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Poles of Artin L-functions and the strong Artin conjecture

By Andrew R. Booker*

Abstract

We show that if the L-function of an irreducible 2-dimensional complex Galois representation over $\mathbb{Q}$ is not automorphic then it has infinitely many poles. In particular, the Artin conjecture for a single representation implies the corresponding strong Artin conjecture.

Introduction

Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible continuous representation of the absolute Galois group of $\mathbb{Q}$. Brauer [2] proved that the Artin L-function $L(s,\rho)$ associated to ρ has meromorphic continuation to the complex plane and satisfies a functional equation of the form

$$\gamma(s)L(s,\rho) = \varepsilon N^{1/2-s}\gamma(1-s)L(1-s,\bar{\rho}), \tag{1}$$

where $\bar{\rho}$ is the conjugate representation, $|\varepsilon| = 1$, N is a positive integer, and $\gamma(s)$ is a certain product of Γ functions canonically associated to ρ. The famous *Artin conjecture* [1] asserts that $L(s,\rho)$ is entire, with the exception of a pole at $s = 1$ if ρ is trivial. Moreover, Langlands' modularity conjecture, also called the *strong Artin conjecture*, predicts that $L(s,\rho)$ is automorphic, i.e. equal to $L(s,\pi)$ for some cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$. Taken as global statements, these conjectures, which are not settled in any dimension $n \geq 2$, are equivalent in dimensions 2 and 3; this follows from the converse theorems of Weil [20] and Jacquet, Piatetski-Shapiro and Shalika [11], [12], together with a calculation of the ε-factors carried out by Langlands [15] for GL(2) and later simplified by Deligne [8]. However, for a single representation, the strong Artin conjecture appears strictly stronger in general, given our current state of knowledge.

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We are interested in 2-dimensional representations ρ, in which case $\gamma(s) = \pi^{-s}\Gamma\left(\frac{s+a}{2}\right)^2$ with $a = 0$ or 1 if ρ is even (meaning $\det(\rho)(c) = 1$, where c denotes complex conjugation), and $\gamma(s) = (2\pi)^{-s}\Gamma(s)$ if ρ is odd ($\det(\rho)(c) = -1$). We may further assume that ρ is icosahedral, i.e. its image in $\mathrm{PGL}_2(\mathbb{C})$ is isomorphic to A_5, as all other 2-dimensional cases have been shown by Langlands [16] and Tunnell [19] to be automorphic. In this article we investigate what happens should $L(s,\rho)$ or one of its twists by a Dirichlet character have a pole. Our main result is:

THEOREM. *If some twist $L(s,\rho\otimes\chi)$ of $L(s,\rho)$ by a Dirichlet character χ has a pole then $L(s,\rho)$ has infinitely many poles.*

Combining this with the GL(2) converse theorem [20], we have

COROLLARY. *If $L(s,\rho)$ is not automorphic then it has infinitely many poles. In particular, the Artin conjecture for ρ implies the strong Artin conjecture for ρ.*

For brevity we will describe in detail the argument for even representations and give a summary of the differences in the odd case. There are two reasons for focusing on even representations. First, there are a few technical difficulties in the even case that do not arise in the odd case. Second, there is already much in the way of evidence, both theoretical and numerical, in support of the strong Artin conjecture for odd representations (see [4], [3], [9], [14], [13], [5]); there are so far no known examples in the even case.

Proceeding, consider the sum

$$\sum_{s_0 \text{ pole}} \mathrm{Res}_{s=s_0} L(s,\rho)\left(\alpha e^{i(\pi/2-\delta)}\right)^{1/2-s}\pi^{-s}\Gamma\left(\frac{s+a}{2}\right)^2 (s-1/2)^{m-a}, \tag{2}$$

running over the poles s_0 of $L(s,\rho)$, where $0<\delta<\pi/2$, α is a positive rational number, and m is a positive integer to be chosen later. Suppose that (2) converges (as it must, for example, if there are finitely many poles). Then it follows from the fact that $L(s,\rho)$ is the ratio of entire functions of order 1 (as given by Brauer) and the Phragmén-Lindelöf principle that (2) may be expressed as the difference of two contour integrals along vertical lines. Doing so and using the functional equation (1) for $L(s,\rho)$, we arrive at

$$\begin{aligned}\frac{1}{2\pi i}\int\Bigg[L(s,\rho)\left(\alpha e^{i(\pi/2-\delta)}\right)^{1/2-s} - \varepsilon(-1)^{m-a}L(s,\bar\rho)\left(\frac{1}{N\alpha}e^{-i(\pi/2-\delta)}\right)^{1/2-s}\Bigg] \\ \cdot\,\pi^{-s}\Gamma\left(\frac{s+a}{2}\right)^2 (s-1/2)^{m-a}\,ds.\end{aligned} \tag{3}$$

