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Quasiconformal homeomorphisms and the convex hull boundary

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Abstract

We investigate the relationship between an open simply-connected region $\Omega \subset \mathbb{S}^2$ and the boundary Y of the hyperbolic convex hull in $\mathbb{H}^3$ of $\mathbb{S}^2 \setminus \Omega$. A counterexample is given to Thurston's conjecture that these spaces are related by a 2-quasiconformal homeomorphism which extends to the identity map on their common boundary, in the case when the homeomorphism is required to respect any group of Möbius transformations which preserves Ω. We show that the best possible universal lipschitz constant for the nearest point retraction $r : \Omega \to Y$ is 2. We find explicit universal constants $0 < c_2 < c_1$, such that no pleating map which bends more than c_1 in some interval of unit length is an embedding, and such that any pleating map which bends less than c_2 in each interval of unit length is embedded. We show that every K-quasiconformal homeomorphism $\mathbb{D}^2 \to \mathbb{D}^2$ is a $(K, a(K))$-quasi-isometry, where $a(K)$ is an explicitly computed function. The multiplicative constant is best possible and the additive constant $a(K)$ is best possible for some values of K.

1. Introduction

The material in this paper was developed as a by-product of a process which we call "angle doubling" or, more generally, "angle scaling". An account of this theory will be published elsewhere. Although some of the material developed in this paper was first proved by us using angle-doubling, we give proofs here which are independent of that theory.

Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ be a simply connected region. Let $X = \mathbb{S}^2 \setminus \Omega$ and let $\mathcal{CH}(X)$ be the corresponding hyperbolic convex hull. The relative boundary $\partial\mathcal{CH}(X) \subset \mathbb{H}^3$ faces Ω. It is helpful to picture a domed stadium—see Figure 5 in Section 3—such as one finds in Minneapolis, with Ω its *floor* and the *dome* given by $\mathrm{Dome}(\Omega) = \partial\mathcal{CH}(X)$.

The dome is canonically associated with the floor, and gives a way of studying problems concerning classical functions of a complex variable defined on Ω by using methods of two and three-dimensional hyperbolic geometry.

In this direction the papers of C. J. Bishop (see [7], [4], [6] and [5]) were particularly significant in stimulating us to do the research reported on here.

Conversely, the topic was developed in the first place (see [28] and [29]) in order to use methods of classical complex variable theory to study 3-dimensional manifolds.

The discussion begins with the following result of Bill Thurston.

THEOREM 1.1. *The hyperbolic metric in* $\mathbb{H}^3$ *induces a path metric on the dome, referred to as its hyperbolic metric. There is an isometry of the dome with its hyperbolic metric onto* $\mathbb{D}^2$ *with its hyperbolic metric.*

A proof of this can be found in [17].

1.2. In one special case, which we call the *folded* case, some interpretation is required. Here Ω is equal to $\mathbb{C}$ with the closed positive x-axis removed, and the convex hull boundary is a hyperbolic halfplane. In this case, we need to interpret Dome(Ω) as a hyperbolic plane which has been folded in half, along a geodesic. Let $r : \Omega \to \mathrm{Dome}(\Omega)$ be the nearest point retraction. By thinking of the two sides of the hyperbolic halfplane as distinct, for example, redefining a point of Dome(Ω) to be a pair (x, c) consisting of a point x in the convex hull boundary plus a choice c of a component of $r^{-1}(x) \subset \Omega$, we recover Theorem 1.1 in a trivially easy case.

The main result in the theory is due to Sullivan (see [28] and [17]); here and throughout the paper K refers either to the *maximal dilatation* of the indicated quasiconformal mapping, or to the supremum of such maximal dilatations over some class of mappings, which will be clear in its context. In other words, when there is a range of possible values of K which we might mean, we will always take the smallest possible such value of K.

THEOREM 1.3 (SULLIVAN, EPSTEIN-MARDEN). *There exists* K *such that, for any simply connected* $\Omega \neq \mathbb{C}$, *there is a* K*-quasiconformal map* $\Psi : \mathrm{Dome}(\Omega) \to \Omega$, *which extends continuously to the identity map on the common boundary* $\partial\Omega$.

QUESTION 1.4. If $\Omega \subset \mathbb{S}^2$ is not a round disk, can $\Psi : \mathrm{Dome}(\Omega) \to \Omega$ be conformal?

In working with a kleinian group which fixes Ω setwise, and therefore Dome(Ω), one would normally want the map Ψ to be equivariant. Let K be the smallest constant that works for all Ω in Theorem 1.3, without regard to any group preserving Ω. Let K_{eq} be the best universal maximal dilatation for quasiconformal homeomorphisms, as in Theorem 1.3, which are equivariant under the group of Möbius transformations preserving Ω. Then $K \leq K_{\mathrm{eq}}$, and it is unknown whether we have equality.

In [17] it is shown that $K_{\mathrm{eq}} < 82.7$. Using some of the same methods, but dropping the requirement of equivariance, Bishop [4] improved this to $K \leq 7.82$. In addition, Bishop [7] suggested a short proof of Theorem 1.3, which however does not seem to allow a good estimate of the constant. Another proof and estimate, which works for the equivariant case as well, follows from Theorem 4.14. This will be pursued elsewhere.

By explicit computation in the case of the slit plane, one can see that $K \geq 2$ for the nonequivariant case. In [29, p. 7], Thurston, discussing the equivariant form of the problem, wrote *The reasonable conjecture seems to be that the best* K *is* 2, *but it is hard to find an angle for proving a sharp constant.* In our notation, Thurston was suggesting that the best constants in Theorem 1.3 might be $K_{\mathrm{eq}} = K = 2$. This has since become known as Thurston's $K = 2$ Conjecture. In this paper, we will show that $K_{\mathrm{eq}} > 2$. That is, Thurston's Conjecture is false in its equivariant form. Epstein and Markovic have recently shown that, for the complement of a certain logarithmic spiral, $K > 2$.

Complementing this result, after a long argument we are able to show in particular (see Theorem 4.2) the existence of a universal constant $C > 0$ with the following property: Any positive measured lamination $(\Lambda, \mu) \subset \mathbb{H}^2$ with norm $\|\mu\| < C$ (see 4.0.5) is the bending measure of the dome of a region Ω which satisfies the equivariant $K = 2$ conjecture. This improves the recent result of Šarič [24] that given μ of finite norm, there is a constant $c = c(\mu) > 0$ such that the pleated surface corresponding to $(\Lambda, c\mu)$ is embedded.

We prove (see Theorem 3.1) that the nearest point retraction $r : \Omega \to \mathrm{Dome}(\Omega)$ is a continuous, 2-lipschitz mapping with respect to the induced hyperbolic path metric on the dome and the hyperbolic metric on the floor. Our result is sharp. It improves the original result in [17, Th. 2.3.1], in which it is shown that r is 4-lipschitz. In [12, Cor. 4.4] it is shown that the nearest point retraction is homotopic to a $2\sqrt{2}$-lipschitz, equivariant map. In [11], a study is made of the constants obtained under certain circumstances when the domain Ω is not simply connected.

Any K-quasiconformal mapping of the unit disk $\mathbb{D}^2 \to \mathbb{D}^2$ is automatically a (K, a)-quasi-isometry with additive constant $a = K \log 2$ when $1 < K \leq 2$ and $a = 2.37(K-1)$ otherwise (see Theorem 5.1). This has the following consequence (see Corollary 5.4): If K is the least maximal dilatation, as we vary over quasiconformal homeomorphisms in a homotopy class of maps $R \to S$ between two Riemann surfaces of finite area, then the infimum of the constants for lipschitz homeomorphisms in the same class satisfies $L \leq K$.

We are most grateful to David Wright for the limit set picture Figure 3 and also Figure 2. A nice account by David Wright is given in `http://klein.math.okstate.edu/kleinian/epstein`.

2. The once punctured torus

In this section, we prove that the best universal equivariant maximal dilatation constant in Theorem 1.3 is strictly greater than two. The open subset $\Omega \subset \mathbb{S}^2$ in the counterexample is one of the two components of the domain of discontinuity of a certain quasifuchsian group (see Figure 3). In fact, we have counterexamples for all points in a nonempty open subset of the space of quasifuchsian structures on the punctured torus.

This space can be parametrized by a single complex coordinate, using complex earthquake coordinates. This method of constructing representations and the associated hyperbolic 3-manifolds and their conformal structures at infinity is due to Thurston. It was studied in [17], where complex earthquakes were called quakebends. In [21], Curt McMullen proved several fundamental results about the complex earthquake construction, and the current paper depends essentially on his results.

A detailed discussion of complex earthquake coordinates for quasifuchsian space will require us to understand the standard action of $\mathrm{PSL}(2,\mathbb{C})$ on upper halfspace $\mathbb{U}^3$ by hyperbolic isometries. We construct quaternionic projective space as the quotient of the nonzero quaternionic column vectors by the nonzero quaternions acting on the right. In this way we get an action by $\mathrm{GL}(2,\mathbb{C})$ acting on the left of one-dimensional quaternionic projective space, and therefore an action by $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$. (However, note that general nonzero complex multiples of the identity matrix in $\mathrm{GL}(2,\mathbb{C})$ do not act as the identity.)

If $(u,v) \neq (0,0)$ is a pair of quaternions, this defines

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} .[u:v] = [au+bv : cu+dv],$$

so that $u = [u:1]$ is sent to $(au+b)(cu+d)^{-1}$, provided $cu+d \neq 0$. A quaternion $u = x+iy+jt = [u:1]$ with $t > 0$ is sent to a quaternion of the same form. The set of such quaternions can be thought of as upper halfspace $\mathbb{U}^3 = \{(x,y,t) : t > 0\} \equiv \mathbb{H}^3$, and we recover the standard action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{U}^3$. The subgroup $\mathrm{PSL}(2,\mathbb{R})$ preserves the vertical halfplane based on $\mathbb{R}$, namely $\{(x,0,t) : t > 0\}$, where we now place $\mathbb{U}^2$.

The basepoint of our quasifuchsian-space is the square once-punctured torus T_0. This means that on T_0 there is a pair of oriented simple geodesics α and β, crossing each other once, which are mutually orthogonal at their point of intersection, and that have the same length. A picture of a fundamental domain in the upper halfplane $\mathbb{U}^2$ is given in Figure 1.

2.1. For each $z = x+iy \in \mathbb{C}$, we will define the map $\mathbb{C}E_z : \mathbb{U}^2 \to \mathbb{U}^3$. We think of $\mathbb{U}^2 \subset \mathbb{U}^3$ as the vertical plane lying over the real axis in $\mathbb{C} \subset \partial\mathbb{U}^3$. Our starting point is this standard inclusion $\mathbb{C}E_0 : \mathbb{U}^2 \to \mathbb{U}^2 \subset \mathbb{U}^3$. Given $z = x+iy$, $\mathbb{C}E_z$ is defined in terms of a complex earthquake along α: We perform a right

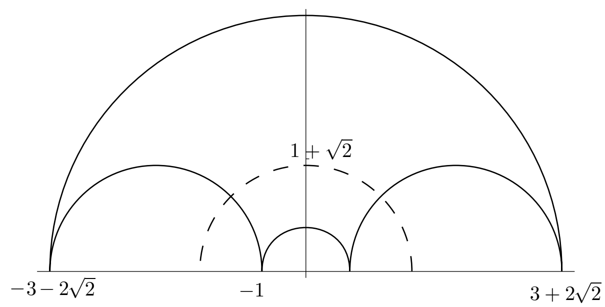

Figure 1: A fundamental domain for the square torus. The dotted semicircle is the axis of B_0. The vertical line is the axis of A.

earthquake along α through the signed distance x, and then bend through a signed rotation of y radians. $\mathbb{U}^2$ is cut into countably many pieces by the lifts of α under the covering map $\mathbb{U}^2 \to T_0$. The map $\mathbb{C}E_z : \mathbb{U}^2 \to \mathbb{U}^3$ is an isometry on each piece and, unless $x = 0$, is discontinuous along the lifts of α. We normalize by insisting that $\mathbb{C}E_z = \mathbb{C}E_0$ on the piece immediately to the left of the vertical axis.

