Annals of Mathematics, **159** (2004), 1–52

On the Julia set of a typical quadratic polynomial with a Siegel disk

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To the memory of Michael R. Herman (1942–2000)

Abstract

Let $0 < \theta < 1$ be an irrational number with continued fraction expansion $\theta = [a_1, a_2, a_3, \ldots]$, and consider the quadratic polynomial $P_\theta : z \mapsto e^{2\pi i\theta} z + z^2$. By performing a trans-quasiconformal surgery on an associated Blaschke product model, we prove that if

$$\log a_n = \mathcal{O}(\sqrt{n}) \text{ as } n \to \infty,$$

then the Julia set of P_θ is locally connected and has Lebesgue measure zero. In particular, it follows that for almost every $0 < \theta < 1$, the quadratic P_θ has a Siegel disk whose boundary is a Jordan curve passing through the critical point of P_θ. By standard renormalization theory, these results generalize to the quadratics which have Siegel disks of higher periods.

Contents

1. Introduction

Consider the quadratic polynomial $P_\theta : z \mapsto e^{2\pi i\theta} z + z^2$, where $0 < \theta < 1$ is an irrational number. It has an *indifferent* fixed point at 0 with multiplier $P'_\theta(0) = e^{2\pi i\theta}$, and a unique finite critical point located at $-e^{2\pi i\theta}/2$. Let $A_\theta(\infty)$ be the basin of attraction of infinity, $K_\theta = \mathbb{C} \smallsetminus A_\theta(\infty)$ be the filled Julia set, and $J_\theta = \partial K_\theta$ be the Julia set of P_θ. The behavior of the sequence of iterates $\{P_\theta^{\circ n}\}_{n\geq 0}$ near J_θ is intricate and highly nontrivial. (For a comprehensive account of iteration theory of rational maps, we refer to [CG] or [M].)

The quadratic polynomial P_θ is said to be *stable* near the indifferent fixed point 0 if the family of iterates $\{P_\theta^{\circ n}\}_{n\geq 0}$ restricted to a neighborhood of 0 is normal in the sense of Montel. In this case, the largest neighborhood of 0 with this property is a simply connected domain Δ_θ called the (maximal) *Siegel disk* of P_θ. The unique conformal isomorphism $\psi_\theta : \Delta_\theta \xrightarrow{\simeq} \mathbb{D}$ with $\psi_\theta(0) = 0$ and $\psi_\theta'(0) > 0$ linearizes P_θ in the sense that $\psi_\theta \circ P_\theta \circ \psi_\theta^{-1}(z) = R_\theta(z) := e^{2\pi i\theta} z$ on $\mathbb{D}$.

Consider the continued fraction expansion $\theta = [a_1, a_2, a_3, \ldots]$ with $a_n \in \mathbb{N}$, and the rational convergents $p_n/q_n := [a_1, a_2, \ldots, a_n]$. The number θ is said to be of *bounded type* if $\{a_n\}$ is a bounded sequence. A celebrated theorem of Brjuno and Yoccoz [Yo3] states that the quadratic polynomial P_θ has a Siegel disk around 0 if and only if θ satisfies the condition

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty,$$

which holds almost everywhere in $[0, 1]$. But this theorem gives no information as to what the global dynamics of P_θ should look like. The main result of this paper is a precise picture of the dynamics of P_θ for almost every irrational θ satisfying the above Brjuno-Yoccoz condition:

THEOREM A. *Let $\mathcal{E}$ denote the set of irrational numbers $\theta = [a_1, a_2, a_3, \ldots]$ which satisfy the arithmetical condition*

$$\log a_n = \mathcal{O}(\sqrt{n}) \quad \textit{as} \quad n \to \infty.$$

If $\theta \in \mathcal{E}$, then the Julia set J_θ is locally connected and has Lebesgue measure zero. In particular, the Siegel disk Δ_θ is a Jordan domain whose boundary contains the finite critical point.

This theorem is a rather far-reaching generalization of a theorem which proves the same result under the much stronger assumption that θ is of bounded type [P2]. It is immediate from the definition that the class $\mathcal{E}$ contains all irrationals of bounded type. But the distinction between the two arithmetical classes is far more remarkable, since $\mathcal{E}$ has full measure in $[0, 1]$ whereas numbers of bounded type form a set of measure zero (compare Corollary 2.2).

The foundations of Theorem A was laid in 1986 by several people, notably Douady [Do]. Their idea was to construct a model map F_θ for P_θ by performing surgery on a cubic Blaschke product f_θ. Along with the surgery, they also proved a meta theorem asserting that F_θ and P_θ are quasiconformally conjugate if and only if f_θ is quasisymmetrically conjugate to the rigid rotation R_θ on $\mathbb{S}^1$.

Soon after, Herman used a cross ratio distortion inequality of Światek [Sw] for critical circle maps to give this meta theorem a real content. He proved that f_θ (or any real-analytic critical circle map with rotation number θ for that matter) is quasisymmetrically conjugate to R_θ if and only if θ is of bounded type [H2]. In 1993, Petersen showed that the "Julia set" $J(F_\theta)$ is locally connected for every irrational θ, and has measure zero for every θ of bounded type [P2]. The measure zero statement was soon extended by Lyubich to all irrational θ. It follows from Herman's theorem that J_θ is locally connected and has measure zero when θ is of bounded type. In this case, the Siegel disk Δ_θ is a quasidisk in the sense of Ahlfors and its boundary contains the finite critical point.

The idea behind the proof of Theorem A is to replace the technique of quasiconformal surgery by a *trans-quasiconformal surgery* on a cubic Blaschke product f_θ. Let us give a brief sketch of this process.

We fix an irrational number $0 < \theta < 1$ and following [Do] we consider the degree 3 Blaschke product

$$f_\theta : z \mapsto e^{2\pi i t}\, z^2 \left(\frac{z-3}{1-3z}\right),$$

which has a double critical point at $z = 1$. Here $0 < t = t(\theta) < 1$ is the unique parameter for which the critical circle map $f_\theta|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ has rotation number θ (see subsection 2.4). By a theorem of Yoccoz [Yo1], there exists a unique homeomorphism $h_\theta : \mathbb{S}^1 \to \mathbb{S}^1$ with $h_\theta(1) = 1$ such that $h_\theta \circ f_\theta|_{\mathbb{S}^1} = R_\theta \circ h_\theta$. Let $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be any homeomorphic extension of h_θ and define

$$F_\theta(z) = F_{\theta,H}(z) := \begin{cases} f_\theta(z) & \text{if } |z| \geq 1 \\ (H^{-1} \circ R_\theta \circ H)(z) & \text{if } |z| < 1. \end{cases}$$

Then F_θ is a degree 2 topological branched covering of the sphere. It is holomorphic outside of $\overline{\mathbb{D}}$ and is topologically conjugate to the rigid rotation R_θ on $\overline{\mathbb{D}}$. This is the candidate model for the quadratic map P_θ.

By way of comparison, if there is any correspondence between P_θ and F_θ, the Siegel disk for P_θ should correspond to the unit disk for F_θ, while the other bounded Fatou components of P_θ should correspond to other iterated F_θ-preimages of the unit disk, which we call *drops*. The basin of attraction of infinity for P_θ should correspond to a similar basin $A(\infty)$ for F_θ (which is the immediate basin of attraction of infinity for f_θ). By imitating the case of polynomials, we define the "filled Julia set" $K(F_\theta)$ as $\mathbb{C} \smallsetminus A(\infty)$ and the "Julia set" $J(F_\theta)$ as the topological boundary of $K(F_\theta)$, both of which are independent of the homeomorphism H (compare Figure 2).

By the results of Petersen and Lyubich mentioned above, $J(F_\theta)$ is locally connected and has measure zero for all irrational numbers θ. Thus, the local-connectivity statement in Theorem A will follow once we prove that for $\theta \in \mathcal{E}$ there exists a homeomorphism $\varphi_\theta : \mathbb{C} \to \mathbb{C}$ such that $\varphi_\theta \circ F_\theta \circ \varphi_\theta^{-1} = P_\theta$.

The measure zero statement in Theorem A will follow once we prove φ_θ is absolutely continuous.

The basic idea described by Douady in [Do] is to choose the homeomorphic extension H in the definition of F_θ to be quasiconformal, which by Herman's theorem is possible if and only if θ is of bounded type. Taking the Beltrami differential of H on $\mathbb{D}$, and spreading it by the iterated inverse branches of F_θ to all the drops, one obtains an F_θ-invariant Beltrami differential μ on $\mathbb{C}$ with bounded dilatation and with the support contained in the filled Julia set $K(F_\theta)$. The measurable Riemann mapping theorem shows that μ can be integrated by a quasiconformal homeomorphism which, when appropriately normalized, yields the desired conjugacy φ_θ.

To go beyond the bounded type class in the surgery construction, one has to give up the idea of a quasiconformal surgery. The main idea, which we bring to work here, is to use extensions H which are trans-quasiconformal, i.e., have unbounded dilatation with controlled growth. What gives this approach a chance to succeed is the theorem of David on integrability of certain Beltrami differentials with unbounded dilatation [Da]. David's integrability condition requires that for all large K, the area of the set of points where the dilatation is greater than K be dominated by an exponentially decreasing function of K (see subsection 2.5 for precise definitions). An orientation-preserving homeomorphism between planar domains is a *David homeomorphism* if it belongs to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ and its Beltrami differential satisfies the above integrability condition. Such homeomorphisms are known to preserve the Lebesgue measure class.

To carry out a trans-quasiconformal surgery, we have to address two fundamental questions:

Question 1. Under what optimal arithmetical condition $\mathcal{E}_{\mathrm{DE}}$ on θ does the linearization h_θ admit a David extension $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$?

Question 2. Under what optimal arithmetical condition $\mathcal{E}_{\mathrm{DI}}$ on θ does the model F_θ admit an invariant Beltrami differential satisfying David's integrability condition in the plane?

It turns out that the two questions have the same answer, i.e., $\mathcal{E}_{\mathrm{DE}} = \mathcal{E}_{\mathrm{DI}}$. Clearly $\mathcal{E}_{\mathrm{DE}} \supseteq \mathcal{E}_{\mathrm{DI}}$, but the other inclusion is a nontrivial result, which we prove in this paper by means of the following construction.

Define a measure ν supported on $\overline{\mathbb{D}}$ by summing up the push forward of Lebesgue measure on all the drops. In other words, for any measurable set $E \subset \mathbb{D}$, set

$$\nu(E) := \operatorname{area}(E) + \sum_g \operatorname{area}(g(E)),$$

where the summation is over all the univalent branches $g = F_\theta^{-k}$ mapping $\mathbb{D}$ to various drops. Evidently ν is absolutely continuous with respect to Lebesgue measure on $\mathbb{D}$. However, we prove a much sharper result:

THEOREM B. *The measure ν is dominated by a universal power of Lebesgue measure. In other words, there exist a universal constant $0 < \beta < 1$ and a constant $C > 0$ (depending on θ) such that*

$$\nu(E) \leq C\,(\text{area}(E))^\beta$$

for every measurable set $E \subset \mathbb{D}$.

It follows immediately from this key estimate that the F_θ-invariant Beltrami differential μ constructed above satisfies David's integrability condition if $\mu|_{\mathbb{D}}$ does, or equivalently, if there is a David extension H for h_θ.

Theorem B can be used to prove that a conjugacy φ_θ between F_θ and P_θ exists whenever h_θ admits a David extension to the disk. The following theorem proves the existence of David extensions for circle homeomorphisms which arise as linearizations of critical circle maps with rotation numbers in $\mathcal{E}$. This theorem, as formulated here in the context of our trans-quasiconformal surgery, is new. However, we should emphasize that all the main ingredients of its constructive proof are already present in a manuscript of Yoccoz [Yo2].

THEOREM C. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a critical circle map whose rotation number $\theta = [a_1, a_2, a_3, \ldots]$ belongs to the arithmetical class $\mathcal{E}$. Then the normalized linearizing map $h : \mathbb{S}^1 \to \mathbb{S}^1$, which satisfies $h \circ f = R_\theta \circ h$, admits a David extension $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ so that*

$$\text{area}\left\{ z \in \mathbb{D} : \left| \frac{\overline{\partial} H(z)}{\partial H(z)} \right| > 1 - \varepsilon \right\} \leq M\, e^{-\frac{\alpha}{\varepsilon}} \;\text{ for all }\; 0 < \varepsilon < \varepsilon_0.$$

Here $M > 0$ is a universal constant, while in general the constant $\alpha > 0$ depends on $\limsup_{n \to \infty} (\log a_n)/\sqrt{n}$ and the constant $0 < \varepsilon_0 < 1$ depends on f.

Let us point out that Theorem C proves $\mathcal{E} \subset \mathcal{E}_{\text{DE}}$, where $\mathcal{E}_{\text{DE}}$ is the arithmetical condition in Question 1. We have reasons to suspect that the above inclusion might in fact be an equality, but so far we have not been able to prove this.

When θ is of bounded type, the boundary of the Siegel disk Δ_θ is a quasicircle, so it clearly has Hausdorff dimension less than 2. McMullen has proved that in this case the entire Julia set J_θ has Hausdorff dimension less than 2 [Mc2], a result which improves the measure zero statement in Petersen's theorem. The situation when θ belongs to $\mathcal{E}$ but is not of bounded type might be quite different. In this case, the proof of Theorem A shows that the boundary of Δ_θ is a *David circle*, i.e., the image of the round circle under a David homeomorphism. It can be shown that, unlike quasiconformal maps, David homeomorphisms do not preserve sets of Hausdorff dimension 0 or 2, and in fact there are David circles of Hausdorff dimension 2 [Z2]. So, *a priori*, the boundary of Δ_θ might have Hausdorff dimension 2 as well. Motivated by these remarks, we ask:

Question 3. What can be said about the Hausdorff dimension of J_θ when θ belongs to $\mathcal{E}$ but is not of bounded type? Does there exist such a θ for which J_θ, or even $\partial\Delta_\theta$, has Hausdorff dimension 2?

The use of trans-quasiconformal surgery in holomorphic dynamics was pioneered by Haïssinsky who showed how to produce a parabolic point from a pair of attracting and repelling points when the repelling point is not in the ω-limit set of a recurrent critical point [Ha]. In contrast, our maps have a recurrent critical point whose orbit is dense in the boundary of the disk on which we perform surgery.

The idea of constructing rational maps by quasiconformal surgery on Blaschke products has been taken up by several authors; for instance Zakeri, who in [Z1] models the one-dimensional parameter space of cubic polynomials with a Siegel disk of a given bounded type rotation number. Also this idea is central to the work of Yampolsky and Zakeri in [YZ], where they show that any two quadratic Siegel polynomials P_{θ_1} and P_{θ_2} with bounded type rotation numbers θ_1 and θ_2 are mateable provided that $\theta_1 \neq 1 - \theta_2$. We believe adaptations of the ideas and techniques developed in the present paper will give generalizations of those results to rotation numbers in $\mathcal{E}$.

Acknowledgements. The first author would like to thank the Mathematics Department of Cornell University for its hospitality and IMFUA at Roskilde University for its financial support. The second author is grateful to IMS at Stony Brook for supporting part of this research through NSF grant DMS 9803242 during the spring semester of 1999. Further thanks are due to the referee whose suggestions improved our presentation of puzzle pieces in Section 4, and to P. Haïssinsky whose comment prompted us to add Lemma 5.5 to our early version of this paper.

2. Preliminaries

2.1. *General notation.* We will adopt the following notation throughout this paper:

- $\mathbb{T}$ is the quotient $\mathbb{R}/\mathbb{Z}$.
- $\mathbb{S}^1$ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$; we often identify $\mathbb{T}$ and $\mathbb{S}^1$ via the exponential map $x \mapsto e^{2\pi i x}$ without explicitly mentioning it.

- $|I|$ is the Euclidean length of a rectifiable arc $I \subset \mathbb{C}$.
- For $x, y \in \mathbb{T}$ or $\mathbb{S}^1$ which are not antipodal, $[x, y] = [y, x]$ (resp. $]x, y[=]y, x[$) denotes the shorter closed (resp. open) interval with endpoints x, y.
- $\mathrm{diam}(\cdot)$, $\mathrm{dist}(\cdot,\cdot)$ and $\mathrm{area}(\cdot)$ denote the Euclidean diameter, Euclidean distance and Lebesgue measure in $\mathbb{C}$.
- For a hyperbolic Riemann surface X, $\ell_X(\cdot)$, $\mathrm{diam}_X(\cdot)$ and $\mathrm{dist}_X(\cdot)$ denote the hyperbolic arclength, diameter and distance in X.
- In a given statement, by a *universal* constant we mean one which is independent of all the parameters/variables involved. Two positive numbers a, b are said to be *comparable* up to a constant $C > 1$ if $b/C \leq a \leq bC$. For two positive sequences $\{a_n\}$ and $\{b_n\}$, we write $a_n \preccurlyeq b_n$ if there exists a universal constant $C > 1$ such that $a_n \leq C\, b_n$ for all large n. We define $a_n \succcurlyeq b_n$ in a similar way. We write $a_n \asymp b_n$ if $b_n \preccurlyeq a_n \preccurlyeq b_n$, i.e., if there exists a universal constant $C > 1$ such that $b_n/C \leq a_n \leq C\, b_n$ for all large n. Any such relation will be called an *asymptotically universal bound*. Note that for any such bound, the corresponding inequalities hold for every n if C is replaced by a larger constant (which may well depend on our sequences and no longer be universal).

  Another way of expressing an asymptotically universal bound, which we will often use, is as follows: When $a_n \preccurlyeq b_n$, we say that a_n/b_n is bounded from above by a constant which is asymptotically universal. Similarly, when $a_n \asymp b_n$, we say that a_n and b_n are comparable up to a constant which is asymptotically universal.

  Finally, let $\{a_n = a_n(x)\}$ and $\{b_n = b_n(x)\}$ depend on a parameter x belonging to a set X. Then we say that $a_n \asymp b_n$ *uniformly in* $x \in X$ if there exists a universal constant $C > 1$ and an integer $N \geq 1$ such that $b_n(x)/C \leq a_n(x) \leq C b_n(x)$ for all $n \geq N$ and all $x \in X$.

2.2. *Some arithmetic.* Here we collect some basic facts about continued fractions; see [Kh] or [La] for more details. Let $0 < \theta < 1$ be an irrational number and consider the continued fraction expansion

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [a_1, a_2, a_3, \ldots],$$

with $a_n = a_n(\theta) \in \mathbb{N}$. The *n-th convergent* of θ is the irreducible fraction $p_n/q_n := [a_1, a_2, \ldots, a_n]$. We set $p_0 := 0$, $q_0 := 1$. It is easy to verify the recursive relations

$$p_n = a_n\, p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n\, q_{n-1} + q_{n-2} \tag{2.1}$$

for $n \geq 2$. The denominators q_n grow exponentially fast; in fact it follows easily from (2.1) that

$$q_n \geq (\sqrt{2})^n \quad \text{for } n \geq 2.$$

Elementary arithmetic shows that

$$\frac{1}{q_n(q_n+q_{n+1})} < \left|\theta - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}, \tag{2.2}$$

which implies $p_n/q_n \to \theta$ exponentially fast.

Various arithmetical conditions on irrational numbers come up in the study of indifferent fixed points of holomorphic maps. Of particular interest are:

- The class $\mathcal{D}_d$ of *Diophantine numbers of exponent* $d \geq 2$. An irrational θ belongs to $\mathcal{D}_d$ if there exists some $C > 0$ such that $|\theta - p/q| \geq Cq^{-d}$ for all rationals p/q. It follows immediately from (2.2) that for any $d \geq 2$

$$\theta \in \mathcal{D}_d \Leftrightarrow \sup_n \frac{q_{n+1}}{{q_n}^{d-1}} < +\infty \Leftrightarrow \sup_n \frac{a_{n+1}}{{q_n}^{d-2}} < +\infty. \tag{2.3}$$

- The class $\mathcal{D} := \bigcup_{d\geq 2} \mathcal{D}_d$ of *Diophantine* numbers. From (2.3) it follows that

$$\theta \in \mathcal{D} \Leftrightarrow \sup_n \frac{\log q_{n+1}}{\log q_n} < +\infty.$$

- The class $\mathcal{D}_2$ of Diophantine numbers of exponent 2. Again by (2.3)

$$\theta \in \mathcal{D}_2 \Leftrightarrow \sup_n a_n < +\infty.$$

  For this reason, any such θ is called a number of *bounded type*.

- The class $\mathcal{B}$ of numbers of *Brjuno type*. By definition,

$$\theta \in \mathcal{B} \Leftrightarrow \sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty.$$

We have the proper inclusions

$$\mathcal{D}_2 \subsetneq \mathcal{D}_d \subsetneq \mathcal{D} \subsetneq \mathcal{B}$$

for any $d > 2$. Diophantine numbers of any exponent $d > 2$ have full measure in $[0, 1]$ while numbers of bounded type form a set of measure zero.

The following theorem characterizes the asymptotic growth of the sequence $\{a_n\}$ for random irrational numbers:

THEOREM 2.1. *Let $\psi : \mathbb{N} \to \mathbb{R}$ be a given positive function.*

(i) *If $\sum_{n=1}^{\infty} \frac{1}{\psi(n)} < +\infty$, then for almost every irrational $0 < \theta < 1$ there are only finitely many n for which $a_n(\theta) \geq \psi(n)$.*

(ii) *If $\sum_{n=1}^{\infty} \frac{1}{\psi(n)} = +\infty$, then for almost every irrational $0 < \theta < 1$ there are infinitely many n for which $a_n(\theta) \geq \psi(n)$.*

This theorem is often attributed to E. Borel and F. Bernstein, at least in the case ψ is increasing. For a proof of the general case, see Khinchin's book [Kh].

