{-1}\langle\!\langle F^{2\tau}_{\mathcal{G}},\ \omega(r_n)|\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n))|^{1-\kappa/2} F^{2\tau}_{\mathcal{G}}\rangle\!\rangle. \tag{5.5}
$$

Proof. Recall the assumptions (4.14), (4.29) and the change of variables

$$
\begin{aligned}
&u_+ = p_n + p_{n+1}, \quad u_- = p_n - p_{n+1}, \qquad \sqrt{2}x = r_n - r_{n+1}, \quad \sqrt{2}y = s_n - s_{n+1}\\
&b^2 = \omega(e_2\cdot u_+) + \boldsymbol{\omega}(\mathbf{s}_{n-1}), \quad a^2 = \omega(e_1\cdot u_+) + \boldsymbol{\omega}(\mathbf{r}_{n-1}).
\end{aligned} \tag{5.6}
$$

Now, we can bound the diagonal term from below by

$$
\begin{aligned}
\langle\!\langle F,\ \mathcal{K}_n^{\kappa,\mathcal{G}^\tau} F \rangle\!\rangle \geq \frac{C}{(n+1)!} & \int_{\sum_{j=1}^{n-1} p_j + u_+ = 0} \prod_{j=1}^{n-1} dp_j \\
& \times \int du_+ \, \omega(e_1 \cdot u_+) \mathcal{G}^\tau(\mathbf{p}_{n+1}) \left| F(p_1, \cdots, p_{n-1}, u_+) \right|^2 \\
& \times \int \int dx dy \, \Big[\lambda + a^2 + b^2 + x^2 + y^2 + (a^2 + x^2) \\
& \qquad \times |\log(\lambda + a^2 + b^2 + x^2 + y^2)|^\kappa \Big]^{-1}.
\end{aligned}
$$

We now impose the condition $x^2 + y^2 \leq |\log \lambda|^{-2\tau}/2$ to have a lower bound. Since

$$
\mathcal{G}^{2\tau}(p_1, \cdots, p_{n-1}, u_+) \left\{ x^2 + y^2 \leq |\log \lambda|^{-2\tau}/2 \right\} \subset \mathcal{G}^\tau(\mathbf{p}_{n+1}),
$$

we can replace $\mathcal{G}^\tau(\mathbf{p}_{n+1})$ by $\left\{ x^2 + y^2 \leq |\log \lambda|^{-2\tau}/2 \right\}$ and F by $F_{\mathcal{G}}^{2\tau}$ to have a lower bound. The lemma now follows from the lower bound of Lemma 4.2. □

6. Off-diagonal terms

Our goal in this section is to prove the following estimate on the off-diagonal terms.

LEMMA 6.1. *Recall that* κ, τ *and* n *satisfy the assumptions* (3.9) *and* (3.10). *The first and second off-diagonal terms are bounded by*

(6.1)

$$
\begin{aligned}
& \left| \langle\!\langle F,\ \Phi_n^{\kappa,\mathcal{G}^\tau} F \rangle\!\rangle \right| + \left| \langle\!\langle F,\ \Psi_n^{\kappa,\mathcal{G}^\tau} F \rangle\!\rangle \right| \\
& \qquad \leq C n^{-1} |\log\log\lambda|^{1+1/2} \int d\mu_n(\mathbf{p}_n) \, \omega(r_n) \left| F_{\mathcal{B}}^{2\tau}(\mathbf{p}_n) \right|^2 \\
& \qquad\quad + C n^{-5} \int d\mu_n(\mathbf{p}_n) \, \omega(r_n) |\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n)|^{1-\kappa/2} \left| F_{\mathcal{G}}^{2\tau}(\mathbf{p}_n) \right|^2.
\end{aligned}
$$

Proof. The first off-diagonal term is bounded by

$$
\begin{aligned}
& \left| \langle\!\langle F,\ \Phi_n^{\kappa,\mathcal{G}^\tau} F \rangle\!\rangle \right| \\
& \quad \leq C \int d\mu_{n+1}(\mathbf{p}_{n+1}) \, \mathcal{G}^\tau(\mathbf{p}_{n+1}) \Theta_\kappa(\mathbf{p}_{n+1}) \left| (e^{ir_1} - e^{-ir_{n+1}})(e^{ir_2} - e^{-ir_{n+1}}) \right| \\
& \quad\quad \times \Big| F(p_1 + p_{n+1}, p_2 \cdots, p_n) F(p_1, p_2 + p_{n+1}, \cdots, p_n) \Big|.
\end{aligned}
$$

