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# Equivariant de Rham torsions

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## Abstract

The purpose of this paper is to give an explicit local formula for the difference of two natural versions of equivariant analytic torsion in de Rham theory. This difference is the sum of the integral of a Chern-Simons current and of a new invariant, the $V$-invariant of an odd dimensional manifold equipped with an action of a compact Lie group. The $V$-invariant localizes on the critical manifolds of invariant Morse-Bott functions.

The results in this paper are shown to be compatible with results of Bunke, and also our with previous results on analytic torsion forms.

## Contents



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## Introduction

In a previous paper [BGo1], we have established a comparison formula for two natural versions of the holomorphic equivariant analytic torsion. This comparison formula is related to a similar formula obtained in [Go] for $\eta$-invariants. In this paper, we establish a corresponding formula, where we compare two natural versions of equivariant analytic torsion in de Rham theory. On one hand the classical equivariant version [LoRo] of the Ray-Singer analytic torsion [RS] appears. On the other hand, we construct an adequately normalized version of the infinitesimal equivariant torsion, by imitating the construction of the Chern analytic torsion forms of [BGo2], which are themselves a renormalized version of the analytic torsion forms of Bismut-Lott [BLo]. Our equivariant infinitesimal torsion is a renormalized version of the torsion suggested by Lott [Lo].

The difference of these two torsions is expressed as the integral of local quantities. One of these is an apparently new invariant of odd-dimensional manifolds equipped with the action of a Lie group. This invariant localizes naturally on the critical manifolds of an invariant Morse-Bott function.

Now, we will explain our results in more detail. Let $X$ be a compact manifold, and let $\left(F, \nabla^F\right)$ be a flat vector bundle on $X$. Let $\left(\Omega^\cdot(X,F), d^X\right)$ be the de Rham complex of $F$-valued smooth differential forms on $X$, and let $N$ be the number operator of $\Omega^\cdot(X,F)$. Let $H^\cdot(X,F)$ be the cohomology of $\left(\Omega^\cdot(X,F), d^X\right)$. Let $g^{TX}, g^F$ be metrics on $TX, F$. Let $d^{X,*}$ be the adjoint of $d^X$ with respect to the obvious $L_2$ Hermitian product on $\Omega^\cdot(X,F)$.

Let $G$ be a compact Lie group acting on $X$, whose action lifts to $F$, and which preserves $\nabla^F, g^{TX}, g^F$. Then $G$ acts on $\left(\Omega^\cdot(X,F), d^X\right)$ and on $H^\cdot(X,F)$. If $g \in G$, set

$$\vartheta_g\left(g^{TX}, \nabla^F, g^F\right)(s) = -\operatorname{Tr_s}\left[Ng\left(D^{X,2}\right)^{-s}\right]. \tag{0.1}$$

Then $\vartheta_g\left(g^{TX}, \nabla^F, g^F\right)(s)$ extends to a meromorphic function of $s \in \mathbf{C}$, which is holomorphic at 0. The quantity $\frac{\partial}{\partial s}\vartheta_g\left(g^{TX}, \nabla^F, g^F\right)(0)$, introduced in [LoRo], is called the equivariant analytic torsion or the equivariant de Rham torsion. It extends the classical Ray-Singer analytic torsion. Using this analytic torsion, an equivariant Ray-Singer metric on the equivariant determinant of $H^\cdot(X,F)$ was defined in [BZ2]. In [BZ2], anomaly formulas were established for $\|\ \|_{\lambda_G(F)}$, and the result of Lott-Rothenberg [LoRo] comparing equivariant Reidemeister and Ray-Singer metrics for unitarily flat vector bundles was extended to arbitrary flat vector bundles. The results of [BZ2] were the obvious extension to the equivariant case of the results of [BZ1], where the theorems of Cheeger [C] and Müller [Mü1, 2] were extended to arbitrary flat vector bundles. Also Bunke [Bu1] showed that for equivariant unitarily flat vector bundles, the equivariant analytic torsion can be determined by counting the cells of a $G-CW$ decomposition of $X$, up to a locally constant function on $G$.

Let $\pi : M \to S$ be a submersion with compact fibre $X$, and let $\left(F, \nabla^F\right)$ be a flat vector bundle on $M$. Then $H^{\cdot}(X,F)$ is a vector bundle on $S$, equipped with a flat connection $\nabla^{H^{\cdot}(X,F)}$. In this situation, Bismut and Lott [BLo] proved a Riemann-Roch Grothendieck formula. Namely, by [BLo], if $h(x)$ is an odd holomorphic function, one can construct odd cohomology classes $h\left(\nabla^F\right)$ on $M$. Let $e(TX)$ be the Euler class of $TX$. Then the Riemann-Roch formula of [BLo] takes the form,

$$h\left(\nabla^{H^{\cdot}(X,F)}\right) = \int_X e(TX)\, h\left(\nabla^F\right) \text{ in } H^{\mathrm{odd}}(S, \mathbf{R}). \tag{0.2}$$

In [BLo], equation (0.2) was refined at the level of differential forms. Namely, a Chern-Weil formalism was developed to represent the classes $h\left(\nabla^F\right)$ by explicit closed differential forms $h\left(\nabla^F, g^F\right)$. Let $T^H M$ be a horizontal subbundle of $TM$. With $h(x) = xe^{x^2}$, an even differential form $\mathcal{T}_h\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ was constructed on $S$, such that

$$d\mathcal{T}_h\left(T^H M, g^{TX}, \nabla^F, g^F\right) = \int_X e\left(TX, \nabla^{TX}\right) h\left(\nabla^F, g^F\right) - h\left(\nabla^{H^{\cdot}(X,F)}, g_{L_2}^{H^{\cdot}(X,F)}\right). \tag{0.3}$$

In (0.3), $\nabla^{TX}$ is a Euclidean connection on $\left(TX, g^{TX}\right)$ associated naturally to $\left(T^H M, g^{TX}\right)$, $e\left(TX, \nabla^{TX}\right)$ is the Chern-Weil representative of the Euler class $e(TX)$, and $g_{L_2}^{H^{\cdot}(X,F)}$ is the metric on $H^{\cdot}(X,F)$ obtained by identification with the corresponding fibrewise harmonic forms.

In [BGo2], the results of [BLo] were extended to an equivariant situation. Namely we assume that $G$ acts as before on $M, F$, and besides that it preserves the fibres $X$. Also we assume, as we may, that all the above objects, like $T^H M$ are $G$-invariant. If $g \in G$, let $M_g$ be the fixed-point manifold of $g$, which fibres on $S$ with fibre $X_g$. In [BGo2], we defined on $M_g$ obvious equivariant analogues $h_g\left(\nabla^F\right), h_g\left(\nabla^F, g^F\right)$ of $h\left(\nabla^F\right), h\left(\nabla^F, g^F\right)$. With $h(x) = xe^{x^2}$, we constructed in [BGo2] even forms $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$, which are such that

$$d\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right) = \int_{X_g} e\left(TX_g, \nabla^{TX_g}\right) h_g\left(\nabla^F, g^F\right) - h_g\left(\nabla^{H^{\cdot}(X,F)}, g_{L_2}^{H^{\cdot}(X,F)}\right). \tag{0.4}$$

Also, in [BGo2], we obtained what we claimed to be the 'right' normalization of the analytic torsion forms $\mathcal{T}_h\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ and

$\mathcal{T}_{h,g}\left(T^HM, g^{TX}\nabla^F, g^F\right)$, the Chern analytic torsion forms. They were denoted $\mathcal{T}_{\mathrm{ch}}\left(T^HM, g^{TX}, \nabla^F, g^F\right)$ and $\mathcal{T}_{\mathrm{ch},g}\left(T^HM, g^{TX}, \nabla^F, g^F\right)$. If $\mathrm{ch}_g^\circ\left(\nabla^F, g^F\right)$ is the odd secondary Chern form obtained in [BLo, Prop. 1.14] and in [BGo2, §2.7], then (0.4) is replaced by

$$d\mathcal{T}_{\mathrm{ch},g}\left(T^HM, g^{TX}, \nabla^F, g^F\right) = \int_{X_g} e\left(TX_g, \nabla^{TX_g}\right)\mathrm{ch}_g^\circ\left(\nabla^F, g^F\right) - \mathrm{ch}_g^\circ\left(\nabla^{H^\cdot(X,F)}, g_{L_2}^{H^\cdot(X,F)}\right). \tag{0.5}$$

In [Lo], Lott suggested the construction of an equivariant infinitesimal torsion by imitating the construction of the forms $\mathcal{T}_{h,g}\left(T^HM, g^{TX}, \nabla^F, g^F\right)$. Indeed when the structure group of the fibration $\pi : M \to S$ is the compact Lie group $G$, the torsion forms $\mathcal{T}_h\left(T^HM, g^{TX}, \nabla^F, g^F\right)$ appear as formal power series on $\mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of $G$, the argument $K$ should then be replaced by $-\Theta/2i\pi$, where $\Theta$ is the curvature of a connection on the corresponding $G$-bundle.

One purpose of this paper is to make the above construction of Lott noninfinitesimal. Namely, if $g \in G$, if $Z(g) \subset G$ is the centralizer of $g$, and if $\mathfrak{z}(g)$ is its Lie algebra, we construct in Section 2.7 an equivariant infinitesimal analytic torsion $\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX}, \nabla^F, g^F\right)$, which is a real-analytic function of $K \in \mathfrak{z}(g)$ on a neighbourhood of 0. This torsion is obtained by normalizing a corresponding $\mathcal{T}_{h,g}\left(T^HM, g^{TX}, \nabla^F, g^F\right)$. An important property of $\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX}, \nabla^F, g^F\right)$, established in (2.119) is that

$$\mathcal{T}_{\mathrm{ch},g,0}\left(g^{TX}, \nabla^F, g^F\right) = \frac{1}{2}\frac{\partial}{\partial s}\vartheta_g\left(g^{TX}, \nabla^F, g^F\right)(0). \tag{0.6}$$

The second main purpose of this paper is to give a local formula for

$$\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX}, \nabla^F, g^F\right) - \mathcal{T}_{\mathrm{ch},ge^K,0}\left(g^{TX}, \nabla^F, g^F\right).$$

One can indeed conjecture that such a formula may hold, in view of the anomaly formulas of [BZ2] and Section 2, which show that the variation of this difference with respect to $g^{TX}, g^F$ is computable locally.

A similar program was followed in [BGo1] for the holomorphic torsion where the corresponding difference was expressed as the integral of a natural equivariant Bott-Chern current, and as an exotic genus $I(\theta, \theta', x)$. Compatibility to the immersion results for Quillen metrics [Q2] and their equivariant analogues [BL], [B11] were key tests of the validity of the formula of [BGo1].

In the context of flat vector bundles on real manifolds, much less is known. In particular, there is no natural theory of cycles, which would be a geometric counterpart for the Riemann-Roch-Grothendieck formula of [BLo]. The comparison formula for the two versions of equivariant de Rham torsion is then *a priori* more mysterious.

On the other hand, as explained in [BGo1], the comparison formula for holomorphic torsion is one of the ways one can *understand* the true, if elusive, nature of holomorphic torsion. A similar expectation could then be justified in the context of de Rham torsion.

Also Bunke [Bu2] showed that for odd-dimensional oriented manifolds equipped with the trivial flat vector bundle, up to a locally constant term, Lott's equivariant torsion for the trivial vector bundle can be computed by counting cells of a $G-CW$ decomposition. The similarity of this last result with Bunke's previous results [Bu1] on classical equivariant torsion suggests that the two torsions should be related by an explicit formula.

Take $g \in G, K_0 \in \mathfrak{z}(g)$, and assume that $K = zK_0$, with $z \in \mathbf{R}^*$. The main result of this paper takes the following form:

THEOREM 0.1. *For* $z \in \mathbf{R}^*$, *if* $|z|$ *is small enough, the following identity holds*:

$$\begin{aligned}(0.7)\qquad \mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX},\nabla^F,g^F\right) - \mathcal{T}_{\mathrm{ch},ge^K,0}\left(g^{TX},\nabla^F,g^F\right) \\ = \int_{X_g} e_K\left(TX_g,\nabla^{TX_g}\right)F_K\left(X_g,g^{TX_g}\right)\mathrm{ch}_g^\circ\left(\nabla^F,g^F\right) \\ + \mathrm{Tr}^{F|_{X_g}}[g]\,V_K\left(X_g\right).\end{aligned}$$

Let us briefly describe the objects which appear in the right-hand side of (0.7). The first term is a contribution of the even-dimensional components of the fixed point manifold $X_g$ under $g$. The form $e_K\left(TX_g,\nabla^{TX_g}\right)$ is the equivariant Euler form of $TX_g$, the current $F_K\left(X_g,g^{TX_g}\right)$ is of Chern-Simons type on $X_g$. This first term represents the 'predictable' part of the formula, given what is known by the anomaly formulas.

The second term is much more mysterious. Only the odd-dimensional components of $X_g$ contribute to $V_K(X_g)$, which is a locally computable diffeomorphism invariant of $X_g$ equipped with the action of $K \in \mathfrak{z}(g)$. The fact that it is an invariant makes it is impossible to guess from the anomaly formulas.

Still, the results of [BGo2] gave us grounds to believe that such a term had to appear. In [BGo2], when the fibres $X$ of the fibration $\pi : M \to S$ carry a Morse-Smale vector field, we expressed the de Rham torsion forms $\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX},\nabla^F,g^F\right)$ in terms of corresponding combinatorial objects, and of an exotic genus $J(\theta,x)$. The results of [BGo2] led us to establish some natural properties of the term $V_K(X_g)$, or should have, even if we had no idea how it should appear. The results of [BGo2] play the same role in the present paper as the immersion formulas of [BL], [B11] in [BGo1].

In fact, a third purpose of this paper is to construct the $V$-invariants from scratch, without any reference to torsion. This program is carried through in Section 3. In the context of equivariant fibrations with odd-dimensional fibres, the $V$-invariants are even cohomology classes on the base of the fibration. In Section 4, we show that the $V$-invariants localize on the critical fibrations associated to fibrewise Morse-Bott functions, and we study their behaviour with respect to multifibrations. In the formulas involving fibrewise Morse-Bott functions, a genus $J^{\theta}(x)$ appears, which is directly related to the genus $J(\theta, x)$ of [BGo2]. These two properties are indeed critical to demonstrate the compatibility of formula (0.7) to the results of [BGo2] on analytic torsion forms, and also to the results of [Ma] on the functoriality of analytic torsion forms.

A remarkable feature of formula (0.7) is that it shows that $\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX}, \nabla^F, g^F\right)$ is indeed the correct normalization of the infinitesimal torsion. In Theorem 5.13, we also give an extension of Theorem 0.1 to the case where $X$ is the generic fibre of an equivariant fibration, so that (0.7), instead of being an equality of complex numbers is now an equality of classes of forms on the base $S$ of the fibration.

Also, we show that our results lead to a refinement of Bunke's results [Bu1], [Bu2] in arbitrary dimensions.

Now we describe the main techniques which are used in this paper. As in previous work on related subjects [BLo], [B11], [BZ1, 2], [BGo1], our main result is obtained by integrating a closed form on a domain, and by pushing the boundaries of the domain to infinity. However, while in the above references, the considered domains were 2-dimensional, here the dimension of the domain is 3. This reflects the fact that the forms $\mathrm{ch}^{\circ}\left(\nabla^F, g^F\right)$ are Chern-Simons forms, which are obtained by integration along a 1-dimensional path of connections, while torsion forms are obtained by a transgression mechanism *above* the forms $\mathrm{ch}^{\circ}\left(\nabla^F, g^F\right)$, and in fact are obtained by integration over a domain of dimension 2.

Local index theory in the context of families [B3], [BeGeV] plays an important role. In particular the Getzler rescaling [Ge] is used in the whole paper. As in [BZ1, 2], two kinds of Clifford variables appear in the analysis, and they are rescaled in different and subtly interrelated ways. Also, we use the Berezin integration formalism of Mathai-Quillen [MQ], which plays a prominent role in our local index computations.

Let us also point out that (0.7) only holds for small $|z|$. This fact is reflected in the analysis. In [BGo1], a similar difficulty appeared in the context of holomorphic torsion. In the present paper, we have used arguments taken from [BGo1] every time the difficulties were formally identical.

Finally, finite propagation speed of solutions of hyperbolic equations plays an important role, to establish that certain estimates can be localized.

This paper is organized as follows. In Section 1, we construct the classical equivariant de Rham torsion, and the corresponding equivariant Ray-Singer metric on $\det(H^{\cdot}(X,F))$. In Section 2, we define the Chern equivariant infinitesimal torsion, in relation to the Chern equivariant analytic torsion forms of [BGo2]. In Section 3, we define the $V$-invariants attached to equivariant fibrations. Their construction uses mysterious identities verified by the curvature tensor of a natural connection on $TX$. In Section 4, we give a localization formula for the $V$-invariants with respect to fibrewise Morse-Bott functions, and we evaluate the $V$-invariants of multifibrations. In Section 5, we state the main result of this paper, in a form equivalent to Theorem 0.1, and we verify that this result is compatible with other known results on analytic torsion and analytic torsion forms, in particular with the results of Bunke [Bu1, 2] where various torsions are evaluated for $G-\mathrm{CW}$ complexes, with the results of Ma [Ma] on the behaviour of analytic torsion forms with respect to multifibrations, and with our own previous results in [BGo2].

Sections 6–12 are devoted to the proof of Theorem 0.1. In Section 6, we introduce a fundamental closed 2-form on part of $\mathbf{R}^3$. In Section 7, using five intermediate results, whose proof is delayed to the next sections, we establish Theorem 0.1.

Sections 8–12 are devoted to the proof of these five intermediate results. They contain the bulk of the mainly analytic and algebraic arguments in the proof. Section 8 only contains short elementary arguments. In Sections 9–11, we essentially establish convergence results of global quantities to locally computable expressions. While the local algebraic arguments are specific to the situation which is considered here, the analytic arguments and the required estimates are taken from [BGo1], with minor changes.

Section 12 contains the bulk of the strictly analytic arguments. Its purpose is to establish a uniform estimate in a range of parameters not covered by [BGo1]. The estimates in Section 12 are in fact the ones which are needed to establish the corresponding estimates in the proofs in Sections 9–11, so that our paper is indeed self-contained.

In the whole paper, we use the superconnection formalism of Quillen [Q1]. If $\mathcal{A}$ is a $\mathbf{Z}_2$-graded algebra, if $A,B\in\mathcal{A}$, $[A,B]$ denotes the supercommutator of $A$ and $B$, i.e.

$$[A,B]=AB-(-1)^{\deg A \deg B}BA. \tag{0.8}$$

The results contained in this paper were announced in [BGo3].

## 1. The classical equivariant de Rham torsion

The purpose of this section is to recall the main properties of the classical equivariant analytic torsion in de Rham theory, and of the corresponding Ray-Singer equivariant metrics.

This section is organized as follows. In 1.1 and 1.2, we introduce the basic conventions on Clifford algebras and Berezin integrals, which will be used in the whole paper. In 1.3, we construct the equivariant Ray-Singer metrics using the equivariant Ray-Singer analytic torsion, whose non equivariant form was introduced in [RS]. In 1.4, we recall a simple formula for the Ray-Singer analytic torsion established in [BLo]. Finally, in 1.5, we recall the anomaly formulas of [BZ2] for Ray-Singer metrics.

1.1. *Real vector spaces and Clifford algebras.* Let $V$ be a finite dimensional real Euclidean vector space of dimension $n$. We denote by $\langle\ \rangle$ the scalar product on $V$. We identify $V$ and $V^*$ by the scalar product $\langle\ \rangle$. Let $c(V)$ be the Clifford algebra of $V$, i.e. the algebra spanned over $\mathbf{R}$ by $1, X \in V$ and the relations for $X, Y$,

$$XY + YX = -2\langle X, Y\rangle. \tag{1.1}$$

If $A \in V$, let $A^*$ correspond to $A \in V$. Set

$$c(A) = A^* \wedge -i_A, \qquad \hat{c}(A) = A^* \wedge +i_A. \tag{1.2}$$

The operators $c(A)$ and $\hat{c}(A)$ act naturally as odd operators on $\Lambda(V^*)$. If $A, B$ in $V$,

$$[c(A), c(B)] = -2\langle A, B\rangle, \quad [\hat{c}(A), \hat{c}(B)] = 2\langle A, B\rangle, \quad [c(A), \hat{c}(B)] = 0. \tag{1.3}$$

Then (1.3) says that $A \to c(A)$ and $A \to i\hat{c}(A)$ give two supercommuting representations of the Clifford algebra $c(V)$.

Also $c(V)$ acts naturally on itself by multiplication on the left and on the right, and these two actions commute. They will be denoted respectively by $c^l$ and $c^r$. Classically, there is a $\mathbf{Z}$-graded isomorphism of vector spaces $c(V) \simeq \Lambda(V^*)$. Let $\tau$ be the operator on $\Lambda(V^*)$, which is 1 on $\Lambda^{\mathrm{even}}(V^*)$, $-1$ on $\Lambda^{\mathrm{odd}}(V^*)$. Then one verifies easily that under the above isomorphism, if $A \in V$,

$$c(A) = c^l(A), \qquad \hat{c}(A) = \tau c^r(A). \tag{1.4}$$

In the sequel, we will often use the notation $c(V)$ and $\hat{c}(V)$ for the algebras generated respectively by the $c(A)$ and by the $\hat{c}(A)$.

If $H \in \mathrm{End}(V)$, then $H$ acts naturally as a derivation on $\Lambda(V^*)$. Let $e_1, \dots, e_n$ be an orthonormal basis of $V$. Then one verifies easily that if $H$ is antisymmetric,

$$H|_{\Lambda(V^*)} = \frac{1}{4}\langle He_i, e_j\rangle \left(c(e_i)c(e_j) - \widehat{c}(e_i)\,\widehat{c}(e_j)\right). \tag{1.5}$$

If $S \in \mathrm{End}\left(\Lambda\left(V^*\right)\right)$, its supertrace $\mathrm{Tr_s}\left[S\right]$ is given by

$$\mathrm{Tr_s}\left[S\right] = \mathrm{Tr}\left[\tau S\right]. \tag{1.6}$$

Now we state a simple result established in [BZ1, Prop. 4.9].

PROPOSITION 1.1. *Among the monomials in the* $c\left(e_i\right), \widehat{c}\left(e_j\right)$, *up to permutation,* $c\left(e_1\right)\widehat{c}\left(e_1\right)\ldots c\left(e_n\right)\widehat{c}\left(e_n\right)$ *is the only one whose supertrace does not vanish. It is given by the formula*

$$\mathrm{Tr_s}\left[c\left(e_1\right)\widehat{c}\left(e_1\right)\ldots c\left(e_n\right)\widehat{c}\left(e_n\right)\right] = \left(-2\right)^n. \tag{1.7}$$

1.2. *The Berezin integral.* Let $E$ and $V$ be real finite dimensional vector spaces of dimension $n$ and $m$. Let $g^E$ be a Euclidean metric on $E$. We will often identify $E$ and $E^*$ by the metric $g^E$. Let $e_1, \cdots, e_n$ be an orthonormal basis of $E$, and let $e^1, \cdots, e^n$ be the corresponding dual basis of $E^*$.

Let $\Lambda^{\cdot}\left(E^*\right)$ be the exterior algebra of $E^*$. It will be convenient to introduce another copy $\widehat{\Lambda}^{\cdot}\left(E^*\right)$ of this exterior algebra. If $e \in E^*$, we will denote by $\hat{e}$ the corresponding element in $\widehat{\Lambda}^{\cdot}\left(E^*\right)$.

Suppose temporarily that $E$ is oriented and that $e_1, \cdots, e_n$ is an oriented basis of $E$. Let $\int^{\widehat{B}}$ be the linear map from $\Lambda^{\cdot}\left(V^*\right)\widehat{\otimes}\widehat{\Lambda}^{\cdot}\left(E^*\right)$ into $\Lambda\left(V^*\right)$, such that if $\alpha \in \Lambda\left(V^*\right), \beta \in \widehat{\Lambda}\left(E^*\right)$,

$$\int^{\widehat{B}} \alpha\beta = 0 \text{ if } \deg\beta < \dim E, \tag{1.8}$$
$$\int^{\widehat{B}} \alpha\widehat{e}^1 \wedge \cdots \wedge \widehat{e}^n = \left(-1\right)^{n(n+1)/2}\alpha.$$

More generally, let $o(E)$ be the orientation line of $E$. Then $\int^{\widehat{B}}$ defines a linear map from $\Lambda^{\cdot}\left(V^*\right)\widehat{\otimes}\widehat{\Lambda}^{\cdot}\left(E^*\right)$ into $\Lambda^{\cdot}\left(V^*\right)\widehat{\otimes}\, o\left(E\right)$, which is called a Berezin integral.

Let $A$ be an antisymmetric endomorphism of $E$. We identify $A$ with the element of $\Lambda\left(E^*\right)$,

$$A = \frac{1}{2}\sum_{1\le i,j\le n}\left\langle e_i, Ae_j\right\rangle \widehat{e}^i \wedge \widehat{e}^j. \tag{1.9}$$

By definition, the Pfaffian $\mathrm{Pf}\left[\frac{A}{2\pi}\right]$ of $\frac{A}{2\pi}$ is given by

$$\int^{\widehat{B}} \exp\left(-A/2\pi\right) = \mathrm{Pf}\left[\frac{A}{2\pi}\right]. \tag{1.10}$$

Then $\mathrm{Pf}\left[\frac{A}{2\pi}\right]$ lies in $o(E)$. Moreover $\mathrm{Pf}\left[\frac{A}{2\pi}\right]$ vanishes if $n$ is odd.

1.3. *Equivariant Ray-Singer metrics.* Let $X$ be a compact manifold of dimension $n$. Let $F$ be a complex vector bundle on $X$, equipped with a flat connection $\nabla^F$. Let $H^{\cdot}(X,F)$ be the cohomology of the sheaf of locally flat sections of $F$.

Let $G$ be a compact Lie group, and let $\mathfrak{g}$ be its Lie algebra. We assume that $G$ acts on the left on $X$, and that this action lifts to $F$, and preserves $\nabla^F$. Then $G$ acts on $H^{\cdot}(X,F)$.

Let $(\Omega^{\cdot}(X,F), d^X)$ be the de Rham complex of smooth sections of $\Lambda^{\cdot}(T^*X)\widehat{\otimes}F$ on $X$. Then

$$H^{\cdot}(\Omega^{\cdot}(X,F), d^X) \simeq H^{\cdot}(X,F). \tag{1.11}$$

Clearly $G$ acts on $(\Omega^{\cdot}(X,F), d^X)$ by the formula

$$(gs)(x) = g_* s(g^{-1}x). \tag{1.12}$$

Then (1.11) is an identity of $G$-spaces.

We define the Lefschetz number $\chi_g(F)$ and the derived Lefschetz number $\chi'_g(F)$ by

(1.13)

$$\chi_g(F) = \sum_{i=0}^{n} (-1)^i \operatorname{Tr}^{H^i(X,F|_X)}[g], \quad \chi'_g(F) = \sum_{i=0}^{n} (-1)^i i \operatorname{Tr}^{H^i(X,F|_X)}[g].$$

Take $g \in G$. Let $X_g$ be the fixed point set of $g$ in $X$. Then $X_g$ is a totally geodesic submanifold of $X$. Let $e(TX_g)$ be the Euler class of $TX_g$. Then the Lefschetz fixed point formula asserts that

$$\chi_g(F) = \int_{X_g} e(TX_g) \operatorname{Tr}^F[g]. \tag{1.14}$$

Let $g^{TX}$, $g^F$ be smooth $G$-invariant metrics on $TX$, $E$. Let $dv_X$ be the corresponding volume form on $X$. Let $\langle\ \rangle_{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F}$ be the corresponding Hermitian product on $\Lambda^{\cdot}(T^*X)\widehat{\otimes}F$. If $s, s' \in \Omega^{\cdot}(X,F)$, put

$$\langle s, s'\rangle = \int_X \langle s, s'\rangle_{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F} \frac{dv_X}{(2\pi)^{\dim X}}. \tag{1.15}$$

Then (1.15) is a $G$-invariant Hermitian product on $\Omega^{\cdot}(X,F)$. We denote by $g^{\Omega^{\cdot}(X,F)}$ the corresponding Hermitian metric on $\Omega^{\cdot}(X,F)$.

Let $d^{X,*}$ be the formal adjoint of $d^X$ with respect to (1.15). Put

$$D^X = d^X + d^{X,*}. \tag{1.16}$$

Then $D^X$ is a first order elliptic operator. By Hodge theory,

$$\ker D^X \simeq H^{\cdot}(X,F). \tag{1.17}$$

Also $D^X$ commutes with $G$, so that $G$ acts on $\ker D^X$. Then (1.17) is an identification of $G$-spaces. Also $\ker D^X$ inherits a $G$-invariant Hermitian product from the Hermitian product (1.15) on $\Omega^{\cdot}(X,F)$. Let $g_{L_2}^{H^{\cdot}(X,F)}$ be the corresponding Hermitian metric on $H^{\cdot}(X,F)$.

Let $\widehat{G}$ be the set of equivalence classes of irreducible representations of $G$. If $W\in\widehat{G}$, let $\chi_W$ be the character of $G$ associated to $W$. Then we have the isotypical decomposition

$$H^{\cdot}(X,F)=\oplus_{W\in\widehat{G}}\mathrm{Hom}_G(W,H^{\cdot}(X,F))\otimes W, \tag{1.18}$$

which is orthogonal with respect to $g_{L_2}^{H^{\cdot}(X,F)}$. If $W\in\widehat{G}$, put

$$\lambda_W(F)=\det(\mathrm{Hom}_G(W,H^{\cdot}(X,F))\otimes W). \tag{1.19}$$

Then $\lambda_W(F)$ is a complex line. Set

$$\lambda_G(F)=\oplus_{W\in\widehat{G}}\lambda_W(F). \tag{1.20}$$

Let $|\ |_{\lambda_W(F)}$ be the metric induced by $g_{L_2}^{H^{\cdot}(X,F)}$ on $\lambda_W(F)$.

*Definition* 1.2. Set

$$\log\left(|\ |_{\lambda_G(F)}\right)=\sum_{W\in\widehat{G}}\log(|\ |_{\lambda_W(F)})\frac{\chi_W}{\mathrm{rk}(W)}. \tag{1.21}$$

The symbol $|\ |_{\lambda_G(F)}$ will be called an equivariant $L_2$ metric on $\lambda_G(F)$.

Let $\ker(D^X)^{\perp}$ be the orthogonal vector space to $\ker(D^X)$ in $\Omega^{\cdot}(X,F)$. Then $D^{X,2}$ acts as an invertible operator on $\ker(D^X)^{\perp}$. Let $(D^{X,2})^{-1}$ denote the inverse of $D^{X,2}$ acting on $\ker(D^X)^{\perp}$.

Let $N$ be the number operator of $\Omega^{\cdot}(X,F)$, i.e. $N$ acts by multiplication by $k$ on $\Omega^k(X,F)$. By standard heat equation methods [Gi], [BeGeV], there exists $\ell$ with $2\ell\in\mathbf{N}^*$ such that as $t\to 0$, for any $k\in\mathbf{N}$,

$$\begin{aligned}\mathrm{Tr_s}[Ng\exp(-tD^{X,2}]=\frac{a_\ell}{t^\ell}+\dots a_0+a_{1/2}t^{1/2}+\dots\\ \dots+a_{k-1/2}t^{k-1/2}+a_kt^k+o(t^k).\end{aligned} \tag{1.22}$$

*Definition* 1.3. For $g\in G, s\in\mathbf{C}, \mathrm{Re}(s)>\ell$, put

$$\vartheta_g(g^{TX},\nabla^F,g^F)(s)=-\mathrm{Tr_s}[Ng(D^{X,2})^{-s}]. \tag{1.23}$$

By (1.22), $\vartheta_g(g^{TX},\nabla^F,g^F)(s)$ extends to a meromorphic function of $s\in\mathbf{C}$, which is holomorphic at $s=0$. In particular, $g\in G\mapsto\frac{\partial}{\partial s}\vartheta_g(g^{TX},\nabla^F,g^F)(0)\in\mathbf{C}$ is a central function. When $g=1$, it was introduced by Ray and Singer [RS]. This function is called the Ray-Singer equivariant analytic torsion.

*Definition* 1.4. For $g\in G$, put

$$\log\left(\|\ \|_{\lambda_G(F)}\right)(g)=\log\left(|\ |_{\lambda_G(F)}\right)(g)+\frac{1}{2}\frac{\partial}{\partial s}\vartheta_g(g^{TX},\nabla^F,g^F)(0). \tag{1.24}$$

The symbol $\| \|_{\lambda_G(F)}$ will be called an equivariant Ray-Singer metric on $\lambda_G(F)$.

1.4. *A formula for the Ray-Singer equivariant torsion.* Put

$$A^X = \frac{1}{2}\left(d^{X,*} + d^X\right), \qquad B^X = \frac{1}{2}\left(d^{X,*} - d^X\right). \tag{1.25}$$

Then

$$D^{X,2} = 4A^{X,2} = -4B^{X,2}. \tag{1.26}$$

Let $\nabla^{TX}$ be the Levi-Civita connection on $TX$, let $\nabla^{\Lambda^{\cdot}(T^*X)}$ be the corresponding connection on $\Lambda^{\cdot}(T^*X)$.

Put

$$\omega\left(\nabla^F, g^F\right) = \left(g^F\right)^{-1} \nabla^F g^F. \tag{1.27}$$

Then $\omega\left(\nabla^F, g^F\right)$ is a 1-form on $X$ with values in self-adjoint elements in $\operatorname{End}(F)$, which is such that

$$\nabla^F \omega\left(\nabla^F, g^F\right) = -\omega^2\left(\nabla^F, g^F\right). \tag{1.28}$$

Let $\nabla^{F,u}$ be the connection on $F$,

$$\nabla^{F,u} = \nabla^F + \frac{1}{2}\omega\left(\nabla^F, g^F\right). \tag{1.29}$$

Then one verifies easily that $\nabla^{F,u}$ is unitary, and that its curvature $R^{F,u}$ is given by

$$R^{F,u} = -\frac{1}{4}\omega^2\left(\nabla^F, g^F\right). \tag{1.30}$$

Also, by (1.28), (1.29),

$$\nabla^{F,u}\omega\left(\nabla^F, g^F\right) = 0. \tag{1.31}$$

Let $\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes} F,u}$ be the connection on $\Lambda^{\cdot}(T^*X)\widehat{\otimes} F$ induced by $\nabla^{\Lambda^{\cdot}(T^*X)}$ and $\nabla^{F,u}$.

Let $e_1, \dots, e_n$ be an orthonormal basis of $TX$. Then by [BZ1, proof of Proposition 4.12], we have the easy formulas,

$$\begin{aligned} A^X &= \frac{1}{2}c(e_i)\nabla_{e_i}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes} F,u} - \frac{1}{4}\widehat{c}(e_i)\,\omega\left(\nabla^F, g^F\right)(e_i), \\ B^X &= -\frac{1}{2}\widehat{c}(e_i)\nabla_{e_i}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes} F,u} + \frac{1}{4}c(e_i)\,\omega\left(\nabla^F, g^F\right)(e_i). \end{aligned} \tag{1.32}$$

Set

$$h(x) = xe^{x^2}. \tag{1.33}$$

*Definition* 1.5. Put

$$h_g^{\wedge}\left(d^X, g^{\Omega^{\cdot}(X,F)}\right) = \operatorname{Tr_s}\left[\frac{N}{2} g h'\left(B^X\right)\right]. \tag{1.34}$$

For $t>0$, set

$$g_t^{TX}=\frac{g^{TX}}{t}. \tag{1.35}$$

Let $g_t^{\Omega^{\cdot}(X,F)}$ be the corresponding Hermitian metric on $\Omega^{\cdot}(X,F)$. Then

$$g_t^{\Omega^{\cdot}(X,F)}=t^{N-n/2}g^{\Omega^{\cdot}(X,F)}. \tag{1.36}$$

Let $B_t^X$ be the corresponding $B^X$. Clearly

$$B_t^X=\frac{1}{2}\left(d^X-td^{X,*}\right), \tag{1.37}$$

so that

$$B_t^{X,2}=tB^{X,2}. \tag{1.38}$$

By (1.34), (1.38),

$$h_g^{\wedge}\left(d^X,g_t^{\Omega^{\cdot}(X,F)}\right)=\mathrm{Tr_s}\left[\frac{N}{2}gh'\left(\sqrt{t}B^X\right)\right]. \tag{1.39}$$

Now we have the result of [BLo, Th. 3.2], [BGo2, Th. 3.30].

THEOREM 1.6. *As $t\to 0$,*

$$h_g^{\wedge}\left(d^X,g_t^{\Omega^{\cdot}(X,F)}\right)=\frac{1}{4}\dim X\chi_g(F)\,h'(0)+\mathcal{O}\left(\sqrt{t}\right). \tag{1.40}$$

*As $t\to+\infty$,*

$$h_g^{\wedge}\left(d^X,g_t^{\Omega^{\cdot}(X,F)}\right)=\frac{1}{2}\chi_g'(F)\,h'(0)+\mathcal{O}\left(1/\sqrt{t}\right). \tag{1.41}$$

*Definition* 1.7. Set

$$\begin{aligned}\mathcal{T}_{h,g}\left(g^{TX},\nabla^F,g^F\right)=-\int_0^{+\infty}\Bigg[&h_g^{\wedge}\left(d^X,g_t^{\Omega^{\cdot}(X,F)}\right)-\frac{1}{2}\chi_g'(F)\,h'(0)\\&-\left(\frac{1}{4}\dim X\chi_g(F)-\frac{1}{2}\chi_g'(F)\right)h'\left(i\sqrt{t}/2\right)\Bigg]\frac{dt}{t}.\end{aligned} \tag{1.42}$$

Now we have the result established in [BLo, Th. 3.29], [BGo2, Rem. 3.36].

THEOREM 1.8. *The following identity holds*:

$$\mathcal{T}_{h,g}\left(g^{TX},\nabla^F,g^F\right)=\frac{1}{2}\frac{\partial\vartheta_g}{\partial s}\left(g^{TX},\nabla^F,g^F\right)(0). \tag{1.43}$$

1.5. *Anomaly formulas for equivariant Ray-Singer metrics.* Let $Q^X$ be the vector space of smooth forms on $X$, let $Q^{X,0}\subset Q^X$ be the subspace of exact smooth forms on $M$.

Let $\nabla^{TX}$ be the Levi-Civita connection on $\left(TX,g^{TX}\right)$. Since $X_g$ is totally geodesic, $\nabla^{TX}$ induces the Levi-Civita connection $\nabla^{TX_g}$ on $TX_g$. Let $R^{TX_g}$ be the curvature of $\nabla^{TX_g}$. Let $e\left(TX_g, \nabla^{TX_g}\right)$ be the Euler form of $TX_g$ which one associates to the Euclidean connection $\nabla^{TX_g}$ using Chern-Weil theory. Then

$$e\left(TX_g, \nabla^{TX_g}\right) = \mathrm{Pf}\left[\frac{R^{TX_g}}{2\pi}\right]. \tag{1.44}$$

Observe that if $n_g = \dim X_g$ is odd, then

$$e\left(TX_g, \nabla^{TX_g}\right) = 0. \tag{1.45}$$

Also $\mathrm{Tr}\left[g\omega\left(\nabla^F, g^F\right)/2\right]$ is a closed 1-form on $X_g$.

Let now $g'^{TX}, g'^F$ be another couple of $G$-invariant metrics on $TX, F$. One denotes with a superscript $'$ the objects we just constructed, which are associated to this new couple of metrics. Let $\widetilde{e}\left(TX_g, \nabla^{TX_g}, \nabla'^{TX_g}\right) \in Q^{X_g}/Q^{X_g,0}$ be the corresponding Chern-Simons class, so that

$$d\widetilde{e}\left(TX_g, \nabla^{TX_g}, \nabla'^{TX_g}\right) = e\left(TX_g, \nabla'^{TX_g}\right) - e\left(TX_g, \nabla^{TX_g}\right). \tag{1.46}$$

Observe that

$$d\log\left(\det\left(g'^F/g^F\right)^{1/2}\right) = \mathrm{Tr}\left[\omega\left(\nabla^F, g'^F\right)/2\right] - \mathrm{Tr}\left[\omega\left(\nabla^F, g^F\right)/2\right]. \tag{1.47}$$

More generally, by splitting $F|_{X_g}$ as an orthogonal direct sum of vector bundles indexed by the locally constant distinct eigenvalues of $g|_{F|_{X_g}}$, we obtain easily a smooth function

$$\log\left(\det\left(g'^F/g^F\right)^{1/2}\right)(g)$$

on $X_g$, which is such that

$$d\log\left(\det\left(g'^F/g^F\right)^{1/2}\right)(g) = \mathrm{Tr}\left[g\omega\left(\nabla^F, g'^F\right)/2\right] - \mathrm{Tr}\left[g\omega\left(\nabla^F, g^F\right)/2\right]. \tag{1.48}$$

We now state the anomaly formula for equivariant Ray-Singer metrics, which was established in [BZ2, Th. 0.1] (we also refer to [W] for another proof of this formula).

Theorem 1.9. *The following identity holds*:

$$\log\left(\frac{\| \ \|'_{\lambda_G(F)}}{\| \ \|_{\lambda_G(F)}}\right)(g) = \int_{X_g} e\left(TX_g, \nabla^{TX_g}\right)\log\left(\det\left(g'^F/g^F\right)^{1/2}\right)(g) + \int_{X_g} \widetilde{e}\left(TX_g, \nabla^{TX_g}, \nabla'^{TX_g}\right)\mathrm{Tr}\left[g\omega\left(\nabla^F, g'^F\right)/2\right]. \tag{1.49}$$

## 2. The Chern equivariant infinitesimal analytic torsion

The purpose of this section is to construct the Chern equivariant infinitesimal analytic torsion forms. Their construction is at least formally related to the construction of corresponding torsion forms in complex geometry [BGo1]. It is imitated from the Chern normalization of the analytic torsion forms in de Rham theory which was given in [BGo2].

This section is organized as follows. In 2.1, we describe in some detail the Lie derivative operator $L_K$ acting on $\Omega^{\cdot}(X, F)$. In 2.2, we make the fundamental assumption that the action of $G$ on $F$ is flat. In 2.3, we recall simple results on Lefschetz and Kirillov-like formulas for the equivariant Euler characteristic $\chi_g(F)$. In 2.4, we briefly recall the heat equation proof given in [B2] of Kirillov-like formulas. In 2.5, and following [BGo2], we define the equivariant analytic torsion forms associated to a fibration with compact structure group. In 2.6, we construct the equivariant infinitesimal analytic torsion, and in 2.7, we obtain the corresponding Chern analogue. Finally in Section 2.8, we briefly establish anomaly formulas for the associated Ray-Singer metrics.

We make the same assumptions as in Section 1, and we use the corresponding notation.

2.1. *The infinitesimal action of G.* If $K \in \mathfrak{g}$, let $K^X$ be the corresponding vector field on $X$. If $K, K' \in \mathfrak{g}$, then

$$[K^X, K'^X] = -[K, K']^X. \tag{2.1}$$

*Definition* 2.1. If $K \in \mathfrak{g}$, set

$$m^{TX}(K) = \nabla^{TX}_{\cdot} K^X. \tag{2.2}$$

The vector field $K^X$ is Killing. Therefore $m^{TX}(K)$ is a skew-adjoint section of $\mathrm{End}(TX)$, which is also the vertical part with respect to $\nabla^{TX}$ of the lift $K^{TX}$ of $K^X$ to $TX$. We identify $m^{TX}(K)$ with its action on $\Lambda^{\cdot}(T^*X)$. By (1.5),

$$m^{TX}(K) = \frac{1}{4}\left\langle \nabla^{TX}_{e_i} K^X, e_j \right\rangle \left(c(e_i)\, c(e_j) - \widehat{c}(e_i)\, \widehat{c}(e_j)\right). \tag{2.3}$$

Recall that $\nabla^{\Lambda^{\cdot}(T^*X)}$ is the connection on $\Lambda^{\cdot}(T^*X)$ induced by $\nabla^{TX}$. Then $\nabla^{\Lambda^{\cdot}(T^*X)}$ acts on the standard de Rham complex $\left(\Omega^{\cdot}(X), d^X\right)$. Since $\nabla^{TX}$ is torsion-free, the Lie derivative operator $L_K$ is given by

$$L_K = \nabla^{\Lambda^{\cdot}(T^*X)}_{K^X} - m^{TX}(K). \tag{2.4}$$

Let $F$ be a complex vector bundle on $X$ as in Section 1.3, equipped with a flat connection $\nabla^F$, and a Hermitian metric $g^F$. Recall that the action of $G$ on $X$ lifts to $F$ and preserves $\nabla^F$ and $g^F$. Let $K^F$ be the vector field induced by the action of $K$ on $F$. Let $m^F(K) \in \mathrm{End}(F)$ be the vertical part of $K^F$

with respect to the $G$-invariant flat connection $\nabla^F$ in $F$. Then $m^F(K)$ is a section of $\mathrm{End}(F)$, and the action $L_K$ of $K$ on smooth sections of $F$ is given by

$$L_K = \nabla^F_{K^X} - m^F(K). \tag{2.5}$$

Let $\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F}$ be the connection on $\Lambda^\cdot(T^*X)\widehat{\otimes}F$ which is induced by $\nabla^{TX}$ and $\nabla^F$. Then $\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F}$ acts on $\Omega^\cdot(X,F)$. The Lie derivative operator $L_K$ also acts on $\Omega^\cdot(X,F)$. More precisely

$$L_K = \nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F} - m^{TX}(K) - m^F(K). \tag{2.6}$$

Moreover since the connections $\nabla^{TX}$ and $\nabla^F$ are $G$-invariant, by [BeGeV, Chap. 7],

$$\begin{aligned} \nabla^{TX}_\cdot m^{TX}(K) + i_{K^X}R^{TX} &= 0,\\ \nabla^F_\cdot m^F(K) &= 0. \end{aligned} \tag{2.7}$$

The metric $g^F$ gives an identification of $F$ and $\overline{F}^*$. Under this identification, the flat connection $\nabla^{\overline{F}^*}$ is given by

$$\nabla^{\overline{F}^*} = \nabla^F + \omega\left(\nabla^F, g^F\right). \tag{2.8}$$

Then applying the second identity in (2.7) to $\overline{F}^*$,

$$\nabla^F_\cdot m^{F,*}(K) = -\left[\omega\left(\nabla^F, g^F\right), m^{F,*}(K)\right]. \tag{2.9}$$

Since the metric $g^F$ is $G$-invariant,

$$\omega\left(\nabla^F, g^F\right)\left(K^X\right) = -\left(m^F(K) + m^{F,*}(K)\right). \tag{2.10}$$

By (2.9), (2.10),

$$\nabla^F_{K^X} m^{F,*} = -\left[m^F(K), m^{F,*}(K)\right]. \tag{2.11}$$

Also since $\omega\left(\nabla^F, g^F\right)\left(K^X\right)$ is $K^X$-invariant,

$$\nabla^F_{K^X}\omega\left(\nabla^F, g^F\right)\left(K^X\right) = \left[m^F(K), \omega\left(\nabla^F, g^F\right)\left(K^X\right)\right], \tag{2.12}$$

which fits with (2.7), (2.10), (2.11).

Recall that the unitary connection $\nabla^{F,u}$ on $F$ was defined in (1.29) by the formula

$$\nabla^{F,u} = \frac{1}{2}\left(\nabla^{F,*} + \nabla^F\right). \tag{2.13}$$

Equivalently, using (2.8), we get

$$\nabla^{F,u} = \nabla^F + \frac{1}{2}\omega\left(\nabla^F, g^F\right). \tag{2.14}$$

By (1.30), its curvature $R^{F,u}$ is given by

$$R^{F,u} = -\frac{1}{4}\omega^2\left(\nabla^F, g^F\right). \tag{2.15}$$

Clearly the connection $\nabla^{F,u}$ is still $G$-invariant. From (2.7), (2.9), we get

$$\nabla^{F,u}_{\cdot} m^F(K) = \frac{1}{2}\left[\omega\left(\nabla^F, g^F\right), m^F(K)\right], \tag{2.16}$$
$$\nabla^{F,u}_{\cdot} m^{F,*}(K) = -\frac{1}{2}\left[\omega\left(\nabla^F, g^F\right), m^{F,*}(K)\right].$$

Needless to say, since $\nabla^{F,u}$ is unitary, the two equations in (2.16) are equivalent.

Let $m^{F,u}(K) \in \operatorname{End}(F)$ be the vertical part of $K^F$ with respect to the connection $\nabla^{F,u}$. Recall that $\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes} F,u}$ is the connection on $\Lambda^{\cdot}(T^*X)\widehat{\otimes} F$ induced by $\nabla^{TX}$ and $\nabla^{F,u}$. Instead of (2.4), we now have

$$L_K = \nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes} F,u}_{K^X} - m^{TX}(K) - m^{F,u}(K). \tag{2.17}$$

Comparing (2.6), (2.14), (2.17), we get

$$m^{F,u}(K) = \frac{1}{2}\omega\left(\nabla^F, g^F\right)\left(K^X\right) + m^F(K). \tag{2.18}$$

Using (2.10) and (2.18), we obtain

$$m^{F,u}(K) = \frac{1}{2}\left(m^F(K) - m^{F,*}(K)\right). \tag{2.19}$$

Note that, as it should be, $m^{F,u}(K)$ takes its values in skew adjoint sections of $\operatorname{End}(F)$. Also since $\nabla^{F,u}$ is $G$-invariant, as in (2.7), we get

$$\nabla^{F,u} m^{F,u}(K) + i_{K^X} R^{F,u} = 0. \tag{2.20}$$

One verifies easily that (2.20) also follows from (2.15), (2.16), (2.18).

2.2. *A fundamental assumption.* In the sequel, we make the fundamental assumption that for any $K \in \mathfrak{g}$,

$$m^F(K) = 0. \tag{2.21}$$

By (2.10), we find that

$$\omega\left(\nabla^F, g^F\right)\left(K^X\right) = 0. \tag{2.22}$$

By (2.21), we get the identity of operators acting on $\Omega^{\cdot}(X, F)$,

$$L_K = \left[d^X, i_{K^X}\right]. \tag{2.23}$$

By (2.23), we find that if $G_0 \subset G$ is the connected component of the identity, then $G_0$ acts trivially on $H^{\cdot}(X, F)$.

Also by (2.17), (2.19), (2.21), we get

$$L_K = \nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes} F,u}_{K^X} - m^{TX}(K). \tag{2.24}$$

2.3. *Lefschetz and Kirillov formulas.* Recall that by (1.13),

$$\chi_g(F) = \mathrm{Tr_s}^{H^{\cdot}(X,F)}[g]\,. \tag{2.25}$$

The McKean-Singer heat equation formula [MKeS] asserts that for any $t > 0$,

$$\chi_g(F) = \mathrm{Tr_s}[g\exp(-tD^{X,2})]. \tag{2.26}$$

By making $t \to 0$ in (2.26), and using local index theory [ABoP], [Gi], [B1], [BeGeV, Chapter 6], one obtains the Lefschetz formula of (1.14),

$$\chi_g(F) = \int_{X_g} e(TX_g)\,\mathrm{Tr}^F[g]\,. \tag{2.27}$$

Let $(\Omega^{\cdot}(X), d)$ be the standard de Rham complex on $X$, equipped with the de Rham operator $d$. Now we follow Berline-Vergne [BeV], with the only difference that $2\pi i$ in [BeV] has been changed here into $2\pi$. Set

$$d_K = d^X - 2\pi i_{K^X}\,. \tag{2.28}$$

Clearly,

$$d_K^2 = -2\pi L_K. \tag{2.29}$$

Also,

$$[L_K, d_K] = 0. \tag{2.30}$$

By (2.29), (2.30), the operator $d_K$ acts on the set of smooth $K^X$-invariant forms, and its square vanishes. The cohomology groups of $d_K$ are related to the equivariant cohomology of $X$.

Put

$$R_K^{TX} = R^{TX} - 2\pi m^{TX}(K)\,. \tag{2.31}$$

Then $R_K^{TX}$ is called the equivariant curvature of $\nabla^{TX}$. With a similar definition, since $R^F = 0, m^F(K) = 0$, the equivariant curvature of $\nabla^F$ vanishes identically. Finally, using (2.19) and (2.21), we find that the equivariant curvature $R_K^{F,u}$ of $\nabla^{F,u}$ is given by

$$R_K^{F,u} = R^{F,u}. \tag{2.32}$$

Take $g \in G$. Let $Z(g) \subset G$ be the centralizer of $g$, and let $\mathfrak{z}(g)$ be its Lie algebra. Then

$$\mathfrak{z}(g) = \{K \in \mathfrak{g},\, g.K = K\}. \tag{2.33}$$

In the sequel, we always take $g \in G, K \in \mathfrak{z}(g)$. Put

$$X_K = \{x \in X,\, K^X(x) = 0\}. \tag{2.34}$$

Then $X_K$, which is the fixed point set of the group generated by $K$, is a totally geodesic submanifold of $X$. Set

$$X_{g,K} = X_g \cap X_K. \tag{2.35}$$

Then $X_{g,K}$ is a totally geodesic submanifold of $X$. More precisely, if $K_0 \in \mathfrak{z}(g)$ and, for $z \in \mathbf{R}^*$, $K = zK_o$, for $z$ small enough,

$$X_{g,K} = X_{ge^K}. \tag{2.36}$$

Since $K \in \mathfrak{z}(g)$, the vector field $K^X$ is $g$-invariant. In particular $K^X|_{X_g} \in TX_g$. So $K^X|_{X_g}$ is the Killing vector field $K^{X_g}$ on $X_g$. Since $X_g$ is totally geodesic, $m^{TX}(K)|_{X_g}$ preserves $TX_g$. More precisely,

$$m^{TX}(K)|_{TX_g} = m^{TX_g}(K). \tag{2.37}$$

The equivariant curvature $R_K^{TX_g}$ of $\nabla^{TX_g}$ is given by

$$R_K^{TX_g} = R^{TX_g} - 2\pi m^{TX_g}(K). \tag{2.38}$$

*Definition* 2.2. For $K \in \mathfrak{z}(g)$, set

$$e_K\left(TX_g, \nabla^{TX_g}\right) = \mathrm{Pf}\left[\frac{R_K^{TX_g}}{2\pi}\right]. \tag{2.39}$$

Then $e_K\left(TX_g, \nabla^{TX_g}\right)$ is a smooth form on $X_g$. It is called the *equivariant Euler form* of $\left(TX_g, \nabla^{TX_g}\right)$. Moreover by [BeV], [BeGeV, Th. 7.7],

$$d_K e_K\left(TX_g, \nabla^{TX_g}\right) = 0. \tag{2.40}$$

The form $e_K\left(TX_g, \nabla^{TX_g}\right)$ defines an equivariant cohomology class $e_K(TX_g)$. This class does not depend on the metric $g^{TX_g}$ or the connection $\nabla^{TX_g}$.

Observe that the function $\mathrm{Tr}^F[g]$ is constant on $X_g$. Counting degrees, we get

$$\int_{X_g} e_K(TX_g)\,\mathrm{Tr}^F[g] = \int_{X_g} e(TX_g)\,\mathrm{Tr}^F[g]. \tag{2.41}$$

Then the localization formulas of Duistermaat-Heckman [DuH], Berline-Vergne [BeV], or, more simply, the fact that $X_{g,K}$ is the zero set of $K^X|_{X_g}$ show that

$$\int_{X_g} e_K(TX_g)\,\mathrm{Tr}^F[g] = \int_{X_{g,K}} e(TX_{g,K})\,\mathrm{Tr}^F[g]. \tag{2.42}$$

Recall that $m^F(K) = 0$, so that, on $X_{g,K}$,

$$\mathrm{Tr}^F\left[ge^K\right] = \mathrm{Tr}^F[g]. \tag{2.43}$$

From the Lefschetz fixed point formula in (2.27), and from (2.41)–(2.43), we recover a form of the Kirillov formulas as in Berline-Vergne [BeV]. Namely,

$$\chi_{ge^K}(F) = \int_{X_{g,K}} e(TX_{g,K})\,\mathrm{Tr}^F[g], \quad \chi_{ge^K}(F) = \int_{X_g} e(TX_g)\,\mathrm{Tr}^F[g], \tag{2.44}$$
$$\chi_{ge^K}(F) = \int_{X_g} e_K(TX_g)\,\mathrm{Tr}^F[g].$$

On the other hand, we saw after (2.23) that the action of $ge^K$ on $H^{\cdot}(X,F)$ coincides with the action of $g$. In particular,

$$\chi_{ge^K}(F) = \chi_g(F)\,. \tag{2.45}$$

Clearly ((2.44), (2.45) are compatible.

2.4. *The heat equation proof of the Kirillov formulas.* First observe that by (2.15), (2.32) and by proceeding as in [BLo, Prop. 1.3], [BGo2, Prop. 1.6], we have the identity of forms on $X_g$,

$$\mathrm{Tr}\left[g\exp\left(-\frac{R_K^{F,u}}{2\pi}\right)\right] = \mathrm{Tr}^F[g]\,. \tag{2.46}$$

Now, we will briefly explain the direct heat equation proof of the last equation in (2.44),

$$\chi_{ge^K}(F) = \int_{X_g} e_K(TX_g)\,\mathrm{Tr}^F[g]\,. \tag{2.47}$$

This proof is a special case of the results in [B2] providing us with an analogue of the heat equation proof of the Atiyah-Bott-Lefschetz formulas outlined in (2.26), (2.27).

*Definition* 2.3. For $t>0$, put

$$C_{K,t} = \frac{1}{2}\left(\sqrt{t}D^X + \frac{c(K^X)}{\sqrt{t}}\right). \tag{2.48}$$

A first trivial step in the heat equation proof of (2.47) is an obvious extension of the McKean-Singer formula,

$$\chi_{ge^K}(F) = \mathrm{Tr_s}[g\exp(-L_K - C_{K,t}^2)]. \tag{2.49}$$

For $g=1$, we then showed in [B2] that for $|K|$ small enough, 'fantastic cancellations' still occur as $t\to 0$ in the local supertrace of the operator $g\exp(-L_K - C_{K,t}^2)$, so that the limit of this local supertrace exists, and is given by the integrand in the right-hand side of (2.47). For this last step, we also need equation (2.46). We thus get a direct proof of (2.47) in the case $g=1$. The case of a general $g$ is treated as in [BGo1, Th. 7.9].

2.5. *Analytic de Rham torsion forms and compact Lie groups.* In this Section, we recall the construction in [BLo] and in [BGo2] of the equivariant analytic torsion forms and of their Chern normalized version, when the structure group of the given fibration is compact.

Let $S$ be a smooth manifold. Let $p : P \overset{Z(g)}{\to} S$ be a $Z(g)$ principal bundle over $S$. We equip this principal bundle with a connection. Let $\theta$ be the connection form on $P$, and let $\Theta$ be its curvature. Then $\Theta$ is a 2-form with values in the vector bundle $P \times_{Z(g)} \mathfrak{g}$.

Put

$$M = P \times_{Z(g)} X. \tag{2.50}$$

Then $\pi : M \overset{X}{\to} S$ is a fibration with compact fibre $X$. Also $g$ acts fibrewise on $M$. Clearly $Z(g)$ acts on $X_g$. Then if $M_g \subset M$ is the fixed point set by $g$ in $M$, then

$$M_g = P \times_{Z(g)} X_g. \tag{2.51}$$

The connection on $P$ induces a connection on $M$; i.e., we have a splitting

$$TM = T^H M \oplus TX, \tag{2.52}$$

and this splitting is $g$-invariant. Using (2.52), we get the isomorphism

$$\Lambda^{\cdot}(T^*M) \simeq \pi^* \Lambda^{\cdot}(T^*S) \widehat{\otimes} \Lambda^{\cdot}(T^*X). \tag{2.53}$$

Also observe that the given connection on $M$ induces a corresponding connection on $M_g$.

Let $P^{TX} : TM \to TX$ be the obvious projection with respect to the splitting (2.52). If $U \in TS$, let $U^H \in T^H M$ be the horizontal lift of $U$. If $U, V \in TS$, put

$$T^H(U,V) = -P^{TX}[U^H, V^H]. \tag{2.54}$$

One verifies easily that $T^H$ is a tensor.

Proposition 2.4. *The tensor $T^H$ is a 2-form on $S$ with values in Killing vector fields along the fibres $X$. More precisely,*

$$T^H = \Theta^X. \tag{2.55}$$

*Proof.* Equation (2.55) follows from (2.54). □

The vector bundle $P \times_{Z(g)} F$ is a vector bundle on $M$. We still denote it by $F$. Then $F$ is a Hermitian vector bundle on $M$. Since the connection $\nabla^F$ is $G$-invariant, the vector bundle $F$ on $M$ is equipped with a connection, which we still denote by $\nabla^F$. Since $m^F(K) = 0$, one verifies easily that $\nabla^F$ is still flat. The form $\omega\left(\nabla^F, g^F\right)$ on $X$ is $G$-invariant, and so it descends to a 1-form along the fibres of $X$ with values in $\mathrm{End}(F)$, which we still denote by $\omega\left(\nabla^F, g^F\right)$. We identify $\omega\left(\nabla^F, g^F\right)$ with the corresponding vertical 1-form on $M$. This form $\omega\left(\nabla^F, g^F\right)$ is just the obvious analogue for $M$ of the form $\omega\left(\nabla^F, g^F\right)$ on $X$, with respect to the flat connection $\nabla^F$. Similarly $Z(g)$ acts on $\Omega^{\cdot}(X,F)$. Therefore $\Omega^{\cdot}(X,F)$ descends to a vector bundle on $S$, given by $P\times_{Z(g)}\Omega^{\cdot}(X,F)$. The metric $g^{\Omega^{\cdot}(X,F)}$ descends to a Hermitian metric on the vector bundle $\Omega^{\cdot}(X,F)$.

The connection on $P$ induces a Hermitian connection $\nabla^{\Omega^{\cdot}(X,F)}$ on $\Omega^{\cdot}(X,F)$.

The operator $d^X$ being $Z(g)$-invariant descends to an operator acting on the vector bundle $\Omega^{\cdot}(X,F)$. Therefore $\left(\Omega^{\cdot}(X,F),d^X\right)$ is a complex of infinite-dimensional vector bundles on $S$.

Let $\Omega^{\cdot}(S,\Omega^{\cdot}(X,F))$ be the space of smooth sections of $\Lambda^{\cdot}(T^*S)\widehat{\otimes}\Omega^{\cdot}(X,F)$ on $S$. Using (2.53), we have the isomorphism

$$\Omega^{\cdot}(S,\Omega^{\cdot}(X,F))\simeq\Omega^{\cdot}(M,F)\,. \tag{2.56}$$

Via (2.56), the operators $d^X,\nabla^{\Omega^{\cdot}(X,F)},i_{T^H}$ act naturally on $\Omega^{\cdot}(M,F)$. Let $d^M$ be the de Rham operator acting on $\Omega^{\cdot}(M,F)$. One verifies easily that

$$d^M=d^X+\nabla^{\Omega^{\cdot}(X,F)}+i_{T^H}. \tag{2.57}$$

Then

$$d^{M,2}=0. \tag{2.58}$$

As explained in [BLo, §3 (b)], [BGo2, §3.2], by (2.58), $d^M$ can be considered as a flat superconnection $A'$ on $\Omega^{\cdot}(X,F)$.

Recall that $TX$ and $TX^*$ are identified by the metric $g^{TX}$. Then the operators $d^{X,*}$ and $T^H\wedge$ also act on $\Omega^{\cdot}(X,F)$. Let $A''$ be the adjoint of the superconnection $A'$ with respect to $T^HM,g^{\Omega^{\cdot}(X,F)}$ in the sense of [BLo, §1 (d)], [BGo2, §1.3]. Then by [BLo, Prop. 3.7], [BGo2, Prop. 3.8],

$$A''=d^{X,*}+\nabla^{\Omega^{\cdot}(X,F),*}-T^H\wedge\,. \tag{2.59}$$

Then $A''$ is also a flat superconnection on $\Omega^{\cdot}(X,F)$.

Now, we use the formalism of [BLo, §1 (e)], [BGo2, §1.5]. Namely, set

$$A=\frac{1}{2}\left(A''+A'\right),\qquad B=\frac{1}{2}\left(A''-A'\right). \tag{2.60}$$

Then $A$ is a $g$-invariant superconnection on $\Omega^{\cdot}(X,F)$, and $B$ is a smooth $g$-invariant section of $\left(\Lambda^{\cdot}(T^*S)\widehat{\otimes}\operatorname{End}(\Omega^{\cdot}(X,F))\right)^{\mathrm{odd}}$. Moreover, since $A'$ and $A''$ are flat, by [BLo, Prop. 1.2], [BGo2, Prop. 1.5],

$$\begin{gathered}B^2=-A^2,\quad [A,B]=0,\\ \left[A',B^2\right]=0,\quad \left[A'',B^2\right]=0,\quad \left[A,B^2\right]=0.\end{gathered} \tag{2.61}$$

The operator $D^X$ defined in (1.16) still acts on $\Omega^{\cdot}(X,F)$. Let $e_1,\dots,e_n$ be an orthonormal basis of $TX$. Using (2.57), (2.59), (2.60), we get special cases of formulas established in [BLo, Th. 3.14],

$$\begin{aligned}A&=\frac{1}{2}D^X+\nabla^{\Omega^{\cdot}(X,F)}-\frac{1}{2}c\left(T^H\right),\\ B&=-\frac{1}{2}\widehat{c}\left(e_i\right)\left(\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F,u}+\left\langle T^H,e_i\right\rangle\right)+\frac{1}{4}c\left(e_i\right)\omega\left(\nabla^F,g^F\right)\left(e_i\right).\end{aligned} \tag{2.62}$$

As explained in [BLo, Rem. 3.10], the superconnection $A$ is a special case of the Levi-Civita superconnection of [B3].

