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An isoperimetric inequality for logarithmic capacity of polygons

By Alexander Yu. Solynin and Victor A. Zalgaller*

Abstract

We verify an old conjecture of G. Pólya and G. Szegő saying that the regular n-gon minimizes the logarithmic capacity among all n-gons with a fixed area.

1. Introduction

The logarithmic capacity $\operatorname{cap} E$ of a compact set E in $\mathbb{R}^2$, which we identify with the complex plane $\mathbb{C}$, is defined by

$$(1.1)\qquad -\log \operatorname{cap} E = \lim_{z\to\infty} (g(z,\infty) - \log|z|),$$

where $g(z,\infty)$ denotes the Green function of a connected component $\Omega(E) \ni \infty$ of $\overline{\mathbb{C}} \setminus E$ having singularity at $z=\infty$; see [4, Ch. 7], [7, §11.1]. By an n-gon with $n \geq 3$ sides we mean a simply connected Jordan domain $D_n \subset \mathbb{C}$ whose boundary ∂D_n consists of n rectilinear segments called sides of D_n. A closed n-gon will be denoted by $\overline{D}_n$.

Our principal result is

Theorem 1. *For any polygon D_n having a given number of sides $n \geq 3$,*

$$(1.2)\qquad \frac{\operatorname{cap}^2 \overline{D}_n}{\operatorname{Area} D_n} \geq \frac{\operatorname{cap}^2 \overline{D}_n^*}{\operatorname{Area} D_n^*} = \frac{n\tan(\pi/n)\Gamma^2(1+1/n)}{\pi 2^{4/n}\Gamma^2(1/2+1/n)}$$

with the sign of equality only for the regular n-gons.

In Theorem 1 and below, $\Gamma(\cdot)$ denotes the Euler gamma function and D_n^* stands for the regular n-gon centered at $z=0$ with one vertex at $z=1$.

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In other words, Theorem 1 asserts that the regular closed polygon has the minimal logarithmic capacity among all closed polygons with a fixed number of sides and prescribed area. For $n \geq 5$, this solves an old problem posed by G. Pólya and G. Szegő [6]. For $n = 3, 4$, the problem was solved by Pólya and Szegő themselves [6, p.158]. Their method based on Steiner symmetrization allows them to establish similar isoperimetric inequalities for the conformal radius, torsional rigidity, principal frequency, etc. However it fails for $n \geq 5$ since Steiner symmetrization increases dimension (= number of sides) of a polygon in general. In [6, p.159] the authors note that "to prove (or disprove) the analogous theorems for regular polygons with more than four sides is a challenging task".

For the conformal radius this task was solved in [8], where it was shown that the regular n-gon maximizes the conformal radius among all polygons with a given number $n \geq 3$ of sides and with a prescribed area. The present work proves the Pólya-Szegő conjecture for the logarithmic capacity. For the torsional rigidity and principal frequency the problem is still open.

A similar question concerning the minimal logarithmic capacity among all compact sets with a prescribed perimeter is nontrivial only for convex sets. This question was studied by G. Pólya and M. Schiffer and Chr. Pommerenke, see [7, p. 51, Prob. 11], who proved that a needle (rectilinear segment) is a unique minimal configuration of the problem. Since a needle can be viewed as a degenerate n-gon, there is no difference between the convex polygonal case and the general case. Thus the regular n-gons do not minimize the logarithmic capacity over the set of all n-gons with a prescribed perimeter. To the contrary, they provide the maximal value for this problem; see [9, Th. 10].

Any isoperimetric problem for polygons of a fixed dimension can be considered as a discrete version of an isoperimetric problem among all simply connected (or more general) domains. It is interesting to note that solutions to continuous versions for the above mentioned functionals have been known for a long time; cf. [6]. The discrete problems are much harder. The situation here is opposite to the classical isoperimetric area-perimeter problem, where solution to the continuous version requires much stronger techniques than the discrete case.

The idea of the proof in [8], used also in the present paper, traces back to the classical method of finding the area of a polygon: divide a polygon into triangles and use the additivity property of the area. Although the characteristics under consideration are not additive functions of a set, often they admit a certain kind of "semiadditivity", at least for special decompositions. For instance, the reduced module $m(D, z_0)$ of a polygon D at its point $z_0 \in D$, a characteristic linked with the conformal radius and logarithmic capacity, admits an explicit upper bound B given by a weighted sum of the reduced modules of triangles composing D, each of which has a distinguished vertex at z_0. The precise definitions and formulations will be given in Section 2. This explicit bound B is a complicated combination of functions including the Euler gamma function, which depends on the angles and areas of triangles composing D. For the problem on the conformal radius, it was shown in [8] that the corresponding maximum of B taken among all admissible values of the parameters provides the sharp upper bound for the reduced module $m(D, z_0)$ where Area D is fixed.

For the logarithmic capacity when the same method is applied, the situation is different; the explicit upper bound B contains more parameters and the supremum of B among all admissible decompositions of D into triangles is infinite. Even more, for instance for the regular n-gon there is only one decomposition (into equal triangles) that gives the desired upper bound for the reduced module. All other decompositions lead to a bigger upper bound and therefore should be excluded from consideration if we are looking for a sharp result.

So it is important to select a more narrow subclass of decompositions among which the maximal value of B corresponding to the logarithmic capacity is finite and provides the sharp bound for the considered characteristic of D. This is the subject of our study in Section 3. The selected subclass contains decompositions of D into triangles that are proportional in a certain sense. This result is of independent interest. We present it in our Theorem 2 restricting for simplicity of formulation to the case of convex polygons. The general version for the nonconvex case is given by Theorem 4 in Section 3.

Let D_n be a convex n-gon having vertices $A_1, \dots, A_n, A_{n+1} = A_1$ enumerated in the positive direction on ∂D_n. A system of Euclidean triangles $\{T_k\}_{k=1}^n$ is called admissible for D_n if $T_k \cap D_n \neq \emptyset$, T_k has the segment $[A_k, A_{k+1}]$ as its base, and if for all $k = 1, \dots, n$, T_k and T_{k+1} have a common boundary segment which is an entire side of at least one of these triangles but not necessarily of both of them.

In Section 3, we give a more general definition of admissibility for a system of triangles suitable for nonconvex polygons. For a convex polygon, the definition of admissibility presented above and the definition given in Section 3 are equivalent.

Let α_k denote the angle of T_k opposite the base $[A_k, A_{k+1}]$. An admissible system $\{T_k\}_{k=1}^n$ is called *proportional* if the quotient $\alpha_k/\operatorname{Area} T_k$ does not depend on $k = 1, \dots, n$.

THEOREM 2. *For every convex n-gon D_n there is at least one proportional system $\{T_k\}_{k=1}^n$ that covers D_n, i.e.*

$$\bigcup_{k=1}^{n} \overline{T}_k \supset D_n. \tag{1.3}$$

Theorem 2 is sharp in the sense that there are polygons, for instance, triangles and regular n-gons, that have a unique proportional system satisfying (1.3). For triangles, Theorem 2 provides a good exercise for the course of elementary geometry. It is not difficult to show that any rectangle different from a square admits a parametric family of proportional systems satisfying (1.3). Figures 1a)–1c) show possible types of proportional configurations for a rectangle $\mathcal{R}$: a) a proportional system that does not cover $\mathcal{R}$; b) a proportional covering system consisting of disjoint triangles; c) a proportional covering system consisting of overlapping triangles the union of which is strictly larger than $\mathcal{R}$ (if $\mathcal{R}$ is sufficiently long). Figure 1d), which is a slightly modified version of Figure 1c), gives an example of a proportional system of six triangles for a nonconvex hexagon. As we have already mentioned, the precise definitions for the nonconvex case will be given in Section 3.

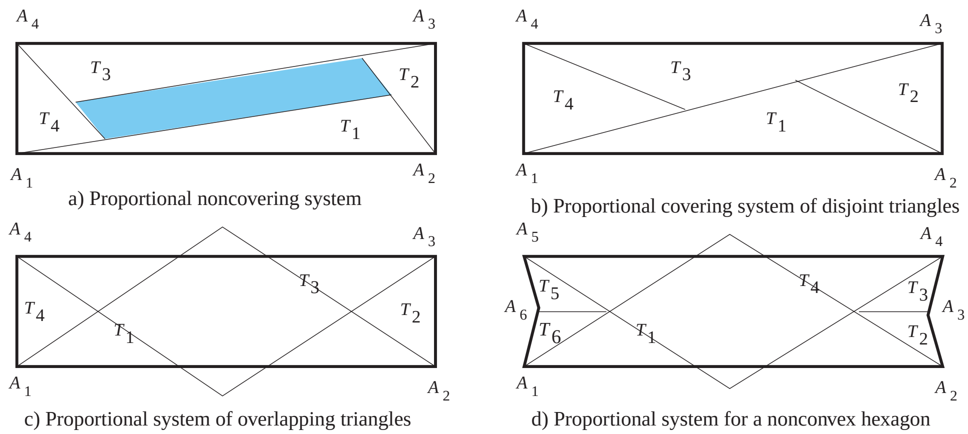

Figure 1. Proportional systems of triangles

To prove a generalization of Theorem 2 for the nonconvex case, we show in Lemma 5 that the family of all proportional systems for D_n admits a natural continuous parametrization. Then the continuity property is used in Lemma 6 to show that at least one system of any continuously parametrized family of admissible systems covers D_n. It is important to note that Theorems 2 and 4 possess counterparts in other cases of proportionality between some two characteristics of a triangle (not necessarily the base angle and the area).

Section 4 finishes the proof of Theorem 1.

The subject of this paper lies at the junction of potential theory, analysis, and geometry. And this work is a natural result of combined efforts of an analyst and a geometer.

We are grateful to the referees for their constructive criticism and many valuable suggestions, which allow us to improve the exposition of our results. In particular, the short proof of Lemma 2 in Section 4 was suggested by one of the referees.

2. Logarithmic capacity and reduced module

There are several other approaches to the measure of a set described by the logarithmic capacity. For example, the geometric concept of transfinite diameter due to M. Fekete and the concept of the Chebyshev's constant from polynomial approximation lead to the same characteristic; cf. [3, § 10.2], [4, Ch. 7].

If a compact set E is connected, then $\Omega(E)$ is a simply connected domain containing the point at ∞. In this case the logarithmic capacity is equal to the *outer radius* $R(E)$ defined as follows. Let

$$f(z) = z + a_0 + a_1 z^{-1} + \dots$$

map $\Omega(E)$ conformally onto $|\zeta| > R$. The radius $R = R(E)$ of the omitted disk is uniquely determined and is called the outer radius of E; see [3, § 10.2], [4, Ch. 7].