COROLLARY 2.2. *Let $\mathcal{E}$ be the set of all irrational numbers $0 < \theta < 1$ for which the sequence $\{a_n = a_n(\theta)\}$ satisfies*

$$\log a_n = \mathcal{O}(\sqrt{n}) \quad \text{as } n \to \infty. \tag{2.4}$$

Then $\mathcal{E}$ has full measure in $[0, 1]$.

The class $\mathcal{E}$ will be the center of focus in the present paper. It is easily seen to be a proper subclass of $\mathcal{D}_d$ for any $d > 2$.

2.3. *Rigid rotations.* We now turn to elementary properties of rigid rotations on the circle. For a comprehensive treatment, we recommend Herman's monograph [H1]. Let $R_\theta : x \mapsto x + \theta \pmod{\mathbb{Z}}$ denote the rigid rotation by the irrational number θ. For $x \in \mathbb{R}$, set $\|x\| := \inf_{n\in\mathbb{Z}} |x - n|$. Then, for $n \geq 2$,

$$\|q_n\theta\| < \|i\theta\| \quad \text{for all } 1 \leq i < q_n.$$

Thus, considering the orbit of $0 \in \mathbb{T}$ under the iteration of R_θ, the denominators q_n constitute the *moments of closest return.* Clearly the same is true for the orbit of every point. It is not hard to verify that

$$\|q_n\theta\| = (-1)^n(q_n\theta - p_n), \tag{2.5}$$

so that the closest returns occur alternately on the left and right sides of 0.

Consider the decreasing sequence $\|q_1\theta\| > \|q_2\theta\| > \|q_3\theta\| > \cdots$ and define the *scaling ratio*

$$s_n := \frac{\|q_n\theta\|}{\|q_{n+1}\theta\|} > 1.$$

By (2.1) and (2.5)

$$s_{n-1} = a_{n+1} + \frac{1}{s_n}.$$

In particular, the two sequences $\{a_{n+2}\}$ and $\{s_n\}$ have the same asymptotic behavior. For example, it follows that the sequence $\{s_n\}$ is bounded if and only if θ is of bounded type.

There are two basic facts about the structure of the orbits of rotations that we will use repeatedly:

- For $i \in \mathbb{Z}$, let x_i denote the iterate $R_\theta^{-i}(0)$ (Caution: We have labelled the orbit of 0 backwards to simplify the subsequent notations; this corresponds to the standard notation for the inverse map R_θ^{-1}). Given two consecutive closest return moments q_n and q_{n+1}, the points in the orbit of 0 occur in the order shown in Figure 1 (the picture shows the case n is odd; for the case n is even simply rotate the picture 180° about 0). Note that $|[0, x_{q_n}]| = |[0, x_{-q_n}]| = \|q_n\theta\|$. Evidently, the orbit of any other point of $\mathbb{T}$ enjoys the same order.

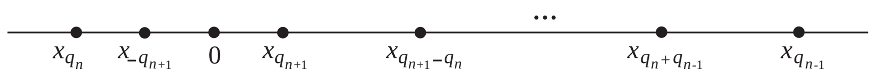

Figure 1. Selected points in the orbit of 0 under the rigid rotation.

- Let $I^n := [0, x_{q_n}]$ be the n-th closest return interval for 0. Then the collection of intervals

$$\Pi^n(R_\theta) := \{R_\theta^{-i}(I^n)\}_{0 \leq i \leq q_{n+1}-1} \cup \{R_\theta^{-i}(I^{n+1})\}_{0 \leq i \leq q_n - 1} \tag{2.6}$$

defines a partition of the circle modulo the common endpoints. We call $\Pi^n(R_\theta)$ the *dynamical partition of level* n for R_θ.

Theorem 2.3 (Poincaré). *Let $f : \mathbb{T} \to \mathbb{T}$ be any circle homeomorphism without periodic points. Then there exists a unique irrational number θ and a continuous degree 1 monotone map $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ f = R_\theta \circ h$.*

The number θ is called the *rotation number* of f and is denoted by $\rho(f)$. The map h is called a *Poincaré semiconjugacy*. It easily follows from this theorem that the combinatorial structure of the orbits of any circle homeomorphism with irrational rotation number θ is the same as the combinatorial structure of the orbit of 0 for R_θ.

2.4. *Critical circle maps.* For our purposes, a *critical circle map* will be a real-analytic homeomorphism of $\mathbb{T}$ with a critical point at 0. It was proved by Yoccoz [Yo1] that for a critical circle map with irrational rotation number, every Poincaré semiconjugacy is in fact a conjugacy:

Theorem 2.4 (Yoccoz). *Let $f : \mathbb{T} \to \mathbb{T}$ be a critical circle map with irrational rotation number $\rho(f) = \theta$. Then there exists a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ f = R_\theta \circ h$. This h is uniquely determined once normalized by $h(0) = 0$.*

We will reserve the notation x_i for the backward iterate $f^{-i}(0)$ of the critical point 0 and $I^n := [0, x_{q_n}]$ for the n-th closest return interval under f^{-1}.

The dynamical partition $\Pi^n(f)$ of level n for f will be defined as $h^{-1}(\Pi^n(R_\theta))$, or equivalently, by (2.6) with R_θ replaced by f.

Herman took the next step in studying critical circle maps by showing that the linearizing map h is quasisymmetric if and only if $\rho(f)$ is irrational of bounded type. The proof of this theorem makes essential use of the existence of *real a priori bounds* developed by Światek and Herman. Here is a version of their result needed in this paper (see [Sw], [H2], [dFdM], or [P4]).

THEOREM 2.5 (Światek-Herman). *Let $f : \mathbb{T} \to \mathbb{T}$ be a critical circle map with $\rho(f)$ irrational. Then*

(i) *There exists an asymptotically universal bound*

$$|[y, f^{\circ q_n}(y)]| \asymp |[y, f^{-q_n}(y)]|$$

which holds uniformly in $y \in \mathbb{T}$.

(ii) *The lengths of any two adjacent intervals in the dynamical partition $\Pi^n(f)$ are comparable up to a bound which is asymptotically universal. In other words,*

$$\max \left\{ \frac{|I|}{|J|} : I, J \in \Pi^n(f) \text{ are adjacent} \right\} \asymp 1.$$

An important corollary of (ii), which exhibits a sharp contrast with the case of rigid rotations, is that the scaling ratio is bounded from above and below by an asymptotically universal constant *regardless of the map f*:

$$s_n(f) := \frac{|I^n|}{|I^{n+1}|} \asymp 1.$$

Remark 2.6. The above (i) and (ii) are presumably the most general statements one can expect when working with the class of *all* critical circle maps. However, stronger versions of these bounds can be obtained by restricting to a special class of such maps. For example, fix a critical circle map f_0 and consider the one-dimensional family

$$\mathcal{F} = \{R_t \circ f_0 : 0 \le t \le 1 \text{ and } \rho(R_t \circ f_0) \text{ is irrational}\}.$$

Then, within this family the above bounds hold for all n (rather than all large n), with the constant depending only on f_0 and not on t. In other words, there exists a constant $C = C(f_0) > 1$ such that

$$\frac{1}{C} \le \frac{|[y, f^{\circ q_n}(y)]|}{|[y, f^{-q_n}(y)]|} \le C \quad \text{for all } n \ge 1,\ y \in \mathbb{T}, \text{ and } f \in \mathcal{F},$$

$$\frac{1}{C} \le \max \left\{ \frac{|I|}{|J|} : I, J \in \Pi^n(f) \text{ are adjacent} \right\} \le C \quad \text{for all } n \ge 1 \text{ and } f \in \mathcal{F}.$$

We will need the following result on the size of the intervals in the dynamical partitions for a critical circle map; it is a direct consequence of real *a priori* bounds (see for example [dFdM, Th. 3.1]):

LEMMA 2.7. *Let $f : \mathbb{T} \to \mathbb{T}$ be a critical circle map, with $\rho(f)$ irrational, and let $\Pi^n(f)$ denote the dynamical partition of level n for f. Then there exist universal constants $0 < \sigma_1 < \sigma_2 < 1$ such that*

$$\sigma_1^n \preccurlyeq |I^n| \leq \max_{I \in \Pi^n(f)} |I| \preccurlyeq \sigma_2^n.$$

2.5. *David homeomorphisms.* An orientation-preserving homeomorphism $\varphi : \Omega \to \Omega'$ between planar domains belongs to the Sobolev class $W^{1,1}_{\text{loc}}(\Omega)$ if the distributional partial derivatives $\partial\varphi$ and $\overline{\partial}\varphi$ exist and are locally integrable in Ω (equivalently, if φ is absolutely continuous on lines in Ω; see for example [A]). In this case, φ is differentiable almost everywhere and the Jacobian $\text{Jac}(\varphi) = |\partial\varphi|^2 - |\overline{\partial}\varphi|^2 \geq 0$ is locally integrable.

A *Beltrami differential* in Ω is a measurable $(-1, 1)$-form $\mu = \mu(z)\, d\overline{z}/dz$ such that $|\mu| < 1$ almost everywhere in Ω. We say that μ is *integrable* if there is a homeomorphism $\varphi : \Omega \to \Omega'$ in $W^{1,1}_{\text{loc}}(\Omega)$ which solves the *Beltrami equation* $\overline{\partial}\varphi = \mu\, \partial\varphi$. The classical quasiconformal mappings arise as the solutions of the Beltrami equation in the case $\|\mu\|_\infty < 1$. However, there are numerous important problems in which one has to study this equation when $\|\mu\|_\infty = 1$. Simple examples show that such a μ is not generally integrable, so one has to seek conditions on the growth of $|\mu|$ which guarantee integrability. One such condition was given by Guy David in [Da], who studied Beltrami differentials satisfying an exponential growth condition. Let us call μ a *David-Beltrami differential* if there exist constants $M > 0$, $\alpha > 0$, and $0 < \varepsilon_0 < 1$ such that

$$\text{area}\{z \in \Omega : |\mu|(z) > 1 - \varepsilon\} \leq M\, e^{-\frac{\alpha}{\varepsilon}} \qquad \text{for all } 0 < \varepsilon < \varepsilon_0. \tag{2.7}$$

This notion can be extended to arbitrary domains on the sphere $\widehat{\mathbb{C}}$; it suffices to replace the Euclidean area with the spherical area in the growth condition (2.7).

David proved that the analogue of the measurable Riemann mapping theorem [AB] holds for the class of David-Beltrami differentials [Da]:

THEOREM 2.8 (David). *Let Ω be a domain in $\mathbb{C}$ and μ be a David-Beltrami differential in Ω. Then μ is integrable. More precisely, there exists an orientation-preserving homeomorphism $\varphi : \Omega \to \Omega'$ in $W^{1,1}_{\text{loc}}(\Omega)$ which satisfies $\overline{\partial}\varphi = \mu\, \partial\varphi$ almost everywhere. Moreover, φ is unique up to postcomposition with a conformal map. In other words, if $\Phi : \Omega \to \Omega''$ is another homeomorphic solution of the same Beltrami equation in $W^{1,1}_{\text{loc}}(\Omega)$, then $\Phi \circ \varphi^{-1} : \Omega' \to \Omega''$ is a conformal map.*

Solutions of the Beltrami equation given by this theorem are called *David homeomorphisms*. They differ from classical quasiconformal maps in many respects. A significant example is the fact that the inverse of a David homeomorphism is not necessarily David. However, they enjoy some convenient properties of quasiconformal maps such as compactness; see [T] for a study of some of these similarities. The following result is particularly important [Da]:

THEOREM 2.9. *Let $\varphi : \Omega \to \Omega'$ be a David homeomorphism. Then φ and φ^{-1} are both absolutely continuous; in other words, for a measurable set $E \subset \Omega$,*

$$\operatorname{area}(E) = 0 \Longleftrightarrow \operatorname{area}(\varphi(E)) = 0.$$

It easily follows that if $\varphi : \Omega \to \Omega'$ is a David homeomorphism, then $\partial\varphi \neq 0$ almost everywhere in Ω. Thus, the *complex dilatation* of φ, defined by the measurable $(-1, 1)$-form

$$\mu_\varphi := \frac{\overline{\partial}\varphi}{\partial\varphi} \, \frac{d\overline{z}}{dz}$$

is a well-defined David-Beltrami differential in the sense of (2.7). Equivalently, the *real dilatation* of φ, given by

$$K_\varphi := \frac{1 + |\mu_\varphi|}{1 - |\mu_\varphi|},$$

satisfies a condition of the form

$$\operatorname{area}\{z \in \Omega : K_\varphi(z) > K\} \leq M\, e^{-\alpha K} \qquad \text{for all } K > K_0 \tag{2.8}$$

for some constants $M > 0$, $\alpha > 0$, and $K_0 > 1$.

2.6. *Extensions of linearizing homeomorphisms.* Let f be a critical circle map with $\rho(f)$ irrational and consider the linearizing map h given by Yoccoz's Theorem 2.4. The problem of extending h to a self-homeomorphism of the disk with nice analytic properties arises in various circumstances in holomorphic dynamics, particularly in the construction of Siegel disks by means of surgery. When $\rho(f)$ is of bounded type, it follows from Theorem 2.5 that h is quasisymmetric. Hence, by the theorem of Beurling-Ahlfors [BA], it can be extended to a quasiconformal map $\mathbb{D} \to \mathbb{D}$ whose dilatation only depends on the quasisymmetric norm of h (which in turn only depends on $\sup_n a_n(\theta)$, where $\theta = \rho(f)$). This allows a quasiconformal surgery (compare [Do], [P2], [Z1], or [YZ]).

On the other hand, when $\rho(f)$ is not of bounded type, again by Theorem 2.5, h fails to be quasisymmetric and hence it admits no quasiconformal extension. Thus, one is forced to give up the idea of quasiconformal surgery.

Still, one can ask if in this case h admits a David extension to $\mathbb{D}$. One way to address this problem is to develop a Beurling-Ahlfors theory for David homeomorphisms of the disk. For example, it is possible to show that a circle homeomorphism whose local distortion has controlled growth admits a David extension. But, to the best of our knowledge, the problem of characterizing boundary values of David homeomorphisms has not yet been solved completely:

Problem. Find necessary and sufficient conditions for a circle homeomorphism to admit a David extension to the unit disk.

Another approach, less general but very effective in our dynamical framework, is to attempt to construct David extensions directly for the circle homeomorphisms which arise as linearizing maps of critical circle maps. This approach turns out to be successful because of the existence of real *a priori* bounds (Theorem 2.5). In fact, using Yoccoz's work in [Yo2], one can prove the following:

THEOREM C. *Let* $f : \mathbb{S}^1 \to \mathbb{S}^1$ *be a critical circle map whose rotation number* $\theta = [a_1, a_2, a_3, \ldots]$ *belongs to the arithmetical class* $\mathcal{E}$ *defined in* (2.4). *Then the linearizing map* $h : \mathbb{S}^1 \to \mathbb{S}^1$, *which satisfies* $h \circ f = R_\theta \circ h$ *and* $h(1) = 1$, *admits a David extension* $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. *Moreover, the constant* M *in condition* (2.7) *is universal, while in general* α *depends on* $\limsup_{n\to\infty}(\log a_n)/\sqrt{n}$ *and* ε_0 *depends on* f.

The proof of this result is rather lengthy and will be presented in the appendix.

3. A Blaschke model

3.1. *Definitions.* Given an irrational number $0 < \theta < 1$, consider the degree 3 Blaschke product

$$f = f_\theta : z \mapsto e^{2\pi i t(\theta)}\, z^2 \left(\frac{z-3}{1-3z}\right), \tag{3.1}$$

which has superattracting fixed points at 0 and ∞ and a double critical point at $z = 1$. Here $0 < t(\theta) < 1$ is the unique parameter for which the critical circle map $f|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ has rotation number θ. By Theorem 2.4, there exists a unique homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ with $h(1) = 1$ such that $h \circ f|_{\mathbb{S}^1} = R_\theta \circ h$. Let $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be *any* homeomorphic extension of h and define

$$F(z) = F_{\theta,H}(z) := \begin{cases} f(z) & \text{if } |z| \geq 1 \\ (H^{-1} \circ R_\theta \circ H)(z) & \text{if } |z| < 1. \end{cases} \tag{3.2}$$

It is easy to see that F is a degree 2 topological branched covering of the sphere which is holomorphic outside of $\overline{\mathbb{D}}$ and is topologically conjugate to a rigid rotation on $\overline{\mathbb{D}}$. By imitating the polynomial case, we define the "filled Julia set" of F by

$$K(F) := \{z \in \mathbb{C} : \text{The orbit } \{F^{\circ n}(z)\}_{n \geq 0} \text{ is bounded}\}$$

and the "Julia set" of F as the topological boundary of $K(F)$:

$$J(F) := \partial K(F).$$

Let $A(\infty)$ be the basin of attraction of ∞ for F. Then $A(\infty)$ is simply-connected and

$$K(F) = \mathbb{C} \smallsetminus A(\infty), \quad J(F) = \partial A(\infty).$$

Let us point out that although the homeomorphism H is by no means canonical, neither $J(F)$ nor $K(F)$ nor any of the definitions to follow depends on a particular choice of H. This is simply because the constructions do not involve the values of F on $\mathbb{D}$. The main purpose of introducing F for the following constructions is to forget about the f-preimages of $\mathbb{D}$ in $\mathbb{D}$. A particular choice of H is only used in the final step of the proof of Theorem A, where we need H to be a David homeomorphism.

The Blaschke product f was introduced by Douady and Herman [Do], using an earlier idea of Ghys, and has been used by various authors in order to study rational maps with Siegel disks; see for example [P2] and [Mc2] for the case of quadratic polynomials, and [Z1] and [YZ] for variants in the case of cubic polynomials and quadratic rational maps.

3.2. *Drops and limbs.* Here we follow the presentations of [P2] and [YZ] with minor modifications. The reader might consult either of these references for a more detailed description.

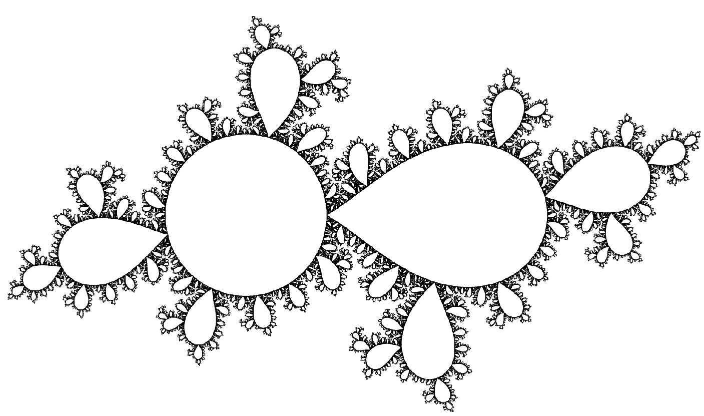

Figure 2. Filled Julia set $K(F)$ for $\theta = [a_1, a_2, a_3, \ldots]$, where $a_n = \lfloor e^{\sqrt{n}} \rfloor$.

By definition, the unique component of $F^{-1}(\mathbb{D}) \smallsetminus \mathbb{D}$ is called the 0-*drop* of F and is denoted by U_0. (In Figure 2, U_0 is the prominently visible Jordan domain attached to the unit disk at $z = 1$.) For $n \geq 1$, any component U of $F^{-n}(U_0)$ is a Jordan domain called an n-*drop*, with n being the *depth* of U. The map $F^{\circ n} = f^{\circ n} : U \to U_0$ is a conformal isomorphism which extends isomorphically to a neighborhood of $\overline{U}$, because $\overline{U}_0$ does not intersect the forward orbit of the critical values. The unique point $F^{-n}(1) \cap \partial U$ is called the *root* of U and is denoted by $x(U)$. The boundary ∂U is a real-analytic Jordan curve except at the root where it has an angle of $\pi/3$. We simply refer to U as a drop when the depth is not important. For convenience, we define $\mathbb{D}$ to be a (-1)-drop, i.e., a drop of depth -1. Note that these drops do not depend on the extension H used to define the map F in (3.2).

Let U and V be distinct drops of depths m and n, respectively, with $m \leq n$. Then either $\overline{U} \cap \overline{V} = \emptyset$ or else $\overline{U} \cap \overline{V} = x(V)$ and $m < n$. In the latter case, we call U the *parent* of V, and V a *child* of U. Every n-drop with $n \geq 0$ has a unique parent which is an m-drop with $-1 \leq m < n$. In particular, the root of this n-drop belongs to the boundary of its parent.

By definition, $\mathbb{D}$ is said to be of *generation* 0. Any child of $\mathbb{D}$ is of generation 1. In general, a drop is of generation k if and only if its parent is of generation $k-1$. Given a point $w \in \bigcup_{n \geq 0} F^{-n}(1)$, there exists a unique drop U with $x(U) = w$. In particular, two distinct children of a parent have distinct roots.