For $t > 0$, we define the metric $g_t^{TX}$ as in (1.35). Also, we use the same notation as in Section 1. Let $A_t''$ be the adjoint of $A'$ with respect to $T^H M, g_t^{\Omega^{\cdot}(X,F)}$. One verifies easily that

$$A_t'' = t^{-N} A'' t^N, \tag{2.63}$$

or equivalently that

$$A_t'' = t d^{X,*} + \nabla^{\Omega^{\cdot}(X,F),*} - \frac{T^H}{t} \wedge . \tag{2.64}$$

Set

$$A_t = \frac{1}{2}\left(A_t'' + A'\right), \qquad B_t = \frac{1}{2}\left(A_t'' - A'\right). \tag{2.65}$$

For $t > 0$, set

$$C_t' = t^{N/2} A' t^{-N/2}, \qquad C_t'' = t^{-N/2} A'' t^{N/2}. \tag{2.66}$$

Then $C_t'$ is a flat superconnection on $\Omega^{\cdot}(X,F)$, and $C_t''$ is its adjoint with respect to $g^{\Omega^{\cdot}(X,F)}$. Set

$$C_t = \frac{1}{2}\left(C_t'' + C_t'\right), \qquad D_t = \frac{1}{2}\left(C_t'' - C_t'\right). \tag{2.67}$$

By (2.66), we get

$$C_t = t^{N/2} A_t t^{-N/2}, \qquad D_t = t^{N/2} B_t t^{-N/2}. \tag{2.68}$$

Of course, all the objects which we just defined are $G$-invariant. By (2.57), (2.64),

$$\begin{aligned} C_t' &= \sqrt{t} d^X + \nabla_{\cdot}^{\Omega^{\cdot}(X,F)} + \frac{i_{T^H}}{\sqrt{t}}, \\ C_t'' &= \sqrt{t} d^{X,*} + \nabla_{\cdot}^{\Omega^{\cdot}(X,F)} - \frac{T^H \wedge}{\sqrt{t}}. \end{aligned} \tag{2.69}$$

Also, by (2.62), (2.68),

(2.70)

$$\begin{aligned} C_t &= \frac{1}{2}\sqrt{t} D^X + \nabla^{\Omega^{\cdot}(X,F)} - \frac{1}{2\sqrt{t}} c\left(T^H\right), \\ D_t &= -\frac{\sqrt{t}}{2} \widehat{c}\left(e_i\right) \left(\nabla_{e_i}^{\Lambda^{\cdot}(T^*X) \widehat{\otimes} F, u} + \left\langle T^H/t, e_i \right\rangle\right) + \frac{\sqrt{t}}{4} c\left(e_i\right) \omega\left(\nabla^F, g^F\right)\left(e_i\right). \end{aligned}$$

Using (2.55), one verifies easily that

$$\nabla^{\Omega^{\cdot}(X,F),2} = -L_{\Theta}. \tag{2.71}$$

By (2.48), (2.70), and (2.71), we get

$$C_t^2 = -L_{\Theta} + C_{-\Theta,t}^2. \tag{2.72}$$

Let $\varphi : \Lambda^{\cdot}(T^*X) \to \Lambda^{\cdot}(T^*X)$ be given by

$$\varphi\alpha = (2\pi)^{-\deg(\alpha)/2}\,\alpha. \tag{2.73}$$

Again, with respect to [BLo] and [BGo2], we replace $2\pi i$ by $2\pi$.

Let $h(x)$ be an odd holomorphic function. Put

$$h_g\left(\nabla^F, g^F\right) = \sqrt{2\pi}\varphi\mathrm{Tr}^F\left[gh\left(\omega\left(\nabla^F, g^F\right)/2\right)\right]. \tag{2.74}$$

We first view $h_g\left(\nabla^F, g^F\right)$ as an odd form on $X_g$. By [BLo, Ths. 1.8 and 1.11], [BGo2, Th. 1.8], we know that this form is closed on $X_g$, and that its cohomology class $h_g\left(\nabla^F\right)$ does not depend on $g^F$. By (2.22), we get

$$i_{K^X} h_g\left(\nabla^F, g^F\right) = 0. \tag{2.75}$$

From (2.75), we obtain

$$d_K h_g\left(\nabla^F, g^F\right) = 0. \tag{2.76}$$

On the other hand, we may also view $F$ as a flat vector bundle on $M$. We saw before that $\omega\left(\nabla^F, g^F\right)$ can also be viewed as a form on $M$, which is in fact the obvious analogue of the corresponding form on $X$. The same arguments as before show that $h_g\left(\nabla^F, g^F\right)$ can also be considered as a form on $M_g$, and that this form is still closed on $M_g$. Observe that this fact can also be derived from the fact that this form is $Z(g)$-invariant , that it is closed along the fibres $X_g$ and that (2.75) holds.

Moreover $Z(g)$ acts on $H^{\cdot}(X,F)$ and the connected component of the identity $Z(g)_0$ acts trivially on $H^{\cdot}(X,F)$. Therefore $H^{\cdot}(X,F)$ descends to the $\mathbf{Z}$-graded flat vector bundle on $S$,

$$P \times_{Z(g)} H^{\cdot}(X,F),$$

which we still denote by $H^{\cdot}(X,F)$. Let $\nabla^{H^{\cdot}(X,F)}$ be the corresponding flat connection on $H^{\cdot}(X,F)$. As in (1.17),

$$\ker D^X \simeq H^{\cdot}(X,F). \tag{2.77}$$

Also $G$ acts on $\ker D^X$, and (2.77) is an identification of $G$-spaces. So we find that the metric $g_{L_2}^{H^{\cdot}(X,F)}$ descends to a flat metric on $H^{\cdot}(X,F)$. In particular,

$$\omega\left(\nabla^{H^{\cdot}(X,F)}, g_{L_2}^{H^{\cdot}(X,F)}\right) = 0, \tag{2.78}$$

so that

$$h_g\left(\nabla^{H^{\cdot}(X,F)}, g_{L_2}^{H^{\cdot}(X,F)}\right) = 0. \tag{2.79}$$

In [BLo, Th. 3.17 ], [BGo2, Th. 3.25], it was shown in a much more general context that

$$h_g\left(\nabla^{H^{\cdot}(X,F)}\right) = \int_{X_g} e_{-\Theta/2\pi}\left(TX_g\right) h_g\left(\nabla^F\right) \text{ in } H^{\mathrm{odd}}(S,\mathbf{C}). \tag{2.80}$$

Now we show how to reconcile (2.79) and (2.80) in this special case. In fact, by (2.40) and (2.76),

$$d_K e_K\left(TX_g, \nabla^{TX_g}\right) = 0, \qquad d_K h_g\left(\nabla^F, g^F\right) = 0. \tag{2.81}$$

Using the localization formula of [DuH], [BeV], we get

$$\int_{X_g} e_K\left(TX_g, \nabla^{TX_g}\right) h_g\left(\nabla^F, g^F\right) = \int_{X_{g,K}} e\left(TX_{g,K}, \nabla^{TX_{g,K}}\right) h_g\left(\nabla^F, g^F\right). \tag{2.82}$$

Since $\deg h_g\left(\nabla^F, g^F\right) \geq 1$, we get

$$\int_{X_{g,K}} e\left(TX_{g,K}\right) h_g\left(\nabla^F\right) = 0. \tag{2.83}$$

From (2.83), we find that the right-hand side of (2.80) vanishes identically, which fits with (2.79).

We still define $\varphi : \Lambda^{\cdot}\left(T^*S\right) \to \Lambda^{\cdot}\left(T^*S\right)$ as in (2.73). As in (1.33), we now set

$$h\left(x\right) = xe^{x^2}. \tag{2.84}$$

*Definition* 2.5. For $t > 0$, set

$$h_g\left(A', g^{\Omega^{\cdot}(X,F)}\right) = \sqrt{2\pi}\varphi \mathrm{Tr_s}\left[gh\left(B\right)\right], \; h_g^{\wedge}\left(A', g^{\Omega^{\cdot}(X,F)}\right) = \varphi \mathrm{Tr_s}\left[\frac{N}{2} gh'\left(B\right)\right]. \tag{2.85}$$

Then by [BLo, Ths. 3.16 and and 3.20] and by [BGo2, Ths. 3.24 and 3.29], the forms $h_g\left(A', g^{\Omega^{\cdot}(X,F)}\right)$ and $h_g^{\wedge}\left(A', g^{\Omega^{\cdot}(X,F)}\right)$ are respectively odd and even.

THEOREM 2.6. *The form* $h_g\left(A', g^{\Omega^{\cdot}(X,F)}\right)$ *vanishes identically. Moreover the form* $h_g^{\wedge}\left(A', g^{\Omega^{\cdot}(X,F)}\right)$ *is closed, and is given by*

$$h_g^{\wedge}\left(A', g^{\Omega^{\cdot}(X,F)}\right) = \mathrm{Tr_s}\left[\frac{N}{2} h'\left(-\frac{1}{2}\widehat{c}\left(e_i\right)\left(\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes} F, u}\left\langle \frac{\Theta^X}{2\pi}, e_i\right\rangle\right) + \frac{1}{4} c\left(e_i\right)\omega\left(\nabla^F, g^F\right)\left(e_i\right)\right)\right]. \tag{2.86}$$

*Proof.* By (2.62), $B$ is an odd endomorphism. Therefore $h\left(B\right)$ is also odd, and so $h_g\left(A', g^{\Omega^{\cdot}(X,F)}\right)$ vanishes. Equation (2.86) follows from (2.55), (2.62) and (2.85). Finally, by Chern-Weil theory, it is clear that the form $h_g^{\wedge}\left(A', g^{\Omega^{\cdot}(X,F)}\right)$ is closed. □

*Remark* 2.7. In a more general context, in [BLo, Th. 3.20], [BGo2, Th. 3.29], it is shown that

$$\frac{\partial}{\partial t} h_g\left(A', g_t^{\Omega^\cdot(X,F)}\right) = d\frac{h_g^\wedge\left(A', g_t^{\Omega^\cdot(X,F)}\right)}{t}. \tag{2.87}$$

Here, (2.87) is a trivial consequence of Theorem 2.6.

In [BLo, Th. 3.21] and in [BGo2, Th. 3.30], it was shown that as $t \to 0$,

$$h_g^\wedge\left(A', g_t^{\Omega^\cdot(X,F)}\right) = \frac{n}{4}\chi_g(F) + \mathcal{O}\left(\sqrt{t}\right), \tag{2.88}$$

and that as $t \to +\infty$,

$$h_g^\wedge\left(A', g_t^{\Omega^\cdot(X,F)}\right) = \frac{1}{2}\chi_g'(F) + \mathcal{O}\left(1/\sqrt{t}\right). \tag{2.89}$$

Now, we follow [BLo, §3 (j)] and [BGo2, §3.12].

*Definition* 2.8. Set

$$\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right) = -\int_0^{+\infty}\left[h_g^\wedge\left(A', g_t^{\Omega^\cdot(X,F)}\right) - \frac{1}{2}\chi_g'(F)\,h'(0) - \left(\frac{n}{4}\chi_g(F) - \frac{1}{2}\chi_g'(F)\right)h'\left(i\sqrt{t}/2\right)\right]\frac{dt}{t}. \tag{2.90}$$

By (2.88), (2.89), the even form $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ is well defined. It is called an equivariant analytic torsion form.

THEOREM 2.9. *The form $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ is closed, and its cohomology class does not depend on $g^{TX}, g^F$, or on the choice of the connection on $P$.*

*Proof.* By Theorem 2.6, it is clear that the form $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ is closed. The fact that its cohomology class does not depend on the data is obvious by functoriality. □

*Remark* 2.10. In the more general context of [BLo] and [BGo2], the even form $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ is in general not closed.

For $s \in \mathbf{R}, \alpha \in \Lambda^p(T^*S)$, set

$$\psi_s \alpha = s^{p/2}\alpha. \tag{2.91}$$

Note that if $p$ is even, $\psi_s\alpha$ is unambiguously defined, and that if $p$ is odd, $\psi_s\alpha/\sqrt{s}$ is also well defined.

If $\alpha \in \Lambda^{\cdot}(T^*S)$, put

$$Q\alpha = \int_0^1 \psi_{4s(1-s)} \alpha ds. \tag{2.92}$$

If $\alpha \in \Lambda^{2p}(T^*S)$, then

$$Q\alpha = \frac{(p!)^2}{(2p+1)!} 4^p \alpha. \tag{2.93}$$

Now, we define the Chern equivariant analytic torsion forms as in [BGo2, Def. 3.46].

*Definition* 2.11. Set

$$\mathcal{T}_{\mathrm{ch},g}\left(T^H M, g^{TX}, \nabla^F, g^F\right) = Q\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right). \tag{2.94}$$

2.6. *The equivariant infinitesimal analytic torsion forms.* We make the same assumptions as in Sections 2.2–2.4. In particular, we only work with a single manifold $X$. Also we use the notation of the previous sections.

Let $K^{X\prime}$ be the 1-form dual to $K^X$ via the metric $g^{TX}$.

*Definition* 2.12. For $K \in \mathfrak{z}(g)$, put

$$\begin{aligned} A'_K &= d^X - i_{K^X}, & A''_K &= d^{X,*} + K^{X\prime}\wedge, \\ A_K &= \frac{1}{2}\left(A''_K + A'_K\right), & B_K &= \frac{1}{2}\left(A''_K - A'_K\right). \end{aligned} \tag{2.95}$$

Since $L_K$ is skew-adjoint, we get easily,

$$A'^2_K = -L_K, \qquad A''^2_K = -L_K. \tag{2.96}$$

From (2.96), we deduce that

$$\begin{aligned} [L_K, A'_K] &= 0, & [L_K, A''_K] &= 0, \\ [L_K, A_K] &= 0, & [L_K, B_K] &= 0. \end{aligned} \tag{2.97}$$

Using (2.96), (2.97), we get

$$\begin{aligned} &A^2_K = -\frac{1}{2}L_K + \frac{1}{4}\left[A'_K, A''_K\right], \qquad B^2_K = -\frac{1}{2}L_K - \frac{1}{4}\left[A'_K, A''_K\right], \\ &\left[A'_K, B^2_K\right] = 0, \qquad \left[A''_K, B^2_K\right] = 0, \qquad [A_K, B_K] = 0. \end{aligned} \tag{2.98}$$

In particular, from (2.98),

$$A^2_K + B^2_K = -L_K. \tag{2.99}$$

Also by (1.2), (1.25), (2.95),

$$A_K = A^X + \frac{1}{2}c\left(K^X\right), \qquad B_K = B^X + \frac{1}{2}\widehat{c}\left(K^X\right). \tag{2.100}$$

When replacing the metric $g^{TX}$ by the metric $g_t^{TX} = g^{TX}/t$, we obtain the analogues of $A''_K, A_K, B_K$, which are denoted $A''_{K,t}, A_{K,t}, B_{K,t}$. Then

$$A''_{K,t} = t^{-N} A''_K t^N. \tag{2.101}$$

Similarly, set

$$C'_{K,t} = t^{N/2} A'_K t^{-N/2}, \qquad C''_{K,t} = t^{-N/2} A''_K t^{N/2}. \tag{2.102}$$

Put

$$C_{K,t} = \frac{1}{2}\left(C''_{K,t} + C'_{K,t}\right), \qquad D_{K,t} = \frac{1}{2}\left(C''_{K,t} - C'_{K,t}\right). \tag{2.103}$$

Then by (2.101)–(2.103),

$$C_{K,t} = t^{N/2} A_{K,t} t^{-N/2}, \qquad D_{K,t} = t^{N/2} B_{K,t} t^{-N/2}. \tag{2.104}$$

Equivalently,

$$C_{K,t} = \sqrt{t} A^X + \frac{1}{2\sqrt{t}} c\left(K^X\right), \qquad D_{K,t} = \sqrt{t} B^X + \frac{1}{2\sqrt{t}} \widehat{c}\left(K^X\right). \tag{2.105}$$

Also our definition for $C_{K,t}$ fits with (2.48).

As in (2.84), we use the notation $h(x) = xe^{x^2}$.

*Definition* 2.13. Put

$$h_g\left(A'_K, g^{\Omega^{\cdot}(X,F)}\right) = \operatorname{Tr_s}\left[g h\left(B_K\right)\right], \qquad h_g^{\wedge}\left(A'_K, g^{\Omega^{\cdot}(X,F)}\right) = \operatorname{Tr_s}\left[g \frac{N}{2} h'\left(B_K\right)\right]. \tag{2.106}$$

By (2.104),

$$\begin{aligned} h_g\left(A'_K, g_t^{\Omega^{\cdot}(X,F)}\right) &= \operatorname{Tr_s}\left[g h\left(B_{K,t}\right)\right] = \operatorname{Tr_s}\left[g h\left(D_{K,t}\right)\right], \\ h_g^{\wedge}\left(A'_K, g_t^{\Omega^{\cdot}(X,F)}\right) &= \operatorname{Tr_s}\left[g \frac{N}{2} h'\left(B_{K,t}\right)\right] = \operatorname{Tr_s}\left[g \frac{N}{2} h'\left(D_{K,t}\right)\right]. \end{aligned} \tag{2.107}$$

Theorem 2.14. *For $t > 0, K \in \mathfrak{z}(g)$, the following identity holds*:

$$h_g\left(A'_K, g^{\Omega^{\cdot}(X,F)}\right) = 0. \tag{2.108}$$

*There exist $\beta > 0, C > 0$ such that if $K \in \mathfrak{z}(g)$ is such that $|K| \leq \beta$, for $t \in ]0,1]$,*

$$\left| h_g^{\wedge}\left(A'_K, g_t^{\Omega^{\cdot}(X,F)}\right) - \frac{n}{4}\chi_g(F) \right| \leq C\sqrt{t}, \tag{2.109}$$

*and for $t \geq 1$,*

$$\left| h_g^{\wedge}\left(A'_K, g_t^{\Omega^{\cdot}(X,F)}\right) - \frac{1}{2}\chi'_g(F) \right| \leq \frac{C'}{\sqrt{t}}. \tag{2.110}$$

*Proof.* By (2.100), $B_K$ is an odd endomorphism. Therefore $h(B_K)$ is also odd, so that $h_g\left(A'_K, g^{\Omega^\cdot(X,F)}\right)$ vanishes.

By proceeding as in [BGo1, proof of Theorem 7.9] and in [BGo2, Th. 3.30], we get (2.109). Also, by proceeding as in [BGo1, proof of Th. 7.1], we get (2.110). □

*Definition* 2.15. For $K \in \mathfrak{z}(g)$, with $|K|$ small enough, set

(2.111)
$$\mathcal{T}_{h,g,K}\left(g^{TX}, \nabla^F, g^F\right) = -\int_0^{+\infty}\left[h_g^\wedge\left(A'_K, g_t^{\Omega^\cdot(X,F)}\right) - \frac{1}{2}\chi'_g(F)\,h'(0) - \left(\frac{n}{4}\chi_g(F) - \frac{1}{2}\chi'_g(F)\right)h'\left(i\sqrt{t}/2\right)\right]\frac{dt}{t}.$$

The quantity $\mathcal{T}_{h,g,K}\left(g^{TX}, \nabla^F, g^F\right)$ will be called the *equivariant infinitesimal analytic torsion*. Comparing with (1.42), (1.43), we get

(2.112)
$$\mathcal{T}_{h,g}\left(g^{TX}, \nabla^F, g^F\right) = \mathcal{T}_{h,g,0}\left(g^{TX}, \nabla^F, g^F\right) = \frac{1}{2}\frac{\partial \vartheta_g}{\partial s}\left(g^{TX}, \nabla^F, g^F\right)(0).$$

Also using Theorem 2.14, we find that for $|K|$ small enough, $\mathcal{T}_{h,g,K}\left(g^{TX}, \nabla^F, g^F\right)$ depends analytically on $K$.

By (2.86), with the notation in (2.90),

$$\mathcal{T}_{h,g}\left(T^HM, g^{TX}, \nabla^F, g^F\right) = \mathcal{T}_{h,g,-\Theta/2\pi}\left(g^{TX}, \nabla^F, g^F\right). \tag{2.113}$$

*Definition* 2.16. For $K \in \mathfrak{z}(g)$ and $|K|$ small enough, set

(2.114)
$$\log\left(\| \|^{\sim}_{\lambda_G(F),h}\right)(g,K) = \log\left(|\ |_{\lambda_G(F)}\right)(ge^K) + \mathcal{T}_{h,g,K}\left(g^{TX}, \nabla^F, g^F\right).$$

The symbol $\| \|^{\sim}_{\lambda_G(F)}$ will be called an *equivariant infinitesimal Ray-Singer metric*. Observe that since $e^K$ acts trivially on $H^\cdot(X,F)$, we can rewrite (2.114) in the form

$$\log\left(\| \|^{\sim}_{\lambda_G(F),h}\right)(g,K) = \log\left(|\ |_{\lambda_G(F)}\right)(g) + \mathcal{T}_{h,g,K}\left(g^{TX}, \nabla^F, g^F\right). \tag{2.115}$$

Using (1.24), Theorem 1.8, (2.113), (2.114), we get

$$\log\left(\| \|_{\lambda_G(F)}\right)(g) = \log\left(\| \|^{\sim}_{\lambda_G(F),h}\right)(g,0). \tag{2.116}$$

2.7. *The Chern equivariant infinitesimal analytic torsion.* Now, by imitating the constructions of [BGo2, Chs. 2 and 3], we will modify our definition of the infinitesimal torsion. If $f(K)$ is an analytic function of $K \in \mathfrak{z}(g)$ defined on a neighbourhood of 0, set

$$Qf(K) = \int_0^1 f(4s(1-s)K)\,ds. \tag{2.117}$$

Then $Qf(K)$ is still analytic near 0.

*Definition* 2.17. For $K \in \mathfrak{z}(g)$, and $|K|$ small enough, set

$$\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX}, \nabla^F, g^F\right) = Q\mathcal{T}_{h,g,K}\left(g^{TX}, \nabla^F, g^F\right). \tag{2.118}$$

Then $\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX}, \nabla^F, g^F\right)$ still depends analytically on $K$. It will be called the *Chern equivariant infinitesimal analytic torsion.* By (2.112),

$$\mathcal{T}_{h,g}\left(g^{TX}, \nabla^F, g^F\right) = \mathcal{T}_{\mathrm{ch},g,0}\left(g^{TX}, \nabla^F, g^F\right). \tag{2.119}$$

Using the notation in (2.94) and by (2.113), we get

$$\mathcal{T}_{\mathrm{ch},g}\left(T^HM, g^{TX}, \nabla^F, g^F\right) = \mathcal{T}_{\mathrm{ch},g,-\Theta/2\pi}\left(g^{TX}, \nabla^F, g^F\right). \tag{2.120}$$

*Definition* 2.18. For $K \in \mathfrak{z}(g)$ and $|K|$ small enough, set

$$\log\left(\widetilde{\|\ \|}_{\lambda_G(F),\mathrm{ch}}\right)(g,K) = \log\left(|\ |_{\lambda_G(F)}\right)(ge^K) + \mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX}, \nabla^F, g^F\right). \tag{2.121}$$

The symbol $\|\ \|_{\lambda_G(F),\mathrm{ch}}$ will be called a *Chern equivariant infinitesimal Ray-Singer metric.*

Observe that (2.116),

$$\log\left(\|\ \|_{\lambda_G(F)}\right)(g) = \log\left(\widetilde{\|\ \|}_{\lambda_G(F),\mathrm{ch}}\right)(g,0). \tag{2.122}$$

2.8. *Anomaly formulas for Chern equivariant infinitesimal Ray-Singer metrics.* If $f(x)$ is a holomorphic function, set

$$Ff(x) = x\int_0^1 f'\left(4s(1-s)x^2\right)ds, \qquad Qf(x) = \int_0^1 f(4s(1-s)x)\,ds. \tag{2.123}$$

Then $Ff(x)$ is an odd holomorphic function. In the sequel, $Fe^{\cdot}(x)$ denotes the function $Ff(x)$, with $f(x) = e^x$.

By following [BGo2, §2.7], set

$$\mathrm{ch}^\circ_g\left(\nabla^F, g^F\right) = \left(Fe^{\cdot}\right)_g\left(\nabla^F, g^F\right). \tag{2.124}$$

Then $\mathrm{ch}^{\circ}_g\left(\nabla^F, g^F\right)$ is a closed odd differential form. It was obtained in [BGo2] as a Chern-Simons class associated to the Chern character. Let $g'^{TX}, g'^F$ be another couple of $G$-invariant metrics on $TX, F$. We denote with a superscript $'$ the objects considered before, which are associated to this new couple of metrics.

Let $\widetilde{e}_K\left(TX_g, \nabla^{TX_g}, \nabla'^{TX_g},\right)$ be the Chern-Simons class of forms on $X_g$ such that

$$d_K\widetilde{e}_K\left(TX_g, \nabla^{TX_g}, \nabla'^{TX_g}\right) = e_K\left(TX_g, \nabla'^{TX_g}\right) - e_K\left(TX_g, \nabla^{TX_g}\right). \tag{2.125}$$

The class $\widetilde{e}_K\left(TX_g, \nabla^{TX_g}, \nabla'^{TX_g},\right)$ is defined modulo the $d_K$ of a smooth $K$-invariant form.

Let $\ell \in [0,1] \to g^F_\ell$ be a smooth family of $G$-invariant metrics on $F$, such that $g^F_0 = g^F, g^F_1 = g'^F$. As in [BGo2, Def. 1.10], we define a form on $X_g$ by the formula

$$\widetilde{h}_g\left(\nabla^F, g^F_\ell\right) = \int_0^1 \varphi \mathrm{Tr_s}\left[g\frac{1}{2}\left(g^F_\ell\right)^{-1}\frac{\partial g^F_\ell}{\partial \ell} h\left(\omega\left(\nabla^F, g^F_\ell\right)/2\right)\right] d\ell. \tag{2.126}$$

Then by [BGo2, Th. 1.11], the class of the form $\widetilde{h}_g\left(\nabla^F, g^F_\ell\right)$ in $Q^{X_g}/Q^{X_g,0}$ does not depend on the path $\ell \to g^F_\ell$, and moreover

$$d\widetilde{h}_g\left(\nabla^F, g^F_\ell\right) = h_g\left(\nabla^F, g'^F\right) - h_g\left(\nabla^F, g^F\right). \tag{2.127}$$

Also, observe that by (2.22),

$$i_{K^X}\widetilde{h}_g\left(\nabla^F, g^F_\ell\right) = 0. \tag{2.128}$$

From (2.127), (2.128), we get

$$d_K\widetilde{h}_g\left(\nabla^F, g^F_\ell\right) = h_g\left(\nabla^F, g'^F\right) - h_g\left(\nabla^F, g^F\right). \tag{2.129}$$

A similar construction is given in [BGo2, §2.7] for the class $\mathrm{ch}^{\circ}$. Namely, set

(2.130)
$$\widetilde{\mathrm{ch}}^{\circ}_g\left(\nabla^F, g^F\right) = \int_0^1 \varphi \mathrm{Tr_s}\left[g\frac{1}{2}\left(g^F_\ell\right)^{-1}\frac{\partial g^F_\ell}{\partial \ell}\left(Fe^{\cdot}\right)'\left(\omega\left(\nabla^F, g^F_\ell\right)/2\right)\right] d\ell.$$

Then, by [BGo2, Th. 2.39],

$$\begin{aligned} d\,\widetilde{\mathrm{ch}}^{\circ}_g\left(A', g^E_\ell\right) &= \mathrm{ch}^{\circ}\left(A', g^E_1\right) - \mathrm{ch}^{\circ}\left(A', g^E_0\right), \\ \widetilde{\mathrm{ch}}^{\circ}_g\left(A', g^E_\ell\right) &= Q\widetilde{h}_g\left(A', g^E_\ell\right). \end{aligned} \tag{2.131}$$

THEOREM 2.19. *For $K \in \mathfrak{z}(g)$, and $|K|$ small enough, the following identity holds,*

(2.132)
$$\log\left(\frac{\widetilde{\| \|'}_{\lambda_G(F),\mathrm{ch}}(g,K)}{\widetilde{\| \|}_{\lambda_G(F),\mathrm{ch}}(g,K)}\right) = \int_{X_g} e_K\left(TX_g, \nabla^{TX_g}\right)\widetilde{\mathrm{ch}_g^\circ}\left(\nabla^F, g_\ell^F\right) + \int_{X_g} \widetilde{e}_K\left(TX_g, \nabla^{TX_g}, \nabla'^{TX_g}\right)\mathrm{ch}_g^\circ\left(\nabla^F, g'^F\right).$$

*Proof.* First, we claim that if, in (2.132), we replace $\| \|_{\lambda_G(F),\mathrm{ch}}$ by $\| \|_{\lambda_G(F),h}$ and $\mathrm{ch}_g^\circ$ by $h_g$, with $h(x)$ still given by (2.84), the corresponding identity holds. In fact, its proof is very similar to [BLo, Th. 3.24], [BGo2, Th. 3.34]. Then we claim that when applying the operator $Q$ to both sides of this identity, we get (2.132). This is of course clear for the left-hand side. Moreover if $m = \dim X_g$, if $A$ is a $(m,m)$ antisymmetric matrix, if $a \in \mathbf{R}$, then

$$\mathrm{Pf}[aA] = a^{m/2}\mathrm{Pf}[A]. \tag{2.133}$$

Using (2.133), we find easily that if $s \in [0,1]$,

$$\begin{aligned}\int_{X_g} e_{4s(1-s)K}\left(TX_g, \nabla^{TX_g}\right)\widetilde{h}_g\left(\nabla^F, g_\ell^F\right) &= \int_{X_g} e_K\left(TX_g, \nabla^{TX_g}\right)\psi_{4s(1-s)}\widetilde{h}_g\left(\nabla^F, g_\ell^F\right),\\ \int_{X_g} \widetilde{e}_{4s(1-s)K}\left(TX_g, \nabla^{TX_g}, \nabla'^{TX_g}\right)\mathrm{ch}^\circ\left(\nabla^F, g^F\right) &= \int_{X_g} \widetilde{e}_K\left(TX_g, \nabla^{TX_g}\right)\sqrt{2\pi}\varphi\\ &\quad \mathrm{Tr_s}\left[g\frac{\omega\left(\nabla^F, g^F\right)}{2}\exp\left(4s(1-s)\,\omega^2\left(\nabla^F, g^F\right)/4\right)\right].\end{aligned} \tag{2.134}$$

By (2.123), (2.124), (2.131), we conclude that when applying the operator $Q$ to the right-hand side of the previously described identity, we get the right-hand side of (2.132). The proof of our theorem is completed. □

## 3. Equivariant fibrations and the classes $V_K(M/S)$

In this section, we construct currents which are naturally attached to an equivariant fibration. In particular, we produce the $V$-invariants which are attached to a $G$-equivariant fibration. If $\mathfrak{g}$ is the Lie algebra of $G$ and if $K \in \mathfrak{g}$, the $V$-invariants are even cohomology classes on the base of the fibration, which depend explicitly on $K$. One of the key properties of these invariants will be established in Section 4.

This section is organized as follows. In 3.1, we construct Chern-Simons equivariant currents $F_K\left(X, g^{TX}\right)$, which refine on the localization formulas of Duistermaat-Heckman [DuH], Berline-Vergne [BeV]. In 3.2, we recall various properties of the Mathai-Quillen equivariant Thom forms [MQ], which are obtained via the Berezin integration formalism of Section 1.2. In 3.3, we construct a current $\psi_K\left(K^X, TX, \nabla^{TX}\right)$ which refines on the equivariant Chern-Gauss-Bonnet formula for $K^X$. In 3.4, we give a comparison formula, which relates the currents $F_K\left(X, g^{TX}\right)$ and $\psi_K\left(K^X, TX, \nabla^{TX}\right)$. In 3.5, we construct currents $\psi_K\left(\nabla f, TX, \nabla^{TX}\right)$, which refine the equivariant Chern-Gauss-Bonnet formula for the gradient field $\nabla f$, when $f$ is a $K^X$-invariant Morse-Bott function. In 3.6, we give a formula which relates the currents $\psi_K\left(K^X, TX, \nabla^{TX}\right)$ and $\psi_K\left(\nabla f, TX, \nabla^{TX}\right)$. These formulas are simpler versions of the intersection formulas with excess of [B8, 9].

In Section 3.7, we consider a proper submersion $\pi : M \to S$, such that $G$ acts along the fibres $X$, which are equipped with a $G$-invariant metric $g^{TX}$. We establish various hidden symmetry properties for the curvature of a natural connection on $TX$. These symmetry properties are related in a fundamental way to the Levi-Civita superconnection introduced in [B3] to establish the local families index theorem. In 3.8, we construct an odd closed form $\delta_K$ on $S$. Using the symmetry properties just described, we show this form is closed. In 3.9, by scaling the metric $g^{TX}$, we prove that the forms $\delta_K$ are exact. By a Chern-Simons transgression argument, we obtain the invariants $V_K\left(M/S, T^HM, g^{TX}\right)$.

3.1. *A Chern-Simons equivariant current.* We make the same assumptions as in Sections 1 and 2.1. Take $K \in \mathfrak{g}$. Set

$$X_K = \left\{x \in X, K^X(x) = 0\right\}. \tag{3.1}$$

Then $X_K$ is totally geodesic in $X$. We identify the normal bundle $N_{X_K/X}$ with the orthogonal bundle to $TX_K$ in $TX|_{X_K}$ with respect to $g^{TX}|_{X_K}$. Then the connection $\nabla^{TX}|_{X_K}$ on $TX_K$ preserves the orthogonal splitting

$$TX|_{X_K} = TX_K \oplus N_{X_K/X}. \tag{3.2}$$

In particular $\nabla^{TX}$ induces the Levi-Civita connection $\nabla^{TX_K}$ on $TX_K$, and a Euclidean connection $\nabla^{N_{X_K/X}}$ on $N_{X_K/X}$. Let $R^{TX_K}, R^{N_{X_K/X}}$ be the curvatures of $\nabla^{TX_K}, \nabla^{N_{X_K/X}}$. Note that $K^X$ acts naturally on $N_{X_K/X}$ as the restriction $m^{N_{X_K/X}}(K)$ of $m^{TX}(K)|_{X_K}$ to $N_{X_K/X}$. Then $m^{N_{X_K/X}}(K)$ is parallel with respect to $\nabla^{N_{X_K/X}}$, antisymmetric and invertible. In particular, $N_{X_K/X}$ is even-dimensional. The corresponding equivariant curvature of $\nabla^{N_{X_K/X}}$ is given by

$$R_K^{N_{X_K/X}} = R^{N_{X_K/X}} - 2\pi m^{N_{X_K/X}}(K). \tag{3.3}$$

The associated equivariant Euler form $e_K\left(N_{X_K/X},\nabla^{N_{X_K/X}}\right)$ is just

$$e_K\left(N_{X_K/X},\nabla^{N_{X_K/X}}\right)=\operatorname{Pf}\left[\frac{R_K^{N_{X_K/X}}}{2\pi}\right]. \tag{3.4}$$

Also $e_K\left(N_{X_K/X},\nabla^{N_{X_K/X}}\right)$ is invertible.

*Definition* 3.1. Let $P^X_{K,X_K}$ be the set of $K^X$-invariant currents on $X$ whose wave front set is included in $N^*_{X_K/X}$. Let $P^{X,0}_{K,X_K}$ be the set of $K^X$-invariant currents $a$, such that there exists a $K^X$-invariant current $b\in P^X_{K,X_K}$ for which $a=d_Kb$.

Note that if $a\in P^{X,0}_{K,X_K}$,

$$d_Ka=0. \tag{3.5}$$

Let $K^{X\prime}$ be the 1-form on $X$ which is dual to $K^X$ by the metric $g^{TX}$. Observe that

$$d_KK^{X\prime}=-2\pi\left|K^X\right|^2+dK^{X\prime}. \tag{3.6}$$

Also, since $g^{TX}$ is $K^X$-invariant,

$$d_K^2K^{X\prime}=0. \tag{3.7}$$

*Definition* 3.2. For $t>0$, set

$$\alpha^X_{K,t}=\exp\left(d_KK^{X\prime}/4\pi t\right),\qquad \beta^X_{K,t}=-\frac{K^{X\prime}}{4\pi t}\exp\left(d_KK^{X\prime}/4\pi t\right). \tag{3.8}$$

Recall that if $s\in\mathbf{R},a\in\Lambda\left(T^*X\right)$, $\psi_s\alpha$ was as defined in (2.91). Now we have the result of [B7, Proposition 5], [B8, Th. 2.3].

THEOREM 3.3. *The forms $\alpha^X_{K,t},\beta^X_{K,t}$ are $K^X$-invariant. The following identity holds*:

$$d_K\alpha^X_{K,t}=0. \tag{3.9}$$

*Moreover*

$$\frac{\partial}{\partial t}\alpha^X_{K,t}=d_K\frac{\beta^X_{K,t}}{t}. \tag{3.10}$$

*Finally, if $z\in\mathbf{R}^*$*,

$$\alpha^X_{zK,t}=\psi_{1/z}\alpha^X_{K,t/z^2},\qquad \beta^X_{zK,t}=\frac{1}{\sqrt{z}}\psi_{1/z}\beta^X_{K,t/z^2}. \tag{3.11}$$

*Proof.* Equation (3.9) follows from (3.7). Equations (3.10) and (3.11) are trivial. ∎

Observe that that $N_{X_K/X}$ is naturally oriented by the condition

$$\operatorname{Pf}\left[m^{N_{X_K/X}}(K)\right] > 0. \tag{3.12}$$

Let $o(TX), o(TX_K)$ be the orientation bundles of $TX, TX_K$. The argument we just gave shows that

$$o(TX)|_{X_K} = o(TX_K). \tag{3.13}$$

By [B4, Th. 1.3], [B8, Th. 2.5 and Rem. 2.6], there are currents $\rho_1, \ldots, \rho_k \ldots$, whose support is included in $X_K$, such that if $\eta$ is a smooth form on $X$ with values in $o(TX)$, for $k \geq 1$, as $t \to 0$,

$$\int_X \eta \alpha^X_{K,t} = \int_{X_K} \frac{\eta}{e_K\left(N_{X_K/X}, \nabla^{N_{X_K/X}}\right)} + \sum_{j=1}^{k} \int_X \eta \rho_j t^j + o\left(t^k\right). \tag{3.14}$$

By (3.8), (3.14), since $K^X$ vanishes on $X_K$, we deduce that as $t \to 0$,

$$\int_X \eta \beta^X_{K,t} = \mathcal{O}(1). \tag{3.15}$$

However, inspection of the proofs in [B4, 8] shows very easily that using the fact that the Gaussian integral of a linear function vanishes, or the fact that $N_{X_K/X}$ is even-dimensional, instead of (3.15), we find that as $t \to 0$,

$$\int_X \eta \beta^X_{K,t} = \mathcal{O}(t). \tag{3.16}$$

Also, by (3.8), as $t \to +\infty$,

$$\int_X \eta \beta^X_{K,t} = \mathcal{O}(1/t). \tag{3.17}$$

*Definition* 3.4. Let $F_K\left(X, g^{TX}\right)$ be the current on $X$,

$$F_K\left(X, g^{TX}\right) = \int_0^{+\infty} \frac{\beta^X_{K,t}}{t} dt. \tag{3.18}$$

Let $\delta_{X_K}$ be the current of integration on the submanifold $X_K$.

THEOREM 3.5. *The odd current $F_K\left(X, g^{TX}\right)$ lies in $P^X_{K,X_K}$. Moreover*

$$d_K F_K\left(X, g^{TX}\right) = 1 - \frac{\delta_{X_K}}{e_K\left(N_{X_K/X}, \nabla^{N_{X_K/X}}\right)}. \tag{3.19}$$

*If $z \in \mathbf{R}^*$,*

$$F_{zK}\left(X, g^{TX}\right) = \frac{1}{\sqrt{z}} \psi_{1/z} F_K\left(X, g^{TX}\right). \tag{3.20}$$

*Proof.* By proceeding as in [B8, Th. 2.5], we find that $F_K\left(X, g^{TX}\right) \in P^X_{K,X_K}$. By (3.10), (3.14), we get (3.19). By (3.11), we get (3.20). □

3.2. *The equivariant Mathai-Quillen Thom forms.* In this section, we use the Berezin integration formalism of Section 1.2 applied to $V = TX, E = TX$. In particular, $\widehat{\Lambda}^{\cdot}(T^*X)$ denotes another copy of $\Lambda^{\cdot}(T^*X)$. Also, most of the time, we will identify $TX$ and $T^*X$ by the metric $g^{TX}$.

Let $e_1, \cdots, e_n$ be an orthonormal basis of $TX$ and let $e^1, \cdots, e^n$ be the corresponding dual basis of $T^*X$. Defining $R_K^{TX}$ as in (2.31), set

$$\dot{R}_K^{TX} = \frac{1}{2}\left\langle e_k, R_K^{TX} e_l \right\rangle \widehat{e}^k \wedge \widehat{e}^l. \tag{3.21}$$

Then $\dot{R}_K^{TX}$ is a section of $\Lambda^{\cdot}(T^*X) \widehat{\otimes} \widehat{\Lambda}^{\cdot}(T^*X)$.

Let $s$ be a $K^X$-invariant section of $TX$. By (2.4),

$$\nabla_{K^X}^{TX} s = m^{TX}(K)\, s. \tag{3.22}$$

*Definition* 3.6. Let $A_{K,s}$ be the section of $\Lambda^{\cdot}(T^*X) \widehat{\otimes} \widehat{\Lambda}^{\cdot}(T^*X)$ such that

$$A_{K,s} = \dot{R}_K^{TX} + \nabla_{\cdot}^{TX} \hat{s} + \frac{1}{2}|s|^2. \tag{3.23}$$

The connection $\nabla^{TX}$ acts as a differential operator on smooth sections of $\Lambda^{\cdot}(T^*X) \widehat{\otimes} \widehat{\Lambda}^{\cdot}(T^*X)$. Set

$$\nabla_K^{TX} = \nabla^{TX} - 2\pi i_{K^X}. \tag{3.24}$$

The interior multiplication $i_{\hat{s}}$ acts naturally on $\widehat{\Lambda}^{\cdot}(T^*X)$, and also as a derivation of the graded algebra $\Lambda^{\cdot}(T^*X) \widehat{\otimes} \widehat{\Lambda}^{\cdot}(T^*X)$. Now we establish an extension of a result established in [MQ, §6] and in [BeGeV, Lemma 1.51 and Props. 1.53 and 1.54].

Theorem 3.7. *For $T > 0$, the following identities hold:*

$$\left[\nabla_K^{TX} + \sqrt{T} i_{\hat{s}}, A_{K,\sqrt{T}s}\right] = 0, \qquad \frac{\partial A_{K,\sqrt{T}s}}{\partial T} = \left[\nabla_K^{TX} + \sqrt{T} i_{\widehat{s}}, \frac{\hat{s}}{2\sqrt{T}}\right]. \tag{3.25}$$

*Proof.* If we replace $R_K^{TX}$ by $R^{TX}$ and $\nabla_K^{TX}$ by $\nabla^{TX}$, our identities are exactly the ones established in the above references. Using (2.7), (3.22), we get the first identity in (3.25). Also, we observe that in the right-hand side of the second identity, we may as well replace $\nabla_K^{TX}$ by $\nabla^{TX}$. The second identity now follows from (3.23). □

*Definition* 3.8. Let $a_{K,s}$ and $b_{K,s}$ be the forms on $TX$,

$$a_{K,s} = \int^{\widehat{B}} \exp\left(-A_{K,s}/2\pi\right), \quad b_{K,s} = \int^{\widehat{B}} \frac{\widehat{s}}{4\pi} \exp\left(-A_{K,s}/2\pi\right). \tag{3.26}$$

In particular, we deduce from (3.26) that

$$\deg b_{K,s} \leq n-1. \tag{3.27}$$

More precisely, the component of top degree $n-1$ in $b_{K,s}$ is just $b_{0,s}$.

For $K=0$, the following result was proved in Mathai-Quillen [MQ, Th. 6.4], and in [BZ1, Th. 3.4].

THEOREM 3.9. *The forms $a_{K,s}, b_{K,s}$ are $K^X$-invariant. The forms $a_{K,s}$ are even, and are such that*

$$d_K a_{K,s} = 0. \tag{3.28}$$

*The forms $b_{K,s}$ are odd. Moreover, for $T \geq 0$,*

$$\begin{aligned} a_{K,\sqrt{T}s}|_{T=0} &= e_K\left(TX, \nabla^{TX}\right), \\ \frac{\partial}{\partial T} a_{K,\sqrt{T}s} &= -d_K \frac{b_{K,\sqrt{T}s}}{T},\ T>0. \end{aligned} \tag{3.29}$$

*Finally, for $z \in \mathbf{R}^*$,*

$$a_{zK,\sqrt{T}s} = z^{n/2}\psi_{1/z} a_{K,\sqrt{T}s}, \quad b_{zK,\sqrt{T}s} = z^{n/2}\frac{1}{\sqrt{z}}\psi_{1/z} b_{K,\sqrt{T}s}. \tag{3.30}$$

*Proof.* By the first identity in (3.25), we get (3.28). Using (1.10), we get the first identity in (3.29). The second identity of (3.29) follows from the second identity in (3.25). The identity (3.30) is trivial. □

3.3. *Convergence of the Mathai-Quillen currents associated to $K^X$.* Now we will use the notation of Section 3.2, with $s = K^X$.

Let $\delta_{X_K}$ be the current of integration on $X_K$. We state a convergence result for the currents $a_{K,\sqrt{T}K^X}, b_{K,\sqrt{T}K^X}$, which was partially proved in [B9, Th. 3.3].

THEOREM 3.10. *There is a constant $C>0$ such that for any smooth form $\mu$ on $X$, for $T \geq 1$,*

$$\begin{aligned} \left| \int_{TX} \mu \left( a_{K,\sqrt{T}K^X} - e\left(TX_K, \nabla^{TX_K}\right)\delta_{X_K}\right)\right| &\leq \frac{C}{\sqrt{T}} \|\mu\|_{C^1(X)}, \\ \left| \int_{TX} \mu b_{K,\sqrt{T}K^X}\right| &\leq \frac{C}{\sqrt{T}} \|\mu\|_{C^1(TX)}. \end{aligned} \tag{3.31}$$

*Proof.* Observe the exact sequence

$$0 \to N_{X_K/X} \to TX|_{X_K} \to TX_K \to 0. \tag{3.32}$$

In (3.32), the map $N_{X_K/X} \to TX|_{X_K}$ is just $dK^X|_{X_K}$, and the second map is the orthogonal projection $TX|_{X_K} \to TX_K$. In fact $TX_K$ is just the orthogonal bundle to $N_{X_K/X}$ in $TX|_{X_K}$, and $\nabla^{TX_K}$ is the orthogonal projection of $\nabla^{TX|_{X_K}}$ on $TX_K$. By proceeding as in [B6, proof of Theorem 5.1] or in [B9, proof of Theorem 1.7], we get the first identity in (3.31). Using the fact that the integral of a linear function with respect to the Gaussian distribution vanishes, we also get the second identity in (3.31). □

*Remark* 3.11. A more precise statement than (3.31) is that the convergence estimates also hold microlocally in $P^X_{K,X_K}$. The proof is the same as in [B9].

*Definition* 3.12. Put

$$\psi_K\left(K^X, TX, \nabla^{TX}\right) = \int_0^{+\infty} b_{K,\sqrt{T}K^X}\frac{dT}{T}. \tag{3.33}$$

Theorem 3.13. *The current* $\psi_K\left(K^X, TX, \nabla^{TX}\right)$ *lies in* $P^X_{K,X_K}$. *Moreover,*

$$d_K\psi_K\left(K^X, TX, \nabla^{TX}\right) = e_K\left(TX, \nabla^{TX}\right) - e\left(TX_K, \nabla^{TX_K}\right)\delta_{X_K}. \tag{3.34}$$

*Also, if* $z \in \mathbf{R}^*$,

$$\psi_{zK}\left(zK^X, TX, \nabla^{TX}\right) = (\operatorname{sgn} z)^n\, z^{n/2}\psi_{1/z}\psi_K\left(K^X, TX, \nabla^{TX}\right). \tag{3.35}$$

*Proof.* By Remark 3.11, the fact that $\psi_K\left(K^X, TX, \nabla^{TX}\right)$ lies in $P^X_{X_K}$ can be proved as in [B8, Ths. 2.5 and 2.12]. Also, by (3.29), (3.31), we get (3.34). Finally (3.35) follows from (3.30). □

*Remark* 3.14. By using (3.27), we find that $\psi_K\left(K^X, TX, \nabla^{TX}\right)$ is of degree $\leq n-1$. More precisely, the component of top degree $n-1$ of $\psi_K\left(K^X, TX, \nabla^{TX}\right)$ is just the current $\psi\left(K^X, TX, \nabla^{TX}\right)$ which is obtained as in (3.33), by replacing $b_{K,\sqrt{T}K^X}$ by $b_{0,\sqrt{T}K^X}$. In particular, instead of (3.34), the current $\psi\left(K^X, TX, \nabla^{TX}\right)$ is such that

$$d\psi\left(K^X, TX, \nabla^{TX}\right) = e\left(TX, \nabla^{TX}\right) - e\left(TX_K, \nabla^{TX_K}\right). \tag{3.36}$$

3.4. *Comparison of the currents* $F_K\left(X, g^{TX}\right)$ *and* $\psi_K\left(K^X, TX, \nabla^{TX}\right)$. Let $i : X_K \to X$ be the obvious embedding. Then by (3.2), we have the identity

$$i^*e_K\left(TX, \nabla^{TX}\right) = e\left(TX_K, \nabla^{TX_K}\right)e_K\left(N_{X_K/X}, \nabla^{N_{X_K/X}}\right). \tag{3.37}$$

Using (3.19) and (3.37), we get

$$d_Ke_K\left(TX, \nabla^{TX}\right)F_K\left(TX, \nabla^{TX}\right) = e_K\left(TX, \nabla^{TX}\right) - e\left(TX_K, \nabla^{TX_K}\right)\delta_{X_K}, \tag{3.38}$$

i.e. the current $e_K\left(TX,\nabla^{TX}\right)F_K\left(TX,\nabla^{TX}\right)$ verifies the same equation as the current $\psi_K\left(K^X,TX,\nabla^{TX}\right)$ in (3.34). In view of (3.5), an explanation for this result is as follows.

THEOREM 3.15. *The following identity holds*:

$$e_K\left(TX,\nabla^{TX}\right)F_K\left(TX,\nabla^{TX}\right)-\psi_K\left(K^X,TX,\nabla^{TX}\right)\in P^{X,0}_{K,X_K}. \tag{3.39}$$

*Proof.* Observe that the forms $b_{K^X,T}$ vanish identically on $X_K$. We claim that by proceeding as in [B8, proof of Th. 3.2], (3.39) follows easily. In fact the results of [B8] refer to Bott-Chern currents on complex manifolds (which solve a current equation with respect to an equivariant version of the $\overline{\partial}\partial$ operator), while here, we only deal with equivariant Chern-Simons type objects. Note that the manifold $X_K$ intersects itself non-transversally, with an excess normal bundle $N_{X_K/X}$. The only important point is to understand why this excess normal bundle does not contribute to the final formula. However this follows easily from the fact that this contribution can be expressed as the integral along the fibre $N_{X_K/X}$ of a form which is odd with respect to $Z\in N_{X_K/X}$, so that it vanishes identically. Details are left to the reader. □

*Remark* 3.16. It follows from Theorem 3.15 that if $X$ is odd-dimensional,

$$\psi_K\left(K^X,TX,\nabla^{TX}\right)\in P^{X,0}_{X_K}. \tag{3.40}$$

Also observe that equations (3.20), (3.35) and (3.39) are compatible. In particular, if $n$ is odd and if $z<0$, there is no sign discrepancy because of (3.40).

3.5. *Convergence of the Mathai-Quillen currents associated to a gradient vector field.* Let now $f:X\to\mathbf{R}$ be a smooth $K^X$-invariant function. Let $\nabla f$ be the gradient field of $f$ with respect to $g^{TX}$. Then $\nabla f$ is a $K^X$-invariant section of $TX$.

We will assume that $f$ is a Morse-Bott function. Let $B$ be the set of critical points of $f$, i.e. the vanishing locus of $\nabla f$. Then $B$ is a smooth compact submanifold of $X$.

Clearly, the Hessian $\nabla^{TX}\nabla_{\cdot}f$ defines a symmetric quadratic form on $TX$. We identify the normal bundle $N_{B/X}$ to the orthogonal bundle to $TB$ in $TX|_B$ with respect to $g^{TX}$. It follows that $\nabla^{TX}\nabla f|_B:N_{B/X}\to N_{B/X}$ is invertible and self-adjoint.

Let $g^{TB}$ be the metric induced by $g^{TX}$ on $TB$. Let $\nabla^{TB}$ be the corresponding Levi-Civita connection on $TB$. Equivalently, $\nabla^{TB}$ is the orthogonal projection of $\nabla^{TX}|_B$ on $TB$.

By (3.22), on $X_K$,

$$m^{TX}(K)\nabla f=0. \tag{3.41}$$

By (3.41), it follows that $\nabla f|_{X_K} \in TX_K$, i.e. $\nabla f|_{X_K}$ is the gradient field for the function $f|_{X_K} : X_K \to \mathbf{R}$ with respect to the metric $g^{TX_K}$.

Since $K^X|_B$ is a Killing vector field on $B$, it follows that its vanishing locus $B_K = B \cap X_K$ is a smooth submanifold of $B$. Therefore $B_K$ is a smooth submanifold of $X_K$. Also $X_K$ and $B$ intersect orthogonally along $B_K$. It follows from the above that $f|_{X_K}$ is a Morse-Bott function on $X_K$, whose critical manifold is just $B_K$.

Let $\operatorname{ind}(f)$ be the locally constant function on $B$ with values in $\mathbf{Z}$, which is the index of the quadratic form $\nabla^{TX}\nabla f$ on $N_{B/X}$, i.e. the number of its negative eigenvalues. We define in the same way the function $\operatorname{ind}(f|_{X_K})$ on $X_K$. Observe that since $\nabla f$ is $K^X$-invariant and $N_{X_K/X}$ is even-dimensional, we have the equality of $\mathbf{Z}$-valued functions on $B_K$,

$$(-1)^{\operatorname{ind}\left(f|_{X_K}\right)} = (-1)^{\operatorname{ind}(f)|_{B_K}} . \tag{3.42}$$

We define the spaces of $K^X$-invariant currents $P^X_{K,B}, P^{X,0}_{K,B}$ on $B$ as in Definition 3.1, by simply replacing $X_K$ by $B$.

Theorem 3.17. *There is a constant $C > 0$ such that for any smooth form $\mu$ on $X$, for $T \geq 1$,*

$$\left|\int_X \mu \left(a_{K,\sqrt{T}\nabla f} - (-1)^{\operatorname{ind}(f)}\, e_K\left(TB, \nabla^{TB}\right)\delta_B\right)\right| \leq \frac{C}{\sqrt{T}} \left\|\mu\right\|_{C^1(X)} , \tag{3.43}$$
$$\left|\int_X \mu b_{K,\sqrt{T}\nabla f}\right| \leq \frac{C}{\sqrt{T}} \left\|\mu\right\|_{C^1(X)} .$$

*Proof.* Instead of (3.32), we have the exact sequence of vector bundles on $B$,

$$0 \to N_{B/X} \to TX|_B \to TB \to 0. \tag{3.44}$$

The map $N_{B/X} \to TX$ is now $\nabla^{TX}\nabla f|_B$, and the map $TX|_B \to TB$ is the orthogonal projection. In fact the image of $N_{B/X}$ by $\nabla^{TX}\nabla f|_B$ is just the orthogonal bundle to $TB$ in $TX|_B$, so that in turn, $TB$ can be identified to the orthogonal bundle to this image. The orthogonal projection of the connection $\nabla^{TX|_B}$ on $TB$ is just the Levi-Civita connection $\nabla^{TB}$. By procedures in [B6, proof of Th. 5.1] or in [B9, proof of Th. 3.3], our Theorem follows. □

*Remark* 3.18. A more precise statement than (3.31) is that the convergence estimates also hold microlocally in $P^X_{K,B}$. The proof is the same as in [B6, B9].

*Definition* 3.19. Put

$$\psi_K\left(\nabla f, TX, \nabla^{TX}\right) = \int_0^{+\infty} b_{K,\sqrt{T}\nabla f}\frac{dT}{T}. \tag{3.45}$$

THEOREM 3.20. *The current* $\psi_K\left(\nabla f, TX, \nabla^{TX}\right)$ *lies in* $P^X_{K,B}$. *Moreover*

$$d_K\psi_K\left(\nabla f, TX, \nabla^{TX}\right) = e_K\left(TX, \nabla^{TX}\right) - (-1)^{\operatorname{ind}(f)} e_K\left(TB, \nabla^{TB}\right)\delta_B. \tag{3.46}$$

*Proof.* By Remark 3.18, the fact that $\psi_K\left(\nabla f, TX, \nabla^{TX}\right)$ lies in $P^X_{K,B}$ can be proved as in [B8, Th. 2.5]. Also, by (3.29), (3.43), we get (3.46). □

By the same method, we can define the current $\psi\left(\nabla f|_{X_K}, TX_K, \nabla^{TX_K}\right)$ on $X_K$, which lies in $P^{X_K}_{B_K}$, and is such that

$$d\psi\left(\nabla f|_{X_K}, TX_K, \nabla^{TX_K}\right) = e\left(TX_K, \nabla^{TX_K}\right) - (-1)^{\operatorname{ind}\left(f|_{X_K}\right)} e\left(TB_K, \nabla^{TB_K}\right)\delta_{B_K}. \tag{3.47}$$

Note that $\psi\left(\nabla f|_{X_K}, TX_K, \nabla^{TX_K}\right)$ is of degree $\dim TX_K - 1$.

3.6. *Comparison of the currents* $\psi_K\left(K^X, \cdot\right)$ *and* $\psi_K(\nabla f, \cdot)$. Using (3.34), (3.42), (3.46) and (3.47), we get

$$\begin{aligned} d_K&\left[\psi_K\left(K^X, TX, \nabla^{TX}\right) + \psi\left(\nabla f|_{X_K}, TX_K, \nabla^{TX_K}\right)\delta_{X_K}\right] \\ &= e_K\left(TX, \nabla^{TX}\right) - (-1)^{\operatorname{ind}(f)} e\left(TB_K, \nabla^{TB_K}\right)\delta_{B_K}, \\ d_K&\left[\psi_K\left(\nabla f, TX, \nabla^{TX}\right) + (-1)^{\operatorname{ind}(f)}\psi_K\left(K^X|_B, TB, \nabla^{TB}\right)\delta_B\right] \\ &= e_K\left(TX, \nabla^{TX}\right) - (-1)^{\operatorname{ind}(f)} e\left(TB_K, \nabla^{TB_K}\right)\delta_{B_K}. \end{aligned} \tag{3.48}$$

Let $P^X_{K,X_K\cup B}$ be the set of $K^X$-invariant currents on $X$ whose wave front set is included in $N^*_{X_K/X} + N^*_{B/X}$, let $P^{X,0}_{K,X_K\cup B}$ be the set of $K^X$-invariant currents $a$ such that there is a $K^X$-invariant current $b \in P^X_{K,X_K\cup B}$ with $a = d_K b$.

A refinement for (3.48) is as follows.

THEOREM 3.21. *The following identity holds*:

$$\begin{aligned} \psi_K&\left(K^X, TX, \nabla^{TX}\right) + \psi\left(\nabla f|_{X_K}, TX_K, \nabla^{TX_K}\right)\delta_{X_K} \\ &= \psi_K\left(\nabla f, TX, \nabla^{TX}\right) \\ &\quad + (-1)^{\operatorname{ind}(f)}\psi_K\left(K^X|_B, TB, \nabla^{TB}\right)\delta_B \text{ in } P^X_{X_K\cup B}/P^{X,0}_{X_K\cup B}. \end{aligned} \tag{3.49}$$

*Proof.* The principle of the proof of our theorem is the same as the proof of [B9, Th. 2.8]. □

*Remark* 3.22. Clearly, we can combine Theorems 3.15 and 3.21.

3.7. *A proper submersion.* Let $\pi : M \to S$ be a submersion of smooth manifolds, with compact fibre $X$ of dimension $n$. Let $TX \subset TM$ be the tangent bundle to the fibres $X$.

Let $G$ be a compact Lie group acting on $M$ along the fibres of $X$, that is if $g \in G$, $\pi g = \pi$. Then $G$ acts on $TM$ and on $TX \subset TM$. Let $T^H M \subset TM$ be a $G$-invariant horizontal subbundle, so that

$$TM = T^H M \oplus TX. \tag{3.50}$$

Observe that since $G$ is compact, such a $T^H M$ always exists. Let $P^{TX} : TM \to TX$ be the projection associated to the splitting (3.50). Observe that

$$T^H M \simeq \pi^* TS. \tag{3.51}$$

Let $g^{TX}$ be a $G$-invariant Euclidean metric on $TX$. In the sequel, we identify $TX$ and $T^*X$ by the metric $g^{TX}$.

By [B3], $\left(T^H M, g^{TX}\right)$ determine a Euclidean connection $\nabla^{TX}$ on $TX$. Let $g^{TS}$ be an Euclidean metric on $TS$. We equip $TM$ with the $G$-invariant metric $g^{TM} = \pi^* g^{TS} \oplus g^{TX}$. Let $\nabla^{TM,L}$ be the Levi-Civita connection on $\left(TM, g^{TM}\right)$. Then the connection $\nabla^{TX}$ on $TX$ is given by

$$\nabla^{TX} = P^{TX} \nabla^{TM,L}, \tag{3.52}$$

and is independent of $g^{TS}$. Let $\nabla^{TM}$ be the connection on $TM$,

$$\nabla^{TM} = \pi^* \nabla^{TS} \oplus \nabla^{TX}. \tag{3.53}$$

Let $T$ be the torsion of $\nabla^{TM}$. Put

$$S = \nabla^{TM,L} - \nabla^{TM}. \tag{3.54}$$

Then $S$ is a 1-form on $M$ with values in antisymmetric elements of $\mathrm{End}\,(TM)$. Classically, if $A, B, C \in TM$,

$$\begin{gathered} S(A)B - S(B)A + T(A,B) = 0, \\ 2\langle S(A)B, C\rangle + \langle T(A,B), C\rangle + \langle T(C,A), B\rangle - \langle T(B,C), A\rangle = 0. \end{gathered} \tag{3.55}$$

By [B3, Th. 1.9], we know that

- The connection $\nabla^{TX}$ preserves the metric $g^{TX}$.
- The connection $\nabla^{TX}$ and the tensors $T$ and $\langle S(\cdot), \cdot, \cdot\rangle$ do not depend on $g^{TS}$.
- The tensor $T$ takes its values in $TX$, and vanishes on $TX \times TX$.
- For any $A \in TM$, $S(A)$ maps $TX$ into $T^H M$.

- For any $A,B\in T^HM$, $S(A)B\in TX$.
- If $A\in T^HM$, $S(A)A=0$.

From (3.55), we find that if $A\in T^HM$, $B,C\in TX$,

$$\langle T(A,B),C\rangle=\langle T(A,C),B\rangle=-\langle S(B)C,A\rangle. \tag{3.56}$$

By construction, all the above objects are $G$-invariant. If $U\in TS$, let $U^H\in T^HM$ be the horizontal lift of $U$ in $T^HM$. If $U$ is a vector field on $TS$, let $L_{U^H}$ be the Lie derivative operator associated to the vector field $U^H$. One verifies easily that $L_{U^H}$ acts on the tensor algebra of $TX$, and that this action defines a corresponding tensor in $U\in TS$. Now, we recall a simple result stated in [B12, Th. 1.1].

Theorem 3.23. *The connection $\nabla^{TX}$ on $\left(TX,g^{TX}\right)$ is characterized by the following two properties*:

- *On each fibre $X$, it restricts to the Levi-Civita connection.*
- *If $U\in TS$,*

$$\nabla^{TX}_{U^H}=L_{U^H}+\frac{1}{2}\left(g^{TX}\right)^{-1}L_{U^H}g^{TX}. \tag{3.57}$$

*Moreover, if $U,V$ are smooth sections of $TS$,*

$$T\left(U^H,V^H\right)=-P^{TX}\left[U^H,V^H\right], \tag{3.58}$$

*and if $U\in TS, A\in TX$,*

$$T\left(U^H,A\right)=\frac{1}{2}\left(g^{TX}\right)^{-1}L_{U^H}g^{TX}A. \tag{3.59}$$

In the sequel, $SP^{TX}S$ and $\nabla^{TX}_{\cdot}S$ are considered as sections of $\Lambda^2(T^*X)\otimes\mathrm{End}(TM)$. The following identity was established in [B3, Th. 4.14], [B5, Th. 2.3].

Proposition 3.24. *If $A,A'\in TX, B,B'\in TM$, then*

$$\begin{aligned}\left\langle R^{TX}(A,A')P^{TX}B,P^{TX}B'\right\rangle+\left\langle SP^{TX}S(A,A')B,B'\right\rangle\\ +\left\langle\left(\nabla^{TX}_{\cdot}S\right)(A,A')B,B'\right\rangle=\left\langle R^{TX}(B,B')A,A'\right\rangle.\end{aligned} \tag{3.60}$$

We will now denote by $\widehat{d}^X$ the fibrewise de Rham operator acting on smooth sections of $\widehat{\Lambda}^{\cdot}(T^*X)$ along the fibres $TX$. We denote by $\widehat{\nabla}^{TX}$ the restriction of the connection $\nabla^{TX}$ to the fibres $X$, which acts on sections of $\widehat{\Lambda}^{\cdot}(T^*X)\otimes TX$. Observe that $\widehat{\nabla}^{TX}$ increases the degree in the exterior algebra $\widehat{\Lambda}^{\cdot}(T^*X)$ by 1. Its curvature, $\widehat{R}^{TX}$, is a smooth section of $\widehat{\Lambda}^2(T^*X)\otimes \operatorname{End}(TX)$. Also $\widehat{\nabla}^{TX}$ acts naturally on smooth sections of $\widehat{\Lambda}^{\cdot}(T^*X) \widehat{\otimes} \Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(T^*S)$ along the fibres.

Let $e_1,\dots,e_n$ be an orthonormal basis of $TX$ and let $e^1,\dots,e^n$ be the corresponding dual basis of $T^*X$, let $f_1,\dots,f_m$ be a basis of $TS$ and let $f^1,\dots,f^m$ be the corresponding dual basis of $T^*S$.

*Definition* 3.25. Set

$$\widehat{T^H}=\frac{1}{2}\left\langle T\left(f_\alpha^H,f_\beta^H\right),e_i\right\rangle \widehat{e}^i\wedge f^\alpha\wedge f^\beta,\qquad T^0=f^\alpha\wedge\widehat{e}^i\wedge T\left(f_\alpha^H,e_i\right). \tag{3.61}$$

Then $\widehat{T^H}$ and $\widehat{d}^X\widehat{T^H}$ are sections of $\Lambda^2(T^*S)\widehat{\otimes}\widehat{\Lambda}^{\cdot}(T^*X)$. Also $T^0$ is a section of $\Lambda^{\cdot}(T^*S)\widehat{\otimes}\widehat{\Lambda}^{\cdot}(T^*X)\widehat{\otimes}TX$. Recall that we identify $TX$ and $T^*X$ by the metric $g^{TX}$. Then $T^0$ can be viewed as the smooth section of $\Lambda^{\cdot}(T^*S)\widehat{\otimes}\widehat{\Lambda}^{\cdot}(T^*X)\widehat{\otimes}T^*X$,

$$T^0=\left\langle T\left(f_\alpha^H,e_i\right),e_j\right\rangle f^\alpha\wedge\widehat{e}^i\wedge e^j. \tag{3.62}$$

By the above, $\widehat{\nabla}^{TX}_{\cdot}T^0$ is well defined. Also the operator $i_{T^0}$ acts on

$$\Lambda^{\cdot}(T^*S)\widehat{\otimes}\Lambda^{\cdot}(T^*X)\widehat{\otimes}\widehat{\Lambda}^{\cdot}(T^*X)$$

by interior multiplication in the variable $e_j$ acting on $\Lambda^{\cdot}(T^*X)$, and exterior product by $f^\alpha$ acting on $\Lambda^{\cdot}(T^*S)$, and by exterior product by $\widehat{e}^i$ acting on $\widehat{\Lambda}^{\cdot}(T^*X)$. In particular, $i_{T^0}$ increases the degree in $\widehat{\Lambda}^{\cdot}(T^*X)$ by 1. Set

$$\left|T^0\right|^2=\left\langle T^0,T^0\right\rangle. \tag{3.63}$$

Equivalently,

$$\left|T^0\right|^2=\sum_{j=1}^n\left(\sum_{\substack{1\le i\le n\\1\le\alpha\le m}}\left\langle T\left(f_\alpha^H,e_i\right),e_j\right\rangle f^\alpha\wedge\widehat{e}^i\right)^2, \tag{3.64}$$

where the square in the right-hand side of (3.64) is taken in $\Lambda^{\cdot}(T^*S)\widehat{\otimes}\widehat{\Lambda}^{\cdot}(T^*X)$. Then $\left|T^0\right|^2$ is a section of $\Lambda^2(T^*S)\widehat{\otimes}\widehat{\Lambda}^2(T^*X)$.