Note that $\mathbb{C}E_z = \Psi_z \circ E_x$, where $E_x : \mathbb{U}^2 \to \mathbb{U}^2$ is a real earthquake and $\Psi_z : \mathbb{U}^2 \to \mathbb{U}^3$ is a pleating map, sometimes known as a bending map. The bending takes place along the images of the lifts of α under the earthquake map, not along the lifts of α, unless $x = 0$. The pleating map is continuous and is an isometric embedding, in the sense that it sends a rectifiable path to a rectifiable path of the same length.

Let F_2 be the free group on the generators α and β. We define the homomorphism $\varphi_z : F_2 \to \mathrm{PSL}(2, \mathbb{C})$ in such a way that $\mathbb{C}E_z$ is φ_z-equivariant, when we use the standard action of F_2 on $\mathbb{U}^2$ corresponding to Figure 1 and the standard action described above of $\mathrm{PSL}(2, \mathbb{C})$ on $\mathbb{U}^3$. We also ensure that, for each $z \in \mathbb{C}$,

$$\mathrm{trace}\varphi_z[\alpha, \beta] = -2.$$

This forces us (modulo some obvious choices) to make the following definitions:

$$\varphi_z(\alpha) = A = \begin{bmatrix} -1+\sqrt{2} & 0 \\ 0 & 1+\sqrt{2} \end{bmatrix} \text{ and}$$

$$\varphi_z(\beta) = B_z = \begin{bmatrix} \sqrt{2}\exp(z/2) & (1+\sqrt{2})\exp(z/2) \\ (-1+\sqrt{2})\exp(-z/2) & \sqrt{2}\exp(-z/2) \end{bmatrix}.$$

Set $G_z = \varphi_z(F_2)$.

The set of values of z, for which φ_z is injective and G_z is a discrete group of isometries, is shown in Figure 2.

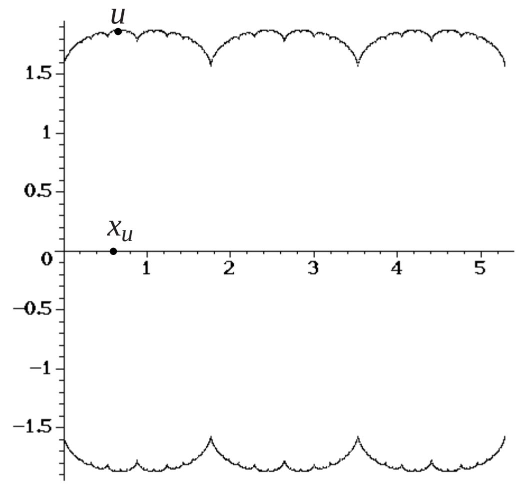

Figure 2: The values of z for which φ_z is injective and G_z is a discrete group of isometries is the region lying between the upper and lower curves. The whole picture is invariant by translation by arccosh(3), which is the length of α in the punctured square torus. The Teichmüller space of T is holomorphically equivalent to the subset of $\mathbb{C}$ above the lower curve. The point marked u is a highest point on the upper curve, and x_u is its x-coordinate. We have here a picture of the part of quasifuchsian space of a punctured torus, corresponding to $\operatorname{trace}(A) = 2\sqrt{2}$. This picture was drawn by David Wright.

Here is an explanation of Figure 2. Changing the x-coordinate corresponds to performing a signed earthquake of size equal to the change in x. Changing the y-coordinate corresponds to bending.

If we start from the fuchsian group on the x-axis and bend by making y nonzero, then at first the group remains quasifuchsian, and the limit set is a topological circle which is the boundary of the pleated surface $\mathbb{C}E_z(\mathbb{U}^2)$. The convex hull boundary of the limit set consists of two pleated surfaces, one of which is $\mathbb{C}E_z(\mathbb{U}^2) = \Psi_z(\mathbb{U}^2)$, which we denote by P_z.

For $z = x + iy$ in the quasifuchsian region, the next assertion follows from our discussion.

Lemma 2.2. *From the hyperbolic metric on P_z given by the lengths of rectifiable paths, as in Theorem 1.1, P_z/G_z has a hyperbolic structure which can be identified with that of $\mathbb{U}^2/G_x$.*

We have $P_z = \operatorname{Dome}(\Omega_z)$, where Ω_z is one of the two domains of discontinuity of G_z. Let Ω'_z be the other domain of discontinuity. Each domain of discontinuity gives rise to an element of Teichmüller space, and we get $T_z = \Omega_z/G_z$ and $T'_z = \Omega'_z/G_z$, two punctured tori. Because of the symmetry of our construction with respect to complex conjugation, $T_z = T'_{\bar{z}}$.

For fixed x, as $y > 0$ increases, the pleated surface $\mathbb{C}E_z(\mathbb{U}^2)$ will eventually touch itself along the limit set. Since the construction is equivariant, touching must occur at infinitely many points simultaneously. For this z, Ω'_z either disappears or becomes the union of a countable number of disjoint disks. In fact the disks are round because the thrice punctured sphere has a unique complete hyperbolic structure. Similarly, as $y < 0$ decreases, the mirror image events occur, the structure T_z disappears, and we reach the boundary of Teichmüller space.

As McMullen shows, T_z continues to have a well-defined projective structure for all z with $y > 0$, and T_z therefore has a well-defined conformal structure.

It may seem from the above explanation that, for fixed x, there should be a maximal interval $a \leq y \leq b$, for which bending results in a proper dome, while no other values of y have this property. Any such hope is quickly dispelled by examining the web pages `http://www.maths.warwick.ac.uk/dbae/papers/EMM/wright.html`. (This is a slightly modified copy of web pages created by David Wright.) One sees that the parameter space is definitely not "vertically convex".

Let $\mathcal{T}$ be the set of $z = x + iy \in \mathbb{C}$ such that either $y > 0$ or such that the complex earthquake with parameter z gives a quasifuchsian structure T_z and a discrete group G_z of Möbius transformations. The following result, fundamental for our purposes, is proved in [21, Th. 1.3].

Theorem 2.3 (McMullen's Disk Theorem). *$\mathcal{T}$ is biholomorphically equivalent to the Teichmüller space of once-punctured tori. Moreover*

$$\mathbb{U}^2 \subset \mathcal{T} \subset \{z = x + iy : y > -i\pi\}\,.$$

In Figure 2, $\mathcal{T}$ corresponds to the set of z above the lower of the two curves. From now on we will think of Teichmüller space as this particular subset of $\mathbb{C}$. We denote by $d_{\mathcal{T}}$ its hyperbolic metric, which is also the Teichmüller metric, according to Royden's theorem [23].

We denote by $\mathcal{QF} \subset \mathcal{T}$ the quasifuchsian space, corresponding to the region between the two curves in Figure 2.

The following result summarizes important features of the above discussion.

Theorem 2.4. *Given $u, v \in \mathcal{QF} \subset \mathcal{T} \subset \mathbb{C}$, let $f : T_u \to T_v$ be the Teichmüller map. Then the maximal dilatation K of f satisfies $d_{\mathcal{T}}(u, v) = \log K$.*

Let $\tilde{f} : \Omega_u \to \Omega_v$ be a lift of f to a map between the components of the ordinary sets associated with u, v. Any F_2-equivariant quasiconformal homeomorphism $h : \Omega_u \to \Omega_v$, which is equivariantly isotopic to $\tilde{f}$, has maximal dilatation at least K; K is uniquely attained by $h = \tilde{f}$.

Let $u = x_u + iy_u$ be a point on the upper boundary of $\mathcal{QF}$, with y_u maximal. An illustration can be seen in Figure 2. Such a point u exists since $\mathcal{QF}$ is periodic. Automatically $\bar{u} = x_u - iy_u$ is a lowest point in $\bar{\mathcal{T}}$.

THEOREM 2.5. *Let u be a fixed highest point in $\overline{\mathcal{QF}}$. Let U be a sufficiently small neighbourhood of u. Then, for any $z = x + iy \in U \cap \mathcal{QF}$, the Teichmüller distance from T_x to T_z satisfies $d_{\mathcal{T}}(x, z) > \log(2)$.*

For any F_2-equivariant K-quasiconformal homeomorphism $\Omega_z \to \mathrm{Dome}(\Omega_z)$ which extends to the identity on $\partial\Omega_z$, $K > 2$. Therefore $K_{\mathrm{eq}} > 2$.

Proof. Let d_- denote the hyperbolic metric in the halfplane

$$H_- = \{t \in \mathbb{C} : \mathrm{Im}(t) > -y_u\}\,.$$

In this metric, $d_-(u, x_u) = \log(2)$, since $\overline{u} = x_u - iy_u \in \partial H_-$. Now $d_-(u, x_u) \leq d_{\mathcal{T}}(u, x_u)$ since $\mathcal{T} \subset H_-$. The inequality is strict because Teichmüller space is a proper subset of H_-. This fact was shown by McMullen in [21]. It can be seen in Figure 2.

Consequently, when U is small enough and $z = x+iy \in U\cap\mathcal{QF}$, $d_{\mathcal{T}}(x, z) > \log(2)$. By Lemma 2.2, T_x represents the same point in Teichmüller space as P_z/G_z, which is one of the two components of the boundary of the convex core of the quasifuchsian 3-manifold $\mathbb{U}^3/G_z$. Up in $\mathbb{U}^3$, $P_z = \mathrm{Dome}(\Omega_z)$, while Ω_z/G_z is equal to T_z in Teichmüller space. The Teichmüller distance from T_z to T_x is equal to $d_{\mathcal{T}}(z, x) > \log(2)$.

By the definition of the Teichmüller distance, the maximal dilatation of any quasiconformal homeomorphism between T_z and T_x, in the correct isotopy class, is strictly greater than 2. Necessarily, any F_2-equivariant quasiconformal homeomorphism between P_z and Ω_z has maximal dilatation strictly greater than 2. In particular, any equivariant quasiconformal homeomorphism which extends to the identity on $\partial\Omega_z$ has maximal dilatation strictly greater than 2.

This completes the proof that $K_{\mathrm{eq}} > 2$. The open set of examples $\{\Omega_z\}$ we have found, that require the equivariant constant to be greater than 2, are domains of discontinuity for once-punctured tori quasifuchsian groups. In particular each is the interior of an embedded, closed quasidisk. □

We end this section with a picture of a domain for which $K_{\mathrm{eq}} > 2$; see Figure 3. Now, Ω_z is a complementary domain of a limit set of a group G_z, with $z \in U \cap \overline{\mathcal{QF}}$.

Curt McMullen (personal communication) found experimentally that the degenerate end of the hyperbolic 3-manifold that corresponds to the "lowest point" $\overline{u}$ appears to have ending lamination equal to the golden mean slope on the torus. That is, the ending lamination is preserved by the Anosov map $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ of the torus.