The outer radius $R(E)$ can be considered as a characteristic of a simply connected domain $\Omega(E)$ at its point at ∞. Another approach due to O. Teichmüller leads to essentially the same characteristic of a simply connected domain. For $R > 0$ big enough, let $\Omega_R(E)$ be a doubly connected domain between E and the circle $C_R = \{z : |z| = R\}$ and let $\operatorname{mod}(\Omega_R(E))$ denote the module of $\Omega_R(E)$ with respect to the family of curves separating the boundary components of $\Omega_R(E)$; see [5, Ch. 2]. Then there is a finite limit

$$m(\Omega(E), \infty) = \lim_{R \to \infty} (\operatorname{mod}(\Omega_R(E)) - (1/2\pi) \log R) \tag{2.1}$$

called *the reduced module of* $\Omega(E)$ *at* $z = \infty$. The reduced module can be defined for any point $a \in \Omega(E)$ finite or not; cf.[5, Ch. 2] but we shall use this notion with $a = \infty$ only. It is well known [2, § 1.3], [4, Ch. 7] that

$$m(\Omega(E), \infty) = -(1/2\pi) \log \operatorname{cap} E. \tag{2.2}$$

Thus, (1.2) holds if and only if $\Omega(\overline{D}_n^*)$ has the maximal reduced module at ∞ among all domains $\Omega(\overline{D}_n)$ corresponding to polygons D_n such that $\operatorname{Area} D_n = \operatorname{Area} D_n^*$.

As mentioned in the introduction, to prove Theorem 1 we apply the method developed in [8], [10] based on a special triangulation of $\Omega(E)$.

By a trilateral $D = D(a_0, a_1, a_2)$ we mean a simply connected domain $D \subset \overline{\mathbb{C}}$ having three distinguished points a_0, a_1, and a_2 called *vertices* on its boundary. Each trilateral will have a distinguished side called *the base*; the opposite vertex and angle will be called the base vertex and the base angle respectively. For our purposes it is enough to deal with trilaterals having the vertex a_0 at ∞ with a piecewise smooth Jordan boundary such that $l_R = D \cap C_R$ contains only one connected component for all $R > 0$ sufficiently large. Let

$D_R = D \cap \mathbb{U}_R$, where $\mathbb{U}_R = \{z : |z| < R\}$. Considering D_R as a quadrilateral with distinguished sides $\widehat{a_1 a_2}$ and l_R, let $\operatorname{mod}(D_R)$ denote the module of D_R with respect to the family of curves separating $\widehat{a_1 a_2}$ from l_R in D_R; cf. [5, Ch. 2]. Let D have an inner angle $0 < \varphi \le 2\pi$ at $a_0 = \infty$. The limit

$$m(D; \infty | a_1, a_2) = \lim_{R \to \infty} (\operatorname{mod}(D_R) - (1/\varphi) \log R), \tag{2.3}$$

provided that it exists and is finite, is called *the reduced module of D at $a_0 = \infty$*. This notion was introduced in [8]. In [11] some sufficient conditions for the existence of the limit in (2.3) are given. In this paper we deal with rectilinear trilaterals only which guarantees existence of all the reduced modules considered below.

Regarding the infinite circular sector $P = P(\rho, \alpha) = \{z : |z| > \rho, 0 < \arg z < \alpha\}$, $\rho > 0$, $0 < \alpha \le 2\pi$ and the upper half-plane $\mathbb{H} = \{z : \Im z > 0\}$ as trilaterals with vertices ∞, ρ, $\rho e^{i\alpha}$ and ∞, 0, ρ, respectively, and computing the corresponding limits in (2.3), we get,

$$m(P; \infty | \rho, \rho e^{i\alpha}) = -(1/\alpha) \log \rho, \quad m(\mathbb{H}; \infty | 0, \rho) = (1/\pi) \log(4/\rho), \tag{2.4}$$

which provides two useful examples of the reduced modules.

The change in the reduced module under conformal mapping can be worked out by means of a standard formula [8], [11]: if a function $f(\zeta) = A\zeta^\alpha (1 + o(1))$ with $\alpha > 0$, $A \ne 0$, and $o(1) \to 0$ as $\zeta \to \infty$ maps the upper half-plane $\mathbb{H}$ conformally onto a trilateral $D = D(a_0, a_1, a_2)$, $a_0 = \infty$ such that $f(\infty) = \infty$, $f(0) = a_1$, $f(1) = a_2$, then

$$m(D; \infty | a_1, a_2) = (1/\pi) \log 4 - (1/(\alpha\pi)) \log |A|. \tag{2.5}$$

Let $T_1, \ldots, T_n$ be pairwise disjoint trilaterals in a simply connected domain D, $\infty \in D \subset \overline{\mathbb{C}}$, such that T_k has a vertex a_0^k at ∞ and the opposite side $\widehat{a_1^k a_2^k}$ on ∂D; see Figure 2, where for simplicity the point at ∞ is represented by a finite point a_0. The next result from [8] linking the reduced module of D with the reduced modules of trilaterals of its decomposition, is basic for our further considerations.

THEOREM 3 ([8]). *Let T_k have an angle $0 < 2\pi\alpha_k < 2\pi$ at the vertex a_0^k and for every $k = 1, \ldots, n$ let the reduced module of T_k at ∞ exist. If $\sum_{k=1}^n \alpha_k = 1$, then*

$$m(D, \infty) \le \sum_{k=1}^{n} \alpha_k^2 m(T_k; \infty | a_1^k, a_2^k). \tag{2.6}$$

Let f map D conformally onto $\mathbb{U}^* = \overline{\mathbb{C}} \setminus \overline{\mathbb{U}}_1$ such that $f(\infty) = \infty$. Equality occurs in (2.6) *if and only if for every $k = 1, \ldots, n$, $f(T_k)$ is an infinite circular sector* (*of opening $2\pi\alpha_k$*) *and if the vertices of T_k correspond under the mapping f to the geometric vertices of this sector.*

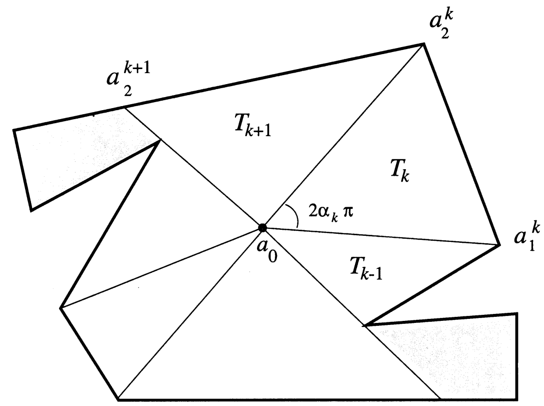

Figure 2. Decomposition into trilaterals

The proof of (2.6) in [8] is based on basic properties of the extremal length. Another approach to more general problems on the extremal decomposition developed by V. N. Dubinin [2] uses the theory of capacities.

Now we consider an instructive example that is important for what then follows. Up to the end of the paper all considered trilaterals will be rectilinear triangles (finite or not) having their geometric vertices as the distinguished boundary points. In this case we shall use the terms "triangle" and "infinite triangle" instead of "trilateral". Thus, everywhere below, "triangle" means a usual Euclidean triangle.

For $\alpha > 0$, $\beta_1 > 0$, $\beta_2 > 0$ such that

$$\beta_1 + \beta_2 = 1 + 2\alpha,$$

and $a > 0$, let $T = T(\alpha, \beta_1, a)$ be the triangle having vertices at $a_0 = 0$, $a_1 = a$, and $a_2 = e^{i2\pi\alpha}(a \sin \pi\beta_1 / \sin \pi\beta_2)$ and the side $[a_1, a_2]$ as its base. Then T has interior angles $2\pi\alpha$, $\pi(1 - \beta_1)$, and $\pi(1 - \beta_2)$ at the vertices a_0, a_1, and a_2, respectively. Let $V_\alpha = \{z : 0 < \arg z < 2\pi\alpha\}$, and let $S(\alpha, \beta_1, a) = V_\alpha \setminus \overline{T}$. Then $S = S(\alpha, \beta_1, a)$ is an infinite rectilinear triangle having vertices at $a_\infty = \infty$, a_1, and a_2, which will be called *the sector associated with* T. In Section 3, the notion of the associated sector will be used in a more general context.

To find the reduced module $m(S; \infty | a_1, a_2)$, we consider the Schwarz-Christoffel function

$$f(\zeta) = a - e^{-i\pi\beta_1}\, C \left(\int_0^\zeta t^{\beta_2 - 1}(1 - t)^{\beta_1 - 1}\, dt - B(\beta_1, \beta_2) \right) \tag{2.7}$$

with

$$C = \frac{a \sin 2\pi\alpha}{\sin \pi\beta_1 B(\beta_1, \beta_2)},$$

where $B(\cdot, \cdot)$ denotes the Euler beta function. The function f maps the upper half-plane $\mathbb{H}$ conformally onto the infinite triangle S such that $f(\infty) = \infty$, $f(1) = a_1$, $f(0) = a_2$. From (2.7),

$$f(\zeta) = (C/2\alpha)\zeta^{2\alpha} + \text{ constant } + o(1), \tag{2.8}$$

where $o(1) \to 0$ as $\zeta \to \infty$.

From (2.5), (2.7), and (2.8), using the second equality in (2.4) with $\rho = 1$, we obtain the desired formula for the reduced module of S:

$$m(S; \infty | a_1, a_2) = \frac{1}{2\pi\alpha} \log \frac{2^{4\alpha+1}\alpha B(\beta_1, \beta_2) \sin \pi\beta_2}{a \sin 2\pi\alpha}. \tag{2.9}$$

Let $s = \operatorname{Area} T$ be the area of the triangle $T = T(\alpha, \beta_1, a)$. Then from elementary trigonometry,

$$a = \left[\frac{2s \sin \pi\beta_2}{\sin 2\pi\alpha \sin \pi\beta_1}\right]^{1/2}.$$

Substituting this in (2.9), we get

$$m(S; \infty | a_1, a_2) = \frac{1}{2\pi\alpha} \log \frac{2^{4\alpha+1}\alpha B(\beta_1, \beta_2)(\sin \pi\beta_1 \sin \pi\beta_2)^{1/2}}{(2s \sin 2\pi\alpha)^{1/2}}. \tag{2.10}$$

For a fixed α, $0 < \alpha < 1/2$, and $s > 0$, let $F(\beta_1)$ denote the right-hand side of (2.10) with $\beta_2 = 1 + 2\alpha - \beta_1$ regarded as a function of β_1, $2\alpha < \beta_1 < 1$. The next lemma shows that F is concave in $2\alpha < \beta_1 < 1$. This implies, in particular, that the isosceles infinite triangle $S(\alpha, 1/2 + \alpha, a)$ has the maximal reduced module among all infinite triangles $S(\alpha, \beta_1, a)$ with fixed angle $2\pi\alpha$ and fixed area s of $T(\alpha, \beta_1, a)$.

Lemma 1. *Let $0 < \alpha < 1/2$ and $s > 0$ be fixed. Then $F(\beta_1)$ is strictly concave in $2\alpha < \beta < 1$ and satisfies the equation $F(\beta_1) = F(\beta_2)$ for $2\alpha < \beta_1 < 1$. In particular,*

$$F(\beta_1) < F(1/2 + \alpha) = \frac{1}{2\pi\alpha} \log \frac{4^\alpha \alpha B(1/2, 1/2 + \alpha)}{(s \tan \pi\alpha)^{1/2}} \tag{2.11}$$

for $2\alpha < \beta_1 < 1$ such that $\beta_1 \neq 1/2 + \alpha$.