Theorem 3.26. *The following identity holds*:

$$\frac{1}{2}\left\langle e_i,R^{TX}e_j\right\rangle\widehat{e}^i\wedge\widehat{e}^j=\frac{1}{2}\left\langle e_i,\widehat{R}^{TX}e_j\right\rangle e^i\wedge e^j +\widehat{\nabla}^{TX}_{\cdot}T^0+\frac{1}{2}\left|T^0\right|^2-\frac{1}{2}\widehat{d}^X\widehat{T^H}. \tag{3.65}$$

*Moreover*,

$$\left(\widehat{\nabla}^{TX}_{\cdot}+i_{T^0}\right)\frac{1}{2}\left\langle e_i,R^{TX}e_j\right\rangle\widehat{e}^i\wedge\widehat{e}^j=0. \tag{3.66}$$

*Proof.* Observe first that when only vertical Grassmann variables $e^i$ are considered, (3.65) is a trivial consequence of the well-known symmetry of the fibrewise Levi-Civita curvature, a form of identity (3.60).

Also we know that if $A \in TM$, $S(A)$ maps $TX$ into $T^HM$. By (3.56), if $A \in TX, B \in TS$,

$$P^{TX} S(A) B^H = T\left(B^H, A\right). \tag{3.67}$$

By (3.67), and the fact that $S(\cdot)$ takes its values in antisymmetric elements of $\mathrm{End}(TM)$, we find that if $A, A' \in TX, B, B' \in TS$,

(3.68)
$$\begin{aligned}&\left\langle \left(SP^{TX}S\right)(A,A') B^H, B'^H\right\rangle\\ &\quad= \left\langle S(A) P^{TX} S(A') B^H, B'^H\right\rangle - \left\langle S(A') P^{TX} S(A) B^H, B'^H\right\rangle\\ &\quad= \left\langle P^{TX} S(A) B^H, P^{TX} S(A') B'^H\right\rangle - \left\langle P^{TX} S(A') B^H, P^{TX} S(A) B'^H\right\rangle\\ &\quad= \left\langle T\left(B^H, A\right), T\left(B'^H, A'\right)\right\rangle - \left\langle T\left(B^H, A'\right), T\left(B'^H, A\right)\right\rangle.\end{aligned}$$

From (3.68), we deduce that

$$\frac{1}{4}\left\langle f_\alpha^H, SP^{TX}S(e_i,e_j) f_\beta^H\right\rangle \widehat{e}^i \wedge \widehat{e}^j \wedge f^\alpha \wedge f^\beta = \frac{1}{2}\left|T^0\right|^2. \tag{3.69}$$

By (3.55), if $A \in TX, B, B' \in TS$,

$$\left\langle S(A) B^H, B'^H\right\rangle = \frac{1}{2}\left\langle T\left(B^H, B'^H\right), A\right\rangle. \tag{3.70}$$

Therefore,

$$\frac{1}{4}\left\langle f_\alpha^H, \nabla_\cdot^{TX} S(e_i,e_j) f_\beta^H\right\rangle \widehat{e}^i \wedge \widehat{e}^j f^\alpha \wedge f^\beta = -\frac{1}{2}\widehat{d}^X \widehat{T}^H. \tag{3.71}$$

Using (3.67), we find that if $B \in TS, A, A', B' \in TX$,

$$\left\langle \nabla_\cdot^{TX} S(A,A') B^H, B'\right\rangle = \left\langle \nabla_\cdot^{TX} T\left(B^H, \cdot\right)(A,A'), B'\right\rangle \tag{3.72}$$

From (3.72), we get

$$\frac{1}{2}\left\langle f_\alpha^H, \nabla^{TX} S(e_i,e_j) e_k\right\rangle \widehat{e}^i \wedge \widehat{e}^j \wedge f^\alpha \wedge e^k = \widehat{\nabla}_\cdot^{TX} T^0. \tag{3.73}$$

By (3.60), (3.68)-(3.73), we get (3.65) in full generality. Clearly,

$$\left(\widehat{\nabla}_\cdot^{TX} + i_{T^0}\right)\widehat{d}^X \widehat{T}^H = \widehat{d}^{X,2}\widehat{T}^H = 0. \tag{3.74}$$

Also, by (3.25), we get

$$\left(\widehat{\nabla}_\cdot^{TX} + i_{T^0}\right)\left(\frac{1}{2}\left\langle e_i, \widehat{R}^{TX} e_j\right\rangle e^i \wedge e^j + \widehat{\nabla}_\cdot^{TX} T^0 + \frac{1}{2}\left|T^0\right|^2\right) = 0. \tag{3.75}$$

By (3.74), (3.75), we get (3.66). The proof of our theorem is completed. □

*Remark* 3.27. Identity (3.66) was proved in [BGo2, Ths. 6.11 and 6.12], by a method closely related to the arguments used in our proof. Moreover, all the above objects are $G$-invariant.

Recall that $\mathfrak{g}$ is the Lie algebra of $G$. If $K \in \mathfrak{g}$, let $K^X$ be the corresponding vector field on $M$. We use the notation $K^X$ instead of $K^M$, because $K^X$ is a section of $TX$, the tangent bundle to the fibres $X$. Let $K^{X\prime} \in T^*X$ be dual to $K^X$ by $g^{TX}$. We identify $K^{X\prime}$ to a vertical 1-form on $M$, i.e. to a 1-form which vanishes on $T^HM$. Then equations (3.6) and (3.7) still hold.

Recall that $\nabla^{TX}$ restricts to the Levi-Civita connection along the fibres $X$. Let $m^{TX}(K)$ be the restriction of $\nabla^{TX}_{\cdot} K^X$ to $TX$. Then $m^{TX}(K)$ is an antisymmetric section of End$(TX)$. Also the group $G$ acts on $TX$. By (2.31), the equivariant curvature $R^{TX}_K$ is still given by

$$R^{TX}_K = R^{TX} - 2\pi m^{TX}(K). \tag{3.76}$$

Take $K \in \mathfrak{g}$. Let $f : M \to \mathbf{R}$ be a $K^X$-invariant smooth function, so that

$$i_{K^X} df = 0. \tag{3.77}$$

From (3.77), we get

$$d_K df = 0. \tag{3.78}$$

Let $\nabla f \in TX$ be the fibrewise gradient field of $f$ with respect to the metric $g^{TX}$. Set

$$d^H f = f^\alpha \nabla_{f^H_\alpha} f. \tag{3.79}$$

Recall that $\widehat{R}^{TX}$ is the curvature of the fibrewise connection $\widehat{\nabla}^{TX}$. As in (2.31), set

$$\widehat{R}^{TX}_K = \widehat{R}^{TX} - 2\pi m^{TX}(K). \tag{3.80}$$

As in (3.21), set

$$\dot{R}^{TX}_K = \frac{1}{2}\left\langle e_k, R^{TX}_K e_l\right\rangle \widehat{e}^k \wedge \widehat{e}^l, \qquad \hat{\dot{R}}^{TX}_K = \frac{1}{2}\left\langle e_i, \widehat{R}^{TX}_K e_j\right\rangle e^i \wedge e^j. \tag{3.81}$$

Then $\dot{R}^{TX}_K \in \Lambda^{\cdot}(T^*M) \widehat{\otimes} \widehat{\Lambda}(T^*X)$, and $\hat{\dot{R}}^{TX}_K \in \widehat{\Lambda}^{\cdot}(T^*X) \widehat{\otimes} \Lambda^{\cdot}(T^*X)$.

Set

$$\nabla^{TX}_K = \nabla^{TX} - 2\pi i_{K^X}, \qquad \widehat{\nabla}^{TX}_K = \widehat{\nabla}^{TX} - 2\pi i_{\widehat{K^X}}. \tag{3.82}$$

Then, as in (2.7),

$$\nabla^{TX}_K \dot{R}^{TX}_K = 0, \qquad \widehat{\nabla}^{TX}_K \hat{\dot{R}}^{TX}_K = 0. \tag{3.83}$$

PROPOSITION 3.28. *The following identities hold*:

(3.84)

$$\nabla^{TX}_{\cdot} \widehat{K^X} = \left(\widehat{\nabla}^{TX}_{\cdot} + i_{T^0}\right) K^X, \quad \nabla^{TX}_{\cdot} \widehat{\nabla f} = -\left(\widehat{\nabla}^{TX}_{\cdot} + i_{T^0}\right) \nabla f - \widehat{d}^X d^H f.$$

*Proof.* By definition,

$$\nabla^{TX}_{\cdot}\widehat{K^X} = \left\langle \nabla^{TX}_{e_i} K^X, e_j \right\rangle e^i \wedge \widehat{e}^j + \left\langle \nabla^{TX}_{f^H_\alpha} K^X, e_i \right\rangle f^\alpha \wedge \widehat{e}^i. \tag{3.85}$$

Since $K^X$ is a fibrewise Killing vector field,

$$\left\langle \nabla^{TX}_{e_i} K^X, e_j \right\rangle = - \left\langle \nabla^{TX}_{e_j} K^X, e_i \right\rangle. \tag{3.86}$$

Also, since $T^H M$ is $K^X$-invariant, by (3.57), (3.59),

$$\nabla^{TX}_{f^H_\alpha} K^X = T\left(f^H_\alpha, K^X\right). \tag{3.87}$$

Using (3.56), (3.87), we get

$$\begin{aligned} \left\langle \nabla^{TX}_{f^H_\alpha} K^X, e_i \right\rangle f^\alpha \wedge \widehat{e}^i &= \left\langle T\left(f^H_\alpha, K^X\right), e_i \right\rangle f^\alpha \wedge \widehat{e}^i \\ &= \left\langle T\left(f^H_\alpha, e_i\right), K^X \right\rangle f^\alpha \wedge \widehat{e}^i = i_{T^0} K^X. \end{aligned} \tag{3.88}$$

By (3.85)–(3.88), we get the first identity in (3.84).

Also the form $df$ is closed, so that if $U, V$ are smooth sections of $TM$,

$$U \langle \nabla f, V \rangle - V \langle \nabla f, U \rangle - \langle \nabla f, [U, V] \rangle = 0. \tag{3.89}$$

Using the fact that $T$ is the torsion of the connection $\nabla^{TM}$, we get the second identity in (3.84). The proof of our proposition is completed. □

3.8. *An odd closed form on $S$.* We make the same assumptions and use the same notation as in Section 3.7. In particular the metric $g^{TX}$ is given on $TX$. This in turn determines the identification $TX \simeq T^*X$, and also the Berezin integral $\int^{\widehat{B}}$.

*Definition* 3.29. Set

$$\begin{aligned} \gamma_K &= \int_X \exp\left(d_K K^{X\prime}/4\pi\right) \int^{\widehat{B}} \exp\left(-\dot{R}^{TX}_K/2\pi\right), \\ \delta_K &= \sqrt{2\pi} \int_X \exp\left(d_K K^{X\prime}/4\pi\right) \int^{\widehat{B}} \widehat{K^{X\prime}} \exp\left(-\dot{R}^{TX}_K/2\pi\right). \end{aligned} \tag{3.90}$$

Then $\gamma_K, \delta_K$ are differential forms on $S$.

In the sequel, we will often interchange the roles of the $e^i$ and of the $\widehat{e}^i$. Namely, we consider now the $\widehat{e}^i$ as standard differential forms on the fibres $X$. Then $\widehat{\int}_X$ denotes the integral along the fibres $X$ of smooth forms generated by the $\widehat{e}^i$. Also $\int^B$ will denote Berezin integration with respect to the variables $e^i$.

Recall that $\widehat{d}^X$ is the de Rham operator along the fibres $X$, which increases the degree in $\widehat{\Lambda}^{\cdot}(T^*X)$ by 1. Set

$$\widehat{d}^X_K = \widehat{d}^X - 2\pi i_{\widehat{K^X}}. \tag{3.91}$$

Let $\chi(X)$ be the Euler characteristic of $X$.

Theorem 3.30. *The form $\gamma_K$ is even and vanishes if* $\dim X$ *is odd, the form $\delta_K$ is odd and vanishes if* $\dim X$ *is even. Moreover, the following identities hold,*

$$\gamma_K = \widehat{\int_X} \exp\left(-\widehat{d}^X_{-K/2\pi}\left(\widehat{K^{X\prime}} - \widehat{T^H}/2\pi\right)\Big/2\right) \int^B \exp\left(-\left(\dot{\widehat{R}}^{TX}_{-K/2\pi} + \left(\widehat{\nabla}^{TX} + i_{\widehat{K^X}}\right)T^0 + \frac{1}{2}\left|T^0\right|^2\right)\Big/2\pi\right), \tag{3.92}$$

$$\delta_K = -\sqrt{2\pi}\widehat{\int_X} \widehat{K^{X\prime}} \exp\left(-\widehat{d}^X_{-K/2\pi}\left(\widehat{K^{X\prime}} - \widehat{T^H}/2\pi\right)/2\right) \int^B \exp\left(-\left(\dot{\widehat{R}}^{TX}_{-K/2\pi} + \left(\widehat{\nabla}^{TX} + i_{\widehat{K^X}}\right)T^0 + \frac{1}{2}\left|T^0\right|^2\right)\Big/2\pi\right).$$

*Also,*

$$\gamma_K = \chi(X), \qquad d\delta_K = 0. \tag{3.93}$$

*Proof.* Observe that if $\dim X$ is odd, $\int^{\widehat{B}} \exp\left(-\dot{R}^{TX}_K/2\pi\right)$ vanishes identically, so that $\gamma_K$ also vanishes. So we may now assume as well that $\dim X$ is even. Let $d^X$ be the standard de Rham operator along the fibre $X$. Since $T$ is the torsion of the connection $\nabla^{TM}$ defined in (3.53), by (3.56) and (3.87), we get

$$\begin{aligned} dK^{X\prime} &= \nabla^{TX}_{\cdot} K^{X\prime} + i_T K^{X\prime} \\ &= d^X K^{X\prime} + \left\langle T\left(f^H_\alpha, K^X\right), e_i\right\rangle f^\alpha \wedge e^i \\ &\quad + \left\langle T\left(f^H_\alpha, e_i\right), K^X\right\rangle f^\alpha \wedge e^i + \left\langle T^H, K^X\right\rangle \\ &= d^X K^{X\prime} + 2\left\langle T\left(f^H_\alpha, K^X\right), e_i\right\rangle f^\alpha \wedge e^i + \left\langle T^H, K^X\right\rangle. \end{aligned} \tag{3.94}$$

We rewrite (3.94) in the form,

$$dK^{X\prime} = d^X K^{X\prime} + i_{\widehat{K^X}}\left(-2T^0 + \widehat{T^H}\right). \tag{3.95}$$

Since $K^X$ is a Killing vector field,

$$\begin{aligned} d^X K^{X\prime} &= \left\langle m^{TX}(K)e_i, e_j\right\rangle e^i \wedge e^j, \\ \widehat{d}^X \widehat{K^{X\prime}} &= \left\langle m^{TX}(K)e_i, e_j\right\rangle \widehat{e}^i \wedge \widehat{e}^j. \end{aligned} \tag{3.96}$$

By (3.65), (3.90), (3.95), (3.96), we get (3.92).

Since the torsion tensor $T$ is $G$-invariant,

$$L_{\widehat{K^X}}\widehat{T^H} = 0, \qquad \left(\widehat{\nabla}^{TX} + i_{\widehat{K^X}}\right)^2 T^0 = \widehat{R}^{TX}_{-K/2\pi}T^0. \tag{3.97}$$

By (3.7) and (3.97), we get

$$\widehat{d}^{X,2}_{-K/2\pi}\left(\widehat{K^{X\prime}} - \widehat{T^H}/2\pi\right) = 0. \tag{3.98}$$

Comparing with (3.23), (3.26), we discover that the Berezin integral with respect to the standard Grassmann variables $e^i$

$$\int^B \exp\left(-\left(\hat{\dot{R}}^{TX}_{-K/2\pi} + \left(\hat{\nabla}^{TX} + i_{\widehat{K^X}}\right)T^0 + \frac{1}{2}\left|T^0\right|^2\right)\Big/2\pi\right)$$

is an extension of the Berezin integrals $a_{s,T}$ considered in (3.26). In fact in (3.26) we assumed that $\widehat{s}$ is a smooth section of $\widehat{TX}$. Here $T^0$ is a smooth even section of $\widehat{\Lambda}\left(T^*X\right)\widehat{\otimes}\Lambda^{\cdot}\left(T^*S\right)\widehat{\otimes}TX$ (recall that here we ultimately integrate on $X$ in the hatted Grassmann variables). By (3.25), (3.97), or by (3.65), (3.66), (3.97), we get

$$\left(\hat{\nabla}^{TX} + i_{\widehat{K^X}+T^0}\right)\left(\hat{\dot{R}}^{TX}_{-K/2\pi} + \left(\hat{\nabla}^{TX} + i_{\widehat{K^X}}\right)T^0 + \frac{1}{2}\left|T^0\right|^2\right) = 0. \tag{3.99}$$

By (3.99), we obtain

$$\widehat{d}^X_{-K/2\pi}\int^B \exp\left(-\left(\hat{\dot{R}}^{TX}_{-K/2\pi} + \left(\hat{\nabla}^{TX} + i_{\widehat{K^X}}\right)T^0 + \frac{1}{2}\left|T^0\right|^2\right)\Big/2\pi\right) = 0. \tag{3.100}$$

By (3.92), (3.98), (3.100), we find that $\gamma_K$ is now a standard integral on the fibre $X$ of a hatted differential form which is $\widehat{d}_{-K/2\pi}$-closed. Since

$$\exp\left(-\widehat{d}^X_{-K/2\pi}\left(\widehat{K^{X\prime}} - \widehat{T^H}/2\pi\right)\Big/2\right) - 1$$

is $\widehat{d}^X_{-K/2\pi}$-exact, we get

$$\gamma_K = \widehat{\int_X}\int^B \exp\left(-\left(\hat{\dot{R}}^{TX}_{-K/2\pi} + \left(\hat{\nabla}^{TX} + i_{\widehat{K^X}}\right)T^0 + \frac{1}{2}\left|T^0\right|^2\right)\Big/2\pi\right). \tag{3.101}$$

By the same argument as in (3.25), (3.29), we can now deform $T^0$ to 0, while leaving $\gamma_K$ unchanged. So we get

$$\gamma_K = \widehat{\int_X}\int^B \exp\left(-\hat{\dot{R}}^{TX}_{-K/2\pi}/2\pi\right). \tag{3.102}$$

Equivalently,

$$\gamma_K = \widehat{\int_X} e_{-K/2\pi}\left(TX, \hat{\nabla}^{TX}\right). \tag{3.103}$$

By (2.44), (2.45), (3.103), we get the first identity in (3.93).

For the same reason as before, the Berezin integral $\int^{\widehat{B}} \widehat{K^{X\prime}} \exp\left(-\dot{R}^{TX}_K/2\pi\right)$ vanishes if $\dim X$ is even. Therefore the form $\delta_K$ vanishes if $\dim X$ is even. So we may as well assume that $\dim X$ is odd. The proof of the second identity in (3.92) is the same as before. By (3.9), (3.83), we get

$$d\delta_K = \sqrt{2\pi}\int_X \exp\left(d_K K^{X\prime}/4\pi\right)\int^{\widehat{B}} \nabla^{TX}_{\cdot}\widehat{K^{X\prime}}\exp\left(-\dot{R}^{TX}_K/2\pi\right). \tag{3.104}$$

Proceeding as in (3.94)–(3.96), using the first equation in (3.84) and (3.104), and we get

$$(3.105)\quad d\delta_K = -\sqrt{2\pi}\,\widehat{\int_X} \exp\left(-\widehat{d}^X_{-K/2\pi}\left(\widehat{K^{X\prime}} - \widehat{T^H}/2\pi\right)\Big/2\right) \int^B \left[\left(\widehat{\nabla}^{TX} + i_{\widehat{K^X}+T^0}\right)K^{X\prime}\right] \exp\left(-\left(\widehat{\dot{R}^{TX}_{-K/2\pi}} + \left(\widehat{\nabla}^{TX} + i_{\widehat{K^X}}\right)T^0 + \frac{1}{2}\left|T^0\right|^2\right)\Big/2\pi\right).$$

By (3.99), (3.105), we obtain

$$(3.106)\quad d\delta_K = -\sqrt{2\pi}\,\widehat{\int_X} \exp\left(-\widehat{d}^X_{-K/2\pi}\left(K^{X\prime} - \widehat{T^H}/2\pi\right)\Big/2\right)\widehat{d}^X_{-K/2\pi} \int^B K^{X\prime} \exp\left(-\left(\widehat{\dot{R}^{TX}_{-K/2\pi}} + \left(\widehat{\nabla}^{TX} + i_{\widehat{K^X}}\right)T^0 + \frac{1}{2}\left|T^0\right|^2\right)\Big/2\pi\right).$$

By (3.98), (3.106), we get the second identity in (3.93). The proof of our theorem is completed. □

*Remark* 3.31. From Theorem 3.30, we find in particular that the cohomology class of the closed form $\delta_K$ does not depend on the metric $g^{TX}$.

3.9. *The $V$-invariant.* We still fix the metric $g^{TX}$ on $TX$. For $t > 0$, we will now construct the form $\delta_{K,t}$ associated to the metric $g^{TX}_t = g^{TX}/t$. Observe that in (3.90), the Berezin integral $\int^{\widehat{B}}$ depends explicitly on the choice of the metric $g^{TX}$. When replacing $g^{TX}$ by $g^{TX}_t$, we should in principle redefine the Berezin integral $\int^{\widehat{B}}$, i.e. introduce a $t$-dependent Berezin integration $\int_t^{\widehat{B}}$. However, for convenience, we will instead keep our definition of the Berezin integral fixed, i.e. independent of $t$. Therefore $\int^{\widehat{B}}$ will still refer to the Berezin integral associated to the metric $g^{TX}$.

Proposition 3.32. *For $t > 0$, the following identity holds*:

$$(3.107)\quad \delta_{K,t} = \sqrt{2\pi}\int_X \exp\left(d_K K^{X\prime}/4\pi t\right)\int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}}\exp\left(-\dot{R}^{TX}_K/2\pi\right).$$

*Proof.* Our identity follows from an obvious computation which is left to the reader. □

*Definition* 3.33. For $t > 0$, set

$$(3.108)\quad \varepsilon_{K,t} = \sqrt{2\pi}\int_X \frac{K^{X\prime}}{4\pi t}\exp\left(d_K K^{X\prime}/4\pi t\right)\int^{\widehat{B}} \frac{\widehat{K^X}}{\sqrt{t}}\exp\left(-\dot{R}^{TX}_K/2\pi\right).$$

Observe that by (3.8), we can rewrite (3.107), (3.108) in the form,

$$\begin{aligned}\delta_{K,t} &= \sqrt{2\pi}\int_X \alpha_{K,t}\int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}}\exp\left(-\dot{R}_K^{TX}/2\pi\right),\\ \varepsilon_{K,t} &= -\sqrt{2\pi}\int_X \beta_{K,t}\int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}}\exp\left(-\dot{R}_K^{TX}/2\pi\right).\end{aligned} \tag{3.109}$$

Recall that if $s \in \mathbf{R}_+^*$, the operator $\psi_s$ acting on $\Lambda^\cdot(T^*S)$ is as defined in (2.91).

THEOREM 3.34. *For $t > 0$, if $\dim X$ is even, the forms $\delta_{K,t}$ and $\varepsilon_{K,t}$ vanish. If $\dim X$ is odd, the forms $\delta_{K,t}$ are odd, and the forms $\varepsilon_{K,t}$ are even.*

*The forms $\delta_{K,t}$ are closed on $S$, and their cohomology class does not depend on $t > 0$. More precisely,*

$$\frac{\partial}{\partial t}\delta_{K,t} = -d\frac{\varepsilon_{K,t}}{t}. \tag{3.110}$$

*If $z \in \mathbf{R}^*$,*

$$\delta_{zK,t} = \frac{\operatorname{sgn} z}{\sqrt{z}}\psi_{1/z}\delta_{K,t/z^2}, \qquad \varepsilon_{zK,t} = \frac{1}{|z|}\psi_{1/z}\varepsilon_{K,t/z^2}. \tag{3.111}$$

*Proof.* By Theorem 3.30, we know that if $\dim X$ is even, $\delta_{K,t}$ vanishes. The same argument holds for $\varepsilon_{K,t}$. As seen in Remark 3.31, the cohomology class of $\delta_{K,t}$ does not depend on $t$.

Now we replace $M$ by $\widetilde{M} = M \times \mathbf{R}_+^*$, and $S$ by $\widetilde{S} = S \times \mathbf{R}_+^*$. Let $\widetilde{\pi} : \widetilde{M} \to \widetilde{S}$ be the obvious projection with fibre $X$. Over $S \times \{t\}$, we equip $TX$ with the metric $g^{TX}/t$. Set $T^H\widetilde{M} = T^HM \oplus T\mathbf{R}_+^*$. We can then use the formalism of Sections 3.7 and 3.8 applied to this new fibration. In particular, we observe that the curvature tensor $\widetilde{R}^{TX}$ is just the pull-back to $\widetilde{M}$ of $R^{TX}$. Let $\widetilde{K^{X\prime}}, \widetilde{d}_K$ be the obvious analogues of $K^{X\prime}, d_K$. One verifies easily that

$$\widetilde{d}_K\widetilde{K^{X\prime}} = d_K K^{X\prime}/t - \frac{dt}{t^2}K^{X\prime}. \tag{3.112}$$

From the above, we get

$$\widetilde{\delta}_K = \delta_{K,t} - \frac{dt}{t}\varepsilon_{K,t}. \tag{3.113}$$

By Theorem 3.30, the form $\widetilde{\delta}_K$ is closed on $\widetilde{S}$. Equation (3.110) is now obvious.

By using in particular (3.11) and (3.30), equation (3.111) follows easily. The proof of our theorem is completed. □

In the sequel, we will write that as $t \to 0$, $\delta_{K,t} = \mathcal{O}\left(\sqrt{t}\right)$ if for any compact subset $A$ of $S$ and for $m \in \mathbf{N}$, there exists $C > 0$ such that for $t \in ]0,1]$, the sup of $\delta_{K,t}$ and its derivatives of order lower than $m$ are dominated by $C\sqrt{t}$. Similar notation will be used when $t \to +\infty$.

THEOREM 3.35. *As* $t \to 0$,

$$\delta_{K,t} = \mathcal{O}\left(\sqrt{t}\right), \qquad \varepsilon_{K,t} = \mathcal{O}\left(\sqrt{t}\right). \tag{3.114}$$

*As* $t \to +\infty$,

$$\delta_{K,t} = \mathcal{O}\left(1/\sqrt{t}\right), \qquad \varepsilon_{K,t} = \mathcal{O}\left(1/t^{3/2}\right). \tag{3.115}$$

*In particular, the cohomology class of the closed forms* $\delta_{K,t}$ *vanishes.*

*Proof.* By proceeding as in [B8, proof of Theorem 1.3] or in [B9, proof of Th. 1.7], and using the fact that the integral of a linear function with respect to a Gaussian density vanishes, we find that as $t \to 0$, there is the convergence of currents on $M$,

$$\exp\left(d_K K^{X\prime}/4\pi t\right) \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \exp\left(-\dot{R}_K^{TX}/2\pi\right) \to 0. \tag{3.116}$$

By using the microlocal estimates similar to the ones given in [B8, eq. (2.18)], we see that the convergence in (3.116) holds microlocally in the space of currents whose wave front set is included in $N^*_{M_K/M}$, with bounds on the microlocal seminorms in the left-hand side of (3.116) of the type $C\sqrt{t}$. Since $\pi^* T^* S \cap N^*_{M_K/M} = \{0\}$, using [Hö, Th. 8.2.13], we get the first identity in (3.114). Let $\widetilde{\delta}_{K,t}$ be the analogue of $\delta_K$ over $\widetilde{S} = S \times \mathbf{R}^*_+$, where the generic element of $\mathbf{R}^*_+$ is now denoted $s$. One verifies easily that

$$\widetilde{\delta}_{K,t} = \delta_{K,st} - \frac{ds}{s}\varepsilon_{K,st}. \tag{3.117}$$

We can now apply the first identity in (3.114) to $\widetilde{\delta}_{K,t}$, to get the second identity. From (3.107), (3.108), we get (3.115).

Finally recall that the cohomology class of the forms $\delta_{K,t}$ does not depend on $t > 0$. From (3.114) or (3.115), we deduce that this cohomology class vanishes. The proof of our theorem is completed. □

Recall that the operator $Q$ acting on differential forms was defined in (2.92).

*Definition* 3.36. Put

$$\begin{aligned} U_K\left(M/S, T^H M, g^{TX}\right) &= \int_0^{+\infty} \varepsilon_{K,t} \frac{dt}{t}, \\ V_K\left(M/S, T^H M, g^{TX}\right) &= Q U_K\left(M/S, T^H M, g^{TX}\right). \end{aligned} \tag{3.118}$$

By (3.114), (3.115), the forms

$$U_K\left(M/S, T^H M, g^{TX}\right), \qquad V_K\left(M/S, T^H M/S, g^{TX}\right)$$

on $S$ are well defined.

Note that in degree 0, $U_K\left(M/S,T^HM,g^{TX}\right)$ and $V_K\left(M/S,T^HM,g^{TX}\right)$ coincide. In the sequel, we will state all our results on the form

$$V_K\left(M/S,T^HM,g^{TX}\right),$$

even if they also hold for $U_K\left(M/S,T^HM,g^{TX}\right)$.

THEOREM 3.37. *The even form* $V_K\left(M/S,T^HM,g^{TX}\right)$ *is closed. It vanishes if* $\dim X$ *is even. Also, its cohomology class does not depend on* $T^HM$ *and on* $g^{TX}$. *Finally, if* $z\in\mathbf{R}^*$,

$$V_{zK}\left(M/S,T^HM,g^{TX}\right)=\frac{1}{|z|}\psi_{1/z}V_K\left(M/S,T^HM,g^{TX}\right). \tag{3.119}$$

*Proof.* The first part of our theorem is a trivial consequence of Theorems 3.34 and 3.35. By functoriality, the second part of our theorem follows tautologically. Finally, using (3.111), we get (3.119). □

*Remark* 3.38. From the above result, it follows in particular that if $\dim X$ is odd, $V_K^{(0)}\left(T^HM,g^{TX}\right)$ is an invariant of $\left(X,K^X\right)$. In the sequel, we will denote by $V_K\left(M/S\right)$ the cohomology class of $V_K(M/S,T^HM,g^{TX})$.

Let now $\pi':M'\to S'$ be another submersion with compact fibre $X'$, which has the same properties as the submersion $\pi:M\to S$. We still assume that $G$ acts on $M'$ and preserves the fibres $X'$. More generally, we suppose that data similar to those just considered are attached to this new submersion. In particular we can define the even cohomology class $V_K\left(M'/S'\right)$ on $S'$.

Set $M''=M\times M'$, $S''=S\times S'$. It is then clear that the projection $\pi'':M''\to S''$ has the same properties as $\pi:M\to S$. Again, we denote with a superscript $''$ the objects natural to this new projection.

THEOREM 3.39. *The following identity holds*:

$$V_K\left(M''/S''\right)=\chi\left(X\right)V_K\left(M'/S'\right)+\chi\left(X'\right)V_K\left(M/S\right)\ \text{ in } H^{\text{even}}\left(S\times S',\mathbf{R}\right). \tag{3.120}$$

*Proof.* Using (3.93) in Theorem 3.30 and proceeding as in (3.113), we get

$$\sqrt{2\pi}\int_X\frac{K^{X\prime}}{4\pi t}\exp\left(d_KK^{X\prime}/4\pi t\right)\int^{\widehat{B}}\exp\left(-\dot{R}_K^{TX}/2\pi\right)=0. \tag{3.121}$$

A similar identity holds for the objects attached to $M'$. Using (3.93), (3.118) and (3.121), we obtain (3.120). □

## 4. Morse-Bott functions, multifibrations and the class $V_K(M/S)$

The purpose of this section is to establish two important properties of the cohomology classes $V_K(M/S)$ which we constructed in Section 3. More precisely, we show that if $f : M \to \mathbf{R}$ is a fibrewise $K^X$-invariant Morse-Bott function, then $V_K(M/S)$ can be expressed in terms of the corresponding $V$-invariants of the fibration defined by the critical points of $f$ along the fibres. Also we study the $V$-invariant of equivariant multifibrations.

This section is organized as follows. In 4.1 we evaluate the $V$-invariants of a $\mathbf{Z}_2$-graded Euclidean vector bundle, equipped with an obvious quadratic fibrewise Morse function. This computation will be used at the final stage of the proof of the comparison formula for Morse-Bott fibrations. In 4.2, we state the comparison formula. In 4.3, 4.4 and 4.5, we derive various consequences of our formula. In particular, we study the behaviour of the $V$-invariants under equivariant surgery.

Sections 4.6–4.10 are devoted to the proof of our formula. In 4.6, we show how to couple $K^X$ and $\nabla f$ in order to extend the forms $\delta_K$ of 3.8. In 4.7, we consider a contour integral in $\mathbf{R}_+^2$. Our main formula will be established by taking the contour to infinity. In 4.8, we make natural simplifying assumptions on the considered metrics. In 4.9, and following [B8], we establish three intermediate results. In 4.10, we compute the asymptotics of the contour integral, as the boundary tends to infinity, and we obtain the comparison formula for Morse-Bott fibrations.

Finally, in Section 4.11, we give a formula for the $V$-invariant of multifibrations.

4.1. *The case of a vector bundle.* Let $V$ be a smooth manifold, and let $E = E_+ \oplus E_-$ be a real $\mathbf{Z}_2$-graded vector bundle on $V$. Let $g^E = g^{E_+} \oplus g^{E_-}$ be a Euclidean metric on $E = E_+ \oplus E_-$, such that $E_+$ and $E_-$ are mutually orthogonal in $E$. Let $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ be a Euclidean connection on $E = E_+ \oplus E_-$, which preserves the splitting, and let $R^E = R^{E_+} \oplus R^{E_-}$ be its curvature.

Let $\mathcal{E}$ be the total space of $E$. Then the connection $\nabla^E$ induces a horizontal subbundle $T^H\mathcal{E}$ of $T\mathcal{E}$, so that

$$T\mathcal{E} = T^H\mathcal{E} \oplus E. \tag{4.1}$$

Recall that $G$ is a compact Lie group and $\mathfrak{g}$ is its Lie algebra. Take $K \in \mathfrak{g}$. Let $m^E(K)$ be an antisymmetric invertible parallel section of $\operatorname{End}(E)$, which preserves the splitting $E = E_+ \oplus E_-$. Then $E_+$ and $E_-$ are of even dimension. Let $Z = (Z_+, Z_-)$ be the generic element of $E = E_+ \oplus E_-$. Clearly, $K^E = m^E(K)\, Z$ is a fibrewise Killing vector field along the fibres $E$.

Let $q^E : \mathcal{E} \to \mathbf{R}$ be given by

$$q^E(Z_+, Z_-) = \frac{1}{2}\left(|Z_+|^2 - |Z_-|^2\right). \tag{4.2}$$

Then $q^E$ is a $K^E$-invariant function on $\mathcal{E}$, which is parallel with respect to $\nabla^E$.

Let $e_1, \ldots, e_n$ be an orthonormal basis of $E$, and let $e^1, \ldots, e^n$ be the corresponding dual basis. The splitting (4.1) enables us to consider $e^1, \ldots, e^n$ can be considered as vertical 1-forms on $\mathcal{E}$.

Now we will use the formalism of Sections 3.7–3.9 in this situation, with $K^E$ replacing $K^X$. In particular, let $K^{E\prime}$ be the 1-form on $\mathcal{E}$ which is dual to $K^E$. We have the easy formula,

$$d_K K^{E\prime} = -\left\langle e_i, m^E(K) e_j\right\rangle e^i \wedge e^j + \left\langle R^E Z, m^E(K) Z\right\rangle - 2\pi \left|m^E(K) Z\right|^2. \tag{4.3}$$

Now we apply the formalism of Section 3.7 to the fibration $\pi : \mathcal{E} \to V$, with fibre $X = E$. The connection $\nabla^{TX}$ attached to $\left(T^H\mathcal{E}, g^E\right)$ is just $\pi^*\nabla^E$.

*Definition* 4.1. Put

$$c_K^E = \exp\left(d_K K^{E\prime}/4\pi\right). \tag{4.4}$$

Then $c_K^E$ is an even form on $\mathcal{E}$, such that

$$d_K c_K^E = 0. \tag{4.5}$$

As in (2.31), (3.3), set

$$R_K^E = R^E - 2\pi m^E(K). \tag{4.6}$$

As in (3.81), set

$$\dot{R}_K^E = \frac{1}{2}\left\langle e_k, R_K^E e_l\right\rangle \widehat{e}^k \wedge \widehat{e}^l. \tag{4.7}$$

Now we define $A^E_{K,\sqrt{T}\nabla q^E}$ as in (3.23), namely:

$$A^E_{K,\sqrt{T}\nabla q^E} = \dot{R}_K^E + \sqrt{T}\nabla^E_{\cdot}\widehat{\nabla q^E} + \frac{T}{2}\left|\nabla q^E\right|^2. \tag{4.8}$$

Equivalently,

$$A^E_{K,\sqrt{T}\nabla q^E} = \dot{R}_K^E + \sqrt{T}\nabla^E_{\cdot}\left(\widehat{Z_+} - \widehat{Z_-}\right) + \frac{T}{2}|Z|^2. \tag{4.9}$$

*Definition* 4.2. For $T \geq 0$, put

$$d^E_{K,T} = \sqrt{2\pi}\int^{\widehat{B}} \widehat{K^E}\frac{\widehat{\sqrt{T}\nabla q^E}}{4\pi}\exp\left(-A^E_{K,\sqrt{T}\nabla q^E}\Big/2\pi\right). \tag{4.10}$$

Theorem 4.3. *The following identity of differential forms on $V$ holds*:

$$\int_E c_K^E d_{K,T}^E = \frac{\sqrt{T}}{2\sqrt{2\pi}} \mathrm{Tr_s}\left[\frac{-m^{E,2}(K)}{\left(\left(m^E(K)R_K^E+T\right)/2\pi\right)^2}\right]. \tag{4.11}$$

*In particular, the form $\int_E c_K^E d_{K,T}^E$ is closed, and its cohomology class does not depend on $g^E$ or $\nabla^E$.*

*Proof.* Let $\tau \in \mathrm{End}(E)$ be the involution defining the $\mathbf{Z}_2$-grading of $E$, i.e. $\tau = \pm 1$ on $E_\pm$. Then

$$\nabla q^E(Z) = \tau Z. \tag{4.12}$$

Using (4.12), we get

$$\int_E c_K^E d_{K,T}^E = -\frac{\partial}{\partial b}\left[\sqrt{2\pi}\int_E c_K^E \left\langle Z, m^E(K)e_k\right\rangle \left\langle Z, \frac{\sqrt{T}}{4\pi}\tau e_l\right\rangle \int^{\widehat{B}} \exp\left(-\left(A_{K,\sqrt{T}\nabla q^E}^E - 2\pi b\widehat{e}^k \wedge \widehat{e}^l\right)/2\pi\right)\right]_{b=0}. \tag{4.13}$$

If $\alpha$ is a differential form on $\mathcal{E}$, we can write $\alpha$ in the form

$$\alpha = \sum e^{i_1}\wedge \ldots \wedge e^{i_p}\alpha_{i_1\ldots i_p}, \tag{4.14}$$

with $\alpha_{i_1\ldots i_p} \in \Lambda^\cdot(T^*V)$. Set

$$\alpha^{\max} = \alpha_{1,\ldots,n}. \tag{4.15}$$

From (4.3), (4.13), we get

$$\begin{aligned}\int_E c_K^E d_{K,T}^E = &-\frac{\sqrt{T}}{2\sqrt{2\pi}} \det\left[m^E(K)R_K^E+T\right]^{-1/2}(2\pi)^{\dim E} \\ &\left\langle m^E(K)e_k, \left(\left(m^E(K)R_K^E+T\right)/2\pi\right)^{-1}\tau e_l\right\rangle \\ &\frac{\partial}{\partial b}\left[\int^{\widehat{B}}\exp\left(-\left(\frac{1}{2}\left\langle e_i, m^E(K)e_j\right\rangle e^i\wedge e^j + \dot{R}_K^E \right.\right.\right. \\ &\left.\left.\left.- 2\pi b\widehat{e}^k\wedge\widehat{e}^l + \sqrt{T}\nabla_\cdot^E\left(\widehat{Z_+}-\widehat{Z_-}\right)\right)/2\pi\right)\right]_{b=0}^{\max}.\end{aligned} \tag{4.16}$$

Set

$$\begin{aligned}\dot{S}_b^E = &\frac{1}{2}\left\langle e_i, m^E(K)e_j\right\rangle e^i\wedge e^j \\ &+\frac{1}{2}\left\langle e_k, R_K^E - 8\pi^2 b m^E(K)\left(m^E(K)R_K^E+T\right)^{-1}\tau e_l\right\rangle \widehat{e}^k\wedge\widehat{e}^l.\end{aligned} \tag{4.17}$$

Then we rewrite (4.16) in the form

$$\int_E c_K^E d_{K,T}^E = \frac{\sqrt{T}}{2\sqrt{2\pi}} \det\left[m^E(K)R_K^E + T\right]^{-1/2} (2\pi)^{\dim E}\left[\frac{\partial}{\partial b}\int^{\widehat{B}} \exp\left(-\left(\dot{S}_b^E + \sqrt{T}\nabla_\cdot^E\left(\widehat{Z_+} - \widehat{Z_-}\right)\right)\Big/2\pi\right)\right]_{b=0}^{\max}. \tag{4.18}$$

One has the easy formula,

$$(2\pi)^{\dim E}\left[\int^{\widehat{B}} \exp\left(-\left(\dot{S}_b^E + \sqrt{T}\nabla_\cdot^E\left(\widehat{Z_+} - \widehat{Z_-}\right)\right)\Big/2\pi\right)\right]^{\max} = \det\left[m^E(K)R_K^E + T - 8\pi^2 b\tau\frac{m^{E,2}(K)}{m^E(K)R_K^E + T}\right]^{1/2}. \tag{4.19}$$

From (4.16), (4.19), we get (4.11). The proof is completed. □

*Definition* 4.4. Set

$$H_K\left(E, g^E, \nabla^E\right) = \int_0^{+\infty}\left\{\int_E c_K^E d_{K,T}^E\right\}\frac{dT}{T}. \tag{4.20}$$

By Theorem 4.3, the form $H_K\left(E, g^E, \nabla^E\right)$ is even, closed, and its cohomology class does not depend on $g^E$ or $\nabla^E$.

*Definition* 4.5. For $\theta \in \mathbf{C}^*, x \in \mathbf{C}, |x| < |\theta|$, put

$$I^\theta(x) = \frac{\pi}{4|\theta|}\left(1 - \frac{ix}{\theta}\right)^{-3/2}, \qquad J^\theta(x) = \frac{\pi}{4|\theta|}\left(1 - \frac{ix}{\theta}\right)^{-1}. \tag{4.21}$$

Recall that the operator $Q$ was defined in (2.92) and in (2.117). We also define a related operator $Q$ as follows. If $f(x)$ is an analytic function defined on a neighbourhood of $0 \in \mathbf{C}$, set

$$Qf(x) = \int_0^1 f(4s(1-s)x)\,ds. \tag{4.22}$$

The next result was established in [BGo2, Prop. 4.34].

PROPOSITION 4.6. *The following identity holds*:

$$QI^\theta(x) = J^\theta(x). \tag{4.23}$$

*Definition* 4.7. Set

$$\mathcal{I}_K\left(E, \nabla^E\right) = \mathrm{Tr_s}\left[I^{-im^E(K)}\left(-\frac{R^E}{2\pi}\right)\right], \tag{4.24}$$

$$\mathcal{J}_K\left(E, \nabla^E\right) = \mathrm{Tr_s}\left[J^{-im^E(K)}\left(-\frac{R^E}{2\pi}\right)\right].$$

Then $\mathcal{I}_K\left(E,\nabla^E\right), \mathcal{J}_K\left(E,\nabla^E\right)$ are closed even forms on $V$. We denote by $\mathcal{I}_K(E), \mathcal{J}_K(E)$ the corresponding cohomology classes.

THEOREM 4.8. *The following identities hold*:

$$\begin{aligned} H_K\left(E,g^E,\nabla^E\right) &= \mathcal{I}_K\left(E,\nabla^E\right), \\ QH_K\left(E,g^E,\nabla^E\right) &= \mathcal{J}_K\left(E,\nabla^E\right). \end{aligned} \tag{4.25}$$

*Proof.* Clearly, for $y>0$,

$$\int_0^{+\infty} \frac{1}{y+T}\frac{dT}{\sqrt{T}} = \frac{\pi}{\sqrt{y}}, \tag{4.26}$$

so that

$$\int_0^{+\infty} \frac{1}{(y+T)^2}\frac{dT}{\sqrt{T}} = \frac{\pi}{2y^{3/2}}. \tag{4.27}$$

By (4.11), (4.20), (4.27), we get the first identity in (4.11). Using now (4.23), we obtain the second identity in (4.25). The proof is complete. □

*Remark* 4.9. The function $I^\theta(x)$ appeared in an entirely different context in [BGo2, Definition 4.28], in the evaluation of the analytic torsion forms associated to a $\mathbf{Z}_2$-graded vector bundle. This is one of the striking elements of evidence demonstrating that the results of [BGo2] are just infinite-dimensional versions of the results which are obtained in this section in a finite-dimensional context.

4.2. *Morse-Bott functions and the class* $V_K(M/S)$. We make the same assumptions as in Sections 3.7–3.9, and we use the corresponding notation. Let $K\in\mathfrak{g}$. Let $f: M\to\mathbf{R}$ be a $K^X$-invariant Morse-Bott function. Let $\nabla f\in TX$ be the fibrewise gradient vector field of $f$ associated to the metric $g^{TX}$. We assume that the zero set $\mathbf{B}$ of $\nabla f$ is a smooth submanifold of $M$, which intersects the fibres $X$ transversally, i.e. which fibres on $S$, with fibre $B\subset X$.

Since $f$ is $K^X$-invariant, the manifold $\mathbf{B}$ is itself $K^X$-invariant. In particular $K^X|_{\mathbf{B}}\in TB$. It follows from the above that the even cohomology classes $V_K(M/S), V_K(\mathbf{B}/S)$ on $S$ are well-defined.

Let $M_K\subset M, \mathbf{B}_K\subset\mathbf{B}$ be the zero sets of $K^X, K^X|_{\mathbf{B}}$. Then $M_K, \mathbf{B}_K$ are smooth submanifolds of $M,\mathbf{B}$, which fibre on $S$, with compact fibres $X_K, B_K$, which are compact submanifolds of $X, B$. As we saw in Section 3.5, $X_K$ and $B$ intersect orthogonally along $B_K$, and $f|_{M_K}: M_K\to\mathbf{R}$ is also a $K^X$-invariant function, which is fibrewise Morse-Bott, and its fibrewise critical set is $B_K$.

Let $\mathrm{ind}(f)$ be the fibrewise index of $f$ along the fibres $B$, i.e. the index of the quadratic form $\nabla^{TX}_{\cdot}\nabla f$ on $N_{B/X}$. Then $\mathrm{ind}(f)$ is a locally constant function on $\mathbf{B}$ with values in $\mathbf{Z}$. By (3.42), we have the identity on $\mathbf{B}_K$,

$$(-1)^{\mathrm{ind}\left(f|_{M_K}\right)} = (-1)^{\mathrm{ind}(f)|_{\mathbf{B}_K}}. \tag{4.28}$$

Let $\widetilde{N}$ be the excess normal bundle which is defined by the exact sequence

$$0 \to N_{B_K/X_K} \oplus N_{B_K/B} \to N_{B_K/X} \to \widetilde{N} \to 0. \tag{4.29}$$

Equivalently, if we identify $N_{B/X}$ to the orthogonal bundle to $TB$ in $TX|_{\mathbf{B}}$, then $m^{TX}(K)|_{\mathbf{B}_K}$ acts naturally on $N_{B/X}$. Now, we have the orthogonal splitting

$$N_{B/X}|_{\mathbf{B}_K} = N_{B_K/X_K} \oplus \widetilde{N}. \tag{4.30}$$

In (4.30), $N_{B_K/X_K}$ is just the kernel of $m^{TX}(K)$, and $\widetilde{N}$ its orthogonal. In particular, $m^{TX}(K)$ acts as an antisymmetric invertible parallel endomorphism of $\widetilde{N}$, so that $\widetilde{N}$ is of even dimension. Since $\nabla f$ is $K^X$-invariant, one verifies easily that $\nabla^{TX}\nabla_{\cdot} f|_{\mathbf{B}_K}$, which acts on $N_{B/X}|_{\mathbf{B}_K}$, preserves the splitting (4.30). Let

$$\widetilde{N} = \widetilde{N}^s \oplus \widetilde{N}^u \tag{4.31}$$

be the corresponding splitting of $\widetilde{N}$ into its stable and unstable part; i.e., the vector bundles $\widetilde{N}^s$ and $\widetilde{N}^u$ are the direct sums of the vector subspaces of $\widetilde{N}$ associated to positive and negative eigenvalues of $\nabla^{TX}\nabla_{\cdot} f|_{\mathbf{B}_K}$. Observe that as a $\mathbf{Z}_2$-graded vector bundle on $\mathbf{B}_K$, $\widetilde{N}$ verifies the assumptions which were verified by $E$ in Section 4.1.

THEOREM 4.10. *The following identity holds,*

$$\begin{aligned} V_K(M/S) = {} & (-1)^{\mathrm{ind}(f)} V_K(\mathbf{B}/S) \\ & - \int_{B_K} (-1)^{\mathrm{ind}(f)} e(TB_K)\, \mathcal{J}_K\left(\widetilde{N}\right) \text{ in } H^{\mathrm{even}}(S,\mathbf{R}). \end{aligned} \tag{4.32}$$

*Proof.* This identity will be proved in Sections 4.6–4.10. □

*Remark* 4.11. One verifies easily that our theorem is compatible with Theorem 3.39. Taking into account the fact that $V_K(M/S)$ vanishes if $X$ is even-dimensional, one verifies that (4.32) is unchanged when we change $f$ into $-f$, especially because $\mathcal{J}_K\left(\widetilde{N}\right)$ is changed into $-\mathcal{J}_K\left(\widetilde{N}\right)$. Also if $\mathbf{B}$ is a finite covering of $S$, i.e. if $B$ is a finite subset of $X$, then $B = B_K$. Therefore, (4.32) takes the form,

$$V_K(M/S) = -\sum_{x\in B} (-1)^{\mathrm{ind}(f)(x)} \mathcal{J}_K\left(\widetilde{N}_x\right) \text{ in } H^{\mathrm{even}}(S,\mathbf{R}). \tag{4.33}$$

4.3. *Equivariant surgery and the invariant* $V_K(M/S)$. We make the same assumptions as in Section 4.2. Also we assume that for every fibre $X$, $a \in \mathbf{R}$ is a non critical value of $f|_X$. Set

$$M_a = f^{-1}\{a\}. \tag{4.34}$$

Then the projection $\pi' : M_a \to S$ is a submersion with compact fibre $X_a \subset X$. Also

$$M_a \cap \mathbf{B} = \emptyset. \tag{4.35}$$

Put

$$\mathbf{B}_{>a} = \mathbf{B} \cap f^{-1}(]a, +\infty[), \qquad \mathbf{B}_{<a} = \mathbf{B} \cap f^{-1}(]-\infty, a[). \tag{4.36}$$

Then $\mathbf{B}_{>a}, \mathbf{B}_{<a}$ fibre on $S$ with fibre $B_{>a}, B_{<a}$.

Observe that $g = (f-a)^2$ is also a fibrewise Morse-Bott function, and that its fibrewise critical set is just $\mathbf{B} \cup M_a$.

Now we will assume that $X$ is even-dimensional, so that $X_a$ is odd-dimensional. By applying Theorem 4.10 to the functions $f$ and $g$, we get

(4.37)
$$(-1)^{\mathrm{ind}(f)} V_K(\mathbf{B}/S) - \int_{B_K} (-1)^{\mathrm{ind}(f)} e(TB_K)\, \mathcal{J}_K\big(\widetilde{N}\big) = 0 \text{ in } H^{\mathrm{even}}(S, \mathbf{R}),$$
$$\begin{aligned}(-1)^{\mathrm{ind}(f)} V_K(\mathbf{B}_{>a}/S) &+ (-1)^{\dim N_{B/X} - \mathrm{ind}(f)} V_K(\mathbf{B}_{<a}/S) + V_K(M_a/S) \\ &- \int_{B_{>a,K}} (-1)^{\mathrm{ind}(f)} e(TB_K)\, \mathcal{J}_K\big(\widetilde{N}\big) \\ &+ \int_{B_{<a,K}} (-1)^{\dim N_{B/X} - \mathrm{ind}(f)} e(TB_K)\, \mathcal{J}_K\big(\widetilde{N}\big) = 0 \text{ in } H^{\mathrm{even}}(S, \mathbf{R}).\end{aligned}$$

By (4.37), and the fact that only the components of $\mathbf{B}_{<\mathbf{a}}$ whose fibres $B_{<a}$ are odd-dimensional contribute to $V_K(\mathbf{B}_{<a})$, we obtain

$$\begin{aligned}V_K(M_a/S) = {} & 2(-1)^{\mathrm{ind}(f)} V_K(\mathbf{B}_{<a}/S) \\ & -2(-1)^{\mathrm{ind}(f)} \int_{B_{<a,K}} e(TB_{<a,K})\, \mathcal{J}_K\big(\widetilde{N}\big).\end{aligned} \tag{4.38}$$

Let now $a, b$ be two noncritical values of $f$, with $a < b$. Set

$$\mathbf{B}_{]a,b[} = \mathbf{B} \cap f^{-1}(]a, b[). \tag{4.39}$$

Proposition 4.12. *The following identity holds*:

(4.40)
$$\begin{aligned}V_K(M_b/S) - V_K(M_a/S) = {} & 2(-1)^{\mathrm{ind}(f)} V_K\big(\mathbf{B}_{]a,b[}/S\big) \\ & - 2(-1)^{\mathrm{ind}(f)} \int_{B_{]a,b[,K}} e\big(TB_{]a,b[,K}\big)\, \mathcal{J}_K\big(\widetilde{N}\big).\end{aligned}$$

*Proof.* This is an obvious consequence of (4.38). □

4.4. *The $V_K$-invariant of unit sphere bundles.* Let $E$ be a real vector bundle on $S$, let $g^E$ be a Euclidean metric on $E$, and let $\nabla^E$ be a Euclidean connection on $E$. We assume that the Lie group $G$ acts fibrewise on $E$ by parallel automorphisms. In particular if $K \in \mathfrak{g}$, let $m^E(K)$ be the corresponding antisymmetric parallel section of $\mathrm{End}(E)$.

Set $S^E$ to be the unit sphere bundle in $E$, and let $\mathcal{E}$ be the total space of $S^E$. Then $G$ still acts on $\mathcal{E}$. In particular, the class $V_K(\mathcal{E}) \in H^{\mathrm{even}}(S, \mathbf{R})$ is well-defined. Let $E^0 = \ker m^E(K)$, and let $E^\perp$ be the orthogonal subbundle to $E^0$ in $E$. Then $E^\perp$ is a trivially $\mathbf{Z}_2$-graded vector bundle on $S$, on which $m^E(K)$ acts as a parallel invertible antisymmetric operator.

THEOREM 4.13. *The following identity holds*:

$$V_K(\mathcal{E}/S) = -\left(2 - \chi\left(S^E\right)\right)\mathcal{J}_K\left(E^\perp\right). \tag{4.41}$$

*Proof.* By Theorem 3.37, we may and we will assume that $E$ is even-dimensional, so that $\chi\left(S^E\right) = 0$. Let $S^{E\oplus\mathbf{R}}$ be the unit sphere bundle in $E \oplus \mathbf{R}$, and let $\mathcal{E}'$ be the total space of $S^{E\oplus\mathbf{R}}$. If $(x,t) \in S^{E\oplus\mathbf{R}}$, set

$$f(x,t) = t. \tag{4.42}$$

Then $f$ is a $G$-invariant fibrewise Morse function, whose fibrewise critical set consist of $(0,1), (0,-1)$. Now we use (4.38) and (4.42). □

*Remark* 4.14. If $E$ is instead odd-dimensional, one verifies that when applying Theorem 4.10 to the function $f(x,t) = t$, we obtain a formula for $V_K(\mathcal{E}'/S)$ which is a special case of (4.41).

In the case where $\dim E = 2$, formula (4.41) can be obtained by an interesting direct computation. Note that in this direct computation, the $-$ sign in (4.41) comes from the fact that for $n = 1$, in (1.8), $(-1)^{n(n+1)/2} = -1$.

4.5. *The case of symplectic manifolds.* In this section, we assume that $X$ is a compact manifold of even dimension $2m$, and that $\omega$ is a symplectic form on $X$.

We assume that the compact Lie group $G$ acts on $X$ and preserves the symplectic form $\omega$. Suppose that there is a moment map $\mu : X \to \mathfrak{g}^*$ associated to the action of $G$ on $X$. Namely, $\mu$ is an equivariant map such that for $K \in \mathfrak{g}$, the function $\langle \mu, K\rangle$ is a Hamiltonian for $K^X$; i.e.,

$$d\langle \mu, K\rangle - 2\pi i_{K^X}\omega = 0. \tag{4.43}$$

Let $T_{\mathbf{R}}X$ be the tangent bundle of $X$. Let $g^{T_{\mathbf{R}}X}$ be a $G$-invariant metric on $T_{\mathbf{R}}X$, and let $J$ be an almost complex structure on $T_{\mathbf{R}}X$, such that if $U, V \in TX$,

$$\omega(U,V) = \langle U, JV\rangle. \tag{4.44}$$

Let $TX \subset T_{\mathbf{R}}X \otimes_{\mathbf{R}} \mathbf{C}$ be the $i$ eigenbundle of $J$. Then $TX$ is a complex vector bundle on $X$, equipped with a Hermitian metric $g^{TX}$. Also $G$ acts naturally on $TX$. Let $\nabla^{TX}$ be a $G$-invariant unitary connection on $TX$, and let $R^{TX}$ be its curvature. If $K \in \mathfrak{g}$, we define the equivariant curvature $R_K^{TX}$ as in (4.6).

If $A$ is an $(m,m)$ matrix, put

$$c_{\max}(A) = \det(A), \qquad c'_{\max}(A) = \frac{\partial}{\partial b}\det(A+b)|_{b=0}. \tag{4.45}$$

Take $K \in \mathfrak{g}$. Put

$$c_K\left(TX, \nabla^{TX}\right) = c_{\max}\left(-R_K/2i\pi\right), \qquad c'_K\left(TX, \nabla^{TX}\right) = c'_{\max}\left(-R_K/2i\pi\right). \tag{4.46}$$

We denote by $c_K(TX), c'_K(TX)$ the corresponding equivariant cohomology classes on $X$. Degree considerations show that

$$\int_X c'_K(TX) = 0. \tag{4.47}$$

Here $X_K$ is a compact symplectic almost complex submanifold. The complex structure $J$ acts on the normal bundle to $X_K$. Let $N_{X_K/X}$ denote the corresponding $i$ eigenbundle. We identify $N_{X_K/X}$ with the orthogonal vector bundle to $TX_K$ in $TX|_{X_K}$. We define $c_{\max}\left(N_{X_K/X}\right)$ as in (4.46) and the sentence which follows.

Using (4.47) and the localization formulas of Duistermaat-Heckman [DuH] and Berline-Vergne [BeV], we deduce from (4.47) that

$$\int_{X_K} \frac{c'_K(TX)}{c_K\left(N_{X_K/X}\right)} = 0. \tag{4.48}$$

Let $T_{\mathbf{R}}X_K$ be the real tangent bundle to $X_K$. Again, by degree considerations, equation (4.48) is easily seen to be equivalent to

$$\int_{X_K} e\left(T_{\mathbf{R}}X_K\right) \frac{c'_K\left(N_{X_K/X}\right)}{c_K\left(N_{X_K/X}\right)} = 0. \tag{4.49}$$

Now observe that the function $f = \langle \mu, K\rangle$ is $K^X$-invariant. Also it is a Morse-Bott function, and its critical set $B$ is just $X_K$. Also its index on $X_K$ is always even. Now we use Theorem 4.10 applied to the trivial fibration with a single fibre $X$. Since $X$ and $X_K$ are even dimensional, their corresponding $V_K$-invariants vanish. So by (4.32), we get

$$\int_{X_K} e\left(T_{\mathbf{R}}X_K\right) \mathcal{J}_K\left(N_{X_K/X,\mathbf{R}}\right) = 0. \tag{4.50}$$

It is now an easy exercise to verify that (4.49) and (4.50) are equivalent.

4.6. *Coupling of $K^X$ and of $\nabla f$.* Now, we construct the objects which will permit us to establish Theorem 4.10.

As in (3.23), set

$$A^{TX}_{K,\nabla f} = \dot{R}^{TX}_{K} + \nabla^{TX}_{\cdot}\widehat{\nabla f} + \frac{1}{2}\left|\nabla f\right|^{2}. \tag{4.51}$$

*Definition* 4.15. Put

$$\gamma_{K,\nabla f} = \int_{X} \exp\left(d_K K^{X\prime}/4\pi\right) \int^{\widehat{B}} \exp\left(-A^{TX}_{K,\nabla f}\Big/2\pi\right), \tag{4.52}$$
$$\delta_{K,\nabla f} = \sqrt{2\pi} \int_{X} \exp\left(d_K K^{X\prime}/4\pi\right) \int^{\widehat{B}} \widehat{K^{X\prime}} \exp\left(-A^{TX}_{K,\nabla f}\Big/2\pi\right).$$

Also, define $\left|T^0 - \nabla f\right|^2$ by a trivial modification of (3.63), (3.64).

THEOREM 4.16. *The form $\gamma_{K,\nabla f}$ is even, and the form $\delta_{K,\nabla f}$ is odd. The following identities hold*:

(4.53)

$$\gamma_{K,\nabla f} = (-1)^n \widehat{\int_X} \exp\left(-\widehat{d}^X_{-K/2\pi}\left(\widehat{K^{X\prime}} - \widehat{T^H}/2\pi - d^H f/\pi\right)\Big/2\right)$$
$$\int^B \exp\left(-\left(\widehat{\dot{R}}^{TX}_{-K/2\pi} + \left(\widehat{\nabla}^{TX} + i_{\widehat{K^X}}\right)\left(T^0 - \nabla f\right) + \frac{1}{2}\left|T^0 - \nabla f\right|^2\right)\Big/2\pi\right),$$
$$\delta_{K,\nabla f} = (-1)^n \sqrt{2\pi} \widehat{\int_X} \widehat{K^{X\prime}} \exp\left(-\widehat{d}^X_{-K/2\pi}\left(\widehat{K^{X\prime}} - \widehat{T^H}/2\pi - d^H f/\pi\right)/2\right)$$
$$\int^B \exp\left(-\left(\widehat{\dot{R}}^{TX}_{-K/2\pi} + \left(\widehat{\nabla}^{TX} + i_{\widehat{K^X}}\right)\left(T^0 - \nabla f\right) + \frac{1}{2}\left|T^0 - \nabla f\right|^2\right)/2\pi\right).$$

*Moreover*,

$$\gamma_{K,\nabla f} = \chi\left(X\right), \qquad d\delta_{K,\nabla f} = 0. \tag{4.54}$$

*Proof.* By the second equation in (3.84), the proof of (4.53) is a trivial modification of the proof of (3.92) in Theorem 3.30. Instead of (3.99), we now have

$$\left(\widehat{\nabla}^{TX} + i_{\widehat{K^X} + T^0 - \nabla f}\right)\left(\widehat{\dot{R}}^{TX}_{-K/2\pi} + \left(\widehat{\nabla}^{TX} + i_{\widehat{K^X}}\right)\left(T^0 - \nabla f\right) + \frac{1}{2}\left|T^0 - \nabla f\right|^2\right) = 0. \tag{4.55}$$

By (4.55), we get the analogue of (3.100). The proof of the first equation in (4.54) continues as in the proof of Theorem 3.30. Also using (3.77), we can rewrite the first equation in (3.84) in the form

$$\nabla^{TX}_{\cdot}\widehat{K^X} = \left(\widehat{\nabla}^{TX}_{\cdot} + i_{\widehat{K^X} + T^0 - \nabla f}\right) K^X. \tag{4.56}$$

The obvious analogue of (3.106) then holds. The proof of the second identity in (4.54) continues as in Theorem 3.30. □

Now we replace $M$ by $M^* = M \times \mathbf{R}_+^* \times \mathbf{R}_+$, $S$ by $S^* = S \times \mathbf{R}_+^* \times \mathbf{R}_+$. Let $\pi^* : M^* \to S^*$ be the obvious projection with fibre $X$. Clearly $G$ still acts fibrewise on $M^*$. Let $f^* : M^* \to \mathbf{R}$ be given by

$$f^*(x,t,T) = \left(\sqrt{\frac{T}{t}}\right) f(x). \tag{4.57}$$

Then $f^*$ is a $K^X$-invariant function on $M^*$.

Also over $S \times \{t\} \times \mathbf{R}$, we equip $TX$ with the metric $g^{TX}/t$. The corresponding metric $g^{TX,*}$ on $TX$ is $G$-invariant. Let $T^H M^*$ be the obvious lift of $T^H M$. The associated connection $\nabla^{TX,*}$ is given by

$$\nabla^{TX,*} = \nabla^{TX} + dt\left(\frac{\partial}{\partial t} - \frac{1}{2t}\right) + dT\frac{\partial}{\partial T}. \tag{4.58}$$

As before, all Berezin integrals will be expressed with respect to the fixed given metric $g^{TX}$.

Let $\delta^*_{K,\nabla f^*}$ be the analogue of $\delta_{K,\nabla f}$. Then $\delta^*_{K,\nabla f^*}$ is a closed odd form on $S^*$.