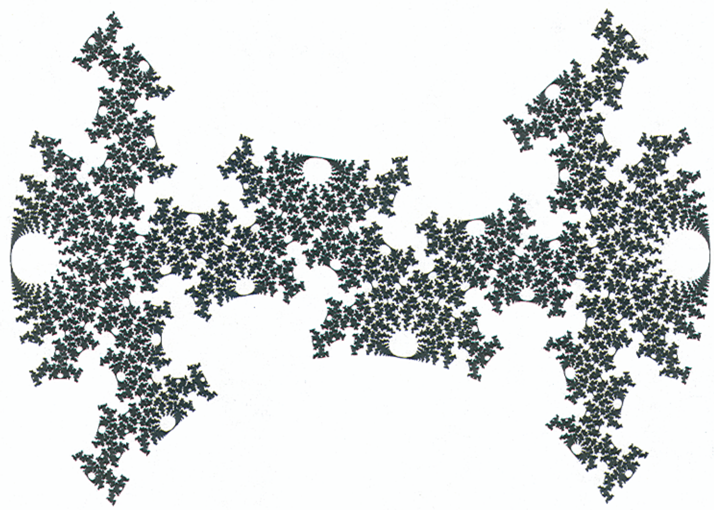

Figure 3: The complement in $\mathbb{S}^1$ of the limit set shown here is a counterexample to the equivariant $K = 2$ conjecture. The picture shows the limit set of G_u, where u is a highest point in $\mathcal{QF} \subset \mathcal{T} \subset \mathbb{C}$. This seems to be a one-sided degeneration of a quasifuchsian punctured torus group. This would mean that, mathematically, the white part of the picture is dense. However, according to Bishop and Jones (see [8]), the limit set of such a group must have Hausdorff dimension two, so the blackness of the nowhere dense limit set is not surprising. In fact, the small white round almost-disks should have a great deal of limit set in them; this detail is absent because of intrinsic computational difficulties. This picture was drawn by David Wright.

3. The nearest point retraction is 2-lipschitz

Let $\Omega \subset \mathbb{C}$ be simply connected and not equal to $\mathbb{C}$. We recall Thurston's definition of the nearest point retraction $r : \Omega \to \mathrm{Dome}(\Omega)$: given $z \in \Omega$, expand a small horoball at z. Denote by $r(z) \in \mathrm{Dome}(\Omega) \subset \mathbb{H}^3$ the (unique) point of first contact.

In this section we prove the following result.

THEOREM 3.1. *The nearest point retraction* $r : \Omega \to \mathrm{Dome}(\Omega)$ *is* 2-*lipschitz in the respective hyperbolic metrics. The result is best possible.*

QUESTION 3.2. What are the best constants for the quasi-isometry

$$r^{-1} : \mathrm{Dome}(\Omega) \to \Omega?$$

Note that r^{-1} is a relation, not a map.

Proof of Theorem 3.1. First we look at the folded case, described in §1.2. The Riemann mapping $z \mapsto z^2$ maps the upper halfplane onto a slit plane Ω, obtained by removing the closed positive x-axis from $\mathbb{C}$. This enables us to work out hyperbolic distances in Ω. The nearest point retraction r sends the negative x-axis to the vertical geodesic over $0 \in \mathbb{U}^3$. These are geodesics in the hyperbolic metric on Ω and the hyperbolic metric on $\mathrm{Dome}(\Omega)$ respectively, and r exactly doubles distances. It follows that, in the statement of Theorem 3.1, we can do no better than the constant 2. At the other extreme, if Ω is a round disk, then r is an isometry. We now show that the lipschitz constant of r is at most 2.

It suffices to consider the case that $S = \mathrm{Dome}(\Omega)$ is finitely bent and no two of its bending lines have a common end point. For we may approximate Ω by a finite union Ω_n of round disks so that no three of the boundary circles of Ω_n meet at a point [17]. Given points $z_1, z_2 \in \Omega$ we may arrange the approximations so that, for all n and for $i = 1, 2$, $r(z_i) = r_n(z_i) \in S_n = \mathrm{Dome}(\Omega_n)$. Then the hyperbolic distances $d_{\Omega_n}(z_1, z_2)$ and $d_{S_n}(r_n(z_1), r_n(z_2))$ are arbitrarily close to $d_\Omega(z_1, z_2)$ and $d_S(r(z_1), r(z_2))$ respectively.

Therefore if we can prove that for all n, $d_{S_n}(r_n(z_1), r_n(z_2)) \leq 2d_{\Omega_n}(z_1, z_2)$, then in the limit $d_S(r(z_1), r(z_2)) \leq 2d_\Omega(z_1, z_2)$, which is what we need to prove.

So we may assume that $\mathrm{Dome}(\Omega)$ is finitely bent such that no two bending lines have a common end point. We may also assume that Ω is not a slit plane or a round disk.

Now $S = \mathrm{Dome}(\Omega)$ is a finite union of flat pieces and bending lines. A flat piece F is a polygon in some hyperbolic plane $H \subset \mathbb{H}^3$. The circle $\partial H \subset \mathbb{S}^2$ is the common boundary of two open round disks in $\mathbb{S}^2$ exactly one of which, say D, lies in Ω. The disk D is *maximal* in the sense that it is not contained in any larger disk lying in Ω. Since D is associated with a flat piece, $\partial D \cap \partial\Omega$ consists of at least three distinct points.

If $\ell \subset S$ is a bending line, then the inverse $r^{-1}(\ell)$, which is a closed set, is a *crescent* with vertex angle α where α is the exterior bending angle of S at ℓ. Here we are using the term crescent in the following sense.

DEFINITION 3.3. A *crescent* in $\mathbb{S}^2$ is a region bounded by two arcs of round circles. It is equivalent under a Möbius transformation to a region in the plane lying between two straight rays from the origin to infinity.

The open regions in Ω which are exterior to the union of crescents coming from bending lines are called *gaps*. Thus if G is a gap, r is a conformal

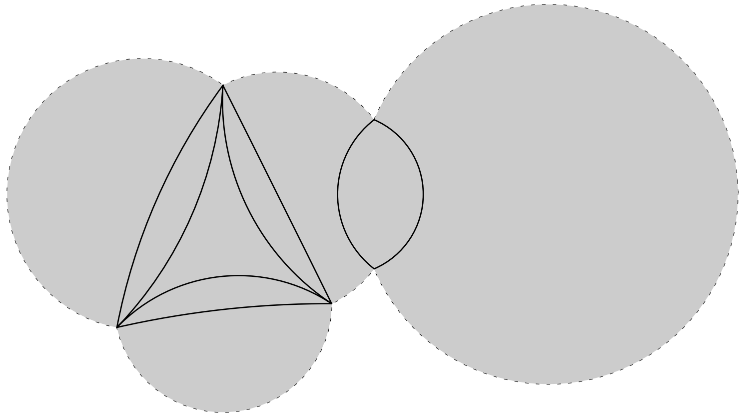

Figure 4: Ω is the union of four disks. Dome(Ω) is the union of five flat pieces as can be seen in Figure 5. The fifth piece is a hyperbolic triangle in the hyperbolic plane represented by a circle lying in the union of three of the original disks. The dome has four bending lines, as shown in Figure 5. The crescents shown are the inverse images of the bending lines under the nearest point retraction. Notice that each boundary component of a crescent is orthogonal to the appropriate circle.

isomorphism of G onto a flat piece F; the set of inverses $\{r^{-1}(\mathrm{int}(F))\}$ of flat pieces F is the set of gaps.

Given a flat piece F, it lies in a unique hyperbolic plane. The boundary of this plane is a circle in $\mathbb{S}^2$, which bounds an open disk $D \subset \Omega$. Let $G \subset D$ be the closure of the inverse image of the interior of F. So $G = \overline{r^{-1}(\mathrm{int}(F))}$, where we are taking the closure in D. Then $r : G \to F$ is an isometry if we use the hyperbolic metrics on S and D. The inverse image in Ω of a bending line is a crescent in $\mathbb{C}$.

Definition 3.4. A set like G above is called a *gap*. We also use "gap" to denote a component of the complement of the bending lamination in the hyperbolic plane. Figure 4 illustrates the situation.

Each gap G is contained in a maximal disk D: the flat piece $F \subset \mathbb{H}^3$ corresponding to G lies in a hyperbolic plane $H \subset \mathbb{H}^3$, and H corresponds to $D \subset \mathbb{S}^2$. The hyperbolic metric on H is isometric to the Poincaré metric on D, and the isometry induces the identity on the common boundary $\partial D = \partial H$. The relative boundary $\partial G \cap D$ is a nonempty finite union of geodesics in the hyperbolic metric of D. Each component c of $\partial G \cap D$ is an edge of a crescent $C \subset \Omega$. The other edge c' of C is a geodesic in another maximal disk D' of Ω and D' corresponds to a flat piece F' that is adjacent to F along a bending

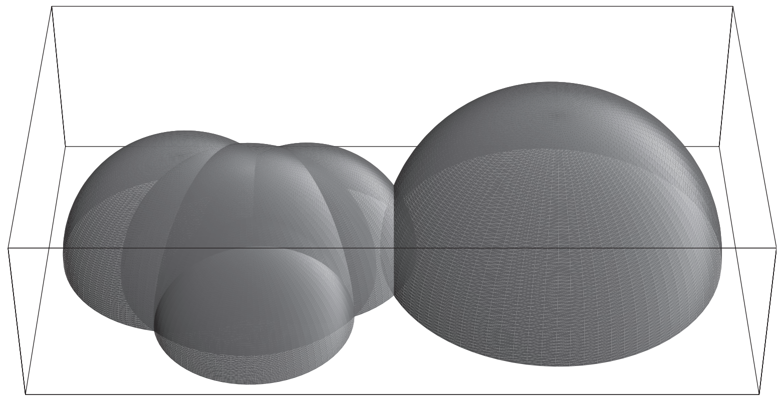

Figure 5: Dome(Ω), where Ω is shown in Figure 4. The dome is placed in the upper halfspace model, and is viewed from inside the convex hull of the complement of Ω, using Euclidean perspective. The space under the dome lies between Ω and Dome(Ω). Since the upper halfspace model is conformal, the angle between disks in Figure 4 is equal to the angle between flat pieces shown in Figure 5.

line ℓ; the exterior bending angle satisfies $0 < \alpha < \pi$ (since S is not folded). The set of vertices of C is equal to $\partial D \cap \partial D'$. This is also the set of endpoints of ℓ. The vertex angle of C is α. The nearest point retraction r sends C onto ℓ. Overall, Ω is the union of gaps and crescents, as shown in Figure 4.

Lemma 3.5. *Suppose $\Omega \subset \mathbb{C}$ is simply connected $\neq \mathbb{C}$ and* Dome(Ω) *is finitely bent, such that no two bending lines have a point at infinity in common. Let $D \subset \Omega$ be a maximal disk and let $G \subset D$ be a gap. Then the hyperbolic metrics $\rho_D|dz|$ of D and $\rho_\Omega|dz|$ of Ω satisfy*

$$\forall z \in G,\ \rho_\Omega(z) \leq \rho_D(z) \leq 2\rho_\Omega(z). \tag{3.5.1}$$

Proof. The left-hand side of Inequality 3.5.1 is immediate. We need to prove the righthand inequality.

Let $\xi \in \partial G$ be a point that lies on an edge c of a crescent C associated with the intersecting maximal open disks D, D', with $c \subset D$. We will prove $\rho_D(\xi) \leq 2\rho_\Omega(\xi)$. Since the inequality is invariant under Möbius transformations, we may assume that C is a wedge, with one vertex at 0 and the other at infinity. Then D and D' become euclidean halfplanes and Ω contains the union of these two halfplanes. The picture is shown in Figure 6.

Denoting euclidean distance by d, we have

$$d(\xi, \partial\Omega) = d(\xi, \partial D) = |\xi| = d(\xi, 0).$$

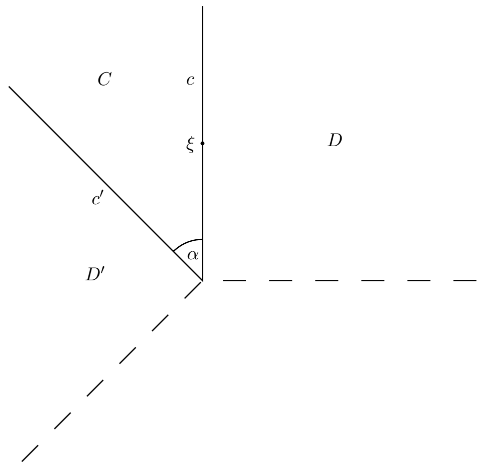

Figure 6: This illustrates the first part of the proof of Lemma 3.5. The dotted lines are part of the boundary of Ω.