Proof. Since $B(\beta_1, \beta_2) = \Gamma(\beta_1)\Gamma(\beta_2)/\Gamma(\beta_1 + \beta_2)$ and $\beta_1 + \beta_2 = 1 + 2\alpha$, (2.10) implies

$$F(\beta_1) = \frac{1}{2\pi\alpha} \log \frac{2^{4\alpha+1}\alpha\Gamma(\beta_1)\Gamma(\beta_2)\,(\sin \pi\beta_1 \, \sin \pi\beta_2)^{1/2}}{\Gamma(1 + 2\alpha)\,(2s \sin 2\pi\alpha)^{1/2}}. \tag{2.12}$$

Using the reflection formula

$$\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z,$$

from (2.12) we obtain

$$F(\beta_1)=\frac{1}{4\pi\alpha}\log\frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1-2\alpha)\Gamma(\beta_2-2\alpha)}+\frac{1}{2\pi\alpha}\log\frac{2^{4\alpha+1}\pi\alpha}{\Gamma(1+2\alpha)\,(2s\sin 2\pi\alpha)^{1/2}},$$

where $\beta_2=1+2\alpha-\beta_1$ and the second term does not depend on β_1. Differentiating twice, we find

$$F''(\beta_1)=\frac{1}{4\pi\alpha}\left[\psi'(\beta_1)-\psi'(\beta_1-2\alpha)+\psi'(\beta_2)-\psi'(\beta_2-2\alpha)\right]<0, \tag{2.13}$$

which is negative because $\psi'(z)=\sum_{k=0}^{\infty}(t+k)^{-2}$ strictly decreases for $t>0$ ([1, p. 45]). Here and below, ψ denotes the logarithmic derivative of the Euler gamma function. Inequality (2.13) shows that $F(\beta_1)$ is strictly concave.

Since $\beta_2=1+2\alpha-\beta_1$, the symmetry formula $F(\beta_1)=F(1+2\alpha-\beta_1)$ follows immediately from (2.10). Symmetry and concavity properties imply that F takes its maximal value at $\beta_1=1/2+\alpha$. Substituting $\beta_1=1/2+\alpha$ in (2.10) and using the formula $B(1/2+\alpha,1/2+\alpha)=4^{-\alpha}B(1/2,1/2+\alpha)$, we get (2.11), and the lemma follows. □

Let $S_n=S(1/n,1/2+1/n,a)$ with $a=(2s/\sin(2\pi/n))^{1/2}$. Then (2.11) with $\alpha=1/n$, $\beta_1=1/2+1/n$ gives

$$m(S_n;\infty|a_1,a_2)=\frac{n}{4\pi}\log\frac{\pi 4^{2/n}\Gamma^2(1/2+1/n)}{s\tan(\pi/n)\Gamma^2(1/n)}.$$

The latter relation combined with the assertion on the equality cases in Theorem 3 leads to the well-known formula for the reduced module of the exterior of the regular n-gon $D_n^*(A)$ having the area A; cf. [6, p.273]:

$$m(\Omega(\overline{D}_n^*(A)),\infty)=\frac{1}{4\pi}\log\frac{\pi}{A}\frac{4^{2/n}n\Gamma^2(1/2+1/n)}{\Gamma^2(1/n)\tan(\pi/n)}. \tag{2.14}$$

The next lemma treats $m(\Omega(\overline{D}_n^*(A)),\infty)$ as a function of the number of sides of $D_n^*(A)$.

Lemma 2. *For a fixed area A, the reduced module $m(\Omega(\overline{D}_n^*(A)),\infty)$ is strictly increasing in n.*

Lemma 2 easily follows from the concavity result of Lemma 7, and the proof is given in Section 4.

3. Triangular covers of a polygon

To prove Theorem 1 for the nonconvex polygons, we need a generalization of Theorem 2 for this case. First we fix terminology and necessary notation. Let T be a triangle with the base $[a_1, a_2]$ and the base vertex a_0. Let V be the smallest infinite sector with the vertex at a_0 that contains T. Then the infinite triangle $S = V \setminus \overline{T}$ having the base $[a_1, a_2]$ and the base vertex $a_\infty = \infty$ will be called *the sector associated with* T.

Let D be an n-gon with vertices $A_1, \dots, A_n$. Throughout this section we use the following conventions concerning numbering:

i) Cyclic convention: if a system $\{x_k\}_{k=1}^n$ contains $n \geq 1$ elements, then $x_{n+1} := x_1$, $x_0 := x_n$, etc.

ii) Positive orientation convention: numeration of geometric objects, e.g. vertices, angles, sides of a polygon D, triangles covering D, etc. agrees with the positive orientation on ∂D.

A triangle T having an associated sector S is called *admissible* for D if $T \cap D \neq \emptyset$, the base of T lies on ∂D (the base of T need not consist of an entire side of D), $S \cap D = \emptyset$, and if each (closed) side (not base!) of S contains at least one vertex of D. Of course, the first condition follows from the second and third conditions.

A system of triangles $\{T_i\}_{i=1}^m$ is called *admissible* for D if each T_i is admissible, the associated sectors S_i are pairwise disjoint and if $\cup_{i=1}^m \overline{S}_i$ covers the complement of the convex hull $\hat{D}$ of D.

If T_i has the base angle α_i, which is equal to the angle of S_i at $z = \infty$, the latter conditions imply that $\sum_{i=1}^m \alpha_i = 2\pi$.

It is important to emphasize that for the case of convex polygons this definition of admissibility is equivalent to the definition of the admissibility of a system of triangles given in the introduction.

An admissible system $\{T_i\}_{i=1}^m$ is called *regular* if for $i = 1, \dots, m$, each side of S_i contains only one vertex of D.

Let α_i and σ_i denote the base angle and area of T_i. By the *coefficient* of T_i we mean the quotient $k_i = \alpha_i / \sigma_i$. An admissible system $\{T_i\}_{i=1}^m$ is called *proportional* if $k_1 = \ldots = k_m$.

In this terminology, the system of triangles shown in Figure 1d) is admissible, regular, and proportional, which covers the hexagon for which it is constructed.

The purpose of this section is to prove the following theorem, which includes Theorem 2 as a special case.

THEOREM 4. *For every n-gon D whose convex hull $\hat{D}$ has $\hat{n} \geq 3$ sides, there is at least one proportional system $\{T_i\}_{i=1}^m$, $\hat{n} \leq m \leq n$, that covers D,* *i.e.*

$$\bigcup_{i=1}^{m} \overline{T}_i \supset D. \tag{3.1}$$

In particular,

$$\sum_{i=1}^{m} \operatorname{Area} T_i \geq \operatorname{Area} D.$$

The proof of Theorem 4 will be given after Lemmas 5 and 6 which study the family of all proportional systems $\{T_i\}$ admissible for a given n-gon D in general position. The latter means that no three vertices of D belong to the same straight line and no side or diagonal is parallel to any other side or diagonal. For such D, we show in Lemma 5 that the set of all proportional systems admits a natural continuous parametrization.

To prove Lemma 5, we need the following variant of the standard implicit function theorem.

LEMMA 3. *Let u_i, $i = 1, \dots, n+1$, be real-valued functions having continuous partial derivatives in a neighborhood of $x^0 \in \mathbb{R}^{n+1}$. Let $u_i = u_i(x_{i-1}, x_i, x_{i+1})$, $i = 2, \dots, n+1$, $u_1 = u_1(x_1, x_2)$ and suppose that the u_i do not depend on the other variables. If for $x = x^0$,*

$$u_1(x) = u_2(x) = \ldots = u_{n+1}(x) \tag{3.2}$$

and if the partial derivatives $u_{i,j}$ satisfy

$$u_{i,i+1}(x^0) > 0, \quad u_{i,j}(x^0) \leq 0 \tag{3.3}$$

for all i and $j \neq i+1$, then for every x_1 in some small interval $(x_1^0 - \delta, x_1^0 + \delta)$ equations (3.2) define a unique solution $x_2 = x_2(x_1), \dots, x_{n+1} = x_{n+1}(x_1)$ continuously differentiable in $(x_1^0 - \delta, x_1^0 + \delta)$ and such that $x_i(x_1^0) = x_i^0$.

Let the functions u_i satisfy the following additional assumptions:

1) *For every $u_i(x_{i-1}, x_i, x_{i+1})$ depending on three parameters, the neighbors $u_{i-1}(x_{i-1}, x_i)$ and $u_{i+1}(x_{i+1}, x_{i+2})$ depend on two parameters each and $u_{i-1,i-1}(x^0) = 0$, $u_{i,i-1}(x^0) < 0$.*

2) *If $u_i(x_i, x_{i+1})$ and $u_{i+1}(x_{i+1}, x_{i+2})$ depend on two parameters each, then $u_{i+1,i+1}(x^0) < 0$.*

Then

$$x_j'(x_1) > 0 \tag{3.4}$$

for every index j such that $u_j = u_j(x_j, x_{j+1})$ depends on two parameters and every $x_1 \in (x_1^0 - \delta, x_1^0 + \delta)$.

Proof. Setting $v_i := u_{i+1} - u_1$, we consider the equations

$$v_i(x_1, \dots, x_{n+1}) = 0, \quad i = 1, \dots, n. \tag{3.5}$$

The assumptions of the lemma imply that the Jacobian $\Delta_n = |v_{i,j}(x^0)|$ in variables $x_2, \dots, x_{n+1}$ has the following form

$$\Delta_n = \begin{vmatrix} - & + & & & & & & & & \\ - & \div & + & & & & & & & \\ - & \div & \div & + & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ - & \div & \cdot & \cdot & \cdot & \cdot & \div & \div & + & \\ - & \div & \cdot & \cdot & \cdot & \cdot & \cdot & \div & \div & + \\ - & \div & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \div & \div \end{vmatrix} \tag{3.6}$$

Here $+$, $-$, and $\div$ represent positive, negative, and nonpositive elements, respectively. The blank spaces are supposed to be filled with zeros.

We claim that

$$\Delta_n \neq 0 \quad \text{and} \quad \Delta_n / |\Delta_n| = (-1)^n. \tag{3.7}$$

The proof is by induction. For $n = 1, 2, 3$ the result is obvious. Assume the assumption holds true for the dimension $n-1$. Then expanding Δ_n in its last column, we get

$$\Delta_n = c_{n,n}\Delta'_{n-1} - c_{n-1,n}\Delta''_{n-1}, \tag{3.8}$$

where $c_{n,n} = v_{n,n+1}(x^0) \leq 0$, $c_{n-1,n} = v_{n-1,n+1}(x^0) > 0$ and Δ'_{n-1}, Δ''_{n-1} are $(n-1)$-dimensional determinants of the form (3.6). The inductive assumption and (3.8) imply (3.7), which shows that $v_1, \dots, v_n$ satisfy the assumption of the standard implicit function theorem. Therefore (3.5), or equivalently (3.2), defines in $(x_1^0 - \delta, x_1^0 + \delta)$ the unique continuously differentiable functions $x_i = x_i(x_1)$, $i = 2, \dots, n+1$.