Theorem 4.17. *The following identity holds*:

$$\delta^*_{K,\nabla f^*} = \sqrt{2\pi}\int_X \exp\left(\left(d_K\frac{K^{X\prime}}{t} - dt\frac{K^{X\prime}}{t^2}\right)/4\pi\right) \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \exp\left(-\left(A^{TX}_{K,\sqrt{T}\nabla f} + \frac{dT}{2\sqrt{T}}\widehat{\nabla f}\right)\Big/2\pi\right). \tag{4.59}$$

*Proof.* Clearly,

$$K^{X,\prime,*} = K^{X\prime}/t. \tag{4.60}$$

Therefore

$$d_K^* K^{X,\prime,*} = d_K K^{X\prime}/t - \frac{dt}{t^2}K^{X\prime}. \tag{4.61}$$

Also, taking into account our conventions on Berezin integration, we find that $\widehat{K^{X\prime}}$ should now be $\widehat{K^{X\prime}}/\sqrt{t}$. By (4.58), the contribution of $R_K^{TX,*}$ to the Berezin integral is the same as before. Also, with the previous conventions,

$$\nabla_\cdot^{TX,*}\nabla^{\widehat{TX,*}} f^* = \frac{1}{\sqrt{t}}\left(\nabla^{TX} + dt\left(\frac{\partial}{\partial t} - \frac{1}{2t}\right) + dT\frac{\partial}{\partial T}\right)\sqrt{tT}\widehat{\nabla f}. \tag{4.62}$$

Equivalently

$$\nabla_\cdot^{TX,*}\nabla^{\widehat{TX,*}} f^* = \sqrt{T}\nabla^{TX}\widehat{\nabla f} + \frac{dT}{2\sqrt{T}}\widehat{\nabla f}. \tag{4.63}$$

Equation (4.59) follows from the above considerations. □

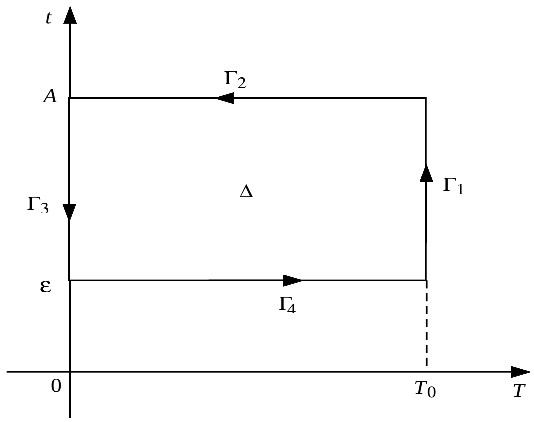


Figure 4.1

We define $\alpha^M_{K,t}, \beta^M_{K,t}$ as in (3.8). By (4.59), one finds easily that there is a smooth odd form $r_{K,t,T}$ on $S$ such that

(4.64)
$$\begin{aligned}\delta^*_{K,\nabla f^*} = {}& \sqrt{2\pi}\int_X \alpha^M_{K,t}\int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \exp\left(-A^{TX}_{K,\sqrt{T}\nabla f}\Big/2\pi\right)\\ &+\sqrt{2\pi}\left[\left(\left(\int_X \beta^M_{K,t}\int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \exp\left(-A^{TX}_{K,\sqrt{T}\nabla f}\Big/2\pi\right)\right)\frac{dt}{t}\right.\right.\\ &+\left.\left(\int_X \alpha^M_{K,t}\int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \frac{\widehat{\nabla f}}{4\pi\sqrt{T}} \exp\left(-A^{TX}_{K,\sqrt{T}\nabla f}\Big/2\pi\right)\right)dT\right] + r_{K,t,T}dtdT.\end{aligned}$$

4.7. *A contour integral.* Take $0<\varepsilon<1<A<+\infty$, $0<T_0<+\infty$. Let $\Gamma=\Gamma_{\varepsilon,A,T_0}$ be the oriented contour indicated in Figure 4.1. This contour is made of oriented pieces $\Gamma_1,\dots,\Gamma_4$. Let $\Delta$ be the interior of $\Gamma$.

THEOREM 4.18. *The following identity of even forms holds on $S$:*

$$\int_\Gamma \delta^*_{K,\nabla f^*} = -d\int_\Delta \delta^*_{K,\nabla f^*}. \tag{4.65}$$

*Proof.* Since the form $\delta^*_{K,\nabla f^*}$ is odd and closed, equation (4.65) follows from Stokes formula. □

Let $P^S$ be the vector space of smooth forms on $S$, let $P^{S,0}\subset P^S$ be the vector space of smooth exact forms on $S$.

Put

$$I^0_k = \int_{\Gamma_k} \delta^*_{K,\nabla f^*}. \tag{4.66}$$

By (4.65), we find that

$$\sum_{k=1}^{4} I_k^0 = 0 \text{ in } P^S/P^{S,0}. \tag{4.67}$$

4.8. *Some simplifying assumptions.* Recall that the function $f : M \to \mathbf{R}$ is fibrewise Morse-Bott. We identify $N_{B/X}$ to the orthogonal bundle to $TB$ in $TX|_B$. Using the families form of the Morse lemma, there is an orthogonal splitting of the normal bundle $N_{B/X}$ into

$$N_{B/X} = N^s_{B/X} \oplus N^u_{B/X}, \tag{4.68}$$

and a fibrewise $G$-equivariant identification of a neighbourhood of the zero-section of $N_{B/X}$ with a tubular neighbourhood of $B$ in $X$, such that if $x \in \mathbf{B}, Z = (Z_+, Z_-) \in N_{B/X,x}$, for $|Z|$ small enough,

$$f(x, Z) = f(x) + \frac{1}{2}\left(|Z_+|^2 - |Z_-|^2\right). \tag{4.69}$$

The vector bundles $N^s_{B/X}$ and $N^u_{B/X}$ are called the stable and unstable parts of $N_{B/X}$.

Let $T^H\mathbf{B} \subset T\mathbf{B}$ be a horizontal bundle on $\mathbf{B}$. Let $\nabla^{N_{B/X}}$ be a Euclidean connection on $N_{B/X}$, which preserves the orthogonal splitting (4.68). We may and we will assume that near $\mathbf{B}$, $T^HM$ is just the obvious horizontal lift of $T^H\mathbf{B}$ with respect to the connection $\nabla^{N_{B/X}}$. Observe that the given metric $g^{TB}$ on $TB$ and the metric $g^{N_{B/X}}$ induce a metric on the total space of $N_{B/X}$, so that the horizontal bundle with respect to $\nabla^{N_{B/X}}$ is orthogonal to the fibres $N_{B/X}$. Using a partition of unity, we will assume that near $\mathbf{B}$, the metric $g^{TX}$ is just this metric.

Let $\nabla^{TB}$ be the Euclidean connection on $TB$ which is associated to $\left(T^H\mathbf{B}, g^{TB}\right)$ as in Section 3.7. Similarly $\nabla^{TX}$ denotes the Euclidean connection on $TX$ associated to $\left(T^HM, g^{TX}\right)$. Then one verifies easily that

$$\nabla^{TX}|_{\mathbf{B}} = \nabla^{TB} \oplus \nabla^{N_{B/X}}. \tag{4.70}$$

In particular, the fibres $B$ are totally geodesic in the fibres $X$.

In the sequel, we assume that the above simplifying assumptions are in force.

As before, we identify the normal bundles to $X_K$ in $X$, to $X_K$ in $B_K$ ... with the orthogonal bundles to the corresponding tangent bundles. By (4.29),

$$\widetilde{N} = N_{B/X}|_{B_K} \cap N_{X_K/X}|_{B_K}. \tag{4.71}$$

By (4.30), the normal bundle $N_{B_K/X}$ splits orthogonally as

$$N_{B_K/X} = N_{B_K/X_K} \oplus N_{B_K/B} \oplus \widetilde{N}. \tag{4.72}$$

Then $m^{TX}(K)|_{B_K}$ acts on $N_{B_K/X}$ and preserves the splitting (4.72). It acts as the zero map on $N_{B_K/X_K}$ and as an invertible antisymmetric map on $N_{B_K/B}$ and on $\widetilde{N}$. Moreover recall that the connection $\nabla^{TX}$ preserves $TB$ and $TX_K$. Therefore the connection $\nabla^{TX}$ induces a corresponding connection on $N_{B_K/X}$, which preserves the splitting (4.72). As in (4.30),

$$N_{B/X}|_{\mathbf{B}_K} = N_{B_K/X_K} \oplus \widetilde{N}. \tag{4.73}$$

Using (4.68), (4.69) and the fact that $f$ is $K^X$ invariant, we deduce that the splitting (4.69) of $N_{B/X}$ induces corresponding orthogonal splittings,

$$N_{B_K/X_K} = N^s_{B_K/X_K} \oplus N^u_{B_K/X_K}, \qquad \widetilde{N} = \widetilde{N}^s \oplus \widetilde{N}^u. \tag{4.74}$$

Let $q^{N_{B/X}} : N_{B/X} \to \mathbf{R}$ be given by

$$q^{N_{B/X}}(Z_+, Z_-) = \frac{1}{2}\left(|Z_+|^2 - |Z_-|^2\right). \tag{4.75}$$

Now we use the notation in (4.4) and (4.10). Namely, set

(4.76)

$$c_K^{N_{X_K/X}|_{\mathbf{B}_K}} = \exp\left(d_{K^{N_{X_K/X}|_{B_K}}} K^{N_{X_K/X}|_{B_K}\prime}/4\pi\right),$$

$$d_{K,T}^{N_{B/X}|_{B_K}} = \sqrt{2\pi}\int^{\widehat{B}} \widehat{K}^{N_{B/X}|_{B_K}} \frac{\sqrt{T}\widehat{\nabla q^{N_{B/X}}}}{4\pi} \exp\left(-A^{N_{B/X}|_{B_K}}_{K,\sqrt{T}\nabla q^{N_{B/X}|_{B_K}}}/2\pi\right).$$

Then $c_K^{N_{X_K/X}|_{\mathbf{B}_K}}$ is a smooth form on the total space of $N_{X_K/X}|_{\mathbf{B}_K}$, and $d_{K,T}^{N_{B/X}|_{\mathbf{B}_K}}$ is a smooth form on the total space of $N_{B/X}|_{\mathbf{B}_K}$.

Let $p_1 : N_{B_K/X} \to N_{X_K/X}|_{\mathbf{B}_K}, p_2 : N_{B_K/X} \to N_{B/X}|_{\mathbf{B}_K}$ be the obvious orthogonal projections.

4.9. *Three intermediate results.*

THEOREM 4.19. *There exists $C > 0$ such that for any $t \in ]0,1], T \in [0,1/t]$,*

$$\left|\int_X \alpha^M_{K,t} \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \sqrt{T}\frac{\widehat{\nabla f}}{4\pi} \exp\left(-A^{TX}_{K,\sqrt{T}\nabla f}/2\pi\right)\right| \leq C\left(t\left(1+T\right)\right)^{1/2}. \tag{4.77}$$

*Proof.* Up to irrelevant modifications, the proof of this result is essentially the same as the proof of [B8, Th. 3.8], which was given in [B8, §3 h)]. ☐

THEOREM 4.20. *For $T > 0$, the following identity holds*:

$$\lim_{t\to 0}\sqrt{2\pi}\int_X \alpha^M_{K,t}\int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}}\sqrt{\frac{T}{t}}\frac{\widehat{\nabla} f}{4\pi}\exp\left(-A^{TX}_{K,\sqrt{T/t}\nabla f}/2\pi\right) \tag{4.78}$$
$$= \int_{B_K} e\left(TB_K, \nabla^{B_K}\right)\int_{N_{B_K/X}} p_1^* c_K^{N_{X_K/X}|_{\mathbf{B}_K}} p_2^* d_{K,T}^{N_{B/X}|_{\mathbf{B}_K}}.$$

*Proof.* The proof is essentially the same as the proof of [B8, Th. 3.9, §3], which was given in [B8, §3 i)]. □

THEOREM 4.21. *There exists $C > 0$ such that for any $t \in ]0,1]$ and any $T \geq 1$,*

$$\left| \int_X \alpha_{K,t}^M \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \sqrt{\frac{T}{t}} \frac{\widehat{\nabla} f}{4\pi} \exp\left(-A_{K,\sqrt{T/t}\nabla f}/2\pi\right) \right| \leq \frac{C}{\sqrt{T}}. \tag{4.79}$$

*Proof.* The proof of our Theorem is the same as the proof of [B8, Th. 3.10], which was given in [B8, §3 )]. □

4.10. *A proof of Theorem* 4.10. Now we will study the asymptotics of the $I_k^0$ as $A \to +\infty, T_0 \to +\infty, \varepsilon \to 0$.

1) The term $I_1^0$. Clearly,

$$I_1^0 = \int_\varepsilon^A \left\{ \sqrt{2\pi} \int_X \beta_{K,t}^M \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \exp\left(-A_{K,\sqrt{T_0}\nabla f}^{TX}/2\pi\right) \right\} \frac{dt}{t}. \tag{4.80}$$

$\alpha$) $A \to +\infty$. Clearly, as $A \to +\infty$,

$$I_1^0 \to I_1^1 = \int_\varepsilon^{+\infty} \left\{ \sqrt{2\pi} \int_X \beta_{K,t}^M \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \exp\left(-A_{K,\sqrt{T_0}\nabla f}^{TX}/2\pi\right) \right\} \frac{dt}{t}. \tag{4.81}$$

$\beta$ $T_0 \to +\infty$. As in the proof of Theorem 3.17, as $T_0 \to |\infty$,

$$I_1^0 \to I_1^1 = (-1)^{\mathrm{ind}(f)} \int_\varepsilon^{+\infty} \left\{ \sqrt{2\pi} \int_B \beta_{K,t}^{\mathbf{B}} \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \exp\left(-R_K^{TB}/2\pi\right) \right\} \frac{dt}{t}. \tag{4.82}$$

$\gamma$) $\varepsilon \to 0$. By (3.114) in Theorem 3.35, as $\varepsilon \to 0$,

$$I_1^2 \to I_1^3 = (-1)^{\mathrm{ind}(f)} \int_0^{+\infty} \left\{ \sqrt{2\pi} \int_B \beta_{K,t}^{\mathbf{B}} \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \exp\left(-R_K^{TB}/2\pi\right) \right\} \frac{dt}{t}. \tag{4.83}$$

$\delta$) Evaluation of $I_1^3$.

PROPOSITION 4.22. *The following identity holds*:

$$I_1^3 = -(-1)^{\mathrm{ind}(f)} U_K\left(\mathbf{B}/S, T^H\mathbf{B}, g^{TB}\right). \tag{4.84}$$

*Proof.* This follows from (3.109), (3.118) and (4.83). □

2) The term $I_2^0$. We have the obvious identity,

$$I_2^0 = -\int_0^{T^0} \left\{ \sqrt{2\pi} \int_X \alpha_{K,A}^M \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{A}} \frac{\widehat{\nabla f}}{4\pi} \exp\left(-A^{TX}_{K,\sqrt{T}\nabla f}/2\pi\right) \right\} \frac{dT}{\sqrt{T}}. \tag{4.85}$$

$\alpha$) $A \to +\infty$. Clearly, as $A \to +\infty$,

$$I_2^0 \to 0. \tag{4.86}$$

3) The term $I_3^0$. We have the obvious,

$$I_3^0 = -\int_\varepsilon^A \left\{ \sqrt{2\pi} \int_X \beta_{K,t}^M \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \exp\left(-\dot{R}_K^{TX}/2\pi\right) \right\} \frac{dt}{t}. \tag{4.87}$$

$\alpha$) $A \to +\infty$. As $A \to +\infty$,

$$I_3^0 \to I_3^1 = -\int_\varepsilon^{+\infty} \left\{ \sqrt{2\pi} \int_X \beta_{K,t}^M \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \exp\left(-\dot{R}_K^{TX}/2\pi\right) \right\} \frac{dt}{t}. \tag{4.88}$$

$\beta$) $T_0 \to +\infty$. As $T_0 \to +\infty$, $I_1^3$ remains constant and equal to $I_3^2$.

$\gamma$) $\varepsilon \to 0$. Using again an integration along the fibre version of (3.16), we find that as $\varepsilon \to 0$,

$$I_3^2 \to I_3^3 = \int_0^{+\infty} \left\{ \sqrt{2\pi} \int_X \beta_{K,t}^M \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{t}} \exp\left(-R_K^{TX}/2\pi\right) \right\} \frac{dt}{t}. \tag{4.89}$$

PROPOSITION 4.23. *The following identity holds*:

$$I_3^3 = -U_K\left(M/S, T^H M, g^{TX}\right). \tag{4.90}$$

*Proof.* This follows from (3.109), (3.118) and (4.89). □

4) The term $I_4^0$. Clearly,

$$I_4^0 = \int_0^{T_0} \left\{ \sqrt{2\pi} \int_X \alpha_{K,\varepsilon}^M \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{\varepsilon}} \frac{\widehat{\nabla f}}{2\pi} \exp\left(-A^{TX}_{K,\sqrt{T}\nabla f}/2\pi\right) \right\} \frac{dT}{2\sqrt{T}}. \tag{4.91}$$

$\alpha$) $A \to +\infty$. The term $I_4^0$ remains constant and equal to $I_4^1$.

$\beta$) $T_0 \to +\infty$. Using (3.43) in Theorem 3.17, we find that as $T_0 \to +\infty$,

$$I_4^1 \to I_4^2 = \int_0^{+\infty} \left\{ \sqrt{2\pi} \int_X \alpha_{K,\varepsilon}^M \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{\varepsilon}} \frac{\widehat{\nabla f}}{2\pi} \exp\left(-A^{TX}_{K,\sqrt{T}\nabla f}/2\pi\right) \right\} \frac{dT}{2\sqrt{T}}. \tag{4.92}$$

γ) $\varepsilon \to 0$. Set

(4.93)
$$J_1^0 = \int_0^1 \left\{ \sqrt{2\pi} \int_X \alpha_{K,\varepsilon}^M \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{\varepsilon}} \frac{\sqrt{T}\widehat{\nabla f}}{4\pi} \exp\left(-A_{K,\sqrt{T}\nabla f}^{TX}/2\pi\right) \right\} \frac{dT}{T},$$
$$J_2^0 = \int_\varepsilon^1 \left\{ \sqrt{2\pi} \int_X \alpha_{K,\varepsilon}^M \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{\varepsilon}} \sqrt{\frac{T}{\varepsilon}} \frac{\widehat{\nabla f}}{4\pi} \exp\left(-A_{K,\sqrt{T/\varepsilon}\nabla f}/2\pi\right) \right\} \frac{dT}{T},$$
$$J_3^0 = \int_1^{+\infty} \left\{ \sqrt{2\pi} \int_X \alpha_{K,\varepsilon}^M \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{\varepsilon}} \sqrt{\frac{T}{\varepsilon}} \frac{\widehat{\nabla f}}{4\pi} \exp\left(-A_{K,\sqrt{T/\varepsilon}\nabla f}/2\pi\right) \right\} \frac{dT}{T}.$$

Then

(4.94)
$$I_4^2 = J_1^0 + J_2^0 + J_3^0.$$

By proceeding as in the proof of (3.14), we find that as $\epsilon \to 0$,

(4.95)
$$J_1^0 \to 0.$$

Also by Theorem 4.19, there exists $C > 0$ such that for $\varepsilon \in ]0,1], T \in [\varepsilon, 1]$,

(4.96)
$$\left| \int_X \alpha_{K,\varepsilon}^M \int^{\widehat{B}} \frac{\widehat{K^{X\prime}}}{\sqrt{\varepsilon}} \sqrt{\frac{T}{\varepsilon}} \frac{\widehat{\nabla f}}{4\pi} \exp\left(-A_{K,\sqrt{T/\varepsilon}\nabla f}/2\pi\right) \right| \le C\left(\varepsilon + T\right)^{1/2} \le CT^{1/2}.$$

By Theorem 4.20 and (4.96), we find that as $\varepsilon \to 0$,

(4.97)
$$J_2^0 \to J_2^1 = \int_0^1 \left\{ \int_{B_K} e\left(TB_K, \nabla^{TB_K}\right) \int_{N_{B_K/X}} p_1^* c^{N_{X_K/X}|_{B_K}} p_2^* d_T^{N_{B/X}|_{B_K}} \right\} \frac{dT}{T}.$$

Finally, by Theorems 4.20 and 4.21, as $\varepsilon \to 0$,

(4.98)
$$J_3^0 \to J_3^1 = \int_1^{+\infty} \left\{ \int_{B_K} e\left(TB_K, \nabla^{TB_K}\right) \int_{N_{B_K/X}} p_1^* c^{N_{X_K/X}|_{B_K}} p_2^* d_T^{N_{B/X}|_{B_K}} \right\} \frac{dT}{T}.$$

By (4.94)–(4.98), we find that as $\varepsilon \to 0$,

(4.99)
$$I_4^2 \to I_4^3 = \int_0^{+\infty} \left\{ \int_{B_K} e\left(TB_K, \nabla^{TB_K}\right) \int_{N_{B_K/X}} p_1^* c^{N_{X_K/X}|_{B_K}} p_2^* d_T^{N_{B/X}|_{B_K}} \right\} \frac{dT}{T}.$$

δ) Evaluation of $I_4^3$.

THEOREM 4.24. *The following identity holds*:

(4.100)
$$I_4^3 = \int_{B_K} (-1)^{\operatorname{ind}(f)} e\left(TB_K, \nabla^{TB_K}\right) \mathcal{I}_K\left(\widetilde{N}, \nabla^{\widetilde{N}}\right).$$

*Proof.* Using the fact that the connection $\nabla^{N_{B/X}|_{B_K}}$ preserves the splitting in (4.72), we get

$$\int_{N_{B_K/X}} p_1^* c_K^{N_{X_K/X}|_{\mathbf{B}_K}} p_2^* d_{K,T}^{N_{B/X}|_{\mathbf{B}_K}} \tag{4.101}$$
$$= \int_{N_{B_K/X_K}} \left\{ \int^{\widehat{B}} \exp\left( -A_{K,\sqrt{T}\nabla q^{N_{B_K/X_K}}} / 2\pi \right) \right\}$$
$$\int_{N_{B_K/B}} c_K^{N_{B_K/B}} \left\{ \int^{\widehat{B}} \exp\left( -\dot{R}_K^{N_{B_K/B}} / 2\pi \right) \right\} \int_{\widetilde{N}} c_K^{\widetilde{N}} d_{K,T}^{\widetilde{N}}.$$

Now one has the easy formula,

$$\int_{N_{B_K/X_K}} \left\{ \int^{\widehat{B}} \exp\left( -A_{K,\sqrt{T}\nabla q^{N_{B_K/X_K}}} / 2\pi \right) \right\} \tag{4.102}$$
$$= (-1)^{\mathrm{ind}\left(f|_{M_K}\right)} \int_{N_{B_K/B}} c_K^{N_{B_K/B}} \left\{ \int^{\widehat{B}} \exp\left( -\dot{R}_K^{N_{B_K/B}} / 2\pi \right) \right\} = 1.$$

So by (4.101)–(4.102), we obtain

$$\int_{N_{B_K/X}} p_1^* c_K^{N_{X_K/X}|_{\mathbf{B}_K}} p_2^* d_{K,T}^{N_{B/X}|_{B_K}} = \int_{\widetilde{N}} (-1)^{\mathrm{ind}\left(f|_{M_K}\right)} c_K^{\widetilde{N}} d_{K,T}^{\widetilde{N}}. \tag{4.103}$$

Using (4.28), Theorem 4.3, (4.20), Theorem 4.8, (4.99), (4.101)–(4.103), we get (4.100). The proof of our Theorem is completed. □

Using (4.67) and the results of Section 4.10, we get

$$I_1^3 + I_3^3 + I_4^3 = 0 \text{ in } P^S / P^{S,0}. \tag{4.104}$$

By Propositions 4.22, 4.23 and Theorem 4.24, we get the analogue of (4.32) for $U_K(M/S)$. By applying the operator $Q$ on both sides, we finally get Theorem 4.10.

4.11. *Multifibrations and the invariant* $V_K(M/S)$. Let now $\pi' : P \to S$ be another submersion with compact fibre $Y$, which has the same properties as the fibration $\pi : M \to S$. In particular, $G$ still acts on $P$ along the fibres $Y$. To this fibration, we associate objects similar to the ones we constructed for $\pi : M \to S$. If $K \in \mathfrak{g}$, let $K^Y \in TY$ be the corresponding fibrewise vector field on $P$.

Let $p : M \to P$ be a $G$-equivariant submersion with compact fibre $Z$, such that

$$\pi = \pi' p. \tag{4.105}$$

In particular $p$ induces a fibrewise submersion $p : X \to Y$ with compact fibre $Z$.

Observe that $P_K$ is a smooth submanifold of $P$, and $\pi' : P_K \to S$ is a submersion with compact fibre $Y_K$. Put

$$M^{P_K} = p^{-1}(P_K). \tag{4.106}$$

Then $p : M^{P_K} \to P_K$ is a submersion with compact fibre $Z$. Also $K^X|_{M^{P_K}} \in TZ$. In the sequel, on $M^{P_K}$, we set

$$K^Z = K^X|_{M^{P_K}}. \tag{4.107}$$

In the above situation, given $K \in \mathfrak{g}$, we can construct $V_K(M/S), V_K(P/S) \in H^{\mathrm{even}}(S, \mathbf{R})$, and $V_K\left(M^{P_K}/P_K\right) \in H^{\mathrm{even}}(P_K, \mathbf{R})$.

Theorem 4.25. *The following identity holds*:

(4.108)
$$V_K(M/S) = \chi(Z) V_K(P/S) + \int_{Y_K} e(TY_K) V_K\left(M^{P_K}/P_K\right) \text{ in } H^{\mathrm{even}}(S, \mathbf{R}).$$

*Proof.* Clearly $G$ maps $TX, TZ, TY$ into themselves. Let $g^{TY}, g^{TZ}$ be $G$-invariant metrics on $TY, TZ$. Let $TX = T^H X \oplus TZ$ be a $G$-invariant orthogonal splitting of $TX$. We lift the metric $g^{TY}$ on $TY$ to a metric $g^{T^H X}$ on $T^H X$. For $T \geq 1$, let $g_T^{TX}$ be the $G$-invariant metric on $TX = T^H X \oplus TZ$,

$$g_T^{TX} = g^{T^H X} \oplus \frac{g^{TZ}}{T}. \tag{4.109}$$

In the sequel we will use the notation $g^{TX} = g_1^{TX}$.

Let $T^H P$ be a $G$-invariant horizontal bundle on $P$. Then $p_*^{-1}\left(T^H P\right)$ is a $G$-invariant subbundle of $TM$, which maps onto $TS$. Let $T^H M \subset p_*^{-1}\left(T^H P\right)$ be a $G$-invariant horizontal subbundle of $TM$ for the projection $\pi$. Then $T^H M \oplus T^H X$ is a horizontal bundle associated to the projection $p : M \to P$. Clearly $G$ acts as the identity on $T^H M \simeq \pi^* TS$. Let $\left(T^H X\right)_K \subset T^H X|_{M^{P_K}}$ be the obvious lift of $TY_K$. Clearly,

$$T^H M|_{M^{P_K}} \oplus \left(T^H X\right)_K \subset TM^{P_K}. \tag{4.110}$$

In particular $T^H M|_{M^{P_K}} \oplus \left(T^H X\right)_K$ is a horizontal bundle associated to the projection $p : M^{P_K} \to P_K$.

For $T \geq 1$, let $\nabla_T^{TX}$ be the connection on $TX$ which is associated to $\left(T^H M, g_T^{TX}\right)$, and let $R_{K,T}^{TX}$ be the corresponding equivariant curvature.

If $K \in \mathfrak{g}$, set

$$K^X = K^{X,H} + K^{X,V}, \; K^{X,H} \in T^H X, \, K^{X,V} \in TZ. \tag{4.111}$$

Then $K^{X,H}$ is the lift in $T^H X$ of $K^Y \in TY$. In the sequel, we identify $TX, TZ, TY$ to their duals by the metrics $g^{TX}, g^{TY}, g^{TZ}$. Also we will consider Berezin integrals on $TX, TZ$, or $TY$ with respect to the fixed metrics $g^{TX}, g^{TZ}$ or $g^{TY}$. With these conventions, $K^{X,H\prime} = p^* K^{Y\prime}$. In the sequel we will often write $K^{Y\prime}$ instead of $p^* K^{Y\prime}$.

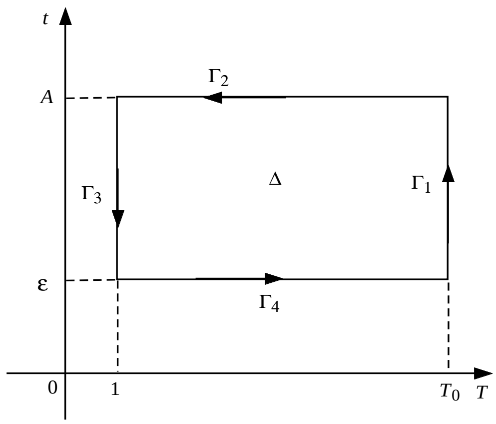


Figure 4.2

Clearly, if $R_T^{TX}$ is the curvature of $\nabla_T^{TX}$, then

$$R_{K,T}^{TX} = R_T^{TX} - 2\pi\nabla_{T,\cdot}^{TX} K^X. \tag{4.112}$$

Recall that we use the Berezin integration formalism with respect to the fixed metric $g^{TX} = g_1^{TX}$. Let $e_1, \ldots, e_m$ be an orthonormal basis of $TY$, let $e_{m+1}, \ldots, e_n$ be an orthonormal basis of $TZ$. With our conventions,

(4.113)

$$\dot{R}_{K,T}^{TX} = \frac{1}{2} \sum_{1\leq i,j\leq m} \left\langle e_i^H, R_{K,T}^{TX} e_j^H \right\rangle \widehat{e}^i \wedge \widehat{e}^j \\ +\frac{1}{2} \sum_{m+1\leq i,j\leq n} \left\langle e_i, R_{K,T}^{TX} e_j \right\rangle \widehat{e}^i \wedge \widehat{e}^j + \frac{1}{\sqrt{T}} \sum_{\substack{1\leq i\leq m \\ m+1\leq j\leq n}} \left\langle e_i, R_{K,T}^{TX} e_j \right\rangle \widehat{e}^i \wedge \widehat{e}^j.$$

We replace $M$ by $M^* = M \times \mathbf{R}_+^* \times \mathbf{R}$, $S$ by $S^* = S \times \mathbf{R}_+^* \times \mathbf{R}$. Over $S\times\{t\}\times\{T\}$, we equip $TX$ with the metric $g_T^{TX}/t$. Let $\delta_K^*$ be the corresponding odd closed form on $S^*$ which is defined as in (4.59) with $f = 0$. One then verifies easily the equality,

(4.114)

$$\delta_K^* = \sqrt{2\pi} \int_X \exp\left(\left(d_K - \frac{dt}{t}\right)\left(K^{X,V\prime}/T + K^{Y\prime}\right)/4\pi t - dT K^{X,V\prime}/4\pi t T^2\right) \\ \int^{\widehat{B}} \left(\frac{\widehat{K^{X,V\prime}}}{\sqrt{tT}} + \frac{\widehat{K^{X,H\prime}}}{\sqrt{t}}\right) \exp\left(-\dot{R}_{K,T}^{TX}/2\pi\right).$$

Let $\Gamma$ be the oriented contour indicated in Figure 4.2, and let $\Delta$ be its interior. As in (4.65), we get

$$\int_\Gamma \delta_K^* = -d\int_\Delta \delta_K^*. \tag{4.115}$$

As in (4.66), set

$$I_k^0 = \int_{\Gamma_k} \delta_K^*. \tag{4.116}$$

By (4.115),

$$\sum_{k=1}^{4} I_k^0 = 0 \text{ in } P^S/P^{S,0}. \tag{4.117}$$

Now we briefly study the asymptotics of the $I_k^0$ as $A \to +\infty$, $T_0 \to +\infty, \varepsilon \to 0$.

1) The term $I_1^0$. Clearly,

(4.118)
$$I_1^0 = -\int_\varepsilon^A \left\{ \sqrt{2\pi} \int_X \exp\left( d_K \left( K^{X,V\prime}/T_0 + K^{Y\prime} \right)/4\pi t \right) \left( \frac{K^{X,V\prime}}{4\pi t T_0} + \frac{K^{X,H\prime}}{4\pi t} \right) \right. \\ \left. \int^{\widehat{B}} \left( \frac{\widehat{K^{X,V\prime}}}{\sqrt{tT_0}} + \frac{\widehat{K^{X,H\prime}}}{\sqrt{t}} \right) \exp\left( -\dot{R}_{K,T_0}^{TX}/2\pi \right) \right\} \frac{dt}{t}.$$

As $A \to +\infty$, $I_1^0$ converges to $I_1^1$, where $A$ in (4.118) is simply replaced by $+\infty$.

As $T_0 \to +\infty$, one verifies easily that the $G$-invariant connection $\nabla_{T_0}^{TX}$ converges to a $G$-invariant connection $\nabla_\infty^{TX}$ on $TX$, which has the following two properties:

- The connection $\nabla_\infty^{TX}$ preserves $TZ$. More precisely its restriction to $TZ$ is just the connection $\nabla^{TZ}$ associated to the projection $p : M \to P$, to the horizontal bundle $T^H X \oplus T^H M$ and to the metric $g^{TZ}$ on $TZ$.
- The projection of $\nabla_\infty^{TX}$ on $T^H X$ with respect to the splitting $TX = T^H X \oplus TZ$ is the pull back to $T^H X$ of the connection $\nabla^{TY}$ associated to $\left(T^H P, g^{TY}\right)$.

From the above, one then finds easily that for $t > 0$, as $T_0 \to +\infty$,

(4.119)
$$\int^{\widehat{B}} \left( \frac{\widehat{K^{X,V\prime}}}{\sqrt{tT_0}} + \frac{\widehat{K^{X,H\prime}}}{\sqrt{t}} \right) \exp\left( -\dot{R}_{K,T_0}^{TX}/2\pi \right) \\ \to e_K\left(TZ, \nabla^{TZ}\right) p^* \int^{\widehat{B}} \frac{\widehat{K^{Y\prime}}}{\sqrt{t}} \exp\left( -\dot{R}_K^{TY}/2\pi \right).$$

By (4.118), (4.119) and dominated convergence, as $T_0 \to +\infty$,

(4.120)
$$I_1^1 \to I_1^2 = -\chi(Z) \int_\varepsilon^{+\infty} \left\{ \sqrt{2\pi} \int_Y \exp\left( d_K K^{Y\prime}/4\pi t \right) \right. \\ \left. \frac{K^{Y\prime}}{4\pi t} \int^{\widehat{B}} \frac{\widehat{K^{Y\prime}}}{\sqrt{t}} \exp\left( -\dot{R}_K^{TY}/2\pi \right) \right\} \frac{dt}{t}.$$

By (3.114), as $\varepsilon \to 0$, $I_1^1$ converges to $I_1^2$, which is given by the right-hand side of (4.120), where $\varepsilon$ is replaced by 0. Comparing with (3.118), we get

$$I_1^3 = -\chi(Z)\, U_K\left(P/S, T^H P, g^{TY}\right). \tag{4.121}$$

<u>2) The term $I_2^0$.</u> Clearly,

$$I_2^0 = \int_1^{T_0}\left\{\int_X \exp\left(d_K\left(K^{X,V\prime}/T + K^{Y\prime}\right)/4\pi A\right)\frac{K^{X,V\prime}}{4\pi A T^2}\right. \\ \left.\int^{\widehat{B}}\left(\frac{\widehat{K^{X,V\prime}}}{\sqrt{AT}} + \frac{\widehat{K^{X,H\prime}}}{\sqrt{A}}\right)\exp\left(-\dot{R}_{K,T}^{TX}/2\pi\right)\right\}dT. \tag{4.122}$$

As $A \to +\infty$,

$$I_2^0 \to 0. \tag{4.123}$$

<u>3) The term $I_3^0$.</u> We have the obvious equality

$$I_3^0 = \int_\varepsilon^A\left\{\sqrt{2\pi}\int_X \beta_{K,t}^M \int^{\widehat{B}}\frac{\widehat{K^{X\prime}}}{\sqrt{t}}\exp\left(-\dot{R}_K^{TX}/2\pi\right)\right\}\frac{dt}{t}. \tag{4.124}$$

As $A \to +\infty$, by (3.115), $I_3^0$ converges to $I_3^1$, which is obtained by making $A = +\infty$ in (4.124). Also $I_3^1$ does not depend on $T_0$ and remains equal to $I_3^2$. Finally as $\varepsilon \to 0$, by (3.114), $I_3^2$ converges to $I_3^3$, where in (4.124), $\varepsilon$ is now made equal to 0. Comparing with (3.118), we get

$$I_3^3 = U_K\left(M/S, T^H M, g^{TX}\right). \tag{4.125}$$

<u>4) The term $I_4^0$.</u> We have the obvious identity

$$I_4^0 = -\int_1^{T_0}\left\{\sqrt{2\pi}\int_X \exp\left(d_K\left(K^{X,V\prime}/T + K^{Y\prime}\right)/4\pi\varepsilon\right)\frac{K^{X,V\prime}}{4\pi\varepsilon T}\right. \\ \left.\int^{\widehat{B}}\left(\frac{\widehat{K^{X,V\prime}}}{\sqrt{\epsilon T}} + \frac{\widehat{K^{X,H\prime}}}{\sqrt{\epsilon}}\right)\exp\left(-\dot{R}_{K,T}^{TX}/2\pi\right)\right\}\frac{dT}{T}. \tag{4.126}$$

As $A \to +\infty$, $I_4^0$ remains constant and equal to $I_4^1$. As $T_0 \to +\infty$, the arguments used after (4.118) show that $I_4^1$ converges to $I_4^2$, which is equal to the right-hand side of (4.126), with $T_0$ replaced by $+\infty$.

Clearly

$$I_4^2 = \int_\varepsilon^{+\infty}\left\{\sqrt{2\pi}\int_X \exp\left(d_K\left(K^{X,V\prime}/T + K^{Y\prime}/\varepsilon\right)/4\pi\right)\frac{K^{X,V\prime}}{4\pi T}\right. \\ \left.\int^{\widehat{B}}\left(\frac{\widehat{K^{X,V\prime}}}{\sqrt{T}} + \frac{\widehat{K^{X,H\prime}}}{\sqrt{\epsilon}}\right)\exp\left(-\dot{R}_{K,T/\epsilon}^{TX}/2\pi\right)\right\}\frac{dT}{T}. \tag{4.127}$$

By proceeding as in the proof of (3.114), one verifies easily that given $T>0$, as $\varepsilon \to 0$,

$$\begin{aligned}(4.128)\quad &\int_X \exp\left(d_K\left(K^{X,V\prime}/T+K^{Y\prime}/\varepsilon\right)/4\pi\right)\frac{K^{X,V\prime}}{4\pi T}\\ &\qquad\int^{\widehat{B}}\left(\frac{\widehat{K^{X,V\prime}}}{\sqrt{T}}+\frac{\widehat{K^{X,H\prime}}}{\sqrt{\epsilon}}\right)\exp\left(-\dot{R}^{TX}_{K,T/\epsilon}/2\pi\right)\\ &\qquad\to \int_{Y_K} e\left(TY_K,\nabla^{TY_K}\right)\int_Z \beta^{M^{P_K}}_{K,T}\int^{\widehat{B}}\frac{\widehat{K^{Z\prime}}}{\sqrt{T}}\exp\left(-\dot{R}^{TZ}_K/2\pi\right).\end{aligned}$$

By proceeding as in the proof of (3.114), we find that there exists $C>0$ such that if $0<t\le 1, T\ge 1$,

$$\begin{aligned}(4.129)\quad &\Bigg|\int_X \exp\left(d_K\left(K^{X,V\prime}+TK^{X,H\prime}\right)/4\pi t\right)\frac{K^{X,V\prime}}{4\pi t}\\ &\qquad\int^{\widehat{B}}\left(\frac{\widehat{K^{X,V\prime}}}{\sqrt{t}}+\sqrt{\frac{T}{t}}\widehat{K^{X,H\prime}}\right)\exp\left(-\dot{R}^{TX}_{K,T}/2\pi\right)\Bigg|\le C\sqrt{t}.\end{aligned}$$

By (4.128), and by (4.129) which is used with $t$ replaced by $T$, and $T$ replaced by $T/\varepsilon$, we find that as $\varepsilon\to 0$,

$$\begin{aligned}(4.130)\quad &\int_\varepsilon^1\Bigg\{\sqrt{2\pi}\int_X \exp\left(d_K\left(K^{X,V\prime}/T+K^{Y\prime}/\varepsilon\right)/4\pi\right)\frac{K^{X,V\prime}}{4\pi T}\\ &\qquad\int^{\widehat{B}}\left(\frac{\widehat{K^{X,V\prime}}}{\sqrt{T}}+\frac{\widehat{K^{X,H\prime}}}{\sqrt{\epsilon}}\right)\exp\left(-\dot{R}^{TX}_{K,T/\epsilon}/2\pi\right)\Bigg\}\frac{dT}{T}\\ &\qquad\to\int_0^1\Bigg\{\sqrt{2\pi}\int_{Y_K} e\left(TY_K,\nabla^{TY_K}\right)\int_Z\beta^{M^{P_K}}_{K,T}\\ &\qquad\qquad\int^{\widehat{B}}\frac{\widehat{K^{Z\prime}}}{\sqrt{T}}\exp\left(-\dot{R}^{TZ}_K/2\pi\right)\Bigg\}\frac{dT}{T}.\end{aligned}$$

Also, we find easily that there exists $C>0$ such that for $t\in\,]0,1], T\ge 1$,

$$\begin{aligned}(4.131)\quad &\Bigg|\int_X \exp\left(d_K\left(K^{X,V\prime}/T+K^{Y\prime}/t\right)/4\pi\right)\frac{K^{X,V\prime}}{4\pi T}\\ &\qquad\int^{\widehat{B}}\left(\frac{\widehat{K^{X,V\prime}}}{\sqrt{T}}+\frac{\widehat{K^{X,H\prime}}}{\sqrt{t}}\right)\exp\left(-\dot{R}^{TX}_{K,T/t}/2\pi\right)\Bigg|\le\frac{C}{T}.\end{aligned}$$

By (4.128), (4.131), as $\varepsilon \to 0$,

$$\begin{aligned}(4.132)\quad &\int_1^{+\infty}\Bigg\{\sqrt{2\pi}\int_X \exp\left(d_K\left(K^{X,V\prime}/T+K^{Y\prime}/\varepsilon\right)/4\pi\right)\frac{K^{X,V\prime}}{4\pi T}\\ &\int^{\widehat{B}}\left(\frac{\widehat{K^{X,V\prime}}}{\sqrt{T}}+\frac{\widehat{K^{X,H\prime}}}{\sqrt{\epsilon}}\right)\exp\left(-\dot{R}^{TX}_{K,T/\epsilon}/2\pi\right)\Bigg\}\frac{dT}{T}\\ &\to \int_1^{+\infty}\Bigg\{\sqrt{2\pi}\int_{Y_K} e\left(TY_K,\nabla^{TY_K}\right)\int_Z \beta^{M^{P_K}}_{K,T}\int^{\widehat{B}}\\ &\frac{\widehat{K^{Z\prime}}}{\sqrt{T}}\exp\left(-\dot{R}^{TZ}_K/2\pi\right)\Bigg\}\frac{dT}{T}.\end{aligned}$$

Using (3.118), (4.127), (4.130), (4.132), as $\varepsilon \to 0$,

$$(4.133)\qquad I_4^2 \to I_4^3 = \int_{Y_K} e\left(TY_K,\nabla^{TY_K}\right)U_K\left(M^{P_K}/P_K, T^H M^{P_K}, g^{TZ}\right).$$

By (4.117), and by the above results, we get

$$(4.134)\qquad I_1^3 + I_3^3 + I_4^3 = 0 \text{ in } P^S/P^{S,0}.$$

By (4.121), (4.123), (4.133), (4.134), we get the analogue of (4.108) from $U_K(M/S)$. By applying the operator $Q$ on both sides, we finally get Theorem 4.25. □

We make the same assumptions as in Section 3.7. Suppose that $G$ acts freely on $M$. Then $\pi' : M/G \to S$ is a submersion with compact fibre $X/G$.

Proposition 4.26.

$$(4.135)\qquad V_K(M/S) = \int_{X/G} e(TX/G)\, V_K(M/(M/G)) \text{ in } H^{\text{even}}(S,\mathbf{R}).$$

*Proof.* We will use Theorem 4.25, with $P = M/G$. Since $G$ acts trivially on $M/G$,

$$(4.136)\qquad V_K((M/G)/S) = 0.$$

One can instead observe that $\chi(G) = 0$, so that using (4.108), we get (4.135). □

4.12. *$V$-invariants and symplectic cuts.* We make the same assumptions as in Section 4.5, with $G = S^1$, so that $\mathfrak{g} = \mathbf{R}$. Then the moment map $\mu$ takes its value in $\mathbf{R}$. Also, since there is no risk of confusion, $TX$ will denote the ordinary real tangent bundle on $X$. Other tangent bundles will be denoted in the same way.

As observed in Section 4.5, if $K \in \mathbf{R}^*$, $\mu$ is a Morse-Bott function on $X$ whose critical set $B$ is just $X_K$, which is even-dimensional.

Assume that 0 is a regular value of $\mu$, and put $X_0 = \mu^{-1}\{0\}$. In the sequel we write $V_K(X_0)$ instead of $V_K(X_0/\mathrm{pt})$. Put $X_{K,>0} = X_K \cap \mu^{-1}(]0,+\infty[)$. By (4.40), we get

$$V_K(X_0) = 2\int\int_{X_{K,>0}} e(TX_K)\,\mathcal{J}_K\left(N_{X_{K/X}}\right). \tag{4.137}$$

Clearly $S^1$ acts locally freely on $X_0$. Assume that this action is free. By combining (4.135) and (4.137), we get

$$\int_{X_0/S^1} e\left(TX_0/S^1\right) V_K\left(X_0/\left(X_0/S^1\right)\right) = 2\int_{X_{K,>0}} e(TX_K)\,\mathcal{J}_K\left(N_{X_{K/X}}\right). \tag{4.138}$$

Degree considerations show that (4.138) is equivalent to

$$V_K\left(S^1\right)\chi\left(X_0/S^1\right) = 2\int_{X_{K,>0}} e(TX_K)\,\mathcal{J}_K\left(N_{X_{K/X}}\right). \tag{4.139}$$

We claim that (4.138) follows directly from the arguments of Section 4.5. In fact, using a symplectic cut argument, we can compactify the open symplectic manifold $X_{>0}$ into a compact symplectic manifold $Y_+$, equipped an action of $S^1$ and a moment map $\mu_+$, whose restriction to $X_{>0}$ coincides with the given ones. The fibre $Y_{+,0} = \mu_+^{-1}\{0\}$ is fixed by $S^1$ and coincides with $X_0/S^1$. The normal bundle $N_{Y_{+,0}/Y_+}$ is the two-dimensional vector bundle associated to the circle bundle $X_0 \to X_0/S^1$.

By Theorem 4.13,

$$\int_{X_0/S^1} e\left(TX_0/S^1\right) V_K\left(X_0/\left(X_0/S^1\right)\right) = -2\int_{Y_0} e(TY_0)\,\mathcal{J}_K\left(N_{Y_0/Y,\mathbf{R}}\right). \tag{4.140}$$

Using (4.50) and (4.140), and taking into account that 0 is the minimum of $\mu_+$ on $Y_+$, so that $N_{Y_{+,0}/Y} = N^s_{Y_{+,0}/Y_+}$, we recover (4.138).

## 5. A comparison formula for the equivariant torsions

In this section, which contains the main result of this paper, we give a formula comparing the equivariant classical and infinitesimal equivariant analytic torsions in de Rham theory. This formula is an analogue of a corresponding formula in [BGo1, Th. 5.1] for the holomorphic torsions. Also we show that our formula is compatible to our previous results in [BGo2], and also with results of Bunke [Bu1, Bu2].

This section is organized as follows. In 5.1, we state our main result. In 5.2, we recall the definition in [BGo2] of the genus $J(\theta, x)$, and relate this genus to the function $J^\theta(x)$. In Section 5.3, we show the compatibility of our main result with the results of [BZ2] and [BGo2], in relation with invariant Morse functions. In 5.4, we consider the case of Morse-Bott functions. Then we show that our results refine results of Bunke [Bu1, 2]. Finally in Section 5.5, we state without proof the obvious extension of our comparison formula to the case of analytic torsion forms, our main result being the degree 0 part of this more general equality.

In this section, we make the same assumptions and we use the same notation as in Sections 2.2, 2.3, 2.6, and 2.8. In particular we assume that equation (2.21) holds.

5.1. *The main result*. For convenience, we state again the main result of this paper, already given in Theorem 0.1. Here we take $g \in G, K_0 \in \mathfrak{z}(g)$. If $z \in \mathbf{R}^*$, we take $K = zK_0$. Note that by (1.24), (2.119), (2.121), for $|z|$ small enough,

$$\log \left( \frac{\widetilde{\| \ \|}_{\lambda_G(F),\mathrm{ch}}(g,K)}{\| \ \|_{\lambda_G(F)}(ge^K)} \right) = \mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX},\nabla^F,g^F\right) - \mathcal{T}_{\mathrm{ch},ge^K,0}\left(g^{TX},\nabla^F,g^F\right). \tag{5.1}$$

THEOREM 5.1. *For $z \in \mathbf{R}^*$, if $|z|$ is small enough, the following identity holds*:

$$\begin{aligned}&\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX},\nabla^F,g^F\right) - \mathcal{T}_{\mathrm{ch},ge^K,0}\left(g^{TX},\nabla^F,g^F\right)\\ &= \int_{X_g} e_K\left(TX_g,\nabla^{TX_g}\right) F_K\left(TX_g,g^{TX_g}\right)\mathrm{ch}_g^{\circ}\left(\nabla^F,g^F\right) + \mathrm{Tr}^{F|_{X_g}}[g]\, V_K(X_g).\end{aligned} \tag{5.2}$$

*Remark* 5.2. Using (3.38), one verifies easily that Theorem 5.1 is compatible with the anomaly formulas of Theorems 1.9 and 2.19. Also observe that the first term in the right-hand side of (5.2) vanishes on the odd-dimensional components of $X_g$, and that the second term vanishes on the even-dimensional components of $X_g$.

5.2. *The genus $J(\theta,x)$*.

*Definition* 5.3. For $y \in \mathbf{R}, s \in \mathbf{C}, \mathrm{Re}(s) > 1$, set

$$\zeta(y,s) = \sum_{n=1}^{+\infty} \frac{\cos(ny)}{n^s}, \qquad \eta(y,s) = \sum_{n=1}^{+\infty} \frac{\sin(ny)}{n^s}. \tag{5.3}$$

Then $\zeta(y,s)$ and $\eta(y,s)$ are the real and imaginary parts of the Lerch series $L(y,s)=\sum_{n=1}^{+\infty}\frac{e^{iny}}{n^s}$ introduced in [Le]. If $y\notin 2\pi\mathbf{Z}$, $s\mapsto\zeta(y,s)$ extends to a holomorphic function on $\mathbf{C}$, if $y\in 2\pi\mathbf{Z}$, $s\mapsto\zeta(y,s)$ extends to a meromorphic function on $\mathbf{C}$ with a simple pole at $s=1$. Also $s\mapsto\eta(y,s)$ extends to a holomorphic function on $\mathbf{C}$. Moreover

$$\zeta(0,s)=\zeta(s),\qquad \eta(0,s)=0. \tag{5.4}$$

*Definition* 5.4. For $\theta\in\mathbf{R}^*$, $x\in\mathbf{C}$, $|x|<|2\pi|$ if $\theta\in 2\pi\mathbf{Z}$, $|x|<\inf_{k\in\mathbf{Z}}|\theta+2k\pi|$ if $\theta\notin 2\pi\mathbf{Z}$, put

$$J(\theta,x)=\frac{1}{2}\left[\sum_{\substack{p\in\mathbf{N}\\ p\text{ even}}}\frac{\partial\zeta}{\partial s}(\theta,-p)\,\frac{x^p}{p!}+i\sum_{\substack{p\in\mathbf{N}\\ p\text{ odd}}}\frac{\partial\eta}{\partial s}(\theta,-p)\,\frac{x^p}{p!}\right]. \tag{5.5}$$

By [BGo2, Ths. 4.29, 4.30 and 4.35], the series in (5.5) converges.

Recall that the function $J^\theta(x)$ was defined in Definition 4.5. In the sequel,

$$\sum_{k\in\mathbf{Z}}{}'\left(J^{2k\pi+\theta}(x)-J^{2k\pi}(0)\right)$$

denotes a sum over $k\in\mathbf{Z}$, with the convention that if $2k\pi+\theta$ or if $2k\pi$ vanish, then $J^{2k\pi+\theta}(x)$ or $J^{2k\pi}(0)$ is replaced by 0. The following results were established in [BGo2, Ths. 4.35 and 4.38].

THEOREM 5.5. *The following identity holds*:

$$J(\theta,x)-J(0,0)=\sum_{k\in\mathbf{Z}}{}'\left(J^{2k\pi+\theta}(x)-J^{2k\pi}(0)\right). \tag{5.6}$$

*If* $\theta\in\mathbf{R}\setminus 2\pi\mathbf{Z},\theta'\in\mathbf{R},x\in\mathbf{C}$, *if* $|\theta'|,|x|$ *are small enough*,

$$J(\theta+\theta',x)=J(\theta,x+i\theta'). \tag{5.7}$$

*Also for* $\theta'\in]-2\pi,2\pi[\setminus\{0\}$, *for* $x\in\mathbf{C},|x|<\inf_{k\in\mathbf{Z}}|\theta'+2k\pi|$, *then*

$$J(\theta',x)=J(0,x+i\theta')+J^{\theta'}(x). \tag{5.8}$$

Put

$${}^0J(\theta,x)=J(\theta,x)-J(0,0). \tag{5.9}$$

Take $g\in G$. Let $V$ be a manifold. We assume that $Z(g)$ acts on $V$, and that $V$ is fixed by $g$.

Let $E=E_+\oplus E_-$ be a real $\mathbf{Z}_2$-graded vector bundle on $V$, which is equipped with a Euclidean metric $g^E=g^{E_+}\oplus g^{E_-}$, and a unitary connection $\nabla^E=\nabla^{E_+}\oplus\nabla^{E_-}$. Let $R^E=R^{E_+}\oplus R^{E_-}$ be the curvature of $\nabla^E$.

We assume that $g \in G$ acts on $E$ as an even unitary parallel automorphism of $E$. We also assume that the action of $Z(g)$ on $V$ lifts to an even unitary action on $E$, which commutes with $g$, and preserves the connection $\nabla^E$. If $K \in \mathfrak{z}(g)$, let $m^E(K) \in \mathrm{End}^{\mathrm{even}}(E)$ be the vertical component of the action of $K$ with respect to $\nabla^E$. Then $m^E(K)$ is an antisymmetric section of $\mathrm{End}^{\mathrm{even}}(E)$, which commutes with $g$. Since $\nabla^E$ is $Z(g)$-invariant, as in (2.7), we get

$$\nabla^E_{\cdot} m^E(K) + i_{K^V} R^E = 0. \tag{5.10}$$

As in (2.31), (3.3), the equivariant curvature $R^E_K$ is given by

$$R^E_K = R^E - 2\pi m^E(K). \tag{5.11}$$

If $-1$ is not an eigenvalue of $g$, there is $B \in \mathrm{End}^{\mathrm{even}}(E)$, which is antisymmetric, parallel, which commutes with $m^E(K)$ and is such that

$$g|_E = e^B. \tag{5.12}$$

Suppose that the action of $g$ on $E$ is given by $-1$. In this case, we write

$$g = e^{i\pi}. \tag{5.13}$$

By the above, we can always write $g|_E$ in the form (5.12), with $B \in \mathrm{End}^{\mathrm{even}}(E) \otimes_{\mathbf{R}} \mathbf{C}$, which commutes with $m^E(K)$.

Put

$${}^0J_{g,K}\left(E, \nabla^E\right) = \mathrm{Tr_s}\left[{}^0J\left(-iB, -R^E_K/2\pi\right)\right]. \tag{5.14}$$

By [BeV], [BeGeV, Th. 7.7],

$$d_K {}^0J_{g,K}\left(E, \nabla^E\right) = 0. \tag{5.15}$$

Let $J_{g,K}(E)$ be the equivariant cohomology class of $J_{g,K}\left(E, \nabla^E\right)$ on $V$. If $K = 0$, we will write instead ${}^0J_g(E)$, which is an ordinary cohomology class on $V$.

5.3. *Equivariant infinitesimal Ray-Singer metrics and Morse functions.* We make the same assumptions as in Section 2. In particular $g^{TX}, g^F$ denote $G$-invariant metrics on $TX, F$.

Let $f : X \to \mathbf{R}$ be a $G$-invariant Morse function. Let $h^{TX}$ be a $G$-invariant metric on $TX$, and let $\nabla f$ be the gradient field on $f$ with respect to $h^{TX}$. We will assume that $Y = -\nabla f$ is a Morse-Smale vector field [Sm1, 2]; i.e., the stable and unstable cells for $Y$ intersect transversally. Let $B$ be the critical set of $f$, i.e. the finite set of zeroes of $Y$. Then $G$ acts on $B$. In particular, if $K \in \mathfrak{g}$, $K^X$ vanishes on $B$.

If $x \in B$, let $o_x^u$ be the orientation line of $TW_x^u$. Let $(C^{\cdot}(W^u, F), \partial)$ be the Thom-Smale complex with coefficients in $F$, which is associated to the vector field $Y$. Then

$$(5.16) \qquad C^{\cdot}(W^u, F) = \bigoplus_{x\in B} F_x \otimes o_x^u, \qquad C^i(W^u, F) = \bigoplus_{\substack{x\in B \\ \mathrm{ind}(f)(x)=i}} F_x \otimes o_x^u.$$

A detailed description of the chain map $\partial$ is given in [BZ1, §1 c)] and [BGo2, Ch. 5.1]. Let us just mention here that, if $x \in B$, $\partial$ maps $F_x \otimes o_x^u$ into the direct sum of $F_y \otimes o_y^u$, with $x$ connected to $y$ along integral curves of $-Y$, so that $f(y) > f(x)$. The main point is that there is a canonical isomorphism,

$$(5.17) \qquad H^{\cdot}(C^{\cdot}(W^u, F), \partial) \simeq H^{\cdot}(X, F).$$

Clearly $G$ acts naturally on $(C^{\cdot}(W^u, F), \partial)$ and on $H^{\cdot}(X, F)$. Then (5.17) is an isomorphism of $\mathbf{Z}$-graded $G$ vector spaces. By (2.21), the connected component of the identity $G_0 \subset G$ acts trivially on $C^{\cdot}(W^u, F)$. Using (5.17), this last result fits with the fact that, as we saw in Section 2, $G_0$ acts trivially on $H^{\cdot}(X, F)$. Since $G$ is compact, $G/G_0$ is a finite group.

Then $G/G_0$ acts on $(C^{\cdot}(W^u, F), \partial)$ and on $H^{\cdot}(X, F)$ and (5.17) is an isomorphism of $\mathbf{Z}$-graded $G/G_0$ vector spaces.

Let $\widehat{G/G_0}$ be the set of equivalence classes of irreducible representations of $G/G_0$. If $w \in \widehat{G/G_0}$, let $\chi_w$ be the character of $G/G_0$ associated to $w$. As in (1.19), if $w \in \widehat{G/G_0}$, set

$$(5.18) \qquad \lambda_w(C^{\cdot}(W^u, F)) = \det\Big(\mathrm{Hom}_{G/G_0}(w, C^{\cdot}(W^u, F)) \otimes w\Big).$$

Put

$$(5.19) \qquad \lambda_{G/G_0}(C^{\cdot}(W^u, F)) = \oplus_{w\in\widehat{G/G_0}} \lambda_w(C^{\cdot}(W^u, F)).$$

Let $|\ |_{\lambda_w(C^{\cdot}(W^u,F))}$ be the metric induced by $g^F|_B$ on $\lambda_w(C^{\cdot}(W^u, F))$.

*Definition* 5.6. Set

$$(5.20) \qquad \log\Big(|\ |_{\lambda_{G/G_0}(C^{\cdot}(W^u,F))}\Big) = \sum_{w\in\widehat{G/G_0}} \log\Big(|\ |_{\lambda_w(C^{\cdot}(W^u,F))}\Big)\frac{\chi_w}{\mathrm{rk} w}.$$

Similarly, we can define $\lambda_w(H^{\cdot}(X, F)), \lambda_{G/G_0}(H^{\cdot}(X, F))$ as in (5.18)–(5.20). By [KnMu], we have canonical isomorphisms,

$$(5.21) \qquad \lambda_w(H^{\cdot}(X, F)) \simeq \lambda_w(C^{\cdot}(W^u, F)), \quad \lambda_{G/G_0}(H^{\cdot}(X, F)) \simeq \lambda_{G/G_0}(C^{\cdot}(W^u, F)).$$

Recall that $\lambda_G(F)$ was defined in (1.20). Note that in general, the sum in the right-hand side of (1.20) contains an infinite number of terms. However, if $W \in \widehat{G}$ does not appear in $H^{\cdot}(X, F)$, the corresponding Hermitian line $\lambda_W(F)$ is canonically trivial. Since $G/G_0$ is a finite group, the analogue of

(5.19) for $\lambda_{G/G_0}(H^{\cdot}(X,F))$ contains only a finite number of terms. Since the representation of $G$ on $H^{\cdot}(X,F)$ factors through $G/G_0$, up to irrelevant canonically trivial terms,

$$\lambda_G(F) \simeq \lambda_{G/G_0}(H^{\cdot}(X,F)). \tag{5.22}$$

*Definition* 5.7. Let $\log\left(\| \|^{\nabla f}_{\lambda_G(F)}\right)$ be obtained from $\log\left(| |_{\lambda_{G/G_0}(C^{\cdot}(W^u,F))}\right)$ via the canonical isomorphisms (5.21) and (5.22).

Observe that $TX|_B$ splits naturally as

$$TX|_B = TX^s|_B \oplus TX^u|_B. \tag{5.23}$$

In (5.23), $TX^s|_B, TX^u|_B$ are the tangent spaces to the stable (ascending) and unstable (descending) cells. In the sequel $TX|_B$ is considered as a $\mathbf{Z}_2$-graded vector bundle.

Let $B_g = B \cap X_g$. If $g \in G$, $g$ preserves the splitting in (5.23). Now we state a result directly related to in [BGo2, Th. 7.4]. This result was established by Lott-Rothenberg in [LoRo] for $K=0$, when $g^F$ is flat, and in [BZ2, Th. 0.2] in the case where $K=0$ and $g^F$ is arbitrary.

THEOREM 5.8. *For* $K \in \mathfrak{z}(g)$ *and* $|K|$ *small enough, the following identity holds*:

(5.24)

$$\log\left(\frac{\widetilde{\| \|_{\lambda_G(F),\mathrm{ch}}}(g,K)}{\| \|^{\nabla f}_{\lambda_G(F)}(g)}\right) = -\int_{X_g} \mathrm{ch}^{\circ}_g\left(\nabla^F, g^F\right)\psi_K\left(\nabla f|_{X_g}, TX_g, \nabla^{TX_g}\right) + \sum_{x \in B_g} (-1)^{\mathrm{ind}(f)(x)} \mathrm{Tr}^{F_x \otimes o^u_x}[g]\, {}^0J_{g,K}(T_xX).$$

*Proof.* In [BGo2, §7], a proof of our theorem is given in a families setting, i.e. in a more general context than in Section 2.5. To establish (5.24), which is an equality of complex numbers, one can either adapt the methods of the proof given in [BGo2], or observe from the results there that (5.24) holds as an equality of power series in the variable $z$, when replacing $K$ by $zK_0$. Using analyticity in the variable $z$, we get (5.24) in full generality. Details are left to the reader. □

*Remark* 5.9. Observe that in the right-hand side of (5.24), only the component of degree 0 of ${}^0J_{g,K}(T_xX)$ appears, so that, with the notation in (5.14),

$${}^0J_{g,K}(T_xX) = \mathrm{Tr_s}\left[{}^0J\left(-iB, m^E(K)\right)\right]. \tag{5.25}$$

Now we will explain the compatibility of Theorems 5.1 and 5.8. For $z \in \mathbf{R}^*, K = zK_0$ and $|z|$ small enough, $X_{ge^K} = X_{g,K}$. Also since the critical points of $f$ are isolated, $B \subset X_K$, so that $B_g = B_{ge^K}$. Note that by (2.21), (2.74), (2.124), we have the equality of forms on $X_{g,K}$,

$$\mathrm{ch}^{\circ}_{ge^K}\left(\nabla^F, g^F\right) = \mathrm{ch}^{\circ}_g\left(\nabla^F, g^F\right). \tag{5.26}$$

By (2.122), Theorem 5.8 and (5.26), for $|z|$ small enough,

(5.27)
$$\begin{aligned}\log\left(\frac{\widetilde{\| \|}_{\lambda_G(F),\mathrm{ch}}(g,K)}{\| \|_{\lambda_G(F)}(ge^K)}\right) &= -\int_{X_g} \mathrm{ch}^{\circ}_g\left(\nabla^F, g^F\right) \psi_K\left(\nabla f|_{X_g}, TX_g, \nabla^{TX_g}\right) \\ &+ \int_{X_{g,K}} \mathrm{ch}^{\circ}_g\left(\nabla^F, g^F\right) \psi\left(\nabla f|_{X_{g,K}}, TX_{g,K}, \nabla^{TX_{g,K}}\right) \\ &+ \sum_{x\in B_g} (-1)^{\mathrm{ind}(f)(x)} \mathrm{Tr}^{F_x\otimes o^u_x}[g]\left({}^0J_{g,K}(T_xX) - {}^0J_{ge^K,0}(T_xX)\right).\end{aligned}$$

By (4.24) and (5.7)–(5.9),

$$\begin{aligned}\sum_{x\in B_g} (-1)^{\mathrm{ind}\ f(x)} \mathrm{Tr}^{F_x\otimes o^u_x}[g]\left({}^0J_{g,K}(T_xX) - {}^0J_{ge^K,0}(T_xX)\right) \\ = -\sum_{x\in B_g} (-1)^{\mathrm{ind}(f)(x)} \mathrm{Tr}^{F_x\otimes o^u_x}[g]\,\mathcal{J}_K\left(N_{X_{g,K}/X_g}\right).\end{aligned} \tag{5.28}$$

We claim that (5.27) and (5.28) are equivalent to (5.2) in Theorem 5.1. Recall that by (2.76),

$$d_K \mathrm{ch}^{\circ}_g\left(\nabla^F, g^F\right) = 0. \tag{5.29}$$

- Using Theorems 3.15 and 3.21 and (5.29), we find that the first terms in the right-hand sides of (5.2) and (5.27) coincide. This is clear for the connected components of $X_g$ which are even-dimensional. In this case, the corresponding $X_{g,K}$ is also even-dimensional, the currents $\psi_K\left(\nabla f|_{X_g}, TX_g, \nabla^{TX_g}\right)$ and $\psi\left(\nabla f|_{X_{g,K}}, TX_{g,K}, \nabla^{TX_{g,K}}\right)$ are odd, and so they anticommute with $\mathrm{ch}^{\circ}_g\left(\nabla^F, g^F\right)$. If the considered connected component of $X_g$ is odd dimensional, the contribution of this component to the first term in the right-hand side of (5.2) vanishes. Again, by Theorems 3.15 and 3.21, this is also the case for the corresponding terms in the right-hand side of (5.27).
- By Theorem 4.10,

$$V_K(X_g) = -\sum_{x\in B_g} (-1)^{\mathrm{ind}\left(f|_{X_g}\right)(x)} \mathcal{J}_K\left(N_{X_{g,K}/X_g}\right). \tag{5.30}$$

Also if $x \in B_g$,

$$\operatorname{Tr}^{o^u_x}[g] = \det{}^{T_x X^u}[g]. \tag{5.31}$$

Using (5.31), one deduces easily that if $x \in B_g$,

$$(-1)^{\operatorname{ind}(f)(x)} \operatorname{Tr}^{o^u_x}[g] = (-1)^{\operatorname{ind}\left(f|_{X_g}\right)(x)}. \tag{5.32}$$

By (5.28)–(5.32), we find that the last terms in the right-hand sides of (5.2) and (5.27) also correspond.