By definition $F = F_{\mathcal{G}}^{2\tau} + F_{\mathcal{B}}^{2\tau}$. Thus the last term is equal to

$$C \int d\mu_{n+1}(\mathbf{p}_{n+1})\, \mathcal{G}^{\tau}(\mathbf{p}_{n+1})\Theta_{\kappa}(\mathbf{p}_{n+1})\big|(e^{ir_1} - e^{-ir_{n+1}})(e^{ir_2} - e^{-ir_{n+1}})\big|$$
$$\times \Big|\big(F_{\mathcal{G}}^{2\tau} + F_{\mathcal{B}}^{2\tau}\big)(p_1 + p_{n+1}, p_2, \cdots, p_n)$$
$$\times \big(F_{\mathcal{G}}^{2\tau} + F_{\mathcal{B}}^{2\tau}\big)(p_2 + p_{n+1}, p_1, p_3, \cdots, p_n)\Big|.$$

From the Schwarz inequality, the cross term is bounded by

$$\begin{aligned}
(6.2)\quad & C \int d\mu_{n+1}(\mathbf{p}_{n+1})\, \mathcal{G}^{\tau}(\mathbf{p}_{n+1})\Theta_{\kappa}(\mathbf{p}_{n+1})\big|(e^{ir_1} - e^{-ir_{n+1}})(e^{ir_2} - e^{-ir_{n+1}})\big| \\
&\quad \times \Big|F_{\mathcal{G}}^{2\tau}(p_1 + p_{n+1}, p_2, \cdots, p_n) F_{\mathcal{B}}^{2\tau}(p_2 + p_{n+1}, p_1, p_3, \cdots, p_n)\Big| \\
&\le C\delta \int d\mu_{n+1}(\mathbf{p}_{n+1})\, \mathcal{G}^{\tau}(\mathbf{p}_{n+1})\Theta_{\kappa}(\mathbf{p}_{n+1})|(e^{ir_1} - e^{-ir_{n+1}})|^2 \\
&\quad \times \Big|F_{\mathcal{G}}^{2\tau}(p_1 + p_{n+1}, p_2, \cdots, p_n)\Big|^2 \\
&\quad + C\delta^{-1} \int d\mu_{n+1}(\mathbf{p}_{n+1})\, \mathcal{G}^{\tau}(\mathbf{p}_{n+1})\Theta_{\kappa}(\mathbf{p}_{n+1})|(e^{ir_2} - e^{-ir_{n+1}})|^2 \\
&\quad \times \Big|F_{\mathcal{B}}^{2\tau}(p_2 + p_{n+1}, p_1, p_3, \cdots, p_n)\Big|^2.
\end{aligned}$$

We first bound the last term. Clearly, in the region

$$\mathcal{G}^{\tau}(\mathbf{p}_{n+1})\mathcal{B}^{2\tau}\{p_2 + p_{n+1}, p_1, p_3, \cdots, p_n\}$$

we have

$$|p_2 - p_{n+1}|^2 \le |\log\lambda|^{4\tau}\big[\,|p_2 + p_{n+1}|^2 + \omega(p_1) + \omega(p_3) + \cdots \omega(p_n)\,\big].$$

Thus we can apply Lemma 4.3. Let $\delta = |\log\log\lambda|^{-1/2}$. We can bound the last term in (6.2) by

$$(6.3)\qquad Cn^{-1}|\log\log\lambda|^{1+1/2} \int d\mu_n(\mathbf{p}_n)\, \omega(r_n)\, \big|F_{\mathcal{B}}^{2\tau}(\mathbf{p}_n)\big|^2.$$

The first term on the right side of (6.2) can be bounded as in the upper bound section. Using Lemma 4.1, we bound it by

$$(6.4)\qquad Cn^{-1}|\log\log\lambda|^{-1/2} \int d\mu_n(\mathbf{p}_n)\, \omega(r_n)|\log(\lambda + \mathbf{p}_n^2)|^{1-\kappa/2}\, \big|F_{\mathcal{G}}^{2\tau}(\mathbf{p}_n)\big|^2.$$

The contribution from the term with $F_{\mathcal{B}}^{2\tau}F_{\mathcal{B}}^{2\tau}$ can be estimated similarly. Finally, we consider the contribution from $F_{\mathcal{G}}^{2\tau}F_{\mathcal{G}}^{2\tau}$. To estimate this term, we need the following lemma which will be proved in the next section.