We give a symbolic description of drops by assigning addresses to them. Set $U_\emptyset := \mathbb{D}$, where $\emptyset$ is the empty index. For $n \geq 0$, let $x_n := F^{-n}(1) \cap \mathbb{S}^1$ and let U_n be the n-drop of generation 1 with root x_n. Let $\iota = \iota_1, \iota_2, \ldots, \iota_k$ be any multi-index of length $k \geq 1$, where each ι_j is a nonnegative integer. We recursively define the $(\iota_1 + \iota_2 + \cdots + \iota_k)$-drop $U_{\iota_1,\iota_2,\ldots,\iota_k}$ of generation k with root $x(U_{\iota_1,\iota_2,\ldots,\iota_k}) = x_{\iota_1,\iota_2,\ldots,\iota_k}$ as follows. We have already defined these for $k = 1$. Suppose that we have defined $x_{\iota_1,\iota_2,\ldots,\iota_{k-1}}$ for all multi-indices $\iota_1, \iota_2, \ldots, \iota_{k-1}$ of length $k-1$. Then, we define

$$x_{\iota_1,\iota_2,\ldots,\iota_k} := F^{-(1+\iota_1)}(x_{\iota_2,\ldots,\iota_k}) \cap \partial U_{\iota_1,\iota_2,\ldots,\iota_{k-1}}.$$

The drop $U_{\iota_1,\iota_2,\ldots,\iota_k}$ will be determined by the condition of having $x_{\iota_1,\iota_2,\ldots,\iota_k}$ as its root. By the way these drops have been given addresses, we have

$$F(U_{\iota_1,\iota_2,\ldots,\iota_k}) = \begin{cases} U_{\iota_2,\ldots,\iota_k} & \text{if } \iota_1 = 0 \\ U_{\iota_1-1,\iota_2,\ldots,\iota_k} & \text{if } \iota_1 > 0. \end{cases}$$

Let us fix a drop $U_{\iota_1,\ldots,\iota_k}$. By definition, the *limb* $L_{\iota_1,\ldots,\iota_k}$ is the closure of the union of this drop and all its descendants, i.e., children, grandchildren, etc.:

$$L_{\iota_1,\ldots,\iota_k} := \overline{\bigcup U_{\iota_1,\ldots,\iota_k,\cdots}}\ .$$

The integers k and $\iota_1 + \cdots + \iota_k$ are called generation and depth of the limb $L_{\iota_1,\ldots,\iota_k}$, respectively. Any two limbs are either disjoint or nested. Moreover, for any limb $L_{\iota_1,\ldots,\iota_k}$, we have

$$F(L_{\iota_1,\ldots,\iota_k}) = \begin{cases} L_{\iota_2,\ldots,\iota_k} & \text{if } \iota_1 = 0 \\ L_{\iota_1-1,\iota_2,\ldots,\iota_k} & \text{if } \iota_1 > 0. \end{cases}$$

In particular, every limb eventually maps to L_0 and then to the entire filled Julia set $L_\emptyset = K(F)$.

3.3. *Main results on $J(F)$.* The Julia set $J(F) = J(F_{\theta,H})$ serves as a model for the Julia set J_θ of the quadratic polynomial $P_\theta : z \mapsto e^{2\pi i\theta} z + z^2$ when J_θ is locally connected. In fact, it follows from the next theorem that F and P_θ are topologically conjugate if and only if J_θ is locally connected:

THEOREM 3.1 (Petersen). *For every irrational $0 < \theta < 1$ the Julia set $J(F)$ is locally connected.*

See [P2] for the original proof as well as [Ya] and [P3] for a simplified version of it. The central theme of the proof is the fact that the Euclidean diameter of a limb $L_{\iota_1,\ldots,\iota_k}$ tends to 0 as its depth $\iota_1 + \cdots + \iota_k$ tends to ∞.

Another issue is the Lebesgue measure of these Julia sets:

THEOREM 3.2 (Petersen, Lyubich). *For every irrational $0 < \theta < 1$ the Julia set $J(F)$ has Lebesgue measure zero.*

This theorem was first proved in [P2] for θ of bounded type. The proof of the general case, suggested by Lyubich, can be found in [Ya].

4. Puzzle pieces and a priori area estimates

4.1. *The dyadic puzzle.* This subsection outlines the construction of puzzle pieces and recalls their basic properties. Much of the material here can be found in greater detail in [P2] and [P3].

Let $\mathcal{R}_0$ denote the closure of the fixed external ray landing at the repelling fixed point $\beta \in \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ of F. Similarly, let $\mathcal{R}_{1/2} := F^{-1}(\mathcal{R}_0) \smallsetminus \mathcal{R}_0$ denote the closure of the external ray landing at the preimage of β (for landing of (pre)periodic rays, see for example [DH1], [P1], or [TY]). Let E be the equipotential $\{z : G(z) = 1\}$, where $G : A(\infty) \to \mathbb{R}$ is the Green's function on the basin of infinity. The set

$$\mathbb{C} \smallsetminus (\mathcal{R}_0 \cup \mathcal{R}_{1/2} \cup E \cup \overline{\mathbb{D} \cup U_0 \cup U_{00} \cup U_{000} \cup \cdots \cup U_1 \cup U_{10} \cup U_{100} \cup \cdots})$$

has two bounded connected components which are Jordan domains. Let $P_{1,0}$ be the closure of that component which intersects the external rays with angles in $]0,1/2[$. Call the closure of the other component $P_{1,1}$, i.e., the one which intersects the external rays with angles in $]1/2,1[$ (see Figure 3). We call these two sets the *puzzle pieces of level* 1. They form the basis of a *dyadic puzzle* as follows. For $n \geq 2$, define the *puzzle pieces of level* n as the set of homeomorphic (univalent in the interior) preimages $F^{-(n-1)}(P_{1,0})$ and $F^{-(n-1)}(P_{1,1})$. There are exactly 2^n puzzle pieces of level n. The collection of all puzzle pieces of all levels ≥ 1 is the dyadic puzzle.

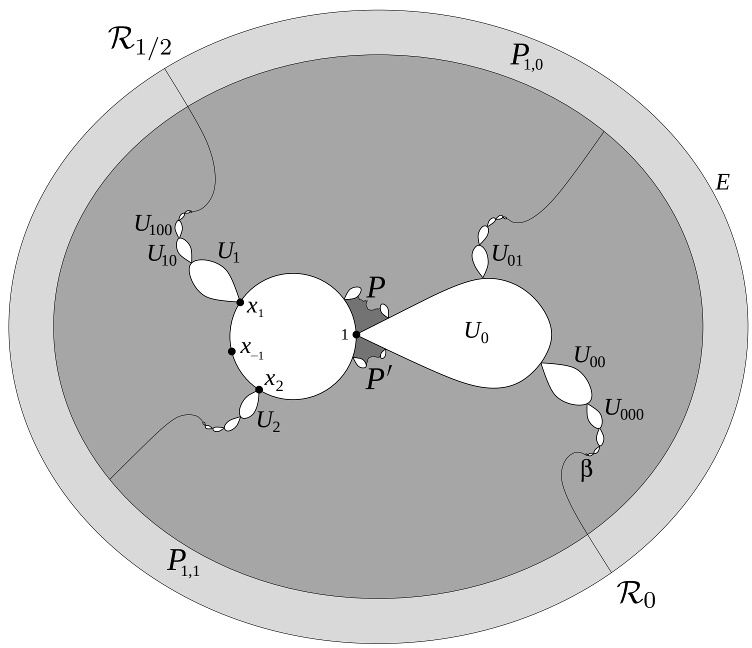

Figure 3. The two puzzle pieces $P_{1,0}$ and $P_{1,1}$ of level 1, together with their four preimages, which form the puzzle pieces of level 2. Also shown (in dark shades) are two critical puzzle pieces P and P' which are "above" and "below" the critical point 1, respectively.

Let P and P' be two distinct puzzle pieces of levels m and n, respectively, with $m \leq n$. Then either P and P' are interiorly disjoint or else $P' \subsetneq P$ and $m < n$. Moreover, for any puzzle piece P and any drop U, either $P \cap U = \emptyset$ or else P contains a neighborhood of $\overline{U} \smallsetminus \{x(U)\}$, where $x(U)$ is the root of U. The boundary of each puzzle piece P consists of a rectifiable arc in $A(\infty)$ and a rectifiable arc in $J(F)$. The latter arc starts at an iterated preimage of β, follows along the boundaries of drops passing from child to parent until it reaches the boundary of a drop U of minimal generation. It then follows the boundary of U along a nontrivial arc I. Finally, it returns along the boundaries of another chain of descendants of U until it reaches a different iterated preimage of β. We call $I = I(P) \subset \partial U$ the *base arc* of the puzzle piece P.

A puzzle piece P is called *critical* if it contains the critical point $x_0 = 1$. The critical puzzle piece $P_{1,0}$ is said to be "above" (the critical point 1), because its intersection with a small disk around 1 is contained in the closed upper half-plane; similarly $P_{1,1}$ is said to be "below". More generally, a critical puzzle piece P is "above" if $P \subset P_{1,0}$ and "below" if $P \subset P_{1,1}$ (compare Figure 3).

Recall that $x_j := F^{-j}(1) \cap \mathbb{S}^1$ for all $j \in \mathbb{Z}$. The base arc $I(P)$ of a critical puzzle piece P is an arc $[x_j, 1] \subset \mathbb{S}^1$, where $j = aq_{n+1} + q_n$ for some $n \geq 0$ and some $0 \leq a < a_{n+2}$, as is easily seen by induction. In fact, this holds trivially for the puzzle pieces $P_{1,0}$ and $P_{1,1}$ (in which case $n = a = 0$). Suppose P is a critical puzzle piece with $I(P) = [x_j, 1]$, where $j = aq_{n+1} + q_n$ and $0 \leq a < a_{n+2}$. Then for every $0 < k < q_{n+1}$ the puzzle piece $F^{-k}(P)$ with base arc $F^{-k}(I(P)) \subset \mathbb{S}^1$ is not critical. But $F^{-q_{n+1}}(P)$ is the union of two critical puzzle pieces: The *Swap* of P, which is on the opposite side of 1 as P is, and the *Gain* of P, which is on the same side of 1 as P. We denote these puzzle pieces by P_{S} and P_{G}, respectively (see Figure 4 right). A brief computation shows that the base arc of P_{S} is $I(P_{\mathrm{S}}) = [1, x_{q_{n+1}}] = [1, x_{j_{\mathrm{S}}}]$, and the base arc of P_{G} is $I(P_{\mathrm{G}}) = [x_{j+q_{n+1}}, 1] = [x_{j_{\mathrm{G}}}, 1]$. Here $j_{\mathrm{S}} := q_{n+1} = 0q_{n+2} + q_{n+1}$ and $j_{\mathrm{G}} := j + q_{n+1} = (a+1)q_{n+1} + q_n \leq q_{n+2}$, with equality if and only if $a = a_{n+2} - 1$ in which case $j_{\mathrm{G}} = q_{n+2} = 0q_{n+3} + q_{n+2}$. It follows that a Swap increases n by 1 and a Gain either preserves n or increases it by 2. The base arcs satisfy $I(P_{\mathrm{S}}) \cap I(P) = \{1\}$ and $I(P_{\mathrm{G}}) \subset I(P)$. As puzzle pieces are either interiorly disjoint or nested, we immediately obtain $P_{\mathrm{S}} \cap P = \{1\}$ and $P_{\mathrm{G}} \subset P$.

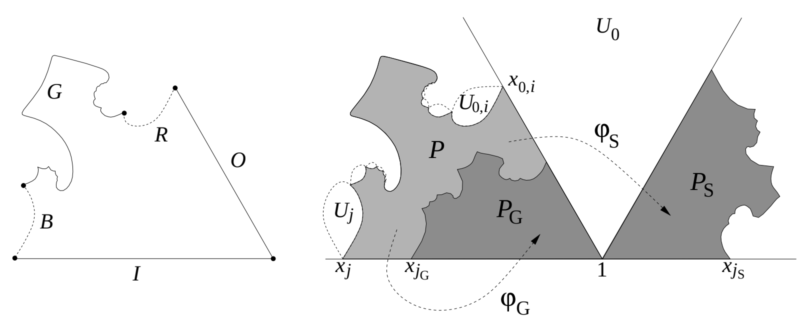

Figure 4. Right: a critical puzzle piece P together with its Gain P_{G} and its Swap P_{S} and the corresponding moves φ_{G} and φ_{S}. Left: the boundary coloring of P.

We use the notations φ_{S} and φ_{G} for the two inverse branches of $F^{-q_{n+1}}$ mapping P homeomorphically to P_{S} and P_{G}, respectively. These will be called the *moves* from P. We also call φ_{S} a Swap and φ_{G} a Gain (see pages 180–181 of [P2]). We use the iterative notation $\varphi_{\mathrm{S}}^{\circ k}$ (resp. $\varphi_{\mathrm{G}}^{\circ k}$) to indicate the effect of k consecutive Swaps (resp. Gains).

In order to make precise references to the constructions in [P2], we need to reproduce the definition of "boundary coloring" here. This is a partition of the boundary of each critical puzzle piece P into five closed and interiorly disjoint arcs I, O, B, R and G defined as follows (compare Figure 4 left):

- The base arc $I = I(P) = P \cap \mathbb{S}^1 = [x_j, 1]$, with $j = aq_{n+1} + q_n$ and $0 \leq a < a_{n+2}$, has already been defined.
- The *Orange* arc $O = O(P) := P \cap \partial U_0 = [1, x_{0,i}]$, where $i = bq_n + q_{n-1} - 1$, $1 \leq b \leq a_{n+1}$, and n is given by j as above. Here and in what follows, the notation $[1, x_{0,i}]$ indicates the shorter subarc of ∂U_0 with endpoints 1 and $x_{0,i}$. (For a comparison, note that in [P2] the point $x_{0,i}$ is denoted by y_{i+1}.)
- The *Blue* arc $B = B(P) := P \cap \partial U_j$, with j as above.
- The *Red* arc $R = R(P) := P \cap \partial U_{0,i}$, with i as above.
- Finally, the *Green* arc $G = G(P)$ is the closure of the complementary arc $\partial P \smallsetminus (I \cup O \cup B \cup R)$.

In what follows, $P(I, O, B, R, G)$ will denote the critical puzzle piece with boundary arcs I, O, B, R, G. Note that the arcs R and G of any critical puzzle piece are compact subsets of $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$.

The relation between boundary colorings and moves is as follows. Suppose

$$\varphi : P'(I', O', B', R', G') \to P(I, O, B, R, G)$$

is a move from the critical puzzle piece P' to the critical puzzle piece P. Then

$$\varphi(I') = I \cup O \qquad \varphi(R' \cup G') = G.$$

Moreover, if $\varphi = \varphi_{\mathrm{S}}$ is a Swap, then

$$\varphi_{\mathrm{S}}(O') = B \qquad \varphi_{\mathrm{S}}(B') = R,$$

while if $\varphi = \varphi_{\mathrm{G}}$ is a Gain, then

$$\varphi_{\mathrm{G}}(O') = R \qquad \varphi_{\mathrm{G}}(B') = B.$$

One can use the above relations to verify that neither I, O nor even I, O, B, R can determine a puzzle piece P uniquely. In fact, if P is a critical puzzle piece with $I(P) = [x_{q_n}, 1]$, it follows from the definitions of Swap and Gain that the two puzzle pieces $P_1 = \varphi_{\mathrm{S}}^{\circ 2}(P)$ and $P_2 = \varphi_{\mathrm{G}}^{\circ a_{n+2}}(P)$ are distinct but have identical base arcs $I(P_1) = I(P_2) = [x_{q_{n+2}}, 1]$. On the other hand, if P_1 and P_2 are two distinct critical puzzle pieces with the same base arc $I(P_1) = I(P_2)$, the above relations show that the two puzzle pieces $\varphi_{\mathrm{S}}^{\circ 3}(P_1)$ and $\varphi_{\mathrm{S}}^{\circ 3}(P_2)$ are distinct but have identical I, O, B, R boundary arcs.

4.2. *A sequence of good puzzle pieces.* Following [P2], we describe how to choose a sequence of critical puzzle pieces with bounded geometry and good combinatorics. The discussion culminates in Theorem 4.3, which is essential in the proofs of both Theorems A and B.

Let us introduce a binary tree $\mathcal{T}$ whose vertices are labeled by critical puzzle pieces and whose edges are labeled by the moves Swap and Gain. (In [P2], the vertices are labeled by the boundaries of the critical puzzle pieces, not the pieces themselves.) Let P^0 denote the level 1 critical puzzle piece which does not contain the critical value x_{-1}. It is easy to check that $P^0 = P_{1,1}$ if $0 < \theta < \frac{1}{2}$ and $P^0 = P_{1,0}$ if $\frac{1}{2} < \theta < 1$. The root of the binary tree $\mathcal{T}$ is the critical puzzle piece P^0. The children of P^0 are the two critical puzzle pieces $(P^0)_{\mathrm{S}}$ and $(P^0)_{\mathrm{G}}$, and the joining edges are labeled by the corresponding moves φ_{S} and φ_{G}. The infinite binary tree $\mathcal{T}$ is then defined by repeating this procedure inductively at each vertex.

Our main goal is to choose an infinite path $P^0 \overset{\varphi^0}{\to} P^1 \overset{\varphi^1}{\to} P^2 \overset{\varphi^2}{\to} \cdots$ in $\mathcal{T}$ whose vertices P^n have bounded geometry and good combinatorics. A natural choice for this path is given by $\varphi^n = \varphi_{\mathrm{S}}$ for all n, which amounts to defining each P^n to be the Swap child of its parent P^{n-1}. This choice is combinatorially compatible with the standard renormalization of critical circle maps, and fulfills some of the geometric estimates we need. For example, [Ya] and [YZ] give asymptotically universal estimates on the diameter and area of such P^n, by an argument simpler than the one given in [P2]. However, more sophisticated bounds on the perimeter or inner radius of puzzle pieces, as in Theorem 4.3 below, do not follow directly from that argument. This is one of the reasons why we adopt the original construction of [P2] in what follows.

Here is the strategy of this construction: For the above simple choice of the P^n, it is not easy to estimate the hyperbolic length of the Green arc $G(P^n)$, and this will sharply affect the perimeter and inner radius bounds. To remedy this problem, instead of choosing the Swap child at every step, we allow *isolated* occurrences of Gain children in our infinite path. Formally, we define a subtree $\mathcal{G}^* \subset \mathcal{T}$ by removing any Gain child of a Gain parent and all its descendants. In other words, if we picture $\mathcal{T}$ as an infinite binary tree with its root at the bottom, growing upward, and having Gain branches to the left and Swap branches to the right at every vertex, then $\mathcal{G}^*$ is the maximal subtree of $\mathcal{T}$ containing P^0 and with no pair of consecutive left branches. We initially construct an infinite path $\{\varphi^n : \widetilde{P}^n \to \widetilde{P}^{n+1}\}_{n \geq 0}$ within the subtree $\mathcal{G}^*$; the freedom acquired by allowing isolated Gains makes it easy to prove that $\{\widetilde{P}^n\}$ has bounded geometry (Theorem 4.2). A slight modification of this path then leads to our final choice of the sequence of puzzle pieces $\{P^n\}$ which has the right combinatorics also (Theorem 4.3).

We remark in passing that many of the estimates in [P2] are in fact proved for a larger subtree $\mathcal{G} \supset \mathcal{G}^*$, in which several consecutive Gains may occur.

Definition 4.1. For an open interval $J \subsetneq \mathbb{S}^1$, define the hyperbolic domain

$$\mathbb{C}^*_J := (\mathbb{C}^* \smallsetminus \mathbb{S}^1) \cup J. \tag{4.1}$$

The simplified notation $\ell^*_J(\cdot) = \ell_{\mathbb{C}^*_J}(\cdot)$, $\mathrm{diam}^*_J(\cdot) = \mathrm{diam}_{\mathbb{C}^*_J}(\cdot)$ and $\mathrm{dist}^*_J(\cdot) = \mathrm{dist}_{\mathbb{C}^*_J}(\cdot)$ will be used for the hyperbolic arclength, diameter and distance in $\mathbb{C}^*_J$.

For $n \geq 0$, let

$$J^n :=]\, x_{-q_{n+1}+q_n}, x_{-q_n} [\qquad \text{and} \qquad J^n_+ :=]\, x_{-q_{n+1}+q_n}, 1[.$$

Note that

$$I^n \smallsetminus \{1\} = [x_{q_n}, 1[\subsetneq J^n_+ \subsetneq J^n.$$

The main technical tool in [P2] is the following collection of estimates on the length of the boundary arcs of critical puzzle pieces.

THEOREM 4.2. *Let $P(I, O, B, R, G)$ be a critical puzzle piece with the base arc $I = [x_j, 1]$, where $j = aq_{n+1} + q_n$ and $0 \leq a < a_{n+2}$. Let $J = J^n$ and $J_+ = J^n_+$. Then the following asymptotically universal bounds hold*:

(i) $|O| \asymp |I|$ *and* $\ell^*_J(O) \asymp \ell^*_J(I) \asymp 1$.

Moreover, if P is a vertex of $\mathcal{G}^*$, then

(ii) $\ell^*_J(B) \preccurlyeq \ell^*_{J_+}(B) \preccurlyeq 1$,

(iii) $\ell_{\mathbb{C} \smallsetminus \overline{\mathbb{D}}}(R) \preccurlyeq 1$.