We have then proved the compatibility of Theorem 5.1 to the results of [BZ2] and [BGo2].

5.4. *Equivariant torsions and Morse-Bott functions.* Let $f : X \to \mathbf{R}$ be a $G$-invariant Morse-Bott function. Let $B$ be the critical set of $f$. Then $B$ is a union of disjoint connected compact submanifolds of $X$, and $G$ acts on $B$. Let $N_{B/X}$ be the normal bundle to $B$ in $X$. We identify $N_{B/X}$ to the orthogonal vector bundle to $TB$ in $TX|_B$. Let $g^{TB}, g^{N_{B/X}}$ be the metrics on $TB, N_{B/X}$ induced by $g^{TX}$ on $TB, N_{B/X}$. Then $G$ acts on $TX, TB, N_{B/X}$ and preserves the corresponding metrics.

If $x \in B$, then $d^2 f(x)$ is a nondegenerate quadratic form on $N_{B/X,x}$. The vector bundle $N_{B/X}$ splits orthogonally as

$$N_{B/X} = N^s_{B/X} \oplus N^u_{B/X}. \tag{5.33}$$

In (5.33), $N^s_{B/X}, N^u_{B/X}$ are the direct sums of the eigenbundles of $d^2 f$ which are associated to positive and negative eigenvalues with respect to the metric $g^{N_{B/X}}$. Since $f$ is $G$-invariant, the splitting (5.33) of $N_{B/X}$ is also $G$-invariant.

Let $o\left(N^u_{B/X}\right)$ be the orientation bundle of $N^u_{B/X}$. Then $o\left(N^u_{B/X}\right)$ is a $\mathbf{Z}_2$-line bundle. In the sequel we will consider $o\left(N^u_{B/X}\right)$ as a complex Hermitian flat line bundle. In particular $F|_B \otimes o\left(N^u_{B/X}\right)$ is a complex vector bundle on $B$, equipped with a Hermitian metric $g^{F|_B \otimes o\left(N^u_{B/X}\right)}$ and a flat connection $\nabla^{F|_B \otimes o\left(N^u_{B/X}\right)}$.

Clearly, $f$ is locally constant on $B$. Let $B_1, \dots, B_q$ be the connected components of $B$. A trivial perturbation argument shows that we may assume that $f$ takes the value $p$ on $B_p$.

For $p \in \mathbf{N}$, set

$$U_p = f^{-1}[p - 1/2, +\infty[. \tag{5.34}$$

Then the $U_p$ define a decreasing filtration on the complex $\left(\Omega^{\cdot}(X,F), d^X\right)$, so that $F^p\Omega^{\cdot}(X,F)$ is the set of $s \in \Omega^{\cdot}(X,F)$ whose support is included in $U_p$. Let $(E_r, d_r), r \geq 0$, be the corresponding spectral sequence. Using Morse-Bott theory, one finds easily that

$$E_1^{p,q} = H^{p+q-\operatorname{ind}(f)}\left(B_p, F|_{B_p} \otimes o\left(N^u_{B/X}\right)\right). \tag{5.35}$$

In particular, for $r \geq 1$, the $(E_r, d_r)$ are finite-dimensional $\mathbf{Z}$-graded complexes. In (5.35), $E_1$ is $\mathbf{Z}$-graded by $p+q$. Moreover,

$$E_\infty \simeq \operatorname{Gr} H^{\cdot}(X,F)\,. \tag{5.36}$$

Clearly $G$ acts on the spectral sequence $(E_r, d_r)\,, r \geq 0$. Using the argument we gave after (2.23), and also (5.35), we find that $G_0$, the connected component of the identity in $G$, acts trivially on the $(E_r, d_r)\,, r \geq 1$.

Now, we proceed as in Sections 1.3 and 5.3. For $r \geq 1$, we define the direct sums of lines $\lambda_{G/G_0}(E_r)$ as in (5.19). By the obvious analogue of (5.21), for $r \geq 1$, we have the canonical isomorphism,

$$\lambda_{G/G_0}(E_r) \simeq \lambda_{G/G_0}(E_{r+1})\,. \tag{5.37}$$

Also classically, we have the canonical isomorphism,

$$\lambda_{G/G_0}(H^{\cdot}(X,F)) \simeq \lambda_{G/G_0}(\operatorname{Gr} H^{\cdot}(X,F))\,. \tag{5.38}$$

By (5.36)–(5.38), we conclude there is a canonical isomorphism,

$$\lambda_{G/G_0}(E_1) \simeq \lambda_{G/G_0}(H^{\cdot}(X,F))\,. \tag{5.39}$$

Moreover, by (5.35),

$$\lambda_{G/G_0}(E_1) = \bigotimes_{p=1}^{q} \lambda_{G/G_0}\left(H^{\cdot}\left(B_p, F|_B \otimes o\left(N^u_{B/X}\right)\right)\right)^{(-1)^{\operatorname{ind}(f)}}. \tag{5.40}$$

Recall that $g^{H^{\cdot}(X,F)}_{L_2}$ denotes the $L_2$ metric on $H^{\cdot}(X,F)$ which was defined in Section 1.3. This metric is associated to the metrics $g^{TX}, g^F$. Let $g^{H^{\cdot}\left(B,F|_B \otimes o\left(N^u_{B/X}\right)\right)}_{L_2}$ be the $L_2$ metric on $H^{\cdot}\left(B, F|_B \otimes o\left(N^u_{B/X}\right)\right)$ which is associated to the metrics $g^{TB}, g^{F|_B \otimes o\left(N^u_{B/X}\right)}$. Let $g^{E_1}$ be the corresponding metric on $E_1$ via the canonical isomorphism in (5.35). Note that the $E_1^{p,q}$ are mutually orthogonal with respect to $g^{E_1}$.

Since $E_2$ is the cohomology of the complex $(E_1, d_1)$, by identifying $E_2$ to the harmonic elements in $E_1$, $E_2$ inherits a metric $g^{E_2}$. By recursion, the $E_r$ inherit metrics $g^{E_r}$. Let $|\ |_{\lambda_{G/G_0}(E_r)}$ be the corresponding metric on $\lambda_{G/G_0}(E_r)$.

Let $d_r^*$ be the adjoint of $d_r$ with respect to $g^{E_r}$. Set

$$D_r = d_r + d_r^*. \tag{5.41}$$

As in Section 1.3, $D_r^2$ acts as an invertible operator on the vector space $E_{r+1}^{\perp} \subset E_r$ orthogonal to $E_{r+1} \simeq \ker D_r$ in $E_r$. Let $\left(D_r^2\right)^{-1}$ denote the corresponding inverse. Let $N_r$ be the operator defining the $\mathbf{Z}$-grading of $E_r$. If $g \in G/G_0$, as in (1.23), set

$$\vartheta_g(E_r)(s) = -\operatorname{Tr_s}\left[N_r g \left(D_r^2\right)^{-s}\right]. \tag{5.42}$$

Proposition 5.10. *For $g \in G$, the following identity holds*:

$$\log\left(\frac{|\ |_{\lambda_{G/G_0}(E^1)}}{|\ |_{\lambda_{G/G_0}(E_\infty)}}\right) = \sum_{r\geq 1}\frac{1}{2}\frac{\partial}{\partial s}\vartheta_g\left(E_r\right)(0)\,. \tag{5.43}$$

*Proof.* By [BGS, Prop. 1.5], for $r \geq 1$, we get

$$\log\left(\frac{|\ |_{\lambda_{G/G_0}}\left(E_r\right)}{|\ |_{\lambda_{G/G_0}}\left(E_{r+1}\right)}\right) = \frac{1}{2}\frac{\partial}{\partial s}\vartheta_g\left(E_r\right)(0)\,, \tag{5.44}$$

from which our proposition follows. □

We take $g \in G, K_0 \in \mathfrak{z}(g)$ and $K = zK_0$, with $z \in \mathbf{R}^*$. If $g \in G$, we still denote by $g$ its image in $G/G_0$. Clearly, $f|_{X_g}$ is a Morse-Bott function on $X_g$. We define $\psi_K\left(\nabla f|_{X_g}, TG_g, \nabla^{TX_g}\right)$ as in Definition 3.19. When $K = 0$, this current will be denoted $\psi\left(\nabla f|_{X_g}, TG_g, \nabla^{TX_g}\right)$.

In the sequel, we make the assumption that if $g \in G, x \in B_g$, then $g$ acts as the identity on $N^u_{B/X,x}$. Equivalently, we assume that for any $g \in G, x \in B_g$,

$$N_{X_g/X,x} \cap N_{B/X,x} \subset N^s_{B/X,x}. \tag{5.45}$$

Observe that we did not make this assumption in Section 5.3.

Finally note that, in the sequel, $TX, TB$ and $N_{B/X}$ will be considered as ordinary vector bundles, i.e. as even vector bundles. In other words, any possible $\mathbf{Z}_2$-grading will be forgotten.

Now we establish a result, which refines results of Bunke [Bu1, 2].

Theorem 5.11. *For $z \in \mathbf{R}$, with $|z|$ small enough,*

$$\begin{aligned}
&\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX},\nabla^F,g^F\right)\\
&\quad -(-1)^{\mathrm{ind}(f)}\,\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TB},\nabla^{F|_B\otimes o\left(N^u_{B/X}\right)},g^{F|_B\otimes o\left(N^u_{B/X}\right)}\right)\\
&\quad +\log\left(\frac{|\ |_{\lambda_{G/G_0}(H^\cdot(X,F))}}{|\ |_{\lambda_{G/G_0}(E^1)}}\right)(g)\\
&= -\int_{X_g}\mathrm{ch}^\circ_g\left(\nabla^F,g^F\right)\psi_K\left(\nabla f|_{X_g},TX_g,\nabla^{TX_g}\right)\\
&\quad +\int_{B_g}(-1)^{\mathrm{ind}(f)}\,\mathrm{Tr}^{F|_B\otimes o\left(N^u_{B/X}\right)}[g]\,e_K\left(TB_g\right)^0 J_{g,K}\left(N_{B/X}\right).
\end{aligned} \tag{5.46}$$

*Equivalently, for $z \in \mathbf{R}$ with $|z|$ small enough,*

$$\begin{aligned}
&\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX},\nabla^F,g^F\right)\\
&\quad -(-1)^{\mathrm{ind}(f)}\,\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TB},\nabla^{F|_B\otimes o\left(N^u_{B/X}\right)},g^{F|_B\otimes o\left(N^u_{B/X}\right)}\right)\\
&\quad +\log\left(\frac{|\ |_{\lambda_{G/G_0}(H^\cdot(X,F))}}{|\ |_{\lambda_{G/G_0}(E_1)}}\right)(g)
\end{aligned} \tag{5.47}$$

$$= -\int_{X_g} \mathrm{ch}_g^\circ\left(\nabla^F, g^F\right) \psi_K\left(\nabla f|_{X_g}, TX_g, \nabla^{TX_g}\right)$$
$$+ \int_{X_g} \mathrm{Tr}^F[g]\, e_K(TX_g)\, {}^0J_{g,K}(TX)$$
$$- \int_{B_g} (-1)^{\mathrm{ind}(f)} \mathrm{Tr}^{F|_B \otimes o\left(N^u_{B/X}\right)}[g]\, e_K(TB_g)\, {}^0J_{g,K}(TB)\,.$$

*Proof.* First we will prove (5.46) with $z = 0$. Note that in the right-hand side of (5.46) , since the degree of $\psi\left(\nabla f|_{X_g}, TX_g, \nabla^{TX_g}\right)$ is equal to $\dim X_g - 1$, we may as well replace $\mathrm{ch}_g^\circ\left(\nabla^F, g^F\right)$ by $\mathrm{Tr}\left[\omega\left(\nabla^F, g^F\right)/2\right]$. By using the anomaly formulas in Theorems 1.9 and 2.19, and also the equation of currents (3.34), one verifies easily that given $g \in G$, we only need to establish (5.46) with $z = 0$ for one given set of $g$-invariant metrics $g^{TX}, g^F$.

Assume that $g \in G$ is of finite order. Let $\Gamma \subset G$ be the finite group generated by $g$. By use of a result of Ilman [I] as in [BZ2, Th. 1.10], there is a $\Gamma$-invariant Morse function $f' : B \to \mathbf{R}$ such that if $B' \subset B$ is the finite set of critical points of $f'$, if $x \in B'_g$, $g$ acts as the identity on $T^u_x B'$.

Using geodesic coordinates, for $\varepsilon > 0$ small enough, we can identify an $\varepsilon$-neighbourhood of $B$ in $N_{B/X}$ to a tubular neighbourhood $U_\varepsilon$ of $B$ in $X$, and this $G$-equivariantly. Let $\pi$ be the obvious projection $U_\varepsilon \to B$. Let $\gamma : \mathbf{R} \to [0,1]$ be such that

$$\begin{aligned} \gamma(s) &= 1 \text{ if } |s| \le \varepsilon/4, \\ &= 0 \text{ if } |s| \ge \varepsilon/2. \end{aligned} \tag{5.48}$$

Then the function $Z \in N_{B/X} \to \gamma(|Z|)$ can be considered as a smooth function on $X$ with support in $U_{\varepsilon/2}$. For $\lambda \in \mathbf{R}^*_+$, put

$$f_\lambda = f + \lambda\gamma(|Z|)\, f' \circ \pi. \tag{5.49}$$

One verifies easily that for $|\lambda|$ small enough, $f_\lambda$ is a $\Gamma$-invariant Morse function on $M$, whose critical set coincides with $B'$. We fix such a $\lambda$, and set $\overline{f} = f_\lambda$. By using an obvious notation, if $x \in B'$, we have

$$T_xX^s = T_xB^s \oplus N^s_{B/X,x}, \qquad T_xX^u = T_xB^u \oplus N^u_{B/X,x}. \tag{5.50}$$

If $x \in B'$, let $o^u_x$ be the orientation line of $T_xX^u$.

The restriction to $B$ of the 1-form $d\overline{f}$ vanishes on $N_{B/X}$. It follows from our assumptions that if $g' \in \Gamma, x \in B'_{g'}$, then $g'$ acts as the identity on $T^u_xX$.

By [BZ2, Th. 1.10], there exists a $\Gamma$-invariant metric $g'^{TB}$ on $TB$, which coincides with $g^{TB}$ near $B'$, which is such that $Y' = -\nabla f'| \in TB$ verifies the Smale transversality conditions. We extend the metric $g'^{TB}$ to a $\Gamma$-invariant metric $g'^{TX}$ on $TX$, such that $N_{B/X}$ is still orthogonal to $TB$, and the restriction of $g'^{TX}$ to $N_{B/X}$ coincides with $g^{N_{B/X}}$. In particular the splitting (5.33) of $N_{B/X}$ is still orthogonal with respect to $g'^{TX}$. Let $\nabla\overline{f}$ be the gradient of $\overline{f}$ with respect to $g'^{TX}$. Put

$$Y = -\nabla\overline{f}. \tag{5.51}$$

It follows from the above that $Y|_B = Y'$, so that $Y|_B \in TB$. Also one verifies that it is possible to choose $g'^{TX}$ so that $Yf$ is negative on $X \setminus B$.

Take $p \in \mathbf{N}$, and let $x, x' \in B_p$. We claim that the stable and unstable cells for $Y$ in $X$ which are centered at $x$ and $x'$ intersect transversally. This is so because since $f$ is constant on $B_p$, and since $Yf < 0$ on $X \setminus B$, the intersection can only occur on $B$. Since the corresponding stable and unstable cells for $Y'$ are transverse in $B$, we have established our claim.

By proceeding as in [Mi, Ths. 4.4 and 5.2] and in [BZ2, Lemma 1.7 and Th. 1.8], we can modify the metric $g'^{TX}$ away from $B$ so that $Y$ itself verifies the Smale transversality conditions. To establish (5.46) with $z = 0$, we may as well replace the $G$-invariant metric $g^{TX}$ by the $\Gamma$-invariant metric $g'^{TX}$.

Let $(C^{\cdot}(W^u, F), \partial)$ be the Thom-Smale complex associated to the vector field $Y$ as in Section 5.3. Then $\Gamma$ acts naturally on $(C^{\cdot}(W^u, F), \partial)$. As we saw in Section 5.3, the chain map $\partial$ maps $F_x \otimes o^u_x$ into a direct sum of $F_y \otimes o^u_y$, where $x$ connects to $y$ by an integral curve of $-Y$. Since $Yf \leq 0$, it follows that $f(y) \geq f(x)$.

Put

$$F^p C^{\cdot}(W'^u, F) = \bigoplus_{\substack{x \in B' \\ f(x) \geq p-1/2}} F_x \otimes o^u_x. \tag{5.52}$$

It follows from the above considerations that $F$ is a filtration on $(C^{\cdot}(W'^u, F), \partial)$. Let $(E'_r, d'_r)$ be the corresponding spectral sequence.

Recall that $Y|_B \in TB$. Then $(E'_0, d'_0)$ is just the Thom-Smale complex

$$\left(C^{\cdot}\left(W'^u, F|_B \otimes o\left(N^u_{B/X}\right)\right), \partial\right)$$

of the connected components of $B$, which is associated to the gradient field $Y'$, so that

$$E'^{(p,q)}_0 = C^{p+q-\mathrm{ind}(f)}\left(W'^u|_{B_p}, F|_{B_p} \otimes o\left(N^u_{B_p/X}\right)\right). \tag{5.53}$$

By (5.35) and (5.53), we get

$$E_1 \simeq E'_1. \tag{5.54}$$

Recall that by a result of Laudenbach [BZ1, Appendix], if near $B'$ the metric $g'^{TX}$ is flat in coordinates where $\overline{f}$ can be written quadratically, there is a canonical quasi-isomorphism $P_\infty$ from $\left(\Omega^{\cdot}(X, F), d^X\right)$ into $(C^{\cdot}(W'^u, F), \partial)$. The map $P_\infty$ is a map of $\mathbf{Z}$-graded filtered complexes. By (5.54), a classical result of homological algebra [CaE, Ch. XIII, Th. 3.2] shows that (5.54) extends to a canonical isomorphism,

$$E_r \simeq E'_r \text{ for } r \geq 1. \tag{5.55}$$

We can always perturb the metric $g'^{TX}$ in a small way so that the above assumption holds. Since the complex $(C^{\cdot}(W^u, F), \partial)$ is unchanged, (5.55) holds for any $g \in G$.

We define $\lambda_\Gamma(F) = \lambda_\Gamma(H^{\cdot}(X,F))$ as in (1.20). Also $\lambda_\Gamma(E_1)$ can be defined as in (5.40). We will now apply [BZ2, Th. 0.2] to the manifolds $X$ and $B$. This is just Theorem 5.8 with $K=0$. Observe that in (5.24), $TX|_B$ is considered as a $\mathbf{Z}_2$-graded vector bundle, the $\mathbf{Z}_2$-grading being defined as in (5.23). However, observe that by (5.9),

$$ {}^0J(0,0) = 0. \tag{5.56} $$

Moreover, by our fundamental assumption, if $x \in B'_g$, then $T_xX^u$ is fixed by $g$. Therefore,

$$ {}^0J_g(T_x^uX) = 0. \tag{5.57} $$

Equivalently, to evaluate ${}^0J_g(T_xX)$, we may as well forget about the $\mathbf{Z}_2$-grading of $T_xX$; i.e., we can consider $T_xX$ as trivially $\mathbf{Z}_2$-graded. The same considerations apply to $T_xB$.

Thus, by using (5.24), we obtain

$$ \log\left(\frac{\|\ \|_{\lambda_\Gamma(F)}}{\|\ \|^{\nabla\overline{f}}_{\lambda_\Gamma(F)}}\right)(g) = -\int_{X_g} \mathrm{ch}_g^\circ\left(\nabla^F, g^F\right)\psi\left(\nabla\overline{f}|_{X_g}, TX_g, \nabla^{TX_g}\right) + \sum_{x\in B'_g}(-1)^{\mathrm{ind}(\overline{f})(x)}\mathrm{Tr}^{F_x\otimes o_x^u}[g]\,{}^0J_g(T_xX), \tag{5.58} $$

$$ \log\left(\frac{\|\ \|_{\lambda_\Gamma(E_1)}}{\|\ \|^{\nabla f'}_{\lambda_\Gamma(E_1)}}\right)(g) = -\int_{B_g}(-1)^{\mathrm{ind}(f)}\mathrm{Tr}^{o\left(N^u_{B/X}\right)}[g]\,\mathrm{ch}_g^\circ\left(\nabla^F, g^F\right) \psi\left(\nabla f'|_{B_g}, TB_g, \nabla^{TB_g}\right) + \sum_{x\in B'_g}(-1)^{\mathrm{ind}(\overline{f})(x)}\mathrm{Tr}^{F_x\otimes o_x^u}[g]\,{}^0J_g(T_xB). $$

By (5.39), we have a canonical isomorphism,

$$ \lambda_\Gamma(F) \simeq \lambda_\Gamma(E_1). \tag{5.59} $$

We claim that under this isomorphism,

$$ \|\ \|^{\nabla\overline{f}}_{\lambda_\Gamma(F)} = \|\ \|^{\nabla f'}_{\lambda_\Gamma(E_1)}. \tag{5.60} $$

Note that equation (5.60) is a result on finite-dimensional filtered complexes. A proof of (5.59) was given in [BL, §2] in a more difficult context. The proof given in [BL] refers to complex manifolds, but can be used as such by taking the manifold $X$ in [BL] to be a point.

From (1.24), (1.43), (2.119) and (5.58), one deduces easily that, by still considering $N_{B/X}$ as trivially $\mathbf{Z}_2$-graded,

(5.61)
$$\begin{aligned}&\mathcal{T}_{\mathrm{ch},g,0}\left(g^{TX},\nabla^F,g^F\right)-(-1)^{\mathrm{ind}(f)}\,\mathcal{T}_{\mathrm{ch},g,0}\left(g^{TB},\nabla^{F|_B\otimes o\left(N^u_{B/X}\right)},g^{F|_B\otimes o\left(N^u_{B/X}\right)}\right)\\&\quad+\log\left(\frac{|\ |_{\lambda_\Gamma(H^\cdot(X,F))}}{|\ |_{\lambda_\Gamma(E^1)}}\right)(g)\\&=-\int_{X_g}\mathrm{ch}^\circ_g\left(\nabla^F,g^F\right)\psi\left(\nabla\overline{f}|_{X_g},TX_g,\nabla^{TX_g}\right)\\&\quad+\int_{B_g}(-1)^{\mathrm{ind}(f)}\,\mathrm{Tr}^{o\left(N^u_{B/X}\right)}[g]\,\mathrm{ch}^\circ_g\left(\nabla^F,g^F\right)\psi\left(\nabla f'|_{B_g},TB_g,\nabla^{TB_g}\right)\\&\quad+\sum_{x\in B'_g}(-1)^{\mathrm{ind}\overline{f}(x)}\,\mathrm{Tr}^{F_x\otimes o^u_x}[g]\,{}^0J_g\left(N_{B/X,x}\right).\end{aligned}$$

The following identities, which are valid respectively on $B'_g$ and $B_g$, are the obvious analogues of (5.32),

$$(-1)^{\mathrm{ind}(f')}\,\mathrm{Tr}^{o(T_xB^u)}[g]=(-1)^{\mathrm{ind}\left(f'|_{B_g}\right)},\qquad (-1)^{\mathrm{ind}(f)}\,\mathrm{Tr}^{o\left(N^u_{B/X}\right)}[g]=(-1)^{\mathrm{ind}\left(f|_{X_g}\right)}.\tag{5.62}$$

Now, by proceeding as in [B8, proof of Theorem 3.2], and using (5.62), we get

$$\begin{aligned}&-\int_{X_g}\mathrm{ch}^\circ_g\left(\nabla^F,g^F\right)\psi\left(\nabla\overline{f}|_{X_g},TX_g,\nabla^{TX_g}\right)\\&\quad+\int_{B_g}(-1)^{\mathrm{ind}(f)}\,\mathrm{Tr}^{o\left(N^u_{B/X}\right)}[g]\,\mathrm{ch}^\circ_g\left(\nabla^F,g^F\right)\psi\left(\nabla f'|_{B_g},TB_g,\nabla^{TB_g}\right)\\&=-\int_{X_g}\mathrm{ch}^\circ_g\left(\nabla^F,g^F\right)\psi\left(\nabla f|_{X_g},TX_g,\nabla^{TX_g}\right).\end{aligned}\tag{5.63}$$

Using (5.62) again, we find that

$$\begin{aligned}&\sum_{x\in B'_g}(-1)^{\mathrm{ind}\left(\overline{f}\right)(x)}\,\mathrm{Tr}^{F_x\otimes o^u_x}[g]\,{}^0J_g\left(N_{B/X,x}\right)\\&\quad=\sum_{x\in B'_g}(-1)^{\mathrm{ind}\left(f'|_{B_g}\right)(x)}(-1)^{\mathrm{ind}(f)}\,\mathrm{Tr}^{F_x\otimes o\left(N^u_{B/X,x}\right)}[g]\,{}^0J_g\left(N_{B/X,x}\right).\end{aligned}\tag{5.64}$$

By (5.64), we obtain

$$\begin{aligned}&\sum_{x\in B'_g}(-1)^{\mathrm{ind}\ \left(\overline{f}\right)(x)}\,\mathrm{Tr}^{F_x\otimes o^u_x}[g]\,{}^0J_g\left(N_{B/X,x}\right)\\&\quad=\int_{B_g}(-1)^{\mathrm{ind}(f)}\,\mathrm{Tr}^{F|_B\otimes o\left(N^u_{B/X}\right)}[g]\,e\left(TB_g\right){}^0J_g\left(N_{B/X}\right).\end{aligned}\tag{5.65}$$

By (5.61)–(5.65), we get (5.46) when $g$ is of finite order. More generally, let $g \in G$, and let $G' \subset G$ be the closed Lie subgroup generated by $g$. Then $G'$ keeps $X_g$ fixed. Also $G'$ is an extension of a torus by a finite group. In particular, elements of finite order are dense in $G'$. If $g' \in G'$ of finite order is close enough to $g$ in $G$, its fixed point set $X_{g'}$ coincides with $X_g$. Now (5.46) holds for $g'$. Also one verifies easily that since $X_{g'} = X_g$, as $g' \to g$, both sides of (5.46) converge to the corresponding expression with $g' = g$. Therefore we have established (5.46) for all $g \in G$ when $z = 0$.

Now we will prove (5.46) for $z \in \mathbf{R}$, with $|z|$ small enough. By (5.46), we may as well assume that $z \in \mathbf{R}^*$. If $|z|$ is small enough, then $X_{ge^K} = X_{g,K} = X_{g,K_0}$. Let $i : X_{g,K} \to X_g$ be the obvious embedding. Using Theorem 5.1, (5.46) applied to $ge^K$, and also (5.26), we get

(5.66)
$$\begin{aligned}
&\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX},\nabla^F,g^F\right)\\
&\quad-(-1)^{\mathrm{ind}(f)}\,\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TB},\nabla^{F|_B\otimes o\left(N^u_{B/X}\right)},g^{F|_B\otimes o\left(N^u_{B/X}\right)}\right)\\
&\quad+\log\left(\frac{|\ |_{\lambda_{G/G_0}(H^\cdot(X,F))}}{|\ |_{\lambda_{G/G_0}(E^1)}}\right)(g)\\
&=-\int_{X_{g,K}}\mathrm{ch}^\circ_g\left(\nabla^F,g^F\right)\psi\left(\nabla f|_{X_{g,K}},TX_{g,K},\nabla^{TX_{g,K}}\right)\\
&\quad+\int_{X_g}e_K\left(TX_g,\nabla^{TX_g}\right)F_K\left(TX_g,g^{TX_g}\right)\mathrm{ch}^\circ_g\left(\nabla^F,g^F\right)\\
&\quad-\int_{B_g}(-1)^{\mathrm{ind}(f)}\,\mathrm{Tr}^{o\left(N^u_{B/X}\right)}[g]\,e_K\left(TB_g,\nabla^{TB_g}\right)F_K\left(TB_g,g^{B_g}\right)\mathrm{ch}^\circ_g\left(\nabla^F,g^F\right)\\
&\quad+\mathrm{Tr}^{F|_{X_g}}[g]\,V_K\left(X_g\right)-(-1)^{\mathrm{ind}(f)}\,\mathrm{Tr}^{F|_B\otimes o\left(N^u_{B/X}\right)|_{B_g}}[g]\,V_K\left(B_g\right)\\
&\quad+\int_{B_{g,K}}(-1)^{\mathrm{ind}(f)}\,\mathrm{Tr}^{F|_B\otimes o\left(N^u_{B/X}\right)}[g]\,e\left(TB_{g,K}\right){}^0J_{ge^K}\left(N_{B/X}\right).
\end{aligned}$$

Using Theorems 3.15, 3.21, (5.29) and (5.62) and proceeding as in Remark 5.9, we obtain

(5.67)
$$\begin{aligned}
&-\int_{X_{g,K}}\mathrm{ch}^\circ_g\left(\nabla^F,g^F\right)\psi\left(\nabla f|_{X_{g,K}},TX_{g,K},\nabla^{TX_{g,K}}\right)\\
&\quad+\int_{X_g}e_K\left(TX_g,\nabla^{TX_g}\right)F_K\left(TX_g,g^{TX_g}\right)\mathrm{ch}^\circ_g\left(\nabla^F,g^F\right)\\
&\quad-\int_{B_g}(-1)^{\mathrm{ind}(f)}\,\mathrm{Tr}^{o\left(N^u_{B/X}\right)}[g]\,e_K\left(TB_g,\nabla^{TB_g}\right)F_K\left(TB_g,g^{B_g}\right)\mathrm{ch}^\circ_g\left(\nabla^F,g^F\right)\\
&=-\int_{X_g}\mathrm{ch}^\circ_g\left(\nabla^F,g^F\right)\psi_K\left(\nabla f|_{X_g},TX_g,\nabla^{TX_g}\right).
\end{aligned}$$

Let $\widetilde{N}$ be the subbundle of $N_{B_g/X_g}|_{B_{g,K}}$ where $K$ acts as an invertible operator. Using our main assumption, we find, using the notation in (4.30), with respect to $X_g$, that $\widetilde{N} = \widetilde{N}^s$; i.e., $\widetilde{N}$ is trivially $\mathbf{Z}_2$-graded. By applying Theorem 4.10 to $X_g$ and using (5.62), we get

$$\begin{aligned} &\operatorname{Tr}^F[g]\, V_K(X_g) - (-1)^{\operatorname{ind}(f)} \operatorname{Tr}^{F|_B \otimes o\left(N^u_{B/X}\right)}[g]\, V_K(B_g) \\ &\quad = -\int_{B_{g,K}} (-1)^{\operatorname{ind}(f)} \operatorname{Tr}^{F|_B \otimes o\left(N^u_{B/X}\right)}[g]\, e(TB_{g,K})\, \mathcal{J}_K\left(\widetilde{N}\right). \end{aligned} \tag{5.68}$$

Using (5.7) and (5.8) in Theorem 5.5, on $B_{g,K}$, we obtain

$$^0J_{ge^K}\left(N_{B/X}\right) - \mathcal{J}_K\left(\widetilde{N}\right) = {}^0J_{g,K}\left(N_{B/X}\right). \tag{5.69}$$

By (5.66)–(5.69), we see that

$$\begin{aligned} &\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX}, \nabla^F, g^F\right) \\ &\quad - (-1)^{\operatorname{ind}(f)} \mathcal{T}_{\mathrm{ch},g,K}\left(g^{TB}, \nabla^{F|_B \otimes o\left(N^u_{B/X}\right)}, g^{F|_B \otimes o\left(N^u_{B/X}\right)}\right) \\ &\quad + \log\left(\frac{|\ |_{\lambda_{G/G_0}(H^\cdot(X,F))}}{|\ |_{\lambda_{G/G_0}(E^1)}}\right)(g) \\ &= -\int_{X_g} \operatorname{ch}^\circ_g\left(\nabla^F, g^F\right) \psi_K\left(\nabla f|_{X_g}, TX_g, \nabla^{TX_g}\right) \\ &\quad + \int_{B_{g,K}} (-1)^{\operatorname{ind}(f)} \operatorname{Tr}^{F|_B \otimes o\left(N^u_{B/X}\right)}[g]\, e(TB_{g,K})\, {}^0J_{g,K}\left(N_{B/X}\right). \end{aligned} \tag{5.70}$$

Finally, by using the localization formulas in equivariant cohomology of [BeV], we get

$$\begin{aligned} &\int_{B_{g,K}} (-1)^{\operatorname{ind}(f)} \operatorname{Tr}^{F|_B \otimes o\left(N^u_{B/X}\right)}[g]\, e(TB_{g,K})^0\, J_{g,K}\left(N_{B/X}\right) \\ &\quad = \int_{B_g} (-1)^{\operatorname{ind}(f)} \operatorname{Tr}^{F|_B \otimes o\left(N^u_{B/X}\right)}[g]\, e_K(TB_g)\, {}^0J_{g,K}\left(N_{B/X}\right). \end{aligned} \tag{5.71}$$

By (5.70) and (5.71), we get (5.46).

Also observe that

$$^0J_{g,K}\left(N_{B/X}\right) = {}^0J_{g,K}(TX|_B) - {}^0J_{g,K}(TB). \tag{5.72}$$

Finally, using (3.46) and (5.62), we get

$$\begin{aligned} &\int_{B_g} (-1)^{\operatorname{ind}(f)} \operatorname{Tr}^{F|_B \otimes o\left(N^u_{B/X}\right)}[g]\, e_K(TB_g)\, {}^0J_{g,K}(TX) \\ &\quad = \int_{X_g} \operatorname{Tr}^F[g]\, e_K(TX_g)\, {}^0J_{g,K}(TX). \end{aligned} \tag{5.73}$$

By (5.46), (5.72), (5.73), we get (5.47). The proof of our theorem is completed. □

*Remark* 5.12. Inspection of the proof shows that in (5.46), the vector bundle $N_{B/X}$ could be treated as well as a $\mathbf{Z}_2$-graded vector bundle. The fact is that here, $N^u_{B/X}$ does not contribute to the integral in the right-hand side in (5.46).

Observe that a special case of (5.46) in Theorem 5.11 was already established in infinitesimal form in [BGo2, Th. 16.3], in the context of unit sphere bundles.

If one subtracts (5.46) from (5.46) at $K = 0$ , we get an identity in which the term

$$\log\left(\frac{|\ |_{\lambda_{G/G_0}(H^\cdot(X,F))}}{|\ |_{\lambda_{G/G_0}(E^1)}}\right)(g)$$

has disappeared. One can give a direct proof of such an identity, which is valid if $f$ is an arbitrary Morse-Bott function, which does not necessarily verify the assumptions given after (5.33). To prove this more general statement, one needs to combine the techniques of [BL] with the techniques of [BZ1, 2], [BGo2]. Needless to say, for this more general case, $N_{B/X}$ has to be treated as a $\mathbf{Z}_2$-graded vector bundle.

If $n(K)$ is a function $\mathfrak{z}(g) \to \mathbf{C}$, put

$$n(K)^{(>0)} = n(K) - n(0). \tag{5.74}$$

Let us now suppose the assumptions of Theorem 5.11 to be in force, i.e. that (5.45) holds. Assume temporarily that $g^F$ is flat. By (5.47), we get

$$\begin{aligned}
&\left[\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX},\nabla^F,g^F\right) - \int_{X_g}\mathrm{Tr}^F[g]\,e_K(TX_g)\,{}^0J_{g,K}(TX_g)\right]^{(>0)} \\
&\quad = \Bigg[(-1)^{\mathrm{ind}(f)}\,\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TB},\nabla^{F|_B\otimes o\left(N^u_{B/X}\right)},g^{F|_B\otimes o\left(N^u_{B/X}\right)}\right) \\
&\qquad - \int_{B_g}(-1)^{\mathrm{ind}(f)}\,\mathrm{Tr}^{F|_B\otimes o\left(N^u_{B/X}\right)}[g]\,e_K(TB_g)\,{}^0J_{g,K}(TB_g)\Bigg]^{(>0)}.
\end{aligned} \tag{5.75}$$

Now in [Bu2], Bunke considers the case where $X$ is odd-dimensional and oriented. Assume that $X$ is a $G$-space of the $G$-homotopy type of a $G-\mathrm{CW}$ complex in the sense of Lück [Lü]. In [Bu2], the author proves that if $g = 1$, the germ of the analytic $K \in \mathfrak{g} \to \mathcal{T}_{\mathrm{ch},1,K}\left(g^{TX},\nabla F,g^F\right)^{(>0)}$ can be expressed as the sum of universal contributions of strata $G/H$ defining the $G-\mathrm{CW}$ structure, independently of the way the strata are embedded in $X$. Note that the $G/H$ should be thought of as being the critical manifold of a $G$-invariant Morse-Bott function verifying the assumptions of Theorem 5.11. Also, in [Bu2], Bunke does not use the ch normalization of infinitesimal torsion, but instead Lott's normalization [Lo].

Equation (5.75), which is valid in full generality, explains Bunke's result in [Bu2]. Indeed if $X$ is odd-dimensional, the integral in the left-hand side of (5.75) vanishes identically. As to the right-hand side, it is a sum of terms correcting the infinitesimal torsion of the strata by the integrals

$$\left[\int_{B_g} (-1)^{\mathrm{ind}(f)} \operatorname{Tr}^{F|_B \otimes o\left(N^u_{B/X}\right)} [g] \, e_K \left(TB_g\right) {}^0 J_{g,K} \left(TB_g\right)\right]^{(>0)} .$$

In [Bu2, Remark, p. 401], Bunke observes that his main result is not valid in even dimensions. Equation (5.75) again explains why this is so, since the integral in the left-hand side of (5.75) no longer vanishes. Equation (5.75) was used in [BGo2, Ch. 16] to evaluate the torsion of odd-dimensional unit sphere bundles.

In a previous paper, Bunke [Bu1] has established a corresponding result for the equivariant Reidemeister torsion in the sense of Lott-Rothenberg [LoRo], valid when $g^F$ is flat. By [LoRo], [BZ2, Th. 0.1], equation (5.47) refines this result of Bunke as well. In fact, by (5.58), when $g$ is of finite order, the correction to $\mathcal{T}_{\mathrm{ch},g,0}\left(g^{TX}, \nabla^F, g^F\right)$ in the left-hand side (5.75) is exactly the one needed to recover the equivariant Reidemeister torsion.

5.5. *An extension of Theorem* 5.1 *to equivariant analytic torsion forms.* We now make the same assumptions as in Section 3.7. In particular $\pi : M \to S$ is a submersion with compact fibre $X$, so that $G$ acts on $M$ and preserves the fibres $X$. Let $T^H M \subset TM$ be a horizontal bundle as in (3.50).

Let $F$ be a complex vector bundle on $M$, let $\nabla^F$ be a flat connection on $F$. As before, we assume that the action of $G$ lifts to $F$, and that the fundamental assumption in (2.21) still holds. Let $g^F$ be a $G$-invariant Hermitian metric on $F$.

Let $H^{\cdot}(X,F)$ be the fibrewise cohomology of $X$ with coefficients in $F|_X$. Let $g^{H^{\cdot}(X,F)}_{L_2}$ be the $L_2$ metric on $H^{\cdot}(X,F)$ one constructs via fibrewise Hodge theory as in Section 1.3 on $H^{\cdot}(X,F)$

Take $g \in G$. As explained in Section 2.5, in [BGo2, §3.17], analytic torsion forms $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ on $S$ were defined. These are even forms on $S$, which are such that

$$d\mathcal{T}_{\mathrm{ch},g}\left(T^H M, g^{TX}, \nabla^F, g^F\right) = \int_{X_g} e\left(TX_g, \nabla^{TX_g}\right) \mathrm{ch}^{\circ}_g\left(\nabla^F, g^F\right) - \mathrm{ch}^{\circ}_g\left(\nabla^{H^{\cdot}(X,F)}, g^{H^{\cdot}(X,F)}_{L_2}\right). \tag{5.76}$$

Take now $g \in G, K \in \mathfrak{z}(g)$. For $|K|$ small enough, by proceeding as in [BGo2, §3] and in Section 2.5, we construct even forms

$$\mathcal{T}_{h,g,K}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$$

as in (2.90), which depend analytically on $K \in \mathfrak{z}(g)$ near $K = 0$.

Then we combine the two definitions of the operator $Q$ in (2.92) and in (2.117). Namely, if $f(K)$ is an even form on $S$ which is analytic in $K \in \mathfrak{g}$, we set

$$Qf(K) = \int_0^1 \psi_{4s(1-s)} f(4s(1-s)K)\, ds. \tag{5.77}$$

As in [BGo2, Def. 3.46], and in (2.94), we set

$$\mathcal{T}_{\mathrm{ch},g,K}\left(T^H M, g^{TX}, \nabla^F, g^F\right) = Q\mathcal{T}_{h,g,K}\left(T^H M, g^{TX}, \nabla^F, g^F\right). \tag{5.78}$$

We can then prove that the obvious analogue of [BGo2, Th. 3.47] holds. Namely,

$$\begin{aligned} d\mathcal{T}_{\mathrm{ch},g,K}&\left(T^H M, g^{TX}, \nabla^F, g^F\right) \\ &= \int_{X_g} e_K\left(TX_g, \nabla^{TX_g}\right) \mathrm{ch}_g^\circ\left(\nabla^F, g^F\right) - \mathrm{ch}_g^\circ\left(\nabla^{H^\cdot(X,F)}, g_{L_2}^{H^\cdot(X,F)}\right). \end{aligned} \tag{5.79}$$

Let $P^S$ be the vector space of smooth forms on $S$, let $P^{S,0} \subset P^S$ be the subspace of exact smooth forms on $S$.

We state, without a proof, the following extension of Theorem 5.1 in arbitrary degree. Although we have chosen not to prove this result, we hope that inspection of [BGo2] and of the present paper gives all the necessary tools for proceeding to establish it.

THEOREM 5.13. *The following identity holds*:

$$\begin{aligned} \mathcal{T}_{\mathrm{ch},g,K}&\left(T^H M, g^{TX}, \nabla^F, g^F\right) - \mathcal{T}_{\mathrm{ch},ge^K,0}\left(T^H M, g^{TX}, \nabla^F, g^F\right) \\ &= \int_{X_g} e_K\left(TX_g, \nabla^{TX_g}\right) F_K\left(TX_g, \nabla^{TX_g}\right) \mathrm{ch}_g^\circ\left(\nabla^F, g^F\right) \\ &\quad + \mathrm{Tr}^{F|_{X_g}}[g]\, V_K(M_g/S) \ \text{ in } P^S/P^{S,0}. \end{aligned} \tag{5.80}$$

*Remark* 5.14. By proceeding as in Section 5.3, one can check easily that Theorem 5.80 is still compatible with the results of [BGo2, §7], use now of Theorem 4.10 in arbitrary degree.

We claim that Theorem 5.13 is compatible with the known functoriality results on the behaviour of analytic torsion forms under composition of two projections. Such a result has been obtained by Ma [Ma] for the ordinary analytic torsion forms. However the techniques of Ma can be applied also to the equivariant analytic torsion forms, and to their infinitesimal version.

On $P_g$, $g$ acts along the fibres $Z$ of $p$. Let $\chi_g(Z, F|_Z)$ be the Lefschetz number of the fibre $Z$, which was defined in (1.13). Then by (1.14),

$$\chi_g(Z, F) = \mathrm{Tr}^{F|_{M_g}}[g]\, \chi(X_g). \tag{5.81}$$

Using (5.81), Theorem 4.25 is ensures the compatibility of Theorem 5.13 to the results of [Ma].

5.6. *The asymptotics of the equivariant torsion forms.* We make the same assumptions as in Section 5.5. By Theorems 3.5, 3.37 and 5.13, we find that for $z \in \mathbf{R}^*$, and $|z|$ small enough,

$$\begin{aligned}(5.82)\quad &\mathcal{T}_{\mathrm{ch},g,zK_0}\left(T^H M, g^{TX}, \nabla^F, g^F\right) - \mathcal{T}_{\mathrm{ch},ge^{zK_0},0}\left(T^H M, g^{TX}, \nabla^F, g^F\right) \\ &= \psi_{1/z}\int_{X_g} e_{K_0}\left(TX_g, \nabla^{TX_g}\right) F_{K_0}\left(TX_g, \nabla^{TX_g}\right)\frac{1}{\sqrt{z}}\psi_z \mathrm{ch}_g^\circ\left(\nabla^F, g^F\right) \\ &\quad + \mathrm{Tr}^{F|_{X_g}}[g]\frac{1}{|z|}\psi_{1/z} V_{K_0}(M_g/S) \text{ in } P^S/P^{S,0}.\end{aligned}$$

Equation (5.82) is interesting. In fact, as $z \to 0$, $\mathcal{T}_{\mathrm{ch},g,zK_0}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ converges to $\mathcal{T}_{\mathrm{ch},g,0}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$. Therefore (5.82) gives us the asymptotic expansion of $\mathcal{T}_{\mathrm{ch},ge^{zK_0},0}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ as $z \in \mathbf{R}^*, z \to 0$.

In particular, if $S$ is a point, i.e. if the assumptions of Section 5.1 are verified, as $z \to 0$,

$$\begin{aligned}(5.83)\quad &\mathcal{T}_{\mathrm{ch},ge^{zK_0},0}\left(g^{TX}, \nabla^F, g^F\right) \\ &= -\frac{1}{|z|}\mathrm{Tr}^{F|_{X_g}}[g] V_{K_0}(X_g) \\ &\quad - \int_{X_g} e_{K_0}\left(TX_g, \nabla^{TX_g}\right) F_{K_0}\left(TX_g, \nabla^{TX_g}\right)\mathrm{Tr}\left[\frac{\omega\left(\nabla^F, g^F\right)}{2}\right] \\ &\quad + \mathcal{T}_{\mathrm{ch},g,0}\left(g^{TX}, \nabla^F, g^F\right) + \mathcal{O}(z).\end{aligned}$$

Also by (2.75), (2.76) and by Theorem 3.15,

$$\begin{aligned}(5.84)\quad &\int_{X_g} e_{K_0}\left(TX_g, \nabla^{TX_g}\right) F_{K_0}\left(TX_g, \nabla^{TX_g}\right)\mathrm{Tr}\left[\frac{\omega\left(\nabla^F, g^F\right)}{2}\right] \\ &= \int_{X_g} \psi_{K_0}\left(K_0^{X_g}, TX_g, \nabla^{TX_g}\right)\mathrm{Tr}\left[\frac{\omega\left(\nabla^F, g^F\right)}{2}\right].\end{aligned}$$

Using Remark 3.14, we get

$$\begin{aligned}(5.85)\quad &\int_{X_g} \psi_{K_0}\left(K_0^{X_g}, TX_g, \nabla^{TX_g}\right)\mathrm{Tr}\left[\frac{\omega\left(\nabla^F, g^F\right)}{2}\right] \\ &= \int_{X_g} \psi\left(K_0^{X_g}, TX_g, \nabla^{TX_g}\right)\mathrm{Tr}\left[\frac{\omega\left(\nabla^F, g^F\right)}{2}\right].\end{aligned}$$

By (5.83)–(5.85), we find that as $z \to 0$,

(5.86)
$$\begin{aligned}\mathcal{T}_{\mathrm{ch},ge^{zK_0},0}\left(g^{TX},\nabla^F,g^F\right) = & -\frac{1}{|z|}\mathrm{Tr}^{F|_{X_g}}[g]\,V_{K_0}(X_g) \\ & -\int_{X_g}\psi\left(K_0^{X_g},TX_g,\nabla^{TX_g}\right)\mathrm{Tr}\left[\frac{\omega\left(\nabla^F,g^F\right)}{2}\right] \\ & +\mathcal{T}_{\mathrm{ch},g,0}\left(g^{TX},\nabla^F,g^F\right)+\mathcal{O}(z).\end{aligned}$$

Incidentally, note that if $X_g$ is even-dimensional, $V_{K_0}(X_g)=0$, and that if $X_g$ is odd-dimensional, the integral in the right-hand side of (5.87) vanishes.

## 6. A fundamental closed form

In this section, we construct a closed form $\kappa_{g,K}$ on $]0,1[\times\mathbf{R}_+^*\times\mathbf{R}$. In Section 7, by integrating this form on the boundary of a well chosen three-dimensional domain, we will establish Theorem 5.1. Also we prove here an important Lichnerowicz formula.

This section is organized as follows. In 6.1 and 6.2, using the local families index formalism of [B3], we obtain a closed Chern character form $\kappa_{g,K}$ on $]0,1[\times\mathbf{R}_+^*\times\mathbf{R}$ . This form is expressed in terms of a differential operator $\mathcal{L}$. In Section 6.3, we give a fundamental Lichnerowicz formula for $L_K+\mathcal{L}$. In Section 6.4, we introduce a simple rescaling of the variable $t\in\mathbf{R}_+^*$.

We make the same assumptions and we use the same notation as in Section 5. Also, in the sequel, we fix $g\in G$, and $K\in\mathfrak{z}(g)$.

6.1. *A fundamental superconnection.* Let $\mathcal{M}$ be the set of smooth $G$-invariant metrics on $TX$. Set

$$\widetilde{M}=]0,1[\times\mathcal{M}\times\mathbf{R}\times X,\qquad \widetilde{S}=]0,1[\times\mathcal{M}\times\mathbf{R}. \tag{6.1}$$

The generic point of $\widetilde{S}$ will be denoted $\left(s,g^{TX},u\right)$. Let $q:\widetilde{M}\to\widetilde{S}$ be the obvious projection. Clearly, the vector bundle $TX$ is equipped with the tautological $G$-invariant metric $\widetilde{g}^{TX}$. Also, we fix a $G$-invariant metric $g^F$ on $F$.

Then $\Omega^\cdot(X,F)$ can be considered as a trivial vector bundle on $\widetilde{S}$. It is equipped with the trivial connection $\nabla^{\Omega^\cdot(X,F)}$ over $S$. Also the operator $d^X$ acts fibrewise on $\Omega^\cdot(X,F)$.

*Definition* 6.1. Set

$$\widetilde{A}'=d^X-ui_{K^X}+\nabla^{\Omega^\cdot(X,F)}. \tag{6.2}$$

Then $\widetilde{A}'$ is a superconnection on $\Omega^\cdot(X,F)$.

Let $g^{\Omega^{\cdot}(X,F)}$ be the fibrewise Hermitian product on $\Omega^{\cdot}(X,F)$ defined in (1.15). Let $d^{X,*}$ be the fibrewise adjoint of $d^X$ with respect to $g^{\Omega^{\cdot}(X,F)}$. Set

$$\dot{g}^{TX} = \left(g^{TX}\right)^{-1} d^{\mathcal{M}} g^{TX}. \tag{6.3}$$

Then $\dot{g}^{TX}$ is a 1-form on $\mathcal{M}$ with values in self-adjoint sections of $\mathrm{End}(TX)$. Also $\dot{g}^{TX}$ acts as a derivation on $\Lambda^{\cdot}(T^*X)$. If $e_1, \dots, e_n$ is an orthonormal basis of $TX$ with respect to $g^{TX}$, the action $\dot{g}^{TX}|_{\Lambda^{\cdot}(T^*X)}$ of $\dot{g}^{TX}$ on $\Lambda^{\cdot}(T^*X)$ is given by

$$\dot{g}^{TX}|_{\Lambda^{\cdot}(T^*X)} = -\left\langle \dot{g}^{TX} e_i, e_j \right\rangle e^i \wedge i_{e_j}. \tag{6.4}$$

Set

$$\dot{g}^{\Omega^{\cdot}(X,F)} = \left(g^{\Omega^{\cdot}(X,F)}\right)^{-1} d^{\mathcal{M}} g^{\Omega^{\cdot}(X,F)}. \tag{6.5}$$

One verifies easily that

$$\dot{g}^{\Omega^{\cdot}(X,F)} = \dot{g}^{TX}|_{\Lambda^{\cdot}(T^*X)} + \frac{1}{2}\mathrm{Tr}\left(\dot{g}^{TX}\right). \tag{6.6}$$

Equivalently,

$$\dot{g}^{\Omega^{\cdot}(X,F)} = -\frac{1}{2}\left\langle \dot{g}^{TX} e_i, e_j \right\rangle c(e_i)\, \widehat{c}(e_j). \tag{6.7}$$

Then the adjoint connection $\nabla^{\Omega^{\cdot}(X,F),*}$ to the connection $\nabla^{\Omega^{\cdot}(X,F)}$ is given by

$$\nabla^{\Omega^{\cdot}(X,F),*} = \nabla^{\Omega^{\cdot}(X,F)} + \dot{g}^{\Omega^{\cdot}(X,F)}. \tag{6.8}$$

Recall that $TX$ and $T^*X$ are identified by the metric $g^{TX}$. Let $\widetilde{A}''$ be the superconnection on $\Omega^{\cdot}(X,F)$ given by

$$\widetilde{A}'' = d^{X,*} + uK^{X\prime}\wedge + \nabla^{\Omega^{\cdot}(X,F),*}. \tag{6.9}$$

*Definition* 6.2. Let $\widetilde{A}$ be the superconnection on $\Omega^{\cdot}(X,F)$,

$$\widetilde{A} = (1-s)\,\widetilde{A}' + s\widetilde{A}''. \tag{6.10}$$

Recall that the operators $A_K, B_K$ were defined in (2.95). The curvature $\widetilde{A}^2$ of $\widetilde{A}$ is a section of $\left(\Lambda^{\cdot}\left(T^*\widetilde{S}\right)\widehat{\otimes}\,\mathrm{End}\left(\Omega^{\cdot}(X,F)\right)\right)^{\mathrm{even}}$.

Theorem 6.3. *The following identity holds*:

$$\begin{aligned}\widetilde{A}^2 = &-uL_K - 4s(1-s)\left(B_{uK} + \dot{g}^{\Omega^{\cdot}(X,F)}/2\right)^2 \\ &+ 2ds\left(B_{uK} + \dot{g}^{\Omega^{\cdot}(X,F)}/2\right) + du\left(-(1-s)\,i_{K^X} + sK^{X\prime}\wedge\right).\end{aligned} \tag{6.11}$$

*Proof.* By (6.10), we get

$$\widetilde{A}^2 = (1-s)\,\widetilde{A}'^2 + s\widetilde{A}''^2 - s(1-s)\left(\widetilde{A}'' - \widetilde{A}'\right)^2 + ds\left(\widetilde{A}'' - \widetilde{A}'\right). \tag{6.12}$$

Using (2.96), we get

$$\widetilde{A}'^2 = -uL_K - du\, i_{K^X}, \qquad \widetilde{A}''^2 = -uL_K + du K^{X\prime}\wedge. \tag{6.13}$$

Using (2.95), (6.2), (6.8) and (6.9), we also have

$$\widetilde{A}'' - \widetilde{A}' = 2B_{uK} + \dot{g}^{\Omega^{\cdot}(X,F)}. \tag{6.14}$$

By (6.12)–(6.14), we get (6.11). The proof of our theorem is completed. □

*Definition* 6.4. Set

$$\eta_{g,K} = \mathrm{Tr_s}\left[g \exp\left(-L_K - \widetilde{A}^2\right)\right]. \tag{6.15}$$

Then $\eta_{g,K}$ is an even form on $\widetilde{S}$. Let $\eta_{g,K}^{(0)}$ be its component of degree 0.

THEOREM 6.5. *The form $\eta_{g,K}$ is closed on $\widetilde{S}$. Moreover*

$$\eta_{g,K}^{(0)} = \chi_g(F). \tag{6.16}$$

*Proof.* The first part of our Theorem follows from [B3, Th. 2.6]. Also, by (2.45), (2.49), (2.105) and (6.11), we get (6.16). □

6.2. *Scaling the metric $g^{TX}$.* Set

$$M = ]0,1[\times \mathbf{R}_+^* \times \mathbf{R} \times X, \qquad S = ]0,1[\times \mathbf{R}_+^* \times \mathbf{R}. \tag{6.17}$$

Let $\pi : M \to S$ be the obvious projection. In the sequel, $(s,t,u)$ denotes the generic element of $S$.

Now we fix once and for all $G$-invariant metrics $g^{TX}, g^F$ on $TX, F$. As in (1.35), set

$$g_t^{TX} = \frac{g^{TX}}{t}. \tag{6.18}$$

Let $i : \mathbf{R}_+^* \to \mathcal{M}$ be the embedding $t \to g_t^{TX}$. Then $i$ extends to the obvious embeddings $M \to \widetilde{M}, S \to \widetilde{S}$.

Set

$$\kappa_{g,K} = i^* \eta_{g,K}. \tag{6.19}$$

By Theorem 6.5, $\kappa_{g,K}$ is an even closed form on $S$, and also

$$\kappa_{g,K}^{(0)} = \chi_g(F). \tag{6.20}$$

Let $e_1, \dots, e_n$ be an orthonormal basis of $TX$. Recall that $N$ is the number operator of $\Lambda^{\cdot}(T^*X)\widehat{\otimes} F$. Set

$$\overline{N} = N - \frac{n}{2}. \tag{6.21}$$

One verifies easily that

$$\overline{N} = \frac{1}{2} c\left(e_i\right) \widehat{c}\left(e_i\right). \tag{6.22}$$

*Definition* 6.6. Put

$$\mathcal{L} = t^{\overline{N}/2} i^* \widetilde{A}^2 t^{-\overline{N}/2}. \tag{6.23}$$

Then

$$\kappa_{g,K} = \operatorname{Tr_s}\left[g \exp\left(-L_K - \mathcal{L}\right)\right]. \tag{6.24}$$

Proposition 6.7. *The following identity holds*:

$$\begin{aligned} \mathcal{L} = & -uL_K - 4s\left(1-s\right)t\left(B_{uK/t} + \frac{\overline{N}dt}{2t^{3/2}}\right)^2 \\ & +2ds\sqrt{t}\left(B_{uK/t} + \frac{\overline{N}dt}{2t^{3/2}}\right) + \frac{du}{\sqrt{t}}\left(sK^{X\prime}\wedge - \left(1-s\right) i_{K^X}\right). \end{aligned} \tag{6.25}$$

*Proof.* Observe that

$$i^* \dot{g}^{TX} = -\frac{dt}{t}. \tag{6.26}$$

By (6.7), (6.22), (6.26),

$$i^* \dot{g}^{\Omega^{\cdot}(X,F)} = \frac{\overline{N}dt}{t}. \tag{6.27}$$

Our proposition is now a trivial consequence of Theorem 6.3. □

6.3. *A Lichnerowicz formula.* In the sequel, $e_1, \ldots, e_n$ denotes a locally defined smooth orthonormal basis of $TX$. We use the notation

$$\nabla_{e_i}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes} F,u,2} = \sum_1^n \nabla_{e_i}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes} F,u,2} - \nabla_{\sum_1^n \nabla_{e_i}^{TX} e_i}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes} F,u}. \tag{6.28}$$

The operator in (6.28) does not depend on the choice of the smooth basis. A similar notation will be used for other connections than $\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes} F,u}$.

Let $H$ be the scalar curvature of $X$. Put

$$\begin{aligned} c\left(\omega\left(\nabla^F, g^F\right)\right) &= c\left(e_i\right)\omega\left(\nabla^F, g^F\right)\left(e_i\right), \\ \widehat{c}\left(\omega\left(\nabla^F, g^F\right)\right) &= \widehat{c}\left(e_i\right)\omega\left(\nabla^F, g^F\right)\left(e_i\right), \\ \left|\omega\left(\nabla^F, g^F\right)\right|^2 &= \sum_{i=1}^n \left(\omega\left(\nabla^F, g^F\right)\left(e_i\right)\right)^2. \end{aligned} \tag{6.29}$$

THEOREM 6.8. *The following identity holds*:

(6.30)
$$
\begin{aligned}
L_K+\mathcal{L}=&-s(1-s)t\bigg(\nabla_{e_i}^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,u}-\frac{1}{t}\Big(u+\frac{1-u}{2s(1-s)}\Big)\Big\langle K^X,e_i\Big\rangle\\
&\qquad\qquad+c(e_i)\frac{dt/t}{2\sqrt{t}}-\widehat{c}(e_i)\frac{ds}{2s(1-s)\sqrt{t}}\bigg)^2\\
&+\frac{1-u}{4s(1-s)t}\Big(1-u(2s-1)^2\Big)\Big|K^X\Big|^2+\frac{s(1-s)t}{4}H\\
&+\frac{1-u}{2\sqrt{t}}\Big(-c\Big(K^X\Big)\frac{dt}{t}+\widehat{c}\Big(K^X\Big)\frac{ds}{s(1-s)}\Big)\\
&-c\Big(\omega\Big(\nabla^F,g^F\Big)\Big)\frac{\sqrt{t}ds}{2}+\widehat{c}\Big(\omega\Big(\nabla^F,g^F\Big)\Big)\frac{s(1-s)\sqrt{t}dt}{2t}\\
&-\Big(c\Big(K^X\Big)+(2s-1)\widehat{c}\Big(K^X\Big)\Big)\frac{du}{2\sqrt{t}}\\
&+\frac{s(1-s)t}{8}\Big\langle e_k,R^{TX}(e_i,e_j)e_l\Big\rangle c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_l)\\
&+\frac{1}{4}\Big\langle\nabla_{e_i}^{TX}K^X,e_j\Big\rangle\big((u-1)c(e_i)c(e_j)\\
&+\Big(1-u(2s-1)^2\Big)\widehat{c}(e_i)\widehat{c}(e_j)\Big)\\
&+4s(1-s)t\bigg(\frac{1}{32}\big(-c(e_i)c(e_j)+\widehat{c}(e_i)\widehat{c}(e_j)\big)\omega^2\Big(\nabla^F,g^F\Big)(e_i,e_j)\\
&\qquad+\frac{1}{16}\Big|\omega\Big(\nabla^F,g^F\Big)\Big|^2-\frac{1}{8}c(e_i)\widehat{c}(e_j)\nabla_{e_i}^{F,u}\omega\Big(\nabla^F,g^F\Big)(e_j)\bigg).
\end{aligned}
$$

*Proof.* By (1.32) and (2.100), we get

$$
B_K=-\frac{1}{2}\widehat{c}(e_i)\Big(\nabla_{e_i}^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,u}-\Big\langle K^X,e_i\Big\rangle\Big)+\frac{1}{4}c\Big(\omega\Big(\nabla^F,g^F\Big)\Big). \tag{6.31}
$$

Using (1.3), (1.30), (6.31) and Lichnerowicz's formula, we get

$$
\begin{aligned}
-B_K^2=&-\frac{1}{4}\Big(\nabla_{e_i}^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,u}-\Big\langle K^X,e_i\Big\rangle\Big)^2+\frac{H}{16}\\
&+\frac{1}{32}\Big\langle e_k,R^{TX}(e_i,e_j)e_l\Big\rangle c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_l)\\
&+\frac{1}{4}\Big\langle\nabla_{e_i}^{TX}K^X,e_j\Big\rangle\widehat{c}(e_i)\widehat{c}(e_j)\\
&+\frac{1}{32}\big(-c(e_i)c(e_j)+\widehat{c}(e_i)\widehat{c}(e_j)\big)\omega^2\Big(\nabla^F,g^F\Big)(e_i,e_j)\\
&+\frac{1}{16}\Big|\omega\Big(\nabla^F,g^F\Big)\Big|^2\\
&+\frac{1}{8}\widehat{c}(e_i)c(e_j)\nabla_{e_i}^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,u}\omega\Big(\nabla^F,g^F\Big)(e_j).
\end{aligned} \tag{6.32}
$$

By (1.31),

$$\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,u}_{e_i}\omega\left(\nabla^F,g^F\right)(e_j)=\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,u}_{e_j}\omega\left(\nabla^F,g^F\right)(e_i)\,. \tag{6.33}$$

By (1.3), (2.3), (2.24), (6.32), (6.33), we obtain

$$\begin{aligned}
&(1-u)\,L_K-4s\,(1-s)\,tB^2_{uK/t}\\
&=-s\,(1-s)\,t\left(\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,u}_{e_i}-\frac{1}{t}\left(u+\frac{1-u}{2s\,(1-s)}\right)\left\langle K^X,e_i\right\rangle\right)^2\\
&\quad+\frac{s\,(1-s)}{t}\left(\left(u+\frac{1-u}{2s\,(1-s)}\right)^2-u^2\right)\left|K^X\right|^2+\frac{s\,(1-s)\,t}{4}H\\
&\quad+\frac{s\,(1-s)\,t}{8}\left\langle e_k,R^{TX}\left(e_i,e_j\right)e_l\right\rangle c\left(e_i\right)c\left(e_j\right)\widehat{c}\left(e_k\right)\widehat{c}\left(e_l\right)\\
&\quad+\frac{1}{4}\left\langle\nabla^{TX}_{e_i}K^X,e_j\right\rangle\left((u-1)\,c\left(e_i\right)c\left(e_j\right)+\left(1-u\,(2s-1)^2\right)\widehat{c}\left(e_i\right)\widehat{c}\left(e_j\right)\right)\\
&\quad+4s\,(1-s)\,t\Bigg(\frac{1}{32}\left(-c\left(e_i\right)c\left(e_j\right)+\widehat{c}\left(e_i\right)\widehat{c}\left(e_j\right)\right)\omega^2\left(\nabla^F,g^F\right)\left(e_i,e_j\right)\\
&\qquad\qquad+\frac{1}{16}\left|\omega\left(\nabla^F,g^F\right)\right|^2-\frac{1}{8}c\left(e_i\right)\widehat{c}\left(e_j\right)\nabla^{F,u}_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right)\Bigg).
\end{aligned}\tag{6.34}$$

By Proposition 6.7 and by (6.34), we find that (6.30) holds in degree 0 with respect to the Grassmann variables $ds, dt, du$.

Clearly,

$$sK^{X\prime}\wedge-(1-s)\,i_{K^X}=\frac{1}{2}\left(c\left(K^X\right)+(2s-1)\,\widehat{c}\left(K^X\right)\right). \tag{6.35}$$

By Proposition 6.7 and by (6.35), (6.30) holds also for the terms containing $du$. By (6.31), the term of degree 1 containing $ds$ in the right-hand side of (6.30) is given by

$$\begin{aligned}
ds\Bigg(&-\sqrt{t}\widehat{c}\left(e_i\right)\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,u}_{e_i}+\frac{1}{\sqrt{t}}\left(u+\frac{1-u}{2s\,(1-s)}\right)\widehat{c}\left(K^X\right)\\
&-\frac{1-u}{2s\,(1-s)\sqrt{t}}\widehat{c}\left(K^X\right)+\frac{\sqrt{t}}{2}c\left(\omega\left(\nabla^F,g^F\right)\right)\Bigg)=2ds\sqrt{t}B_{uK/t}.
\end{aligned}\tag{6.36}$$

By Proposition 6.7 and by (6.36), (6.30) holds for the term of degree 1 containing $ds$. Using (1.32), we find that the term of degree 1 containing $dt$ in the right-hand side of (6.30) is given by

$$(6.37)\quad dt\Bigg(\frac{s(1-s)}{\sqrt{t}}c(e_i)\nabla_{e_i}^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,u}-\frac{s(1-s)}{t^{3/2}}\left(u+\frac{1-u}{2s(1-s)}\right)c\left(K^X\right)$$
$$+\frac{1-u}{2t^{3/2}}c\left(K^X\right)-\frac{s(1-s)}{2\sqrt{t}}\widehat{c}\left(\omega\left(\nabla^F,g^F\right)\right)\Bigg)$$
$$=2s(1-s)\,dt\left(\frac{A^X}{\sqrt{t}}-\frac{u}{2t^{3/2}}c\left(K^X\right)\right).$$

Moreover, in the right-hand side of (6.25), the term of degree 1 containing $dt$ is given by

$$(6.38)\quad -4s(1-s)\,t\left[B_{uK/t},\frac{\overline{N}}{2t^{3/2}}dt\right]=2s(1-s)\,dt\left(\frac{A^X}{\sqrt{t}}-\frac{u}{2t^{3/2}}c\left(K^X\right)\right),$$

which coincides with (6.37). Therefore (6.30) also holds for this term. Finally, in the right-hand side of (6.30), the term containing $dsdt$ is given by $c(e_i)\,\widehat{c}(e_i)\,dsdt/2t=\overline{N}dsdt/t$, which is equal to the corresponding term in the right-hand side of (6.25).