Now we shall prove the additional assertion (3.4). To check the sign of $y_{i-1} = x_i'(x_1^0)$, we note that $y_1, \dots, y_n$ satisfy the system of linear equations

$$(v_{i,j}(x^0))\,(y_i) = (-v_{i,1}(x^0)), \tag{3.9}$$

where $(v_{i,j}(x^0))$ is the Jacobi matrix corresponding Δ_n. By Cramer's rule,

$$y_k = \Delta_n^k / \Delta_n, \tag{3.10}$$

where the determinant Δ_n^k is obtained from Δ_n by replacing its k-th column with the column $(-v_{i,1}(x^0))$.

1) If $u_2 = u_2(x_2, x_3)$ depends on two parameters, then

$$\begin{aligned} v_{1,1}(x^0) &= \ldots = v_{n-1,1}(x^0) = -u_{1,1}(x^0) > 0, \\ v_{n,1}(x^0) &= u_{n+1,1}(x^0) - u_{1,1}(x^0) > 0. \end{aligned} \tag{3.11}$$

This shows that the determinant Δ_n^1 has the form (3.6). Therefore $\Delta_n^1 \neq 0$, $\Delta_n^1/|\Delta_n^1| = (-1)^n$. This combined with (3.7) and (3.10) implies that $y_1 = x_2'(x_1^0) > 0$.

2) If u_2 depends on three parameters, then by the assumptions of the lemma, $u_3 = u_3(x_3, x_4)$ depends on two parameters and

$$\begin{aligned} v_{1,1}(x^0) &= u_{2,1}(x^0) - u_{1,1}(x^0) = u_{2,1}(x^0) < 0, \\ v_{2,1}(x^0) &= \ldots = v_{n-1,1}(x^0) = 0, \quad v_{n,1}(x^0) = u_{n+1,1}(x^0) > 0. \end{aligned} \tag{3.12}$$

Therefore, Δ_n^2 has the form (3.6) in the case under consideration. Hence, $x_3'(x_1^0) = \Delta_n^2/\Delta_n > 0$.

Note that the assumptions of the lemma allow us to apply the same arguments for the "shifted" functions:

$$v_i^2 := u_{i+1} - u_2, \quad i = 2, \ldots, n+1, \tag{3.13}$$

if u_2 satisfies additional assumption 1), or for the functions

$$v_i^3 := u_{i+1} - u_3, \quad i = 3, \ldots, n+2, \tag{3.14}$$

if u_2 and u_3 satisfy additional assumption 2). Since (3.13) and (3.14) are equivalent to (3.5), each of them defines the same system of solutions $x_1, x_2, \ldots, x_{n+1}$ in a neighborhood of x^0. Considerations above show that $x_3'(x_1) = x_3'(x_2)x_2'(x_1) > 0$ in the case corresponding to (3.13) and $x_4'(x_1) = x_4'(x_2)x_2'(x_1) > 0$ in the case corresponding to (3.14).

Repeating these arguments, after a finite number of steps we get the desired assertion (3.4). □

The next geometrically obvious lemma will be used in the proofs of Lemma 5 and Theorem 4. Let T be a triangle with the base $[a_1, a_2]$. A system of triangles $\{T_i\}_{i=1}^m$ is called admissible for T if:

1) T_i has a base $[a_{1,i}, a_{2,i}]$, such that the segments $[a_{1,i}, a_{2,i}]$, $i = 1, \ldots, m$, constitute a disjoint decomposition of the base $[a_1, a_2]$;

2) T_i and T_{i+1} are disjoint and have a common boundary segment that is an entire side of at least one of these triangles but not necessarily of both of them;

3) T_1 and T_m each has a common boundary segment with $\partial T \setminus [a_1, a_2]$.

Lemma 4. *If* $\{T_i\}_{i=1}^m$ *is admissible for* T*, then* $\cup_{i=1}^m \overline{T}_i \supset T$.

It is important to emphasize that all the triangles under consideration are the usual Euclidean triangles. An elementary inductive proof of the lemma is left to the readers.

Let D be an n-gon in general position with vertices $A_1, \dots, A_n$. Let the convex hull $\hat{D}$ of D have vertices $A'_1 = A_1, A'_2, \dots, A'_{\hat{n}}$. If $A'_1 = 0$ and $A'_2 > 0$, we say that D is in standard position. Let $\{T_i\}_{i=1}^m$ be an admissible system for D. Let $[a_{1,i}, a_{2,i}]$ and $a_{0,i}$ denote the base and base vertex of T_i. Let $\gamma_{1,i}$ and $\gamma_{2,i}$ be the closed sides of the associated sector S_i starting at the points $a_{1,i}$ and $a_{2,i}$, respectively. Since D is in general position, $\gamma_{k,i}$ contains one or two vertices of D. We shall denote them by $B_{k,i}$ and $B'_{k,i}$, where the second vertex, if it exists, lies between $B_{k,i}$ and $a_{k,i}$. Let $l_{k,i}$ and $l'_{k,i}$ denote the rays outgoing from $B_{k,i}$ and $B'_{k,i}$ each containing the side $[a_{k,i}, a_{0,i}]$ of T_i.

The angles $\varphi_{k,i}$, $\varphi'_{k,i}$ formed by $l_{k,i}$ or $l'_{k,i}$ with the positive horizontal direction will be called *the inclinations* of $l_{k,i}$, $l'_{k,i}$. Although $\varphi_{k,i} = \varphi'_{k,i}$, these inclinations will be considered as independent parameters. It is important to note that each vertex A'_j of $\hat{D}$ serves as the origin for one of the rays $l_{k,i}$.

Everywhere in this section, T_1 will denote the triangle such that the corresponding ray $l_{1,1}$ has its origin at the vertex A'_1. Figures 3 and 4 show some notation used in this section.

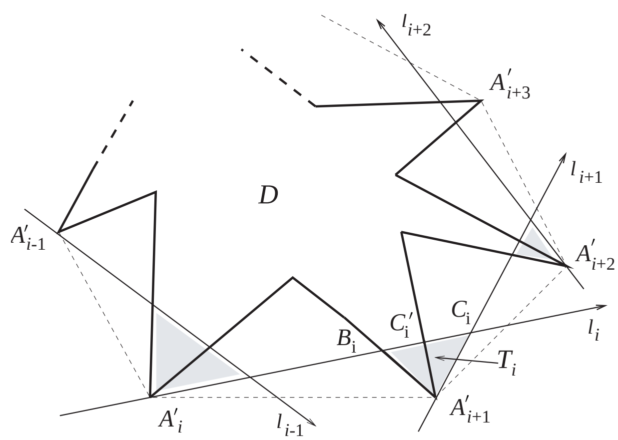

Figure 3. Regular proportional system for small θ

For the main parameter θ, we choose the inclination of $l_{1,1}$: $\theta = \varphi_{1,1}$. Let θ^* be the angle formed by the sides $[A'_1, A'_2]$ and $[A'_1, A'_{\hat{n}}]$ of the convex hull $\hat{D}$, then $0 < \theta < \theta^*$.

Lemma 5. *For any* n*-gon* D *in standard position there are a finite number of intervals* (θ_{j-1}, θ_j), $0 = \theta_0 < \theta_1 < \ldots < \theta_{s+1} = \theta^*$, *such that for each interval* (θ_{j-1}, θ_j) *there is a number* m_j, $\hat{n} \le m_j \le n$ *and a one parameter fam-*

ily of proportional admissible systems $\{T_i(\theta)\}_{i=1}^{m_j}$, *which continuously depend on* θ, $\theta_{j-1} < \theta < \theta_j$ *and satisfy the following conditions*:

a) *The inclinations* $\varphi_i(\theta)$ *of* $l_{1,i}(\theta)$, $i = 1, \dots, m_j$, *strictly increase in* $\theta_{j-1} < \theta < \theta_j$.

b) *If* $\theta \to \theta_j - 0$ *or* $\theta \to \theta_{j-1} + 0$, $j = 1, \dots, s$, *then each triangle* $T_i(\theta)$ *converges to a limit triangle* $T_i^-(\theta_j)$ *or* $T_i^+(\theta_{j-1})$, *some of which but not all can degenerate to certain nondegenerate segments. For every* $j = 1, \dots, s$, *the sets of nondegenerate limit configurations* $\{T_i^-(\theta_j)\}$ *and* $\{T_i^+(\theta_j)\}$ *coincide.*

The function $\theta \mapsto \{T_i(\theta)\}_{i=1}^{m(\theta)}$ *establishes a one-to-one continuous correspondence between the interval* $(0, \theta^*)$ *and the set of all proportional systems admissible for* D.

Proof. 1) First we show that a regular proportional system, if it exists for some θ^0, $0 < \theta^0 < \theta^*$, exists also for all θ in some neighborhood of θ^0. Let $\bar{\varphi}^0 = (\varphi_1^0, \dots, \varphi_m^0)$ denote the vector of inclinations for the regular proportional system $\{T_i^0\}_{i=1}^m$ corresponding to θ^0. Turning all the rays $l_{1,i} = l_{1,i}(\bar{\varphi}^0)$ onto small angles $\varepsilon_i = \varphi_i - \varphi_i^0$ around the origin of $l_{1,i}(\bar{\varphi}^0)$, we get a new system of triangles $\{T_i(\bar{\varphi})\}_{i=1}^m$ corresponding to the vector of inclinations $\bar{\varphi} = (\varphi_1, \dots, \varphi_m)$. This new system is regular and admissible for D, but not proportional in general.

Simple computations show that the coefficient $k_i(\bar{\varphi})$ of $T_i(\bar{\varphi})$, $i = 1, \dots, m$, is a differentiable function depending only on φ_i and φ_{i+1} such that $\partial k_i/\partial \varphi_i < 0$, $\partial k_i/\partial \varphi_{i+1} > 0$. Thus $k_i(\bar{\varphi})$, $i = 1, \dots, m$, satisfy assumptions of Lemma 3. This implies that for each $\theta = \varphi_1$ in some small interval $\theta^0 - \delta < \theta < \theta^0 + \delta$ there are unique inclinations $\varphi_i(\theta)$, $i = 2, \dots, m$ such that the system $\{T_i(\theta)\}_{i=1}^m$ with $T_i(\theta) = T_i(\bar{\varphi}(\theta))$ is regular and proportional. Moreover, since $d\varphi_i(\theta^0)/d\theta > 0$ by inequality (3.4) of Lemma 3, all rays $l_{1,i}(\theta)$ turn in the same direction as $l_{1,1}(\theta)$ does.

Let $\theta' < \theta < \theta''$ be the maximal interval containing θ^0 that carries a regular proportional system. If $\theta' > 0$ then at least one of the $2m$ limit rays $l_{1,i}(\theta')$, $l_{2,i}(\theta')$, $i = 1, \dots, m$ has two vertices of D. Similarly, if $\theta'' < \theta^*$ then at least one of the $2m$ limit rays $l_{1,i}(\theta'')$, $l_{2,i}(\theta'')$ has two vertices of D. Note that the limit system of rays always exists since the inclinations $\varphi_i(\theta)$ are monotone in θ. In other words, the limit systems $\{T_i(\theta')\}_{i=1}^m$ and $\{T_i(\theta'')\}_{i=1}^m$ are singular (=nonregular).

Indeed, if for instance, $\theta' > 0$, $\{T_i(\theta')\}_{i=1}^m$ is regular, and all limit triangles $T_i(\theta')$ do not degenerate, we can continue construction of the system as above into some right neighborhood of the point θ'.