LEMMA 6.2. *Recall that* κ, τ *and* n *satisfy the assumptions* (3.9) *and* (3.10). *Then we have the following two estimates exist*:

$$
\begin{aligned}
(6.5) \quad Q_1 &= \int d\mu_{n+1}(\mathbf{p}_{n+1})\ \Theta_\kappa(\mathbf{p}_{n+1}) \big|(e^{ir_1} - e^{-ir_3})(e^{ir_2} - e^{-ir_3})\big| \\
&\quad \times \left| F_{\mathcal{G}}^{2\tau}(p_1 + p_3, p_2, p_4, \cdots, p_{n+1}) F_{\mathcal{G}}^{2\tau}(p_2 + p_3, p_1, p_4, \cdots, p_{n+1}) \right| \\
&\quad \le C n^{-5} \int d\mu_n(\mathbf{p}_n)\ \omega(r_n) |\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n))|^{1-\kappa/2} \left| F_{\mathcal{G}}^{2\tau}(\mathbf{p}_n) \right|^2
\end{aligned}
$$

$$
\begin{aligned}
(6.6) \quad Q_2 &= \int d\mu_{n+1}(\mathbf{p}_{n+1})\ \Theta_\kappa(\mathbf{p}_{n+1}) \big|(e^{ir_1} - e^{-ir_2})(e^{ir_3} - e^{-ir_4})\big| \\
&\quad \times \left| F_{\mathcal{G}}^{2\tau}(p_1 + p_2, p_3, p_4, \cdots, p_{n+1}) F_{\mathcal{G}}^{2\tau}(p_3 + p_4, p_1, p_2, p_5, \cdots, p_{n+1}) \right| \\
&\quad \le C n^{-5} \int d\mu_n(\mathbf{p}_n)\ \omega(r_n) |\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n))|^{1-\kappa/2} \left| F_{\mathcal{G}}^{2\tau}(\mathbf{p}_n) \right|^2 .
\end{aligned}
$$

We now collect all our efforts. The cross terms are bounded by (6.3) and (6.4). The contribution from $F_{\mathcal{B}}^{2\tau} F_{\mathcal{B}}^{2\tau}$ can be estimated similarly. Finally the contribution from $F_{\mathcal{G}}^{2\tau} F_{\mathcal{G}}^{2\tau}$ is bounded as shown in the last lemma. Thus we have proved the estimate on $\Phi_n^{\kappa, \mathcal{G}^\tau}$ in Lemma 6.1. The estimate on $\Psi_n^{\kappa, \mathcal{G}^\tau}$ can be proved in a similar way by using instead the equation (6.6). This proves Lemma 6.1.

6.1. *Proof of the lower bound.* Recall the condition (3.10) on the size of n. Combining the lower bound on the diagonal term in Lemma 5.1 and the estimate on the off-diagonal terms in Lemma 6.1, we have

$$
\begin{aligned}
& n^2 \langle\!\langle F,\ \mathcal{K}_n^{\kappa, \mathcal{G}^\tau} F \rangle\!\rangle - n^3 \big| \langle\!\langle F,\ \Phi_n^{\kappa, \mathcal{G}^\tau} F \rangle\!\rangle \big| - n^4 \big| \langle\!\langle F,\ \Psi_n^{\kappa, \mathcal{G}^\tau} F \rangle\!\rangle \big| \\
&\quad \ge C n^2 \int d\mu_n(\mathbf{p}_n)\ \omega(r_n) |\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n))|^{1-\kappa/2} \left| F_{\mathcal{G}}^{2\tau}(\mathbf{p}_n) \right|^2 \\
&\quad\quad - C n^4 |\log\log\lambda|^{1+1/2} \int d\mu_n(\mathbf{p}_n)\ \omega(r_n) \left| F_{\mathcal{B}}^{2\tau}(\mathbf{p}_n) \right|^2 \\
&\quad\quad - C \big[\, n^{-1} + n^3 |\log\log\lambda|^{-1/2} \,\big] \\
&\quad\quad \times \int d\mu_n(\mathbf{p}_n) \omega(r_n) |\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n))|^{1-\kappa/2} \left| F_{\mathcal{G}}^{2\tau}(\mathbf{p}_n) \right|^2 .
\end{aligned}
$$

The last term can be absorbed into the first term on the right side with a change of constant. The middle term on the right side gives the estimate on the bad set. This proves the lower bound for Theorem 3.1.