Finally, there exists an infinite path $\{\varphi^k : \widetilde{P}^k \to \widetilde{P}^{k+1}\}_{k \geq 0}$ in $\mathcal{G}^*$, starting at the root $\widetilde{P}^0 = P^0$, such that

(iv) $\ell_{\mathbb{C} \smallsetminus \overline{\mathbb{D}}}(\widetilde{G}^k) \preccurlyeq 1$,

where $\widetilde{G}^k = G(\widetilde{P}^k)$ is the Green arc of $\partial \widetilde{P}^k$.

Proof. The bounds in (i) are immediate consequences of real *a priori* bounds (Theorem 2.5) and the fact that f has a cubic critical point at 1 (compare the proof of Theorem 2.2(1) in [P2] as well as the following proof of (ii)).

The bounds in (ii) are essentially proved in Lemma 3.3 of [P2]; we shall however sketch a proof here. As $\mathbb{C}^*_{J_+} \subset \mathbb{C}^*_J$, the Schwarz lemma implies that $\ell^*_J(\cdot) \leq \ell^*_{J_+}(\cdot)$ so we need only prove the bound $\ell^*_{J_+}(B) \preccurlyeq 1$. Let $\varphi : P'(I', O', B', R', G') \to P(I, O, B, R, G)$ be the move to P from its parent P'. Then φ is a branch of $F^{-q_n} = f^{-q_n}$. We divide the proof into two cases depending on whether $\varphi = \varphi_{\mathrm{S}}$ is a Swap or $\varphi = \varphi_{\mathrm{G}}$ is a Gain.

Assume first that φ is a Swap, so that $B = \varphi(O')$. Let $K := f^{\circ q_n}(J_+) =]x_{-q_{n+1}}, x_{-q_n}[$. Then $W := f^{-q_n}(\mathbb{C}^*_K)$ is a proper subdomain of $\mathbb{C}^*_{J_+}$, so by the Schwarz lemma the inclusion $i : W \hookrightarrow \mathbb{C}^*_{J_+}$ contracts the hyperbolic metrics. On the other hand, the critical values of $f^{\circ q_n}$ are located at $0, \infty, x_{-1}, \ldots, x_{-q_n}$, none of which belongs to $\mathbb{C}^*_K$. This shows $f^{\circ q_n} : W \to \mathbb{C}^*_K$ is an unbranched covering map, hence a local isometry by the Schwarz lemma. Thus $\varphi = i \circ f^{-q_n}$ is a contraction with respect to the hyperbolic metrics on $\mathbb{C}^*_K$ and $\mathbb{C}^*_{J_+}$, so that

$$\ell^*_{J_+}(B) = \ell^*_{J_+}(\varphi(O')) \leq \ell^*_K(O'),$$

and we need only prove that $\ell^*_K(O') \preccurlyeq 1$. Since the arc O' is contained in ∂U_0 which makes an angle of $\pi/3$ with $\mathbb{S}^1$ at 1, it suffices to show that

$$|O'| \asymp \min\{\, |[1, x_{-q_n}]|\, ,\, |[1, x_{-q_{n+1}}]|\, \}. \tag{4.2}$$

For this, observe that $O' = [1, x_{0,i'}]$, where $i' = b'q_{n-1} + q_{n-2} - 1$ and $1 \leq b' \leq a_n$, so that

$$[1, x_{0,q_n - 1}] \subset O' \subset [1, x_{0,q_{n-2}-1}].$$

By real *a priori* bounds (Theorem 2.5) and the fact that f has a cubic critical point at 1, we have

$$|[1, x_{0,q_{n-2}-1}]| \asymp |[1, x_{0,q_n-1}]| \asymp |[1, x_{-q_n}]| \asymp |[1, x_{-q_{n+1}}]|,$$

which proves (4.2).

Assume next that φ is a Gain and let

$$\varphi' : P''(I'', O'', B'', R'', G'') \to P'(I', O', B', R', G')$$

denote the move to P' from its parent P''. Then φ' is a Swap because P is a vertex of $\mathcal{G}^*$. Hence $B = \varphi(\varphi'(O''))$. From this point on, the proof is similar to the Swap case treated above, and further details will be left to the reader.

The bound in (iii) is Theorem 2.2(4) in [P2]; note that $\mathcal{G}^* \subset \mathcal{G}$.

Finally, the existence of an infinite path $\{\varphi^k : \widetilde{P}^k \to \widetilde{P}^{k+1}\}_{k \geq 0}$ in $\mathcal{G}^*$ satisfying (iv) is proved in pages 188–189 of [P2]. Let us just give a brief outline here: Suppose $P(I, O, B, R, G)$ is a vertex of $\mathcal{G}^*$, so that $\ell_{\mathbb{C} \smallsetminus \overline{\mathbb{D}}}(R) \leq L$ for some asymptotically universal $L > 0$ by (iii). Let $P'(I', O', B', R', G')$ and $P''(I'', O'', B'', R'', G'')$ be the two children of P. Then the moves from P to P' and P'' contract the hyperbolic metric on $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$, because $F^{-1}(\mathbb{C} \smallsetminus \overline{\mathbb{D}}) \subset \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ and $F = f$ has no critical values in $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$. Since these moves map $R \cup G$ to G' and G'', we obtain

$$\max\{\ell_{\mathbb{C} \smallsetminus \overline{\mathbb{D}}}(G'), \ell_{\mathbb{C} \smallsetminus \overline{\mathbb{D}}}(G'')\} \leq \ell_{\mathbb{C} \smallsetminus \overline{\mathbb{D}}}(G) + L. \tag{4.3}$$

A more careful application of the Schwarz lemma (see Lemma 1.11 of [P2]) shows that there is an asymptotically universal $\varepsilon > 0$ such that

$$\min\{\ell_{\mathbb{C} \smallsetminus \overline{\mathbb{D}}}(G'), \ell_{\mathbb{C} \smallsetminus \overline{\mathbb{D}}}(G'')\} \leq (1 - \varepsilon)(\ell_{\mathbb{C} \smallsetminus \overline{\mathbb{D}}}(G) + L). \tag{4.4}$$

To define the sequence $\{\widetilde{P}^k\}$ it suffices to specify the move φ^k at each vertex, starting with $\widetilde{P}^0 = P^0$ already defined. Set $\varphi^0 = \varphi^1 = \varphi_{\rm S}$, so that $\widetilde{P}^1 = (\widetilde{P}^0)_{\rm S}$ and $\widetilde{P}^2 = (\widetilde{P}^1)_{\rm S}$. Assuming $k \geq 2$ and $\widetilde{P}^k$ is defined, we consider two cases: If $\widetilde{P}^k$ is a Gain child, by the definition of $\mathcal{G}^*$ we must choose $\varphi^k = \varphi_{\rm S}$. On the other hand, if $\widetilde{P}^k$ is a Swap child, then we have a choice between Swap and Gain, and we define φ^k to be the move which introduces a definite contraction in (4.4). It follows that the length $\ell_k := \ell_{\mathbb{C}\smallsetminus\overline{\mathbb{D}}}(\widetilde{G}^k)$ undergoes a contraction of the form (4.3) for all k and a definite contraction of the form (4.4) for at least every other k. It follows that

$$\ell_{k+2} \leq (1-\varepsilon)(\ell_k + L) + L$$

for all k. Evidently this implies that $\{\ell_k\}$ is bounded by a constant $C = C(L, \varepsilon)$. Since L and ε are asymptotically universal, the same must be true for C and this finishes the proof of (iv). $\square$

The following theorem gives us a sequence of critical puzzle pieces with bounded geometry and good combinatorics. The existence of such a sequence was the crucial step in the proof of local connectivity in [P2], and will be fully utilized in the next two subsections. In what follows, by the inner and outer radius of a critical puzzle piece $P(I, O, B, R, G)$ is meant

$$\begin{aligned} r_{\rm in}(P) &:= \min\{|1-z| : z \in B \cup R \cup G\} \\ r_{\rm out}(P) &:= \max\{|1-z| : z \in B \cup R \cup G\}. \end{aligned}$$

THEOREM 4.3. *There exists a sequence $\{P^n\}_{n\geq 0}$ of critical puzzle pieces with*

$$I(P^n) = I^n := [1, x_{q_n}], \qquad O(P^n) = O^n := [1, x_{0,q_n+q_{n-1}-1}], \tag{4.5}$$

$$I((P^n)_{\rm S}) = I^{n+1}, \qquad O((P^n)_{\rm S}) = O^{n+1} \tag{4.6}$$

which satisfies the following asymptotically universal bounds:

$$\operatorname{diam}^*_{J^n}(P^n) \asymp \ell^*_{J^n}(\partial P^n) \asymp 1 \tag{4.7}$$

$$\operatorname{diam}^*_{J^{n+1}}((P^n)_{\rm S}) \asymp \ell^*_{J^{n+1}}(\partial (P^n)_{\rm S}) \asymp 1 \tag{4.8}$$

$$r_{\rm in}(P^n) \asymp |I^n| \asymp r_{\rm out}(P^n) \tag{4.9}$$

$$r_{\rm in}((P^n)_{\rm S}) \asymp |I^{n+1}| \asymp r_{\rm out}((P^n)_{\rm S}). \tag{4.10}$$

Proof. The following essentially repeats the construction in Proposition and Definition 3.1 of [P2]. It is not hard to see from the definition of Swap as well as the boundary coloring that if $I(P) = I^n$ for some n, then $I(P_{\rm S}) = I^{n+1}$ and $O(P_{\rm S}) = O^{n+1}$. This observation is the key to the following construction. By definition, $\widetilde{P}^1 = (\widetilde{P}^0)_{\rm S}$ and $\widetilde{P}^2 = (\widetilde{P}^1)_{\rm S}$. We set $P^n := \widetilde{P}^n$ for $n = 0, 1, 2$. For $n \geq 3$, we look for a k with $I(\widetilde{P}^k) = I^n$ and $O(\widetilde{P}^k) = O^n$. If such a k exists, we define $P^n := \widetilde{P}^k$; otherwise we look for a k with $I(\widetilde{P}^k) = I^{n-1}$. If such a k

exists, we define $P^n := (\widetilde{P}^k)_{\mathrm{S}}$; otherwise we take a k with $I(\widetilde{P}^k) = I^{n-2}$. Such a k must exist, because the sequence of n's associated with $\{\widetilde{P}^k\}_{k\geq 0}$ cannot omit two or more consecutive integers. In this last case, we define $P^n := ((\widetilde{P}^k)_{\mathrm{S}})_{\mathrm{S}}$, that is, the critical puzzle piece obtained from two consecutive Swaps of $\widetilde{P}^k$. By the construction, $\{P^n\}_{n\geq 0}$ defined this way satisfies (4.5) and (4.6).

Let us now prove (4.7); the proof of (4.8) is similar. First note that

$$\ell^*_{J^n}(\partial P^n) \geq 2\,\mathrm{diam}^*_{J^n}(P^n) \geq 2\ell^*_{J^n}(I^n) \asymp 1,$$

where the last bound comes from Theorem 4.2(i). It follows that

$$\ell^*_{J^n}(\partial P^n) \succcurlyeq \mathrm{diam}^*_{J^n}(P^n) \succcurlyeq 1. \tag{4.11}$$

To prove the upper bounds, let P be a puzzle piece in the sequence $\{\widetilde{P}^k\}_{k\geq 0}$ given by Theorem 4.2(iv), where as before $I(P) = [x_j, 1]$, $j = aq_{n+1} + q_n$, and $0 \leq a < a_{n+2}$. Then the bound

$$\ell^*_{J^n}(\partial P) \preccurlyeq 1 \tag{4.12}$$

simply follows from Theorem 4.2 because $\mathbb{C} \smallsetminus \overline{\mathbb{D}} \subset \mathbb{C}^*_{J^n}$ implies $\ell^*_{J^n}(\cdot) \leq \ell_{\mathbb{C}\smallsetminus\overline{\mathbb{D}}}(\cdot)$. Furthermore, any move $\varphi : P(I, O, B, R, G) \to P'(I', O', B', R', G')$ contracts the hyperbolic metric on $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$. Since $G' = \varphi(R \cup G)$, we obtain from Theorem 4.2(iii), (iv)

$$\ell_{\mathbb{C}\smallsetminus\overline{\mathbb{D}}}(G') \leq \ell_{\mathbb{C}\smallsetminus\overline{\mathbb{D}}}(R) + \ell_{\mathbb{C}\smallsetminus\overline{\mathbb{D}}}(G) \preccurlyeq 1.$$

In particular, when $P' = P_{\mathrm{S}}$ is the Swap of P, it follows from Theorem 4.2 together with $\ell^*_{J^{n+1}}(\cdot) \leq \ell_{\mathbb{C}\smallsetminus\overline{\mathbb{D}}}(\cdot)$ that

$$\ell^*_{J^{n+1}}(\partial P_{\mathrm{S}}) \preccurlyeq 1. \tag{4.13}$$

By the construction, every P^n is of the form $\widetilde{P}^k$ or $(\widetilde{P}^k)_{\mathrm{S}}$ or $((\widetilde{P}^k)_{\mathrm{S}})_{\mathrm{S}}$ for some k. This, together with (4.12) and (4.13), implies

$$\mathrm{diam}^*_{J^n}(P^n) \preccurlyeq \ell^*_{J^n}(\partial P^n) \preccurlyeq 1. \tag{4.14}$$

Now (4.7) follows by combining (4.11) and (4.14).

Finally, let us prove (4.9); the proof of (4.10) is similar. Since $\mathrm{diam}^*_{J^n}(P^n) \asymp 1$, we have the Euclidean bound $\mathrm{diam}(P^n) \asymp |J^n|$. It follows from Theorem 4.2(i) that for any $z \in B^n \cup R^n \cup G^n$,

$$|1 - z| \leq \mathrm{diam}(P^n) \asymp |J^n| \asymp |I^n|,$$

so that

$$r_{\mathrm{out}}(P^n) \preccurlyeq |I^n|. \tag{4.15}$$

On the other hand, by Theorem 4.2(ii)–(iv) and the fact that $\ell^*_{J^n_+}(\cdot) \leq \ell_{\mathbb{C}\smallsetminus\overline{\mathbb{D}}}(\cdot)$ we have $\ell^*_{J^n_+}(B^n \cup R^n \cup G^n) \preccurlyeq 1$. Hence, every $z \in B^n \cup R^n \cup G^n$ satisfies $\mathrm{dist}^*_{J^n_+}(z, x_{q_n}) \preccurlyeq 1$. It follows that

$$|1 - z| \succcurlyeq |J^n_+| \asymp |I^n|,$$

which gives

$$r_{\mathrm{in}}(P^n) \succcurlyeq |I^n| \tag{4.16}$$

Putting (4.15) and (4.16) together, we obtain (4.9). □

As a final preparation, it will be convenient to consider each puzzle piece P^n as a compact subset of a simply connected domain rather than the multiply connected domain $\mathbb{C}^*_{J^n}$. This can be done at an asymptotically negligible expense by introducing appropriate cuts in $\mathbb{C}^*_{J^n}$ (alternatively, we could take the maximal embedded hyperbolic disk in $\mathbb{C}^*_{J^n}$ centered at 1).

Definition 4.4. Let $\mathcal{R}$ be the closure of the unique external ray landing at the critical value x_{-1} and let $\widehat{\mathcal{R}}$ be the image of $\mathcal{R}$ under the reflection $z \mapsto 1/\overline{z}$. For an open interval $J \subsetneq \mathbb{S}^1 \smallsetminus \{x_{-1}\}$, define the *simply connected domain*

$$\mathbb{C}_J := \mathbb{C}^*_J \smallsetminus (\mathcal{R} \cup \widehat{\mathcal{R}}) = (\mathbb{C} \smallsetminus (\mathbb{S}^1 \cup \mathcal{R} \cup \widehat{\mathcal{R}})) \cup J. \tag{4.17}$$

As before, the notation $\ell_J(\cdot) = \ell_{\mathbb{C}_J}(\cdot)$, $\mathrm{diam}_J(\cdot) = \mathrm{diam}_{\mathbb{C}_J}(\cdot)$ and $\mathrm{dist}_J(\cdot) = \mathrm{dist}_{\mathbb{C}_J}(\cdot)$ will be reserved for the hyperbolic arclength, diameter and distance in $\mathbb{C}_J$.

Corollary 4.5. *The sequence of critical puzzle pieces* $\{P^n\}_{n\geq 0}$ *in* Theorem 4.3 *also satisfies the following asymptotically universal bounds*:

$$\begin{aligned} \mathrm{diam}_{J^n}(P^n) &\asymp \ell_{J^n}(\partial P^n) \asymp 1, \\ \mathrm{diam}_{J^{n+1}}((P^n)_{\mathrm{S}}) &\asymp \ell_{J^{n+1}}(\partial (P^n)_{\mathrm{S}}) \asymp 1. \end{aligned}$$

Proof. The hyperbolic distance $\mathrm{dist}^*_{J^n}(1, \mathcal{R} \cup \widehat{\mathcal{R}})$ tends to infinity as $n \to \infty$. Hence on the hyperbolic ball in $\mathbb{C}^*_{J^n}$ of a fixed radius centered at 1, the hyperbolic metrics of $\mathbb{C}^*_{J^n}$ and $\mathbb{C}_{J^n}$ are asymptotically equal as $n \to \infty$. □

4.3. *Some new sets.* For $0 \leq j < q_{n+1}$, define I^n_j and J^n_j as the iterated preimages $(F|_{\mathbb{S}^1})^{-j}(I^n)$ and $(F|_{\mathbb{S}^1})^{-j}(J^n)$, respectively. Observe that $I^n_j \subsetneq J^n_j$, and that the collection

$$\{I^n_j\}_{j=0}^{q_{n+1}-1} \cup \{I^{n+1}_j\}_{j=0}^{q_n-1}$$

induces the dynamical partition $\Pi^n(f)$ as defined in subsection 2.4.

Based on the sequence $\{P^n\}_{n\geq 0}$ given by Theorem 4.3, we shall define several new sets, which will be the basis of the proofs of our main theorems. In what follows the integer $n \geq 0$ will be fixed.

(i) Define $P^n_0 := P^n$ and $P^n_{q_{n+1}} := (P^n)_{\mathrm{S}}$. For $0 \leq j < q_{n+1}$, let P^n_j be the unique puzzle piece with base arc I^n_j which maps isomorphically to P^n by $F^{\circ j}$, and for $q_{n+1} \leq j < q_{n+1} + q_n$, let P^n_j be the unique puzzle piece with base arc $I^{n+1}_{j-q_{n+1}}$ which maps isomorphically to $P^n_{q_{n+1}}$ by $F^{\circ j - q_{n+1}}$.

(ii) For $0 \leq j < q_{n+1} + q_n$, we define the reflected puzzle piece $\widehat{P}_j^n \subset \overline{\mathbb{D}}$ as the image of P_j^n under $z \mapsto 1/\overline{z}$. By abuse of the language, these reflected puzzle pieces and their iterated F-preimages outside $\mathbb{D}$ will also be called "puzzle pieces". To emphasize this distinction, the original elements of the dyadic puzzle will sometimes be referred to as the *exterior* puzzle pieces.

(iii) Let $Q_0^n \subset \overline{U_0}$ be the unique puzzle piece which satisfies $F(Q_0^n) = f(\widehat{P}^n_{q_{n+1}}) = \widehat{P}^n_{q_{n+1}-1}$. For $0 \leq j < q_{n+1} + q_n$, define Q_j^n to be the unique puzzle piece in $\overline{U_j}$ which maps isomorphically to Q_0^n by $F^{\circ j}$. Similarly, for $0 \leq j < q_{n+1} + q_n$, define $\widehat{Q}_j^n \subset \overline{\mathbb{D}}$ to be the image of Q_j^n under the reflection $z \mapsto 1/\overline{z}$ (see Figure 5).

(iv) For $0 \leq j < q_{n+1} + q_n,\ j \neq q_{n+1} - 1$, let $P_{0,j}^n$ be the unique puzzle piece whose base arc is on ∂U_0 and satisfies $F(P_{0,j}^n) = P_j^n$. Similarly, we define $\widehat{P}_{0,j}^n \subset \overline{U_0}$ as the reflection of $P_{0,j}^n$ in ∂U_0, i.e., the unique puzzle piece with the same base arc as $P_{0,j}^n$ which satisfies $F(\widehat{P}_{0,j}^n) = \widehat{P}_j^n$.

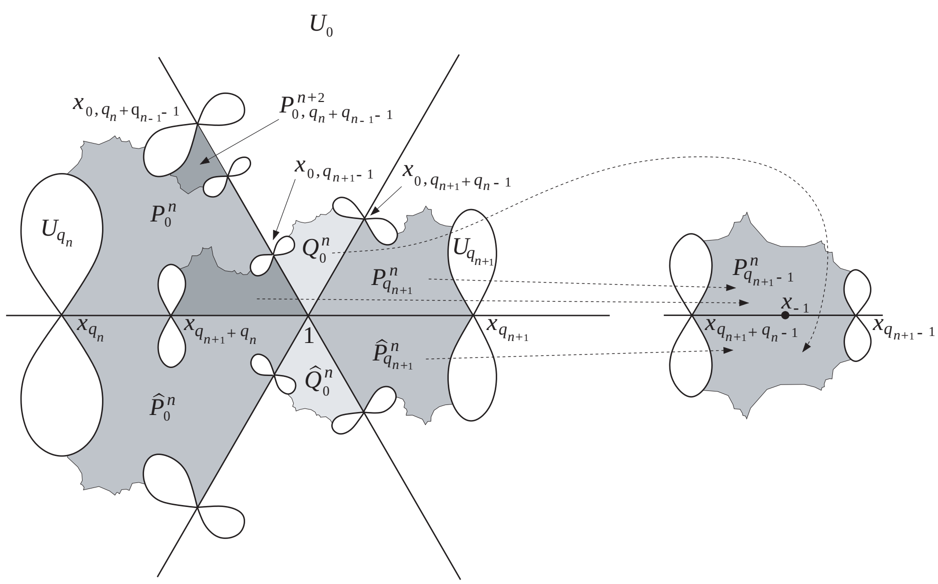

Figure 5. Some other puzzle pieces.