(Unless otherwise noted, all integrals are along a vertical line far to the right.) Note that $e^{-i\pi s/2}$ cancels the decay of the Γ-factor, so that (2) runs essentially over poles with imaginary part between 0 and about $1/\delta$; if there were finitely many poles then it would be bounded as $\delta \to 0$. We will show under the hypotheses of the theorem that (2) is typically large, for an appropriate choice of input data.

The main device in the proof is a technique developed by Conrey and Ghosh [7] to investigate simple zeros of the L-function of Ramanujan's Δ. The technique, the details of which are presented in Lemmas 2 and 3, transforms the two parts of (3) into integrals of additive twists of $L(s,\rho)$ against (essentially) δ^{-s}. More precisely, the twist corresponding to the left half is

$$L(s,\rho,\alpha) = \sum_{n=1}^{\infty} a_n e(-n\alpha) n^{-s}, \tag{4}$$

where α is as above and a_n are the Dirichlet coefficients of $L(s,\rho)$. This twist has nice analytic properties, in particular meromorphic continuation to the plane (Lemma 1), so we may shift contours. If our chosen twist has a pole in the critical strip, then after some work we eventually apply Mellin inversion (Lemma 4) to conclude that (2) is large. The main difficulty arises from the fact that the two parts of (3) may in principle cancel out.

Now, suppose that some Dirichlet twist $L(s,\rho\otimes\chi)$ has a pole. It is known, by nonvanishing results for Hecke L-functions [10], that $L(s,\rho\otimes\chi)$ is holomorphic in $\Re s \geq 1$, and by the functional equation, in $\Re s \leq 0$. Thus, any pole must have real part strictly between 0 and 1.

Next, the Fourier inversion formula

$$\chi(n) = \frac{\tau(\chi)}{q} \sum_{t=1}^{q} \bar{\chi}(t) e(-tn/q), \tag{5}$$

where q is the conductor of χ and τ is the Gauss sum, shows that $L(s,\rho\otimes\chi)$ may be written, with the Euler factors for primes dividing q removed, as a combination of the additive twists $L(s,\rho,t/q)$. Since the Ramanujan conjecture is true in this case, each local factor polynomial vanishes only on $\Re s = 0$ or to the left; thus removal of finitely many Euler factors cannot cancel the pole of $L(s,\rho\otimes\chi)$. Note that this is not an issue for the corollary, since in the converse theorem one may restrict attention to characters such that $(q,N) = 1$, for which the relevant Euler factors are 1. Therefore, not knowing the Ramanujan conjecture is not an obstacle to generalizing the result (see Remark 3 below). In any case, we see that at least one $L(s,\rho,t/q)$ must also have a pole, and the theorem follows by taking $\alpha = t/q$ in the above.

Remarks. 1. It is natural to try to generalize to higher-dimensional representations or to number fields. In both cases higher degree Γ-factors occur. As a result one no longer gets additive twists, but twists of the form $e(\alpha n^{2/d})$, where d is the degree. Unfortunately, we have no analogue of Lemma 1 (below) to help in this case.

2. The statement about infinitely many poles can be strengthened to a slowly growing lower bound for the number of poles with imaginary part between 0 and T; we will not carry this out. In any case, the result is ineffective as it depends on the location of a hypothetical pole of $L(s,\rho,\alpha)$ (which probably does not exist!).

3. The fact that we consider the L-function of a Galois representation not essential to the argument; the technique outlined above may be used to prove more generally a stronger version of the general GL(2) converse theorem [21], requiring that the twisted L-functions be given by Euler products, have meromorphic continuation and precise functional equations, are expressible as ratios of entire functions of finite order (this is what replaces "bounded in vertical strips" of the usual converse theorem), and that at least one twist has at most finitely many poles. This may have other applications.