6.2. *Proof of Lemma* 6.2. We first bound Q_1. Consider the two cases.

Case 1. Some $p_i, i = 1, 2, 3$, dominates, say,

$$|p_1| \ge 2(|p_2| + |p_3|).$$

Then $|p_1 - p_3| \le 4|p_1 + p_3|$. From the Schwarz inequality

$$
\begin{aligned}
(6.7)\qquad & \int d\mu_{n+1}(\mathbf{p}_{n+1}) \,\big\{|p_1 - p_3| \le 4|p_1 + p_3|\big\} \\
& \quad \times\ \Theta_\kappa(\mathbf{p}_{n+1})\big|(e^{ir_1} - e^{-ir_3})(e^{ir_2} - e^{-ir_3})\big| \\
& \quad \times \Big|F_{\mathcal{G}}^{2\tau}(p_1 + p_3, p_2, p_4, \cdots, p_{n+1})F_{\mathcal{G}}^{2\tau}(p_2 + p_3, p_1, p_4, \cdots, p_{n+1})\Big| \\
& \le \delta^{-1} \int d\mu_{n+1}(\mathbf{p}_{n+1}) \,\big\{|p_1 - p_3| \le 4|p_1 + p_3|\big\} \\
& \quad \times\ \Theta_\kappa(\mathbf{p}_{n+1})|e^{ir_1} - e^{-ir_3}|^2\big|F_{\mathcal{G}}^{2\tau}(p_1 + p_3, p_2, p_4, \cdots, p_{n+1})\big|^2 \\
& \quad + \delta \int d\mu_{n+1}(\mathbf{p}_{n+1})\ \ \Theta_\kappa(\mathbf{p}_{n+1})|e^{ir_2} - e^{-ir_3}|^2 \\
& \quad \times \big|F_{\mathcal{G}}^{2\tau}(p_2 + p_3, p_1, p_4, \cdots, p_{n+1})\big|^2.
\end{aligned}
$$

The last term on the right side of (6.7) can be bounded using Lemma 4.1. To estimate the first term, we drop $\boldsymbol{\omega}(\mathbf{r}_{n+1})|\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_{n+1}))|^\kappa$ in $\Theta_\kappa(\mathbf{p}_{n+1})$ and integrate $p_1 - p_3$. The integration can be estimated easily by

$$
(6.8)\qquad \int d(p_1 - p_3)\ \big\{|p_1 - p_3| \le 4|p_1 + p_3|\big\}\ |\Theta_\kappa(\mathbf{p}_{n+1})| \le C.
$$

We now choose $\delta = n^{-5}$ and use

$$
n^{10} \le |\log\log\lambda|^{1/2} \le C\big|\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n))\big|^{1-\kappa/2}
$$

if $\boldsymbol{\omega}(\mathbf{p}_n) \le |\log\lambda|^{-4\tau}$ and $0 \le \kappa \le 1$. The left side of (6.7) is thus bounded above by

$$
Cn^{-5} \int d\mu_n(\mathbf{p}_n)\omega(r_n)\big|\log(\lambda + \boldsymbol{\omega}(\mathbf{p}_n))\big|^{1-\kappa/2}\big|F_{\mathcal{G}}^{2\tau}(\mathbf{p}_n)\big|^2.
$$

Here we have changed variables so that the variable of the function $F_{\mathcal{G}}^{2\tau}$ is of the standard form.

Case 2. $|p_1| \sim |p_2| \sim |p_3|$.

In this case, we have $|p_1 - p_3| \le 16|p_2|$. Similar arguments prove the same bound in this region. This proves (6.5).

We now estimate Q_2. We can assume without loss of generality that

$$
\omega(p_1 - p_2) \le \omega(p_3 - p_4).
$$

Again, we bound it by the Schwarz inequality to have

$$
\begin{aligned}
Q_2 \le & \delta^{-1} \int d\mu_{n+1}(\mathbf{p}_{n+1})\ \Theta_\kappa(\mathbf{p}_{n+1})|e^{ir_1} - e^{-ir_2}|^2 \\
& \times \big\{\omega(p_1 - p_2) \le \omega(p_3 - p_4)\big\}\big|F_{\mathcal{G}}^{2\tau}(p_1 + p_2; p_3, p_4, \cdots, p_{n+1})\big|^2 \\
& + \delta \int d\mu_{n+1}(\mathbf{p}_{n+1})\ \Theta_\kappa(\mathbf{p}_{n+1})|e^{ir_3} - e^{-ir_4}|^2 \\
& \times \big|F_{\mathcal{G}}^{2\tau}(p_3 + p_4; p_1, p_2, p_5, \cdots, p_{n+1})\big|^2.
\end{aligned}
$$