(v) For $0 \leq j < q_{n+1} + q_n$, let $Q_{0,j}^n$ and $\widehat{Q}_{0,j}^n$ denote the unique puzzle pieces containing $x_{0,j}$ which map by F to Q_j^n and $\widehat{Q}_j^n$, respectively.

(vi) We define the closed annuli

$$A^n \quad := \quad \bigcup_{j=0}^{q_{n+1}+q_n-1} \left(P_j^n \cup Q_j^n \right),$$

$$\begin{aligned}\widehat{A}^n &:= \bigcup_{j=0}^{q_{n+1}+q_n-1} \left(\widehat{P}_j^n \cup \widehat{Q}_j^n\right),\\ \mathcal{A}^n &:= A^n \cup \widehat{A}^n.\end{aligned}$$

It is easy to check that $\mathcal{A}^n$ is a closed topological annulus whose interior contains the unit circle.

(vii) Similarly, we define the closed "rectangles"

$$\begin{aligned}A_0^n &:= \bigcup_{j=0, j\neq q_{n+1}-1}^{q_{n+1}+q_n-1} P_{0,j}^n \cup \bigcup_{j=0}^{q_{n+1}+q_n-1} Q_{0,j}^n,\\ \widehat{A}_0^n &:= \bigcup_{j=0, j\neq q_{n+1}-1}^{q_{n+1}+q_n-1} \widehat{P}_{0,j}^n \cup \bigcup_{j=0}^{q_{n+1}+q_n-1} \widehat{Q}_{0,j}^n,\\ \mathcal{A}_0^n &:= A_0^n \cup \widehat{A}_0^n.\end{aligned}$$

It is easy to check that $\mathcal{A}_0^n$ is a closed topological disk which does not contain the critical point $x_0 = 1$. Moreover, $\mathcal{A}^n \cup \mathcal{A}_0^n$ contains an open neighborhood of the union $\mathbb{S}^1 \cup \partial U_0$ (see Figure 6). Note also that $F^{-1}(\widehat{A}^n) \cap \overline{U_0} = \widehat{A}_0^n \cup Q_0^n$.

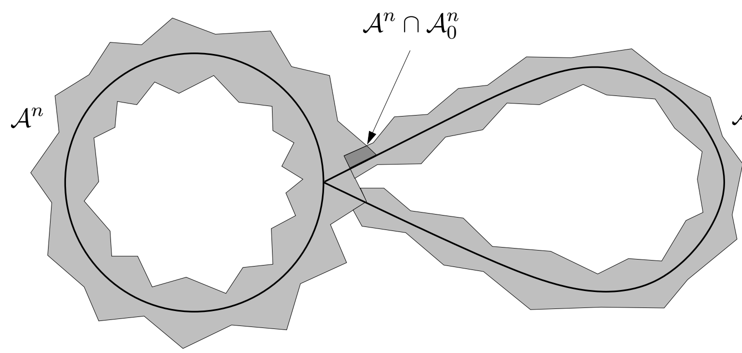

Figure 6. Schematic picture of the annulus $\mathcal{A}^n$ and the "rectangle" $\mathcal{A}_0^n$.

(viii) Finally, pull these annuli and rectangles back to define the sets

$$\mathcal{Z}_{-1}^n := \widehat{A}^n, \qquad \mathcal{Z}_k^n := \widehat{A}^n \cup \bigcup_{m=0}^{k} F^{-m}(\widehat{A}_0^n \cup Q_0^n),$$

$$\mathcal{Z}^n := \widehat{A}^n \cup \bigcup_{m=0}^{\infty} F^{-m}(\widehat{A}_0^n \cup Q_0^n),$$

$$\mathcal{Y}_{-1}^n := \mathcal{A}^n, \qquad \mathcal{Y}_k^n := \mathcal{A}^n \cup \bigcup_{m=0}^{k} F^{-m}(\mathcal{A}_0^n), \qquad \mathcal{Y}^n := \mathcal{A}^n \cup \bigcup_{m=0}^{\infty} F^{-m}(\mathcal{A}_0^n).$$

Observe that $\mathcal{Z}_k^n$ and $\mathcal{Z}^n = \lim_{k\to\infty} \mathcal{Z}_k^n$ are subsets of the filled Julia set $K(F)$. Moreover, we have the inclusions

$$\begin{array}{lll} \mathcal{Z}_k^{n+2} \subset \mathcal{Z}_k^n, & \mathcal{Y}_k^{n+2} \subset \mathcal{Y}_k^n, & \mathcal{Z}_k^n \subset \mathcal{Y}_k^n \\ \mathcal{Z}^{n+2} \subset \mathcal{Z}^n, & \mathcal{Y}^{n+2} \subset \mathcal{Y}^n, & \mathcal{Z}^n \subset \mathcal{Y}^n. \end{array}$$

In what follows we use the generic symbol P for any of the puzzle pieces P or Q defined in the items (i)–(v) above, as well as their iterated preimages under F. Similarly, the generic symbol $\widehat{P}$ will be used for any of the puzzle pieces $\widehat{P}$ or $\widehat{Q}$ defined in (i)–(v) and their preimages. Note that puzzle pieces always come in pairs $(P, \widehat{P})$ which are the reflections of one another through the boundary of some drop U, with $P \cap U = \emptyset$, and $\widehat{P} \subset \overline{U}$.

By an abuse of language, we say that a puzzle piece P *belongs* to one of the sets defined in items (vi)–(viii) above if P appears as a puzzle piece in one of the unions used in the definition of that set. We use the notation $\triangleleft$ to express this relation. As an example, P_0^n belongs to A^n and we write $P_0^n \triangleleft A^n$. Note that the relation $\triangleleft$ implies the set-theoretic $\subset$, but not vice versa. For instance, $\widehat{P}_0^{n+2} \subset \mathcal{Z}^n$ but $\widehat{P}_0^{n+2} \triangleleft \mathcal{Z}^n$ does not hold.

4.4. *Supporting lemmas.* This subsection will prove several *a priori* estimates on the geometry of the puzzle pieces and the sets defined above. As is common in dynamics, distortion estimates for long compositions of f or its inverse play a central role. We use the following version of the classical Köebe distortion theorem (see for example [Po]):

THEOREM 4.6. *Let $\phi : U \to \mathbb{C}$ be a univalent map on a simply-connected domain $U \subsetneq \mathbb{C}$ and let $K \subset U$ be compact with hyperbolic diameter d. Then*

$$\chi(\phi, K) := \sup \left\{ \left| \frac{\phi'(z)}{\phi'(w)} \right| \; : z, w \in K \right\} \leq e^{4d}.$$

LEMMA 4.7. *The following asymptotically universal bounds exist*:

$$\begin{aligned} \operatorname{diam}_{J_0^n}(P_0^n) &= \operatorname{diam}_{J_0^n}(\widehat{P}_0^n) \asymp 1, \\ \operatorname{diam}_{J_0^{n+1}}(P_{q_{n+1}}^n) &= \operatorname{diam}_{J_0^{n+1}}(\widehat{P}_{q_{n+1}}^n) \asymp 1, \\ \operatorname{diam}_{J_0^n}(Q_0^n) &= \operatorname{diam}_{J_0^n}(\widehat{Q}_0^n) \asymp 1. \end{aligned}$$

Proof. The first two come from Corollary 4.5. To prove the third bound, observe that $\operatorname{diam}(\widehat{P}_{q_{n+1}}^n) \asymp \operatorname{diam}(Q_0^n)$ and hence by the second bound we have the (Euclidean) asymptotically universal bound

$$|J_0^n| \asymp |J_0^{n+1}| \asymp \operatorname{diam}(\widehat{P}_{q_{n+1}}^n) \asymp \operatorname{diam}(Q_0^n).$$

Since Q_0^n is a subset of U_0 whose boundary makes an angle of $\pi/3$ with the unit circle at $z = 1$, the third bound follows. $\square$

Combining Lemma 4.7 with Theorem 4.6, we immediately obtain

Corollary 4.8. *Let g be any univalent branch of f^{-k} defined on the simply connected domain $\mathbb{C}_{J_0^n}$. Then, there exist the asymptotically universal distortion bounds*

$$\chi(g, P_0^n \cup \widehat{P}_0^n) \asymp \chi(g, P_{q_n}^{n-1} \cup \widehat{P}_{q_n}^{n-1}) \asymp \chi(g, Q_0^n \cup \widehat{Q}_0^n) \asymp 1$$

uniformly in g.

Lemma 4.9. *There exists the following asymptotically universal bound*:

$$\operatorname{area}(P_0^n) \asymp |I^n|^2.$$

Proof. This is an immediate consequence of (4.9) in Theorem 4.3 and the fact that ∂U_0 makes an angle of $\pi/3$ with $\mathbb{S}^1$ at 1. □

Lemma 4.10. *There exist the following asymptotically universal bounds*:

$$\begin{aligned}
\operatorname{area}(P_0^n \smallsetminus A^{n+2}) &\asymp \operatorname{area}(\widehat{P}_0^n \smallsetminus \widehat{A}^{n+2}) \asymp \operatorname{area}(P_0^n \cup \widehat{P}_0^n),\\
\operatorname{area}(P_{q_{n+1}}^n \smallsetminus A^{n+2}) &\asymp \operatorname{area}(\widehat{P}_{q_{n+1}}^n \smallsetminus \widehat{A}^{n+2}) \asymp \operatorname{area}(P_{q_{n+1}}^n \cup \widehat{P}_{q_{n+1}}^n),\\
\operatorname{area}(Q_0^n \smallsetminus A^{n+2}) &\asymp \operatorname{area}(\widehat{Q}_0^n \smallsetminus \widehat{A}^{n+2}) \asymp \operatorname{area}(Q_0^n \cup \widehat{Q}_0^n).
\end{aligned}$$

Proof. We prove the first bound, the other two being similar. Clearly,

$$\operatorname{area}(P_0^n \smallsetminus A^{n+2}) \asymp \operatorname{area}(\widehat{P}_0^n \smallsetminus \widehat{A}^{n+2}) \preccurlyeq \operatorname{area}(P_0^n \cup \widehat{P}_0^n),$$

and so we need only prove the reverse bound. With $i = q_n + q_{n-1} - 1$, the base arc of the puzzle piece $P_{0,i}^{n+2}$ satisfies $I(P_{0,i}^{n+2}) = [x_{0,i+q_{n+2}}, x_{0,i}] \subset [1, x_{0,i}] = O(P^n)$. It follows that $P_{0,i}^{n+2} \subset P_0^n \smallsetminus A^{n+2}$ (see Figure 5). Observe that by real *a priori* bounds, Corollary 4.8 and Lemma 4.9,

$$\begin{aligned}
|I^n|^2 &\asymp |[x_{q_{n+2}+q_n+q_{n-1}}, x_{q_n+q_{n-1}}]|^2\\
&\asymp |[x_{0,q_{n+2}+i}, x_{0,i}]|^2 = |I(P_{0,i}^{n+2})|^2\\
&\asymp \operatorname{area}(P_{0,i}^{n+2}).
\end{aligned}$$

Hence by another application of Lemma 4.9,

$$\operatorname{area}(P_0^n \cup \widehat{P}_0^n) \asymp |I^n|^2 \asymp \operatorname{area}(P_{0,i}^{n+2}).$$

Since $\operatorname{area}(P_0^n \smallsetminus A^{n+2}) \geq \operatorname{area}(P_{0,i}^{n+2})$, we obtain the reverse bound

$$\operatorname{area}(P_0^n \smallsetminus A^{n+2}) \succcurlyeq \operatorname{area}(P_0^n \cup \widehat{P}_0^n).$$ □

Our next task is to estimate the area of the sets $\mathcal{Z}^n$ and $\mathcal{Y}^n$ defined above. We use Corollary 4.8 to prove two distortion lemmas which will be essential in the proof of Theorem 4.15. The first lemma deals with the pull-backs of the critical puzzle pieces to $\mathcal{A}^n$ and $\mathcal{A}_0^n$ only:

LEMMA 4.11. *Every pair $(P, \widehat{P}) \lhd \mathcal{A}^n$ or $\mathcal{A}_0^n$ is a bounded distortion pull-back of the corresponding pair of critical puzzle pieces in $\mathcal{A}^n$. More precisely, let g be the univalent branch of f^{-j} which maps the pair of critical puzzle pieces $(P', \widehat{P}') \lhd \mathcal{A}^n$ to $(P, \widehat{P})$, where $(P', \widehat{P}') = (P_0^n, \widehat{P}_0^n)$ or $(P_{q_{n+1}}^n, \widehat{P}_{q_{n+1}}^n)$ or $(Q_0^n, \widehat{Q}_0^n)$. Then*

$$\chi(g, P' \cup \widehat{P}') \asymp 1$$

uniformly in g.

Proof. Let us first assume $(P, \widehat{P}) \lhd \mathcal{A}^n$. It suffices to consider the case $(P, \widehat{P}) = (P_j^n, \widehat{P}_j^n)$ for some $0 \leq j < q_{n+1}$, because the other two cases are similar. The critical values of $f^{\circ j}$ are located at $0, \infty, x_{-1}, \ldots, x_{-j}$, none of which belongs to $\mathbb{C}_{J_0^n}$ since $j < q_{n+1}$. Hence the univalent branch $g = f^{-j}$ which maps $P_0^n \cup \widehat{P}_0^n$ to $P_j^n \cup \widehat{P}_j^n$ extends univalently to the simply-connected domain $\mathbb{C}_{J_0^n}$, and the claim follows from Corollary 4.8.

Now let us assume $(P, \widehat{P}) \lhd \mathcal{A}_0^n$. Then either $(P, \widehat{P}) = (P_{0,j}^n, \widehat{P}_{0,j}^n)$ for some $0 \leq j < q_{n+1} - 1$, or for some $q_{n+1} - 1 < j < q_{n+1} + q_n$, or else $(P, \widehat{P}) = (Q_{0,j}^n, \widehat{Q}_{0,j}^n)$ for some $0 \leq j < q_{n+1} + q_n$. Again, let us consider only the first case, the others being similar. In this case, the branch $g = f^{-j-1}$ which maps $P_0^n \cup \widehat{P}_0^n$ to $P_{0,j}^n \cup \widehat{P}_{0,j}^n$ has a univalent extension to $\mathbb{C}_{J_0^n}$ since by $j+1 < q_{n+1}$ the latter set does not contain any critical value of $f^{\circ j+1}$. Hence, again, the claim follows from Corollary 4.8. $\square$

The second distortion lemma considers further pull-backs of puzzle pieces. First, it will be convenient to include the following:

Definition 4.12. For an integer $k \geq 0$ and a k-drop U, consider the branch of $f^{-k} = F^{-k}$ mapping U_0 isomorphically to U. It is easy to see that this branch has a univalent extension to the simply connected domain $\mathbb{C} \smallsetminus (\overline{\mathbb{D}} \cup \mathcal{R})$, where $\mathcal{R}$ is the closure of the external ray landing at the critical value x_{-1}. We denote this univalent extension by g_U. Furthermore, we define

$$\mathcal{H}_k := \{g_U : U \text{ is a drop of depth } k\},$$

and we set $\mathcal{H} := \bigcup_{k=0}^{\infty} \mathcal{H}_k$. Note that $\mathcal{H}_0 = \{\mathrm{id}\}$.

LEMMA 4.13. *For every pair of puzzle pieces $(P, \widehat{P}) \lhd \mathcal{A}_0^n$, there exists the asymptotically universal distortion bound*

$$\chi(g, P \cup \widehat{P}) \asymp 1$$

which holds uniformly in $(P, \widehat{P})$ and $g \in \mathcal{H}$.

Proof. Note that every $g \in \mathcal{H}$ is defined on $\mathcal{A}_0^n$ since the domain $\mathbb{C} \smallsetminus (\overline{\mathbb{D}} \cup \mathcal{R})$ certainly contains $\mathcal{A}_0^n$. Fix a pair $(P, \widehat{P}) \lhd \mathcal{A}_0^n$. By the proof of Lemma 4.11, the univalent branch g_1 of f^{-j-1} which maps the pair of critical puzzle pieces $(P', \widehat{P}') \lhd \mathcal{A}^n$ to $(P, \widehat{P})$ has a univalent extension to $\mathbb{C}_{J_0^n}$ or $\mathbb{C}_{J_0^{n+1}}$ (depending on which of the three possible types $(P', \widehat{P}')$ is). Let us assume we have the first case, the other two cases being similar. If $\Omega := g_1(\mathbb{C}_{J_0^n})$, it follows that the hyperbolic diameter $\mathrm{diam}_\Omega(P \cup \widehat{P})$ is equal to $\mathrm{diam}_{J_0^n}(P' \cup \widehat{P}')$ which is $\asymp 1$ by Lemma 4.7. Let Ω' be the connected component of $\Omega \smallsetminus (\overline{\mathbb{D}} \cup \mathcal{R})$ containing the pair $(P, \widehat{P})$. It is easy to see that Ω' is simply connected and $\mathrm{diam}_{\Omega'}(P \cup \widehat{P}) \asymp 1$. It follows that g is univalent in Ω' and Theorem 4.6 implies $\chi(g, P \cup \widehat{P}) \asymp 1$ as claimed. $\square$

LEMMA 4.14. (i) *Let $k \geq 0$, U be a k-drop, P' be an exterior puzzle piece, and $n \geq 0$. If $U \cap P' \neq \emptyset$, then $U \subset P'$ and $g_U(\mathcal{A}_0^n) \subset P'$.*

(ii) *Let $k \geq 0$ and U be a k-drop. Then either $g_U(\mathcal{A}_0^n) \subset \mathcal{Y}_{k-1}^n$ or $g_U(\widehat{A}_0^n) \cap \mathcal{Y}_{k-1}^n = \emptyset$.*

(iii) *For every $k \geq 0$, there is the equality*

$$\mathcal{Y}_k^n = \mathcal{Y}_{k-1}^n \cup \bigcup \{P \cup \widehat{P} : \widehat{P} \lhd \mathcal{H}_k(\widehat{A}_0^n) \text{ and } \widehat{P} \cap \mathcal{Y}_{k-1}^n = \emptyset\}.$$

(iv) *For every $\widehat{P} \lhd \mathcal{H}_k(\widehat{A}_0^n)$ with $\widehat{P} \cap \mathcal{Y}_{k-1}^n = \emptyset$, $\widehat{P} \smallsetminus \mathcal{Y}_k^{n+2} = \widehat{P} \smallsetminus \mathcal{Z}_k^{n+2}$.*

Proof. (i) As was remarked in subsection 4.1, $U \cap P' \neq \emptyset$ implies $U \subset P'$ so that $\widehat{P} \subset P'$ for every $\widehat{P} \lhd g_U(\widehat{A}_0^n)$. If $P \lhd g_U(A_0^n)$, then the interior of P intersects the interior of P' since P' contains a neighborhood of $\overline{U} \smallsetminus \{x(U)\}$. It follows from the nested property of puzzle-pieces that $P \subset P'$.

(ii) For $k = 0$ the claim is clear since $\widehat{A}_0^n \cap \mathcal{Y}_{-1}^n = \emptyset$. Assume $k \geq 1$ and consider a k-drop U. If $g_U(\widehat{A}_0^n) \cap \mathcal{Y}_{k-1}^n \neq \emptyset$, then for some external puzzle piece $P' \lhd \mathcal{Y}_{k-1}^n$ we must have $U \cap P' \neq \emptyset$. It follows from (i) that $U \subset P'$ and $g_U(\mathcal{A}_0^n) \subset P'$. This proves $g_U(\mathcal{A}_0^n) \subset \mathcal{Y}_{k-1}^n$.

(iii) From the definition $\mathcal{Y}_k^n = \mathcal{Y}_{k-1}^n \cup \mathcal{H}_k(\mathcal{A}_0^n)$. Hence the inclusion $\supset$ is trivial. For the reverse inclusion, suppose that $z \in \mathcal{Y}_k^n \smallsetminus \mathcal{Y}_{k-1}^n$. Then $z \in \mathcal{H}_k(\mathcal{A}_0^n)$, which means $z \in P \cup \widehat{P}$, where $\widehat{P} \lhd g_U(\widehat{A}_0^n)$ for some k-drop U. Since $z \in g_U(\mathcal{A}_0^n) \smallsetminus \mathcal{Y}_{k-1}^n$, by (ii) we must have $g_U(\widehat{A}_0^n) \cap \mathcal{Y}_{k-1}^n = \emptyset$, so that $\widehat{P} \cap \mathcal{Y}_{k-1}^n = \emptyset$.