4. It is expected that analyticity and one functional equation are sufficient to imply modularity. Our result shows that this is the case for 2-dimensional Galois representations. See [6], [17] and [18] for some general ideas towards this end and results for small level.

Proof for even representations

LEMMA 1. *Let α be a rational number. Then $L(s,\rho,\alpha)$ has meromorphic continuation to the complex plane, with poles possible only in the strip $0 < \Re s < 1$, and is expressible as the ratio of two entire functions of order* 1.

Proof. Let V be the vector space of Dirichlet series spanned by $q^{-s}L(s,\rho\otimes\chi_0)$ for positive integers q and Dirichlet characters χ_0. As each $L(s,\rho\otimes\chi_0)$ satisfies the conclusion of the lemma (note in particular that there is no pole at $s=1$ since $\rho\otimes\chi_0$ is icosahedral), so do the functions in V.

Clearly $L(s,\rho)\in V$. Also, by the Chinese remainder theorem, the additive twist by α can be built out of twists by c/p^m for p prime not dividing c. Thus, it suffices to show that V is stable under such twists. To that end, let q and χ_0 be given, and put $L(s,\rho\otimes\chi_0)=\sum_{n=1}^{\infty}a_n n^{-s}$. Then the twist of $q^{-s}L(s,\rho\otimes\chi_0)$ by c/p^m is

$$\sum_{n=1}^{\infty} a_n e(-cqn/p^m)(qn)^{-s}. \tag{6}$$

Let us assume, without loss of generality, that all common factors have been canceled between q and p^m. Then writing $n = p^k r$ with $(r,p) = 1$, we have

$$q^{-s} \sum_{k=0}^{m-1} \frac{a_{p^k}}{p^{ks}} \sum_{(r,p)=1} a_r e(-cqr/p^{m-k}) r^{-s} + q^{-s} \sum_{k=m}^{\infty} \frac{a_{p^k}}{p^{ks}} \sum_{(r,p)=1} a_r r^{-s}. \tag{7}$$

Now, by Fourier analysis, the exponential in (7) may be written as a combination $\sum c_\chi \chi(r)$ of characters χ to modulus p^{m-k}, including imprimitive ones. The inner sum on the right is $E_p(s)L(s, \rho \otimes \chi_0)$ where $E_p(s) = 1 - a_p p^{-s} + \dots$ is the local factor polynomial at p. Thus, we have

$$\begin{aligned} q^{-s} \sum_{k=0}^{m-1} \frac{a_{p^k}}{p^{ks}} \sum_{\chi \,(\mathrm{mod}\ p^{m-k})} c_\chi \sum_{(r,p)=1} a_r \chi(r) r^{-s} \\ + q^{-s} L(s, \rho \otimes \chi_0) E_p(s) \left(\frac{1}{E_p(s)} - \sum_{k=0}^{m-1} \frac{a_{p^k}}{p^{ks}} \right). \end{aligned} \tag{8}$$

Finally $\sum a_r \chi(r) r^{-s}$ is $L(s, \rho \otimes \chi_0 \otimes \chi)$, with the Euler factor at p removed. That factor and the terms involving a_{p^k} and $E_p(s)$ amount to polynomials in p^{-s}. This completes the proof. □

LEMMA 2. *Let α be a positive rational number, $\nu = \pm 1$ and $0 < \delta < \pi/2$. Then*

$$\begin{aligned} &\frac{1}{2\pi i} \int L(s,\rho) \left(\alpha e^{i\nu(\pi/2-\delta)} \right)^{1/2-s} (2\pi)^{-s} \Gamma(s+c)\, ds \\ &\quad = \frac{1}{2\pi i} \int L(s, \rho, \nu\alpha) \alpha^{1/2-s} e^{i\nu\left(\frac{\delta}{2}(s-c-1) + \frac{\pi}{2}(c+1/2)\right)} \\ &\quad\quad \cdot \left(2 \sin \frac{\delta}{2} \right)^{-(s+c)} (2\pi)^{-s} \Gamma(s+c)\, ds. \end{aligned} \tag{9}$$