The degeneracy of $T_i(\theta')$ belonging to the regular limit system may occur in two cases. First, if $l_{1,i}(\theta')$ is parallel to $l_{2,i}(\theta')$: Since the system $\{T_i(\theta')\}_{i=1}^m$ is regular, $l_{1,i}(\theta')$ and $l_{2,i}(\theta')$ do not lie on the same straight line. This implies that $\alpha_i = \alpha_i(\theta) \to 0$ and $\sigma_i = \sigma_i(\theta) \to \infty$ as $\theta \to \theta' + 0$. Since the system $\{T_i(\theta)\}_{i=1}^m$ is proportional for $\theta' < \theta < \theta''$, the latter implies that $\alpha_j(\theta) \to 0$ for all $j = 1, \dots, m$ as $\theta \to \theta' + 0$. This contradicts the condition $\sum_{i=1}^m \alpha_i(\theta) = 2\pi$.

The second type of degeneracy can happen when some triangle $T_i(\theta')$ shrinks to a point $C \in \partial D$ different from the vertices of D. Since the considered limit system of triangles is regular, the limit angle $\alpha_i(\theta')$ cannot be zero. Hence the limit coefficients $k_j(\theta') = k_i(\theta') = \infty$. This yields that all the limit triangles shrink to some points on ∂D different from the vertices of D. This certainly cannot happen, since for every $i = 1, \dots, \hat{n}$ and every θ, A'_i belongs to the boundary of some triangle under consideration.

Thus our analysis show that the limit systems $\{T_i(\theta')\}$ and $\{T_i(\theta'')\}$ are singular. Note that each of them contains at least $\hat{n}$ nondegenerate triangles.

2) Now we show that a regular proportional system exists for some $\theta > 0$ small enough. We shall consider rays $l_i = l_i(\varepsilon_i)$, $i = 1, \dots, \hat{n}$, outgoing from the vertices A'_i of the convex hull $\hat{D}$ with inclinations $\varphi_i = \tilde{\varphi}_i + \varepsilon_i$, where $\tilde{\varphi}_i$ is the inclination of the side $[A'_i, A'_{i+1}]$ and $0 < \varepsilon_i \le \varepsilon_i^0$. Here $\varepsilon_i^0 > 0$ are fixed and small enough such that for $0 < \varepsilon_i \le \varepsilon_i^0$ the ray $l_i = l_i(\varepsilon_i)$ does not contain vertices of D except A'_i.

Since D is in general position, it follows that l_i cuts off from D a triangle with vertices A'_{i+1}, B_i and C'_i, where B_i precedes C'_i along l_i; see Figure 3.

Let $T_i = T_i(\varepsilon_i, \varepsilon_{i+1})$ denote the triangle with vertices A'_{i+1}, B_i, and C_i, where C_i is the point of intersection of l_i and l_{i+1}. Then $\{T_i(\bar{\varepsilon})\}_{i=1}^{\hat{n}}$ with $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{\hat{n}})$, is a regular system admissible for D. Let α_i, and σ_i denote the base angle and area of T_i. As we observed above, the coefficient $k_i(\bar{\varepsilon}) = \alpha_i(\bar{\varepsilon})/\sigma_i(\bar{\varepsilon})$ depends only on ε_i and ε_{i+1}. In addition, $k_i(\bar{\varepsilon})$ is continuous and strictly increases to ∞ as ε_i decreases to 0. Therefore the maximal coefficient

$$k = k(\bar{\varepsilon}) = \max_i k_i(\bar{\varepsilon})$$

is continuous in $\bar{\varepsilon}$ and $k(\bar{\varepsilon}) \to \infty$ if at least one of the parameters ε_i goes to 0. This implies that the minimum

$$p = \min k(\bar{\varepsilon}) \quad \text{over the set} \quad 0 < \varepsilon_i \le \varepsilon_i^0, \quad i = 1, \dots, \hat{n}, \tag{3.15}$$

is achieved at some point $\bar{\varepsilon}^* = (\varepsilon_1^*, \dots, \varepsilon_{\hat{n}}^*)$ with $0 < \varepsilon_i^* \le \varepsilon_i^0$ for all $i = 1, \dots, \hat{n}$.

Let us show that $k_i^* = k_i(\bar{\varepsilon}^*)$ equals p for all $i = 1, \dots, \hat{n}$. If not, we consider the set $Q_1(\bar{\varepsilon}^*)$ of vertices A'_i such that $k_i^* = p$ and the set $Q_2(\bar{\varepsilon}^*) \ne \emptyset$ of vertices A'_i such that $k_i^* < p$. There is an index j such that $k_j < p$ and $k_{j+1} = p$. Decreasing ε_j slightly, we get a configuration $\{T_i(\bar{\varepsilon})\}_{i=1}^{\hat{n}}$ for which $k(\bar{\varepsilon}) \le p$ and the set $Q_1(\bar{\varepsilon})$ corresponding to this new configuration contains one vertex less than the set $Q_1(\bar{\varepsilon}^*)$.

If $Q_1(\bar{\varepsilon})$ is empty, we get a contradiction to (3.15). If not, we can repeat the previous procedure. After a finite number of steps we get a system with $k_i < p$ for all $i = 1, \dots, \hat{n}$, contradicting (3.15). This proves that for some $\theta > 0$ small enough and, by the same arguments, for some θ close enough to θ^*, there is a regular proportional system.

3) For a nonconvex polygon, analysis in 1) and 2) shows that there are two intervals $(0, \theta_1]$ and $[\theta', \theta^*)$, each of which carries a parametric family of proportional systems $\{T_i(\theta)\}_{i=1}^{\hat{n}}$ with $0 < \theta \le \theta_1$ and $\theta' \le \theta < \theta^*$. Note that for every θ in these intervals the system consists of $\hat{n}$ triangles and the limit systems $\{T_i(\theta_1)\}_{i=1}^{\hat{n}}$ and $\{T_i(\theta')\}_{i=1}^{\hat{n}}$ are singular.

Next we show that any proportional singular system $\{T_i(\theta_k)\}_{i=1}^{m}$ with $0 < \theta_k < \theta^*$ and m depending on k, can be continued into some right neighborhood of θ_k. First we complete the system $\{T_i(\theta_k)\}_{i=1}^{m}$ with certain *degenerate triangles* $\hat{T}_s$ corresponding to singular triangles T_{i_s} that have two vertices $B_{1,s}$ and $B'_{1,s}$ on the ray $\tilde{l}_{1,s} = l_{1,i_s}(\theta_k)$. All possible singular configurations having this property are depicted in Figure 4 below. The shaded areas in these figures belong to the polygon D. The configurations shown in Figure 4: 1, 2, 5, 6, 7, 13, 14, 15, and 16 can also have a second vertex $B'_{2,s}$ on the ray l_{2,i_s}. The angle of D corresponding to this possible second vertex is shown in the dashed line. Thus, the number of all possible configurations depicted in Figure 4 equals 26.

For configurations in Figure 4: 15, 16, and 17, $\hat{T}_s$ will denote a degenerate triangle having its degenerate base at the point $C_s \in \partial D$, where the ray $\tilde{l}_{1,s}$ enters into D for the first time, and the base vertex at the point $\hat{a}_s$ that follows C_s on $\tilde{l}_{1,s}$ and satisfies the following condition:

$$k(\hat{T}_s) := 2|\hat{a}_s - C_s|^{-2} = k(\theta_k), \tag{3.16}$$

where $k(\theta_k) = k_i(\theta_k)$ is the coefficient of the limit system $\{T_i(\theta_k)\}_{i=1}^{m}$.

For configurations in Figure 4: 1, 2, 3, 4, 12, 13 and 14, we put $C_s = B'_{1,s}$, and then define the point $\hat{a}_s$, triangle $\hat{T}_s$, and the coefficient $k(\hat{T}_s)$ as above.

Finally, for configurations in Figure 4: 5–11, $\hat{T}_s$ has the base $[B_{1,s}, B'_{1,s}]$ and the base vertex at the point $\hat{a}_s$ that follows $B'_{1,s}$ on $\tilde{l}_{1,s}$ and satisfies the condition:

$$k(\hat{T}_s) := 2|(\hat{a}_s - B_{1,s})(\hat{a}_s - B'_{1,s})|^{-1} = k(\theta_k). \tag{3.17}$$

It is important to note, that in all cases the point $\hat{a}_s$, and therefore the triangle $\hat{T}_s$, exists and is uniquely determined by condition (3.16) or (3.17).

We combine all the triangles $T_i(\theta_k)$ and $\hat{T}_s$ into a new system $\{R_i\}_{i=1}^{\tilde{m}}$, $m < \tilde{m} \le n$, keeping our usual convention concerning numbering. The latter means, in particular, that a singular triangle T_{i_s} and the corresponding degenerate triangle $\hat{T}_s$ get the indices $i_s + s - 1$ and $i_s + s$ in this new system.

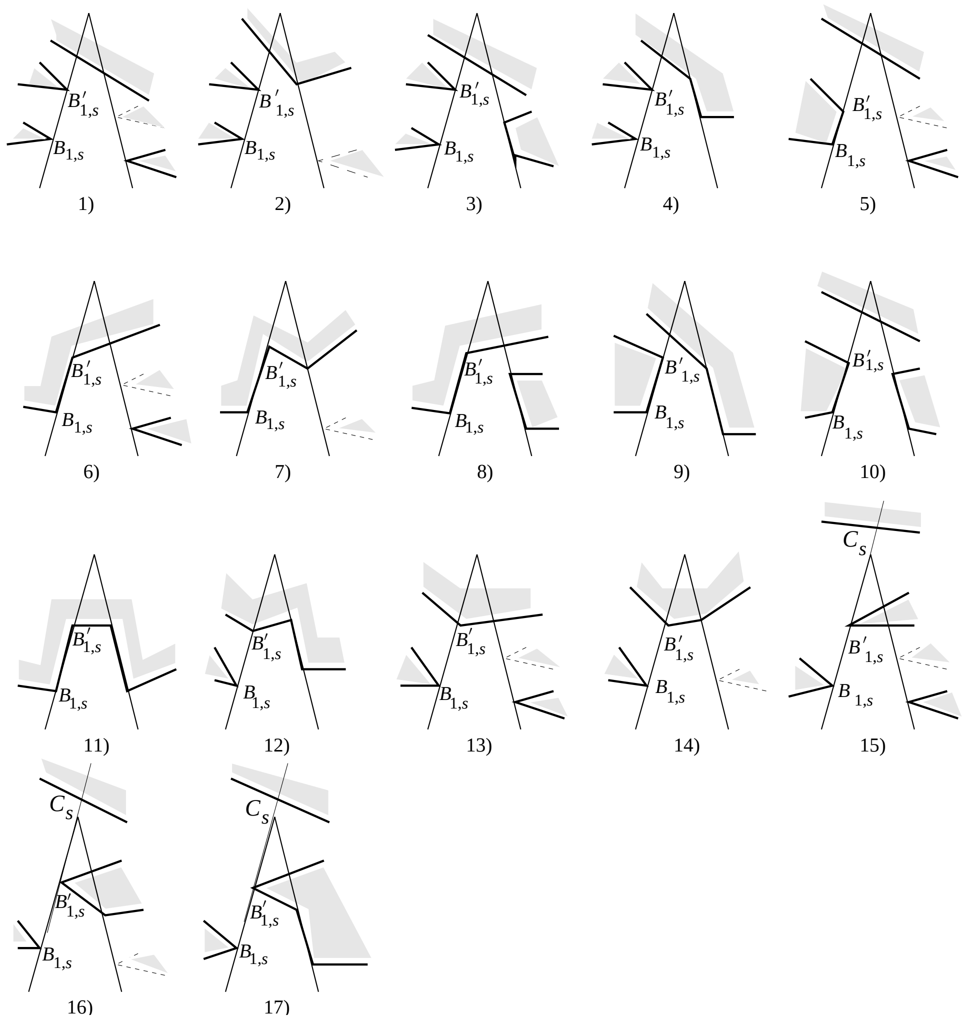

Figure 4. Singular configurations

To each R_i we associate two rays r_i and r_{i+1} according to the following rules. Each regular triangle T_i comes into the new system with the corresponding rays $l_{1,i}$ and $l_{2,i}$. If $R_i = T_j$ is singular, then $r_i = l'_{1,j}$, $r_{i+1} = l_{2,j}$. If $R_i = \hat{T}_s$ is degenerate corresponding to a singular triangle $\tilde{T}_s = T_{i_s}$, then $r_i = l_{1,i_s}$, $r_{i+1} = l'_{1,i_s}$.