(iv) As $\widehat{P} \cap \mathcal{Y}_{k-1}^n = \emptyset$ implies $\widehat{P} \cap \mathcal{Y}_{k-1}^{n+2} = \emptyset$,

$$\widehat{P} \smallsetminus \mathcal{Y}_k^{n+2} = \widehat{P} \smallsetminus \mathcal{H}_k(\mathcal{A}_0^{n+2}) = \widehat{P} \smallsetminus \mathcal{H}_k(\widehat{A}_0^{n+2} \cup Q_0^{n+2}) = \widehat{P} \smallsetminus \mathcal{Z}_k^{n+2},$$

where the last equality holds since $\widehat{P} \cap \mathcal{Z}_{k-1}^{n+2} = \emptyset$. $\square$

The following is one of the main technical results of this paper. It is this estimate which allows us to show that the pull-back of a David-Beltrami differential on $\mathbb{D}$ to the union of all drops is a David-Beltrami differential on $\mathbb{C}$ (compare Theorem B).

THEOREM 4.15. *The following asymptotically universal bound exists*:

$$\operatorname{area}(\mathcal{Y}^n \smallsetminus \mathcal{Y}^{n+2}) \asymp \operatorname{area}(\mathcal{Y}^n). \tag{4.18}$$

As a result, there is a universal constant $0 < \delta < 1$ *such that*

$$\operatorname{area}(\mathcal{Z}^n) \leq \operatorname{area}(\mathcal{Y}^n) \preccurlyeq \delta^n. \tag{4.19}$$

Proof. Combining Lemma 4.10 with Lemma 4.11, we obtain a universal constant $0 < \lambda' < 1$ and an integer $N' \geq 1$ such that for every $n \geq N'$,

$$\begin{aligned} &\operatorname{area}(\widehat{P} \smallsetminus \widehat{A}^{n+2}) \geq \lambda' \operatorname{area}(P \cup \widehat{P}) \quad \text{for all } \widehat{P} \lhd \widehat{A}^n, \\ &\operatorname{area}(\widehat{P} \smallsetminus \widehat{A}_0^{n+2}) \geq \lambda' \operatorname{area}(P \cup \widehat{P}) \quad \text{for all } \widehat{P} \lhd \widehat{A}_0^n. \end{aligned} \tag{4.20}$$

This, together with Lemma 4.13, shows that there exist a universal constant λ and an integer N, with $0 < \lambda < \lambda' < 1$ and $N \geq N'$, such that for every $n \geq N$, $k \geq 0$, and $\widehat{P} \lhd \mathcal{H}_k(\widehat{A}_0^n)$,

$$\operatorname{area}(\widehat{P} \smallsetminus \mathcal{Z}_k^{n+2}) \geq \lambda \operatorname{area}(P \cup \widehat{P}). \tag{4.21}$$

We shall prove by induction on $k \geq -1$ that for every $n \geq N$,

$$\operatorname{area}(\mathcal{Z}_k^n \smallsetminus \mathcal{Y}_k^{n+2}) \geq \lambda \operatorname{area}(\mathcal{Y}_k^n). \tag{4.22}$$

For the induction basis, note that the puzzle pieces which belong to $\widehat{A}^n$ have disjoint interiors. Thus, summing up the first estimate in (4.20) over all $\widehat{P} \lhd \widehat{A}^n$, we obtain

$$\operatorname{area}(\mathcal{Z}_{-1}^n \smallsetminus \mathcal{Y}_{-1}^{n+2}) = \operatorname{area}(\mathcal{Z}_{-1}^n \smallsetminus \mathcal{Z}_{-1}^{n+2}) \geq \lambda' \operatorname{area}(\mathcal{A}^n) > \lambda \operatorname{area}(\mathcal{Y}_{-1}^n).$$

For the induction step, assume (4.22) holds for some $k-1 \geq -1$. Writing $\mathcal{Z}_k^n = \mathcal{Z}_{k-1}^n \cup \mathcal{H}_k(\widehat{A}_0^n \cup Q_0^n)$, we have the following estimates in which the sums are taken over all puzzle pieces $\widehat{P} \lhd \mathcal{H}_k(\widehat{A}_0^n)$ which do not intersect $\mathcal{Y}_{k-1}^n$:

$$\begin{aligned} \operatorname{area}(\mathcal{Z}_k^n \smallsetminus \mathcal{Y}_k^{n+2}) &\geq \operatorname{area}(\mathcal{Z}_{k-1}^n \smallsetminus \mathcal{Y}_k^{n+2}) + \sum_{\widehat{P}} \operatorname{area}(\widehat{P} \smallsetminus \mathcal{Y}_k^{n+2}) \\ \text{(by Lemma 4.14(iv))} \quad &= \operatorname{area}(\mathcal{Z}_{k-1}^n \smallsetminus \mathcal{Y}_{k-1}^{n+2}) + \sum_{\widehat{P}} \operatorname{area}(\widehat{P} \smallsetminus \mathcal{Z}_k^{n+2}) \\ \text{(by (4.21) and (4.22))} \quad &\geq \lambda \operatorname{area}(\mathcal{Y}_{k-1}^n) + \sum_{\widehat{P}} \lambda \operatorname{area}(P \cup \widehat{P}) \\ \text{(by Lemma 4.14(iii))} \quad &\geq \lambda \operatorname{area}(\mathcal{Y}_k^n). \end{aligned}$$

This completes the induction step. It now follows from (4.22) that for every $k \geq -1$ and $n \geq N$,

$$\operatorname{area}(\mathcal{Y}^n \smallsetminus \mathcal{Y}_k^{n+2}) \geq \operatorname{area}(\mathcal{Y}_k^n \smallsetminus \mathcal{Y}_k^{n+2}) \geq \operatorname{area}(\mathcal{Z}_k^n \smallsetminus \mathcal{Y}_k^{n+2}) \geq \lambda \operatorname{area}(\mathcal{Y}_k^n).$$

Taking the limit as $k \to \infty$ yields $\operatorname{area}(\mathcal{Y}^n \smallsetminus \mathcal{Y}^{n+2}) \geq \lambda \operatorname{area}(\mathcal{Y}^n)$, which is equivalent to (4.18).

The proof of (4.19) is now immediate. Let $0 < \eta := 1 - \lambda < 1$ and let N be as above. Then by induction we obtain

$$\operatorname{area}(\mathcal{Y}^n) \leq \eta^{\frac{n-N-1}{2}} \operatorname{area}(\mathcal{Y}^1) \leq \eta^{\frac{n-N-1}{2}} \operatorname{area}\{z : G(z) \leq 1\},$$

where $G : A(\infty) \to \mathbb{R}$ is the Green's function on the basin of infinity. Since this last area is bounded by a universal constant $C > 0$, we obtain

$$\operatorname{area}(\mathcal{Y}^n) \leq C \, \eta^{\frac{n}{3}}$$

for all $n \geq 3N + 3$, which proves (4.19) with $\delta := \eta^{\frac{1}{3}}$. □

5. Proofs of Theorems A and B

In this section we prove Theorems A and B cited in the introduction. As indicated there, Theorem B implies that a David-Beltrami differential supported on $\mathbb{D}$ extends to an F-invariant David-Beltrami differential on $\mathbb{C}$ by pull-back. Note that the statement of this theorem is completely independent of the arithmetic of the rotation number θ. Thus, with Theorem B in hand, it follows that Theorem A is true for any arithmetical condition on θ for which the more elementary Theorem C holds (compare Questions 1 and 2 in the introduction and the discussion there).

5.1. *Concentrating Lebesgue measure.* Consider the Blaschke product $f = f_\theta$ of (3.1) and the modified map $F = F_{\theta,H}$ of (3.2) for any irrational $0 < \theta < 1$ and any homeomorphism H. We will associate to f a measure $\nu = \nu_\theta$ depending only on θ, supported on the closed unit disk and with total mass equal to $\operatorname{area}(K(F))$. This measure is obtained by summing up the push forward of Lebesgue measure on each drop U by the minimal iterate of f mapping U to $\mathbb{D}$. More explicitly, let $g_0 : \mathbb{D} \to U_0$ denote the univalent branch of $f^{-1} = F^{-1}$ and let $\mathcal{H}$ be as in Definition 4.12. Then, for any measurable set $E \subset \mathbb{D}$,

$$\nu(E) := \operatorname{area}(E) + \sum_{g \in \mathcal{H}} \operatorname{area}(g \circ g_0(E)). \tag{5.1}$$

Evidently ν is absolutely continuous with respect to Lebesgue measure on $\mathbb{D}$ so that $\nu(E) \to 0$ as $\operatorname{area}(E) \to 0$. Remarkably, it is possible to control the rate of this convergence by the following power law:

Theorem B. *The measure $\nu = \nu_\theta$ is dominated by a universal power of Lebesgue measure. In other words, there exist a universal constant $0 < \beta < 1$ and a constant $C > 0$ (depending on θ) such that*

$$\nu(E) \leq C\,(\mathrm{area}(E))^\beta$$

for every measurable set $E \subset \mathbb{D}$.

Following the notation of Section 4, we consider the sequence of puzzle pieces $\{\widehat{P}^n_{q_{n+1}-1}\}_{n\geq 1}$ in $\overline{\mathbb{D}}$ containing the critical value x_{-1} (compare Figure 7). For simplicity, we set $\Delta^n := \widehat{P}^n_{q_{n+1}-1}$, and thus obtain the nest of puzzle pieces

$$\Delta^1 \supset \Delta^2 \supset \cdots \supset \Delta^n \supset \cdots \supset \{x_{-1}\}.$$

Note that $\mathrm{diam}(\Delta^n) \asymp |I^n_{q_{n+1}-1}|$ by Lemma 4.7 and Lemma 4.11. In particular, $\mathrm{diam}(\Delta^n) \to 0$ as $n \to \infty$, and hence

$$\bigcap_{n\geq 1} \Delta^n = \{x_{-1}\}.$$

Define

$$D^0 := \overline{\mathbb{D}} \smallsetminus \Delta^1 \qquad \text{and} \qquad D^n := \Delta^n \smallsetminus \Delta^{n+1} \qquad \text{for} \qquad n \geq 1.$$

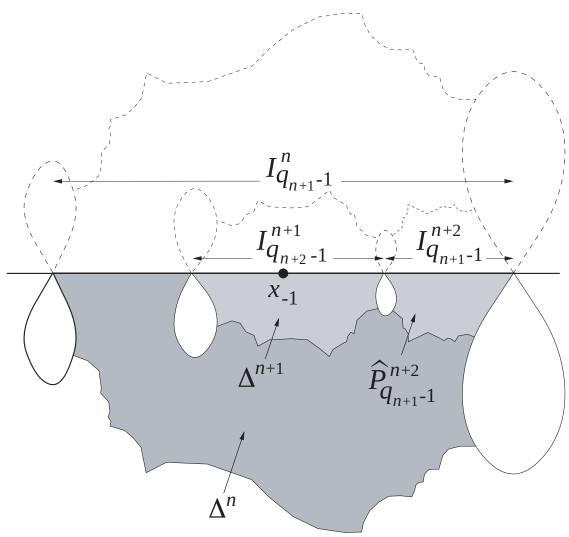

Figure 7. Puzzle pieces Δ^n and Δ^{n+1} near the critical value x_{-1}.

Using the *a priori* area estimates we developed in the previous section, it is quite easy to prove Theorem B in the special case where $E = D^n$ for large n:

Lemma 5.1. *There exists a universal constant $0 < \beta_1 < 1$ such that the following asymptotically universal bound holds*:

$$\nu(D^n) \preccurlyeq (\mathrm{area}(D^n))^{\beta_1}.$$

Proof. Since

$$D^n \cup \bigcup_{g\in\mathcal{H}} g \circ g_0(D^n) \subset \mathcal{Z}^n$$

for all $n \geq 1$, it follows from (5.1) and Theorem 4.15 that

$$\nu(D^n) \leq \operatorname{area}(\mathcal{Z}^n) \leq \operatorname{area}(\mathcal{Y}^n) \preccurlyeq \delta^n. \tag{5.2}$$

We claim that

$$\operatorname{area}(D^n) \asymp \operatorname{area}(\Delta^n).$$

In fact, $I^{n+2}_{q_{n+1}-1}$ and $I^{n+1}_{q_{n+2}-1}$ are interiorly disjoint subintervals of $I^n_{q_{n+1}-1}$, and by Theorem 2.5

$$|I^{n+2}_{q_{n+1}-1}| \asymp |I^{n+1}_{q_{n+2}-1}| \asymp |I^n_{q_{n+1}-1}|.$$

It follows in particular that $\widehat{P}^{n+2}_{q_{n+1}-1} \subset \overline{D^n}$ (compare Figure 7). By Lemma 4.9 and Lemma 4.11,

$$\operatorname{area}(\Delta^n) \geq \operatorname{area}(D^n) \geq \operatorname{area}(\widehat{P}^{n+2}_{q_{n+1}-1}) \asymp |I^{n+2}_{q_{n+1}-1}|^2 \asymp |I^n_{q_{n+1}-1}|^2 \asymp \operatorname{area}(\Delta^n),$$

which proves our claim. It follows that

$$\operatorname{area}(D^n) \asymp \operatorname{area}(\Delta^n) \asymp |I^n_{q_{n+1}-1}|^2 \succcurlyeq \sigma_1^{6n}, \tag{5.3}$$

where $0 < \sigma_1 < 1$ is the universal constant given by Lemma 2.7. Now by (5.2) and (5.3), any positive constant $\beta_1 < (\log\delta)/(6\log\sigma_1)$ will satisfy the condition of the lemma. □

LEMMA 5.2. *There exists the following asymptotically universal bound*:

$$\operatorname{dist}(x_{-1}, \partial P^n_{q_{n+1}-1} \smallsetminus \mathbb{S}^1) \asymp |I^n_{q_{n+1}-1}|.$$

Proof. Note that

$$P^n_{q_{n+1}-1} = f(P^n_{q_{n+1}}) = f((P^n)_{\mathrm{S}}).$$

For any $w \in \partial P^n_{q_{n+1}-1} \smallsetminus \mathbb{S}^1$ there is a unique z in the union $B \cup R \cup G \subset \partial(P^n)_{\mathrm{S}}$ such that $w = f(z)$. Since f has a cubic critical point at 1, it follows from (4.10) in Theorem 4.3 that

$$|x_{-1} - w| \asymp |1-z|^{\frac{1}{3}} \asymp |I^{n+1}|^{\frac{1}{3}} \asymp |I^n_{q_{n+1}-1}|.$$ □

COROLLARY 5.3. *Let* $T := \{rx_{-1} : r \geq 1\}$ *be the radial line segment going from the critical value out to infinity. Then there is the following asymptotically universal bound*:

$$\operatorname{diam}_{\mathbb{C}\smallsetminus T}(D^n) \asymp 1.$$

Proof. This easily follows from the above Lemma 5.2 since $D^n = \Delta^n \smallsetminus \Delta^{n+1}$, $|I^n_{q_{n+1}-1}| \asymp |I^{n+1}_{q_{n+2}-1}|$, and the hyperbolic metric of $\mathbb{C} \smallsetminus T$ at z is comparable to $1/\operatorname{dist}(z, T)$. □

Let W denote the connected component of $F^{-1}(\mathbb{C} \smallsetminus T)$ which contains U_0. Then $W \subset \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ and g_0 extends to a biholomorphic map $g_0 : \mathbb{C} \smallsetminus T \to W$. Hence each $g \circ g_0$ for $g \in \mathcal{H}$ has a univalent extension to $\mathbb{C} \smallsetminus T$ and we obtain the following result by applying Theorem 4.6 and Corollary 5.3:

COROLLARY 5.4. *There exists the asymptotically universal distortion bound*

$$\chi(g \circ g_0, D^n) \asymp 1,$$

which holds uniformly in $g \in \mathcal{H}$.

Proof of Theorem B. Choose positive constants C_1, C_2 and C_3 (all depending on θ), such that for all $n \geq 0$ and all $g \in \mathcal{H}$,

$$\begin{aligned} \nu(D^n) &\leq C_1\, (\mathrm{area}(D^n))^{\beta_1}, \\ \chi(g \circ g_0, D^n) &\leq C_2, \\ \mathrm{area}(D^n) &\leq C_3\, \delta^n. \end{aligned}$$

The existence of these constants is assured by Lemma 5.1, Corollary 5.4 and the estimate (5.2), respectively. Fix a measurable set $E \subset \mathbb{D}$ and decompose it into the disjoint union

$$E = E^0 \cup E^1 \cup E^2 \cup \cdots,$$

where $E^n := D^n \cap E$ for $n \geq 0$. Then,

$$\begin{aligned} \nu(E^n) = &\ \mathrm{area}(E^n) + \sum_{g \in \mathcal{H}} \mathrm{area}(g \circ g_0(E^n)) \\ \leq &\ C_2^2\, \frac{\mathrm{area}(E^n)}{\mathrm{area}(D^n)} \left(\mathrm{area}(D^n) + \sum_{g \in \mathcal{H}} \mathrm{area}(g \circ g_0(D^n)) \right) \\ = &\ C_2^2\, \frac{\mathrm{area}(E^n)}{\mathrm{area}(D^n)}\, \nu(D^n) \\ = &\ C_2^2\, \frac{\nu(D^n)}{(\mathrm{area}(D^n))^{\beta_1}} \left(\frac{\mathrm{area}(E^n)}{\mathrm{area}(D^n)} \right)^{1-\beta_1} (\mathrm{area}(E^n))^{\beta_1}, \end{aligned}$$

which gives

$$\nu(E^n) \leq C_1\, C_2^2\, (\mathrm{area}(E^n))^{\beta_1} \quad \text{for all } n \geq 0. \tag{5.4}$$

Choose any $0 < \beta < \beta_1$ and let $\beta_2 := \beta_1 - \beta$. Then, it follows from (5.4) and Hölder's inequality that

$$\begin{aligned}\nu(E) \leq\ & C_1\, C_2^2 \sum_{n=0}^{\infty} (\mathrm{area}(E^n))^{\beta+\beta_2} \\ \leq\ & C_1\, C_2^2 \left(\sum_{n=0}^{\infty} (\mathrm{area}(E^n))^{\frac{\beta_2}{1-\beta}} \right)^{1-\beta} \left(\sum_{n=0}^{\infty} \mathrm{area}(E^n) \right)^{\beta} \\ \leq\ & C_1\, C_2^2\, C_3^{\beta_2} \left(\sum_{n=0}^{\infty} \delta^{\frac{n\beta_2}{1-\beta}} \right)^{1-\beta} (\mathrm{area}(E))^{\beta}.\end{aligned}$$

This completes the proof of Theorem B. □

5.2. *Main Theorem.* Now we are ready to prove the main result of this work:

THEOREM A. *Let $\mathcal{E}$ denote the set of irrational numbers $\theta = [a_1, a_2, a_3, \ldots]$ which satisfy the arithmetical condition*

$$\log a_n = \mathcal{O}(\sqrt{n}) \quad \text{as } n \to \infty.$$

If $\theta \in \mathcal{E}$, then the Julia set of the quadratic polynomial $P_\theta : z \mapsto e^{2\pi i\theta} z + z^2$ is locally connected and has Lebesgue measure zero. In particular, the Siegel disk Δ_θ of P_θ is a Jordan domain whose boundary contains the finite critical point of P_θ.

Recall that $\mathcal{E}$ has full measure in $[0, 1]$ by Corollary 2.2.

Proof. Fix an irrational $\theta \in \mathcal{E}$ and consider the Blaschke product f_θ in (3.1). By Theorem C (see the end of subsection 2.6 as well as the appendix), there exists a David homeomorphism $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, with $H \circ f_\theta \circ H^{-1} = R_\theta$ on $\mathbb{S}^1$, so that

$$\mathrm{area}\{z \in \mathbb{D} : |\mu_H|(z) > 1 - \varepsilon\} \leq M\, e^{-\frac{\alpha}{\varepsilon}} \ \text{ for all } \ 0 < \varepsilon < \varepsilon_0.$$

Here $M > 0$ is universal, $\alpha > 0$ depends on $\limsup_{n\to\infty} (\log a_n)/\sqrt{n}$, and $0 < \varepsilon_0 < 1$ depends on θ. Let $F = F_{\theta,H}$ denote the Blaschke map modified by H as in (3.2). We define an F-invariant measurable Beltrami differential μ on $\mathbb{C}$ as follows: First, on the unit disk $\mathbb{D}$, let

$$\mu := \mu_H = \frac{\overline{\partial} H}{\partial H} \frac{d\overline{z}}{dz}$$

be the pull-back of the zero Beltrami differential by H. Then, pull $\mu|_{\mathbb{D}}$ back to the union of all drops by the univalent branches $g \circ g_0$ for $g \in \mathcal{H}$. Finally, on the rest of the plane, define μ to be the zero Beltrami differential. By the very construction, $F^*(\mu) = \mu$. Also, the branches $g \circ g_0$ used to spread μ around are all conformal, so they do not change $|\mu|$. It follows that

$$\text{area}\{z \in \mathbb{C} : |\mu|(z) > 1 - \varepsilon\} = \nu\{z \in \mathbb{D} : |\mu|(z) > 1 - \varepsilon\},$$

where ν is the measure defined in (5.1). By Theorem B, there is a universal constant $0 < \beta < 1$ and a constant $C > 0$ depending on θ such that $\nu(E) \leq C\,(\text{area}(E))^\beta$ for all $E \subset \mathbb{D}$. It follows that for all $0 < \varepsilon < \varepsilon_0$,

$$\text{area}\{z \in \mathbb{C} : |\mu|(z) > 1-\varepsilon\} \leq C\ (\text{area}\{z \in \mathbb{D} : |\mu|(z) > 1 - \varepsilon\})^\beta \leq C\, M^\beta\, e^{-\frac{\alpha\beta}{\varepsilon}}.$$

One can actually get rid of the constants in front of the exponential by making ε_0 smaller. In fact, choose any ω such that $0 < \omega < \alpha\,\beta$ and define

$$\varepsilon_1 := \min\left\{\varepsilon_0, \frac{\alpha\,\beta - \omega}{\log(C\,M^\beta)}\right\}.$$

Then a brief computation shows that if $0 < \varepsilon < \varepsilon_1$,

$$\text{area}\{z \in \mathbb{C} : |\mu|(z) > 1 - \varepsilon\} \leq e^{-\frac{\omega}{\varepsilon}}. \tag{5.5}$$

This shows that μ is a David-Beltrami differential, hence integrable by Theorem 2.8. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be the solution to the Beltrami equation $\mu_\varphi = \mu$, normalized by $\varphi(H^{-1}(0)) = 0$, $\varphi(1) = -e^{2\pi i\theta}/2$. Let $P := \varphi \circ F \circ \varphi^{-1}$ denote the conjugate map. Assuming for a moment that P is a quadratic polynomial (see Lemma 5.5 below), we see clearly that $\Delta = \varphi(\mathbb{D})$ is a Siegel disk of rotation number θ for P. By the way we normalized φ, we must have $P = P_\theta$. Local connectivity of J_θ now follows from Theorem 3.1 and the fact that φ is a homeomorphism. That $\text{area}(J_\theta) = 0$ follows from Theorems 3.2 and 2.9. $\square$

It remains to prove the following

Lemma 5.5. *The conjugate map $P := \varphi \circ F \circ \varphi^{-1}$ is a quadratic polynomial.*

Note that this trivially follows from Weyl's lemma when μ has bounded dilatation on $\mathbb{C}$, or equivalently when φ is quasiconformal.