Proof. Let $F(z) = z^c e^{-z}$ for $\Re z > 0$. Recall the Mellin transform identity

$$F(z) = \frac{1}{2\pi i} \int \Gamma(s+c) z^{-s}\, ds. \tag{10}$$

Also, put $L(s,\rho) = \sum_{n=1}^{\infty} a_n n^{-s}$. We substitute these into the left-hand side of (9) to get

(11)

$$\begin{aligned} &e^{\frac{i\nu}{2}(\pi/2-\delta)} \alpha^{1/2} \sum_{n=1}^{\infty} a_n F\left(2\pi\alpha n e^{i\nu(\pi/2-\delta)} \right) \\ &\quad = e^{\frac{i\nu}{2}(\pi/2-\delta+c(\pi-2\delta))} \alpha^{1/2} \sum_{n=1}^{\infty} a_n e(-\nu\alpha n)(2\pi\alpha n)^c \exp\left(-2\pi i \nu\alpha n (e^{-i\nu\delta} - 1) \right) \end{aligned}$$

$$= e^{\frac{i\nu}{2}(\pi/2-\delta+c(\pi-\delta))}\alpha^{1/2}\left(2\sin\frac{\delta}{2}\right)^{-c}$$
$$\cdot \sum_{n=1}^{\infty} a_n e(-\nu\alpha n) F\left(2\pi\alpha n e^{-i\nu\delta/2}\cdot 2\sin\frac{\delta}{2}\right)$$
$$= e^{\frac{i\nu}{2}(\pi/2-\delta+c(\pi-\delta))}\alpha^{1/2}\left(2\sin\frac{\delta}{2}\right)^{-c}\frac{1}{2\pi i}\int L(s,\rho,\nu\alpha)$$
$$\cdot \left(2\pi\alpha e^{-i\nu\delta/2}\cdot 2\sin\frac{\delta}{2}\right)^{-s}\Gamma(s+c)\,ds,$$

which is the right-hand side. □

Lemma 3. *Let m be given. Then there are numbers b_k, with $b_m > 0$, such that for any n*

$$\pi^{-s}\Gamma\left(\frac{s+a}{2}\right)^2 (s-1/2)^{m-a} = \sum_{k=n}^{m} b_k (2\pi)^{-s}\Gamma(s+k-1/2) + (2\pi)^{-s}\Gamma(s+n-1/2)E_n(s), \tag{12}$$

where $E_n(s)$ is holomorphic and $O(1/s)$ in $\Re s > 1$.

Proof. Stirling's formula gives, for $\Re s \geq 1$,

$$\frac{\Gamma(s/2)^2}{2^{-s}\Gamma(s-1/2)} = \sqrt{8\pi}\left(1+\frac{c_1}{s}+\ldots+\frac{c_n}{s^n}+O\left(\frac{1}{s^{n+1}}\right)\right) \tag{13}$$

for certain numbers c_k. Thus, we have

$$\frac{\pi^{-s}\Gamma\left(\frac{s+a}{2}\right)^2(s-1/2)^{m-a}}{(2\pi)^{-s}\Gamma(s+m-1/2)} \tag{14}$$
$$= 2^{-a}\sqrt{8\pi}\frac{(s-1/2)^m}{(s-1/2)\cdots(s+m-3/2)}$$
$$\cdot\left(1+\frac{c_1}{s+a}+\ldots+\frac{c_n}{(s+a)^n}+O\left(\frac{1}{(s+a)^{n+1}}\right)\right)$$
$$= 2^{-a}\sqrt{8\pi}+\frac{b_{m-1}}{s+m-3/2}+\cdots$$
$$\cdots+\frac{b_n+O(1/s)}{(s+m-3/2)\cdots(s+n-1/2)}.$$

□

LEMMA 4. *Let $\phi(s)$ be meromorphic in the complex plane, and holomorphic and of rapid decay in vertical strips in a right-half plane. If $\phi(s)$ has a pole at $s = \beta + i\tau$, then*

$$\frac{1}{2\pi 1}\int \phi(s)x^{-s}\,ds = \Omega_\varepsilon(x^{-(\beta-\varepsilon)}) \qquad \textit{as } x \to 0 \tag{15}$$

for all $\varepsilon > 0$, where the notation $f = \Omega(g)$ means $f \neq O(g)$.