Since $\infty \notin \Omega$, by the Koebe inequality (see [2]),

$$\rho_\Omega(\xi) \geq \frac{1}{2d(\xi, \partial\Omega)}.$$

Since D is a halfplane,

$$\rho_D(\xi) = \frac{1}{d(\xi, \partial D)} = \frac{1}{d(\xi, \partial\Omega)}.$$

We conclude that $\rho_D(\xi)/\rho_\Omega(\xi) \leq 2$. This holds for all points $\xi \in \partial G \cap D$, where D is the open halfplane or disk defined above.

Next consider a component c of $\partial G \cap \partial D \subset \partial\Omega$. For the purpose of proving the inequality, we may assume that D is equal to the upper halfplane, and that c is equal to the positive x-axis.

Fix $\varepsilon > 0$. We choose a horizontal euclidean strip R (see Figure 7) in the upper halfplane, so that R is a neighbourhood of c in $\overline{G}$. For all points $\xi = (x, y) \in G \cap R$ with $x \geq 0$, the orthogonal projection of ξ to $\mathbb{R}$ is the closest point of $\partial\Omega$, while if $\xi \in G \cap R$, with $x < 0$, the closest point in $\partial\Omega$ is 0. Making R sufficiently thin, we can ensure that $d(\xi, \partial\Omega) \leq y(1 + \varepsilon)$.

We conclude as before that, for all ξ sufficiently close (in the euclidean sense) to c,

$$\rho_\Omega(\xi) \geq \frac{1}{2d(\xi, \partial\Omega)} \geq \frac{1}{2d(\xi, \partial D)(1 + \varepsilon)} = \frac{\rho_D(\xi)}{2(1 + \varepsilon)}.$$

We have shown, for all points ξ on or near ∂G, that

$$\rho_D(\xi) \leq 2(1 + \varepsilon)\rho_\Omega(\xi).$$

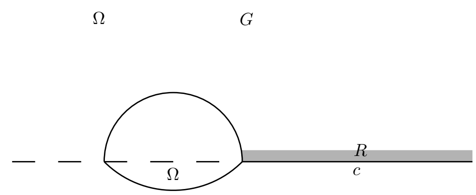

Figure 7: This illustrates the second part of the proof of Lemma 3.5. The label Ω appears twice in order to indicate that Ω encompasses the upper arc shown. G, on the other hand, lies entirely above the upper arc. The dotted line indicates the boundary of D.

Now

$$\Delta \log \rho_D = {\rho_D}^2 \text{ and } \Delta \log \rho_\Omega = {\rho_\Omega}^2.$$

Here Δ is the euclidean laplacian. The first expression can be seen by direct calculation with D equal to the upper halfplane. The second follows immediately upon changing coordinates since holomorphic functions are harmonic.

We conclude that, for all $z \in D$,

$$\Delta\tau(z) > 0, \text{ where } \tau(z) = \log \frac{\rho_D}{\rho_\Omega}(z),$$

since we have excluded the case that Ω is a round disk. That is, τ is subharmonic in D. Consequently τ cannot have a maximum in D. So it cannot have a maximum in the interior of G. Now $\tau(\xi) \leq \log 2$ on $\partial G \cap D$ and, near $\partial G \cap \partial\Omega$, $\tau(z) \leq \log(1+\varepsilon) + \log 2$. Since $\varepsilon > 0$ is arbitrary, this establishes Inequality 3.5.1 for G. □

We now continue with the proof of Theorem 3.1 where no two bending lines meet at infinity and the convex hull boundary is finitely bent.

Suppose G is a gap, that is, a region bounded by bending lines. We normalize so that $r(G) \subset S = \mathrm{Dome}(\Omega)$ is contained in a vertical halfplane in $\mathbb{U}^3$. Then $r|G$ is a euclidean rotation and, for $z \in G$, $|r'(z)| = 1$, where r' refers to the euclidean derivative. Consequently, from Inequality 3.5.1, for $z \in G$ we have

$$\rho_S(r(z))|r'(z)||dz| = \rho_S(r(z))|dz| = \rho_D(z)|dz| \leq 2\rho_\Omega(z)|dz|,$$

where the extreme terms give a form that is invariant under Möbius transformations.

Next consider a crescent C. Normalize so that its endpoints are $0, \infty$. Then C is a wedge of vertex angle $\alpha < \pi$. The euclidean halfplanes D, D' are adjacent to C along the two edges of C, with the earlier notation. Therefore, given $z \in C$, the closest euclidean distance $d(z, \partial\Omega)$ is $|z|$, the distance to $z = 0$.

Consequently

$$\rho_\Omega(z) \geq \frac{1}{2d(z, \partial\Omega)} = \frac{1}{2|z|}.$$

The bending line ℓ corresponding to C becomes the vertical halfline ending at $0 \in \mathbb{U}^3$. The nearest point retraction $r : C \to \ell$ is a euclidean isometry on each line in C through 0. In particular $r : C \to \ell$ preserves euclidean distances to $z = 0$. Also, for $z \in C$, $|r'(z)| \leq 1$. Consequently

$$\rho_S(r(z))|r'(z)| \leq \rho_S(r(z)) = \frac{1}{|z|} \leq 2\rho_\Omega(z).$$

We conclude that, for all $z \in \Omega$,

$$\rho_S(r(z))|r'(z)| \leq 2\rho_\Omega(z). \tag{3.5.2}$$

Upon integration, we find, for any two points $z_1, z_2 \in \Omega$,

$$d_S(r(z_1), r(z_2)) \leq 2d_\Omega(z_1, z_2). \tag{3.5.3}$$

□

4. Embedded pleated surfaces

Let (Λ, μ) be a measured lamination on the hyperbolic plane. We allow μ to be a real-valued signed measure; the only restriction is that it should be a Borel measure, supported on the space of leaves of Λ. In particular, the measure of any compact transverse interval is finite.

Following Thurston, there is a *pleating map* $\Psi_{(\Lambda,\mu)} : \mathbb{H}^2 \to \mathbb{H}^3$, which sends rectifiable curves to rectifiable curves of the same length, such that the signed bending along any short geodesic open interval $C \subset \mathbb{H}^2$ is $\mu(C)$ (see [17, p. 209–215]). In subsection 2.1, we had a similar situation, but the pleating map was denoted by Ψ_{iy}.

More generally, we have the complex earthquake

$$\mathbb{C}E_{(\Lambda,z\mu)} : \mathbb{H}^2 \to \mathbb{H}^3$$

based on $(\Lambda, z\mu)$, where Λ is a geodesic lamination of $\mathbb{H}^2$, μ is a transverse signed measure, and $z = x + iy \in \mathbb{C}$. To define this, we first apply a (real) earthquake $E_{(\Lambda,x\mu)} : \mathbb{H}^2 \to \mathbb{H}^2$. This is discontinuous along any leaf with atomic measure and is continuous otherwise. It extends to a homeomorphism $\partial\mathbb{H}^2 \to \partial\mathbb{H}^2$. The earthquake sends the geodesic lamination Λ to a new geodesic lamination Λ', with transverse measure μ', induced from μ. We now define the pleating map $\Psi_{(\Lambda',y\mu')} : \mathbb{H}^2 \to \mathbb{H}^3$, sending rectifiable paths to rectifiable paths of the same length. Then we define the complex earthquake as the composition

$$\mathbb{C}E_{(\Lambda,z\mu)} = \Psi_{(\Lambda',y\mu')} \circ E_{(\Lambda,x\mu)}. \tag{4.0.4}$$

The pleated surface $P_{(\Lambda,z\mu)} \subset \mathbb{H}^3$ is defined to be the image of the discontinuous map $\mathbb{C}E_{(\Lambda,z\mu)}$. This image is also equal to the image of the isometric map $\Psi_{(\Lambda',y\mu')}$.

Consider the norm

$$\|\mu\| = \|(\Lambda,\mu)\| = \sup_C |\mu|(C) \ \in [0,\infty] \tag{4.0.5}$$

as C varies over all open transverse geodesic intervals C of unit length. (We get the same supremum if we vary over all half-open intervals of unit length, but the answer may be larger if we vary over closed intervals of unit length.) This is much the same thing as the *average bending* introduced by Martin Bridgeman in [9]; more generally, he considered the quotient of the bending measure deposited on a geodesic interval divided by the length of the interval. The average bending has been used in other works, for example in Bridgeman and Canary (see [11]).

Since $\|(\Lambda,\mu)\| = 0$ if and only if the image of the pleating map is a plane, the norm can be used as a measure of the "roundness" of a simply connected region $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$. We formalize this point of view with the following definition:

Let (Λ,μ) be the bending measure on $\mathbb{H}^2$ for $S = \mathrm{Dome}(\Omega)$.

Definition 4.1. We define the *roundness measure* $\rho(\Omega) = \|(\Lambda,\mu)\|$.

Theorem 4.2. 1. *There is a constant* $c_1 > 0$, *satisfying*

$$\pi + 1 \le c_1 \le 2\pi - \arcsin(1/\cosh(1)) \approx 4.8731,$$

such that for each simply connected $\Omega \neq \mathbb{C}$, $\rho(\Omega) \le c_1$. *Let* c_1 *be the smallest such constant. The upper bound for* c_1 *is due to Bridgeman in* [9] *and* [10].

2. *There is a number* $c_2 > 0$ *with the following property. Let* (Λ,μ) *be any measured lamination on* $\mathbb{H}^2$, *where* μ *is a signed Borel measure with* $\|(\Lambda,\mu)\| < c_2$. *Then the pleating map* $\Psi_{(\Lambda,\mu)} : \mathbb{H}^2 \to \mathbb{H}^3$ *is a bilipschitz embedding which extends to an embedding of* $\mathbb{S}^1 = \partial\mathbb{H}^2$ *as a quasicircle in* $\mathbb{S}^2 = \partial\mathbb{H}^3$. *Let* $c_2 = 0.73$.

3. *If* c_2 *is the largest constant satisfying* 4.2.2, *then*

$$c_2 \le 2\arcsin(\tanh(1/2)) \approx 0.96.$$

For any number $c > 2\arcsin(\tanh(1/2))$, *there is a geodesic lamination* Λ *and a nonnegative measure* μ, *such that* $\|(\Lambda,\mu)\| = c$ *and the pleating map* $\Psi_{(\Lambda,\mu)} : \mathbb{H}^2 \to \mathbb{H}^3$ *is not an embedding.*

4. *If* $\rho = \rho(\Omega) < c_2$, *then there is a* $K(\rho)$*-quasiconformal homeomorphism* $f_\Omega : \Omega \to \mathrm{Dome}(\Omega)$, *which extends continuously to the identity on* $\partial\Omega$.

Figure 8: Take Ω to be the central region shown in the diagram. The vertical (euclidean) distance between the horizontal intervals is exactly twice the length of the horizontal intervals. The vertical lines go to infinity.

Here $K : [0, c_2) \to [1, 2]$ is a continuous, monotonic increasing function with $K(0) = 1$.

If G is the group of Möbius transformations which preserves Ω, then f_Ω is G-equivariant. Thus Ω satisfies the equivariant $K = 2$ conjecture.

REMARK 4.3. We conjecture that $c_2 = 2 \arcsin(\tanh(1/2))$.