The proof of our theorem is completed. □

6.4. *A rescaling of the $t$-coordinate.* Set

$$(6.39)\qquad t'=s(1-s)\,t.$$

Then

$$(6.40)\qquad \frac{dt'}{t'}=\frac{dt}{t}+\frac{ds}{s}-\frac{ds}{1-s}.$$

Let $d$ be the de Rham operator acting in the $s,t,u$ variables.

PROPOSITION 6.9. *The following identities hold*:

$$(6.41)\quad \left(\frac{s}{1-s}t\right)^{\overline{N}/2}i^*(1-s)\,\widetilde{A}'\left(\frac{s}{1-s}t\right)^{-\overline{N}/2}$$
$$=\sqrt{t'}d^X-\frac{(1-s)^2}{\sqrt{t'}}ui_{K^X}+(1-s)\left(d-\frac{\overline{N}}{2}\left(\frac{dt'}{t'}+2\frac{ds}{1-s}\right)\right),$$
$$\left(\frac{s}{1-s}t\right)^{\overline{N}/2}i^*s\widetilde{A}''\left(\frac{s}{1-s}t\right)^{-\overline{N}/2}$$
$$=\sqrt{t'}d^{X,*}+\frac{s^2}{\sqrt{t'}}uK^{X\prime}\wedge+s\left(d+\frac{\overline{N}}{2}\left(\frac{dt'}{t'}-2\frac{ds}{s}\right)\right).$$

*In particular,*

(6.42)
$$\left(\frac{s}{1-s}t\right)^{\overline{N}/2} i^*\widetilde{A}\left(\frac{s}{1-s}t\right)^{-\overline{N}/2} = \sqrt{t'}d^X - \frac{(1-s)^2}{\sqrt{t'}}ui_{K^X} + \sqrt{t'}d^{X,*} + \frac{s^2}{\sqrt{t'}}uK^{X\prime}\wedge + d + \frac{\overline{N}}{2}\left((2s-1)\frac{dt'}{t'} - 4ds\right).$$

*Proof.* Clearly,

(6.43)
$$\begin{aligned}\left(\frac{s}{1-s}t\right)^{\overline{N}/2}(1-s)\left(d^X - ui_{K^X}\right)\left(\frac{s}{1-s}t\right)^{-\overline{N}/2} &= \sqrt{t'}d^X - \frac{(1-s)^2}{\sqrt{t'}}ui_{K^X},\\ \left(\frac{s}{1-s}t\right)^{\overline{N}/2} s\left(td^{X,*} + \frac{u}{t}K^{X\prime}\wedge\right)\left(\frac{s}{1-s}t\right)^{-\overline{N}/2} &= \sqrt{t'}d^{X,*} + \frac{s^2u}{t'}K^{X\prime}\wedge,\\ \left(\frac{s}{1-s}t\right)^{\overline{N}/2} d\left(\frac{s}{1-s}t\right)^{-\overline{N}/2} &= d - \frac{\overline{N}}{2}\left(\frac{dt'}{t'} + 2\frac{ds}{1-s}\right).\end{aligned}$$

From (6.41), (6.43), we get (6.42). □

THEOREM 6.10. *The following identity holds*:

(6.44)
$$\begin{aligned}L_K + \mathcal{L} = &-t'\Bigg(\nabla_{e_i}^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,u} - \frac{1}{t'}\left(s(1-s)u + \frac{1-u}{2}\right)\left\langle K^X, e_i\right\rangle\\ &+ \frac{\sqrt{s(1-s)}}{2\sqrt{t'}}c(e_i)\frac{dt}{t} - \frac{1}{2\sqrt{s(1-s)t'}}\widehat{c}(e_i)\,ds\Bigg)^2\\ &+\frac{1-u}{4t'}\left(1-u(2s-1)^2\right)\left|K^X\right|^2\\ &+\frac{t'}{4}H + \frac{1-u}{2\sqrt{t'}}\sqrt{s(1-s)}\left(-c\left(K^X\right)\frac{dt}{t} + \widehat{c}\left(K^X\right)\frac{ds}{s(1-s)}\right)\\ &-c\left(\omega\left(\nabla^F, g^F\right)\right)\frac{\sqrt{t'}ds}{2\sqrt{s(1-s)}} + \widehat{c}\left(\omega\left(\nabla^F, g^F\right)\right)\frac{\sqrt{s(1-s)t'}}{2}\frac{dt}{t}\\ &-\left(c\left(K^X\right) + (2s-1)\widehat{c}\left(K^X\right)\right)\sqrt{s(1-s)}\frac{du}{2\sqrt{t'}}\\ &+\frac{t'}{8}\left\langle e_k, R^{TX}(e_i,e_j)e_l\right\rangle c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_l)\\ &+\frac{1}{4}\left\langle \nabla_{e_i}^{TX}K^X, e_j\right\rangle\left((u-1)c(e_i)c(e_j) + \left(1-u(2s-1)^2\right)\widehat{c}(e_i)\widehat{c}(e_j)\right)\\ &+4t'\Bigg(\frac{1}{32}\left(-c(e_i)c(e_j) + \widehat{c}(e_i)\widehat{c}(e_j)\right)\omega^2\left(\nabla^F, g^F\right)(e_i,e_j)\\ &+\frac{1}{16}\left|\omega\left(\nabla^F, g^F\right)\right|^2 - \frac{1}{8}c(e_i)\widehat{c}(e_j)\nabla_{e_i}^{F,u}\omega\left(\nabla^F, g^F\right)(e_j)\Bigg).\end{aligned}$$

*Proof.* This is a trivial consequence of Theorem 6.8. □

*Remark* 6.11. A most remarkable aspect of formula (6.44) is that if one makes $ds = 0$, the obtained formula extends by continuity at $s = 0$ and $s = 1$, and that the terms containing $dt$ or $du$ then vanish identically. This fact will play a crucial role in Section 7.

## 7. A proof of the comparison formula

In this section, we establish the main result result of this paper, stated in the Theorem 5.1. The idea is to integrate the form $\kappa_{g,K}$ of Section 6.2 on an adequately chosen 2-dimensional polyhedral domain in $\mathbf{R}^3$, which is then 'pushed' to infinity. This strategy is closely related to the strategy used in [BL], [B11], [BGo1] in the context of holomorphic torsion, and in [BZ1, 2], [BGo2] for de Rham torsion. The main difference is that in the above references, the polyhedron was just 1-dimensional.

This section is organized as follows. In 7.1, we show that the form $\kappa_{g,K}$ can be continued to $s = 0$ and $s = 1$. In 7.2, we construct the polyhedron $\Gamma$, which depends on two parameters $a, A$ with $0 < a < 1 < A < +\infty$, on which the integral of $\kappa_{g,K}$ vanishes identically. So we get an identity written in the form $\sum_{k=1}^{3} I_k^0 = 0$. In 7.3, we state five intermediate results, whose proof is delayed to Sections 8–12. In 7.4, we study the asymptotics of the $I_k^0$ as $A \to +\infty, a \to 0$, by using the above results. Finally in 7.5, we obtain an identity, which is shown to be equivalent to Theorem 5.1.

Here, we make the same assumptions and use the same notation as in Sections 5 and 7. Again, we fix $g \in G$ and $K \in \mathfrak{z}(g)$.

7.1. *An extension of the form $\kappa_{g,K}$.* Put

$$t' = s(1-s)\,t, \qquad u = 1 - \frac{t'}{v}. \tag{7.1}$$

In the sequel, $(s, v, t')$ denotes the generic element of $\mathbf{R}^3$. Also we give to $\mathbf{R}^3$ its natural orientation. Let $j$ be the embedding of the affine hyperplanes $s = 0$ and $s = 1$ into $\mathbf{R}^3$.

Clearly, the form $\kappa_{g,K}$ is well defined for $(s, v, t') \in ]0, 1[\times \mathbf{R} \times \mathbf{R}_+^*$.

Theorem 7.1. *The form $\kappa_{g,K}$ extends to a closed smooth form on $\mathbf{R} \times \mathbf{R} \times \mathbf{R}_+^*$. Moreover,*

$$j^* \kappa_{g,K}^{(2)} = 0. \tag{7.2}$$

*Proof.* Clearly, we only need to establish our theorem for the component of degree 2, since $\kappa_{g,K}^{(0)}$ is necessarily a constant (which, by (6.20), is equal to $\chi_g(F)$).

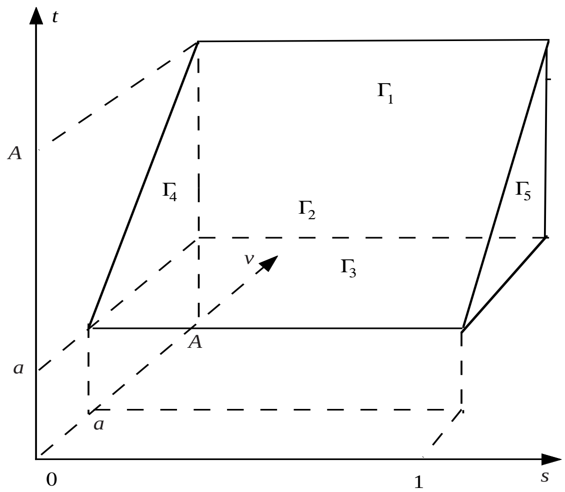


Figure 7.1

We first consider the component of $\kappa_{g,K}$ which contains the 2-form $dt'du$, i.e. which is obtained from $\kappa^{(2)}_{g,K}$ by making $ds = 0$. By (6.40),

$$\frac{dt}{t} = \frac{dt'}{t'} - \frac{ds}{s} + \frac{ds}{1-s}. \tag{7.3}$$

Observe that in (6.44), $\sqrt{s\,(1-s)}$ is a factor of both $dt/t$ and $du$. This guarantees that the component of $\kappa^{(2)}_{g,K}$ which contains $dt'du$ extends to a smooth form on $\mathbf{R} \times \mathbf{R} \times \mathbf{R}^*_+$, and that its restriction vanishes on $s = 0$ or $s = 1$.

Now we consider the components of $\kappa^{(2)}_{g,K}$ which contain either $dsdt'$ or $dsdu$. These components are unchanged by the rescaling

$$ds \to \sqrt{s\,(1-s)}ds, dt' \to dt'/\sqrt{s\,(1-s)}, du \to du/\sqrt{s\,(1-s)}.$$

One verifies easily that under this rescaling, the right-hand side of (6.44) still extends into a smooth function of its parameters.

Finally since the form $\kappa_{g,K}$ is closed on its domain of definition, i.e. for $s \in ]0,1[$, by analyticity in the variable $s$, it is still closed for $s \in \mathbf{R}$. The proof of our theorem is completed. □

7.2. *An application of Stokes's formula.* Take $a, A \in \mathbf{R}^*_+$ such that $0 < a < 1 \leq A < +\infty$. Let $\Gamma = \Gamma_{a,A}$ be the oriented polyhedron indicated in Figure 7.1. The polyhedron $\Gamma$ has five oriented faces $\Gamma_1, \ldots, \Gamma_5$. To avoid any ambiguity, we mention that $\Gamma_1$ is the face with $v = A$, and $\Gamma_2$ the face with $t' = v$.

Set

$$I^0_k = \int_{\Gamma_k} \kappa_{g,K},\ 1 \leq j \leq 3. \tag{7.4}$$

THEOREM 7.2. *The following identity holds*:

$$\sum_{k=1}^{3} I_k^0 = 0. \tag{7.5}$$

*Proof.* Since the form $\kappa_{g,K}$ is closed,

$$\int_\Gamma \kappa_{g,K} = 0. \tag{7.6}$$

By Theorem 7.1, for $k = 4$ and $k = 5$,

$$\int_{\Gamma_k} \kappa_{g,K} = 0. \tag{7.7}$$

From (7.6), (7.7), we get (7.5). The proof of our theorem is completed. □

In the sequel, we will make $A \to +\infty, a \to 0$ in this order in the identity (7.5), and we will ultimately obtain Theorem 5.1.

7.3. *Five intermediate results.* In the sequel, we fix $K_0 \in \mathfrak{z}(g)$. For $z \in \mathbf{R}^*$, put $K = zK_0$.

If $\alpha \in \Lambda^{\cdot}(\mathbf{R}^3)$, let $\alpha^{(0)} \in \mathbf{R}, \alpha^{ds} \in \Lambda^1(\mathbf{R}^3), \dots, \alpha^{dsdt'} \in \Lambda^2(\mathbf{R}^3) \dots$ be the real multiples of $1, ds, \dots, dsdt' \dots$ such that

$$\alpha = \alpha^{(0)} + \alpha^{ds} + \dots + \alpha^{dsdt'} + \dots . \tag{7.8}$$

Similarly for $0 \le j \le 3$, $\alpha^{(j)}$ denotes the component of $\alpha$ in $\Lambda^j(\mathbf{R}^3)$.

For $t > 0$, we define the form $\alpha_{K,t}^{X_g}, \beta_{K,t}^{X_g}$ on $X_g$ as in (3.8).

*Definition* 7.3. For $h \in \mathbf{R}, v \in \mathbf{R}_+^*$, set

$$\begin{aligned} m_{h,v} &= -2 \int_{X_g} e_K\left(TX_g, \nabla^{TX_g}\right) \beta_{K,v/2}^{X_g} \\ &\qquad \mathrm{Tr}\left[g \frac{\omega\left(\nabla^F, g^F\right)}{2} \exp\left(h\omega^2\left(\nabla^F, g^F\right)/2\pi\right)\right], \\ n_v &= \sqrt{\pi} \int_{X_g} \mathrm{Tr}^{F|_{X_g}}[g] \frac{v}{2} \beta_{K,v/2}^{X_g} \int^{\widehat{B}} \widehat{K^{X_g\prime}} \exp\left(-\dot{R}_K^{TX_g}/2\pi\right), \\ o_{h,v} &= -\int_{X_g} \alpha_{K,v/2}^{X_g} \frac{1}{4\sqrt{\pi}} \mathrm{Tr}\left[g\omega\left(\nabla^F, g^F\right) \exp\left(h\omega^2\left(\nabla^F, g^F\right)/2\pi\right)\right] \\ &\qquad \int^{\widehat{B}} \widehat{K^{X_g\prime}} \exp\left(-\dot{R}_K^{TX_g}/2\pi\right). \end{aligned} \tag{7.9}$$

Observe that if $\varepsilon_{K,v/2}$ is the invariant of $X_g$ associated to $K^{X_g}$ as in (3.109), then

$$n_v = -\frac{1}{\sqrt{2}} \mathrm{Tr}^{F|_{X_g}}[g] (v/2)^{3/2} \varepsilon_{K,v/2}. \tag{7.10}$$

Moreover, if $X_g$ is odd-dimensional, $m_{h,v}$ vanishes, and if $X_g$ is even-dimensional, $n_v$ and $o_{h,v}$ vanish.

By (3.114), (7.10), for $v \in ]0,1]$,

$$n_v = \mathcal{O}\left(v^2\right), \tag{7.11}$$

and moreover, $n_v$ is uniformly bounded on $\mathbf{R}_+$. Using (3.14), for bounded $h \in \mathbf{R}$, for $v \in [0,1]$,

$$o_{h,v} = \mathcal{O}\left(v\right), \tag{7.12}$$

and $o_{h,v}$ remains uniformly bounded for $v \in \mathbf{R}_+$.

In the sequel, we will use the notation

$$w = \frac{4v^2}{4s\left(1-s\right)v + \left(2s-1\right)^2 a}. \tag{7.13}$$

Observe that there exist $C > 0, C' > 0$ such that for $(s,v) \in ]0,1[\times[a,1]$,

$$Cv \leq w \leq C' \inf\left\{\frac{v}{s\left(1-s\right)}, \frac{v^2}{a}\right\}. \tag{7.14}$$

Note that $w$ depends explicitly on $a$. We will often write

$$w = w\left(a,s,v\right).$$

Let $i_a : [0,1] \times \mathbf{R}_+^* \to [0,1] \times \mathbf{R}_+^*$ be given by $i_a\left(s,v\right) = \left(s, av\right)$. Let $j_a, j'_a$ : $[0,1] \times \mathbf{R}_+ \to [0,1] \times \mathbf{R}_+$ be given by $j_a\left(s,v\right) = \left(\sqrt{a}s, \sqrt{a}v\right), j'_a\left(s,v\right) = \left(1-\sqrt{a}s, \sqrt{a}v\right)$. In the sequel, $k_a$ will denote either $j_a$ or $j'_a$.

If $\gamma$ is a smooth form on $\mathbf{R}^3$, $\gamma_{t'=a}$ denotes its restriction to the hyperplane $(t'=a)$. So $\gamma_{t'=a}$ is a form on this hyperplane.

Recall that $\chi_g\left(F\right), \chi'_g\left(F\right)$ were defined in (1.13).

Theorem 7.4. *There exist $C > 0, \beta > 0$ such that for $z \in \mathbf{R}, |z| \leq \beta$, $t' \in [1,+\infty[, s \in [0,1], u \in [0,1]$,*

$$\begin{aligned} &\left|t'\mathrm{Tr_s}\left[g\exp\left(-L_K - \mathcal{L}\right)\right]^{dsdt'} + \left(\chi'_g\left(F\right) - \frac{n}{2}\chi_g\left(F\right)\right)dsdt'\right| \leq \frac{C}{\sqrt{t'}}, \\ &\left|\mathrm{Tr_s}\left[g\exp\left(-L_K - \mathcal{L}\right)\right]^{dsdu}\right| \leq \frac{C}{\sqrt{t'}}. \end{aligned} \tag{7.15}$$

Theorem 7.5. *For $z \in \mathbf{R}^*$ and $|z|$ small enough, for any $(s,v) \in ]0,1[\times\mathbf{R}_+^*$, as $a \to 0$,*

$$\mathrm{Tr_s}\left[g\exp\left(-L_K - \mathcal{L}\right)\right]^{(2)}_{t'=a} \to m_{s(1-s),v/s(1-s)}\frac{dsdv}{v}. \tag{7.16}$$

*For $z \in \mathbf{R}^*$ and $|z|$ small enough, there exists $C > 0$ such that for $a \in ]0,1], (s,v) \in [0,1] \times [1,+\infty[$,*

$$\left|\mathrm{Tr_s}\left[g\exp\left(-L_K - \mathcal{L}\right)\right]^{(2)}_{t'=a}\right| \leq \frac{C}{v^2}. \tag{7.17}$$

THEOREM 7.6. *For $z \in \mathbf{R}^*$ and $|z|$ small enough, as $a \to 0$,*

$$k_a^* \mathrm{Tr_s} \left[g \exp\left(-L_K - \mathcal{L}\right)\right]^{(2)}_{t'=a} \to 2n_{4v^2/4sv+1} \frac{dsdv}{v^3}. \tag{7.18}$$

THEOREM 7.7. *For $z \in \mathbf{R}^*$ and $|z|$ small enough, as $a \to 0$,*

$$i_a^* \mathrm{Tr_s} \left[g \exp\left(-L_K - \mathcal{L}\right)\right]^{(2)}_{t'=a} \to 0. \tag{7.19}$$

In the next theorem, we emphasize that $w$ still depends explicitly on $a$.

THEOREM 7.8. *For $z \in \mathbf{R}^*$ and $|z|$ small enough, there exist $C > 0$, $\gamma \in \left]0, 1/2\right]$ such that for $(s, v) \in \left]0, 1\right[ \times [a, 1]$,*

$$\begin{aligned} &\Bigg| \mathrm{Tr_s} \left[g \exp\left(-L_K - \mathcal{L}\right)\right]^{(2)}_{t'=a} \\ &\quad - \left( \frac{m_{v/w,w}}{v} + \left(1 + (2s-1)^2\right) \sqrt{a} \frac{n_w}{v^3} + (2s-1) \sqrt{a} \frac{o_{v/w,w}}{v^2} \right) dsdv \Bigg| \\ &\quad \leq C \left(\frac{a}{v}\right)^\gamma \frac{w}{v^2}. \end{aligned} \tag{7.20}$$

*In particular, for $z \in \mathbf{R}^*$, and $|z|$ small enough, given $\varepsilon \in \left]0, 1/2\right[$, there exist $C > 0, \gamma \in \left]0, 1\right]$ such that for $a \in \left]0, 1\right], (s, v) \in [\varepsilon, 1-\varepsilon] \times [a, 1]$,*

$$\left| \mathrm{Tr_s} \left[g \exp\left(-L_K - \mathcal{L}\right)\right]^{(2)}_{t'=a} - m_{s(1-s), v/s(1-s)} \frac{dsdv}{v} \right| \leq \frac{C}{v} \left(a/v\right)^\gamma. \tag{7.21}$$

*Remark* 7.9. Theorem 7.4 will be proved in Section 8, Theorem 7.5 in Section 9, Theorem 7.6 in Section 10, Theorem 7.7 in Section 11, and Theorem 7.8 in Section 12.

We claim that (7.20) in Theorem 7.8 implies (7.16) in Theorem 7.5, and also Theorem 7.6. In fact, as $a \to 0$,

$$\begin{aligned} m_{v/w,w} &\to m_{s(1-s), v/s(1-s)}, \\ \sqrt{a} n_w / v^3 &\to 0, \\ \sqrt{a} \frac{o_{v/w,w}}{v^2} &\to 0, \\ \sqrt{a} m_{\sqrt{a}v/w\left(a, \sqrt{a}s, \sqrt{a}v\right), w\left(a, \sqrt{a}s, \sqrt{a}v\right)} &\to 0, \\ \left(1 + \left(2\sqrt{a}s - 1\right)^2\right) n_{w\left(a, \sqrt{a}s, \sqrt{a}v\right)} &\to 2n_{4v^2/(4sv+1)}, \\ \left(2\sqrt{a}s - 1\right) \sqrt{a} o_{\sqrt{a}v/w\left(a, \sqrt{a}s, \sqrt{a}v\right), w\left(a, \sqrt{a}s, \sqrt{a}v\right)} &\to 0, \end{aligned} \tag{7.22}$$

from which the above implications follow immediately. Also, by (3.14), (3.16), (3.114), (7.10),

$$\begin{aligned} m_{av/w(a,s,av), w(a,s,av)} &\to 0, \\ n_{w(a,s,av)} / a^{3/2} &\to 0, \\ o_{av/w(a,s,av), w(a,s,av)} / \sqrt{a} &\to 0. \end{aligned} \tag{7.23}$$

By (7.20) in Theorem 7.8 and by (7.23), we deduce that given $(s,v) \in ]0,1[\times [1,+\infty[$, as $a \to 0$, $i_a^* \mathrm{Tr_s}\left[g \exp\left(-L_K - \mathcal{L}\right)\right]$ remains uniformly bounded. Of course, Theorem 7.7 says that as $a \to 0$, these forms converge to 0.

Also (7.21) is an easy consequence of (7.20). In fact, take $\varepsilon \in ]0, 1/2[$, and assume that $a \in ]0,1], (s,v) \in [\varepsilon, 1-\varepsilon] \times [a,1]$. By (7.14), $v/w, w/v$ and $w$ remain uniformly bounded. Using (7.20), (7.11), (7.12), we get (7.21).

Let us point out that we have chosen to present the proofs of Theorems 7.5 and 7.6 before proving Theorem 7.8, because their proofs are easier, and also to introduce more naturally the various tools which are needed in the proof of Theorem 7.8. Finally note that (7.21) is not explicitly used later in the paper, and is only given for completeness.

7.4. *The asymptotics of the $I_k^0$.* Now we study the $I_k^0$. It will be understood in the sequel that in all our statements, $z \in \mathbf{R}^*$ will be such that $|z|$ is small enough.

1) The term $I_1^0$. We orient the plane containing $\Gamma_1$ by the 2-form $dsdt'$. As a part of $\Gamma$, $\Gamma_1$ inherits the opposite orientation. Therefore,

$$\text{(7.24)} \quad I_1^0 = -\int_{\substack{(s,t') \in [0,1] \times [a,A] \\ u = 1 - t'/A}} \Big[ \mathrm{Tr_s}\left[g \exp\left(-L_K - \mathcal{L}\right)\right] + \left(h'(0) - h'\left(i\sqrt{t'}\right)\right)\left(\chi_g'(F) - \frac{n}{2}\chi_g(F)\right)\frac{dsdt'}{t'}\Big] + \int_a^A \left(h'(0) - h'\left(i\sqrt{t'}\right)\right)\frac{dt'}{t'}\left(\chi_g'(F) - \frac{n}{2}\chi_g(F)\right).$$

$\alpha)$ $A \to +\infty$. When integrating the form $\mathrm{Tr_s}\left[g \exp\left(-L_K - \mathcal{L}\right)\right]$ in the right-hand side of (7.24), $du$ should be replaced by $-dt'/A$. Using Theorem 7.4 and dominated convergence, we find that as $A \to +\infty$,

$$\text{(7.25)} \quad I_1^0 - \left(\chi_g'(F) - \frac{n}{2}\chi_g(F)\right)\log(A) \to I_1^1 = -\int_{\substack{(s,t') \in [0,1] \times [a,+\infty[ \\ u=1}} \Big[ \mathrm{Tr_s}\left[g \exp\left(-L_K - \mathcal{L}\right)\right] + \left(h'(0) - h'\left(i\sqrt{t'}\right)\right)\left(\chi_g'(F) - \frac{n}{2}\chi_g(F)\right)\frac{dsdt'}{t'}\Big] + \left[\int_a^1 \left(h'(0) - h'\left(i\sqrt{t'}\right)\right)\frac{dt'}{t'} - \int_1^{+\infty} h'\left(i\sqrt{t'}\right)\frac{dt'}{t'}\right]\left(\chi_g'(F) - \frac{n}{2}\chi_g(F)\right).$$

By Proposition 6.7,

(7.26)
$$\left(L_K+\mathcal{L}\right)|_{u=1}=-4t'\left(B_{\frac{s(1-s)}{t'}K}+\frac{\overline{N}dt}{2t^{3/2}}\right)^2+2ds\sqrt{t}\left(B_{\frac{s(1-s)}{t'}K}+\frac{\overline{N}dt}{2t^{3/2}}\right).$$

By (7.26), and the fact that supertraces vanish on supercommutators,

(7.27)
$$\operatorname{Tr_s}\left[g\exp\left(-L_K-\mathcal{L}\right)\right]^{dsdt'}_{u=1}=\operatorname{Tr_s}\Bigg[g\exp\left(\left(2\sqrt{t'}B_{\frac{s(1-s)}{t'}K}+\overline{N}\sqrt{s\left(1-s\right)}\frac{dt}{t}\right)^2\right)$$
$$\left(2\sqrt{t'}B_{\frac{s(1-s)}{t'}K}+\overline{N}\sqrt{s\left(1-s\right)}\frac{dt}{t}\right)\Bigg]^{dt'}\frac{ds}{\sqrt{s\left(1-s\right)}}.$$

Now, using (7.3), we observe that, in (7.27), $dt/t$ can be replaced by $dt'/t'$. By making the transformation
$$ds\to\sqrt{s\left(1-s\right)}ds,dt'\to\frac{dt'}{\sqrt{s\left(1-s\right)}},$$
from (7.27), we get

$$\begin{aligned}\text{(7.28)}\quad \operatorname{Tr_s}\left[g\exp\left(-L_K-\mathcal{L}\right)\right]^{dsdt'}_{u=1}&=\operatorname{Tr_s}\left[gh\left(2\sqrt{t'}B_{\frac{s(1-s)}{t'}K}+\overline{N}\frac{dt'}{t'}\right)\right]^{dt'}ds\\&=\operatorname{Tr_s}\left[\overline{N}gh'\left(2\sqrt{t'}B_{\frac{s(1-s)}{t'}K}\right)\right]\frac{dt'}{t'}ds.\end{aligned}$$

By (7.25), (7.28), we obtain

(7.29)
$$\begin{aligned}I_1^1=\int_0^1ds\Bigg\{\int_{4a}^{+\infty}\Bigg[&\operatorname{Tr_s}\left[g\overline{N}h'\left(\sqrt{t}B_{\frac{4s(1-s)}{t}K}\right)\right]\\&-\left(h'\left(0\right)-h'\left(i\sqrt{t}/2\right)\right)\left(\chi'_g\left(F\right)-\frac{n}{2}\chi_g\left(F\right)\right)\Bigg]\frac{dt}{t}\Bigg\}\\+\Bigg(\int_{4a}^1\left(h'\left(0\right)-h'\left(i\sqrt{t}/2\right)\right)\frac{dt}{t}&-\int_1^{+\infty}h'\left(i\sqrt{t}/2\right)\frac{dt}{t}+2\log\left(2\right)\Bigg)\\\left(\chi'_g\left(F\right)-\frac{n}{2}\chi_g\left(F\right)\right).\end{aligned}$$

$\beta)$ $a\to 0$. By Theorem 2.14, as $a\to0$,

$$\begin{aligned}\text{(7.30)}\quad I_1^1\to I_1^2=\int_0^1ds\Bigg\{\int_0^{+\infty}\Bigg[&\operatorname{Tr_s}\left[g\overline{N}h'\left(\sqrt{t}B_{\frac{4s(1-s)}{t}K}\right)\right]\\&-\left(h'\left(0\right)-h'\left(i\sqrt{t}/2\right)\right)\left(\chi'_g\left(F\right)-\frac{n}{2}\chi_g\left(F\right)\right)\Bigg]\frac{dt}{t}\Bigg\}\end{aligned}$$

$$+\left(\int_0^1 \left(h'(0)-h'\left(i\sqrt{t}/2\right)\right)\frac{dt}{t} - \int_1^{+\infty} h'\left(i\sqrt{t}/2\right)\frac{dt}{t}+2\log(2)\right)\left(\chi_g'(F)-\frac{n}{2}\chi_g(F)\right).$$

$\gamma$) Evaluation of $I_1^2$.

THEOREM 7.10. *The following identity holds*:

$$I_1^2 = -2\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX},\nabla^F,g^F\right)+\left(2-\Gamma'(1)\right)\left(\chi_g'(F)-\frac{n}{2}\chi_g(F)\right). \tag{7.31}$$

*Proof.* A trivial computation shows that

$$\int_0^1 \left(h'\left(i\sqrt{t}/2\right)-h'(0)\right)\frac{dt}{t}+\int_1^{+\infty} h'\left(i\sqrt{t}/2\right)\frac{dt}{t}=\Gamma'(1)+2\left(\log(2)-1\right). \tag{7.32}$$

Moreover, by proceeding as in the proof of (2.49), one finds that

$$\mathrm{Tr_s}\left[gh'\left(\sqrt{t}B_{\frac{4s(1-s)}{t}K}\right)\right]=\chi_g(F). \tag{7.33}$$

Our theorem now follows from (2.111), (2.117), (2.118) and from (7.30), (7.32), and (7.33). □

2) The term $I_2^0$. We still orient $\Gamma_2$ by the form $dsdt'$. As a part of $\Gamma$, $\Gamma_2$ inherits precisely this orientation. Then

$$I_2^0 = \int_{\substack{(s,t')\in[0,1]\times[a,A]\\ u=0}}\Bigg[\mathrm{Tr_s}\left[g\exp\left(-L_K-\mathcal{L}\right)\right] + \left(h'(0)-h'\left(i\sqrt{t'}\right)\right)\left(\chi_g'(F)-\frac{n}{2}\chi_g(F)\right)\frac{dsdt'}{t'}\Bigg] - \int_a^A \left(h'(0)-h'\left(i\sqrt{t'}\right)\right)\frac{dt'}{t'}\left(\chi_g'(F)-\frac{n}{2}\chi_g(F)\right). \tag{7.34}$$

$\alpha$) $A\to+\infty$. By Proposition 6.7,

$$\mathcal{L}|_{u=0} = -4t'\left(B^X+\frac{\overline{N}dt}{2t^{3/2}}\right)^2+2ds\sqrt{t}\left(B^X+\frac{\overline{N}dt}{2t^{3/2}}\right). \tag{7.35}$$

By proceeding as in (7.26)–(7.28), we get

$$\mathrm{Tr_s}\left[g\exp\left(-L_K-\mathcal{L}\right)\right]^{dsdt'}|_{u=0} = \mathrm{Tr_s}\left[ge^K\overline{N}h'\left(2\sqrt{t'}B^X\right)\right]\frac{dt'}{t'}ds. \tag{7.36}$$

Using Theorem 1.6 or Theorem 7.4, and also (7.34), (7.36), we see that as $A \to +\infty$,

$$I_2^0 + \left(\chi_g'(F) - \frac{n}{2}\chi_g(F)\right)\log(A) \to I_2^1 = -\int_{4a}^{+\infty}\Big[\operatorname{Tr_s}\left[ge^K\overline{N}h'\left(\sqrt{t}B^X\right)\right] - \left(h'(0) - h'\left(i\sqrt{t}/2\right)\right)\left(\chi_g'(F) - \frac{n}{2}\chi_g(F)\right)\Big]\frac{dt}{t} - \left(\int_a^1\left(h'(0) - h'\left(i\sqrt{t}\right)\right)\frac{dt}{t} - \int_1^{+\infty} h'\left(i\sqrt{t}\right)\frac{dt}{t}\right)\left(\chi_g'(F) - \frac{n}{2}\chi_g(F)\right). \tag{7.37}$$

$\beta)$ $a \to 0$. By Theorem 2.14, as $a \to 0$,

$$I_2^1 \to I_2^2 = -\int_0^{+\infty}\Big[\operatorname{Tr_s}\left[ge^K\overline{N}h'\left(\sqrt{t}B^X\right)\right] - \left(h'(0) - h'\left(i\sqrt{t}/2\right)\right)\left(\chi_g'(F) - \frac{n}{2}\chi_g(F)\right)\Big]\frac{dt}{t} - \left(\int_0^1\left(h'(0) - h'\left(i\sqrt{t}/2\right)\right)\frac{dt}{t} - \int_1^{+\infty} h'\left(i\sqrt{t}/2\right)\frac{dt}{t} + 2\log(2)\right)\left(\chi_g'(F) - \frac{n}{2}\chi_g(F)\right). \tag{7.38}$$

$\gamma)$ Evaluation of $I_2^2$.

Theorem 7.11. *The following identity holds*:

$$I_2^2 = 2\mathcal{T}_{\mathrm{ch},ge^K,0}\left(g^{TX}, \nabla^F, g^F\right) + \left(\Gamma'(1) - 2\right)\left(\chi_g'(F) - \frac{n}{2}\chi_g(F)\right). \tag{7.39}$$

*Proof.* Our identity follows from (1.39), (1.42), from (7.32) and from (7.38). □

3) The term $I_3^0$. We orient $\Gamma_3$ by the the 2-form $dsdv$. As a part of $\Gamma$, $\Gamma_3$ inherits the opposite orientation. Then

$$I_3^0 = -\int_{\substack{(s,v)\in[0,1]\times[a,A]\\ t'=a}} \operatorname{Tr_s}\left[g\exp\left(-L_K - \mathcal{L}\right)\right]. \tag{7.40}$$

$\alpha)$ $A \to +\infty$. As $A \to +\infty$, using (6.44), we find easily that

$$I_3^0 \to I_3^1 = -\int_{\substack{(s,v)\in[0,1]\times[a,+\infty[\\ t'=a}} \operatorname{Tr_s}\left[g\exp\left(-L_K - \mathcal{L}\right)\right]. \tag{7.41}$$

$\underline{\beta)\ a \to 0}$. Set

(7.42)
$$
\begin{aligned}
J_1 &= -\int_{(s,v)\in[0,1]\times[a,1]} m_{v/w,w}\frac{dsdv}{v},\\
J_2 &= -\int_{(s,v)\in[0,1]\times[a,1]} \left(1+(2s-1)^2\right) n_w \frac{\sqrt{a}dsdv}{v^3},\\
J_3 &= -\int_{(s,v)\in[0,1]\times[a,1]} o_{v/w,w}\frac{\sqrt{a}dsdv}{v^2},\\
J_4 &= -\int_{\substack{(s,v)\in[0,1]\times[1,+\infty[\\ t'=a}} \mathrm{Tr_s}\left[g\exp\left(-L_K-\mathcal{L}\right)\right],\\
J_5 &= -\int_{\substack{(s,v)\in[0,1]\times[a,1]\\ t'=a}} \Bigg(\mathrm{Tr_s}\left[g\exp\left(-L_K-\mathcal{L}\right)\right]\\
&\quad -\left(\frac{m_{v/w,w}}{v}+\left(1+(2s-1)^2\right)\frac{\sqrt{a}n_w}{v^3}+(2s-1)\frac{\sqrt{a}o_{v/w,w}}{v^2}\right)dsdv\Bigg),
\end{aligned}
$$

so that

(7.43)
$$I_3^1 = J_1+J_2+J_3+J_4+J_5.$$

For $s\in]0,1[$, put $\overline{w} = w_{(a,s,s(1-s)v)}$, i.e.

(7.44)
$$\overline{w} = \frac{4v^2}{4v+(2s-1)^2 a/\left(s\left(1-s\right)\right)^2}.$$

We will often write $\overline{w} = \overline{w}_{a,s}\left(v\right)$. Clearly,

(7.45)
$$J_1 = -\int_{(s,v)\in[0,1]\times[a/s(1-s),1/s(1-s)]} m_{s(1-s)v/\overline{w},\overline{w}}\frac{dsdv}{v}.$$

Observe that given $a\in]0,1], s\in\ ]0,1[$, the map $v\to\overline{w}_{a,s}\left(v\right)$ is strictly increasing, and is one-to-one from $\mathbf{R}_+$ into itself. Let $\overline{\mathfrak{v}}_{a,s}$ be the corresponding inverse map. By (7.44), we get

(7.46)
$$\frac{d\overline{w}_{a,s}}{\overline{w}_{a,s}} = \left(1+\frac{(2s-1)^2 a}{4\left(s\left(1-s\right)\right)^2 v+(2s-1)^2 a}\right)\frac{dv}{v}.$$

By (7.46), we deduce that there exists a smooth function $k_{a,s}:\mathbf{R}_+^*\to[0,1]$ such that

(7.47)
$$\frac{dv}{v} = k_{a,s}\left(\overline{w}\right)\frac{d\overline{w}}{\overline{w}},$$

and moreover, as $a\to 0$,

(7.48)
$$k_{a,s}\to 1 \text{ on } \mathbf{R}_+^*.$$

By (7.45), (7.47), we get

(7.49)
$$J_1 = -\int_0^1\left\{\int_{\overline{w}_{a,s}(a/s(1-s))}^{\overline{w}_{a,s}(1/s(1-s))} m_{s(1-s)\overline{\mathfrak{v}}_{a,s}(w)/w,w}k_{a,s}\left(w\right)\frac{dw}{w}\right\}ds.$$

Observe that by (7.14), $s(1-s)\overline{\mathfrak{v}}_{a,s}(w)/w$ remains uniformly bounded. Also by (3.16) and (3.17), for bounded $h \in \mathbf{R}$,

$$m_{h,w} = \mathcal{O}(w),\, w \in ]0,1], \qquad m_{h,w} = \mathcal{O}(1/w),\, w \geq 1. \tag{7.50}$$

By the above considerations, we can use dominated convergence in the integral in the right-hand side of (7.49). Then we find that as $a \to 0$,

$$J_1 \to J_1^1 = -\int_0^1 \left\{ \int_0^{1/s(1-s)} m_{s(1-s),v} \frac{dv}{v} \right\} ds. \tag{7.51}$$

Also,

$$J_2 = -2 \int_{(s,v)\in[0,1/2\sqrt{a}]\times[\sqrt{a},1/\sqrt{a}]} \left(1 + \left(2\sqrt{a}s - 1\right)^2\right) n_{w\left(a,\sqrt{a}s,\sqrt{a}v\right)} \frac{dsdv}{v^3}. \tag{7.52}$$

Clearly, if $\overline{w}' = w(a, \sqrt{a}s, \sqrt{a}v)$,

$$\overline{w}' = \frac{4v^2}{4s\left(1-\sqrt{a}s\right)v + \left(2\sqrt{a}s-1\right)^2}. \tag{7.53}$$

In the sequel, we will also write $\overline{w}' = \overline{w}'_{a,s}(v)$.

Given $a \in ]0,1], s \in ]0, 1/2\sqrt{a}[$, the map $v \to \overline{w}'_{a,s}(v)$ is strictly increasing, and is one-to-one from $\mathbf{R}_+$ into itself. Let $\overline{\mathfrak{v}}'_{a,s}$ be the corresponding inverse map. As in (7.46), we get

$$\frac{d\overline{w}'_{a,s}}{\overline{w}'_{a,s}} = \left(1 + \frac{\left(2\sqrt{a}s-1\right)^2}{4s\left(1-\sqrt{a}s\right)v + \left(2\sqrt{a}s-1\right)^2}\right)\frac{dv}{v}. \tag{7.54}$$

By (7.54), there exists a smooth function $k'_{a,s}(\overline{w}')$ with values in $[0,1]$ such that

$$\frac{dv}{v} = k'_{a,s}(\overline{w}')\frac{d\overline{w}'}{\overline{w}'}. \tag{7.55}$$

By (7.52), (7.55), we obtain

$$J_2 = -2 \int_{(s,w)\in[0,1/2\sqrt{a}]\times\left[\overline{w}'_{a,s}\left(\sqrt{a}\right),\overline{w}'_{a,s}\left(1/\sqrt{a}\right)\right)} \left(1 + \left(2\sqrt{a}s-1\right)^2\right) \frac{n_w}{w} \frac{k'_{a,s}(w)}{\overline{\mathfrak{v}}'^2_{a,s}(w)} dsdw. \tag{7.56}$$

By (7.53),

$$v^2 - s\left(1-\sqrt{a}s\right)\overline{w}'v - \frac{\left(2\sqrt{a}s-1\right)^2}{4}\overline{w}' = 0. \tag{7.57}$$

From (7.57), we deduce that there exists $C > 0$, such that for $s \in [0, 1/2\sqrt{a}]$, $w \geq 0$, if $v = \overline{\mathfrak{v}}'_{a,s}(w)$,

$$v \geq C\left(sw + \left(1 - 2\sqrt{a}s\right)\sqrt{w}\right). \tag{7.58}$$

By (7.58), we deduce that there is $C' > 0$ such that if $s \in [0, 1/2\sqrt{a}]$, $w \geq 16a$,

$$v \geq C'\left(sw + \sqrt{w}\right). \tag{7.59}$$

Also,

$$\int_0^{+\infty} \frac{ds}{(sw+\sqrt{w})^2} = \frac{1}{w^{3/2}}. \tag{7.60}$$

By the statement containing (7.11), we find that $n_w/w^{5/2}$ is integrable on $\mathbf{R}_+^*$. Since $k'_{a,s}$ takes its values in $[0,1]$, it follows from (7.52) and from the above that the dominated convergence theorem can be used on the piece of the integral (7.56) defining $J_2$ such that $w \geq 16a$.

Moreover, using (7.58) again, we find that there exists $C > 0$ such that for $w > 0$,

$$\int_0^{1/2\sqrt{a}} \frac{1}{\overline{\mathfrak{v}}'^2_{a,s}(w)} ds \leq \frac{C}{w^{3/2}}. \tag{7.61}$$

By (7.11), (7.61),

$$\left| \int_{(s,w)\in[0,1/2\sqrt{a}]\times[0,16a]} \left(1+\left(2\sqrt{a}s-1\right)^2\right) \frac{n_w}{w} \frac{k'_{a,s}(w)}{\overline{\mathfrak{v}}'^2_{a,s}(w)} dsdw \right| \leq C\sqrt{a}. \tag{7.62}$$

It follows from the above that as $a \to 0$,

$$J_2 \to J_2^1 = -4 \int_{(s,v)\in\mathbf{R}_+^{*2}} n_{4v^2/4sv+1} \frac{dsdv}{v^3}. \tag{7.63}$$

By the statement containing (7.12) and by (7.42), we get

$$|J_3| \leq C \int_{(s,v)\in[0,1]\times[a,1]} \frac{\sqrt{a}w}{v^2} dsdv. \tag{7.64}$$

Moreover,

$$\begin{aligned}\int_{(s,v)\in[0,1]\times[a,1]} \frac{\sqrt{a}w}{v^2} dsdv &\leq C \int_{(s,v)\in[0,1]\times[a,1]} \frac{\sqrt{a}dsdv}{sv+a} \\ &\leq C \int_{(s,v)\in[0,1]\times[1,1/a]} \frac{\sqrt{a}dsdv}{sv+1} \\ &= C\sqrt{a} \int_1^{1/a} \log(v+1) \frac{dv}{v}.\end{aligned} \tag{7.65}$$

By (7.64), (7.65), we find that as $a \to 0$,

$$J_3 \to 0. \tag{7.66}$$

By (7.16), (7.17) in Theorem 7.5, as $a \to 0$,

$$J_4 \to J_4^1 = -\int_{(s,v)\in[0,1]\times[1,+\infty[} m_{s(1-s),v/s(1-s)} \frac{dsdv}{v}. \tag{7.67}$$

Clearly,

$$J_4^1 = -\int_0^1 \left\{ \int_{1/s(1-s)}^{+\infty} m_{s(1-s),v} \frac{dv}{v} \right\} ds. \tag{7.68}$$

Also

$$J_5 = -\int_{(s,v)\in[0,1]\times[1,1/a]}\Bigg(i_a^*\mathrm{Tr_s}\left[g\exp\left(-L_K-\mathcal{L}\right)\right]^{(2)}_{t'=a} - \Big(\frac{m_{av/w_{a,s}(av),w_{a,s}(av)}}{v} + \left(1+(2s-1)^2\right)\frac{n_{w_{a,s}(av)}}{a^{3/2}v^3} + (2s-1)\frac{o_{av/w_{a,s}(av),w_{a,s}(av)}}{\sqrt{a}v^2}\Big)dsdv\Bigg). \tag{7.69}$$

By (7.20) in Theorem 7.8, for $(s,v)\in[0,1]\times[1,1/a]$, the integrand in the right-hand side of (7.69) is dominated by

$$C\left(\frac{1}{v}\right)^{\gamma}\frac{w_{a,s}(av)}{av^2}. \tag{7.70}$$

Moreover

$$\frac{w_{a,s}(av)}{av^2} = \frac{4}{4s(1-s)v+(2s-1)^2}. \tag{7.71}$$

For $s$ near $1/2$, there is $C'>0$ such that if $v\geq 1$,

$$\frac{4}{4s(1-s)v+(2s-1)^2} \leq \frac{C'}{v}. \tag{7.72}$$

Also if $\varepsilon\in\left]0,1/2\right[$, there exists $C''>0$ such that if $(s,v)\in([0,\varepsilon]\cup[1-\varepsilon,1])\times[1,+\infty[$,

$$\frac{4}{4s(1-s)v+(2s-1)^2} \leq \frac{C''}{s(1-s)v+1}. \tag{7.73}$$

Moreover

$$\begin{aligned}\int_{(s,v)\in([0,\varepsilon]\cup[1-\varepsilon,1])\times[1,+\infty[}&\frac{1}{s(1-s)v+1}(1/v)^{\gamma}dsdv\\ &\leq C\int_{(s,v)\in[0,\varepsilon]\times[1,+\infty[}\frac{1}{sv+1}(1/v)^{\gamma}dsdv\\ &\leq C\int_{(s,v)\in[0,\varepsilon v]\times[1,+\infty[}\frac{1}{s+1}(1/v)^{\gamma+1}dsdv\\ &\leq C'\int_{v\in[1,+\infty[}(1+\log(v))(1/v)^{\gamma+1}dv<+\infty.\end{aligned} \tag{7.74}$$

Equations (7.70)–(7.74) show that we can use the dominated convergence theorem in the integral which appears in the right-hand side of (7.69).

So using (7.19) in Theorem 7.7, (7.23) and (7.69), we find that as $a\to 0$,

$$J_5\to J_5^1=0. \tag{7.75}$$

By (7.43), (7.51), (7.63), (7.66)–(7.68), (7.75), we find that as $a\to 0$,

$$I_3^1\to I_3^2=J_1^1+J_2^1+J_4^1. \tag{7.76}$$

$\gamma$) Evaluation of $I_3^2$.

THEOREM 7.12. *The following identity holds*:

$$I_3^2 = 2\int_{X_g} e_K\left(TX_g, \nabla^{TX_g}\right) F_K\left(X_g, g^{TX_g}\right) \mathrm{ch}_g^\circ\left(\nabla^F, g^F\right) + 2\mathrm{Tr}^{F|_{X_g}}[g]\, V_K(X_g). \tag{7.77}$$

*Proof.* By (7.51), (7.68),

$$J_1^1 + J_4^1 = -\int_{(s,v)\in[0,1]\times[0,+\infty[} m_{s(1-s),v} \frac{dsdv}{v}. \tag{7.78}$$

By (3.18), (2.123), (2.124), (7.9), (7.78),

$$J_1^1 + J_4^1 = 2\int_{X_g} e_K\left(TX_g, \nabla^{TX_g}\right) F_K\left(X_g, g^{TX_g}\right) \mathrm{ch}_g^\circ\left(\nabla^F, g^F\right). \tag{7.79}$$

Let $\psi : \mathbf{R}_+^* \to \mathbf{R}$ be a continuous function with compact support. Then, by making first the change of variables $4sv \to s$, and then $2v^2/(s+1) \to v$, we get

$$\begin{aligned} &\int_{(s,v)\in\mathbf{R}_+^*\times\mathbf{R}_+^*} \psi_{2v^2/(4sv+1)} \frac{dsdv}{v^3} \\ &\quad = \frac{1}{2\sqrt{2}}\int_0^{+\infty} \frac{ds}{(s+1)^{3/2}} \int_0^{+\infty} \psi_v \frac{dv}{v^{5/2}} = \frac{1}{\sqrt{2}}\int_0^{+\infty} \psi_v \frac{dv}{v^{5/2}}. \end{aligned} \tag{7.80}$$

By (3.109), (3.118), (7.10), (7.63) and (7.80), we get

$$J_2^1 = 2\mathrm{Tr}^{F|_{X_g}}[g]\, V_K(X_g). \tag{7.81}$$

By (7.76), (7.79) and (7.81), we get (7.77). The proof of our theorem is completed. □

7.5. *A proof of Theorem* 5.1. Using (7.5), (7.25), (7.37), (7.41), we get

$$\sum_{k=1}^3 I_k^1 = 0. \tag{7.82}$$

By (7.30), (7.38), (7.76),

$$\sum_{k=1}^3 I_k^2 = 0. \tag{7.83}$$

By Theorems 7.10–7.12 and by (7.83), we obtain

$$\begin{aligned} &-\mathcal{T}_{\mathrm{ch},g,K}\left(g^{TX}, \nabla^F, g^F\right) + \mathcal{T}_{\mathrm{ch},ge^K,0}\left(g^{TX}, \nabla^F, g^F\right) \\ &\quad + \int_{X_g} e_K\left(TX_g, \nabla^{TX_g}\right) F_K\left(X_g, g^{TX_g}\right) \mathrm{ch}_g^\circ\left(\nabla^F, g^F\right) \\ &\quad + \mathrm{Tr}^{F|_{X_g}}[g]\, V_K(X_g) = 0. \end{aligned} \tag{7.84}$$

Using (1.24), (1.43), (2.121) and (7.84), we get (5.2). The proof of Theorem 5.1 is completed. □

## 8. A proof of Theorem 7.4

We use the notation in Section 6. Set

$$E = \sqrt{t'}\left(d^X + d^{X,*}\right) + \frac{1}{\sqrt{t'}}\left(s^2 u K^{X\prime} \wedge - (1-s)^2 u i_{K^X}\right) + d + \frac{\overline{N}}{2}\left((2s-1)\, dt' - 4ds\right). \tag{8.1}$$

Observe that $E$ is obtained from the right-hand side of (6.42) by replacing $dt'$ by $t'dt'$. By (6.23), (6.24), (6.42), (8.1), we get

$$\begin{aligned} t'\mathrm{Tr_s}\left[g\exp\left(-L_K - \mathcal{L}\right)\right]^{dsdt'} &= \mathrm{Tr_s}\left[g\exp\left(-L_K - E^2\right)\right]^{dsdt'}, \\ \mathrm{Tr_s}\left[g\exp\left(-L_K - \mathcal{L}\right)\right]^{dsdu} &= \mathrm{Tr_s}\left[g\exp\left(-L_K - E^2\right)\right]^{dsdu}. \end{aligned} \tag{8.2}$$

By (1.17),

$$\ker\left(d^X + d^{X,*}\right) \simeq H^{\cdot}(X,F). \tag{8.3}$$

Also

$$\left(d + \frac{\overline{N}}{2}\left((2s-1)\, dt' - 4ds\right)\right)^2 = \overline{N} ds dt'. \tag{8.4}$$

Moreover the connection $d + \frac{\overline{N}}{2}\left((2s-1)\, dt' - 4ds\right)$ restricts to a connection on $H^{\cdot}(X,F)$. Using (8.1), (8.3), (8.4) and proceeding as in [BGo1, Th. 7.1], we find that there exist $C>0, \beta>0$ such that for $z \in \mathbf{R}, |z| \le \beta, t' \in [1,+\infty[$, $s \in [0,1], u \in [0,1]$,

$$\left|\mathrm{Tr_s}\left[g\exp\left(-L_K - E^2\right)\right] - \mathrm{Tr_s}^{H^{\cdot}(X,F)}\left[ge^K \exp\left(-\overline{N} dsdt'\right)\right]\right| \le \frac{C}{\sqrt{t'}}. \tag{8.5}$$

By using in particular the argument after (2.23), we get

$$\mathrm{Tr_s}^{H^{\cdot}(X,F)}\left[ge^K \exp\left(-\overline{N} dsdt'\right)\right] = \chi_g(F) - \left(\chi'_g(F) - \frac{n}{2}\chi_g(F)\right) dsdt'. \tag{8.6}$$

By (8.2), (8.5), we get (7.15). The proof of Theorem 7.4 is completed. □

## 9. A proof of Theorem 7.5

The purpose of this section is to establish Theorem 7.5, and in particular to evaluate the limit as $a \to 0$ of $\mathrm{Tr_s}\left[g\exp\left(-L_K - \mathcal{L}\right)\right]^{(2)}_{t'=a}$. We organize it as follows. In 9.1, we establish a Lichnerowicz formula for a suitable modification of the operator $L_K + \mathcal{L}$. In Section 9.2, we describe a natural coordinate system near $X_g$. In 9.3, given $x \in X_g$, we locally replace $X$ by $T_xX$, and we compute the limit as $a \to 0$ of the operators $L_K + \mathcal{L}$ after a Getzler rescaling.

In 9.5, we evaluate the fundamental solution for the limit operator, and certain corresponding supertraces. Finally in Sections 9.6 and 9.7, we complete the proof of Theorem 7.5.

We have kept strictly analytic arguments to a minimum, since they are essentially the same as in [BGo1, §7]. For more details, we refer to the more difficult Section 12 in the present paper.

Using the notation of Sections 5, 6, and 7, we take $K_0 \in \mathfrak{z}(g)$ and $K = zK_0$, with $z \in \mathbf{R}$.

9.1. *A Lichnerowicz formula.* In the sequel, the operator $L_K + \mathcal{L}$ will be written in the coordinates $(s, v, t')$.

*Definition* 9.1. Given $a \in ]0,1]$, let $N^a$ be the operator obtained from $L_K + \mathcal{L}$ by making $t' = a$, by replacing $ds, dv$ by $\sqrt{s(1-s)}ds, dv/\sqrt{s(1-s)}$.

PROPOSITION 9.2. *The following identity holds*:

(9.1)

$$\begin{aligned}
N^a = -a\Bigg(&\nabla_{e_i}^{\Lambda^\cdot(T^*X)\widehat{\otimes} F,u} - \frac{1}{a}\left(s(1-s)(1-a/v) + \frac{a}{2v}\right)\left\langle K^X, e_i\right\rangle \\
&+ (2s-1)\frac{c(e_i)}{2\sqrt{a}}ds - \frac{\widehat{c}(e_i)}{2\sqrt{a}}ds\Bigg)^2 + \left(4s(1-s) + (2s-1)^2\frac{a}{v}\right)\frac{\left|K^X\right|^2}{4v} \\
&+\frac{a}{4}H + \frac{\sqrt{a}}{2v}\left(-(2s-1)c\left(K^X\right) + \widehat{c}\left(K^X\right)\right)ds \\
&-\sqrt{a}c\left(\omega\left(\nabla^F, g^F\right)\right)ds/2 + (2s-1)\sqrt{a}\widehat{c}\left(\omega\left(\nabla^F, g^F\right)\right)ds/2 \\
&-\sqrt{a}\left(c\left(K^X\right) + (2s-1)\widehat{c}\left(K^X\right)\right)dv/2v^2 \\
&+\frac{a}{8}\left\langle e_k, R^{TX}(e_i, e_j)e_l\right\rangle c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_l) \\
&+\frac{1}{4}\left\langle \nabla_{e_i}^{TX}K^X, e_j\right\rangle\Bigg(-\frac{a}{v}c(e_i)c(e_j) \\
&\qquad\qquad + \left(4s(1-s) + (2s-1)^2\frac{a}{v}\right)\widehat{c}(e_i)\widehat{c}(e_j)\Bigg) \\
&+4a\Bigg(\frac{1}{32}\left(-c(e_i)c(e_j) + \widehat{c}(e_i)\widehat{c}(e_j)\right)\omega^2\left(\nabla^F, g^F\right)(e_i, e_j) \\
&+\frac{1}{16}\left|\omega\left(\nabla^F, g^F\right)\right|^2 - \frac{1}{8}c(e_i)\widehat{c}(e_j)\nabla_{e_i}^{F,u}\omega\left(\nabla^F, g^F\right)(e_j)\Bigg).
\end{aligned}$$

*Proof.* This is a trivial consequence of (6.44) in Theorem 6.10, of (7.1) and of the fact that if $t' = a$, by (7.3),

$$\frac{dt}{t} = \frac{2s-1}{s(1-s)}ds. \tag{9.2}$$

□

Clearly

$$\operatorname{Tr_s}\left[g\exp\left(-L_K-\mathcal{L}\right)\right]^{dsdv}_{t'=a}=\operatorname{Tr_s}\left[g\exp\left(-N^a\right)\right]^{dsdv}, \tag{9.3}$$

so that to establish Theorem 7.5, we may as well replace $L_K+\mathcal{L}$ in (7.16), (7.17) by $N^a$.

9.2. *A trivialization near* $X_g$. If $x\in X,\alpha>0$, let $B^X(x,\alpha)$ be the open ball of centre $x$ and radius $\alpha$. Similarly, $B^{T_xX}(0,\alpha)$ denotes the open ball of centre 0 and radius $\alpha$ in $T_xX$.

Let $a_X$ be the injectivity radius of $X$. Take $x\in X_g$. Then the exponential map $Z\in T_xX,|Z|\le a_X/2\to \exp_x(Z)\in X$ identifies $B^{T_xX}(0,a_X/2)$ and $B^X(x,a_X/2)$.

We identify the normal bundle $N_{X_g/X}$ with the orthogonal bundle to $TX_g$ in $TX|_{X_g}$.

Given $\varepsilon>0$, let $\mathcal{U}_\varepsilon$ be the $\varepsilon$-neighbourhood of $X_g$ in $N_{X_g/X}$. There exist $\varepsilon_0\in\,]0,a_X/32]$ such that if $\varepsilon\in\,]0,16\varepsilon_0]$, the map $(x,Z)\in N_{X_g/X}\to\exp^X_x(Z)$ is a diffeomorphism of $\mathcal{U}_\varepsilon$ on the tubular neighbourhood $\mathcal{V}_\varepsilon$ of $X_g$ in $X$. In the sequel, we identify $\mathcal{U}_\varepsilon$ and $\mathcal{V}_\varepsilon$. This identification is $g$-equivariant.

In the sequel, $\Lambda^\cdot(\mathbf{R}^2)$ denotes the exterior algebra of $\mathbf{R}^2$. Here $\Lambda^1(\mathbf{R}^2)$ is spanned by $ds,dv$.

*Definition* 9.3. Let ${}^1\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2),u,a}$ be the connection on $\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2)$ over $X$,

$$\begin{aligned}{}^1\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2),u,a}_\cdot&=\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,u}_\cdot\\&-\frac{1}{a}\left(s(1-s)(1-a/v)+\frac{a}{2v}\right)K^{X\prime}(\cdot)+(2s-1)\frac{c(\cdot)}{2\sqrt{a}}ds-\frac{\widehat{c}(\cdot)}{2\sqrt{a}}ds.\end{aligned} \tag{9.4}$$

Note that in (9.4), the dependence of ${}^1\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2),u,a}$ on $(s,v)$ is not explicitly written. Also ${}^1\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2),u,a}$ can also be considered as a superconnection on $\Lambda(T^*X)\widehat{\otimes}F$ in the sense of Quillen [Q1]. Recall that $R^{F,u}$ was obtained in (1.30). By (1.5), the curvature ${}^1\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2),u,a,2}$ of the connection ${}^1\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2),u,a}$ is given by

$$\begin{aligned}{}^1\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2),u,a,2}&=\frac14\left\langle R^{TX}e_i,e_j\right\rangle\left(c(e_i)c(e_j)-\widehat{c}(e_i)\widehat{c}(e_j)\right)\\&+R^{F,u}-\frac1a\left(s(1-s)(1-a/v)+\frac{a}{2v}\right)dK^{X\prime}.\end{aligned} \tag{9.5}$$

In the sequel, if $x\in X_g,Z\in T_xX,|Z|\le a_X/2$, we trivialize

$$\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot\left(\mathbf{R}^2\right)$$

by parallel transport along $h\in[0,1]\to hZ\in X$ with respect to the connection ${}^1\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2),u,a}$. In particular, if $B$ is a smooth section of $TX$, by (1.3), we get

$$
\begin{aligned}
(9.6)\qquad & {}^{1}\nabla_{\cdot}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2),u,a} c(B) = c\left(\nabla_{\cdot}^{TX} B\right) + \frac{2s-1}{\sqrt{a}}\langle B,\cdot\rangle\, ds,\\
& {}^{1}\nabla_{\cdot}^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2),u,a} \widehat{c}(B) = \widehat{c}\left(\nabla_{\cdot}^{TX} B\right) + \frac{1}{\sqrt{a}}\langle B,\cdot\rangle\, ds.
\end{aligned}
$$

Recall that $c(TX)$ and $\widehat{c}(TX)$ inherit a **Z**-grading from the grading of $\Lambda^{\cdot}(T^*X)$. The bundle of algebras $\left(c(TX)\widehat{\otimes}\widehat{c}(TX)\widehat{\otimes}\operatorname{End}(F)\right)_x\widehat{\otimes}\Lambda(\mathbf{R}^2)$ is then naturally **Z**-graded by the sum of the gradings on $c(TX), \widehat{c}(TX)$ and $\Lambda^{\cdot}(T^*X)$.

Let ${}^{1}\Gamma^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2)}$ be the connection form for ${}^{1}\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2),u,a}$ in the above trivialization. Using (9.6), as in [B12, proof of Th. 11.11], we find easily that in the above trivialization, ${}^{1}\Gamma^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2)}$ is of length $\leq 2$ in the algebra $\left(c(TX)\widehat{\otimes}\widehat{c}(TX)\widehat{\otimes}\operatorname{End}(F)\widehat{\otimes}\Lambda(\mathbf{R}^2)\right)_x$. By [ABoP, Prop. 3.7],

$$
(9.7)\qquad {}^{1}\Gamma^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2)} = \frac{1}{2}{}^{1}\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2),u,a,2}(Z,\cdot) + \mathcal{O}\left(|Z|^2\right),
$$

and $\mathcal{O}\left(|Z|^2\right)$ has length $\leq 2$.

9.3. *Replacing $X$ by $T_xX$.* Let $\gamma(s)$ be a smooth even function from **R** into $[0,1]$ such that

$$
(9.8)\qquad \begin{aligned}
\gamma(s) &= 1 \text{ if } |s| \leq 1/2,\\
&= 0 \text{ if } |s| \geq 1.
\end{aligned}
$$

If $Z \in T_xX$, put

$$
(9.9)\qquad \rho(Z) = \gamma\left(\frac{|Z|}{4\varepsilon_0}\right).
$$

Then

$$
(9.10)\qquad \begin{aligned}
\rho(Z) &= 1 \text{ if } |Z| \leq 2\varepsilon_0,\\
&= 0 \text{ if } |Z| \geq 4\varepsilon_0.
\end{aligned}
$$

For $x \in X_g$, let $\mathbf{H}_x$ be the vector space of smooth sections of

$$
\left(\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}\left(\mathbf{R}^2\right)\right)_x
$$

over $T_xX$. Let $\Delta^{TX}$ be the standard Laplacian on the fibres of $TX$.

*Definition* 9.4. Let $N_{x,K}^{1,a}$ be the differential operator,

$$
(9.11)\qquad N_{x,K}^{1,a} = -\left(1-\rho^2(Z)\right)a\Delta^{TX} + \rho^2(Z)\,N^a.
$$

9.4. *The Getzler rescaling.* Let $\mathrm{Op}_x$ be the set of scalar differential operators acting on $\mathbf{H}_x$. Then

$$N^{1,a}_{x,K} \in \mathrm{End}\left(\Lambda^{\cdot}\left(T^*X\right)\widehat{\otimes} F \widehat{\otimes} \Lambda^{\cdot}\left(\mathbf{R}^2\right)\right)_x \widehat{\otimes}\, \mathrm{Op}_x. \tag{9.12}$$

For $t>0$, let $H_a : \mathbf{H}_x \to \mathbf{H}_x$ be the linear map

$$H_a h(Z) = h(Z/\sqrt{a}). \tag{9.13}$$

*Definition* 9.5. Let $N^{2,a}_{x,K}$ be the differential operator acting on $\mathbf{H}_x$,

$$N^{2,a}_{x,K} = H_a^{-1} N^{1,a}_{x,K} H_a. \tag{9.14}$$

Then

$$N^{2,a}_{x,K} \in \left(c\left(TX\right)\widehat{\otimes}\,\widehat{c}\left(TX\right)\widehat{\otimes}\,\mathrm{End}\left(F\right)\widehat{\otimes}\,\Lambda\left(\mathbf{R}^2\right)\right)_x \widehat{\otimes}\, \mathrm{Op}_x. \tag{9.15}$$

Put $m = \dim X_g$. Let $e_1,\dots,e_m$ be an orthonormal basis of $T_xX_g$, let $e_{m+1},\dots,\ e_n$ be an orthonormal basis of $N_{X_g/X,x}$, so that $e_1,\dots,e_n$ is an orthonormal basis of $T_xX$. For $a>0, U \in T_xX$, set

$$c_a\left(U\right) = \sqrt{2/a}U^* \wedge -\sqrt{a/2}i_U. \tag{9.16}$$

*Definition* 9.6. Let $N^{3,a}_{x,K}$ be the operator obtained from $N^{2,a}_{x,K}$ by replacing $c\left(e_j\right)$ by $c_a\left(e_j\right)$ for $1\leq j\leq m$, while leaving the $c\left(e_j\right)$ unchanged for $m+1\leq j\leq n$.