Proof. Let $S(x)$ denote the integral in (15). By shifting the contour to the right we see that $S(x)$ is of rapid decay as $x \to \infty$. Further, by Mellin inversion,

$$\phi(s) = \int_0^\infty S(x)x^{s-1}\,dx. \tag{16}$$

Suppose for some fixed $\varepsilon > 0$ we have $S(x) = O(x^{-(\beta-\varepsilon)})$. Then (16) defines $\phi(s)$ holomorphically for $\Re s > \beta - \varepsilon$. This proves the contrapositive. $\square$

Now, by Lemma 3 with $n = -1$, we may replace the Γ-factor in (3) by

$$\sum_{k=-1}^{m} b_k(2\pi)^{-s}\Gamma(s+k-1/2) \tag{17}$$

with error terms essentially of the form

$$\int L(s,\rho)e^{i(\pi/2-\delta)s}(2\pi)^{-s}\Gamma(s-3/2)E_{-1}(s)\,ds. \tag{18}$$

Shifting the contour of this integral to $\Re s = 3/2 + \Delta$, with $0 < \Delta < 1/2$, the integrand is $O\Big(\frac{1}{|s|^{3/2-\Delta}}\Big)$, independently of δ, and thus the error is $O(1)$ as $\delta \to 0$.

Applying Lemma 2 with $c = k - 1/2$, (3) is

$$\begin{aligned} O(1) + \frac{1}{2\pi i}\int \sum_{k=-1}^{m} & b_k(2\pi)^{-s}\Gamma(s+k-1/2)\Big(2\sin\frac{\delta}{2}\Big)^{-(s+k-1/2)} \\ \cdot\Big[& L(s,\rho,\alpha)\alpha^{1/2-s}e^{i\left(\frac{\delta}{2}(s-k-1/2)+\frac{\pi k}{2}\right)} \\ & -\varepsilon(-1)^{m-a}L(s,\bar\rho,-1/N\alpha)(1/N\alpha)^{1/2-s}e^{-i\left(\frac{\delta}{2}(s-k-1/2)+\frac{\pi k}{2}\right)}\Big]\,ds. \end{aligned} \tag{19}$$

We may shift the contour of the $k = -1$ term to the left, taking into account the pole at $s = 3/2$, to see that it is also $O(1)$.

Next we consider the $k = m$ term. Let $f(s,\delta)$ be the expression in brackets with $k = m$. Put

$$f_0(s) = f(s,0) = i^m\Big(L(s,\rho,\alpha)\alpha^{1/2-s} - \varepsilon(-1)^a L(s,\bar\rho,-1/N\alpha)(1/N\alpha)^{1/2-s}\Big)$$

and define $f_1(s,\delta)$ so that $f(s,\delta) = f_0(s) + \delta f_1(s,\delta)$; note that since $f(s,\delta)$ is holomorphic in δ, $f_1(s,\delta)$ extends to a holomorphic function near $\delta = 0$ as well.

Now, by hypothesis, $L(s,\rho,\alpha)$ has a pole somewhere in the critical strip, say at $s = \beta + i\tau$. We expect then that $f_0(s)$ does as well. If that is the case then by Lemma 4 the term involving $f_0(s)$ is $\Omega_\varepsilon((1/\delta)^{\beta+m-1/2-\varepsilon})$. As for the remaining terms $f_1(s,\delta)$ and those with $k < m$, we shift the contour just to the right of 1 to see that they are $O_\varepsilon((1/\delta)^{m-1/2+\varepsilon})$. Since $\beta > 0$, the theorem follows.

The above argument breaks down only if $f_0(s)$ is entire, i.e. all poles of $L(s,\rho,\alpha)$ are canceled by poles of $L(s,\bar{\rho},-1/N\alpha)$. In this case, we replace $\varepsilon(-1)^a L(s,\bar{\rho},-1/N\alpha)(1/N\alpha)^{1/2-s}$ in (19) by $L(s,\rho,\alpha)\alpha^{1/2-s} - i^{-m} f_0(s)$ to get, for m even,

$$O(1) + \frac{1}{\pi}\int \sum_{k=0}^{m} b_k(2\pi)^{-s}\Gamma(s+k-1/2)\left(2\sin\frac{\delta}{2}\right)^{-(s+k-1/2)} \cdot\left[L(s,\rho,\alpha)\alpha^{1/2-s}\sin\left(\frac{\delta}{2}(s-k-1/2)+\frac{\pi k}{2}\right) + \frac{(-1)^{m/2}}{2i} f_0(s) e^{-i\left(\frac{\delta}{2}(s-k-1/2)+\frac{\pi k}{2}\right)}\right] ds. \tag{20}$$