Proof of Theorem 4.2(1). Every explicit example of a convex hull boundary gives a lower bound for c_1. The inequality $c_1 \geq \pi + 1$ follows by taking $\Omega \subset \mathbb{C}$ as in Figure 8. The dome in upper halfspace is the union of two vertical planes, with half-disks removed, together with the boundary of a half-cylinder which joins the two vertical planes. The path along the top of the half-cylinder is an interval of hyperbolic length 1. The bending along this interval is $\pi + 1$, proving that $c_1 \geq \pi + 1$.

We next discuss the upper bound for c_1. It is easy to show that c_1 is bounded above, if one is not bothered by the size of the bound. An explicit upper bound for c_1, that is not too huge, can be estimated using [17, Prop. 2.14.2]. The best available technique for estimating c_1 is due to Bridgeman (see [9, Th. 1]). This is further explained in [10], where it is shown that $c_1 \leq 2\pi - \arcsin(1/\cosh(1)) \approx 4.8731$. $\square$

For the following key lemma we have to choose a pair of numbers $\varepsilon, \theta_0 > 0$ such that $\theta_0 + \varepsilon < \pi/2$ while $\sin(\theta_0) > \varepsilon$. Thus ε, θ_0 must satisfy the inequality

$$\varepsilon < \sin(\theta_0) < \cos(\varepsilon).$$

Lemma 4.4. *Choose ε, θ_0 as above. Let $\gamma : [0, \infty)$ be a piecewise geodesic in $\mathbb{H}^3$, parametrized by pathlength, such that the sum of the bending angles along each half-open subinterval of length 1 is at most ε. Let $\theta(t)$ be the angle between $\gamma'(t)$ (when it exists) and the geodesic ray from $\gamma(0)$ through $\gamma(t)$. Then, for all $t \geq 0$, $0 \leq \theta(t) \leq \theta_0 + \varepsilon < \pi/2$.*

Proof. Suppose first $\gamma : \mathbb{R} \to \mathbb{D}^2$ is a geodesic parametrized by hyperbolic arclength and let $p \in \mathbb{H}^2$. We write $s = s(t) = d(p, \gamma(t))$. We set $\theta(t)$ equal to the angle between the oriented geodesic from p through $\gamma(t)$ and γ, where $0 < \theta < \pi$. Then

$$s'(t) = \cos(\theta(t)) \text{ and } \theta'(t) = -\frac{\sin(\theta(t))}{\tanh(s(t))} < -\sin(\theta(t)).$$

In particular $\theta(t)$ is strictly decreasing along a geodesic, unless it is identically zero.

Now let $\gamma : [0, \infty) \to \mathbb{H}^3$ be as in the statement of Lemma 4.4, $p = \gamma(0)$. As before define $\theta(t)$ at the smooth points to be the angle which the geodesic ray from p to $\gamma(t)$ makes with the tangent vector $\gamma'(t)$. At a bending point s of γ, instead define $\theta(s)$ to be the limit of $\theta(t)$ as t increases to s. At a bending point, θ may jump upwards, but at other points it is strictly decreasing, provided $\theta(t) > 0$. The initial segment of γ has $\theta(t) = 0$.

We will prove by contradiction that

$$\forall\, t \in [0, \infty),\ 0 \leq \theta(t) \leq \theta_0 + \varepsilon < \frac{\pi}{2}.$$

So suppose this is false. Then we can define

$$t_2 = \inf\{t : \theta(t) > \theta_0 + \varepsilon\}.$$

Since the initial segment of γ has $\theta(t) = 0$, we must have $t_2 > 0$.

Again because of the initial segment, we can define

$$t_1 = \sup\{t : 0 < t < t_2 \text{ and } \theta(t) < \theta_0\}$$

and see that $t_1 > 0$. So, for $t_1 < t \leq t_2$, we have

$$\theta(t_1) < \theta_0 \leq \theta(t) < \theta_0 + \varepsilon.$$

It follows that, at smooth points in (t_1, t_2),

$$\theta'(t) = -\sin(\theta(t)) < -\sin(\theta_0) < -\varepsilon.$$

We must have $t_2 \geq t_1 + 1$, because θ can jump upwards a distance at most ε in an open interval of length 1.

By integration, and taking into account that $\theta(t)$ is defined at bending points by a limiting process, we find that

$$\theta(t_1 + 1) - \theta(t_1) < -\sin(\theta_0) + \varepsilon < 0.$$

Therefore $\theta(t_1 + 1) < \theta(t_1) < \theta_0$. This contradicts the definition of t_1 and completes the proof of Lemma 4.4. □

From Lemma 4.4, we have

$$\forall\, t \in [0, \infty),\ \cos(\theta_0 + \varepsilon) \leq \cos(\theta(t)) \leq 1,$$

while at the smooth points, $\cos(\theta(t)) = s'(t)$. Therefore

$$\cos(\theta_0 + \varepsilon) \leq s'(t) \leq 1,$$

and upon integrating, we obtain

$$\cos(\theta_0 + \varepsilon)(t_2 - t_1) \leq d(\gamma(t_1), \gamma(t_2)) \leq (t_2 - t_1).$$

This shows that γ is a bilipschitz map onto its image, where the image has the induced metric from $\mathbb{H}^3$.

Corollary 4.5. *In Lemma* 4.4, $\varepsilon = 0.73$, *and* $\theta_0 = 0.83$ *can be chosen, since* $\theta_0 + \varepsilon < \pi/2$ *and* $\sin(.83) > 0.73$. *For these choices*, $\cos(\theta_0 + \varepsilon) > 1/100$, *and*

$$|t_1 - t_2|/100 \leq d(\gamma(t_1), \gamma(t_2)) \leq |t_1 - t_2|.$$

Corollary 4.6. *There exists* $\varepsilon > 0$, *with the following property. Suppose that* (Λ, μ) *is a measured lamination of the hyperbolic plane with* $\|(\Lambda, \mu)\| \leq \varepsilon$. (*The measure is allowed to be signed.*) *Then the pleating map* $\Psi_{(\Lambda,\mu)} : \mathbb{H}^2 \to \mathbb{H}^3$ *is bilipschitz so that its image is embedded. Moreover* $\Psi_{(\Lambda,\mu)}$ *has a continuous extension to* $\partial\mathbb{H}^2$ *which maps* $\partial\mathbb{H}^2$ *onto a quasicircle.*

Explicitly, choose ε, θ_0 *so as to satisfy Lemma* 4.4; *for example set* $\varepsilon = 0.73, \theta_0 = 0.83$. *Then, in the respective hyperbolic metrics*,

$$\cos(\theta_0 + \varepsilon) d(u, v) \leq d(\Psi_{(\Lambda,\mu)}(u), \Psi_{(\Lambda,\mu)}(v)) \leq d(u, v).$$

Proof. Choose ε so that there is a pair ε, θ_0 that satisfies Lemma 4.4. Consider first the case that the bending lines are isolated, as we have been assuming. The proof of Lemma 4.4 shows that the pleating mapping is bilipschitz, with the inequality as given.

For a general lamination, we approximate by a finite lamination. The relevant inequalities remain the same.

It is a well-known fact that a bilipschitz map of $\mathbb{D}^2$ onto itself extends to a homeomorphism of $\overline{\mathbb{D}}^2$ which is quasisymmetric on $\partial\mathbb{D}^2$. In fact, even a quasi-isometric map extends to a quasisymmetric map on $\mathbb{S}^1$. The same argument applies to the map of $\mathbb{D}^2$ to a pleated surface in $\mathbb{D}^3$.

This type of phenomenon was first uncovered by Efremovič, Tihomirova and others in the Russian school in the early 1960's (see [15], [16] and [13]). Margulis ([20]) independently proved a version which was similar to De-Spiller's. At much the same time, Mostow ([22]) proved a more general version, where the map on the disk is assumed only to be a quasi-isometry, and does not even have to be continuous. Mostow used such results to prove his famous Rigidity Theorem.

For those who are unfamiliar with the interesting argument, we give a quick sketch of the proof of the Mostow version of these results, at least in the special form in which we need it in our paper. A geodesic in the domain maps to a quasigeodesic in the range. Each such quasigeodesic is at a uniformly bounded Hausdorff hyperbolic distance from some geodesic. This gives a well-defined injective extension $\partial\mathbb{D}^2 \to \partial\mathbb{D}^3$. Given four points in $\mathbb{S}^2$, the six geodesics joining them define an ideal tetrahedron. The shape of the tetrahedron, which is the same as the cross-ratio of the four points, is determined by the hyperbolic distances between opposite edges and similarly for four points in $\mathbb{S}^1$. For a quasi-isometry, large hyperbolic distances in $\mathbb{H}^3$ are controlled up to a multiplicative constant by the corresponding large hyperbolic distances in $\mathbb{H}^2$. For example continuity on the boundary can be proved by fixing three points, $u, v, w \in \partial\mathbb{D}^2$. A point $z \in \overline{\mathbb{D}^2}$ converges to u if and only if the distance from the geodesic zu to the fixed geodesic vw tends to infinity. This implies that the distances between the image quasigeodesics tend to infinity, and therefore that the image of z converges to the image of u. Quasisymmetry can be defined in terms of the effect on cross-ratios. $\square$

Proof of Theorem 4.2(2). We have just proved that Theorem 4.2(2) holds with $c_2 = 0.73$. $\square$

Proof of 4.2(3). By abuse of notation, we are now using c_2 to denote the supremum of all values of c_2 for which 4.2(2) is true. We have shown that $c_2 \geq 0.73$. All that remains is to prove the stated upper bound for c_2.

Return to the number $\theta_0 = 2\arcsin(\tanh(1/2))$ used in Corollary 4.5. We claim that $c_2 \leq \theta_0$. To show this, consider a horizontal horocycle in the upper halfplane. Mark points along this horocycle such that consecutive points are a unit distance apart. Join each pair of consecutive points by a geodesic arc. Then the exterior bending angle between successive geodesic arcs is θ_0; the bending along any half-open interval of length 1 that is transverse to the bending lines is the same.

The relevant region Ω is the region in the complex plane lying between such a curve and its complex conjugate. Then Dome(Ω) is embedded, but it touches itself at infinity. Any increase in angle will result in a self-intersection. $\square$

The proof of 4.2.4 will come at the end of this section. First we must get in position to apply the λ-lemma.

Given a finite lamination (Λ, μ) and $\lambda \in \mathbb{C}$, we have the complex earthquake map $\mathbb{C}E_\lambda = \mathbb{C}E_{(\Lambda,\lambda\mu)} : \overline{\mathbb{D}^2} \to \overline{\mathbb{D}^3}$. This map is discontinuous on the bending lines if and only if the real part of λ is nonzero, but the extension to $\mathbb{S}^1 \to \mathbb{S}^2$ is always continuous. Let $\lambda = x + iy \in \mathbb{C}$. Let (Λ', μ') be the image of (Λ, μ) under the earthquake $E_{(\Lambda,x\mu)}$. We recall that the complex earthquake

$\mathbb{C}E_{(\Lambda,\lambda\mu)} : \mathbb{D}^2 \to \mathbb{D}^3$, is quasi-isometric and in general discontinuous, whereas the pleating map $\Psi_{(\Lambda',y\mu')} : \mathbb{D}^2 \to \mathbb{D}^3$ is bilipschitz.

LEMMA 4.7. *Let (Λ, μ) be a measured lamination. Suppose $U \subset \mathbb{C}$ is an open subset such that, for all $\lambda \in U$, $\|(\Lambda, \lambda\mu)\| < c_2$. Then, for each $z \in \mathbb{S}^1$, $\mathbb{C}E_{(\Lambda,\lambda\mu)}(z)$ is a holomorphic function of $\lambda \in U$.*

In Corollary 4.13 we will produce a fixed, simply-connected U containing $\mathbb{R}$ which is independent of Λ and μ, and for which the hypotheses of the lemma hold whenever $\|\mu\| \leq 1$.