Proof. P is a degree 2 branched covering of the sphere with $P^{-1}(\infty) = \infty$, so it will be a quadratic polynomial once we show that it is holomorphic. To this end, we prove that $\varphi \circ F \in W^{1,1}_{\text{loc}}(\mathbb{C} \smallsetminus \{1\})$. Then, on a small neighborhood U of every regular point of F, both φ and $\varphi \circ F$ pull the zero Beltrami differential back to $\mu|_U$, and it follows from Theorem 2.8 that they must coincide up to a conformal map, meaning in particular that P must be holomorphic in $\varphi(U)$. This argument will show that P is holomorphic on $\mathbb{C} \smallsetminus \{\varphi(1)\}$, hence on the entire plane.

So let us prove $\varphi \circ F \in W^{1,1}_{\text{loc}}(\mathbb{C} \smallsetminus \{1\})$. Evidently $\varphi \circ F \in W^{1,1}_{\text{loc}}(\mathbb{C} \smallsetminus \overline{\mathbb{D}})$ simply because F is holomorphic in $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$ and $\varphi \in W^{1,1}_{\text{loc}}(\mathbb{C})$. On the other hand, on $\mathbb{D}$ we can write $\varphi\circ F = \varphi\circ H^{-1}\circ R_\theta\circ H$. Here $\varphi\circ H^{-1}$ is conformal by another application of Theorem 2.8 since both H and φ are homeomorphisms in $W^{1,1}_{\mathrm{loc}}(\mathbb{D})$ pulling the zero Beltrami differential back to $\mu|_{\mathbb{D}}$. It follows that $\varphi\circ F$ is given in $\mathbb{D}$ by postcomposing H with a conformal map, hence is in $W^{1,1}_{\mathrm{loc}}(\mathbb{D})$.

It remains to show that $\varphi\circ F\in W^{1,1}_{\mathrm{loc}}(U)$ for every small open disk U whose center belongs to $\mathbb{S}^1\smallsetminus\{1\}$. Note that by the above argument $\varphi\circ F$ is differentiable almost everywhere in U, which shows

$$\int_U \mathrm{Jac}(\varphi\circ F)\le \mathrm{area}((\varphi\circ F)(U))<+\infty,$$

so that $\mathrm{Jac}(\varphi\circ F)\in L^1(U)$. Moreover, on every compact subset of $U\smallsetminus\mathbb{S}^1$ the ordinary partial derivatives $\partial(\varphi\circ F)$ and $\overline{\partial}(\varphi\circ F)$ are integrable and coincide with the distributional derivatives of $\varphi\circ F$. If we show that $\partial(\varphi\circ F)$, and hence $\overline{\partial}(\varphi\circ F)=\mu_{\varphi\circ F}\,\partial(\varphi\circ F)$, is in $L^1(U)$, it follows from a standard approximation argument that $\partial(\varphi\circ F)$ and $\overline{\partial}(\varphi\circ F)$ are the distributional partial derivatives in U, and hence $\varphi\circ F\in W^{1,1}_{\mathrm{loc}}(U)$. But $\partial(\varphi\circ F)\in L^1(U)$ is fairly easy to see: since $\mu_{\varphi\circ F}=\mu$ almost everywhere in U, we have

$$|\partial(\varphi\circ F)|^2=\frac{\mathrm{Jac}(\varphi\circ F)}{1-|\mu_{\varphi\circ F}|^2}\le\frac{\mathrm{Jac}(\varphi\circ F)}{1-|\mu_{\varphi\circ F}|}=\frac{\mathrm{Jac}(\varphi\circ F)}{1-|\mu|},$$

so that

$$|\partial(\varphi\circ F)|\le \mathrm{Jac}(\varphi\circ F)^{\frac{1}{2}}(1-|\mu|)^{-\frac{1}{2}}.$$

By the exponential growth condition (5.5), the measurable function $(1-|\mu|)^{-1}$ belongs to $L^1(\mathbb{C})$. It follows then from Hölder inequality that $\partial(\varphi\circ F)\in L^1(U)$. $\square$

We would like to draw the reader's attention to the following corollary which is implicit in the above proof. It describes how the conjugating map in the above construction depends on various parameters; this point may be of interest in possible future investigations, when one considers a family of such David conjugacies as θ varies in $\mathcal{E}$:

Corollary 5.6. *Let $\theta\in\mathcal{E}$ and let φ be the conjugating homeomorphism between F_θ and P_θ given by Theorem* A. *Then φ is a David homeomorphism on $\mathbb{C}$ so that its dilatation satisfies an exponential condition of the form* (2.7). *Moreover, the constant M in* (2.7) *can be chosen to be* 1, *but in general α depends on $\limsup_{n\to\infty}(\log a_n)/\sqrt{n}$ and ε_0 depends on θ.*

By [P5], the boundary of the Siegel disk of P_θ is a quasicircle containing the critical point if and only if θ belongs to the class $\mathcal{D}_2$ of bounded type irrational numbers.

Corollary 5.7. *Let θ belong to the full measure set $\mathcal{E} \smallsetminus \mathcal{D}_2$. Then the boundary of the Siegel disk of the quadratic polynomial P_θ is a Jordan curve of measure zero containing the critical point, but it is not a quasicircle.*

As a final remark, let us briefly sketch how to generalize Theorem A to the case of Siegel disks of higher periods (we assume familiarity with the theory of polynomial-like maps). Let $P : z \mapsto z^2 + c$ be a quadratic polynomial with a Siegel disk Δ of period $n > 1$ and rotation number $\theta \in \mathcal{E}$. Douady and Hubbard proved that P is renormalizable, i.e., there exist open topological disks U and V, with $\Delta \subset U \subset \overline{U} \subset V$, such that $P^{\circ n}|_U : U \to V$ is a degree 2 proper holomorphic map (compare [Z1, Th. 4.2]). According to [DH2], $P^{\circ n}|_U$ is a hybrid equivalent to the quadratic polynomial P_θ. In particular, the "little Julia set" $J := \partial\{z \in U : P^{\circ nk}(z) \in U \text{ for all } k \geq 1\}$ is quasiconformally homeomorphic to the Julia set J_θ. It follows from Theorem A that J is locally connected and has measure zero. From this, it is not hard to draw the same conclusions for the "big Julia set" $J(P)$. The fact that local connectivity of J implies that of $J(P)$ is standard in renormalization theory (see for example [P3]). That $J(P)$ has measure zero follows from the general principle (see [Ly] or [Mc1]) that the orbit of almost every $z \in J(P)$ converges to the postcritical set of P, which is the union $\partial\Delta \cup \cdots \cup P^{\circ n-1}(\partial\Delta)$ in this case. Thus, up to a set of measure zero, $J(P) = \bigcup_{k \geq 0} P^{-k}(J)$, which shows area$(J(P)) = 0$.

6. Appendix: A proof of Theorem C

In this appendix we present a proof of Theorem C, which is substantially based on Yoccoz's work in the unpublished manuscript [Yo2]. The idea of the proof is to construct two combinatorially equivalent dynamically defined cell decompositions for the upper half-plane using the critical circle map and the corresponding rigid rotation. The cells in these decompositions have bounded geometry and are labelled by an integer, called their *level*. The closer the cell is chosen to the boundary of $\mathbb{H}$, the higher its level and the smaller its Euclidean diameter will be. The required extension quasiconformally maps each cell of level n of the first decomposition to a unique cell of level n of the second decomposition, with the dilatation depending only on the $(n+1)$-st term a_{n+1} of the continued fraction expansion of the rotation number θ. The cell decompositions of bounded geometry, a fundamental inequality of Yoccoz (Theorem 6.6 below), and a construction of Strebel (Lemma 6.10 below) are the main ingredients of the proof.

6.1. *Two cell decompositions for the upper half-plane.* As in subsection 2.4, let $f : \mathbb{T} \to \mathbb{T}$ be a critical circle map with a critical point at 0 and irrational rotation number $\theta = [a_1, a_2, a_3, \ldots]$ with convergents p_n/q_n. We set $x_n := f^{-n}(0)$ for all $n \in \mathbb{Z}$.

Consider the dynamical partition $\Pi^n(f)$ as defined in subsection 2.4. It is easy to see that the collection of endpoints of the intervals in $\Pi^n(f)$ is precisely the set $\{x_j : 0 \leq j < q_{n+1} + q_n\}$. By Theorem 2.5, these points chop the circle up into comparable adjacent pieces. Unfortunately, this is not true for the corresponding partition for the rigid rotation R_θ unless θ is of bounded type. To circumvent this problem in our forthcoming arguments, we choose a slightly different partition as follows.

For every integer $n \geq 0$, consider the collection of points

$$\mathcal{Q}_n := \{x_j : 0 \leq j < q_n\}$$

on $\mathbb{T}$, so that $\mathcal{Q}_0 = \{0\}$. It is not hard to see that

$$\mathbb{T} \smallsetminus \mathcal{Q}_n = \left(\bigcup_{0 \leq j < q_n - q_{n-1}}]x_{j+q_{n-1}}, x_j[\right) \cup \left(\bigcup_{0 \leq j < q_{n-1}}]x_j, x_{j+q_n-q_{n-1}}[\right).$$

Thus x_j and x_k, with $j < k$, are adjacent in $\mathcal{Q}_n$ if and only if either $k = j + q_{n-1}$ and $0 \leq j < q_n - q_{n-1}$, or $k = j + q_n - q_{n-1}$ and $0 \leq j < q_{n-1}$. It follows that in the first case

$$[x_k, x_j] \cap \mathcal{Q}_{n+1} = \{x_k, x_{k+q_n}, x_{k+2q_n}, \ldots, x_{k+(a_{n+1}-1)q_n} = x_{j+q_{n+1}-q_n}, x_j\}, \tag{6.1}$$

and in the second case

$$[x_j, x_k] \cap \mathcal{Q}_{n+1} = \{x_j, x_{j+q_n}, x_{j+2q_n}, \ldots, x_{j+a_{n+1}q_n} = x_{k+q_{n+1}-q_n}, x_k\}. \tag{6.2}$$

As a result, we see that x_j and x_k, with $j < k$, are adjacent in both $\mathcal{Q}_n$ and $\mathcal{Q}_{n+1}$ if and only if $a_{n+1} = 1$, $k = j + q_{n-1}$, and $0 \leq j < q_n - q_{n-1}$.

Using the canonical projection $\mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$, we lift the set $\mathcal{Q}_n$ to the translation-invariant set $\widetilde{\mathcal{Q}}_n := \mathcal{Q}_n + \mathbb{Z}$ in $\mathbb{R}$. By the above construction, for $n \geq 1$, the closure of each interval in $\mathbb{T} \smallsetminus \mathcal{Q}_n$ is either an interval or the union of two adjacent intervals in $\Pi^n(f)$. Hence, by the lift to $\mathbb{R}$, Theorem 2.5(ii) implies the following:

Lemma 6.1. *Any two adjacent intervals in $\mathbb{R} \smallsetminus \widetilde{\mathcal{Q}}_n$ have lengths comparable up to a bound which is asymptotically universal. In other words,*

$$\max \left\{ \frac{|I|}{|J|} : I, J \text{ are adjacent in } \mathbb{R} \smallsetminus \widetilde{\mathcal{Q}}_n \right\} \asymp 1.$$

For $n \geq 0$ and $x \in \widetilde{\mathcal{Q}}_n$, let

$$M_n(x) := \frac{1}{2}(x_r - x_\ell),$$

where x_r and x_ℓ are the points in $\widetilde{\mathcal{Q}}_n$ immediately to the right and left of x. Evidently, $M_n(x) > M_{n+1}(x)$ unless x_r and x_ℓ are adjacent to x in $\widetilde{\mathcal{Q}}_{n+1}$ also, in which case $M_n(x) = M_{n+1}(x)$. Observe that $M_0(x) = 1$ for all $x \in \widetilde{\mathcal{Q}}_0 = \mathbb{Z}$. Define

$$z_n(x) := x + i\, M_n(x) \qquad n \geq 0,\ x \in \widetilde{\mathcal{Q}}_n.$$

Using the sequence $\{z_n\}$, we shall define an imbedded graph Γ in the upper half-plane $\mathbb{H}$ as follows: The vertices of Γ are the points $\{z_n(x) : n \geq 0 \text{ and } x \in \widetilde{\mathcal{Q}}_n\}$. Note that $z_n(x) = z_{n+1}(x)$ if and only if $M_n(x) = M_{n+1}(x)$, in which case the corresponding vertex of Γ is doubly labelled. The edges of Γ are the *vertical* segments

$$\{[z_n(x), z_{n+1}(x)] : n \geq 0 \ \text{ and } \ x \in \widetilde{\mathcal{Q}}_n \ \text{ with } \ M_n(x) \neq M_{n+1}(x)\}$$

as well as the *nonvertical* segments

$$\{[z_n(x), z_n(y)] : n \geq 0 \text{ and } x, y \text{ are adjacent in } \widetilde{\mathcal{Q}}_n\}.$$

By a *cell* of Γ we mean the closure of any bounded connected component of $\mathbb{H} \smallsetminus \Gamma$. Any cell γ of Γ is uniquely determined by a pair of adjacent points $x < y$ in $\widetilde{\mathcal{Q}}_n$ with the property that either $M_n(x) \neq M_{n+1}(x)$ or $M_n(y) \neq M_{n+1}(y)$. The integer $n \geq 0$ will be called the *level* of γ, or we say that γ is an *n-cell*. The *top* of the n-cell γ is formed by the nonvertical edge $[z_n(x), z_n(y)]$ while its *bottom* is formed by the union of nonvertical edges

$$[z_{n+1}(t_0), z_{n+1}(t_1)] \cup [z_{n+1}(t_1), z_{n+1}(t_2)] \cup \ldots \cup [z_{n+1}(t_{k-1}), z_{n+1}(t_k)],$$

where the points $x = t_0 < t_1 < \ldots < t_k = y$ form the intersection $[x, y] \cap \widetilde{\mathcal{Q}}_{n+1}$. The *sides* of γ are formed by the vertical edge $[z_n(x), z_{n+1}(x)]$ (which collapses to a single point if $M_n(x) = M_{n+1}(x)$) as well as $[z_n(y), z_{n+1}(y)]$ (which similarly collapses to a single point if $M_n(y) = M_{n+1}(y)$). If $k = 1$ so that x, y are also adjacent in $\widetilde{\mathcal{Q}}_{n+1}$, then γ is either a triangle or a trapezoid. Otherwise $k \geq 2$ and by (6.1) or (6.2), γ is a $(k + 3)$-gon, where k is either a_{n+1} or $a_{n+1} + 1$.

Note that for $m \geq n$, any m-cell γ is contained in the horizontal strip

$$\{z \in \mathbb{H} : 0 \leq \operatorname{Im}(z) \leq \max_{x \in \widetilde{\mathcal{Q}}_n} M_n(x)\}.$$

Hence, Lemma 2.7 implies the following

LEMMA 6.2. *Fix any integer $n \geq 0$. Then the union of all the m-cells of Γ for all $m \geq n$ is contained in a horizontal strip $\{z \in \mathbb{H} : 0 \leq \operatorname{Im}(z) \leq \ell\}$ whose height satisfies an asymptotically universal bound $\ell \preccurlyeq \sigma_2^n$, where $0 < \sigma_2 < 1$ is the universal constant given by* Lemma 2.7.

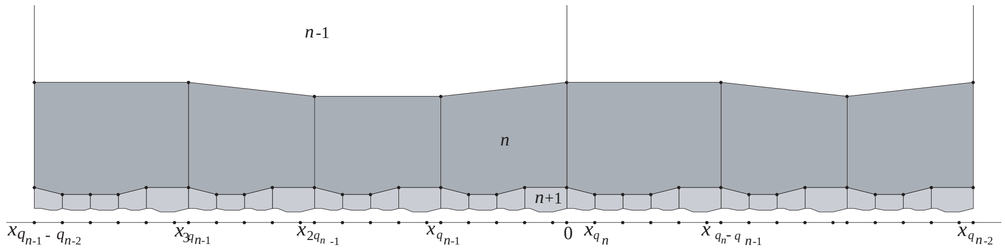

Figure 8. The imbedded graph Γ' for the rigid rotation with selected cells and points in $\mathcal{Q}'_n$. In this picture $a_n = 3$ and $a_{n+1} = 4$. The labels on cells denote their level.

The next lemma is a straightforward consequence of the construction of Γ and Lemma 6.1:

Lemma 6.3. *The cells of Γ have "bounded geometry" in the following sense: There is a constant $C > 1$ such that the top, bottom, and sides of any n-cell γ of Γ have lengths comparable up to C. Moreover, the slopes of nonvertical edges of γ are bounded by C. The constant C is asymptotically universal.*

In a similar fashion, we can construct the above objects for the rigid rotation R_θ, for which we choose similar but "primed" notation. Thus, we have the backward iterate x'_j of the point 0, the sets $\mathcal{Q}'_n$ and $\widetilde{\mathcal{Q}}'_n$, the functions $M'_n(\cdot)$ and $z'_n(\cdot)$, and the imbedded graph Γ' with a typical cell γ' (compare Figure 8). Note that in this case any two (adjacent or not) intervals I and J of $\mathbb{R} \smallsetminus \widetilde{\mathcal{Q}}'_n$ satisfy $1/2 < |I|/|J| < 2$. We thus obtain the following analogue of Lemma 6.3 for rigid rotations:

Lemma 6.4. *The cells of Γ' have "bounded geometry" in the following sense: There is a universal constant $C > 1$ such that the top, bottom, and sides of any cell γ' of Γ' have lengths comparable up to C. Moreover, the slopes of nonvertical edges of γ are bounded by $1/2$.*

6.2. *Construction of the extension.* Now let $h : \mathbb{T} \to \mathbb{T}$ denote the conjugacy between the critical circle map f and the rigid rotation R_θ, normalized by $h(0) = 0$, given by Yoccoz's Theorem 2.4. Let $\widetilde{h} : \mathbb{R} \to \mathbb{R}$ be its lift with $\widetilde{h}(0) = 0$. Note that $\widetilde{h}$ fixes the integer points and $\widetilde{h}(\widetilde{\mathcal{Q}}_n) = \widetilde{\mathcal{Q}}'_n$ for all $n \geq 0$. We shall extend $\widetilde{h}$ to a homeomorphism $\widetilde{H}$ between the imbedded graphs Γ and Γ' by mapping each vertex of Γ to the corresponding vertex of Γ' and each edge of Γ *affinely* to the corresponding edge of Γ'. Strictly speaking, for each $n \geq 0$ and $x \in \widetilde{\mathcal{Q}}_n$, we define $\widetilde{H}(z_n(x)) := z'_n(\widetilde{h}(x))$. Then $[z, w]$ is an edge of Γ if and only if $[\widetilde{H}(z), \widetilde{H}(w)]$ is an edge of Γ'. Thus we can extend $\widetilde{H}$ further to a homeomorphism $\Gamma \xrightarrow{\simeq} \Gamma'$ by mapping each such edge $[z, w]$ affinely to

$[\widetilde{H}(z), \widetilde{H}(w)]$. Note that $\widetilde{H}$ defined this way is the identity on the horizontal line $\mathbb{R} + i$ so that we can define $H(z) = z$ for all $z \in \mathbb{H}$ with $\mathrm{Im}(z) \geq 1$. It is easy to check that for each cell γ of Γ, the boundary $\partial\gamma$ is mapped by $\widetilde{H}$ homeomorphically and edgewise affinely onto the boundary $\partial\gamma'$ of a unique cell γ' of Γ'.