Both terms can be estimated by similar arguments used for Q_1. So we obtain (6.6). $\square$

7. Conclusions

From the main estimate Theorem 3.1, we need the relation

$$\kappa_{n-1} = 1 - \kappa_n/2.$$

To satisfy this relation, for any large integers N fixed, we let

$$\kappa_n = 2/3 + (-1)^n 2^{-2N+n}/3, \quad n = 1, \cdots, 2N+1. \tag{7.1}$$

A few terms are given explicitly in the following:

$$\kappa_{2N+1} = 2/3 - 2/3 = 0, \quad \kappa_{2N} = 2/3 + 1/3 = 1, \quad \kappa_{2N-1} = 2/3 - 1/6,$$
$$\kappa_{2N-2} = 2/3 + 1/12, \quad \cdots, \ \kappa_2 = 2/3 + 2^{-2N+2}/3.$$

We first apply Theorem 3.1 to obtain

$$A_+^* D_{2N+1}^{-1} A_+ \le C |\log\log\lambda|^2 \mathbf{U}^{2N}_{\kappa_{2N},\tau}.$$

In order to satisfy the condition $\gamma \le |\log\log\lambda|^{-3}$ later on, we now replace $|\log\log\lambda|^2$ on the right side by $|\log\log\lambda|^3$ to have a further upper bound. Now we apply the lower bound part of Theorem 3.1 to have

$$A_+^* \Big\{ D_{2N} + A_+^* D_{2N+1}^{-1} A_+ \Big\}^{-1} A_+ \ge C|\log\log\lambda|^{-3} \mathbf{V}^{2N-1}_{\kappa_{2N-1},2\tau}.$$

We can repeat this procedure until we have

$$A_+^* \big(D_3 + \cdots \big)^{-1} A_+ \le C|\log\log\lambda|^{2N+4} \mathbf{U}^2_{\kappa_2,\tau}.$$

Thus we have

$$\langle\!\langle w, \Big[D_2 + A_+^* \big(D_3 + \cdots \big)^{-1} A_+ \Big]^{-1} w \rangle\!\rangle \ge \langle\!\langle w, [D_2 + C|\log\log\lambda|^{2N+4} \mathbf{U}^2_{\kappa_2,\tau}]^{-1} w \rangle\!\rangle.$$

The Fourier transform of w is

$$\hat{w}(p_1, p_2) = e^{-ir_2}.$$

Since $p_1 + p_2 = 0$ under the measure $d\mu_2$, we have

$$\begin{aligned}
&\langle\!\langle w, [D_2 + C|\log\log\lambda|^{2N+4} \mathbf{U}^2_{\kappa_2,\tau}]^{-1} w \rangle\!\rangle \\
&\qquad = \frac{1}{2} \int dp_1 \Big\{ \lambda + 2\omega(p_1) + C|\log\log\lambda|^{2N+4} \mathbf{U}^2_{\kappa_2,\tau}(p_1, -p_1)] \Big| \Big\}^{-1}.
\end{aligned}$$

The last integration is the same as the right side of (5.1) with $n = 1$ and $A_+ F$ replaced by one. Following a similar argument, we have

$$\langle\!\langle w, [D_2 + C|\log\log\lambda|^{2N+4} \mathbf{U}^2_{\kappa_2,\tau}]^{-1} w \rangle\!\rangle \ge C|\log\log\lambda|^{-2N-4} K^{\tau}_{\kappa_2}(0,0)$$

where $K^{\tau}_{\kappa_2}(0,0)$ is defined as in Lemma 4.2. From (4.20), we have

$$K^{\tau}_{\kappa_2}(0,0) \ge |\log\lambda|^{\kappa_1}\,, \qquad \kappa_1 = 2/3 - 2^{-2N+1}/3.$$

Thus,

$$\langle\!\langle w, [D_2 + C|\log\log\lambda|^{2N+4}\mathbf{U}^2_{\kappa_2,\tau}]^{-1} w\rangle\!\rangle \ge |\log\log\lambda|^{-2N-4}|\log\lambda|^{\kappa_1}.$$

Therefore, we have the lower bound

$$\begin{aligned}&\langle\!\langle w, \Big[D_2 + A_+^*\big(D_3 + \cdots\big)^{-1}A_+\Big]^{-1} w\rangle\!\rangle \\ &\qquad \ge |\log\lambda|^{2/3}\exp\Big[-\frac{|\log\log\lambda|}{2^{2N-1}3} - (2N+4)|\log\log\log\lambda|\Big].\end{aligned}$$

By choosing

$$N = \alpha|\log\log\log\lambda|$$

with α large enough, together with Lemma 2.1 we have proved the lower bound.