In the sequel, forms like $\omega\left(\nabla^F, g^F\right)$ or $R^{TX}$ will always be restricted to $X_g$.

*Definition* 9.7. Let

$$N^{3,0}_{x,K} \in \left(\Lambda^{\cdot}\left(T^*X_g\right)\widehat{\otimes}\,\widehat{c}\left(TX_g\right)\widehat{\otimes}\,\mathrm{End}\left(\Lambda^{\cdot}\left(N^*_{X_g/X}\right)\right)\right.$$
$$\left.\widehat{\otimes}\,\mathrm{End}\left(F\right)\widehat{\otimes}\,\Lambda^{\cdot}\left(\mathbf{R}^2\right)\right)_x \widehat{\otimes}\,\mathrm{Op}_x$$

be given by

$$\begin{aligned} N^{3,0}_{x,K} = &-\left(\nabla_{e_i} + \frac{1}{2}\left\langle R^{TX}_{2s(1-s)K/2\pi} Z, e_i\right\rangle\right)^2 + \frac{s\left(1-s\right)}{v}\left|K^{X_g}\right|^2 \\ &-\frac{K^{X_g\prime}}{\sqrt{2}}\left(\frac{2s-1}{v}ds + \frac{dv}{v^2}\right) - \frac{dK^{X_g\prime}}{2v} \\ &+\left(\widehat{c}\left(R^{TX}_{2s(1-s)/2\pi K} Z\right) - \frac{\omega\left(\nabla^F, g^F\right)}{\sqrt{2}}\right) ds \\ &+\frac{1}{2}\left\langle e_k, R^{TX}_{2s(1-s)K/2\pi} e_l\right\rangle \widehat{c}\left(e_k\right)\widehat{c}\left(e_l\right) - \frac{1}{2}\omega^2\left(\nabla^F, g^F\right). \end{aligned} \tag{9.17}$$

We will write that a family of differential operators on $T_xX$ converges if its coefficients converge together with their derivatives uniformly over compact subsets of $T_xX$.

THEOREM 9.8. *For* $(s,v)\in[0,1]\times\mathbf{R}_+^*$, *as* $a\to 0$,

$$N_{x,K}^{3,a}\to N_{x,K}^{3,0}. \tag{9.18}$$

*Proof.* We use equation (9.1) for $N^a$. By (9.5), (9.7) and proceeding as in [BeGeV, Ch. 10], [B12, §11.5], [BGo1, §7.7], we find easily that as $a\to 0$, in the given trivialization,

$$\begin{aligned}-a\Bigg(\nabla_{e_i}^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,u}-\frac{1}{a}\left(s\left(1-s\right)\left(1-a/v\right)+\frac{a}{2v}\right)\left\langle K^X,e_i\right\rangle\\+\left(2s-1\right)\frac{c\left(e_i\right)}{2\sqrt{a}}ds-\frac{\widehat{c}\left(e_i\right)}{2\sqrt{a}}ds\Bigg)^2\\\to-\left(\nabla_{e_i}+\frac{1}{2}\left\langle R^{TX}_{2s(1-s)K/2\pi}Z,e_i\right\rangle\right)^2.\end{aligned} \tag{9.19}$$

Observe that since $m^{TX}(K)=\nabla^{TX}_{\cdot}K^X$ is antisymmetric,

$$\left\langle\nabla^{TX}_ZK^X\left(\sqrt{a}Z\right),Z\right\rangle=0. \tag{9.20}$$

By (9.6), (9.20), we find that in the considered trivialization,

$$\begin{aligned}c\left(K^X\right)\left(\sqrt{a}Z\right)&=c\left(K^X_x\right)+\sqrt{a}c\left(\nabla^{TX}_ZK^X\left(x\right)\right)\\&\quad+\left(2s-1\right)\left\langle K^X\left(x\right),Z\right\rangle ds\\&\quad+\mathcal{O}_1\left(a\left|Z\right|^2\right)+\mathcal{O}_0\left(a\left|Z\right|^3\right)ds,\\\widehat{c}\left(K^X\right)\left(\sqrt{a}Z\right)&=\widehat{c}\left(K^X\left(x\right)\right)+\sqrt{a}\widehat{c}\left(\nabla^{TX}_ZK^X\left(x\right)\right)\\&\quad+\left\langle K^X\left(x\right),Z\right\rangle ds+\widehat{\mathcal{O}_1}\left(a\left|Z\right|^2\right)+\widehat{\mathcal{O}}_0\left(a\left|Z\right|^3\right)ds.\end{aligned} \tag{9.21}$$

In (9.21), the lower scripts 1 or 0 indicate the length in $c(T_xX)$ or $\widehat{c}(T_xX)$.

Let $[A]^3_a$ be the expression obtained from $A$ by doing the Getzler rescalings indicated in Definition 9.6. Recalling that $K^X|_{X_g}$ is the vector field $K^{X_g}\subset TX_g$, by (9.21), we get

$$\left[\sqrt{a}c\left(K^{X}\right)\left(\sqrt{a}Z\right)\right]_{a}^{3}=\sqrt{2}K^{X_g\prime}\wedge+\mathcal{O}\left(\sqrt{a}\left(1+|Z|+|Z|^{2}\right)\right), \tag{9.22}$$
$$\left[\sqrt{a}\widehat{c}\left(K^{X}\right)\left(\sqrt{a}Z\right)\right]_{a}^{3}=\mathcal{O}\left(\sqrt{a}\left(1+|Z|\right)\right).$$

If $U \in T_xX, Z \in T_xX, |Z| \leq 4\varepsilon_0$, let $\tau U(Z)$ be the parallel transport of $U$ along the geodesic $h \in [0,1] \to hZ \in X$ with respect to the Levi-Civita connection $\nabla^{TX}$. By (9.6), we get

$$\sqrt{a}\left[c\left(\tau U\right)\left(\sqrt{a}Z\right)\right]_{a}^{3}=\sqrt{2}U^{*}\wedge+\mathcal{O}\left(\sqrt{a}\left(1+|Z|^{2}\right)\right), \tag{9.23}$$
$$\left[\widehat{c}\left(\tau U\right)\left(\sqrt{a}Z\right)\right]_{a}^{3}=\widehat{c}\left(U\right)+\langle U,Z\rangle\, ds+\mathcal{O}\left(\sqrt{a}\,|Z|\right).$$

By (9.1) and by the above considerations, we get (9.18). The proof of our theorem is completed. □

*Remark* 9.9. Incidentally, observe that had we not used (9.20), the estimate in the right-hand side of (9.21) would have been $\mathcal{O}_0\left(\sqrt{a}\,|Z|^2\right)ds$ or $\widehat{\mathcal{O}}_0\left(\sqrt{a}\,|Z|^2\right)ds$. This estimate would have been quite sufficient in the present proof, but the stronger estimate (9.21) will be needed in equation (11.7), in our proof of Theorem 11.5.

9.5. *The heat kernel associated to $N^{3,0}_{x,K}$.* Let $dv_{TX}$ be the volume form on the fibres of $TX$ with respect to the metric $g^{TX}$. Set

$$\overline{N}^{3,0}_{x,K}=N^{3,0}_{x,K}-\widehat{c}\left(R^{TX}_{2s(1-s)K/2\pi}Z\right)ds. \tag{9.24}$$

Recall that $K_0 \in \mathfrak{z}(g)$ is fixed and that $K = zK_0$, with $z \in \mathbf{R}^*$. For $z \in \mathbf{R}^*$ and $|z|$ small enough, let $P^{3,0}_{x,K}(Z,Z'), \overline{P}^{3,0}_{x,K}(Z,Z'), Z, Z' \in T_xX$ be the smooth kernel associated to $\exp\left(-N^{3,0}_{x,K}\right), \exp\left(-\overline{N}^{3,0}_{x,K}\right)$ with respect to $dv_{TX}(Z')/(2\pi)^{n/2}$.

In fact, observe that the operator $N^{3,0}_{x,K}, \overline{N}^{3,0}_{x,K}$ are not lower bounded, so that the above heat kernels are not well defined. An easy way out is to make $z \in i\mathbf{R}$, and extend the corresponding heat kernel by analytic continuation. Details are left to the reader. For an extensive discussion, which will be much more relevant in Section 12, we refer to the introduction of [BGo1] and to [BGo1, §7.13].

Proposition 9.10. *For $z \in \mathbf{R}$, and $|z|$ small enough, the following identity holds*:

(9.25)

$$\overline{P}^{3,0}_{x,K}\left(Z,Z'\right) = 2^{-n/2}\left[\det\left(\frac{R^{TX}_{2s(1-s)K/2\pi}}{\sinh\left(R^{TX}_{2s(1-s)K/2\pi}\right)}\right)\right]^{1/2}$$

$$\exp\Bigg(-\frac{1}{4}\left\langle\frac{R^{TX}_{2s(1-s)K/2\pi}}{\tanh\left(R^{TX}_{2s(1-s)K/2\pi}\right)}Z,Z\right\rangle - \frac{1}{4}\left\langle\frac{R^{TX}_{2s(1-s)K/2\pi}}{\tanh\left(R^{TX}_{2s(1-s)K/2\pi}\right)}Z',Z'\right\rangle + \frac{1}{2}\left\langle\frac{R^{TX}_{2s(1-s)K/2\pi}}{\sinh\left(R^{TX}_{2s(1-s)K/2\pi}\right)}e^{R^{TX}_{2s(1-s)/2\pi K}}Z,Z'\right\rangle\Bigg)$$

$$\exp\left(-\frac{s\left(1-s\right)}{v}\left|K^{X}\right|^{2}+\frac{dK^{X_g\prime}}{2v}+\frac{K^{X_g\prime}}{\sqrt{2}}\left(\frac{2s-1}{v}ds+\frac{dv}{v^2}\right)\right)$$

$$\exp\left(-\frac{1}{2}\left\langle e_k,R^{TX}_{2s(1-s)K/2\pi}e_l\right\rangle\widehat{c}\left(e_k\right)\widehat{c}\left(e_l\right)\right)$$

$$\exp\left(\frac{1}{2}\omega^2\left(\nabla^F,g^F\right)+\frac{\omega\left(\nabla^F,g^F\right)}{\sqrt{2}}ds\right).$$

*Proof.* The proof follows from (9.17) and from Mehler's formula as in [B10, eq. (4.48)]. □

Clearly $g$ acts as a parallel isometry of $N_{X_g/X}$, with no eigenvalue equal to 1. If no eigenvalue of $g$ is equal to $-1$, there is a locally constant $B \in \operatorname{End}\left(N_{X_g/X}\right)$, which is antisymmetric and invertible, and which is such that

$$g|_{N_{X_g/X}} = e^{B}. \tag{9.26}$$

If the eigenbundle $N^{-1}_{X_g/X}$ associated to the eigenvalue $-1$ is not reduced to 0, we can still find a locally constant $B \in \operatorname{End}\left(N_{X_g/X}\right)\otimes\mathbf{C}$, which is invertible, which preserves $N^{-1}_{X_g/X}$ and its orthogonal $N^{-1\perp}_{X_g/X}$, which acts by multiplication by $i\pi$ on $N^{-1}_{X_g/X}$, and like a real antisymmetric matrix on $N^{-1\perp}_{X_g/X}$, so that (9.26) still holds. We may extend $B$ to a section of $\operatorname{End}\left(TX|_{X_g}\right)$ acting as 0 on $TX_g$, so that (9.26) holds on $TX|_{X_g}$.

Proposition 9.11. *For $z \in \mathbf{R}$ and $|z|$ small enough, the following identity holds*:

(9.27)

$$\overline{P}^{3,0}_{x,K}\left(g^{-1}Z,Z\right)=2^{-n/2}\left[\det\left(\frac{R^{TX}_{2s(1-s)K/2\pi}}{\sinh\left(R^{TX}_{2s(1-s)K/2\pi}\right)}\right)\right]^{1/2}$$

$$\exp\left(-\left\langle\frac{R^{TX}_{2s(1-s)K/2\pi}}{\sinh\left(R^{TX}_{2s(1-s)K/2\pi}\right)}\sinh\left(R^{TX}_{2s(1-s)K/2\pi}-B/2\right)\sinh\left(B/2\right)Z,Z\right\rangle\right)$$

$$\exp\left(-\frac{s\left(1-s\right)}{v}\left|K^X\right|^2+\frac{dK^{X\prime}}{2v}+\frac{K^{X\prime}}{\sqrt{2}}\left(\frac{2s-1}{v}ds+\frac{dv}{v^2}\right)\right)$$

$$\exp\left(-\frac{1}{2}\left\langle e_k,R^{TX}_{2s(1-s)K/2\pi}e_l\right\rangle\widehat{c}\left(e_k\right)\widehat{c}\left(e_l\right)\right)$$

$$\exp\left(\frac{1}{2}\omega^2\left(\nabla^F,g^F\right)+\frac{\omega\left(\nabla^F,g^F\right)}{\sqrt{2}}ds\right).$$

*Proof.* Our identity follows from Proposition 9.10. □

*Definition* 9.12. Let $\widehat{\mathrm{Tr_s}}$ be the functional defined on $\widehat{c}\left(TX_g\right)$, with values in $o\left(TX_g\right)$, which vanishes on all monomials in the $\widehat{c}\left(e_i\right)$ whose length is $< m$, and is such that

$$\widehat{\mathrm{Tr_s}}\left[\widehat{c}\left(e_1\right)\ldots\widehat{c}\left(e_m\right)\right]=\left(-1\right)^{m(m+1)/2}. \tag{9.28}$$

In the sequel, we use the Berezin integration formalism of Section 1.2, with $V=TX_g, E=TX_g$. Equivalently we use the notation of Section 3.2, with $X$ replaced by $X_g$.

PROPOSITION 9.13. *The following identities hold*:

$$\widehat{\mathrm{Tr_s}}\left[\exp\left(-\frac{1}{2}\left\langle e_k,R^{TX_g}_{2s(1-s)K/2\pi}e_l\right\rangle\widehat{c}\left(e_k\right)\widehat{c}\left(e_l\right)\right)\right] \tag{9.29}$$

$$=\left[\det\left(\frac{\sinh\left(R^{TX_g}_{2s(1-s)K/2\pi}\right)}{R^{TX_g}_{2s(1-s)K/2\pi}}\right)\right]^{1/2}\int^{\widehat{B}}\exp\left(-\dot{R}^{TX_g}_{2s(1-s)K/2\pi}\right),$$

$$\mathrm{Tr_s}^{\Lambda^{\cdot}\left(N^*_{X_g/X}\right)}\left[g\exp\left(-\frac{1}{2}\left\langle e_k,R^{N_{X_g/X}}_{2s(1-s)K/2\pi}e_l\right\rangle\widehat{c}\left(e_k\right)\widehat{c}\left(e_l\right)\right)\right]$$

$$=\left[\det\left(4\sinh\left(R^{N_{X_g/X}}_{2s(1-s)K/2\pi}-B/2\right)\sinh\left(B/2\right)\right)\right]^{1/2}.$$

*Proof.* The first identity in (9.29) follows from results of Mathai-Quillen [MQ]. The second identity in (9.29) follows from [BGo2, Prop. 4.9]. □

If $\alpha \in \Lambda^{\cdot}(T^*X_g)$, $\alpha^{\max}$ denotes the component of $\alpha$ of maximal degree $m$.

THEOREM 9.14. *The following identity holds*:

(9.30)
$$2^{m/2}\left[\int_{N_{X_g/X}}\widehat{\mathrm{Tr_s}}\left[g\overline{P}^{3,0}_{x,K}\left(g^{-1}Z,Z\right)\right]\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{n/2}}\right]^{dsdv,\max}$$
$$= -\left\{2e_K\left(TX_g,\nabla^{TX_g}\right)\beta^{X_g}_{K,v/2s(1-s)}\right.$$
$$\left.\mathrm{Tr}^{F|_{X_g}}\left[g\frac{\omega\left(\nabla^F,g^F\right)}{2}\exp\left(s\left(1-s\right)\omega^2\left(\nabla^F,g^F\right)\right)/2\pi\right]\right\}^{\max}\frac{dsdv}{v}.$$

*Proof.* By (9.27), (9.29), we get

(9.31)
$$2^{m/2}\int_{N_{X_g/X}}\widehat{\mathrm{Tr_s}}\left[g\overline{P}^{3,0}_{x,K}\left(g^{-1}Z,Z\right)\right]\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{n/2}}$$
$$=\frac{1}{(2\pi)^{m/2}}\exp\left(-\frac{s\left(1-s\right)}{v}\left|K^X\right|^2+\frac{dK^{X\prime}}{2v}+\frac{K^{X\prime}}{\sqrt{2}}\left(\frac{2s-1}{v}ds+\frac{dv}{v^2}\right)\right)$$
$$\int^{\widehat{B}}\exp\left(-\dot{R}^{TX_g}_{2s(1-s)K/2\pi}\right)\mathrm{Tr}^{F|_{X_g}}\left[g\exp\left(\frac{1}{2}\omega^2\left(\nabla^F,g^F\right)\right.\right.$$
$$\left.\left.+\frac{\omega\left(\nabla^F,g^F\right)}{\sqrt{2}}ds\right)\right]. \qquad \square$$

By (9.31), we obtain

(9.32)
$$2^{m/2}\left[\int_{N_{X_g/X}}\widehat{\mathrm{Tr_s}}\left[g\overline{P}^{3,0}_{x,K}\left(g^{-1}Z,Z\right)\right]\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{n/2}}\right]^{dsdv}$$
$$=\frac{1}{(2\pi)^{m/2}}\exp\left(-\frac{s\left(1-s\right)}{v}\left|K^X\right|^2+\frac{dK^{X\prime}}{2v}\right)\frac{K^{X\prime}}{v^2}$$
$$\int^{\widehat{B}}\exp\left(-\dot{R}^{TX_g}_{2s(1-s)K/2\pi}\right)\frac{1}{2}\mathrm{Tr}\left[g\omega\left(\nabla^F,g^F\right)\exp\left(\frac{1}{2}\omega^2\left(\nabla^F,g^F\right)\right)\right]dsdv.$$

In (9.32), we now make the rescaling $e^i\to\sqrt{2s\left(1-s\right)}e^i, \widehat{e}^i\to\frac{1}{\sqrt{2s(1-s)}}\widehat{e}^i, 1\leq i\leq m$. Such a rescaling does not change the term of maximal degree in (9.32).

From (9.32), we obtain

(9.33)

$$2^{m/2}\left[\int_{N_{X_g/X}}\widehat{\mathrm{Tr_s}}\left[g\overline{P}^{3,0}_{x,K}\left(g^{-1}Z,Z\right)\right]\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{n/2}}\right]^{dsdv,\max}$$
$$=\Bigg\{\exp\left(-\frac{s(1-s)}{v}\left|K^X\right|^2+\frac{s(1-s)}{2\pi v}dK^{X\prime}\right)\frac{K^{X\prime}}{2\pi v}$$
$$\int^{\widehat{B}}\exp\left(-\dot{R}^{TX_g}_K/2\pi\right)$$
$$\mathrm{Tr}^{F|_{X_g}}\left[g\omega\left(\nabla^F,g^F\right)\exp\left(s(1-s)\,\omega^2\left(\nabla^F,g^F\right)/2\pi\right)\right]\Bigg\}^{\max}$$
$$s(1-s)\frac{dsdv}{v},$$

which is just (9.30). The proof of our theorem is completed. □

9.6. *A proof of equation* (7.16). Using the trivializations indicated in Section 9.2, and the fixed point techniques in local index theory in [BGo1, §7], and also Theorem 9.14, we find easily that for $z \in \mathbf{R}$, and $|z|$ small enough, as $a \to 0$,

$$\mathrm{Tr_s}\left[g\exp\left(-L_K-\mathcal{L}\right)\right]^{(2)}_{t'=a}$$
$$\to 2^{m/2}\int_{X_g}\left[\int_{N_{X_g/X}}\widehat{\mathrm{Tr_s}}\left[gP^{3,0}_{x,K}\left(g^{-1}Z,Z\right)\right]\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{n/2}}\right]^{dsdv,\max}. \tag{9.34}$$

More precisely, we use the known fact [BZ1, Prop. 4.9] that among the monomials in $c(e_i), \widehat{c}(e_i), 1 \le i \le m$, up to permutation, $c(e_1)\widehat{c}(e_1)\dots c(e_m)\widehat{c}(e_m)$ is the only monomial whose supertrace on $\Lambda^{\cdot}(T^*X_g)$ is nonzero, and moreover

$$\mathrm{Tr_s}\left[c(e_1)\widehat{c}(e_1)\dots c(e_m)\widehat{c}(e_m)\right]=(-2)^m. \tag{9.35}$$

When the Getzler rescaling indicated in (9.16) is done, comparison with (9.35) shows that there is an extra factor $2^{m/2}$ which should be incorporated in the final computation.

If $Z \in TX$, put

$$I(Z)=-Z. \tag{9.36}$$

Then the operator $I^{-1}N^{3,0}_{x,K}I$ is obtained from $N^{3,0}_{x,K}$ by changing

$$\widehat{c}\left(R^{TX}_{2s(1-s)K/2\pi}Z\right)ds$$

into

$$-\widehat{c}\left(R^{TX}_{2s(1-s)K/2\pi}Z\right)ds.$$

It follows that $P^{3,0}_{x,K}(-Z,-Z')$ is obtained from $P^{3,0}_{x,K}(Z,Z')$ by changing $ds$, $\omega\left(\nabla^F,g^F\right)$ into $-ds,-\omega\left(\nabla^F,g^F\right)$. This shows that in the integral on the right-hand side of (9.34), $\widehat{c}\left(R^{TX}_{2s(1-s)K/2\pi}Z\right)ds$ does not contribute to the computation, i.e. we may replace $P^{3,0}_{x,K}$ by $\overline{P}^{3,0}_{x,K}$.

By (7.9), (9.30) and (9.34), we find that for $z\in\mathbf{R}$, and $|z|$ small enough, as $a\to 0$,

$$\operatorname{Tr_s}\left[g\exp\left(-L_K-\mathcal{L}\right)\right]^{(2)}_{t'=a}\to m_{s(1-s),v/s(1-s)}\frac{dsdv}{v},\tag{9.37}$$

i.e. we have established (7.16).

9.7. *A proof of equation* (7.17). Let $\overline{N}^a$ be the operator obtained from $N^a$ by replacing $dv$ by $v^2dv$. By (9.3),

$$v^2\operatorname{Tr_s}\left[g\exp\left(-L_K-\mathcal{L}\right)\right]^{(2)}_{t'=a}=\operatorname{Tr_s}\left[g\exp\left(-\overline{N}^a\right)\right]^{dsdv}_{t'=a}.\tag{9.38}$$

By (9.1), we deduce easily that for $a\in\,]0,1],(s,v)\in[0,1]\times[1,+\infty[$,

$$\operatorname{Tr_s}\left[g\exp\left(-\overline{N}^a\right)\right]^{dsdv}_{t'=a}$$

remains uniformly bounded. A similar argument was given in [BGo1, §8.14]. We find that for $z\in\mathbf{R}$ and $|z|$ small enough, there exists $C>0$ such that for $a\in\,]0,1],(s,v)\in[0,1]\times[1,+\infty[$,

$$\left|\operatorname{Tr_s}\left[g\exp\left(-\overline{N}^a\right)\right]^{dsdv}_{t'=a}\right|\leq C.\tag{9.39}$$

By (9.38), (9.39), we get (7.17). The proof of Theorem 7.5 is complete.

## 10. A proof of Theorem 7.6

The purpose of this section is to establish Theorem 7.6, i.e. to evaluate the limit as $a\to 0$ of $k_a^*\operatorname{Tr_s}\left[g\exp\left(-L_K-\mathcal{L}\right)\right]^{(2)}_{t'=a}$. The strategy is very similar to the one we followed in Section 9.

This Section is organized as follows. In 10.1, we introduce a rescaled version of the operator $N^a$. In 10.2, we introduce a new Getzler rescaling, and evaluate the limit as $a\to 0$ of this rescaled operator. Also, we compute the fundamental solution for the limit operator. Finally, in 10.3, we establish Theorem 7.6.

As in Section 9, we have kept the strictly analytic arguments to a minimum, since they are the same as in [BGo1, §7]. For more details, we refer again to Section 12 of the present paper.

We use the notation of Sections 6, 7 and 9. Also we take $K_0\in\mathfrak{z}(g)$ and $K=zK_0$, with $z\in\mathbf{R}^*$.

10.1. *The operator $N^{a\prime}$.* The operator $N^a$ was defined in Definition 9.1. As we saw in Section 7.3,

$$j_a(s,v) = \left(\sqrt{a}s, \sqrt{a}v\right). \tag{10.1}$$

Under $j_a^*$, the variables $ds, dv$ are scaled into $\sqrt{a}ds, \sqrt{a}dv$.

*Definition* 10.1. Let $N^{a\prime}$ be the operator obtained from $j_a^* N^a$ by the transformation $(ds, dv) \to \left(a^{-1/4}ds, a^{1/4}dv\right)$.

By (9.3),

$$j_a^* \mathrm{Tr_s}\left[g\exp\left(-L_K - \mathcal{L}\right)\right]^{(2)}_{t'=a} = \mathrm{Tr_s}\left[g\exp\left(-N^{a\prime}\right)\right]^{dsdv}. \tag{10.2}$$

By (9.1), we get

(10.3)

$$\begin{aligned}
N^{a\prime} = -a\Bigg(&\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F,u}_{e_i} - \frac{1}{\sqrt{a}}\left(s\left(1-\sqrt{a}s\right)\left(1-\sqrt{a}/v\right) + \frac{1}{2v}\right)\left\langle K^X, e_i\right\rangle \\
&+ \left(2\sqrt{a}s - 1\right)\frac{c(e_i)}{2a^{1/4}}ds - \frac{\widehat{c}(e_i)}{2a^{1/4}}ds\Bigg)^2 \\
&+ \left(4s\left(1-\sqrt{a}s\right) + \left(2\sqrt{a}s-1\right)^2\frac{1}{v}\right)\frac{\left|K^X\right|^2}{4v} \\
&+ \frac{a}{4}H + \frac{a^{1/4}}{2v}\left(-\left(2\sqrt{a}s-1\right)c\left(K^X\right) + \widehat{c}\left(K^X\right)\right)ds \\
&- a^{3/4}c\left(\omega\left(\nabla^F, g^F\right)\right)ds/2 + \left(2\sqrt{a}s-1\right)a^{3/4}\widehat{c}\left(\omega\left(\nabla^F, g^F\right)\right)ds/2 \\
&- a^{1/4}\left(c\left(K^X\right) + \left(2\sqrt{a}s-1\right)\widehat{c}\left(K^X\right)\right)dv/2v^2 \\
&+ \frac{a}{8}\left\langle e_k, R^{TX}(e_i,e_j)e_l\right\rangle c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_l) \\
&+ \frac{1}{4}\left\langle \nabla^{TX}_{e_i}K^X, e_j\right\rangle\Bigg(-\frac{\sqrt{a}}{v}c(e_i)c(e_j) \\
&+ \left(4s\left(1-\sqrt{a}s\right) + \left(2\sqrt{a}s-1\right)^2\frac{1}{v}\right)\sqrt{a}\widehat{c}(e_i)\widehat{c}(e_j)\Bigg) \\
&+ 4a\Bigg(\frac{1}{32}\left(-c(e_i)c(e_j) + \widehat{c}(e_i)\widehat{c}(e_j)\right)\omega^2\left(\nabla^F, g^F\right)(e_i,e_j) \\
&+ \frac{1}{16}\left|\omega\left(\nabla^F, g^F\right)\right|^2 - \frac{1}{8}c(e_i)\widehat{c}(e_j)\nabla^{F,u}_{e_i}\omega\left(\nabla^F, g^F\right)(e_j)\Bigg).
\end{aligned}$$

10.2. *A new Getzler rescaling.* Now we use the notation of Section 9.2. Recall that the connection ${}^1\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2),u,a}$ was defined in Definition 9.4.

*Definition* 10.2. Let ${}^1\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2),u,a\prime}$ be the connection on the vector bundle

$$\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot\left(\mathbf{R}^2\right),$$

over $X$, which is obtained from $j_a^{*1}\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2),u,a}$ by the rescaling $ds\to a^{-1/4}ds$.

By (9.5),

$$\begin{aligned}(10.4)\qquad &{}^1\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2),u,a\prime,2}\\ &=\frac{1}{4}\left\langle R^{TX}e_i,e_j\right\rangle\left(c(e_i)c(e_j)-\widehat{c}(e_i)\widehat{c}(e_j)\right)\\ &\quad+R^{F,u}-\frac{1}{\sqrt{a}}\left(s\left(1-\sqrt{a}s\right)\left(1-\sqrt{a}/v\right)+\frac{1}{2v}\right)dK^{X\prime}.\end{aligned}$$

Then we use the trivializations corresponding to the trivializations in Section 9.2, except that the connection ${}^1\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2),u,a}$ is now replaced by the connection ${}^1\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^\cdot(\mathbf{R}^2),u,a\prime}$.

Recall that the function $\rho(Z)$ was defined in (9.10).

*Definition* 10.3. Let $N_{x,K}^{1,a\prime}$ be the differential operator,

$$N_{x,K}^{1,a\prime}=-\left(1-\rho^2(Z)\right)a\Delta^{TX}+\rho^2(Z)N^{a\prime}. \tag{10.5}$$

We still define $H_a$ as in (9.13).

*Definition* 10.4. Let $N_{x,K}^{2,a\prime}$ be the differential operator acting on $\mathbf{H}_x$,

$$N_{x,K}^{2,a\prime}=H_a^{-1}N_{x,K}^{1,a\prime}H_a. \tag{10.6}$$

We take the orthonormal basis $(e_1,\ldots,e_n)$ of $T_xX$ as in Section 9.2.

Also, we introduce another copy of $\Lambda^\cdot(T^*X_g)$, which we denote by $\widehat{\Lambda}^\cdot(T^*X_g)$. For $b>0, U\in T_xX_g$, set

$$c_b(U)=\sqrt{2/b}U^*\wedge-\sqrt{b/2}i_U,\qquad \widehat{c}_b(U)=\sqrt{2/b}\widehat{U}^*\wedge+\sqrt{b/2}i_{\widehat{U}}. \tag{10.7}$$

We define $\dot{R}_K^{TX_g}$ as in (3.21).

*Definition* 10.5. Let $N_{x,K}^{3,a\prime}$ be the operator obtained from $N_{x,K}^{2,a\prime}$ by replacing $c(e_j),\widehat{c}(e_j)$ by $c_{\sqrt{a}}(e_j),\widehat{c}_{\sqrt{a}}(e_j)$ for $1\le j\le m$, while leaving the $c(e_j),\widehat{c}(e_j)$ unchanged for $m+1\le j\le n$.

*Definition* 10.6. Let

$$\begin{aligned}N_{x,K}^{3,0\prime}\in\Big(&\Lambda^\cdot(T^*X_g)\widehat{\otimes}\widehat{\Lambda}^\cdot(T^*X_g)\\ &\widehat{\otimes}\operatorname{End}\left(\Lambda^\cdot\left(N_{X_g/X}^*\right)\right)\widehat{\otimes}\operatorname{End}(F)\widehat{\otimes}\Lambda^\cdot\left(\mathbf{R}^2\right)\Big)_x\widehat{\otimes}\operatorname{Op}_x\end{aligned}$$

be given by

$$N^{3,0\prime}_{x,K} = -\Delta^{TX} + \frac{4sv+1}{4v^2}\left|K^{X_g}\right|^2 + \frac{1}{\sqrt{2}v}\left(K^{X_g\prime} + \widehat{K^{X_g\prime}}\right)ds - \frac{1}{\sqrt{2}}\left(K^{X_g\prime} - \widehat{K^{X_g\prime}}\right)\frac{dv}{v^2} - \frac{dK^{X_g\prime}}{2v} + 2\dot{R}^{TX_g}_{\frac{1}{2}\frac{4sv+1}{v}K/2\pi}. \tag{10.8}$$

THEOREM 10.7. *As* $a \to 0$,

$$N^{3,a\prime}_{x,K} \to N^{3,0\prime}_{x,K}. \tag{10.9}$$

*Proof.* We use equation (10.3) for $N^{a\prime}$. By (9.7), (10.4), and by proceeding as in [BGo1, §7.7], we find that as $a \to 0$, in the given trivialization,

$$-a\Bigg(\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F,u}_{e_i} - \frac{1}{\sqrt{a}}\left(s\left(1-\sqrt{a}s\right)\left(1-\sqrt{a}/v\right) + \frac{1}{2v}\right)\left\langle K^X, e_i\right\rangle + \left(2\sqrt{a}s - 1\right)\frac{c\left(e_i\right)}{2a^{1/4}}ds - \frac{\widehat{c}\left(e_i\right)}{2a^{1/4}}ds\Bigg)^2 \to -\Delta^{TX}. \tag{10.10}$$

The other terms in the right-hand side of (10.3) can be dealt with as in the proof of Theorem 9.8. Observe that, since the variables $\widehat{c}\left(e_i\right)$ always appear with a power of $a$ which is at least $a^{1/4}$, there are no exotic asymptotic expansions like the second one for $\widehat{c}\left(\tau U\right)\left(\sqrt{a}Z\right)$ in (9.23). The proof of our theorem is completed. □

Let $P^{3,0\prime}_{x,K}\left(Z, Z'\right)$ be the smooth kernel associated to the operator $\exp\left(-N^{3,0\prime}_{x,K}\right)$ with respect to the volume $dv_{TX}\left(Z'\right)/\left(2\pi\right)^{n/2}$. We now have the following trivial identity.

PROPOSITION 10.8. *The following identity holds*:

$$gP^{3,0\prime}_{x,K}\left(g^{-1}Z, Z\right) = \frac{1}{2^{n/2}}\exp\left(-\left|\left(g^{-1}-1\right)Z\right|^2/4\right) \exp\Bigg(-\frac{4sv+1}{4v^2}\left|K^{X_g}\right|^2 + dK^{X_g\prime}/2v - 2\dot{R}^{TX_g}_{\frac{1}{2}\frac{4sv+1}{v}K/2\pi} - \frac{1}{\sqrt{2}v}\left(K^{X_g\prime} + \widehat{K^{X_g\prime}}\right)ds + \frac{1}{\sqrt{2}}\left(K^{X_g\prime} - \widehat{K^{X_g\prime}}\right)dv/v^2\Bigg). \tag{10.11}$$

Let $\widehat{\mathrm{Tr_s}} : \Lambda^{\cdot}\left(T^*X_g\right)\widehat{\otimes}\widehat{\Lambda}^{\cdot}\left(T^*X_g\right) \to \mathbf{R}$ be the functional which vanishes in the monomials in the $e^i, \widehat{e}^i, 1 \le i \le m$ of length $< 2m$, and is such that

$$\widehat{\mathrm{Tr_s}}\left[e^1\widehat{e^1}\dots e^m\widehat{e^m}\right] = (-2)^m. \tag{10.12}$$

As in Section 9.5, we use the Berezin integration formalism of Section 3.2, with $X$ replaced by $X_g$.

Proposition 10.9. *The following identity holds*:

(10.13)

$$\left[\int_{N_{X_g/X}} \int^{\widehat{B}} \widehat{\mathrm{Tr_s}} \left[g P^{3,0\prime}_{x,K}\left(g^{-1}Z, Z\right)\right] \frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{n/2}}\right]^{dsdv,\max} =$$

$$-\left[2\sqrt{\pi}\int_{X_g} \mathrm{Tr}^{F|_{X_g}}[g]\,\frac{K^{X_g\prime}}{4\pi}\alpha_{K,2v^2/(4sv+1)}\int^{\widehat{B}} \widehat{K^{X_g\prime}}\exp\left(-\dot{R}^{TX_g}_K/2\pi\right)\right]^{\max}\frac{dsdv}{v^3}.$$

*Proof.* By Proposition 10.8, we get

$$\left[\int_{N_{X_g/X}} \int^{\widehat{B}} \widehat{\mathrm{Tr_s}} \left[g P^{3,0\prime}_{x,K}\left(g^{-1}Z, Z\right)\right] \frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{n/2}}\right]^{dsdv,\max}$$
$$= -\left[\mathrm{Tr}^{F|_{X_g}}[g]\exp\left(-\frac{4sv+1}{4v^2}\left|K^{X_g}\right|^2 + dK^{X_g\prime}/8\pi v\right)\frac{K^{X_g\prime}}{2\sqrt{\pi}}\right.$$
$$\left.\int^{\widehat{B}} \widehat{K^{X_g\prime}}\exp\left(-\dot{R}^{TX_g}_{\frac{4sv+1}{v}K}/2\pi\right)\right]^{\max}\frac{dsdv}{v^3}. \tag{10.14}$$

In (10.14), we now make the rescaling

$$e^i \to ((4sv+1)/v)^{1/2}e^i, \widehat{e}^i \to \left((4sv+1/v)^{-1/2}\right)\widehat{e}^i,$$

and find that (10.14) is equivalent to (10.13). The proof of our proposition is completed. □

10.3. *A proof of equation* (7.18). Using the trivializations indicated in Section 9.2 and the standard fixed point techniques in local index theory as in [BGo1, §7], by (7.9), (10.2) and by Proposition 10.9, we find that for $z \in \mathbf{R}^*$ and $|z|$ small enough, as $a \to 0$,

$$\mathrm{Tr_s}\left[g\exp\left(-L_K - \mathcal{L}\right)\right]^{dsdv}_{t'=a} \to 2n_{4v^2/4sv+1}\frac{dsdv}{v^3}; \tag{10.15}$$

i.e., we have established (7.18) for $k_a = j_a$.

By replacing $j_a$ by $j'_a$, we also get (7.18) for $k_a = j'_a$. The proof of Theorem 7.6 is completed. □

## 11. A proof of Theorem 7.7

The purpose of this section is to prove Theorem 7.7, i.e. to show that the limit as $a \to 0$ of $\mathrm{Tr_s}\left[g\exp\left(-L_K - \mathcal{L}\right)\right]^{(2)}_{t'=a}$ vanishes.

This section is organized as follows. In 11.1, we describe a coordinate system near $X_{g,K}$, and we introduce an adequate Getzler rescaling. Also, we compute the limit as $a \to 0$ of the rescaled form of the operator $L_K + \mathcal{L}$. In 11.2, we compute the fundamental solution for the limit operator. Finally, in 11.3, we prove Theorem 7.7.

Again, the analytic arguments have been kept to a minimum, since they are essentially the same as in [BGo2, §9].

We use the notation of Sections 9 and 10. In particular, we still assume that $K_0 \in \mathfrak{z}(g)$ and that $K = zK_0$, with $z \in \mathbf{R}^*$.

11.1. *A change of coordinates and a Getzler rescaling.* We take $\varepsilon_0$ as in Section 9.2. Take $y_0 \in X_{g,K}$. If $Z \in T_{y_0}X, |Z| \leq 4\varepsilon_0$, we identify $Z$ to $\exp^X_{y_0}(Z) \in X$. Similarly, we identify $\left(\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2)\right)_Z$ to $\left(\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2)\right)_{y_0}$ by parallel transport with respect to the connection ${}^1\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2),u,a}$ along the curve $h \in [0,1] \to hZ \in X$.

*Definition* 11.1. Let $N^{1,a\prime\prime}_{y_0,K}$ be the differential operator,

$$N^{1,a\prime\prime}_{y_0,K} = -\left(1-\rho^2(Z)\right)a\Delta^{TX} + \rho^2(Z)\, i^*_a N^a. \tag{11.1}$$

We still define $H_a : \mathbf{H}_{y_0} \to \mathbf{H}_{y_0}$ as in (9.13).

*Definition* 11.2. Let $N^{2,a\prime\prime}_{y_0,K}$ be the differential operator acting on $\mathbf{H}_{y_0}$,

$$N^{2,a\prime\prime}_{y_0,K} = H_a^{-1} N^{2,a\prime\prime}_{y_0,K} H_a. \tag{11.2}$$

We identify the normal bundle $N_{X_{g,K}/X_g}$ to the orthogonal bundle to $TX_{g,K}$ in $TX_g|_{X_{g,K}}$. Let $e_1, \dots e_\ell$ be an orthonormal basis of $T_{y_0}X_{g,K}$, let $e_{\ell+1}, \dots, e_m$ be an orthonormal basis of $N_{X_{g,K}/X_g,y_0}$, let $e_{m+1}, \dots, e_n$ be an orthonormal basis of $N_{X_g/X,y_0}$. Recall that when $U \in T_{y_0}X_g$, $c_b(U)$ and $\widehat{c}_b(U)$ are as defined in (10.7).

*Definition* 11.3. Let $N^{3,a\prime\prime}_{y_0,K}$ be the operator obtained from $N^{2,a\prime\prime}_{y_0,K}$ by replacing $c(e_j), \widehat{c}(e_j)$ by $c_{\sqrt{a}}(e_j), \widehat{c}_{\sqrt{a}}(e_j)$ for $1 \leq j \leq \ell$, while leaving the $c(e_j), \widehat{c}(e_j)$ unchanged for $\ell+1 \leq j \leq n$.

In the sequel, the tensors are evaluated at $y_0$.

*Definition* 11.4. Let

$$N^{3,0\prime\prime}_{y_0,K} \in \Big(\Lambda^{\cdot}(T^*X_{g,K})\,\widehat{\otimes}\,\widehat{\Lambda}^{\cdot}(T^*X_{g,K}) \\ \widehat{\otimes}\,\mathrm{End}\left(\Lambda^{\cdot}\left(N^*_{X_{g,K}/X}\right)\widehat{\otimes}F\right)\widehat{\otimes}\Lambda^{\cdot}\left(\mathbf{R}^2\right)\Big)_{y_0}\widehat{\otimes}\,\mathrm{Op}_{y_0}$$

be given by

(11.3)

$$\begin{aligned}N^{3,0\prime\prime}_{y_0,K} = &-\left(\nabla_{e_i} - \left(s\left(1-s\right)\left(1-1/v\right)+\frac{1}{2v}\right)\left\langle \nabla^{TX}_{Z}K^X, e_i\right\rangle\right)^2\\ &+\left(4s\left(1-s\right)+\frac{\left(2s-1\right)^2}{v}\right)\frac{1}{4v}\left|\nabla^{TX}_{Z}K^X\right|^2\\ &-2s\left(1-s\right)\left(1-\frac{1}{v}\right)\widehat{c}\left(\nabla^{TX}_{Z}K^X\right)ds\\ &-\left(c\left(\nabla^{TX}_{Z}K^X\right)+\left(2s-1\right)\widehat{c}\left(\nabla^{TX}_{Z}K^X\right)\right)dv/2v^2+2\dot{R}^{TX_{g,K}}\\ &+\frac{1}{4}\sum_{\ell+1\leq i,j\leq n}\left\langle \nabla^{TX}_{e_i}K^X, e_j\right\rangle\\ &\left(-\frac{1}{v}c\left(e_i\right)c\left(e_j\right)+\left(4s\left(1-s\right)+\frac{\left(2s-1\right)^2}{v}\right)\widehat{c}\left(e_i\right)\widehat{c}\left(e_j\right)\right).\end{aligned}$$

THEOREM 11.5. *For* $(s,v)\in[0,1]\times\mathbf{R}^*_+$,

$$N^{3,a\prime\prime}_{y_0,K}\to N^{3,0\prime\prime}_{y_0,K}. \tag{11.4}$$

*Proof.* We use again equation (9.1) for $N^a$. By (9.5), (9.7), since $K^X$ vanishes on $X_{g,K}$, we find as $a\to 0$,

$$\begin{aligned}-a\Bigg(&\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2)}_{e_i}-\frac{1}{a}\left(s\left(1-s\right)\left(1-1/v\right)+1/2v\right)\left\langle K^X, e_i\right\rangle\\ &+\left(2s-1\right)\frac{c\left(e_i\right)}{2\sqrt{a}}ds-\frac{\widehat{c}\left(e_i\right)}{2\sqrt{a}}ds\Bigg)^2\\ &\to -\left(\nabla_{e_i}-\left(s\left(1-s\right)\left(1-1/v\right)+1/2v\right)\left\langle \nabla^{TX}_{Z}K^X, e_i\right\rangle\right)^2.\end{aligned} \tag{11.5}$$

Moreover, as $a\to 0$,

$$\frac{\left|K^X\left(\sqrt{a}Z\right)\right|^2}{a}\to\left|\nabla^{TX}_{Z}K^X\right|^2. \tag{11.6}$$

By using the same notation as in the proof of Theorem 9.8, we deduce from (9.21) that as $a\to 0$,

$$\begin{aligned}\left[\frac{c\left(K^X\right)}{\sqrt{a}}\right]^3_a &= c\left(\nabla^{TX}_{Z}K^X\right)+\mathcal{O}\left(a^{1/4}\left|Z\right|^2\right)+\mathcal{O}\left(\sqrt{a}\left|Z\right|^2\right),\\ \left[\frac{\widehat{c}\left(K^X\right)}{\sqrt{a}}\right]^3_a &= \widehat{c}\left(\nabla^{TX}_{Z}K^X\right)+\mathcal{O}\left(a^{1/4}\left|Z\right|^2\right)+\mathcal{O}\left(\sqrt{a}\left|Z\right|^2\right).\end{aligned} \tag{11.7}$$

We now use the notation which we introduced after equation (9.22). Since $h \in [0,1] \to hZ$ is a geodesic in $X$, $Z$ is parallel with respect to $\nabla^{TX}$ along this geodesic. By (9.6),

$$c(\tau e_i)(\sqrt{a}Z) = c(e_i) + (2s-1)\langle Z, e_i\rangle ds, \qquad \widehat{c}(\tau e_i)(\sqrt{a}Z) = \widehat{c}(e_i) + \langle Z, e_i\rangle ds. \tag{11.8}$$

From (11.8), we find that for $1 \leq i \leq \ell$,

$$\Big[c_{\sqrt{a}}(\tau e_i)(\sqrt{a}Z)\Big]_a^3 = \frac{\sqrt{2}}{a^{1/4}} e^i \wedge -\frac{a^{1/4}}{\sqrt{2}} i_{e_i} + (2s-1)\langle Z, e_i\rangle ds, \qquad \Big[\widehat{c}_{\sqrt{a}}\tau(e_i)(\sqrt{a}Z)\Big]_a^3 = \frac{\sqrt{2}}{a^{1/4}} \widehat{e}^i \wedge +\frac{a^{1/4}}{\sqrt{2}} i_{\widehat{e_i}} + \langle Z, e_i\rangle ds. \tag{11.9}$$

Also, by the first equation in (2.7), we find that for $1 \leq i, j \leq n$,

$$\Big\langle \nabla^{TX}_{\tau e_i} K^X(\sqrt{a}Z), \tau e_j\Big\rangle = \Big\langle \nabla^{TX}_{e_i} K^X(y_0), e_j\Big\rangle + \mathcal{O}\Big(a|Z|^2\Big). \tag{11.10}$$

Moreover if $1 \leq i \leq \ell, 1 \leq j \leq n$,

$$\Big\langle \nabla^{TX}_{e_i} K^X(y_0), e_j\Big\rangle = 0. \tag{11.11}$$

By (11.9)–(11.11), we find that as $a \to 0$,

$$\Big[\frac{a}{8}\Big\langle e_k, R^{TX}(\sqrt{a}Z)(e_i, e_j)e_l\Big\rangle c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_l)\Big]_a^3 = 2\dot{R}^{TX_{g,K}}(y_0) + \mathcal{O}\Big(a^{1/2}\Big), \tag{11.12}$$

$$\begin{aligned}\Big[\frac{1}{4}\Big\langle \nabla^{TX}_{e_i} K^X(\sqrt{a}Z), e_j\Big\rangle c(e_i)c(e_j)\Big]_a^3 &= \frac{1}{4}\sum_{\ell+1\leq i,j\leq n}\Big\langle \nabla^{TX}_{e_i} K^X(y_0), e_j\Big\rangle c(e_i)c(e_j) \\ &\quad -\frac{1}{2}(2s-1)c\Big(\nabla^{TX}_Z K^X\Big)ds + \mathcal{O}\Big(\sqrt{a}|Z|^2\Big), \\ \Big[\frac{1}{4}\Big\langle \nabla^{TX}_{e_i} K^X(\sqrt{a}Z), e_j\Big\rangle \widehat{c}(e_i)\widehat{c}(e_j)\Big]_a^3 &= \frac{1}{4}\sum_{\ell+1\leq i,j\leq n}\Big\langle \nabla^{TX}_{e_i} K^X(y_0), e_j\Big\rangle \widehat{c}(e_i)\widehat{c}(e_j) \\ &\quad -\frac{1}{2}\widehat{c}\Big(\nabla^{TX}_Z K^X\Big)ds + \mathcal{O}\Big(\sqrt{a}|Z|^2\Big).\end{aligned} \tag{11.13}$$

From (11.5)–(11.12), we get (11.4). The proof of our theorem is completed. □

11.2 *The heat kernel associated to the operator* $N^{3,0\prime\prime}_{y_0,K}$.

*Definition* 11.6. Let $\mathfrak{N}^{3,0\prime\prime}_{y_0,K}$ be the operator,

$$\begin{aligned}\mathfrak{N}^{3,0\prime\prime}_{y_0,K} = & -\left(\nabla_{e_i} - \left(s\left(1-s\right)\left(1-1/v\right)+\frac{1}{2v}\right)\left\langle \nabla^{TX}_{Z} K^X, e_i\right\rangle\right)^2 \\ & + \left(4s\left(1-s\right)+\frac{\left(2s-1\right)^2}{v}\right)\frac{1}{4v}\left|\nabla^{TX}_{Z}K^X\right|^2 \\ & + \frac{1}{4}\sum_{\ell+1\leq i,j\leq n}\left\langle \nabla^{TX}_{e_i}K^X, e_j\right\rangle\left(-\frac{1}{v}c\left(e_i\right)c\left(e_j\right)\right. \\ & \left. + \left(4s\left(1-s\right)+\frac{\left(2s-1\right)^2}{v}\right)\widehat{c}\left(e_i\right)\widehat{c}\left(e_j\right)\right).\end{aligned} \tag{11.14}$$

Observe that $\mathfrak{N}^{3,0\prime\prime}_{y_0,K}$ is obtained from the operator $N^{3,0\prime\prime}_{y_0,K}$ by making $ds=0, dv=0$ and $\dot{R}^{TX_{g,K}}=0$.

Recall that $K_0 \in \mathfrak{z}\left(g\right)$ is fixed and that $K=zK_0$, with $z\in\mathbf{R}^*$.

*Definition* 11.7. For $z\in\mathbf{R}^*$, and $|z|$ small enough, let $\mathfrak{P}^{3,0\prime\prime}_{y_0,K}\left(Z,Z'\right), Z, Z'\in N_{X_{g,K}/X}$ be the smooth kernel associated to the operator $\exp\left(-\mathfrak{N}^{3,0\prime\prime}_{y_0,K}\right)$ with respect to the volume $dv_{N_{X_K/X}}\left(Z'\right)/\left(2\pi\right)^{\dim N_{X_{g,K}/X}/2}$.

Observe that the operator $\mathfrak{N}^{3,0\prime\prime}_{y_0,K}$ is not self-adjoint. As explained at length in the introduction to [BGo1] and also in [BGo1, §7.13], for $z\in\mathbf{R}^*$ and $|z|$ small enough, one can still make sense of the smooth kernel $\mathfrak{P}^{3,0\prime\prime}_{y_0,K}\left(Z,Z'\right)$, either by using analyticity in the variable $z$, or by using a truncation procedure.

Put

$$\begin{aligned}C &= -2\left(s\left(1-s\right)\left(1-1/v\right)+1/2v\right)\nabla^{TX}_{\cdot}K^X, \\ H &= -\left(4s\left(1-s\right)+\left(2s-1\right)^2/v\right)\frac{1}{v}\left(\nabla^{TX}_{\cdot}K^X\right)^2, \\ Q &= 2s\left(1-s\right)\left(1-1/v\right)\nabla^{TX}_{\cdot}K^X.\end{aligned} \tag{11.15}$$

Then

$$C^2+H=Q^2 \tag{11.16}$$

Proposition 11.8. *For* $z\in\mathbf{R}^*$, *and for* $|z|$ *small enough, the following identity holds*:

(11.17)

$$\begin{aligned}g\mathfrak{P}^{3,0\prime\prime}_{x,K}\left(g^{-1}Z,Z\right) = 2^{-\dim N_{X_{g,K}/X}/2}\left[\det\left(\frac{Q}{\sinh\left(Q\right)}\right)\right]^{1/2} \\ \exp\left(-\left\langle\frac{Q}{\sinh\left(Q\right)}\sinh\left(\left(C-B+Q\right)/2\right)\sinh\left(\left(-C+B+Q\right)/2\right)Z,Z\right\rangle\right)\end{aligned}$$

$$\exp\left(-\frac{1}{4}\sum\left\langle\nabla^{TX}_{e_i}K^X,e_j\right\rangle\left(-\frac{1}{v}c\left(e_i\right)c\left(e_j\right)+\left(4s\left(1-s\right)+\frac{\left(2s-1\right)^2}{v}\right)\right.\right.$$
$$\left.\left.\widehat{c}\left(e_i\right)\widehat{c}\left(e_j\right)\right)\right).$$

*Proof.* This identity follows from Mehler's formula. A proof is given in [B10, eqs. (4.49) and (4.50)]. □

PROPOSITION 11.9. *For $z \in \mathbf{R}^*$ and for $|z|$ small enough, the following identity holds*:

$$\begin{aligned}(11.18)\quad &\operatorname{Tr_s}^{\Lambda^{\cdot}\left(N^*_{X_{g,K}/X}\right)}\left[g\exp\left(-\frac{1}{4}\sum\left\langle\nabla^{TX}_{e_i}K^X,e_j\right\rangle\right.\right.\\&\left.\left.\left(-\frac{1}{v}c\left(e_i\right)c\left(e_j\right)+\left(4s\left(1-s\right)+\frac{\left(2s-1\right)^2}{v}\right)\widehat{c}\left(e_i\right)\widehat{c}\left(e_j\right)\right)\right)\right]\\&=\left[\det\left(4\sinh\left(\left(C-B+Q\right)/2\right)\right)\sinh\left(\left(-C+B+Q\right)/2\right)\right]^{1/2}.\end{aligned}$$

*Proof.* Observe that

$$\begin{aligned}(11.19)\qquad C+Q&=-\frac{1}{v}\nabla^{TX}K^X,\\-C+Q&=\left(4s\left(1-s\right)+\frac{\left(2s-1\right)^2}{v}\right)\nabla^{TX}_{\cdot}K^X.\end{aligned}$$

By [BGo2, Prop. 4.9], by proceeding as in Proposition 9.13, using the obvious analogue of (9.35) for $N_{X_{g,K}/X}$, and by (11.19), we get (11.18). □

PROPOSITION 11.10. *For $z \in \mathbf{R}^*$ and $|z|$ small enough, the following identity holds*:

$$(11.20)\qquad \int_{N_{X_{g,K}/X}}\operatorname{Tr_s}^{\Lambda^{\cdot}\left(N^*_{X_{g,K}/X}\right)}\left[g\mathfrak{P}^{3,0\prime\prime}_{x,K}\left(g^{-1}Z,Z\right)\right]\frac{dv_{N_{X_{g,K}/X}}\left(Z\right)}{\left(2\pi\right)^{\dim N_{X_{g,K}/X}/2}}=1.$$

*Proof.* This is a consequence of Propositions 11.8 and 11.9. □

*Definition* 11.11. For $z \in \mathbf{R}^*$ and $|z|$ small enough, let $P^{3,0\prime\prime}_{y_0,K}\left(Z,Z'\right), Z, Z' \in T_{y_0}X$ be the smooth kernel associated to the operator $\exp\left(-N^{3,0\prime\prime}_{y_0,K}\right)$ with respect to $dv_{TX}\left(Z'\right)/\left(2\pi\right)^{n/2}$.

Here, we will not compute the kernel $P^{3,0\prime\prime}_{y_0,K}\left(Z,Z'\right)$ explicitly. However, we will prove the following result.

THEOREM 11.12. *For $z \in \mathbf{R}^*$ and $|z|$ small enough, the following identity holds*:

(11.21)
$$\int_{N_{X_{g,K}/X}} \int^{\widehat{B}} \mathrm{Tr_s}^{\Lambda^{\cdot}\left(N^*_{X_{g,K}/X}\right)} \left[g P^{3,0\prime\prime}_{y_0,K}\left(g^{-1}Z, Z\right)\right]^{dsdv} \frac{dv_{N_{X_{g,K}/X}}(Z)}{(2\pi)^{n/2}} = 0.$$

*Proof.* Let $\overline{\mathfrak{N}}^{3,0\prime\prime}_{y_0,K}$ be obtained from $N^{3,0\prime\prime}_{y_0,K}$ by making $\dot{R}^{TX_{g,K}} = 0$. Let $\overline{\mathfrak{P}}^{3,0\prime\prime}_{x,K}(Z,Z')$, $Z, Z' \in N_{X_{g,K}/X}$ be the smooth kernel associated to $\exp\left(-\overline{\mathfrak{N}}^{3,0\prime\prime}_{y_0,K}\right)$ and the volume form $dv_{N_{X_{g,K}/X}}(Z') / (2\pi)^{\dim N_{X_{g,K}/X}/2}$.

We will use the Berezin integration formalism of Sections 3.2 and 3.3, with $X$ replaced by $X_{g,K}$. By (11.3), we have the obvious

(11.22)
$$\begin{aligned}\int_{N_{X_{g,K}/X}} \int^{\widehat{B}} \mathrm{Tr_s}^{\Lambda^{\cdot}\left(N^*_{X_{g,K}/X}\right)} \left[g P^{3,0\prime\prime}_{y_0,K}\left(g^{-1}Z, Z\right)\right]^{dsdv} \frac{dv_{N_{X_{g,K}/X}}(Z)}{(2\pi)^{n/2}} \\ = e\left(TX_{g,K}, \nabla^{TX_{g,K}}\right) \int_{N_{X_{g,K}/X}} \mathrm{Tr_s}^{\Lambda^{\cdot}\left(N^*_{X_{g,K}/X}\right)} \left[g \overline{\mathfrak{P}}^{3,0\prime\prime}_{x,K}\left(g^{-1}Z, Z\right)\right]^{dsdv} \\ \frac{dv_{N_{X_{g,K}/X}}(Z)}{(2\pi)^{\dim N_{X_{g,K}/X}/2}}.\end{aligned}$$

Recall that we identify $N_{X_g/X}$ to the orthogonal bundle to $TX_g$ in $TX|_{X_g}$. We will use the same conventions for other normal bundles. Let $\widetilde{N}$ be the orthogonal bundle to $N_{X_{g,K}/X_K} \oplus N_{X_{g,K}/X_g}$ in $N_{X_{g,K}/X}$. Then we have the orthogonal splitting,

$$N_{X_{g,K}/X} = N_{X_{g,K}/X_K} \oplus N_{X_{g,K}/X_g} \oplus \widetilde{N}. \tag{11.23}$$

The vector bundles in (11.23) are stable under the action of $g$ or of $\nabla^{TX}_{\cdot} K^X|_{X_{g,K}}$.

By using the involution $I : Z \to -Z$ on each of the corresponding eigenbundles of $g$ and $\nabla^{TX}_{\cdot} K^X|_{X_{g,K}}$, and also Proposition 11.10, one finds easily that the integral appearing in the right-hand side of (11.22) can be expressed as the sum of the integrals of the kernels associated to the operators $\overline{\mathfrak{N}}^{3,0\prime\prime}_{y_0,K}$ corresponding to the three vector bundles which appear in the right-hand side of (11.23).

Since $\nabla^{TX}_{\cdot} K^X|_{X_{g,K}}$ vanishes on $N_{X_{g,K}/X_K}$, the contribution of $N_{X_{g,K}/X_K}$ vanishes identically. Also since $\nabla^{TX}_{\cdot} K^X|_{X_{g,K}}$ acts as an antisymmetric invertible operator on $N_{X_{g,K}/X_g}$ and on $\widetilde{N}$, it follows that these vector bundles are even dimensional, and that the corresponding $-1$ eigenbundle of $g$ in $\widetilde{N}$ is also even dimensional. In particular, $g$ acts as the identity mapping on $N_{X_{g,K}/X_g}$ and as an orientation preserving map on $\widetilde{N}$. Therefore there is a real antisymmetric endomorphism $B = B_{y_0} \in \operatorname{End}\left(\widetilde{N}\right)_{y_0}$ such that

$$g|_{\widetilde{N}_{y_0}} = e^B. \tag{11.24}$$

Of course, we can extend $B$ to an antisymmetric section of $\operatorname{End}(N_{X_{g,K}/X_g} \oplus \widetilde{N})_{y_0}$, which vanishes on $N_{X_{g,K}/X_g, y_0}$, so that

$$g|_{N_{X_{g,K}/X_g,y_0} \oplus \widetilde{N}, y_0} = e^B. \tag{11.25}$$

Let $e_1, \ldots, e_r$ be an orthonormal basis of $\widetilde{N}$. From (11.25), we get

$$g|_{\Lambda^\cdot\left(N^*_{X_{g,K}/X_g} \oplus \widetilde{N}^*\right)} = \exp\left(\frac{1}{4}\langle Be_i, e_j\rangle \left(c\left(e_i\right)c\left(e_j\right) - \widehat{c}\left(e_i\right)\widehat{c}\left(e_j\right)\right)\right). \tag{11.26}$$

From (11.26), we deduce that $g$ is even in the variables $c\left(e_i\right)$ and also in the variables $\widehat{c}\left(e_i\right)$.

It follows from the above that when evaluating

$$\operatorname{Tr_s}^{\Lambda^\cdot\left(N^*_{X_{g,K}/X_g} \oplus \widetilde{N}^*\right)}\left[g\overline{\mathfrak{P}}^{3,0\prime\prime}_{y_0,K}\left(g^{-1}Z, Z\right)\right]^{dsdv},$$

only *odd* monomials in the $c\left(e_i\right)$ and the $\widehat{c}\left(e_i\right)$ contribute to the supertrace. However, since $N_{X_{g,K}/X_g} \oplus \widetilde{N}$ is even-dimensional, by the obvious analogue of (9.35), the supertrace of such monomials vanishes. The proof of our theorem is completed. □

11.3. *A proof of Theorem* 7.7. By proceeding as in [BGo1, §9], we find that for $z \in \mathbf{R}^*$ and $|z|$ small enough, as $a \to 0$,

(11.27)

$$i_a^* \operatorname{Tr_s}\left[g \exp\left(-L_K - \mathcal{L}\right)\right]^{(2)}_{t'=a} \to \int_{X_{g,K}} \int^{\widehat{B}} \operatorname{Tr_s}^{\Lambda^\cdot\left(N^*_{X_{g,K}/X}\right)}\left[gP^{3,0\prime\prime}_{\cdot,K}\left(g^{-1}Z, Z\right)\right]^{dsdv} \frac{dv_{N_{X_{g,K}/X}}(Z)}{(2\pi)^{n/2}}.$$

Using (11.21) in Theorem 11.12 and (11.27), we get (7.19). The proof of Theorem 7.7 is complete.

## 12. A proof of Theorem 7.8

The purpose of this section is to establish Theorem 7.8, i.e. to obtain the asymptotics of $\operatorname{Tr_s}\left[g\exp\left(-L_K - \mathcal{L}\right)\right]$ as $a \to 0$ in the range $(s, v) \in [0,1] \times [a, 1]$. The main difficulty is to obtain a precise estimate on the remainder. As explained in Remark 7.9, our theorem is stronger than Theorems 7.5 and 7.6, which were established in Sections 9 and 10.

In this section, we provide the full analytic machinery, which allows us to use the techniques developed in [BL], and more especially in [BGo1, §9], where a similar problem was considered, in a holomorphic context. We still use the Getzler rescaling techniques introduced in Sections 9–11. However, since our range of parameters covers the cases which were considered in these sections, our proof incorporates the techniques which are used there.

Let us describe here a few features of the proof:

- The Clifford variables of type $c$ and $\widehat{c}$ are both rescaled.
- The cases $w \in \,]0,1]$ and $w \geq 1$ are handled with different techniques. In particular when $w$ is 'small', the term $\left|K^X\right|^2/w$ forces localization near $X_{g,K}$. This phenomenon has to be explicitly taken care of in our estimates.
- Recall that $w$ does depend on $a$. Still, when $a \to 0$, the idea will be to 'freeze' $w$ while taking the limit as $a \to 0$ of all the other terms, in order to establish the appropriate estimates.

This section is organized as follows. In 12.1, we use finite propagation speed to show that our estimates can be localized near $X_g$. In 12.2, we introduce a new Grassmann variable $\overline{ds}$, and a new operator $\mathcal{N}^a$, which is obtained from $N^a$ by a trivial rescaling.

Sections 12.3–12.7 are devoted to the difficult case $w \in \,]0,1]$. In 12.3, we consider a coordinate system near $X_{g,K}$ and a corresponding trivialization, which depends explicitly on the choice of $Z_0 \in N_{X_{g,K}/X,y_0}$, and we introduce Getzler rescalings on the Clifford variables $c$ and $\widehat{c}$. In 12.4, we define a family of norms adapted to the problem which is considered here, in particular with respect to the Getzler rescaling on both kinds of Clifford variables. In 12.5, we show that the rescaled operators $\mathcal{N}^{3,a}_{\sqrt{w}Z_0}$ verify uniform estimates with respect to these norms. In 12.6, we prove a key estimate for a difference operator $\mathcal{N}^{3,a}_{\sqrt{w}Z_0} - \overline{\mathcal{N}}^{3,a}_{\sqrt{w}Z_0}$, one critical feature being that the operator $\overline{\mathcal{N}}^{3,a}_{\sqrt{w}Z_0}$, while being a 'local' operator, still depends on $a$. In 12.7, we prove Theorem 7.8 when $w \in \,]0,1]$. Finally, in Section 12.8, we consider the case $w \geq 1$.

In the whole section, we assume that $(s,v) \in ]0,1[\times[a,1]$. Also, we use the notation of Sections 7–11.

12.1. *Finite propagation speed and localization.*

*Definition* 12.1. Let $\mathcal{A}^a$ be the operator obtained from the operator $aN^a$ by replacing $ds, dv$ by $a^{3/2}ds, \sqrt{a}dv$.

Let $|\ |_0, |\ |_1$ be natural norms on the Sobolev spaces of order $0,1$ of sections of $F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2)$.

THEOREM 12.2. *Given $\beta > 0$, there exist $C_1 > 0, C_2 > 0, C_2'(\beta) > 0$, $C_3(\beta) > 0$, $C_3' > 0, C_4 > 0, C_5(\beta) > 0$ such that if $K \in \mathfrak{g}, |K| \leq \beta, a \in ]0,1]$, if $s, s' \in \Omega^{\cdot}(X, F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2))$,*

$$\begin{aligned} \operatorname{Re}\langle \mathcal{A}^a s, s\rangle &\geq C_1 a^2 |s|_1^2 - \Big(C_2 a^2 + C_2'(\beta)\Big)|s|_0^2, \\ |\operatorname{Im}\langle \mathcal{A}^a s, s\rangle| &\leq a\,(C_3(\beta) + C_3' a)\,|s|_1 |s|_0, \\ |\langle \mathcal{A}^a s, s'\rangle| &\leq C_4(a|s|_1 + C_5(\beta)|s|_0)(a|s'|_1 + C_5(\beta)|s'|_0). \end{aligned} \tag{12.1}$$

*Moreover, as $\beta \to 0$, $C_2'(\beta), C_3(\beta), C_5(\beta) \to 0$.*

*Proof.* The key fact is that, by (9.1), all the coefficients of the operator $\mathcal{A}^a$ remain uniformly bounded for $a \in ]0,1]$. The proof of (12.1) is then similar to the proof of [BGo1, Th. 7.11]. □

*Remark* 12.3. Observe that in [BGo1, Th. 7.11], the term $C_3' a$ does not appear.