(For odd m we arrive at a similar expression, with sine replaced by cosine.) Since $f_0(s)$ is entire, by shifting the contour to the left, using Phragmén-Lindelöf to control $f_0(s)$ in the critical strip, and taking into account the poles of Γ, we see that its contribution to (20) is $O(1)$.

Now, the idea is to show, by a refinement of the previous argument, that even though the poles of $L(s,\rho,\alpha)$ may vanish in the limit as $\delta \to 0$, they must do so to finite order in δ. More precisely, expanding in a power series in δ, we have

$$\sum_{k=0}^{m} b_k(2\pi)^{-s}\Gamma(s+k-1/2)\left(2\sin\frac{\delta}{2}\right)^{m-k}\sin\left(\frac{\delta}{2}\left(s-k-1/2\right)+\frac{\pi k}{2}\right) = \phi_0(s) + \phi_1(s)\delta + \ldots + \phi_n(s)\delta^n + R_n(s,\delta)\delta^{n+1}. \tag{21}$$

By the above, $\phi_0(s)$ vanishes identically. Our goal will be to show for some $n < m$ that $\phi_n(s)$ does not cancel all of the poles of $L(s,\rho,\alpha)$. Suppose for now that this is the case, and let n be the smallest such number. Then for $j < n$, $\phi_j(s)L(s,\rho,\alpha)$ has no pole to the right of $1/2 + j - m$; as in the treatment of $f_0(s)$ above, those terms thus contribute $O(1)$. For the remainder term $R_n(s,\delta)$, shifting the contour just to the right of 1 we get $O_\varepsilon((1/\delta)^{m-n-1/2+\varepsilon})$. By Lemma 4, the term $\phi_n(s)$ contributes $\Omega_\varepsilon((1/\delta)^{\beta+m-n-1/2-\varepsilon})$, where β is the real part of any pole of $L(s,\rho,\alpha)$ not canceled by $\phi_n(s)$. Since $\beta > 0$ and $n < m$, the result follows.

To complete the proof we need look no further than $m = 2$ and $n = 1$. After differentiating and setting $\delta = 0$ we find that this term is

$$\begin{aligned}(22)\qquad \phi_1(s) &= (2\pi)^{-s}\left(b_1\Gamma(s+1/2) - \frac{b_2}{2}\Gamma(s+3/2)(s-5/2)\right)\\ &= -\frac{b_2}{2}(2\pi)^{-s}\Gamma(s+1/2)\left(s^2 - 2s - 5/4 - 2b_1/b_2\right).\end{aligned}$$

This expression can have at most two zeros, depending only on the ratio b_1/b_2. One can determine the numbers explicitly, or more simply observe that $L(s,\chi)^2$, for a Dirichlet character χ with $\chi(-1) = (-1)^a$, is an L-function with Γ-factor $\pi^{-s}\Gamma\left(\frac{s+a}{2}\right)^2$, which has a twist with a double pole at $s = 1$, namely $\zeta(s)^2$, and yet has as most one pole itself. To avoid concluding that it has infinitely many poles, the above polynomial must be $(s-1)^2$. Therefore, $\phi_1(s)$ has no zeros strictly inside the critical strip, and does not cancel the pole of $L(s,\rho,\alpha)$.

Modifications in the odd case

For odd representations, one uses the Γ-factor $(2\pi)^{-s}\Gamma(s)(s-1/2)^m$ in (2). Lemma 3 then takes the form

$$(23)\qquad (2\pi)^{-s}\Gamma(s)(s-1/2)^m = \sum_{k=0}^{m} b_k(2\pi)^{-s}\Gamma(s+k).$$

This is an exact equation, and has a simpler proof not requiring Stirling's formula. Moreover, since all twists $L(s,\rho\otimes\chi)$ have the same Γ-factor, the proof of Lemma 1 shows that the additive twists $L(s,\rho,\alpha)$ and $L(s,\bar{\rho},-1/N\alpha)$ have trivial zeros at $s = 0, -1, -2, \ldots$. Thus, there are no poles at the integers and no need for $O(1)$ terms in (19) and (20). We proceed as before, although now it is most convenient to take simply $n = m = 1$ at the end of the proof. Again we see that $\phi_1(s)$ has a factor of $(s-1)^2$ and no other zeros.