Instead of (7.1), we can choose

$$\kappa_n = 2/3 - (-1)^n 2^{-2N+n+1}/3, \quad n = 1, \cdots, 2N.$$

Explicit examples are

$$\begin{gathered}\kappa_{2N} = 2/3 - 2/3 = 0, \quad \kappa_{2N-1} = 2/3 + 1/3 = 1, \quad \kappa_{2N-2} = 2/3 - 1/6, \\ \kappa_{2N-3} = 2/3 + 1/12, \quad \cdots,\ \kappa_2 = 2/3 - 2^{-2N+3}/3.\end{gathered}$$

With this choice of κ_n, a similar argument proves the upper bound. This concludes the proof of Theorem 1.2.

Acknowledgement. I would like to thank P. Deift, J. Baik and H. Spohn for explaining their results to me. In particular, Spohn has pointed out the relation (1.1) so that the connection between the current across the zero and the diffusion coefficient becomes transparent. I would also like to thank A. Sznitman for his hospitality and invitation to lecture on this subject at ETH.

Courant Institute, New York University, New York, NY
Current address: Stanford University, Stanford, CA
E-mail address: yau@math.Stanford.edu

References

[1] J. Baik, P. A. Deift, and K. Johansson, On the distribution of the length of the longest increasing subsequence in a random permutation, *J. Amer. Math. Soc.* **12** (1999), 1119–1178.

[2] J. Baik and E. M. Rains, Limiting distributions for a polynuclear growth model with external sources, *J. Stat. Phys.* **100** (2000), 523–542.

[3] H. van Beijeren, R. Kutner, and H. Spohn, Excess noise for driven diffusive systems, *Phys. Rev. Lett.* **54** (1985), 2026–2029.

[4] R. Esposito, R. Marra, and H.-T. Yau , Diffusive limit of asymmetric simple exclusion, *Rev. Math. Phys.* **6** (1994), 1233–1267.

[5] ———, Navier-Stokes equations for a stochastic particle system on the lattice, *Comm. Math. Phys.* **182** (1996), 396–456.

[6] K. Johansson, Shape fluctuations and random matrices, *Comm. Math. Phys.* **209** (2000), 437–476.

[7] C. Landim, S. Olla, and H.-T. Yau, Some properties of the diffusion coefficient for asymmetric simple exclusion processes, *Ann. Probab.* **24** (1996), 1779–1808.

[8] C. Landim, J. Quastel, M. Salmhofer, and H.-T. Yau, Superdiffusivity of one and two dimensional asymmetric simple exclusion processes, preprint, 2002.

[9] C. Landim and H.-T. Yau, Fluctuation-dissipation equation of asymmetric simple exclusion processes, *Probab. Theory Related Fields* **108** (1997), 321–356.

[10] M. Prähofer and H. Spohn, Current Fluctuations for the totally asymmetric simple exclusion process, in *In and Out of Equilibrium* (Macbucaba, 2000), 185–204, *Progr. Probab.* **51**, 185–204, Birkhäuser, Boston, 2002.

[11] ———, Universal distribution for growth processes in 1+1 dimensions and random matrices, *Phys. Rev. Lett.* **84** (2000), 4882–4885.

[12] J. Quastel and H.-T. Yau, Lattice gases, large deviations, and the incompressible Navier-Stokes equations, *Ann. of Math.* **148** (1998), 51–108.

[13] F. Rezakhanlou, Hydrodynamic limit for attractive particle systems on $\mathbf{Z}^d$, *Comm. Math. Phys.* **140** (1990), 417-448.

[14] S. Sethuraman, S. R. S. Varadhan, and H.-T. Yau, Diffusive limit of a tagged particle in asymmetric simple exclusion processes, *Comm. Pure Appl. Math.* **53** (2000), 972–1006.

[15] H. Spohn, *Large Scale Dynamics of Interacting Particles*, Springer-Verlag, New York, 1991.

(Received January 31, 2002)