Proof. Let (Λ_n, μ_n) be a sequence of finite measured laminations converging to (Λ, μ), in the sense that, for any continuous function f, with compact support in an interval transverse to Λ, $\int f\, d\mu_n$ converges to $\int f\, d\mu$. We also assume that the norm of each μ_n is less than c_2 and that, for each n, $\Lambda_n \subset \Lambda_{n+1} \subset \Lambda$. We write $\mathbb{C}E_{n,\lambda} = \mathbb{C}E_{(\Lambda_n,\mu_n)}$.

For each $z \in \mathbb{S}^1$, and each $n \in \mathbb{N}$, $\mathbb{C}E_{n,\lambda}(z)$ is a holomorphic function of $\lambda \in U$, since the finite set of Möbius transformations involved in the definition of $\mathbb{C}E_{n,\lambda}$ all depend holomorphically on λ.

Let γ be a geodesic or gap for Λ. For each $z \in \overline{\gamma}$, $\mathbb{C}E_{n,\lambda}(z)$ converges to $\mathbb{C}E_\lambda(z)$. The convergence is known to be uniform on $\gamma \cap K$, where K is a compact subset of $\mathbb{D}^2$ (see, for example [17]). Moreover, the restriction to γ is given by a Möbius transformation. It follows that $\mathbb{C}E_{n,\lambda}(z)$ converges to $\mathbb{C}E_\lambda(z)$, uniformly for $\lambda \in \mathbb{D}^2$ and $z \in \overline{\gamma}$, where the closure of γ is taken in the closed unit disk. Consequently for each fixed $z \in \mathbb{S}^1 \cap \overline{\gamma}$, $\mathbb{C}E_\lambda(z)$ is holomorphic in λ.

Now each map in the family $\{\mathbb{C}E_{n,\lambda}, \mathbb{C}E_\lambda\}$ gives a quasi-isometry $\mathbb{D}^2 \to \mathbb{D}^3$, with constants depending only on $\|\mu\|$. The family $\left\{\mathbb{C}E_{n,\lambda}|\mathbb{S}^1, \mathbb{C}E_\lambda|\mathbb{S}^1\right\}$ is therefore equicontinuous, with constants depending only on $\|\mu\|$. Since the set $\{\overline{\gamma}\}$ is dense in $\mathbb{S}^1$, it follows that $\mathbb{C}E_{n,\lambda}|\mathbb{S}^1$ converges to $\mathbb{C}E_\lambda|\mathbb{S}^1$, and the convergence is uniform in λ and in $z \in \mathbb{S}^1$. Therefore, for $z \in \mathbb{S}^1$, $\mathbb{C}E_\lambda(z)$ is a holomorphic function of λ. □

We now recall the λ-lemma, as treated in Mané–Sad–Sullivan [19], Sullivan–Thurston [27], Bers–Royden [3], and Slodkowski [25]. The strongest of these is Slodkowski's and that has been further expanded to an equivariant form in [14] and, independently, in [26].

DEFINITION 4.8. Let $B \subset \mathbb{S}^2$ be an arbitrary set containing at least three points. Let $\{f_\lambda(z)\}$ denote a family of functions $B \to \mathbb{S}^2$ with parameter $\lambda \in \mathbb{D}^2$. The family $\{f_\lambda\}$ is called a *holomorphic motion* of B if it has the following three properties:

- For each fixed $\lambda \in \mathbb{D}^2$, $f_\lambda : z \in B \mapsto f_\lambda(z) \in \mathbb{S}^2$ is injective.

- For each fixed $z \in B$, the map $\lambda \mapsto f_\lambda(z)$ is a holomorphic map $\mathbb{D}^2 \to \mathbb{S}^2$.
- For each $z \in B$, $f_0(z) = z$.

THEOREM 4.9 (λ-LEMMA). *Let G be a group of Möbius transformations which preserves a subset $B \subset \mathbb{S}^2$. Suppose $\{f_\lambda\}$ is a holomorphic motion of B. Suppose further that, for each $\lambda \in \mathbb{D}^2$, we have an isomorphism $G \to G_\lambda$ with a group of Möbius transformations of $f_\lambda(B)$, such that f_λ is G-equivariant.*

Then

1. *$f_\lambda(z)$ is jointly continuous in λ and z.*
2. *For fixed $\lambda \in \mathbb{D}^2$, $f_\lambda(z)$ ($z \in B$) is the restriction to B of a K^*_λ-quasiconformal and G-equivariant mapping $f^*_\lambda : \mathbb{S}^2 \to \mathbb{S}^2$ where*
$$K^*_\lambda = \frac{1+|\lambda|}{1-|\lambda|}.$$
3. *With $\lambda \in \mathbb{D}^2$ and $z \in \mathbb{S}^2$, $\{f^*_\lambda(z)\}$ gives a holomorphic motion of $\mathbb{S}^2$.*

Note that continuity in z is *not* assumed; instead, continuity is a conclusion. In particular, if $B = \mathbb{D}^2$, then $f_\lambda(B)$ is a quasidisk.

In order to apply the λ-lemma to our situation, we need to identify a set of $\lambda \in \mathbb{C}$ for which $\mathbb{C}E_{(\Lambda,\lambda\mu)}|\mathbb{S}^1$ is injective. This will require some estimates from hyperbolic geometry.

LEMMA 4.10. *Let $\alpha : \mathbb{R} \to \mathbb{H}^2$ and $\beta : \mathbb{R} \to \mathbb{H}^2$ be disjoint geodesics in $\mathbb{H}^2$ with arclength parameters. Let $A = \alpha(0)$ and $B = \alpha(x)$, with $x > 0$. Let $\pi : \mathbb{H}^2 \to \beta$ be orthogonal projection onto the geodesic β. Let $P = \pi(A)$ and $Q = \pi(B)$. Let $u = d(A,P)$ and $v = d(B,Q)$. Then*
$$\sinh(u) \leq e^x \sinh(v), \text{ and } \sinh(v) \leq e^x \sinh(u).$$
Equality is attained in one of the two equalities if and only if α and β meet at infinity.

Proof. Let $\theta = \angle PAB$. Working in the Minkowski model, we find a general identity for a quadrilateral with two right angles,

$$\sinh(v) = \cosh(x)\sinh(u) - \cos(\theta)\sinh(x)\cosh(u). \tag{4.10.6}$$

When α and β meet at ∞, the quadrilateral becomes a right triangle for which $\tanh(u) = \cos(\phi)$. Here $\phi < \pi/2$ is the smallest possible value for θ. Therefore we have also
$$-\tanh(u) \leq \cos(\theta) \leq \tanh(u).$$
With the help of this latter inequality, the claimed result follows from Identity 4.10.6. □

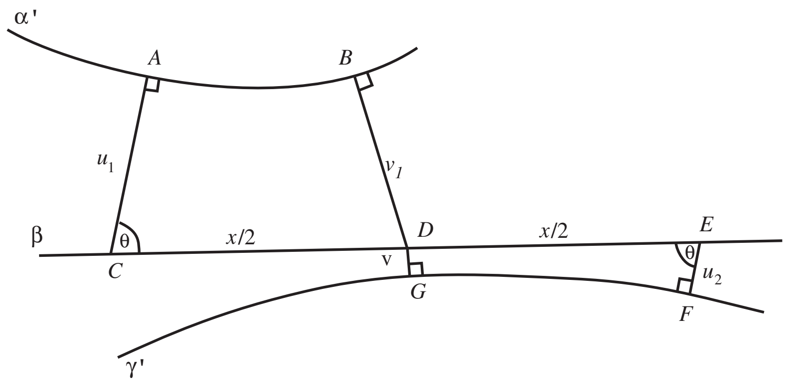

Figure 9: This picture illustrates the proof of Lemma 4.11.

LEMMA 4.11. *Let* α, β *and* γ *be disjoint geodesics, such that* β *separates* α *from* γ. *Let* E *be an earthquake along* β *through a distance* $x \in \mathbb{R}$. *Let* α' *and* γ' *be the images of* α *and* γ *under* E. *Then* $\sinh(d(\alpha', \gamma')) \leq e^{|x|} \sinh(d(\alpha, \gamma))$ *and* $d(\alpha', \gamma') \leq e^{|x|/2} d(\alpha, \gamma)$.

Proof. If α and γ have a common point at infinity, then the inequalities say that $0 \leq 0$. So we assume that α and γ do not have a common endpoint at infinity. In this case, and if in addition $x \neq 0$, the inequalities in the statement are strict.

By reflecting in β if necessary, we may assume that $x > 0$. We take a shortest geodesic segment σ from a point of α to a point of γ. The earthquake breaks σ into two segments, AC of length u_1 and EF of length u_2, where $C, E \in \beta$, $A \in \alpha'$ and $E \in \gamma'$ and $d(C, E) = x$ (see Figure 9). Since σ is a geodesic, we have $\angle ACE = \angle FEC$. Let D be the midpoint of CE. We drop perpendiculars from D to α' and γ', obtaining segments DB and DG of lengths v_1 and v_2 respectively.

Clearly $d(\alpha', \gamma') \leq v_1 + v_2$. By Lemma 4.10, $\sinh(v_i) \leq e^{x/2} \sinh(u_i)$. Also $\cosh^2(v_i) = 1 + \sinh^2(v_i) < e^x \cosh^2(u_i)$, so that $\cosh(v_i) < e^{x/2} \cosh(u_i)$. Putting this together, and using the sum formula for sinh, we obtain

$$\sinh(d(\alpha', \gamma')) \leq \sinh(v_1 + v_2) < e^x \sinh(u_1 + u_2) = e^x \sinh(d(\alpha, \gamma)).$$

To prove the final inequality, note that

$$\sinh(v_i) \leq e^{x/2} \sinh(u_i) < \sinh(e^{x/2} u_i),$$

so that $v_i < e^{x/2} u_i$. $\square$

The next estimate is an improvement on that found in [24].

THEOREM 4.12. *Let α and γ be leaves of a geodesic lamination (Λ, μ) with signed measure μ. Let X be an open transverse geodesic segment, with endpoints on α and γ. Let $x = |\mu|(X)$. Let α' and β' be the images of α and β under the earthquake specified by μ. Then $\sinh(d(\alpha', \beta')) \leq e^x \sinh(d(\alpha, \beta))$, and $\sinh(d(\alpha, \beta)) \leq e^x \sinh(d(\alpha', \beta'))$. Also, $d(\alpha', \beta') \leq e^{x/2} d(\alpha, \beta)$, and $d(\alpha, \beta) \leq e^{x/2} d(\alpha', \beta')$.*

Proof. Only leaves strictly between α and γ make any difference to the computation. We may therefore assume that α and γ carry no atomic measure, and that all other leaves separate α from γ. We may therefore approximate (Λ, μ) by a finite lamination with leaves $\beta_1, \ldots, \beta_n$ which lie strictly between α and γ and are numbered consecutively. Let E_i be the hyperbolic transformation with axis β_i, translating a distance $x_i \in \mathbb{R}$. We may assume that $x = \sum |x_i|$.

We may assume that the earthquake fixes α. We have

$$E_1 \ldots E_n(\gamma) = \gamma'.$$

For $1 \leq i \leq n+1$, we define $\gamma_i = E_i \ldots E_n(\gamma)$. By Lemma 4.11, we have $\sinh(d(\alpha, \gamma_i)) \leq e^{|x_i|} \sinh(d(\alpha, \gamma_{i+1}))$. It follows that

$$\sinh(d(\alpha, \gamma')) \leq e^x \sinh(d(\alpha, \gamma)).$$

The reverse inequality follows since γ' is obtained from γ by applying the inverse earthquake. The proof of the other inequalities is virtually the same. $\square$

We can apply Theorem 4.12 to construct the region U for Lemma 4.7.