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Princeton University, Princeton, NJ
E-mail address: arbooker@math.princeton.edu

References

[1] E. Artin, Zur Theorie der L-Reihen mit allgemeinen Gruppencharakteren, *Abh. Math. Sem. Univ. Hamburg* **8** (1930), 292–306.
[2] R. Brauer, On Artin's L-series with general group characters, *Ann. of Math.* **48** (1947), 502–514.

[3] J. P. Buhler, Icosahedral Galois representations, *Lecture Notes in Mathematics* **654**, Springer-Verlag, New York, 1978.

[4] K. Buzzard, M. Dickinson, N. Shepard-Barron, and R. Taylor, On icosahedral Artin representations, *Duke Math. J.* **109** (2001), 283–318.

[5] K. Buzzard and W. Stein, A mod five approach to modularity of icosahedral Galois representations, *Pacific Jour. Math.* **203** (2002), 265–282.

[6] J. B. Conrey and D. W. Farmer, An extension of Hecke's converse theorem, *Internat. Math. Res. Notices* **9** (1995), 445–463.

[7] J. B. Conrey and A. Ghosh, Simple zeros of the Ramanujan τ-Dirichlet series, *Invent. Math.* **94** (1988), 403–419.

[8] P. Deligne, Les constantes des équations fonctionnelles des fonctions L, in *Modular Functions of One Variable*, II (*Proc. Internat. Summer School, Univ. Antwerp*, *Antwerp*, 1972), 501–597, *Lecture Notes in Math.* **349**, Springer-Verlag, New York, 1973.

[9] G. Fred (editor), On Artin's conjecture for odd 2-dimensional representations, *Lecture Notes in Mathematics* **1585**, Springer-Verlag, New York, 1994.

[10] H. Heilbronn, Zeta functions and L-functions, in *Algebraic Number Theory* (J. W. S. Cassells and A. Frohlich, eds.), Academic Press, New York, 1967.

[11] H. Jacquet, I. Piatetski-Shapiro, and J. Shalika, Automorphic forms on GL(3). I, *Ann. of Math.* **109** (1979), 169–212.

[12] ———, Automorphic forms on GL(3). II, *Ann. of Math.* **109** (1979), 213–258.

[13] A. Jehanne and M. Müller, Modularity of an odd icosahedral representation, *Colloque International de Théorie des Nombres* (*Talence*, 1999), *J. Théor. Nombres Bordeaux*, *Journal de Théorie des Nombres de Bordeaux* **12** (2000), 475–482.

[14] I. Kiming, On the experimental verification of the Artin conjecture for 2-dimensional odd Galois representations over **Q**. Liftings of 2-dimensional projective Galois representations over **Q**, in *On Artin's Conjecture for Odd* 2*-Dimensional Representations*, *Lecture Notes in Math.* **1585** Springer-Verlag, New York, 1994, 1–36.

[15] R. P. Langlands, On the Functional Equation of the Artin L-functions, unpublished, 1970.

[16] ———, Base change for GL(2), *Ann. of Math. Studies* **96**, Princeton Univ. Press, Princeton, N.J., 1980.

[17] W. Li, On a theorem of Hecke-Weil-Jacquet-Langlands, in *Recent Progress in Analytic Number Theory*, vol. 2, 119–152, Academic Press, New York, 1980.

[18] ———, On converse theorems for GL(2) and GL(1), *Amer. J. Math.* **103** (1981), 851–885.

[19] J. Tunnell, Artin's conjecture for representations of octahedral type, *Bull. AMS* **5** (1981), 173–175.

[20] A. Weil, Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, *Math. Ann.* **168** (1967), 149–156.

[21] ———, *Dirichlet Series and Automorphic Forms*, *Lecture Notes in Math.* **189**, Springer-Verlag, New York, 1971.

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