According to our notation, R_i and R_{i+1} have r_{i+1} as a common associate ray and one can easily check that this is actually the case.

Let $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{\tilde{m}})$, where ε_i is small enough and not necessarily positive. We consider a varying system of rays $r_i(\bar{\varepsilon})$, $i = 1, \dots, \tilde{m}$, obtained as follows.

If r_i corresponds to some ray $l_{1,j}$, then $r_i(\bar{\varepsilon})$ is obtained from r_i by rotation onto the angle ε_i around the origin of r_i. Thus $r_i(\bar{\varepsilon}) = r_i(\varepsilon_i)$ depends only on ε_i for such rays.

If r_i corresponds to $l'_{1,j}$, then r_{i-1} corresponds to $l_{1,j}$ and $r_i(\bar{\varepsilon})$ will denote the ray having a common origin with $l'_{1,j}$ that intersects $r_{i-1}(\bar{\varepsilon})$ at the point $a_{0,i}(\bar{\varepsilon})$ such that $|a_{o,i}(\bar{\varepsilon})-B_{1,j}| = |\hat{a}_j - B_{1,j}| - \varepsilon_i$, where $\hat{a}_j$ is defined by (3.16) or (3.17) and $B_{1,j}$ is the origin of $l_{1,j}$. One can easily see that $r_i(\bar{\varepsilon})$ corresponding to $l'_{1,j}$ depends on two parameters: ε_{i-1} and ε_i.

Let q_i denote the straight line passing through the side of D that contains the base of R_i. For configurations shown in Figure 4: 1, 2, 3, 4, 12, 13, and 14 there are two such straight lines. In this case, q_i denotes the one through which the corresponding ray $r_i(\varepsilon_i)$ with $\varepsilon_i > 0$ small enough enters into D for the first time.

Let $R_i(\bar{\varepsilon})$ be the triangle (degenerate or not) having its base on q_i and sides belonging to the rays $r_i(\bar{\varepsilon})$ and $r_{i+1}(\bar{\varepsilon})$. If $R_i(\bar{\varepsilon})$ is degenerate, we assume in addition that $R_i(\bar{\varepsilon})$ has the base vertex at the point $a_{0,i}(\bar{\varepsilon})$ such that $|a_{0,i}(\bar{\varepsilon}) - B_{1,j}| = |\hat{a}_j - B_{1,j}| - \varepsilon_i$, where $\hat{a}_j$ and $B_{1,j}$ are defined above.

If $\varepsilon_i > 0$ for all rays $r_i(\varepsilon_i)$ depending only on one parameter, then it is not difficult to see that $\{R_i(\bar{\varepsilon})\}_{i=1}^{\tilde{m}}$ is a regular system admissible for D. Of course, if at least one such ε_i is negative, the varied system is not admissible. Consider the coefficients $k_i(\bar{\varepsilon})$ of the triangles $R_i(\bar{\varepsilon})$. For the degenerate triangles, $k_i(\bar{\varepsilon})$ is defined by (3.16) or (3.17) with $\hat{a}_j$ replaced by $a_{0,i}(\bar{\varepsilon})$.

By our construction, $k_i(\bar{\varepsilon})$ depends on two or three parameters: ε_i and ε_{i+1} or ε_{i-1}, ε_i, ε_{i+1}, respectively. By direct computation one can easily check that each $k_i(\bar{\varepsilon})$ has continuous partial derivatives near the point $\bar{\varepsilon}^0 = (0, \dots, 0)$. Moreover, $\partial k_i(\bar{\varepsilon}_0)/\partial\varepsilon_i < 0$, $\partial k_i(\bar{\varepsilon}^0)/\partial\varepsilon_{i+1} > 0$ if k_i corresponds to a regular triangle; $\partial k_i(\bar{\varepsilon}^0)/\partial\varepsilon_i = 0$, $\partial k_i(\bar{\varepsilon}^0)/\partial\varepsilon_{i+1} > 0$, if k_i corresponds to a degenerate triangle, and $\partial k_i(\bar{\varepsilon}^0)/\partial\varepsilon_{i-1} < 0$, $\partial k_i(\bar{\varepsilon}^0)/\partial\varepsilon_i = 0$, $\partial k_i(\bar{\varepsilon}^0)/\partial\varepsilon_{i+1} > 0$ if k_i corresponds to a singular triangle.

Therefore, the functions $k_1, \dots, k_{\tilde{m}}$ satisfy the assumptions of Lemma 3. This lemma implies that there is $\delta > 0$ such that for each $\varepsilon_1 \in (-\delta, \delta)$ there are unique continuously differentiable functions $\varepsilon_i(\varepsilon_1)$ solving the equations

$$k_1(\varepsilon_1, \dots, \varepsilon_{\tilde{m}}) = k_2(\varepsilon_1, \dots, \varepsilon_{\tilde{m}}) = \dots = k_{\tilde{m}}(\varepsilon_1, \dots, \varepsilon_{\tilde{m}}) \tag{3.18}$$

such that $\varepsilon_i(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$.

In addition, inequality (3.4) of Lemma 3 shows that each parameter $\varepsilon_i(\varepsilon_1)$ corresponding to some ray $l_{1,j}$ strictly increases when ε_1 does. As we noted above, the latter implies that the system of triangles $\{R_i(\bar{\varepsilon}(\varepsilon_1))\}_{i=1}^{\tilde{m}}$ with $\bar{\varepsilon}(\varepsilon_1) = (\varepsilon_1, \varepsilon_2(\varepsilon_1), \dots, \varepsilon_{\tilde{m}}(\varepsilon_1))$ is admissible for D. By (3.18), this system is proportional.

Thus we have proved that every singular proportional system $\{T_i(\theta_k)\}_{i=1}^{m}$ can be continued into some right neighborhood and, by the same arguments, into some left neighborhood of the parameter θ_k. The arguments above show also that such continuation is unique. Moreover, since continued systems are regular, the arguments in the part 1) show that all inclinations $\varphi_{k,i}(\theta)$ are monotonic.

4) The arguments in 1)–3) show that there is a family $\mathcal{T}(\theta)$ of proportional, admissible for D systems $\{T_i(\theta)\}_{i=1}^{m(\theta)}$ that continuously in the sense of this lemma depend on the parameter $0 < \theta < \theta^*$ and satisfy conditions a) and b).

To prove the last assertion of the lemma, we assume that there is a proportionally admissible for D system $\{\tilde{T}_i\}_{i=1}^{\tilde{m}}$ that is not included in $\mathcal{T}(\theta)$. In this case the arguments above show that there is a second family $\tilde{\mathcal{T}}(\theta)$ of systems $\{\tilde{T}_i(\theta)\}_{i=1}^{\tilde{m}(\theta)}$ satisfying the same assertions of the lemma. The uniqueness of continuation in a neighborhood established in 1) and 3) shows that $\{T_i(\theta_1)\}_{i=1}^{m(\theta_1)} \neq \{\tilde{T}_i(\theta_2)\}_{i=1}^{\tilde{m}(\theta_2)}$ for all $\theta_1, \theta_2 \in (0, \theta^*)$.

Let $\psi_i(\theta)$ and $\tilde{\psi}_i(\theta)$ denote the inclinations of the rays $l_{1,k_i}(\theta)$ and $\tilde{l}_{1,s_i}(\theta)$ corresponding to the triangles $T_{k_i}(\theta)$ and $\tilde{T}_{s_i}(\theta)$ and outgoing from the vertex A_i' of $\hat{D}$. First we show that for every $i = 1, \dots, \hat{n}$,

$$\psi_i(\theta) \to \tilde{\varphi}_i \quad \text{and} \quad \tilde{\psi}_i(\theta) \to \tilde{\varphi}_i \quad \text{as} \quad \theta \to 0, \tag{3.19}$$

where $\tilde{\varphi}_i$ is the inclination of $[A_i', A_{i+1}']$. Suppose for instance, the first relation in (3.19) is not valid. Then there is an index j such that

$$\psi_j(\theta) \to \tilde{\varphi}_j \quad \text{and} \quad \psi_{j-1}(\theta) \to \tilde{\varphi}_{j-1} + \varepsilon_0 \quad \text{as} \quad \theta \to 0, \tag{3.20}$$

with some $\varepsilon_0 > 0$.

Consider the triangle T, possibly infinite, with the base $[A_{j-1}', A_j']$ and sides on the limit rays $l_{1,k_j}(0)$, $l_{1,k_{j-1}}(0)$. (3.20) shows that $T \cap D \neq \emptyset$. Let $T_k(\theta)$ be a nondegenerate triangle of the system $\{T_i(0)\}_{i=1}^{m(0)}$ having the associated sector $S_k(\theta)$ such that $[a_k(\theta), b_k(\theta)] := \bar{S}_k(\theta) \cap [A_{j-1}', A_j']$ is not empty. Let $R_k(\theta)$ be a triangle with the base $[a_k(\theta), b_k(\theta)]$ that has a common base angle with $T_k(\theta)$. For $\theta > 0$ small enough, let $\{T_k(\theta)\}_j$ and $\{R_k(\theta)\}_j$ be the systems of all such triangles $T_k(\theta)$ and $R_k(\theta)$ corresponding to $[A_{j-1}', A_j]$. Since the inclination $\varphi_k(\theta)$ corresponding to $T_k(\theta)$ is a monotonic function of θ, there are limit systems of triangles $\{T_k(0)\}_j$ and $\{R_k(0)\}_j$ as $\theta \to 0$. Since D is in general position and $\mathcal{T}(\theta)$ consists of proportional systems, (3.20) implies that $k_i(\theta) \to \infty$ for all i as $\theta \to 0$. This implies that all limit triangles $T_k(0)$ are degenerate.