COROLLARY 4.13. *Let (Λ, μ) be a nontrivial measured lamination, μ a signed measure and c_2 any constant that satisfies condition* 4.2(2); *for example, $c_2 = 0.73$. Let*

$$f(u, x) = \min\left(\operatorname{arcsinh}(e^{|x|} \sinh(u)), e^{|x|/2} u\right). \tag{4.13.7}$$

Let $U \subset \mathbb{C}$ be the simply connected region given by

$$U = \left\{ x + iy : |y| < \frac{c_2}{f(1, |x|) + 1} \right\}.$$

Then, for each $\lambda \in U$, $\|(\Lambda, \lambda\mu / \|\mu\|)\| < c_2$ and the corresponding pleating map $\Psi_{(\Lambda', \lambda\mu'/\|\mu\|)}$ is a bilipschitz embedding. (*The notation is explained just before Equation* 4.0.4.)

The region U is shown in Figure 10.

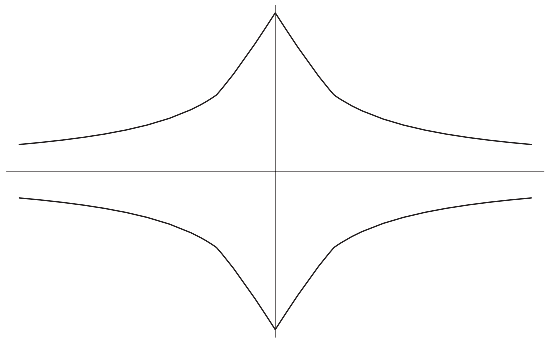

Figure 10: The region U. The highest point in the boundary of the region is $(0, c_2)$ and the lowest point is $(0, -c_2)$.

Proof. Set $\lambda = x + iy$. Let E be the earthquake supported on Λ, induced by the measure $x\mu / \|\mu\|$, which has norm $|x|$. Under E, the image lamination is Λ', with transverse measure $E(\mu)$. The measure used for bending is $\mu' = yE(\mu)/\|\mu\|$. We want to estimate

$$\|\lambda\mu/\|\mu\|\| = \|\mu'\| = |y|\,\|E(\mu)\|/\|\mu\|.$$

We need to find the maximum $E(\mu)$-measure of any half-open interval A' of length 1. By Theorem 4.12, a shortest geodesic interval A, meeting exactly the leaves of Λ corresponding to those leaves of Λ' met by A', has length at most $f(1, |x|)$, where f is defined in Equation 4.13.7. The $|\mu|$-measure carried by A is bounded above by $\|\mu\|$ for each unit of length. This means that the $|E(\mu)|$-measure carried by A' is at most $\|\mu\|\,(f(1,|x|)+1)$. Here we have taken account of the fact that Theorem 4.12 deals with an interval of any length, whereas our knowledge of the measure on a transverse interval is confined to intervals of integral length. (So a better estimate could have been found by using the smallest integer greater than or equal to $\|\mu\|\,f(1, |x|\,\|\mu\|)$, but Figure 10 would not have looked so nice.) So $\|E(\mu)\| \leq \|\mu\|\,(f(1,|x|)+1)$. It follows that $\|\mu'\| \leq |y|(f(1,|x|)+1)$. The region U is determined by the requirement that $\|\mu'\| \leq c_2$. ☐

We summarize our work of this section in the following result, which also proves 4.2.4. In the next result, $\lambda\,\|\mu\|$ plays the role that was played by λ in Corollary 4.13.

THEOREM 4.14. *Let (Λ, μ) be any geodesic lamination with signed measure of finite norm $\|\mu\| = \|(\Lambda, \mu)\|$. Let $\lambda \in \mathbb{C}$ be chosen so that $\lambda\,\|\mu\| \in U$.*

1. *The complex earthquake map $\mathbb{C}E_{(\Lambda,\lambda\mu)} : \mathbb{D}^2 \to \mathbb{D}^3$ extends to a holomorphic motion of $\mathbb{S}^1$ in $\mathbb{S}^2$. By Theorem 4.9, the holomorphic motion extends to a holomorphic motion of $\mathbb{S}^2$ in $\mathbb{S}^2$, which we denote by $\Phi_{(\Lambda,\lambda\mu)}$.*

2. *For* $\lambda \|\mu\| \in U$, $\Phi_{(\Lambda,\lambda\mu)} : \mathbb{S}^2 \to \mathbb{S}^2$ *is a quasiconformal homeomorphism. Its maximal dilatation* K *satisfies*

$$K \leq \frac{1 + |h(\lambda \|\mu\|)|}{1 - |h(\lambda \|\mu\|)|},$$

where $h : U \to \mathbb{D}^2$ *is a Riemann map taking* 0 *to* 0.

3. *If there is a group of Möbius transformations which preserves* (Λ, μ), *then the holomorphic motion* ψ *can be taken so that it commutes with* G.

4. *Suppose in addition that* μ *is a nonnegative measure. Set*

$$V = \{w \in \mathbb{C} : |w| < 1/3\}.$$

If $\lambda \|\mu\| \in U$, *then the image of* $\mathbb{C}E_{(\Lambda,\lambda\mu)}$ *is the dome of a region* Ω_λ. *It is also the image of* $\Psi_{(\Lambda',\lambda\mu')}$. *Any nonnegative real-valued measured lamination* (Λ, μ) *for which* $h(\|\mu\|) \in V$, *determines an embedded pleated surface over a floor* Ω *for which* $K_{\mathrm{eq}} \leq 2$ *holds.*

Proof. It follows from Theorem 4.2(2) that the pleated surface which is the image of $\Psi_{(\Lambda',\lambda\mu')}$ or, equivalently, of $\mathbb{C}E_{(\Lambda,\lambda\mu)}$, is the dome of a simply connected open region $\Omega_\lambda \subset \mathbb{S}^2$.

According to Lemma 4.7, we have a holomorphic motion of $\mathbb{S}^1$ in $\mathbb{S}^2$, with parameter $\lambda \|\mu\|$ in the open topological disk U. After transforming U to the unit disk by a Riemann mapping h, the hypotheses of the λ-Lemma (Theorem 4.9) are satisfied. We obtain a holomorphic motion $\Phi_{(\Lambda,\lambda\mu)} : \mathbb{S}^2 \to \mathbb{S}^2$, where the dependence on λ is holomorphic. According to [14], we can choose this holomorphic motion equivariantly.

If $\lambda = iy$ is pure imaginary, we have

$$\mathbb{C}E_{(\Lambda,\lambda\mu)} = \Psi_{(\Lambda,y\mu)}$$

since the earthquake component of the factorization is the identity. It follows that the holomorphic motion $\Phi_{(\Lambda,\lambda\mu)}$ agrees with extension of the pleating map to the boundary $\mathbb{S}^1$.

According to Theorem 4.9, $\Phi_{(\Lambda,\lambda\mu)}$ is an equivariant quasiconformal homeomorphism with constant at most 2, since $h(|y| \|\mu\|) \in V$. Recall that $\Psi_{(\Lambda,\lambda\mu)} : \mathbb{D}^2 \to \mathrm{Dome}(\Omega_\lambda)$ is an equivariant isometry. Hence

$$\Phi_{(\Lambda,\lambda\mu)} \circ \left(\Psi_{(\Lambda,\lambda\mu)} : \mathbb{D}^2 \to \mathrm{Dome}(\Omega_\lambda)\right)^{-1}$$

is an equivariant quasiconformal homeomorphism which extends to the identity on the boundary, and the constant of quasiconformality is at most 2. This proves that $K_{\mathrm{eq}} \leq 2$ for such regions. $\square$

5. Boundary values

The object of this section is to show that all quasiconformal homeomorphisms are quasi-isometries. This useful general fact is established in Theorem 5.1.

An indication of this phenomenon is seen in the Ahlfors-Beurling and Douady-Earle extensions to $\mathbb{D}^2$ of quasisymmetric homeomorphisms of $\mathbb{S}^1$. These are known to be quasi-isometries, with constants determined by the quasisymmetric constants. Our work also controls constants, and we obtain a sharp result for the multiplicative constant.

In the converse direction, it is clear that we cannot hope for a quasi-isometry to be quasiconformal: a quasi-isometry doesn't even need to be continuous, and quasiconformality is a local property.

In 5.3 we will apply the result to give an upper bound for best lipschitz constants.

THEOREM 5.1. *Suppose $f : \mathbb{D}^2 \to \mathbb{D}^2$ is a K-quasiconformal map. Let d denote the hyperbolic metric. Then there exists $a = a(K) > 0$ such that*

$$\forall\, z, w \in \mathbb{D}^2,\ (d(z,w) - a)/K \leq d(f(z), f(w)) \leq K d(z,w) + a, \tag{5.1.8}$$

where $a = a(K) > 0$. Also, $a(K) = K \log 2$ if $1 < K \leq 2$ and $a(K) = 2.37(K-1)$ if $K > 2$.

Proof. We may assume that $f(0) = 0$. It suffices to estimate $d(0, f(z))$. By the Hersch-Pfluger distortion theorem (see [18, p. 64]),

$$\forall\, z \in \mathbb{D}^2,\ |f(z)| \leq \phi_K(|z|). \tag{5.1.9}$$

Here ϕ_K is the distortion function which only depends on the maximal dilatation K. In particular for $K = 2$,

$$\phi_2(t) = \frac{2\sqrt{t}}{1+t}, \quad 0 \leq t \leq 1.$$

For a 2-quasiconformal map f, Inequality 5.1.9 implies

$$|f(z)| \leq \frac{2\sqrt{|z|}}{1+|z|}. \tag{5.1.10}$$

Recall that in the unit disk model, hyperbolic distance is given by the formula $d(0,z) = \log\left(\frac{1+|z|}{1-|z|}\right)$. Introducing the strictly increasing function $g : [0,1] \to [1,2]$ given by $g(u) = \frac{(1+u)^2}{1+u^2}$ and substituting Inequality 5.1.10, we find

$$d(0, f(z)) - 2d(0,z) \leq 2\log\left(g\left(\sqrt{|z|}\right)\right) < 2\log 2.$$

From this we conclude that

$$d(0, f(z)) \leq 2d(0,z) + 2\log 2. \tag{5.1.11}$$

This proves the right side of Inequality 5.1.8 when $K = 2$. The left side is proved by applying Inequality 5.1.11 to f^{-1}.

To continue, we need the following lemma.

LEMMA 5.2. *Given $L \geq 1$, and a point p on a hyperbolic geodesic γ, there is an L-quasiconformal diffeomorphism $f_L : \mathbb{H}^2 \to \mathbb{H}^2$ which fixes p and sends γ to itself, stretching hyperbolic distances along γ by a factor L, and stretching all other distances by a factor between 1 and L.*

Proof. We use the upper halfplane model. Take $i = p$ and γ to be a vertical geodesic through i. In polar coordinates about the origin define $f_L : (r, \theta) \mapsto (r^L, \theta)$. Its complex dilatation is $\mu = \frac{L-1}{L+1}\frac{z}{\bar{z}}$. □

Continuing with the proof of Theorem 5.1, we separate the proof into two cases. In the case $1 < K < 2$, apply Lemma 5.2 to $\mathbb{D}^2$, with $L = 2/K$, $p = 0$ and γ equal to the geodesic passing through 0 and $f(z)$. Note that $f_L f$ is 2-quasiconformal and fixes 0. We obtain

$$Ld(0, f(z)) = d(0, f_L f(z)) \leq 2d(0, z) + 2\log 2.$$

Therefore $d(0, f(z)) \leq Kd(0, z) + K\log 2$.