The following is the key result of this appendix:

THEOREM 6.5 (Yoccoz). *There exists a constant $C > 0$ with the following property: For any n-cell γ of Γ, the edgewise affine boundary homeomorphism $\widetilde{H} : \partial\gamma \to \partial\gamma'$ extends to a quasiconformal homeomorphism $\widetilde{H} : \gamma \to \gamma'$ whose dilatation is at most $C(1 + (\log a_{n+1})^2)$. The constant C is asymptotically universal.*

Assuming this result for a moment, we show how Theorem C, cited at the end of subsection 2.6, follows:

Proof of Theorem C. Consider the extension $\widetilde{H} : \mathbb{H} \to \mathbb{H}$ obtained by gluing various extensions to cells given by Theorem 6.5. Clearly $\widetilde{H}$ is in $W^{1,1}_{\mathrm{loc}}(\mathbb{H})$ and satisfies $\widetilde{H}(z+1) = \widetilde{H}(z) + 1$ for all $z \in \mathbb{H}$. Since $\log a_n = \mathcal{O}(\sqrt{n})$ by the assumption, there are a constant $C_1 > 0$ and an integer $N_1 \geq 1$, both depending on θ, such that $1 + (\log a_{n+1})^2 \leq C_1\, n$ whenever $n > N_1$. By Theorem 6.5, there is a universal constant $C_2 > 0$ and an integer $N_2 \geq 1$ depending on f such that the dilatation $K_{\widetilde{H}}$ in the interior of any n-cell of Γ is at most $C_2\,(1 + (\log a_{n+1})^2)$ whenever $n > N_2$. Finally, by Lemma 6.2, there is a universal constant $C_3 > 0$ and an integer $N_3 \geq 1$ depending on f such that if $n > N_3$,

$$\bigcup_{m=n}^{\infty} \{\gamma : \gamma \text{ is an } m-\text{cell of } \Gamma\} \subset \{z \in \mathbb{H} : 0 < \mathrm{Im}(z) \leq C_3 \sigma_2^n\}.$$

Set $N := \max\{N_1, N_2, N_3\}$ and define

$$K_0 := \max\{K_{\widetilde{H}}(z) : z \text{ belongs to the interior of an } m\text{-cell of } \Gamma \text{ with } m \leq N\}.$$

If $K_{\widetilde{H}}(z) > K > K_0$, then either $z \in \Gamma$ (which has Lebesgue measure zero), or else z belongs to the interior of an n-cell of Γ with $n \geq N$, so that $K < K_{\widetilde{H}}(z) \leq C_1 C_2\, n$, or equivalently, $n > K/(C_1 C_2)$. It follows that

$$\mathrm{area}\{z \in \mathbb{H} : 0 \leq \mathrm{Re}(z) \leq 1 \text{ and } K_{\widetilde{H}}(z) > K\} \leq C_3\ \sigma_2^{\frac{K}{C_1 C_2}} = C_3\ e^{-\frac{\log \sigma_2}{C_1 C_2} K}.$$

The exponential map $z \mapsto e^{2\pi i z}$ does not change the dilatation and has norm of the derivative bounded by 2π when restricted to the upper half-plane $\mathbb{H}$. Therefore, the induced homeomorphism $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ satisfies

$$\mathrm{area}\{z \in \mathbb{D} : K_H(z) > K\} \leq 4\pi^2\ C_3\ e^{-\frac{\log \sigma_2}{C_1 C_2} K}$$

whenever $K > K_0$. It follows that H is a David homeomorphism as in (2.8), with $M = 4\pi^2 C_3$, $\alpha = (\log \sigma_2)/(C_1 C_2)$, and K_0 defined as above. Moreover, M is universal, α depends on C_1 (which in turn depends on $\limsup_{n\to\infty}(\log a_n)/\sqrt{n}$), and K_0 depends on f. $\square$

6.3. *The proof of Yoccoz's theorem.* It remains to give the proof of Theorem 6.5. Before we proceed, some preliminaries are in order.

Let $n \geq 1$ and x, y be adjacent points in $\mathcal{Q}_n$. Let $\{x = t_0, t_1, \ldots, t_{k-1}, t_k = y\} = [x, y] \cap \mathcal{Q}_{n+1}$. Note that by (6.1) and (6.2), $k = a_{n+1}$ or $a_{n+1} + 1$. By Lemma 6.1, we know that each interval $[t_{j-1}, t_j]$ in this cascade has length comparable to the next one $[t_j, t_{j+1}]$. However, for large values of k, the action of $f^{\circ q_n}$ on this cascade of intervals is uniformly close to the action of a Möbius transformation on its fundamental domains near a parabolic fixed point. This idea led Yoccoz to the following much sharper statement about the relative size of these intervals, a proof of which can be found in [Yo2] or [dFdM]:

Theorem 6.6 (Yoccoz's almost-parabolic bound). *The lengths of the intervals in the above cascade satisfy*

$$|[t_{j-1}, t_j]| \asymp \frac{|[t_0, t_k]|}{\min\{j, k - j + 1\}^2}$$

uniformly in j, $1 \leq j \leq k$.

Möbius transformations with two distinct fixed points on the real line will play a basic role in the proof of Theorem 6.5. For our purposes, it will be convenient to put them in the normal form

$$\zeta_a(z) := \frac{z}{a - (a-1)z}, \quad a \geq 2,\ z \in \widehat{\mathbb{C}}. \tag{6.3}$$

Note that ζ_a preserves the real line, has an attracting fixed point at $z = 0$ with multiplier $D\zeta_a(0) = a^{-1}$ and a repelling fixed point at $z = 1$ with multiplier $D\zeta_a(1) = a$. (Here and in what follows, D is the differentiation operator.)

Lemma 6.7. *The derivative $D\zeta_a(x)$ is monotonically increasing from $1/a$ to a on $0 \leq x \leq 1$. Moreover, for $0 \leq x < x + \varepsilon \leq 1$, there exist the estimates*

$$1 < \frac{D\zeta_a(x+\varepsilon)}{D\zeta_a(x)} \leq (1 + \varepsilon a)^2,$$

$$\frac{\varepsilon^3 a}{(1+\varepsilon a)^2}\ \frac{1}{(1-x)^2} \leq \zeta_a(x+\varepsilon) - \zeta_a(x) \leq \frac{\varepsilon(1+\varepsilon a)^2}{a}\ \frac{1}{(1-x)^2}.$$

In particular, if ε is comparable to $1/a$ so that $1/(Ca) \leq \varepsilon \leq C/a$ for some

$C \geq 1$, then the above estimates take the form

$$1 < \frac{D\zeta_a(x+\varepsilon)}{D\zeta_a(x)} \leq (1+C)^2, \tag{6.4}$$

(6.5)
$$\frac{1}{C^3(1+C)^2} \; \frac{1}{a^2(1-x)^2} \leq \zeta_a(x+\varepsilon) - \zeta_a(x) \leq C(1+C)^2 \; \frac{1}{a^2(1-x)^2}.$$

Proof. This is an elementary computation which will be left to the reader. For the second set of inequalities, it is convenient to estimate $D\zeta_a$ and apply the Mean Value Theorem. □

Finally, let us also recall the following standard result in quasiconformal theory:

LEMMA 6.8. *Let $K > 1$ and $g : \mathbb{R} \to \mathbb{R}$ be a piecewise differentiable homeomorphism such that*

$$\frac{1}{K} \leq Dg(x) \leq K$$

for all x at which g is differentiable. Then the homeomorphic extension $G : \mathbb{H} \to \mathbb{H}$ given by $G(x+iy) := g(x) + iy$ is K-quasiconformal.

The proof of Theorem 6.5 begins as follows. Throughout we may assume $k \geq 4$, for otherwise γ and γ' are m-gons of bounded geometry for some $m \leq 6$ (compare Lemmas 6.3 and 6.4) and evidently there is an extension $\widetilde{H} : \gamma \to \gamma'$ with asymptotically universal dilatation. It will be convenient to normalize both γ and γ' by mapping them to the upper half-plane. Let $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. As before, let the projections on $\mathbb{R}$ of the vertices of the n-cells γ and γ' consist of the points

$$t_0 < t_1 < \ldots < t_k \qquad \text{and} \qquad t'_0 < t'_1 < \ldots < t'_k,$$

where $t'_j = \widetilde{h}(t_j)$. Since $k \geq 4$, we have $M_n(t_0) \neq M_{n+1}(t_0)$ and $M_n(t_k) \neq M_{n+1}(t_k)$. It follows from Lemma 6.3 that the top of γ is bounded by the graph of a positive affine map g_1, with $|Dg_1| \preccurlyeq 1$. The bottom of γ is bounded by the graph of a positive piecewise affine map g_2 with $|Dg_2| \preccurlyeq 1$. Moreover,

$$\sup_{t_0 \leq x \leq t_k} g_1(x) - g_2(x) \asymp \inf_{t_0 \leq x \leq t_k} g_1(x) - g_2(x) \asymp t_k - t_0.$$

Define a homeomorphism $p : \gamma \xrightarrow{\simeq} \mathrm{SQ} := \{x + iy : |x| \leq 1 \text{ and } 0 \leq y \leq 2\}$ by

$$p(x,y) := \left(-1 + 2\,\frac{x - t_0}{t_k - t_0}, 2\,\frac{y - g_2(x)}{g_1(x) - g_2(x)} \right).$$

Note that p is affine in the x-coordinate, is fiberwise affine in the y-coordinate, and maps the corners $z_n(t_0)$, $z_n(t_k)$, $z_{n+1}(t_0)$ and $z_{n+1}(t_k)$ to $-1+2i$, $1+2i$, -1

and 1, respectively (see Figure 9). Since γ has bounded geometry as seen in the above conditions on g_1 and g_2, it is not hard to check that p is a quasiconformal homeomorphism whose maximum dilatation is asymptotically universal.

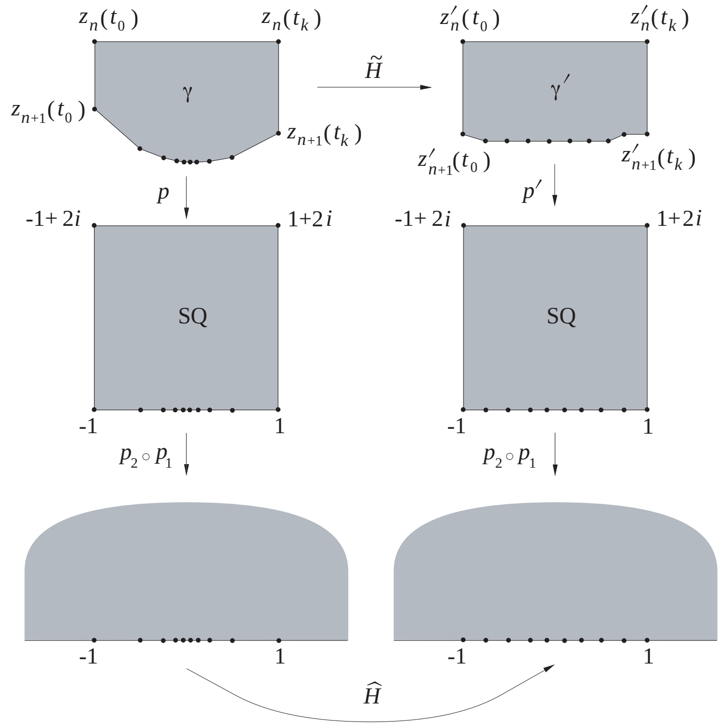

Figure 9. Normalizing the cells γ and γ', where $k = 9$.

Similarly, map γ' onto the square SQ by a quasiconformal homeomorphism p' which is affine in the x-coordinate and is fiberwise affine in the y-coordinate as above. Then, by Lemma 6.4, the maximum dilatation of p' will be bounded by a universal constant.

To finish the normalization process, we should map the square SQ to $\overline{\mathbb{H}}$ in an appropriate way. Let $p_1 : \mathrm{SQ} \xrightarrow{\simeq} \overline{\mathbb{H}}$ be the unique conformal isomorphism, which fixes $-1, 0, 1$. A brief computation shows that p_1 maps the corners $\pm 1 + 2i$ to ± 3. Postcompose p_1 with the quasiconformal homeomorphism $p_2 : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ given by

$$p_2(z) := \begin{cases} 1 + (z-1)(1 - p_1^{-1}(1 - |z-1|)) & \text{if } |z-1| < 1 \\ -1 + (z+1)(1 + p_1^{-1}(-1 + |z+1|)) & \text{if } |z+1| < 1 \\ z & \text{otherwise} \end{cases}$$

It is easy to check that the composition $p_2 \circ p_1$ is a quasiconformal homeomorphism $\mathrm{SQ} \to \overline{\mathbb{H}}$ with $p_2 \circ p_1(t) = t$ for all $-1 \le t \le 1$ (note that both p_1 and p_2 are common for all cells and thus universal). The quasiconformal homeomorphisms

$$\phi := p_2 \circ p_1 \circ p : \gamma \to \overline{\mathbb{H}} \quad \text{and} \quad \phi' := p_2 \circ p_1 \circ p' : \gamma' \to \overline{\mathbb{H}}$$

have maximum dilatations which are asymptotically universal and universal, respectively. They give the required normalizations of γ and γ'.

Define a new homeomorphism $\widehat{H} := \phi' \circ \widetilde{H} \circ \phi^{-1} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, and set $s_j := \phi(z_{n+1}(t_j))$, $s'_j := \phi'(z'_{n+1}(t'_j)) = \widehat{H}(s_j)$. Note that $\widehat{H}$ is the identity when $|x| \geq 1$, and maps the interval $[s_{j-1}, s_j]$ affinely onto the interval $[s'_{j-1}, s'_j]$ for every $1 \leq j \leq k$. From its definition, it is easy to see that the map $\phi \circ z_{n+1} : [t_0, t_k] \to [-1, 1]$ sending $\{t_j\}$ to $\{s_j\}$ is affine. Together with Theorem 6.6, this shows the existence of an asymptotically universal constant $C_0 > 1$ such that for every $1 \leq j \leq k$,

$$\frac{1}{C_0 \min\{j, k-j+1\}^2} \leq s_j - s_{j-1} \leq \frac{C_0}{\min\{j, k-j+1\}^2}. \tag{6.6}$$

Similarly, the map $\phi' \circ z'_{n+1} : [t'_0, t'_k] \to [-1, 1]$ sending $\{t'_j\}$ to $\{s'_j\}$ is affine. Together with the fact that the t'_j are roughly equally spaced, this shows the existence of a universal constant $C_1 > 1$ such that for every $1 \leq j \leq k$,

$$\frac{1}{C_1 k} \leq s'_j - s'_{j-1} \leq \frac{C_1}{k}, \tag{6.7}$$

so that

$$\frac{j}{C_1 k} \leq s'_j + 1 \leq \frac{C_1 j}{k} \quad \text{and} \quad \frac{k-j}{C_1 k} \leq 1 - s'_j \leq \frac{C_1(k-j)}{k}. \tag{6.8}$$

Thus, we have reduced Theorem 6.5 to the situation described in the following:

Lemma 6.9. *Let $\widehat{H} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a homeomorphism such that $\widehat{H}(x) = x$ when $|x| \geq 1$. Suppose that $k \geq 4$ and there are points $s_0 = -1 < s_1 < \ldots < s_{k-1} < s_k = 1$ mapping to $\widehat{H}(s_j) = s'_j$ such that $\widehat{H}$ is affine on each interval $[s_j, s_{j-1}]$. If $\{s_j\}$ and $\{s'_j\}$ satisfy* (6.6) *and* (6.7), *then there exists a quasiconformal extension of $\widehat{H}$ to $\overline{\mathbb{H}}$ whose dilatation is at most $C\,(1+(\log k)^2)$, where $C > 0$ depends only on C_0 and C_1.*

The idea of the proof is to change $\widehat{H}$ up to a quasiconformal factor to make it into a piecewise Möbius transformation on $[-1, 1]$ for which the result is easier to prove. Write $k = a + b$, where a, b are integers such that $2 \leq a \leq b \leq a + 1$. Let ζ_- and ζ_+ be the Möbius transformations defined by

$$\begin{aligned} \zeta_-(z) &:= s'_a - (1 + s'_a)\, \zeta_a\left(\frac{-z + s'_a}{1 + s'_a}\right), \\ \zeta_+(z) &:= s'_a + (1 - s'_a)\, \zeta_b\left(\frac{z - s'_a}{1 - s'_a}\right). \end{aligned}$$

Here ζ_a and ζ_b are the Möbius transformations defined by (6.3). Define a homeomorphism $\psi_1 : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ by

$$\psi_1(x) := \begin{cases} \zeta_-(x) & -1 \leq x \leq s'_a \\ \zeta_+(x) & s'_a \leq x \leq 1 \\ x & \text{otherwise.} \end{cases}$$

Then by (6.5), (6.7) and (6.8) there exists a constant $C_2 > 1$ depending only on C_1 such that for all $1 \leq j \leq k$,

$$\frac{1}{C_2 \min\{j, k-j+1\}^2} \leq \psi_1(s'_j) - \psi_1(s'_{j-1}) \leq \frac{C_2}{\min\{j, k-j+1\}^2}. \tag{6.9}$$

Moreover, let $\psi_2 : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be the piecewise affine map which is the identity when $|x| \geq 1$ and satisfies $\psi_2(s'_j) = \psi_1(s'_j)$. Then by (6.4) there exists a constant $C_3 > 1$ depending only on C_1 such that the homeomorphism $\psi_3 := \psi_1 \circ \psi_2^{-1} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is piecewise differentiable, with $1/C_3 \leq D\psi_3(x) \leq C_3$ for all $x \in \mathbb{R}$. Finally, let $\psi_4 : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be the piecewise affine map which is the identity when $|x| \geq 1$ and satisfies $\psi_4(s_j) = \psi_1(s'_j)$. Then by (6.6) and (6.9) there exists a constant $C_4 > 1$ depending only on C_0 and C_2 such that $1/C_4 \leq D\psi_4(x) \leq C_4$ for all $x \in \mathbb{R}$. Note that

$$\widehat{H} = \psi_2^{-1} \circ \psi_4 = \psi_1^{-1} \circ \psi_3 \circ \psi_4,$$

because $\widehat{H}$ is piecewise affine. By Lemma 6.8, ψ_3 and ψ_4 have quasiconformal extensions whose dilatations are bounded by C_3 and C_4, hence depend only on C_0 and C_1. Thus the proof of Lemma 6.9 will be complete once we show that the piecewise Möbius map ψ_1 has an extension to $\overline{\mathbb{H}}$ with the bound $2(1 + (\log k)^2)$ on its dilatation. But this is a special case of the following lemma due to K. Strebel:

Lemma 6.10. *Let $\psi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be an orientation-preserving homeomorphism which is piecewise Möbius in the following sense: There exist $n \geq 2$ fixed points $x_1 = x_{n+1} < x_2 < \cdots < x_n$ and n Möbius transformations $\zeta^1, \zeta^2, \ldots, \zeta^n$ preserving $\overline{\mathbb{R}}$ such that $\psi|_{[x_j, x_{j+1}]} = \zeta^j$ for $1 \leq j \leq n$. Let $k > 1$ be the largest among the multipliers of the repelling fixed points of the ζ^j. Then ψ has a quasiconformal extension $\Psi : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ whose dilatation is bounded by $2(1 + (\log k)^2)$.*

Proof. Let us first consider a related but easier problem on the horizontal strip $S := \{z \in \mathbb{C} : 0 \leq \mathrm{Im}(z) \leq \pi/2\}$ with $\psi(z) = z$ on the bottom edge $\mathbb{R}$ and $\psi(z) = z + \log \lambda$ on the top edge $\mathbb{R} + i\pi/2$, where $\lambda > 1$. In this case, we can extend ψ to a quasiconformal self-homeomorphism Ψ of S by interpolating linearly:

$$\Psi(z) = z + \frac{2}{\pi} \, \mathrm{Im}(z) \, \log \lambda.$$

It is easy to verify that the dilatation of this Ψ is less than $2(1+(\log\lambda)^2)$. (As an exercise, the reader can show that this is the best possible extension.)

Back to the original situation, consider the hyperbolic convex set D_j bounded by the interval $[x_j, x_{j+1}] \subset \overline{\mathbb{R}}$ and the hyperbolic geodesic Υ_j in $\overline{\mathbb{H}}$ with endpoints x_j and x_{j+1}. Each D_j is conformally isomorphic to the strip S above, with $[x_j, x_{j+1}]$ mapping to $\overline{\mathbb{R}} + i\pi/2$ and Υ_j mapping to $\overline{\mathbb{R}}$. The action of ψ on $[x_j, x_{j+1}]$ corresponds to $z \mapsto z \pm \log\lambda_j$, where $\lambda_j > 1$ is the multiplier of the repelling fixed point of ζ^j. Thus, by the initial construction, ψ can be extended to a quasiconformal homeomorphism $\Psi : D_j \to D_j$ which interpolates between $\psi|_{[x_j, x_{j+1}]}$ and the identity on Υ_j, with dilatation less that $2(1+(\log\lambda_j)^2)$. On $\overline{\mathbb{H}} \smallsetminus \bigcup_{j=1}^{n} D_j$, an ideal hyperbolic n-gon, extend Ψ as the identity map. Evidently the dilatation of Ψ on $\overline{\mathbb{H}}$ is less than $2(1+(\log(\max_j \lambda_j))^2) = 2(1+(\log k)^2)$. $\square$

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(Received August 17, 2000)

(Revised October 24, 2001)