On the other hand, since D is in general position and $[A_{j-1}', A_j']$ is covered by the system $\{\bar{R}_k(0)\}_j$, the latter system contains nondegenerate triangles. It is clear that $\{R_k(0)\}_j$ is an admissible system (in the sense of Lemma 4) for the triangle T defined above. By Lemma 4, $\cup \bar{R}_k(0) \supset T$. Since $T \cap D \neq \emptyset$, the latter implies that for some k, $R_k(0) \cap D \neq \emptyset$. Since $T_k(0)$ is degenerate, the corresponding limit sector $S_k(0)$ should have a nonempty intersection with D. This contradiction shows that for every i, $\psi_i(\theta) \to \tilde{\varphi}_i$ and similarly $\tilde{\psi}_i(\theta) \to \tilde{\varphi}_i$ as $\theta \to 0$.

(3.19) shows that for some $\theta_0 > 0$ small enough the systems $\{T_i(\theta_0)\}_{i=1}^{\hat{n}}$ and $\{\tilde{T}_i(\theta_0)\}_{i=1}^{\hat{n}}$ are of the type considered in part 2) of this proof; cf. Figure 3. In particular, each of them contains $\hat{n}$ triangles.

Let us show that $\{T_i(\theta_0)\}_{i=1}^{\hat{n}} = \{\tilde{T}_i(\theta_0)\}_{i=1}^{\hat{n}}$. If not, then there is an index i_0, $1 \le i_0 \le \hat{n}-1$, such that $\psi_i(\theta_0) = \tilde{\psi}_i(\theta_0)$ for $i = 1,\dots,i_0$ and $\psi_{i_0+1}(\theta_0) \ne \tilde{\psi}_{i_0+1}(\theta_0)$. To be definite, let

$$\psi_{i_0+1}(\theta_0) > \tilde{\psi}_{i_0+1}(\theta_0). \tag{3.21}$$

Then

$$k_{i_0}(\psi_{i_0}(\theta_0), \psi_{i_0+1}(\theta_0)) > \tilde{k}_{i_0}(\psi_{i_0}(\theta_0), \tilde{\psi}_{i_0+1}(\theta_0)), \tag{3.22}$$

where k_i and $\tilde{k}_i$ denote the coefficients of T_i and $\tilde{T}_i$, respectively. Since the functions k_i and $\tilde{k}_i$ strictly decrease in their first parameter and strictly increase in the second one, it follows that $\psi_i(\theta_0) > \tilde{\psi}_i(\theta_0)$ for all $i = i_0+1,\dots,\hat{n}$. This implies that

$$k_{\hat{n}}(\psi_{\hat{n}}(\theta_0), \theta_0) < \tilde{k}_{\hat{n}}(\tilde{\psi}_{\hat{n}}(\theta_0), \theta_0),$$

contradicting (3.22). This contradiction shows that $\{T_i(\theta_0)\}_{i=1}^{\hat{n}} = \{\tilde{T}_i(\theta_0)\}_{i=1}^{\hat{n}}$ and therefore $\mathcal{T}(\theta) = \tilde{\mathcal{T}}(\theta)$ for all $0 < \theta < \theta^*$. This finishes the proof of Lemma 5. □

Let D be a convex n-gon in standard position having the internal angle φ_i at the vertex A_i, $i = 1,\dots,n$. Let $\theta_1 = \theta$. For $i = 2,\dots,n$, let $\theta_i = \theta_i(\theta)$ be functions continuous on $0 \le \theta \le \varphi_1$ such that $\theta_i(0) = 0$, $\theta_i(\varphi_1) = \varphi_i$ and

$$0 < \theta_i(\theta) < \varphi_i, \quad \varphi_{i+1} - \theta_{i+1}(\theta) + \theta_i(\theta) < \pi \quad \text{for all} \quad 0 < \theta < \varphi_1. \tag{3.23}$$

Let $l_i(\theta)$ denote the ray having the angle $\theta_i(\theta)$ with the side $[A_i, A_{i+1}]$ at A_i. If $T_i = T_i(\theta)$ denotes the triangle with the base $[A_i, A_{i+1}]$ having its sides on the rays $l_i(\theta)$ and $l_{i+1}(\theta)$, then (3.23) guarantees that for every $0 < \theta < \varphi_1$ the system $\{T_i(\theta)\}_{i=1}^{n}$ is admissible for D.

Lemma 6. *Let D and $\{T_i(\theta)\}_{i=1}^{n}$ be a convex n-gon and a system of triangles described above. Then there is θ^*, $0 < \theta^* < \varphi_1$, such that $\cup_{i=1}^{n} \overline{T}_i(\theta^*) \supset D$.*

Proof. Let q_i denote the straight line containing the side $[A_i, A_{i+1}]$ and let $d(z, q_i)$ denote the distance from z to q_i. To each q_i we assign the positive weight $p_i = p_i(\theta)$ as follows. We take $p_0 = 1$, then for $i = 1,\dots,n$, we put

$$p_i = p_{i-1} d(z, q_{i-1})/d(z, q_i) = p_{i-1} \sin(\varphi_i - \theta_i(\theta))/\sin\theta_i(\theta) \quad \text{for} \quad z \in l_i(\theta) \setminus A_i. \tag{3.24}$$

Since the quotient $d(z, q_{i-1})/d(z, q_i)$ is constant on $l_i(\theta) \setminus A_i$, the functions $p_i(\theta)$ are well defined and continuous in $0 < \theta < \varphi_1$. Since $\sin(\varphi_i - \theta_i(\theta))/\sin\theta_i(\theta) \to +\infty$ as $\theta \to 0$,

$$p_n(\theta) > \ldots > p_1(\theta) > p_0 = 1$$

for all θ small enough. Similarly,

$$p_n(\theta) < \ldots < p_1(\theta) < p_0 = 1$$

for all θ close enough to φ_1. Since $p_n(\theta)$ is continuous, there is θ^*, $0 < \theta^* < \varphi_1$ such that $p_n(\theta^*) = 1 = p_0$.

We claim that $\{T_i(\theta^*)\}_{i=1}^n$ is a desired cover of D. To show this, we consider components D_i' and D_i'' of $D \setminus l_i(\theta^*)$, where D_i' lies on the left side of $l_i(\theta^*)$, when we walk on $l_i(\theta^*)$ towards A_i. Using (3.24) one can easily show that for $i = 1, \ldots, n$,

$$\begin{aligned} &d(z, q_{i-1})/d(z, q_i) > p_i(\theta^*)/p_{i-1}(\theta^*) \qquad \text{for all} \quad z \in D_i', \\ &d(z, q_{i-1})/d(z, q_i) < p_i(\theta^*)/p_{i-1}(\theta^*) \qquad \text{for all} \quad z \in D_i''. \end{aligned} \tag{3.25}$$

Assuming that a point $\zeta \in D$ is not covered by $\cup_{i=1}^n \overline{T}_i(\theta^*)$, choose an index j such that

$$p_j(\theta^*) d(\zeta, q_j) = \min_i p_i(\theta^*) d(\zeta, q_i).$$

Then,

$$\begin{aligned} d(\zeta, q_j)/d(\zeta, q_{j+1}) &\leq p_{j+1}(\theta^*)/p_j(\theta^*), \\ d(\zeta, q_{j-1})/d(\zeta, q_j) &\geq p_j(\theta^*)/p_{j-1}(\theta^*). \end{aligned} \tag{3.26}$$

(3.25) and (3.26) show that $\zeta \in \overline{D}_{j+1}'' \cap \overline{D}_j'$. Since $\zeta \in D$, the latter implies that $\zeta \in \overline{T}_j$ contradicting our assumption. $\square$

Proof of Theorem 4. 1) Assume first that D is in standard general position. Then by Lemma 5, the set of all proportional systems admissible for D admits a parametrization in terms of the angle θ, $0 < \theta < \varphi_1$, formed by the ray $l_{1,1}(\theta)$ and the segment $[A_1', A_2']$ at the vertex A_1'; see Lemma 5 for the notation. This parametrization of $\{T_i(\theta)\}$ is continuous in the sense of Lemma 5.

Let $\hat{l}_k(\theta)$, $k = 1, \ldots, \hat{n}$ be the ray outgoing from the vertex A_k' of the convex hull $\hat{D}$ that corresponds to some triangle $T_j(\theta)$. As mentioned above, for each θ, every vertex A_k' has one and only one such a ray.

Let $\hat{T}_k(\theta)$ denote the triangle with the base $[A_k', A_{k+1}']$ that has its sides on the rays $\hat{l}_k(\theta)$ and $\hat{l}_{k+1}(\theta)$. It is clear that for every θ, $0 < \theta < \varphi_1$, $\{\hat{T}_k(\theta)\}_{k=1}^{\hat{n}}$ is a system of triangles admissible for the convex hull $\hat{D}$. Moreover, this system is continuously parametrized by θ, $0 < \theta < \varphi_1$, and satisfies all other conditions of Lemma 6 for the $\hat{n}$-gon $\hat{D}$. In particular, the limit relations (3.19) are satisfied as well their counterparts for $\theta \to \varphi_1$. Therefore by this lemma, there is θ^*, $0 < \theta^* < \varphi_1$, such that

$$\bigcup_{k=1}^{\hat{n}} \overline{\hat{T}}_k(\theta^*) \supset \hat{D}. \tag{3.27}$$

Let $S_i(\theta^*)$ and $\hat{S}_k(\theta^*)$ denote the sectors associated with the triangles $T_i(\theta^*)$ and $\hat{T}_k(\theta^*)$, respectively. Let $I(k)$ denote the set of indices i such that $S_i(\theta^*) \cap \hat{S}_k(\theta^*) \neq \emptyset$. Then $I(1), \ldots, I(\hat{n})$ is a disjoint decomposition of the set of all indices corresponding to the parameter θ^*.

For $T_i(\theta^*)$ with $i \in I(k)$, let $T'_i(\theta^*)$ be a triangle with the base on $[A'_k, A'_{k+1}]$ which has a common base angle with $T_i(\theta^*)$. The set of the triangles $\{T'_i(\theta^*) : i \in I(k)\}$ satisfies the assumptions of Lemma 4 for the triangle $\hat{T}_k(\theta^*)$. Therefore,

$$\bigcup_{i \in I(k)} \overline{T}'_i(\theta^*) \supset \hat{T}_k(\theta^*), \quad k = 1, \ldots, \hat{n}. \tag{3.28}$$

(3.27) and (3.28) show that

$$\bigcup_i \overline{T}'_i(\theta^*) \supset \hat{D}. \tag{3.29}$$

Since $T_i(\theta^*) = T'_i(\theta^*) \setminus \overline{S}_i(\theta^*)$ and $S_i(\theta^*) \cap D = \emptyset$, (3.29) yields $\cup_i \overline{T}_i(\theta^*) \supset D$. This proves Theorem 4 for every n-gon in general position.

2) For arbitrary n-gon D with vertices $A_1, \ldots, A_n$, consider a sequence of n-gons $D^1, D^2, \ldots$, each in general position, that converges to D; i.e., if D^k has vertices A_i^k, $i = 1, \ldots, n$, then $A_i^k \to A_i$ as $k \to \infty$. By part 1), for every k there is a proportional system $\{T_i^k\}_{i=1}^{m(k)}$ that covers D^k. Since $\hat{n} \leq m(k) \leq n$, we may assume that $m(k) = m$ is constant.