In the case $K > 2$, find the integer $n \geq 2$ such that $2^{n-1} < K \leq 2^n$. According to [1, p. 100], f can be written as the composition of $K^{1/n}$-quasiconformal maps $f_i : \mathbb{D}^2 \to \mathbb{D}^2$, $1 \leq i \leq n$

$$f = f_1 \circ f_2 \circ \cdots \circ f_n.$$

Apply Inequality 5.1.8 to each f_i, with K replaced by $K^{1/n} \leq 2$. We find a constant a_n such that, for all $z, w \in \mathbb{D}^2$, $d(g(z), g(w)) \leq Kd(z, w) + a_n$. The inductive procedure allows us to find a value for a_n in terms of K:

$$a_n = \frac{K^{1/n}\log(2)\,(K-1)}{K^{1/n}-1} \leq \frac{(K-1)\sqrt{2}\log 2}{\sqrt{2}-1}.$$

Here we are also using the fact that

$$K^{1/n} > 2^{(n-1)/n} \geq \sqrt{2}.$$

So, if $K > 2$, we can take $a(K) = 2.37(K-1)$. □

5.3. *Minimal lipschitz constants.* Suppose $g : R \to S$ is a homeomorphism between complete hyperbolic surfaces of finite area. Let L_g be the lipschitz constant of g if g is lipschitz and ∞ otherwise. Let K_g be the dilatation of g if g is quasiconformal and ∞ otherwise.

We obtain a new proof of Wolpert's Theorem (see [31]).

Corollary 5.4. *Let $f : R \to S$ be a homeomorphism between complete hyperbolic surfaces of finite topological type. Set $K(f) = \inf\{K_g : g \simeq f\}$, and $L(f) = \inf\{L_g : g \simeq f\}$, where g varies over all homeomorphisms which are homotopic to f. Then $L(f) \leq K(f)$.*

We remark that in the definition of $K(f)$, "min" replaces "inf" in the light of Teichmüller's famous theorem. In the case of $L(f)$, likewise "min" can be used in place of "inf" because $L(f)$ is achieved by a (not necessarily unique) lipschitz homeomorphism, as shown by Thurston [30, p. 4].

Proof. We may assume without loss of generality that f is the Teichmüller map with $K_f = K(f)$. According to Thurston's unpublished manuscript [30],

$$L(f) = \inf_{\alpha} \frac{\mathrm{length}[f(\alpha)]}{\mathrm{length}[\alpha]},$$

where α is a simple closed geodesic on R of hyperbolic length length$[\alpha]$ and length$[f(\alpha)]$ is the length on S of the geodesic freely homotopic to $f(\alpha)$.

Choose fuchsian representations G_R, G_S for R and S so that $\pi_R : \mathbb{D}^2 \to \mathbb{D}^2/G_R$ and $\pi_S : \mathbb{D}^2 \to \mathbb{D}^2/G_S$. Let $f^* : \mathbb{D}^2 \to \mathbb{D}^2$ be a lift of f. Let α be an arbitrary simple closed geodesic on R. Choose a component α^* of $\pi_R^{-1}(\alpha)$. We can write α^* as the disjoint union of half-closed arcs $\alpha^* = \cup_j \alpha_j$, where $\pi_R : \alpha_j \to \alpha$ is bijective. Applying Theorem 5.1 we find

$$\mathrm{length}[f^*(\alpha_j \cup \alpha_{j+1})] \leq K_f(\mathrm{length}[\alpha_j \cup \alpha_{j+1}]) + a(K_f).$$

Projecting back to the surfaces we have

$$2\mathrm{length}[f(\alpha)] \leq 2K_f(\mathrm{length}[\alpha]) + a(K_f),$$

or,

$$\mathrm{length}[f(\alpha)] \leq K_f \mathrm{length}[\alpha] + \frac{a(K_f)}{2}.$$

More generally,

$$\mathrm{length}[f(\alpha)] \leq K_f(\mathrm{length}[\alpha]) + \frac{a(K_f)}{n}.$$

Since this holds for all indices,

$$\mathrm{length}[f(\alpha)] \leq K_f(\mathrm{length}[\alpha]).$$

Now α is an arbitrarily chosen simple closed geodesic. Consequently Corollary 5.4 follows directly from Thurston's result above. □

Finally we will generalize Corollary 5.4 to the disk.

Theorem 5.5. *Let $f : \overline{\mathbb{D}^2} \to \overline{\mathbb{D}^2}$ be a homeomorphism from the closed disk to itself and set $K(f) = \inf\{K_g : g \simeq f\}$ where g varies over all quasiconformal maps with the same boundary values as f. Set $L(f) = \inf\{L_g : g \simeq f\}$,* *where g now varies over all continuous mappings $\overline{\mathbb{D}^2} \to \overline{\mathbb{D}^2}$, not necessarily homeomorphisms, which have the same boundary values as f. Then $L(f) \leq K(f)$.*

Proof. We may assume that $K(f) = K_f$ is finite. We normalize by fixing three points on $\mathbb{S}^1$. We will assume that all our maps fix these same three points.

Let μ be the Beltrami differential of f. Choose a large genus n and then a fundamental domain $\mathcal{F}_n \in \mathbb{D}^2$ for a closed surface of genus g. We take $\mathcal{F}_n$ to be a regular polygon with $4n$ sides, centre 0 and vertex angles $\pi/2n$. Let G_n be the group of Möbius transformations preserving the unit disk generated by the transformations that pair opposite sides of $\mathcal{F}_n$. Now, G_n is isomorphic to the fundamental group of a closed surface of genus n. Let μ_n be the result of distributing μ around $\mathbb{D}^2$, using the action of G_n. Solving the Beltrami equation gives a quasiconformal homeomorphism $f_n : \mathbb{D}^2 \to \mathbb{D}^2$ which conjugates G_n to another fuchsian group H_n.

As $n \to \infty$, $\mathcal{F}_n \nearrow \mathbb{D}^2$. Since μ_n converges to μ pointwise on $\mathbb{D}^2$, f_n converges to f uniformly. This is most easily seen by reflecting μ_n, μ to the exterior of $\mathbb{D}^2$ and solving the Beltrami equations on $\mathbb{S}^2$. The normalized solutions automatically map $\mathbb{D}^2$ onto itself. Since $\|\mu_n\| \leq \|\mu\|$, and $\mu_n \to \mu$ pointwise a.e., $f_n \to f$ uniformly on S^2 and hence on $\mathbb{D}^2$ (we have normalized them consistently); see [18, Th. 4.1] or [1, Lemma 1, Ch. V]. In particular the boundary values of $\{f_n\}$ converge to those of f.

For each n, let g_n be a G_n-equivariant lipschitz mapping which minimizes the lipschitz constant L_n. By Corollary 5.4, $L_n \leq K$. Since g_n has the same boundary values as f_n, these converge uniformly to boundary values of f.

By the Ascoli theorem, we have a subsequence of $(g_n)_{n\in\mathbb{N}}$ which converges to a limit $g : \mathbb{D}^2 \to \mathbb{D}^2$ which is lipschitz, with lipschitz constant $L \leq K$. Clearly, g has the same boundary values as f. □

A similar technique can be applied to obtain a corresponding result on an arbitrary hyperbolic surface. It would be interesting to know if in Theorem 5.5 one could take g to be a homeomorphism, or even bilipschitz.

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References

[1] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, *Van Nostrand Math. Studies* **10**, D. Van Nostrand, Co., Inc., New York, 1966.

[2] ———, *Conformal Invariants*: *Topics in Geometric Function Theory*, McGraw-Hill Book Co., New York, 1973.

[3] L. Bers and H. L. Royden, Holomorphic families of injections, *Acta Math.* **157** (1986), 259–286.

[4] C. J. Bishop, An explicit constant for Sullivan's convex hull theorem, *Proc. of the* 2002 *Ahlfor-Bers Colloquium*, *Contemp. Math*, to appear.

[5] ———, Divergence groups have the Bowen property, *Ann. of Math.* **154** (2001), 205–217.

[6] ———, Bilipschitz approximations of quasiconformal maps, *Ann. Acad. Sci. Fenn.* **27** (2002), 97–108.

[7] ———, Quasiconformal Lipschitz maps, Sullivan's convex hull theorem, and Brennan's conjecture, *Ark. Math.* **40** (2002), 1–26.

[8] C. J. Bishop and P. W. Jones, Hausdorff dimension and Kleinian groups, *Acta Math.* **179** (1997), 1–39.

[9] M. Bridgeman, Average bending of convex pleated planes in hyperbolic three-space, *Invent. Math.* **132** (1998), 381–391.

[10] ———, Bounds on the average bending of the convex hull boundary of a Kleinian group, *Michigan Math. J.* **51** (2003), 363–378.

[11] M. Bridgeman and R. D. Canary, From the boundary of the convex core to the conformal boundary, *Geom. Dedicata* **96** (2003), 211–240.

[12] R. Canary, The conformal boundary and the boundary of the convex core, *Duke Math. J.* **106** (2001), 193–207.

[13] D. A. De-Spiller, Equimorphisms and quasi-conformal mappings of the absolute, *Dokl. Akad. Nauk SSSR* **194** (1970), 1006–1009.

[14] C. J. Earle, I. Kra, and S. L. Krushkal, Holomorphic motions and Teichmüller spaces, *Trans. Amer. Math. Soc.* **343** (1994), 927–948.

[15] V. A. Efremovič and E. S. Tihomirova, The continuation of an equimorphism to infinity, *Dokl. Akad. Nauk SSSR* **152** (1963), 1051–1053.

[16] ———, Equimorphisms of hyperbolic spaces, *Izv. Akad. Nauk SSSR Ser. Math.* **28** (1964), 1139–1444.

[17] D. B. A. Epstein and A. Marden, Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces, in *Analytical and Geometric Aspects of Hyperbolic Space* (Coventry/Durham 1984), *London Math. Soc. Lecture Note Ser.* **111**, 113–253, Cambridge, Univ. Press, Cambridge, 1987.

[18] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, 2nd edition, Springer-Verlag, New York, 1973.

[19] R. Mañé, P. Sad, and D. Sullivan, On the dynamics of rational maps, *Ann. Sci. Ecole Norm. Sup.* **16** (1983), 193–217.

[20] G. A. Margulis, Isometry of closed manifolds of constant negative curvature with the same fundamental group, *Dokl. Akad. Nauk SSSR* **192** (1970), 736–737.

[21] C. T. McMullen, Complex earthquakes and Teichmüller theory, *J. Amer. Math. Soc.* **11** (1998), 283–320.

[22] G. D. Mostow, *Strong Rigidity of Locally Symmetric Spaces*, *Ann. of Math. Studies* **78**, Princeton Univ. Press, Princeton, NJ, 1973.

[23] H. L. Royden, Automorphisms and isometries of Teichmüller space, in *Advances in the Theory of Riemann Surfaces*, *Ann. of Math. Studies* **66**, 369–383, Princeton Univ. Press, Princeton, NJ, 1971.

[24] D. Šarič, Complex earthquakes are holomorphic, Ph. D. thesis, CUNY, 2001.

[25] Z. Slodkowski, Holomorphic motions and polynomial hulls, *Proc. Amer. Math. Soc.* **111** (1991), 347–355.

[26] ———, Invariant extensions of holomorphic motions, *Abstracts Amer. Math. Soc.* **13** (1992), 259.

[27] D. Sullivan and W. P. Thurston, Extending holomorphic motions, *Acta Math.* **157** (1986), 243–257.

[28] D. P. Sullivan, Travaux de Thurston sur les groupes quasi-fuchsiens et les variétés hyperboliques de dimension 3 fibrés sur S^1, *Lecture Notes in Math.* **842**, Springer-Verlag, New York, 1981.

[29] W. P. Thurston, Hyperbolic structure on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle; available at http://arXiv.org under arXiv:math.GT/9801045.

[30] ———, Minimal stretch maps between hyperbolic surfaces, available at http://arXiv.org under arXiv:math.GT/9801039.

[31] S. Wolpert, The length spectra as moduli for compact Riemann surfaces, *Ann. of Math.* **103** (1979), 323–351.

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