Recall that $a_X$ is the injectivity radius of $X$. Let $\alpha \in ]0, \frac{a_X}{8}]$. The precise value of $\alpha$ will be fixed later. The constants $C > 0, C' > 0 \ldots$ may depend on the choice of $\alpha$.

Let $f : \mathbf{R} \to [0,1]$ be a smooth even function such that

$$\begin{aligned} f(s) &= 1 \text{ for } |s| \leq \frac{\alpha}{\sqrt{2}}, \\ &= 0 \text{ for } |s| \geq \alpha. \end{aligned} \tag{12.2}$$

Set

$$g(s) = 1 - f(s). \tag{12.3}$$

*Definition* 12.4. For $a > 0, b \in \mathbf{C}$, put

$$\begin{aligned} F_a(b) &= \int_{-\infty}^{+\infty} \exp(isb)\exp\Big(-s^2/4\Big)\, f(\sqrt{a}s)\frac{ds}{\sqrt{4\pi}}, \\ G_a(b) &= \int_{-\infty}^{+\infty} \exp(isb)\exp\Big(-s^2/4\Big)\, g(\sqrt{a}s)\frac{ds}{\sqrt{4\pi}}. \end{aligned} \tag{12.4}$$

Then $F_a(b), G_a(b)$ are even holomorphic functions of $b$ such that

$$\exp(-b^2) = F_a(b) + G_a(b). \tag{12.5}$$

Moreover $F_a$ and $G_a$ both lie in the Schwartz space $S(\mathbf{R})$.

Put

$$I_a(b) = \int_{-\infty}^{+\infty} \exp(isb/a)\exp\Big(-s^2/4a\Big)\, g(s)\frac{ds}{\sqrt{4\pi a}}. \tag{12.6}$$

Then

$$I_a(b) = G_a\left(b/\sqrt{a}\right). \tag{12.7}$$

By (12.2), (12.6), we find that given $m, m' \in \mathbf{N}$, there exist $C > 0, C' > 0$ such that if $a \in \,]0,1], b \in \mathbf{C}, |\mathrm{Im}(b)| \leq \frac{\alpha}{8}$,

$$|a|^m |I_a^{(m')}(b)| \leq C \exp(-C'/a). \tag{12.8}$$

Clearly, there exist uniquely defined holomorphic functions $\widetilde{F}_a(b), \widetilde{G}_a(b), \widetilde{I}_a(b)$ such that

$$F_a(b) = \widetilde{F}_a(b^2), \qquad G_a(b) = \widetilde{G}_a(b^2), \qquad I_a(b) = \widetilde{I}_a(b^2). \tag{12.9}$$

By (12.5), (12.7),

$$\begin{aligned} \exp(-a) &= \widetilde{F}_a(b) + \widetilde{G}_a(b), \\ \widetilde{I}_a(b) &= \widetilde{G}_a(b/a). \end{aligned} \tag{12.10}$$

For $c > 0$, set

$$\begin{aligned} V_c &= \left\{ \lambda \in \mathbf{C}, \mathrm{Re}\,(\lambda) \geq \frac{\mathrm{Im}\,(\lambda)^2}{4c^2} - c^2 \right\}, \\ \Gamma_c &= \left\{ \lambda \in \mathbf{C}, \mathrm{Re}\,(\lambda) = \frac{\mathrm{Im}\,(\lambda)^2}{4c^2} - c^2 \right\}. \end{aligned} \tag{12.11}$$

Then $V_c, \Gamma_c$ are the image of $\{\lambda \in \mathbf{C}, |\mathrm{Im}\,(\lambda)| \leq c\}, \{\lambda \in \mathbf{C}, |\mathrm{Im}\,(\lambda)| = c\}$ by the map $\lambda \to \lambda^2$.

By (12.8), we find that if $\lambda \in V_{\alpha/8}$, then

$$|\lambda|^m |\widetilde{I}_a^{(m')}(\lambda)| \leq C \exp(-C'/a). \tag{12.12}$$

By (12.10),

$$\exp\left(-N^a\right) = \widetilde{F}_a\left(N^a\right) + \widetilde{I}_a\left(aN^a\right). \tag{12.13}$$

If $H$ is a Hilbert space and if $A \in \mathcal{L}\,(H)$ is trace class, set

$$\|A\|_1 = \mathrm{Tr}\left[\left(A^* A\right)^{1/2}\right]. \tag{12.14}$$

Then $\|A\|_1$ is a norm on the vector space of trace class operators.

THEOREM 12.5. *There exist $\beta > 0, C > 0, C' > 0$ such that if $K \in \mathfrak{g}$, $|K| \leq \beta, a \in \,]0,1]$,*

$$\left\| \widetilde{I}_a\left(\mathcal{A}^a\right) \right\|_1 \leq C \exp\left(-C'/a\right). \tag{12.15}$$

*Proof.* By Theorem 12.2, the proof of our theorem is the same as the proof of [BGo1, Th. 7.15]. In fact, for $q \in \mathbf{N}$, let $\widetilde{I}_{a,q}(\lambda)$ be the holomorphic function on $\mathbf{C}$, which is characterized by the following two properties:

$$\begin{aligned} &\lim_{\lambda \to +\infty} \widetilde{I}_{a,q}(\lambda) = 0, \\ &\frac{\widetilde{I}_{a,q}^{(q)}(\lambda)}{(q-1)!} = \widetilde{I}_a(\lambda). \end{aligned} \tag{12.16}$$

By (12.12), (12.16), if $\lambda \in V_{\alpha/8}$,

$$|\lambda|^m |\widetilde{I}_{a,q}(\lambda)| \leq C \exp(-C'/a). \tag{12.17}$$

We claim that, by Theorem 12.2, we can use the same methods as in [BGo1, §7.3] to establish our theorem. Still, as explained in Remark 12.3, the estimates in (12.1) contain the extra constant $C_3'a$ with respect to the corresponding in [BGo1, Th. 7.11]. Still inspection of the proof of the simple estimates [BGo1, equations (7.56)–(7.60)] shows immediately that replacing $C_3(\beta)$ in [BGo1] by $C_3(\beta) + C_3'a$ is irrelevant.

We can then proceed as in [BGo1, Th. 7.15], and find that for $\beta > 0$ and $a \in \,]0,1]$ small enough,

$$\widetilde{I}_a(\mathcal{A}^a) = \frac{1}{2\pi i} \int_{\Gamma_{\alpha/8}} \widetilde{I}_{a,q}(\lambda)(\lambda - \mathcal{A}^a)^{-q} \, d\lambda. \tag{12.18}$$

Using (12.18) and proceeding as in [BGo1], we get (12.15). The proof of our theorem is completed. □

By (9.3), (12.13), (12.15), we find that to establish (7.20) in Theorem 7.8, we may as well replace $\mathrm{Tr_s}\left[g \exp\left(-L_K - N^a\right)\right]^{(2)}_{t'=a}$ by $\mathrm{Tr_s}\left[g\widetilde{F}_a(N^a)\right]^{dsdv}$. This will be done systematically in the sequel.

Let $\widetilde{F}_a(N^a)(x,x')$ be the smooth kernel associated to the operator $\widetilde{F}_a(N^a)$ with respect to the volume $dv_X(x')/(2\pi)^{n/2}$. Then the kernel associated to the operator $g\widetilde{F}_a(N^a)$ is just $g\widetilde{F}_a(N^a)(g^{-1}x, x')$. Moreover,

$$\mathrm{Tr_s}\left[g\widetilde{F}_a(N^a)\right]^{dsdv} = \int_X \mathrm{Tr_s}\left[g\widetilde{F}_a(N^a)\left(g^{-1}x, x\right)\right]^{dsdv} \frac{dv_X(x)}{(2\pi)^{n/2}}. \tag{12.19}$$

By (12.4),

$$\widetilde{F}_a(N^a) = 2\int_0^{+\infty} \cos\left(s\sqrt{N^a}\right) \exp\left(-s^2/4\right) f(\sqrt{a}s) \frac{ds}{\sqrt{4\pi}}. \tag{12.20}$$

The principal symbol of the differential operator $N^a$ is equal to $a|\xi|^2$. Also $f(\sqrt{a}s)$ vanishes for $|\sqrt{a}s| \geq \alpha$.

Using finite propagation speed for solutions of hyperbolic equations [ChP, §7.8], [T, §4.4], we find that given $x \in X$, $\widetilde{F}_a(N^a)(x,.)$ vanishes on $X \setminus B^X(x,\alpha)$, and depends only on the restriction of the operator $N^a$ to the ball $B^X(x,\alpha)$. Therefore, we have shown that the proof of (7.20) can be made local on $X$. Moreover $g\widetilde{F}_a(g^{-1}x,x)$ vanishes if $d^X(g^{-1}x,x) \geq \alpha$.

Now we explain our choice of $\alpha$. We use the notation of Section 9.2, where $\varepsilon_0$ was defined. We will assume that $\alpha \in \,]0,\varepsilon_0]$ is small enough so that if $x \in X, d^X(g^{-1}x,x) \leq \alpha$, then $x \in \mathcal{V}_{\varepsilon_0}$. By (12.19) and by the above considerations, it follows that for $\beta > 0$ small enough, for $K \in \mathfrak{z}(g)$ such that $|K| \leq \beta$, our proof of (7.20) has been localized on the $\varepsilon_0$-neighbourhood $\mathcal{V}_{\varepsilon_0}$ of $X_g$.

12.2. *A new Grassmann variable $\overline{ds}$.* Recall that $w$ was defined in (7.13). Note that

$$s(1-s)(1-a/v)+\frac{a}{2v}=\frac{v}{w}+\frac{a}{4v}. \tag{12.21}$$

Let $\overline{ds}$ be another odd Grassmann variable, which anticommutes with the other odd Grassmann or Clifford variables.

*Definition* 12.6. Let $\mathcal{N}^a$ be the operator,

$$\begin{aligned}\mathcal{N}^a = &-a\Bigg(\nabla_{e_i}^{\Lambda^\cdot(T^*X)\widehat{\otimes}F,u}-\frac{1}{a}\left(\frac{v}{w}+\frac{a}{4v}\right)\left\langle K^X,e_i\right\rangle\\&\qquad+(2s-1)\frac{c(e_i)}{2\sqrt{a}}ds-\frac{\widehat{c}(e_i)}{2\sqrt{a}}ds\Bigg)^2+\frac{\left|K^X\right|^2}{w}\\&+\frac{a}{4}H+\frac{1}{2}\sqrt{a/v}\left(1+(2s-1)^2\right)\widehat{c}\left(K^X/\sqrt{w}\right)\overline{ds}\\&-\sqrt{a}c\left(\omega\left(\nabla^F,g^F\right)\right)ds/2+(2s-1)\sqrt{a}\widehat{c}\left(\omega\left(\nabla^F,g^F\right)\right)ds/2\\&-\frac{1}{2}\sqrt{a/v}\left(c\left(K^X/\sqrt{w}\right)+(2s-1)\widehat{c}\left(K^X/\sqrt{w}\right)\right)dv\\&+\frac{a}{8}\left\langle e_k,R^{TX}(e_i,e_j)e_l\right\rangle c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_l)\\&+\frac{1}{4}\left\langle\nabla_{e_i}^{TX}K^X,e_j\right\rangle\left(-\frac{a}{v}c(e_i)c(e_j)+\frac{4v}{w}\widehat{c}(e_i)\widehat{c}(e_j)\right)\\&+4a\Bigg(\frac{1}{32}\left(-c(e_i)c(e_j)+\widehat{c}(e_i)\widehat{c}(e_j)\right)\omega^2\left(\nabla^F,g^F\right)(e_i,e_j)\\&\qquad+\frac{1}{16}\left|\omega\left(\nabla^F,g^F\right)\right|^2-\frac{1}{8}c(e_i)\widehat{c}(e_j)\nabla_{e_i}^{F,u}\omega\left(\nabla^F,g^F\right)(e_j)\Bigg).\end{aligned} \tag{12.22}$$

*Definition* 12.7. Set

$$\mathbf{E}_1=\mathrm{Tr_s}\left[g\widetilde{F}_a\left(\mathcal{N}^a\right)\right]^{dsdv}\frac{\sqrt{w}}{v^{3/2}},\quad \mathbf{E}_2=\mathrm{Tr_s}\left[g\widetilde{F}_a\left(\mathcal{N}^a\right)\right]^{\overline{ds}dv}\frac{w}{v^2}. \tag{12.23}$$

Let $\mathbf{E}_2|_{\overline{ds}=ds}$ be obtained from $\mathbf{E}_2$ by replacing $\overline{ds}$ by $ds$.

PROPOSITION 12.8. *The following identity holds*:

$$\mathrm{Tr_s}\left[g\widetilde{F}_a\left(N^a\right)\right]^{dsdv}=\mathbf{E}_1+\mathbf{E}_2|_{\overline{ds}=ds}. \tag{12.24}$$

*Proof.* First, we observe that $\mathrm{Tr_s}\left[g\widetilde{F}_a\left(N^a\right)\right]^{dsdv}$ is unchanged if we replace $dv/v$ by $dv/v-(2s-1)\,ds$. Using (12.21), and comparing equations (9.1) and (12.22), we complete the proof. □

12.3 *A local coordinate system and a Getzler rescaling.* In Sections 12.3–12.7, we will assume that $(s,v)\in[0,1]\times[a,1]$ are such that $w\in\,]0,1]$. The much easier case where $w\in[1,+\infty[$ will be dealt with in Section 12.8.

If $x\in X_g$, we still define $\mathbf{H}_x$ as in Section 9.3, except that $\Lambda^{\cdot}(\mathbf{R}^2)$ is now replaced by $\Lambda^{\cdot}(\mathbf{R}^3)$.

Since $X_g$ is totally geodesic in $X$, the Levi-Civita connection $\nabla^{TX}$ induces a Euclidean connection $\nabla^{N_{X_g/X}}$ on the normal bundle $N_{X_g/X}$.

Take $y_0\in X_{g,K}$. If $Z\in T_{y_0}X_g, |Z|\le 4\varepsilon_0$, we identify $Z$ as $\exp_{y_0}^{X_g}(Z)\in X_g$. We trivialize $N_{X_g/X}$ along the geodesic $h\in[0,1]\to hZ\in X_g$ by parallel transport with respect to the connection $\nabla^{N_{X_g/X}}$. Then $(Z,Z')\in T_{y_0}X_g\times N_{X_g/X,y_0}\to \exp^X_{\exp_{y_0}^{X_g}(Z)}(Z')\in X, |Z|,|Z'|\le 4\varepsilon_0$ defines a coordinate system near $y_0$.

If $Z_0\in N_{X_{g,K}/X_g,y_0}, |Z_0|\le\varepsilon_0$, we trivialize $TX$ along the geodesic $h\in[0,1]\to hZ_0\in X_g$ by parallel transport with respect to $\nabla^{TX}$. Then $Z\in T_{y_0}X, |Z|\le 4\varepsilon_0\to \exp^X_{Z_0}(Z)\in X$ is a coordinate system near $Z_0\in X_g$. By an abuse of notation, we will often write $Z_0+Z$ instead of $\exp^X_{Z_0}(Z)$.

Now we fix $Z_0\in N_{X_{g,K}/X_g,y_0}, |Z_0|\le\varepsilon_0$, and take $Z\in T_{y_0}X$, $|Z|\le 4\varepsilon_0$. The curve $h\in[0,1]\to\exp^X_{Z_0}(hZ)$ lies in $B^X_{y_0}(0,5\varepsilon_0)$. Moreover we identify

$$T_{Z_0+Z}X, \left(\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}\left(\mathbf{R}^2\right)\right)_{Z_0+Z}$$

with

$$TX_{Z_0}, \left(\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}\left(\mathbf{R}^2\right)\right)_{Z_0}$$

by parallel transport with respect to the connections

$$\nabla^{TX}, {}^1\nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}(\mathbf{R}^2),u,a}$$

along this curve.

When $Z_0\in N_{X_{g,K}/X_g,y_0}$ varies, we identify

$$T_{Z_0}X, \left(\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}\left(\mathbf{R}^2\right)\right)_{Z_0}$$

with

$$T_{y_0}X, \left(\Lambda^{\cdot}(T^*X)\widehat{\otimes}F\widehat{\otimes}\Lambda^{\cdot}\left(\mathbf{R}^2\right)\right)_{y_0}$$

by parallel transport with respect to the connections $\nabla^{TX}, \nabla^{\Lambda^{\cdot}(T^*X)\widehat{\otimes}F,u}$ along the curve $h\in[0,1]\to hZ_0\in X_g$.

We may and we will assume that $\varepsilon_0$ is small enough so that if $|Z_0|$, $|Z|\le 4\varepsilon_0$, then

$$\frac{1}{2}g^{TX}_{y_0}\le g^{TX}_{Z_0+Z}\le\frac{3}{2}g^{TX}_{y_0}. \tag{12.25}$$

We fix $Z_0\in N_{X_{g,K}/X_g,y_0}, |Z_0|\le\varepsilon_0$. The considered trivializations depend explicitly on $Z_0$. Therefore the action of the operator $\mathcal{N}^a$ depends explicitly on $Z_0$. We denote by $\mathcal{N}^a_{Z_0}$ the action of this operator centred at $Z_0$, i.e. $\mathcal{N}^a$ acts on a section $h$ by the formula,

$$\mathcal{N}^a_{Z_0} h(Z) = \mathcal{N}^a h(Z_0 + Z). \tag{12.26}$$

In (12.26), the operator acts on $\mathbf{H}_{Z_0}$. Also $\mathbf{H}_{Z_0}$ is identified with $\mathbf{H}_{y_0}$, so that ultimately, $\mathcal{N}^a_{Z_0}$ acts on $\mathbf{H}_{y_0}$.

Recall that $\rho(Z)$ was defined in (9.10), and that $H_a$ was defined in (9.13).

*Definition* 12.9. Put

$$\mathcal{N}^{1,a}_{Z_0} = -\left(1-\rho^2(Z)\right) a\Delta^{TX} + \rho^2(Z)\,\mathcal{N}^a_{Z_0}. \tag{12.27}$$

Let $\mathcal{N}^{2,a}_{Z_0} = H_a^{-1}\mathcal{N}^{1,a}_{Z_0} H_a$ be the operator obtained from $\mathcal{N}^{1,a}_{Z_0}$ as in (9.14).

Let $(e_1, \ldots, e_\ell)$, $(e_{\ell+1}, \ldots, e_m)$, $(e_{m+1}, \ldots, e_n)$ be orthonormal bases of $T_{y_0}X_{g,K}$, $N_{X_{g,K}/X_g, y_0}$, $N_{X_g/X, y_0}$. We denote with an upper index the corresponding dual basis.

We use the notation in (10.7).

*Definition* 12.10. Let $\mathcal{N}^{3,a}_{Z_0}$ be the operator obtained from $\mathcal{N}^{2,a}_{Z_0}$ by

- replacing $c(e_i)$ by $c_{a/v}(e_i)$ for $1 \le i \le m$.
- replacing $\widehat{c}(e_i)$ by $\widehat{c}_v(e_i)$ for $1 \le i \le \ell$.
- replacing $\widehat{c}(e_i)$ by $\widehat{c}_{v/w}(e_i)$ for $\ell+1 \le i \le m$.
- keeping $c(e_i)$ and $\widehat{c}(e_i)$ unchanged for $m+1 \le i \le n$.

We denote by $\widetilde{F}_a\left(\mathcal{N}^{3,a}_{Z_0}\right)(Z,Z')$ the kernel associated to the operator $\widetilde{F}_a\left(\mathcal{N}^{3,a}_{Z_0}\right)$ with respect to $dv_{TX}(Z')/(2\pi)^{n/2}$.

In the sequel, $I, I' \ldots$ denote collections of distinct indices taken in $1, \ldots, m$, $e^I, \widehat{e^{I'}}$ denote the corresponding wedge products of the $e^i, \widehat{e}^i$ in $\Lambda^\cdot(T^*X_g)$ and in $\widehat{\Lambda}^\cdot(T^*X_g)$. We use the same notation for products of the associated annihilation operators.

Clearly,

$$\begin{aligned} \widetilde{F}_a\left(\mathcal{N}^{3,a}_{Z_0}\right)(Z,Z') &= \sum e^I \wedge \widehat{e^{I'}} i_{e_J} i_{\widehat{e_{J'}}} Q^{J,J'}_{I,I'}(Z,Z'), \\ Q^{J,J'}_{I,I'}(Z,Z') &\in \left(\operatorname{End}\left(\Lambda^\cdot\left(N^*_{X_g/X}\right)\widehat{\otimes} F\right)\widehat{\otimes}\Lambda^\cdot\left(\mathbf{R}^3\right)\right)_{y_0}. \end{aligned} \tag{12.28}$$

Let $\widetilde{F}_a\left(\mathcal{N}^{3,a}_{Z_0}\right)^{\max}(Z,Z')$ be the operator which appears in (12.28) after

$$e^1 \wedge \ldots \wedge e^m \wedge \widehat{e}^1 \wedge \ldots \wedge \widehat{e}^m,$$

with no annihilation operator.

If $x \in X_g$, in the coordinate system $Z \in T_xX, |Z| \leq 4\varepsilon_0 \to \exp_x^X(Z)$, there is a smooth function $k'_x(Z)$ with values in $\mathbf{R}_+^*$ such that

$$dv_X(Z) = k'_x(Z)\, dv_{TX}(Z)\,, \tag{12.29}$$

and

$$k'_x(0) = 1 \tag{12.30}$$

Proposition 12.11. *For $y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g,y_0}, |Z_0| \leq \varepsilon_0, Z \in N_{X_g/X,y_0}$, $|Z| \leq \varepsilon_0/\sqrt{a}$, the following identity holds*:

(12.31)
$$a^{(n-m)/2}\mathrm{Tr_s}\left[g\widetilde{F}_a\left(\mathcal{N}^a\right)\left(g^{-1}\left(Z_0,\sqrt{a}Z\right),\left(Z_0,\sqrt{a}Z\right)\right)\right]k'_{(y_0,Z_0)}\left(\sqrt{a}Z\right)$$
$$= (-1)^{m(m+1)/2}\frac{1}{w^{(m-\ell)/2}}\mathrm{Tr_s}^{\Lambda^{\cdot}\left(N^*_{X_g/X}\right)}\left[g\widetilde{F}_a\left(\mathcal{N}^{3,a}_{Z_0}\right)\left(g^{-1}Z,Z\right)\right]^{\max}.$$

*Proof.* This is a trivial consequence of (9.35). □

12.4. *A family of norms.* As indicated in Section 12.3, we will assume that $(s,v) \in [0,1]\times[a,1]$ are such that $w \in\ ]0,1]$.

For $0 \leq p \leq \ell, 0 \leq q \leq m-\ell$, set

$$\Lambda^{(p,q)}\left(T^*X_g\right)_{y_0} = \Lambda^p\left(T^*_{y_0}X_{g,K}\right)\widehat{\otimes}\Lambda^q\left(N^*_{X_{g,K}/X_g,y_0}\right). \tag{12.32}$$

The various $\Lambda^{(p,q)}\left(T^*X_g\right)_{y_0}$ are mutually orthogonal in $\Lambda^{\cdot}\left(T^*X_g\right)$. Also $\Lambda^{r,r'}\left(\mathbf{R}^3\right)$ denote the forms in $\Lambda^{\cdot}\left(\mathbf{R}^3\right)$ which have partial degree $r$ in the Grassmann variable $ds$, and $r'$ in the Grassmann variables $\overline{ds}, dv$.

Let $\mathbf{I}_{y_0}$ be the vector space of smooth sections of

$$\left(\Lambda^{\cdot}\left(T^*X_g\right)\widehat{\otimes}\widehat{\Lambda}^{\cdot}\left(T^*X_g\right)\widehat{\otimes}\Lambda^{\cdot}\left(N^*_{X_g/X}\right)\widehat{\otimes}F\widehat{\otimes}\Lambda\left(\mathbf{R}^3\right)\right)_{y_0}$$

on $T_{y_0}X$; let $\mathbf{I}_{y_0}^{(p,p',q',r,r')}$ be the vector space of smooth sections of

$$\left(\Lambda^p\left(T^*X_g\right)\widehat{\otimes}\widehat{\Lambda}^{(p',q')}\left(T^*X_g\right)\widehat{\otimes}\Lambda^{\cdot}\left(N^*_{X_g/X}\right)\widehat{\otimes}F\widehat{\otimes}\Lambda^{r,r'}\left(\mathbf{R}^3\right)\right)_{y_0}$$

on $T_{y_0}X$. Let $\mathbf{I}^0_{y_0}, \mathbf{I}_{y_0}^{(p,p',q',r,r'),0}$ be the corresponding vector spaces of square-integrable sections.

*Definition* 12.12. For $a \in\ ]0,1], (s,v) \in [0,1]\times[a,1]$, $y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g,y_0}$, $|Z_0| \leq \varepsilon_0/\sqrt{w}$, if $h \in \mathbf{I}_{y_0}^{(p,p',q',r,r')}$ has compact support, set

$$|h|^2_{a,s,v,Z_0,0} = \int_{T_{y_0}X}|h(Z)|^2\left(1+\sqrt{v}\,|Z|\,\rho\left(\sqrt{a}Z/2\right)\right)^{2(m-p)} \cdot\left(1+\left(\sqrt{a/v}\,|Z|+|Z_0|\right)\rho\left(\sqrt{a}Z/2\right)\right)^{2(\ell-p'+1-r')} \tag{12.33}$$

$$\cdot\left(1+\sqrt{aw/v}\,|Z|\,\rho\left(\sqrt{a}Z/2\right)\right)^{2(m-\ell-q')}$$
$$\left(1+\left(|Z|+|Z_0|\right)\rho\left(\sqrt{a}Z/2\right)\right)^{2(1-r)}dv_{TX}(Z)\,.$$

Then (12.33) defines a Hermitian product $\langle\ \rangle_{a,s,v,Z_0,0}$ on $\mathbf{I}_{y_0}^{(p,p',q',r,r'),0}$. We equip $\mathbf{I}_{y_0}^{0}$ with the direct sum of these Hermitian products.

Using (7.14), we find that the coefficients $\sqrt{v}$, $\sqrt{a/v}$ and $\sqrt{aw/v}$ are uniformly bounded.

Proposition 12.13. *For $1\le i\le m$, when acting on $\left(\mathbf{I}_{y_0},|\ |_{a,s,v,Z_0,0}\right)$, the following family of operators have uniformly bounded norm*:

$$1_{|\sqrt{a}Z|\le 4\varepsilon_0}\sqrt{a/v}c_{a/v}(e_i)\,,\quad 1_{|\sqrt{a}Z|\le 4\varepsilon_0}\sqrt{a}\,|Z|\,c_{a/v}(e_i)\,. \tag{12.34}$$

*Similarly, for $1\le i\le\ell$, the family of operators*

$$\begin{aligned}&1_{|\sqrt{a}Z|\le 4\varepsilon_0}\sqrt{v}\widehat{c}_v(e_i)\,, && 1_{|\sqrt{a}Z|\le 4\varepsilon_0}\sqrt{a}\,|Z|\,\widehat{c}_v(e_i)\,,\\ &1_{|\sqrt{a}Z|\le 4\varepsilon_0}\sqrt{v}\,|Z_0|\,\widehat{c}_v(e_i)\,, && 1_{|\sqrt{a}Z|\le 4\varepsilon_0}\sqrt{av/w}\,|Z|\,\widehat{c}_v(e_i)\,,\end{aligned} \tag{12.35}$$

*and for $\ell+1\le i\le m$, the family of operators*

$$1_{|\sqrt{a}Z|\le 4\varepsilon_0}\sqrt{v/w}\widehat{c}_{v/w}(e_i)\,,\quad 1_{|\sqrt{a}Z|\le 4\varepsilon_0}\sqrt{a}\,|Z|\,\widehat{c}_{v/w}(e_i)\,, \tag{12.36}$$

*are uniformly bounded. Finally the operators*

$$\begin{aligned}&1_{|\sqrt{a}Z|\le 4\varepsilon_0}ds, && && 1_{|\sqrt{a}Z|\le 4\varepsilon_0}|Z|\,ds,\\ &1_{|\sqrt{a}Z|\le 4\varepsilon_0}\overline{ds}, && 1_{|\sqrt{a}Z|\le 4\varepsilon_0}|Z_0|\,\overline{ds}, && 1_{|\sqrt{a}Z|\le 4\varepsilon_0}\sqrt{a/v}\,|Z|\,\overline{ds},\\ &1_{|\sqrt{a}Z|\le 4\varepsilon_0}dv, && 1_{|\sqrt{a}Z|\le 4\varepsilon_0}|Z_0|\,dv, && 1_{|\sqrt{a}Z|\le 4\varepsilon_0}\sqrt{a/v}\,|Z|\,dv.\end{aligned} \tag{12.37}$$

*are also uniformly bounded.*

*Proof.* By (9.10), if $|\sqrt{a}Z|\le 4\varepsilon_0$, then $\rho\left(\sqrt{a}Z/2\right)=1$. Then the uniform bounds on the family (12.34) follow from the next uniform bounds under the given conditions on $s,v,Z_0,Z$:

$$\begin{aligned}&\frac{1}{1+\sqrt{v}\,|Z|}\le C, && \frac{\sqrt{v}\,|Z|}{1+\sqrt{v}\,|Z|}\le C,\\ &\frac{a}{v}\left(1+\sqrt{v}\,|Z|\right)\le C, && \frac{a}{\sqrt{v}}\,|Z|\left(1+\sqrt{v}\,|Z|\right)\le C.\end{aligned} \tag{12.38}$$

The uniform bounds on the family (12.35) follow from the next uniform bounds, where (7.14) is used,

(12.39)
$$\frac{1}{1+\sqrt{a/v}\,|Z|+|Z_0|}\leq C,\qquad \frac{\sqrt{a/v}\,|Z|}{1+\sqrt{a/v}\,|Z|+|Z_0|}\leq C,$$
$$v\left(1+\sqrt{a/v}\,|Z|+|Z_0|\right)\leq C,\qquad \sqrt{av}\,|Z|\left(1+\sqrt{a/v}\,|Z|+|Z_0|\right)\leq C,$$
$$\frac{|Z_0|}{1+\sqrt{a/v}\,|Z|+|Z_0|}\leq C,\qquad \frac{\sqrt{a/w}\,|Z|}{1+\sqrt{a/v}\,|Z|+|Z_0|}\leq C,$$
$$v\,|Z_0|\left(1+\sqrt{a/v}\,|Z|+|Z_0|\right)\leq C,\quad \sqrt{a/wv}\,|Z|\left(1+\sqrt{a/v}\,|Z|+|Z_0|\right)\leq C.$$

The bounds in (12.36) follow from the next bounds, where (7.14) is used again:

$$\frac{1}{1+\sqrt{aw/v}\,|Z|}\leq C,\qquad \frac{\sqrt{aw/v}\,|Z|}{1+\sqrt{aw/v}\,|Z|}\leq C,$$
$$\frac{v}{w}\left(1+\sqrt{aw/v}\,|Z|\right)\leq C,\qquad \sqrt{av/w}\,|Z|\left(1+\sqrt{aw/v}\,|Z|\right)\leq C. \tag{12.40}$$

Finally the fact that the operators in (12.37) have uniformly bounded norm is trivial. The proof of our proposition is complete. □

*Definition* 12.14. For $a\in\,]0,1],(s,v)\in[0,1]\times[a,1],\ y_0\in X_{g,K},Z_0\in N_{X_{g,K}/X,y_0},\ |Z_0|\leq\varepsilon_0/\sqrt{w}$, if $h\in\mathbf{I}_{y_0}$ has compact support, set

(12.41)
$$|h|^2_{a,s,v,Z_0,1}=|h|^2_{a,s,v,Z_0,0}+\frac{1}{w}\Big|\rho\left(\sqrt{a}Z\right)\Big|K^X\left(\sqrt{w}Z_0+\sqrt{a}Z\right)\Big|\,h\Big|^2_{a,s,v,Z_0,0}+\sum_{i=1}^n|\nabla_{e_i}h|^2_{a,s,v,Z_0,0}\,.$$

Let $(\mathbf{I}^1_{y_0},|\,|_{t,x,1})$ be the Hilbert closure of the above vector space with respect to $|\,|_{a,s,v,Z_0,1}$. Then $(\mathbf{I}^1_{y_0},|\,|_{a,s,v,Z_0,1})$ is densely embedded in $(\mathbf{I}^0_{y_0},|\,|_{a,s,v,Z_0,0})$ with norm smaller than 1. We identify $\mathbf{I}^0_{y_0}$ to its antidual by the Hermitian product $\langle\rangle_{a,s,v,Z_0,0}$. Let $(\mathbf{I}^{-1}_{y_0},|\,|_{a,s,v,Z_0,-1})$ be the antidual of $(\mathbf{I}^1_{y_0},|\,|_{a,s,v,Z_0,1})$. Then $\mathbf{I}^{-1}_{y_0}$ embeds into $\mathbf{I}^1_{y_0}$, and the norm of the embedding is $\leq 1$.

12.5. *Uniform estimates on the operators* $\mathcal{N}^{3,a}_{\sqrt{w}Z_0}$. If $Z_0\in N_{X_{g,K}/X_g,y_0}$, $|Z_0|\leq\varepsilon_0, Z\in T_{y_0}X,|Z|\leq 4\varepsilon_0$, if $U\in T_{y_0}X$, let $\tau^{Z_0}U\left(Z\right)$ be the parallel transport of $U$ along the curve $h\to 2tZ_0, 0\leq h\leq 1/2, h\to\exp_{Z_0}\left(\left(2h-1\right)Z\right), 1/2\leq h\leq 1$ with respect to $\nabla^{TX}$.

Recall that $k=zK_0$. Now we have an analogue of [BGo1, Th. 8.18].

THEOREM 12.15. *There exist constants* $C_1 > 0, \dots, C_4 > 0$ *such that if* $a \in [0,1]$, *if* $(s,v) \in [0,1] \times [a,1]$ *are such that* $w \in \,]0,1]$, *if* $n \in \mathbf{N}$, $y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g, y_0}, |Z_0| \le \varepsilon_0/\sqrt{w}$, *if* $z \in \mathbf{R}, |z| \le 1$, *if* $h, h' \in \mathbf{I}_{y_0}$ *have compact support in* $\{Z \in T_{y_0}X, |Z| \le n\}$, *then*

(12.42)
$$\begin{aligned} \operatorname{Re}\left\langle \mathcal{N}^{3,a}_{\sqrt{w}Z_0} h, h' \right\rangle_{a,s,v,Z_0,0} &\ge C_1 \left|h\right|^2_{a,s,v,Z_0,1} - C_2 \left(1 + |nz|^2\right) |s|^2_{a,s,v,Z_0,0}\,, \\ \left|\operatorname{Im}\left\langle \mathcal{N}^{3,a}_{\sqrt{w}Z_0} h, h' \right\rangle_{a,s,v,Z_0,0}\right| &\le C_3 \left(1 + |nz|\right) |h|_{a,s,v,Z_0,1} \, |h|_{a,s,v,Z_0,0}\,, \\ \left|\left\langle \mathcal{N}^{3,a}_{\sqrt{w}Z_0} h, h' \right\rangle_{a,s,v,Z_0,0}\right| &\le C_4 \left(1 + |nz|^2\right) |h|_{a,s,v,Z_0,1} \, |h'|_{a,s,v,Z_0,1}\,. \end{aligned}$$

*Proof.* Recall that $Z$ is parallel along the geodesic $h \in [0,1] \to Z_0 + hZ$. By (9.6), we find that in the trivialization indicated in Section 12.3, if $U \in T_{y_0}X$,

$$\begin{aligned} c\left(\tau^{Z_0} U\right)(\sqrt{a}Z) &= c\,(U) + (2s-1)\,\langle U, Z\rangle\, ds, \\ \widehat{c}\left(\tau^{Z_0} U\right)(\sqrt{a}Z) &= \widehat{c}\,(U) + \langle U, Z\rangle\, ds. \end{aligned} \tag{12.43}$$

In particular the right-hand side of (12.43) does not depend on $a$.

Observe that by (7.14), if $a, s, v$ are taken as indicated,

$$v/w + a/4v$$

remains uniformly bounded. Then using Proposition 12.13, (12.43) and by proceeding as in [BL, Th. 11.26], [BGo1, Ths. 7.31 and 8.18], we find easily that the term which is part of $\mathcal{N}^{3,a}_{\sqrt{w}Z_0}$,

$$\begin{aligned} -a\Bigg(\nabla^{\Lambda^\cdot(T^*X)\widehat{\otimes} F, u}_{e_i} - \frac{1}{a}\left(\frac{v}{w} + \frac{a}{4v}\right)\left\langle K^X, e_i \right\rangle \\ + (2s-1)\frac{c\,(e_i)}{2\sqrt{a}} ds - \frac{\widehat{c}\,(e_i)}{2\sqrt{a}} ds\Bigg)^2 + \frac{\left|K^X\right|^2}{w} + \frac{a}{4} H \end{aligned}$$

fits with the estimates in (12.42).

Using again Proposition 12.13, we find that the contribution of the terms

$$-\sqrt{a} c\left(\omega\left(\nabla^F, g^F\right)\right) ds/2 + (2s-1)\sqrt{a}\widehat{c}\left(\omega\left(\nabla^F, g^F\right)\right) ds$$

is also harmless for our estimates.

Recall that $w \in \,]0,1]$. By Proposition 12.13, the operators $\sqrt{a/v}c_{a/v}(e_i)$, $1 \le i \le \ell$, $\sqrt{v}\widehat{c}_v(e_i)$, $1 \le i \le \ell$, $\sqrt{v}\widehat{c}_{v/w}(e_i)$, $\ell + 1 \le i \le m$ remain uniformly bounded. Using again (12.43), we find that the term

$$\frac{a}{8}\left\langle e_k, R^{TX}(e_i, e_j)\, e_l \right\rangle c\,(e_i)\, c\,(e_j)\, \widehat{c}\,(e_k)\, \widehat{c}\,(e_l)$$

in $\overline{\mathcal{N}}^a$ is also harmless for our estimates. The same argument applies to

$$4a\Bigg(\frac{1}{32}\left(-c\left(e_i\right)c\left(e_j\right)+\widehat{c}\left(e_i\right)\widehat{c}\left(e_j\right)\right)\omega^2\left(\nabla^F,g^F\right)\left(e_i,e_j\right) + \frac{1}{16}\left|\omega\left(\nabla^F,g^F\right)\right|^2-\frac{1}{8}c\left(e_i\right)\widehat{c}\left(e_j\right)\nabla^{F,u}_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right)\Bigg).$$

Now, we will consider the other remaining terms in the right-hand side of (12.22). Using (12.34) in Proposition 12.13 and the definition of the norm $|\ |_{a,s,v,Z_0,1}$ in (12.41) , the term

$$-\frac{\sqrt{a}}{2\sqrt{v}}c\left(K^X/\sqrt{w}\right)dv$$

is easily dealt with. Moreover,

$$\sqrt{a/v}\widehat{c}\left(K^X/\sqrt{w}\right)=\sum_{1\le i\le n}\left\langle K^X,\tau^{Z_0}e_i\right\rangle\sqrt{a/vw}\widehat{c}\left(\tau^{Z_0}e_i\right). \tag{12.44}$$

By (12.37) in Proposition 12.13 and by (12.43), the contribution of the $i$, $m+1\le i\le n$ in the right-hand side of (12.44) is harmless. Also, by (7.14),

$$a/v\le Cv/w. \tag{12.45}$$

By (12.36), (12.37) in Proposition 12.13, (12.43) and (12.45), the same holds for the contribution of the $i$, $\ell+1\le i\le m$ is also harmless. Also one verifies easily that if $1\le i\le \ell$,

$$\left\langle K^X\left(Z_0+Z\right),\tau^{Z_0}e_i\left(Z\right)\right\rangle$$

and its first derivatives in the variables $Z_0, Z$ vanish at $(0,0)$, so that

$$\left\langle K^X,\tau^{\sqrt{w}Z_0}e_i\right\rangle(\sqrt{w}Z_0+\sqrt{a}Z)=\mathcal{O}\left(\left|\sqrt{w}Z_0\right|^2+\left|\sqrt{a}Z\right|^2\right). \tag{12.46}$$

By (12.46), if $1\le i\le\ell$, we get

$$\begin{aligned}\sqrt{a/vw}&\left\langle K^X\left(\sqrt{w}Z_0+\sqrt{a}Z\right),\tau^{\sqrt{w}Z_0}e_i\left(\sqrt{a}Z\right)\right\rangle\widehat{c}\left(\tau^{\sqrt{w}Z_0}e_i\left(\sqrt{a}Z\right)\right)\\&=\sqrt{aw/v^2}\sqrt{v}\left|Z_0\right|\widehat{c}\left(\tau^{\sqrt{w}Z_0}e_i\left(\sqrt{a}Z\right)\right)\mathcal{O}\left(\left|Z_0\right|\right)\\&\quad+\sqrt{\frac{av}{w}}\left|Z\right|\widehat{c}\left(\tau^{\sqrt{w}Z_0}e_i\left(\sqrt{a}Z\right)\right)\mathcal{O}\left(\frac{a}{v}\left|Z\right|\right).\end{aligned} \tag{12.47}$$

By (7.14), (12.35), (12.37) in Proposition 12.13, by (12.43), by (12.44)–(12.47), we deduce that the term

$$\frac{1}{2}\sqrt{a/v}\widehat{c}\left(K^X/\sqrt{w}\right)\left(\left(1+(2s-1)^2\right)\overline{ds}-(2s-1)\,dv\right)$$

in the right-hand side of (12.22) is compatible with (12.42).

Using (12.34) in Proposition 12.13 and (12.43), we see that the term

$$\frac{1}{4}\left\langle \nabla^{TX}_{e_i} K^X, e_j \right\rangle \frac{a}{v} c\left(e_i\right) c\left(e_j\right)$$

is still harmless.

The final term to be considered is

$$\frac{1}{4}\left\langle \nabla^{TX}_{e_i} K^X, e_j \right\rangle \frac{4v}{w} \widehat{c}\left(e_i\right) \widehat{c}\left(e_j\right).$$

By (7.14), by (12.36), (12.37) in Proposition 12.13 and by (12.43), (12.45), the case where $\ell+1 \leq i, j \leq n$ is easily dealt with. Also, using (2.7), we find that for $1 \leq i \leq \ell$,

$$\nabla^{TX}_{\tau^{\sqrt{w} Z_0} e_i\left(\sqrt{a} Z\right)} K^X \left(\sqrt{w} Z_0 + \sqrt{a} Z\right) = \mathcal{O}\left(w \left|Z_0\right|^2 + a \left|Z\right|^2\right). \tag{12.48}$$

By Proposition 12.13 and by (12.43), (12.48), we find that the contribution of the $i, j$ such that $1 \leq i$ or $j \leq \ell$ is also harmless. The proof of Theorem 12.15 is complete. □

12.6. *An estimate on the difference of two operators.* Now we take $x \in X_g$. In the sequel, the tensors which we will consider are evaluated at $x$. By imitating (9.17), set

(12.49)

$$\begin{aligned}
\mathcal{N}^{3,a\prime}_{x,K} = & -\left(\nabla_{e_i} + \frac{1}{2}\left\langle R^{TX}_{2(v/w)K/2\pi} Z, e_i \right\rangle\right)^2 + \frac{\left|K^{X_g}\right|^2}{w} \\
& - \frac{K^{X_g\prime}}{\sqrt{2vw}} dv - \frac{dK^{X_g\prime}}{2v} + \left(\widehat{c}\left(R^{TX}_{2(v/w)/2\pi K} Z\right) - \frac{\omega\left(\nabla^F, g^F\right)}{\sqrt{2}}\right) ds \\
& + \frac{1}{2}\sqrt{\frac{a}{v}}\, \widehat{c}\left(K^X/\sqrt{w}\right)\left(\left(1 + (2s-1)^2\right) \overline{ds} - (2s-1)\, dv\right) \\
& + \frac{1}{2}\left\langle e_k, R^{TX}_{2(v/w)K/2\pi} e_l \right\rangle \widehat{c}\left(e_k\right) \widehat{c}\left(e_l\right) - \frac{1}{2}\omega^2\left(\nabla^F, g^F\right).
\end{aligned}$$

As the notation emphasizes, the operator $\mathcal{N}^{3,a\prime}_{Z_0}$ depends on $a$, through $w$, but also because $\sqrt{a}$ appears explicitly in the right-hand side of (12.49).

*Definition* 12.16. Let $\overline{\mathcal{N}}^{3,a}_{Z_0}$ be the operator obtained from the operator $\mathcal{N}^{3,a\prime}_{Z_0,K}$ by replacing $e^i$ by $\sqrt{v} e^i, 1 \leq i \leq m$, by replacing $\widehat{c}\left(e_i\right)$ by $\widehat{c}_v\left(e_i\right)$ for $1 \leq i \leq \ell$, by $\widehat{c}_{v/w}\left(e_i\right)$ for $\ell+1 \leq i \leq m$, while leaving the $\widehat{c}\left(e_i\right)$ unchanged for $m+1 \leq i \leq n$.

Let $\overline{\mathcal{N}}^{3,0}_{Z_0}$ be the operator obtained from $\overline{\mathcal{N}}^{3,a}_{Z_0}$ by making $a = 0$, including in $w$, which was defined in (7.13). By Theorem 9.8, we know that as $a \to 0$,

$$\mathcal{N}^{3,a}_{Z_0} \to \overline{\mathcal{N}}^{3,0}_{Z_0}. \tag{12.50}$$

Now we establish an analogue of [BGo1, Th. 8.24].

Theorem 12.17. *There exist* $C>0, r\in \mathbf{N}$ *such that for* $a\in\, ]0,1]$, *if* $(s,v)\in[0,1]\times[a,1]$ *are such that* $w\in\, ]0,1]$, *if* $z\in\mathbf{R}^*, |z|\le 1, n\in\mathbf{N}$, $y_0\in X_{g,K}, Z_0\in N_{X_{g,K}/X,y_0}, |Z_0|\le \varepsilon_0/\sqrt{w}$, *if the support of* $s\in \mathbf{I}_{y_0}$ *is included in* $\{Z\in T_{y_0}X, |Z|\le n\}$, *then*

$$\left|\left(\mathcal{N}^{3,a}_{\sqrt{w}Z_0}-\overline{\mathcal{N}}^{3,a}_{\sqrt{w}Z_0}\right)s\right|_{a,s,v,Z_0,-1} \le C\left(1+n^r\right)\sqrt{\frac{a}{v}}\left(1+|Z_0|^2\right)|s|_{a,v,s,Z_0,1}. \tag{12.51}$$

*Proof.* We need to show that if $s,s'\in\mathbf{I}_{y_0}$ have compact support,

$$\left|\left\langle\left(\mathcal{N}^{3,a}_{\sqrt{w}Z_0}-\overline{\mathcal{N}}^{3,a}_{\sqrt{w}Z_0}\right)s,s'\right\rangle_{a,s,v,Z_0,0}\right| \le C\left(1+n^r\right)\sqrt{\frac{a}{v}}\left(1+|Z_0|^2\right)|s|_{a,v,s,Z_0,1}\left|s'\right|_{a,v,s,Z_0,1}. \tag{12.52}$$

We use again the Lichnerowicz formula for $\mathcal{N}^a$ in (12.22). The first term in the right-hand side of (12.22) does not raise any difficulty with respect to the corresponding term in [BL, Th. 11.35] and of [BGo1, Th. 8.24].

Clearly

$$\begin{aligned}\frac{1}{w}\left(\left|K^X(\sqrt{w}Z_0+\sqrt{a}Z)\right|^2-\left|K^X(\sqrt{w}Z_0)\right|^2\right) &= \frac{1}{w}\Big\langle K^X(\sqrt{w}Z_0+\sqrt{a}Z)-K^X(\sqrt{w}Z_0),\\ &\qquad K^X(\sqrt{w}Z_0+\sqrt{a}Z)+K^X(\sqrt{w}Z_0)\Big\rangle.\end{aligned} \tag{12.53}$$

By (12.53), since $K^X$ vanishes on $X_{g,K}$, so that if $(Z_0,Z)=(0,0)$, then $K^X(\sqrt{w}Z_0+\sqrt{a}Z)=0$, we get

$$\begin{aligned}\frac{1}{w}\left(\left|K^X(\sqrt{w}Z_0+\sqrt{a}Z)\right|^2-\left|K^X(\sqrt{w}Z_0)\right|^2\right) &= \frac{1}{w}\mathcal{O}\left(|\sqrt{a}Z|\left(|\sqrt{w}Z_0|+|\sqrt{a}Z|\right)\right)\\ &= \mathcal{O}\left(\sqrt{a/w}\,|Z|\,|Z_0|+(a/w)\,|Z|^2\right).\end{aligned} \tag{12.54}$$

By (7.14), there is $C>0$ such that $w\ge Ca$, and so, by (12.54), we deduce that for $|Z|\le n$,

$$\left|\frac{1}{w}\left(\left|K^X\left(\sqrt{w}Z_0+\sqrt{a}Z\right)\right|^2-\left|K^X\left(\sqrt{w}Z_0\right)\right|^2\right)\right|\le C\sqrt{\frac{a}{w}}\left(1+n^2\right)\left(1+|Z_0|\right). \tag{12.55}$$

Using (7.14), (12.55), we find that the term $\left|K^X\right|^2/w$ in the right-hand side of (12.22) is also compatible with (12.51).

By Proposition 12.13, the term

$$-\sqrt{a}c\left(\omega\left(\nabla^F,g^F\right)\right)ds/2+(2s-1)\sqrt{a}\widehat{c}\left(\omega\left(\nabla^F,g^F\right)\right)ds/2$$

can be dealt with easily. Using (7.14), (12.34), (12.37) in Proposition 12.13 and (12.43), the term

$$-\frac{1}{2}\sqrt{a/v}c\left(K^X/\sqrt{w}\right)dv$$

can also be easily dealt with.

We still write

$$\sqrt{a/v}\widehat{c}\left(K^X/\sqrt{w}\right)$$

as in (12.44). The same arguments as above and (12.36), (12.37) in Proposition 12.13 allow us to control easily the contribution of the terms with $\ell+1\leq i\leq n$. Moreover, for $1\leq i\leq \ell$, as we saw after (12.45),

$$\left\langle K^X\left(Z_0+Z\right),\tau^{Z_0}e_i\left(Z\right)\right\rangle$$

and its first derivatives in the variables $(Z_0,Z)$ vanish at $(0,0)$. Therefore, for $1\leq i\leq \ell$,

$$\begin{aligned}(12.56)\qquad &\left|\left\langle K^X\left(\sqrt{w}Z_0+\sqrt{a}Z\right),\tau^{\sqrt{w}Z_0}\left(\sqrt{a}Z\right)e_i\right\rangle-\left\langle K^X\left(\sqrt{w}Z_0\right),\tau^{\sqrt{w}Z_0}e_i\right\rangle\right|\\ &\qquad\leq C\sqrt{a}\left|Z\right|\left(\left|\sqrt{w}Z_0\right|+\left|\sqrt{a}Z\right|\right).\end{aligned}$$

Using (12.35), (12.37) in Proposition 12.13 again and (12.43), (12.56), we find that in (12.44), the $i$ such that $1\leq i\leq \ell$ are also harmless.

By (2.7), we get

$$\begin{aligned}(12.57)\qquad &\left\langle\nabla^{TX}_{\tau^{\sqrt{w}Z_0}e_j\left(\sqrt{a}Z\right)}K^X\left(\sqrt{w}Z_0+\sqrt{a}Z\right),\tau^{\sqrt{w}Z_0}e_{j'}\left(\sqrt{a}Z\right)\right\rangle\\ &\quad-\left\langle\nabla^{TX}_{\tau^{\sqrt{w}Z_0}e_j(0)}K^X\left(\sqrt{w}Z_0\right),\tau^{\sqrt{w}Z_0}e_{j'}\left(0\right)\right\rangle\\ &\quad=\mathcal{O}\left(\sqrt{a}\left|Z\right|\right)\mathcal{O}\left(\sqrt{w}\left|Z_0\right|+\sqrt{a}\left|Z\right|\right).\end{aligned}$$

Using Proposition 12.13 and (12.57), we see that the term

$$\frac{1}{4}\left\langle\nabla^{TX}_{e_i}K^X,e_j\right\rangle\left(-\frac{a}{v}c\left(e_i\right)c\left(e_j\right)+\frac{4v}{w}\widehat{c}\left(e_i\right)\widehat{c}\left(e_j\right)\right)$$

can also be dealt with. The remaining terms in the right-hand side of (12.22) are handled easily.

The proof of our theorem is completed. □

*Remark* 12.18. Now we briefly explain why the terms containing $\sqrt{a/v}\widehat{c}\left(K^X/\sqrt{w}\right)$ have been included in (12.49), although they vanish identically for $a=0$. In fact, observe that

$$\sqrt{a/v}\widehat{c}_v\left(e_i\right)=\frac{\sqrt{2a}}{v}\widehat{e}^i-\sqrt{a/2}i_{\widehat{e_i}} \text{ for } 1\le i\le \ell,$$
$$\sqrt{a/v}\widehat{c}_{v/w}\left(e_i\right)=\frac{\sqrt{2aw}}{v}\widehat{e}^i-\sqrt{a/2w}i_{\widehat{e_i}} \text{ for } \ell+1\le i\le m. \tag{12.58}$$

Equation (12.58) makes clear that, if the terms $\sqrt{a/v}\widehat{c}\left(K^X/\sqrt{w}\right)$ had not been included, in the right-hand side of (12.51), $\sqrt{a/v}$ should have been replaced by at least $\sqrt{a}/v$, which is not even bounded.

12.7. *A proof of* (7.20) *when* $w\in\ ]0,1]$. Put

(12.59)
$$\mathfrak{m}_{h,v}=-\left\{(2\pi)^{m/2}\,2e_K\left(TX_g,\nabla^{TX_g}\right)\beta^{X_g}_{K,v/2}\right.$$
$$\left.\operatorname{Tr}\left[g\frac{\omega\left(\nabla^F,g^F\right)}{2}\exp\left(h\omega^2\left(\nabla^F,g^F\right)\right)/2\pi\right]\right\}^{\max},$$
$$\mathfrak{n}_v=\left\{(2\pi)^{m/2}\sqrt{\pi}\operatorname{Tr}^{F|_{X_g}}[g]\,\frac{v}{2}\beta^{X_g}_{K,v/2}\int^{\widehat{B}}\widehat{K^{X_g\prime}}\exp\left(-\dot{R}^{TX_g}_K/2\pi\right)\right\}^{\max},$$
$$\mathfrak{o}_{h,v}=-\left\{(2\pi)^{m/2}\,\alpha^{X_g}_{K,v/2}\frac{1}{4\sqrt{\pi}}\operatorname{Tr}\left[g\omega\left(\nabla^F,g^F\right)\exp\left(h\omega^2\left(\nabla^F,g^F\right)\right)/2\pi\right]\right.$$
$$\left.\int^{\widehat{B}}\widehat{K^{X_g\prime}}\exp\left(-\dot{R}^{TX_g}_K/2\pi\right)\right\}^{\max}.$$

The above forms are the normalized integrands for $m_{h,v},n_v,o_{h,v}$ in (7.9).

Let $k\left(x,Z\right)$ be the smooth function on $\mathcal{U}_{\varepsilon_0}$, such that

$$dv_X\left(x,Z\right)=k\left(x,Z\right)dv_{X_g}\left(x\right)dv_{N_{X_g/X}}\left(Z\right). \tag{12.60}$$

THEOREM 12.19. *There exist* $c\in\ ]0,1],\gamma\in\ ]0,1]$ *such that for* $p\in\mathbf{N}$, *there is* $C>0$ *such that if* $z\in\mathbf{R}^*,|z|\le c,a\in\ ]0,1]$, *if* $(s,v)\in[0,1]\times[a,1]$ *are such that* $w\in\ ]0,1]$, *if* $y_0\in X_{g,K},Z_0\in\mathbf{N}_{X_{g,K}/X_g,y_0},|Z_0|\le\frac{\varepsilon_0}{\sqrt{w}}$, *then*

(12.61)
$$\Bigg|w^{(m-\ell)/2}\Bigg(\int_{\substack{Z\in N_{X_g/X,\mathbf{R},y_0}\\|Z|\le\varepsilon_0}}\operatorname{Tr_s}\Big[g\widetilde{F}_a\left(\mathcal{N}^a\right)\left(g^{-1}\left(y_0,\sqrt{w}Z_0,Z\right),\right.$$
$$\left.\left(y_0,\sqrt{w}Z_0,Z\right)\right)\Big]^{dsdv}k\left(y_0,\sqrt{w}Z_0,Z\right)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{(n-m)/2}}$$
$$-\left(\sqrt{\frac{v}{w}}\mathfrak{m}_{v/w,w}\left(y_0,\sqrt{w}Z_0\right)+(2s-1)\sqrt{a/vw}\mathfrak{o}_{v/w,w}\left(y_0,\sqrt{w}Z_0\right)\right)dsdv\Bigg)\Bigg|$$
$$\le C\frac{(1+|Z_0|)^{\ell+1}}{(1+|zZ_0|)^p}\left(\frac{a}{v}\right)^{\gamma},$$

$$\Bigg| w^{(m-\ell)/2} \Bigg( \int_{\substack{Z \in N_{X_g/X,\mathbf{R},y_0} \\ |Z| \le \varepsilon_0}} \mathrm{Tr_s} \Big[ g \widetilde{F}_a \left( \mathcal{N}^a \right) \Big( g^{-1} \left( y_0, \sqrt{w} Z_0, Z \right), \left( y_0, \sqrt{w} Z_0, Z \right) \Big) \Big]^{\overline{ds}dv} k \left( y_0, \sqrt{w} Z_0, Z \right) \frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{(n-m)/2}} - \Big( 1 + (2s-1)^2 \Big) \frac{\sqrt{a}}{vw} \mathfrak{n}_w \overline{ds} dv \Bigg|$$
$$\le C \frac{(1+|Z_0|)^{\ell+1}}{(1+|zZ_0|)^p} \left( \frac{a}{v} \right)^{\gamma}.$$

*Proof.* We start from Proposition 12.11 and use Theorems 12.15 and 12.17. We will now briefly explain how to use similar results established by Bismut-Goette [BGo1] in a holomorphic context. In fact the above results are the strict analogues of [BGo1, Prop. 8.11, Th. 8.18 and 8.24 ].

Since $X$ is a compact manifold, there exists a finite family of smooth functions $f_1, \ldots, f_r : X \to [0,1]$ which have the following properties:

- $X_K = \bigcap_{j=1}^{r} \{x \in X, f_j(x) = 0.\}$
- On $X_K$, $df_1, \ldots, df_r$ span $N_{X_{g,K}/X}$.

Let $\mathcal{Q}_{a,s,v,Z_0}$ be the family of operators

(12.62)
$$\mathcal{Q}_{a,s,v,Z_0} = \left\{ \nabla_{e_i}, 1 \le i \le 2\ell; \frac{z}{\sqrt{w}} \rho \left( \sqrt{a} Z \right) f_j \left( \sqrt{w} Z_0 + \sqrt{a} Z \right), 1 \le j \le r \right\}.$$

Then commutator estimates similar to the estimates in [BL, Prop. 11.29] and [BGo1, Prop. 8.22] can easily be proved, along the lines of the proof of Theorems 12.15 and12.17. By proceeding as in [BGo1, §8.6–8.11], we obtain in particular the analogue of [BGo1, eq. (8.76)], so that given $p \in \mathbf{N}$, there exist $C > 0, C' > 0$ such that if all the variables are taken as before,

(12.63)
$$\left| \left( \widetilde{F}_a \left( \mathcal{N}^{3,a}_{\sqrt{w} Z_0} \right) - \exp \left( -\overline{\mathcal{N}}^{3,a}_{\sqrt{w} Z_0} \right) \right) \left( g^{-1} Z, Z \right) \right| \le C \left( \frac{a}{v} \right)^{1/4(n+1)} \frac{(1+|Z_0|)^{\ell+1}}{(1+|zZ_0|)^p} \exp \left( -C' |Z|^2 \right).$$

We use the notation in (9.28). By (9.30) in Theorem 9.14 and by (10.13) in Proposition 10.9,

(12.64)
$$2^{m/2} \left[ \int_{N_{X_g/X}} \widehat{\mathrm{Tr_s}} \left[ g \exp \left( -\mathcal{N}^{3,a\prime}_{Z_0,K} \right) \left( g^{-1} Z, Z \right) \right] \frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{(n-m)/2}} \right]^{dsdv,\max} = \left( \sqrt{\frac{v}{w}} \mathfrak{m}_{w/v,w} \left( y_0, Z_0 \right) + (2s-1) \sqrt{a/vw} \mathfrak{o}_{v/w,w} \right) dsdv,$$

$$2^{m/2}\left[\int_{N_{X_g/X}}\widehat{\mathrm{Tr_s}}\left[g\exp\left(-\mathcal{N}^{3,a\prime}_{Z_0}\right)\left(g^{-1}Z,Z\right)\right]\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{(n-m)/2}}\right]^{\overline{ds}dv,\max}$$
$$=\left(1+(2s-1)^2\right)\frac{\sqrt{a}}{vw}\mathfrak{n}_w\overline{ds}dv.$$

Using (12.63), (12.64) and proceeding as in [BGo1, §8.11], we get (12.61). The proof of our theorem is completed. □

Theorem 12.20. *There exist* $c>0, r\in\mathbf{N}, C>0, \gamma\in\,]0,1/2]$ *such that if* $a\in\,]0,1], (s,v)\in[0,1]\times[a,1]$ *are such that* $w\in\,]0,1]$, *if* $z\in\mathbf{R}^*_+, |z|\le c$, *then*

(12.65)
$$|z|^r\left|\mathrm{Tr_s}\left[g\widetilde{F}_a\left(\mathcal{N}^a\right)\right]^{dsdv}-\left(\sqrt{\frac{v}{w}}m_{v/w,w}+(2s-1)\sqrt{a/vw}o_{v/w,w}\right)dsdv\right|$$
$$\le C\left(\frac{a}{v}\right)^\gamma,$$
$$|z|^r\left|\mathrm{Tr_s}\left[g\widetilde{F}_a\left(\mathcal{N}^a\right)\right]^{\overline{ds}dv}-(1+(2s-1))^2\frac{\sqrt{a}}{vw}n_w\overline{ds}dv\right|\le C\left(\frac{a}{v}\right)^\gamma.$$

*Proof.* By Theorem 12.19, the proof of our theorem is the same as the proof of [BGo1, Th. 8.29]. The idea is essentially that the 'hard' estimate of Theorem 12.19 near $X_{g,K}$ can be trivially extended near $X_g$, but away from $X_{g,K}$. □

*Remark* 12.21. By (7.14), (12.23), (12.24) and by (12.65), under the conditions of Theorem 12.20, we get

$$|z|^r\left|\mathrm{Tr_s}\left[g\widetilde{F}_a\left(N^a\right)\right]^{dsdv}-\left(\frac{m_{v/w,w}}{v}+\left(1+(2s-1)^2\right)\sqrt{a}\frac{n_w}{v^3}+(2s-1)\sqrt{a}\frac{o_{v/w,w}}{v^2}\right)dsdv\right|\le C\left(\frac{a}{v}\right)^\gamma\frac{w}{v^2}. \tag{12.66}$$

Using the argument which was given after Theorem 12.5 and (12.66), we find that (7.20) was established when $w\in\,]0,1]$.

12.8. *A proof of* (7.20) *when* $w\ge 1$. Now, we will assume that $(s,v)\in[0,1]\times[a,1]$ are such that $w\ge 1$. Our estimates will now be much simpler, since the fact that $K^X$ vanishes on $X_K$ will not play any role.

Our starting point is still equation (12.24). In Section 12.3, we replace $Z_0\in N_{X_{g,K}/X_g}$ by an arbitrary $x\in X_g$. Otherwise, we use the same trivializations as in this section. We define the operator $\mathcal{N}^{1,a}_x$ as in Definition 12.9.

Let $(e_1, \ldots, e_m)$ be an orthonormal basis of $T_x X_g$, and let $e_{m+1}, \ldots, e_m$ be an orthonormal basis of $N_{X_g/X,x}$. We still define the operator $\mathcal{N}_x^{3,a}$ as in Definition 12.10, with the obvious modification that if $1 \le i \le m$, $\widehat{c}\,(e_i)$ is replaced by $\widehat{c}_v\,(e_i)$; i.e., $w$ does not appear any more.

Then the obvious analogue of equation (12.31) holds, with $w$ replaced by 1. Let now $\mathbf{I}_x^{p,p',r,r'}$ be the vector space of smooth sections of

$$\Lambda^p\,(T^*X_g)\,\widehat{\otimes}\,\widehat{\Lambda}^{p'}\,(T^*X_g)\,\widehat{\otimes}\,\Lambda\left(N^*_{X_g/X}\right)\widehat{\otimes}\,\Lambda^{r,r'}\left(\mathbf{R}^3\right).$$

We use other notation similar to the notation of Section 12.4. Instead of (12.33), if $h \in \mathbf{I}_x^{p,p',r,r'}$ has compact support, set

$$|h|^2_{a,s,v,x,0} = \int_{T_xX} |h\,(Z)|^2 \left(1+\sqrt{v}\,|Z|\,\rho\left(\sqrt{a}Z/2\right)\right)^{2(m-p)} \left(1+\sqrt{\frac{a}{v}}\,|Z|\,\rho\left(\sqrt{a}\,|Z|\,/2\right)\right)^{2(m-p'+1-r')} \left(1+|Z|\,\rho\left(\sqrt{a}Z/2\right)\right)^{2(1-r)} dv_{TX}\,(Z)\,. \tag{12.67}$$

Then an analogue of Proposition 12.13 still holds. Namely the operators in (12.34) are still uniformly bounded. Also the first row of operators in (12.35) is now uniformly bounded, and here $1 \le i \le m$. The fact that the operators $1_{|\sqrt{a}Z|\le 4\varepsilon_0}\sqrt{av/w}\,|Z|\,\widehat{c}_v\,(e_i)\,, 1 \le i \le m$, are uniformly bounded follows from the above.

It is then easy to proceed as in Section 12.7 and to establish (12.66) also in the case where $w \ge 1$.

*Remark* 12.22. It should be pointed out that when $w \ge 1$, in the right-hand side of (12.22), we could have replaced $c\left(K^X/\sqrt{w}\right)$ and $\widehat{c}\left(K^X/\sqrt{w}\right)$ by $c\left(K^X\right), \widehat{c}\left(K^X\right)$, while replacing $\sqrt{w}/v^{3/2}$ in our formula for $\mathbf{E}_1$ in (12.23) by $1/v^{3/2}$. This is because when $w \ge 1$, we have the trivial

$$1/v^{3/2} \le \sqrt{w}/v^{3/2}. \tag{12.68}$$

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