Note that the set of all vertices of all triangles $T_i^k, i = 1, \ldots, m$, $k = 1, 2, \ldots$, is bounded. Therefore we can choose a subsequence k_s, if necessary, such that $T_i^{k_s}$ converge to limit triangles T_i^∞, $i = 1, \ldots, m$, some of which but not all can degenerate. It is clear that $\{T_i^\infty\}_{i=1}^m$ is an admissible proportional system that covers D. □

4. Proof of Theorem 1

Let D_n, $n \geq 3$, be an n-gon and let $A = \operatorname{Area} D_n$. By Theorem 4, there is a proportional admissible for D_n system $\{T_i\}_{i=1}^m$ with $3 \leq m \leq n$ such that

$$\sigma := \sum_{i=1}^m \sigma_i \geq A, \tag{4.1}$$

where $\sigma_i > 0$ denotes the area of the triangle T_i. Let $2\pi\alpha_i$ be the base angle of T_i. Then

$$\alpha_i / \sigma_i = 1/\sigma \quad \text{for all} \quad i = 1, \ldots, m, \tag{4.2}$$

since $\{T_i\}_{i=1}^m$ is proportional. Let $\{S_i\}_{i=1}^m$ be the system of sectors S_i associated with T_i in the sense of Section 3. Since $\{T_i\}_{i=1}^m$ is admissible for D_n, it follows that $\{S_i\}_{i=1}^m$ is a competing system of trilaterals in the sense of Theorem 3 corresponding to a simply connected domain $\Omega(\overline{D}_n) = \overline{\mathbb{C}} \setminus \overline{D}_n$. Therefore by Theorem 3,

$$m(\Omega(\overline{D}_n), \infty) \le \sum_{k=1}^{m} \alpha_k^2 m(S_k; \infty | a_1^k, a_2^k), \tag{4.3}$$

where a_1^k and a_2^k are geometric vertices of S_k different from ∞.

By Lemma 1,

$$\begin{aligned} m(S_k; \infty | a_1^k, a_2^k) &\le \frac{1}{2\pi\alpha_k} \log \frac{4^{\alpha_k} \alpha_k B(1/2, 1/2+\alpha_k)}{(\sigma_k \tan \pi\alpha_k)^{1/2}} \\ &= \frac{1}{2\pi\alpha_k} \log \frac{\pi^{1/2} 4^{\alpha_k} \Gamma(1/2+\alpha_k)}{(\sigma_k \tan \pi\alpha_k)^{1/2} \Gamma(\alpha_k)}. \end{aligned} \tag{4.4}$$

Taking into account the proportionality property (4.2) and (4.4), we get from (4.3)

$$m(\Omega(\overline{D}_n), \infty) \le \frac{1}{4\pi} \sum_{k=1}^{m} \alpha_k \log \frac{\pi 2^{4\alpha_k} \Gamma^2(1/2+\alpha_k)}{\sigma \alpha_k \tan \pi\alpha_k \Gamma^2(\alpha_k)} = \frac{1}{4\pi} \log \frac{\pi}{\sigma} + \frac{1}{4\pi} \sum_{k=1}^{m} H(\alpha_k), \tag{4.5}$$

where

$$H(\alpha) = \alpha \log \frac{2^{4\alpha} \Gamma^2(1/2+\alpha)}{\alpha \tan \pi\alpha \Gamma^2(\alpha)}. \tag{4.6}$$

In Lemma 7 below we shall show that $H(\alpha)$ is strictly concave in $0 < \alpha < 1/2$. Since $\sum_{k=1}^{m} \alpha_k = 1$ and $0 < \alpha_k < 1/2$, (4.5), the concavity property (4.11) and equality (2.14) imply

$$\begin{aligned} m(\Omega(\overline{D}_n), \infty) &\le \frac{1}{4\pi} \log \frac{\pi}{\sigma} + \frac{m}{4\pi} H(\frac{1}{m}) \\ &= \frac{1}{4\pi} \log \frac{\pi 2^{4/m} m \Gamma^2(1/2+1/m)}{\sigma \tan(\pi/m) \Gamma^2(1/m)} = m(\Omega(\overline{D}^*_m(\sigma)), \infty). \end{aligned} \tag{4.7}$$

By Lemma 2, $m(\Omega(\overline{D}^*_k(\sigma)), \infty)$ strictly increases in k and obviously it strictly decreases in σ. Therefore, (4.7) and (4.1) yield

$$m(\Omega(\overline{D}_n), \infty) \le m(\Omega(\overline{D}^*_m(\sigma)), \infty) \le m(\Omega(\overline{D}^*_n(A)), \infty). \tag{4.8}$$

By (2.2), (4.8) is equivalent to (1.2).

To prove the uniqueness assertion of Theorem 1, assume that for D_n considered in the proof above, (1.2) holds with the sign of equality. By (2.2), the latter is equivalent to the equality for the reduced modules:

$$m(\Omega(\overline{D}_n), \infty) = m(\Omega(\overline{D}^*_n(A)), \infty). \tag{4.9}$$

In order to have the sign of equality in (4.9), we must have the sign of equality in all of the relations (4.1)–(4.8). In particular, the sign of equality holds in both inequalities in (4.8), which implies that $m = n$ and $\sigma = A$. The latter equality shows, in particular, that the triangles T_i, $i = 1, \ldots, n$, are mutually disjoint and provide a triangulation of D_n.

Further, equality (4.9) implies that (4.7) holds with the sign of equality. Since $H(\alpha)$ is strictly convex, the latter yields

$$\alpha_1 = \ldots = \alpha_n = 1/n.$$

To have (4.9), we must have the sign of equality in (4.4) for all $k = 1, \ldots, n$. Now Lemma 1 and (4.2) show that for every $k = 1, \ldots, n$, T_k is an isosceles triangle having area A/n and the angle $2\pi/n$ at the base vertex a_0^k. Therefore, for every $k = 1, \ldots, n$, S_k is an isosceles infinite triangle having the angle $2\pi/n$ at ∞, which is associated with the triangle T_k.

Let $f(\zeta)$ map $\mathbb{U}^*$ conformally onto $\Omega(\overline{D}_n)$ such that $f(\infty) = \infty$ and let $G_k = f^{-1}(S_k)$. Since all S_k have the same angle $2\pi/n$ at ∞, the uniqueness assertion of Theorem 3 implies that in the case of equality in (4.3), each G_k is an infinite sector of the form $\{\zeta : |\zeta| > 1, \varphi_k < \arg \zeta < \varphi_k + 2\pi/n\}$. Moreover, geometric vertices of G_k correspond under the mapping f to the geometric vertices of S_k. Now the Schwarz reflection principle implies that f maps $\mathbb{U}^*$ conformally onto the exterior of a regular n-gon. This finishes the proof of Theorem 1. □

Now we justify the concavity result used in the proof above.

Lemma 7. *The function* $H(\alpha)$ *defined by* (4.6) *is strictly concave in* $0 < \alpha < 1/2$. *In particular,*

$$\sum_{j=1}^{n} H(\alpha_i) \leq nH(1/n) \quad \textit{if} \quad 0 < \alpha_j < 1/2 \quad \textit{and} \quad \sum_{j=1}^{n} \alpha_j = 1. \tag{4.10}$$

Proof. Using the recurrence formula $\alpha\Gamma(\alpha) = \Gamma(1+\alpha)$ and applying the reflection formula $\Gamma(\alpha) + \Gamma(1-\alpha) = \pi/\sin \pi\alpha$ to the other factor of $F(\alpha)$ and to one factor of $\Gamma(1/2+\alpha)$, we can express $H(\alpha)$ in a more symmetric form:

$$H(\alpha) = 4\alpha^2 \log 2 + \alpha \log \frac{\Gamma(1/2+\alpha)\Gamma(1/2-\alpha)}{\Gamma(1+\alpha)\Gamma(1-\alpha)}.$$

Differentiating twice, we obtain

$$\begin{aligned} H''(\alpha) \quad &= \quad 8 \log 2 + 2[\psi(1/2+\alpha) - \psi(1+\alpha) - \psi(1-\alpha) + \psi(1/2-\alpha)] \\ &\quad + \alpha[\psi'(1/2+\alpha) - \psi'(1+\alpha) + \psi'(1-\alpha) - \psi'(1/2-\alpha)]. \end{aligned}$$

For $\alpha = 0$ we use the well-known relations [1, p.15,18]

$$\psi(1) = \gamma \quad \text{and} \quad \psi(1/2) = -\gamma - 2\log 2,$$

where γ denotes the Euler constant [1, p.1] to obtain

$$H''(0) = 8\log 2 + 4\psi(1/2) - 4\psi(1) = 0. \tag{4.11}$$

A third differentiation gives:

$$\begin{aligned} H'''(\alpha) &= 3\left[\psi'(1/2+\alpha) - \psi'(1+\alpha) + \psi'(1-\alpha) - \psi'(1/2-\alpha)\right] \\ &\quad + \alpha\left[\psi''(1/2+\alpha) - \psi''(1+\alpha) - \psi''(1-\alpha) + \psi''(1/2-\alpha)\right]. \end{aligned} \tag{4.12}$$

Let $B_1(\alpha)$ and $B_2(\alpha)$ denote expressions in the first and second brackets in (4.12). Then

$$B_1(0) = 0 \quad \text{and} \quad B_1'(\alpha) = B_2(\alpha) < 0 \tag{4.13}$$

since $\psi''(t) = -2\sum_{k=0}^{\infty}(t+k)^{-3}$ increases in $t > 0$. (4.13) shows that $B_1(\alpha) < 0$ and therefore $F'''(\alpha) < 0$ for $0 < \alpha < 1/2$.

The latter and (4.11) imply that $H''(\alpha) < 0$ and therefore $H(\alpha)$ is strictly concave in $0 < \alpha < 1/2$.

It is well known that concavity of H yields (4.10). □

Proof of Lemma 2. The proof follows from Lemma 7 as shown next. Let

$$\Phi(\alpha) = \log \frac{2^{4\alpha}\Gamma^2(1/2+\alpha)}{\alpha\Gamma^2(\alpha)\tan\pi\alpha}.$$

From (2.14) we have

$$m(\Omega(\overline{D}_n^*(A)), \infty) = (1/4\pi)\,\Phi(1/n) + (1/4\pi)\,\log(\pi/A).$$

To show that $\Phi(\alpha)$ strictly decreases in α, we note that Φ is given by a difference quotient of the function H in Lemma 7, as

$$\Phi(\alpha) = \frac{H(\alpha) - H(0)}{\alpha},$$

since $H(0) = 0$. This difference quotient is a strictly decreasing function of $\alpha \in (0, 1/2)$, by the concavity of H. This proves Lemma 2. □

Steklov Institute of Mathematics at St. Petersburg, Russian Academy of Sciences, St. Petersburg, Russia
Current address: Department of Mathematical Sciences, University of Arkansas, Fayetteville, AR
E-mail address: solynin@uark.edu

The Weizmann Institute of Science, Rehovot, Israel
E-mail address: victorzalgaller@weizmann.ac.il

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