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Approximation to real numbers by cubic algebraic integers. II

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Abstract

It has been conjectured for some time that, for any integer $n \geq 2$, any real number $\varepsilon > 0$ and any transcendental real number ξ, there would exist infinitely many algebraic integers α of degree at most n with the property that $|\xi - \alpha| \leq H(\alpha)^{-n+\varepsilon}$, where $H(\alpha)$ denotes the height of α. Although this is true for $n = 2$, we show here that, for $n = 3$, the optimal exponent of approximation is not 3 but $(3 + \sqrt{5})/2 \simeq 2.618$.

1. Introduction

Define the *height* $H(\alpha)$ of an algebraic number α as the largest absolute value of the coefficients of its irreducible polynomial over $\mathbf{Z}$. Thanks to work of H. Davenport and W. M. Schmidt, we know that, for any real number ξ which is neither rational nor quadratic over $\mathbf{Q}$, there exists a constant $c > 0$ such that the inequality

$$|\xi - \alpha| \leq cH(\alpha)^{-\gamma^2},$$

where $\gamma = (1 + \sqrt{5})/2$ denotes the golden ratio, has infinitely many solutions in algebraic integers α of degree at most 3 over $\mathbf{Q}$ (see Theorem 1 of [3]). The purpose of this paper is to show that the exponent γ^2 in this statement is best possible.

Theorem 1.1. *There exists a real number ξ which is transcendental over $\mathbf{Q}$ and a constant $c_1 > 0$ such that, for any algebraic integer α of degree at most 3 over $\mathbf{Q}$, we have*

$$|\xi - \alpha| \geq c_1 H(\alpha)^{-\gamma^2}.$$

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In general, for a positive integer n, denote by τ_n the supremum of all real numbers τ with the property that any transcendental real number ξ admits infinitely many approximations by algebraic integers α of degree at most n over $\mathbf{Q}$ with $|\xi - \alpha| \le H(\alpha)^{-\tau}$. Then, the above result shows that $\tau_3 = \gamma^2 \simeq 2.618$ against the natural conjecture that $\tau_n = n$ for all $n \ge 2$ (see [7, p. 259]). Since $\tau_2 = 2$ (see the introduction of [3]), it leaves open the problem of evaluating τ_n for $n \ge 4$. At present the best known estimates valid for general $n \ge 2$ are

$$\lceil (n+1)/2 \rceil \le \tau_n \le n$$

where the upper bound comes from standard metrical considerations, while the lower bound, due to M. Laurent [4], refines, for even integers n, the preceding lower bound $\tau_n \ge \lfloor (n+1)/2 \rfloor$ of Davenport and Schmidt [3]. Note that similar estimates are known for the analog problem of approximation by algebraic numbers, but in this case the optimal exponent is known only for $n \le 2$ (see [2]).

In the next section we recall the results that we will need from [6]. Then, in Section 3, we present the class of real numbers for which we will prove, in Section 4, that they satisfy the measure of approximation of Theorem 1.1. Section 3 also provides explicit examples of such numbers based on the Fibonacci continued fractions of [5] and [6] (a special case of the Sturmian continued fractions of [1]).

2. Extremal real numbers

The arguments of Davenport and Schmidt in Section 2 of [3] show that, if a real number ξ is not algebraic over $\mathbf{Q}$ of degree at most 2 and has the property stated in Theorem 1.1, then there exists another constant $c_2 > 0$ such that the inequalities

$$1 \le |x_0| \le X, \quad |x_0\xi - x_1| \le c_2 X^{-1/\gamma}, \quad |x_0\xi^2 - x_2| \le c_2 X^{-1/\gamma}, \tag{2.1}$$

have a solution in integers x_0, x_1, x_2 for any real number $X \ge 1$. In [6], we defined a real number ξ to be *extremal* if it is not algebraic over $\mathbf{Q}$ of degree at most 2 and satisfies the latter property of simultaneous approximation. We showed that such numbers exist and form a countable set. Thus, candidates for Theorem 1.1 have to be extremal real numbers.

For each $\mathbf{x} = (x_0, x_1, x_2) \in \mathbf{Z}^3$ and each $\xi \in \mathbf{R}$, we define

$$\|\mathbf{x}\| = \max\{|x_0|, |x_1|, |x_2|\} \quad \text{and} \quad L_\xi(\mathbf{x}) = \max\{|x_0\xi - x_1|, |x_0\xi^2 - x_2|\}.$$

Identifying $\mathbf{x}$ with the symmetric matrix

$$\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix},$$

we also define

$$\det(\mathbf{x}) = x_0x_2 - x_1^2.$$

Then, Theorem 5.1 of [6] provides the following characterization of extremal real numbers.

PROPOSITION 2.1. *A real number ξ is extremal if and only if there exists a constant $c_3 \ge 1$ and an unbounded sequence of nonzero points $(\mathbf{x}_k)_{k\ge 1}$ of $\mathbf{Z}^3$ satisfying, for all $k \ge 1$,*

(i) $c_3^{-1}\|\mathbf{x}_k\|^\gamma \le \|\mathbf{x}_{k+1}\| \le c_3\|\mathbf{x}_k\|^\gamma$,

(ii) $c_3^{-1}\|\mathbf{x}_k\|^{-1} \le L_\xi(\mathbf{x}_k) \le c_3\|\mathbf{x}_k\|^{-1}$,

(iii) $1 \le |\det(\mathbf{x}_k)| \le c_3$,

(iv) $1 \le |\det(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{x}_{k+2})| \le c_3$.

In order to prove our main Theorem 1.1, we will also need the following special case of Proposition 9.1 of [6] where, for a real number t, the symbol $\{t\}$ denotes the distance from t to a closest integer:

PROPOSITION 2.2. *Let ξ be an extremal real number and let $(\mathbf{x}_k)_{k\ge 1}$ be as in Proposition 2.1. Assume that, upon writing $\mathbf{x}_k = (x_{k,0}, x_{k,1}, x_{k,2})$, there exists a constant $c_4 > 0$ such that*

$$\{x_{k,0}\xi^3\} \ge c_4$$

for all $k \ge 1$. Then, for any algebraic integer α of degree at most 3 over $\mathbf{Q}$, we have

$$|\xi - \alpha| \ge c_5 H(\alpha)^{-\gamma^2}$$

for some constant $c_5 > 0$.

Since extremal real numbers are transcendental over $\mathbf{Q}$ (see [6, §5]), this reduces the proof of Theorem 1.1 to finding extremal real numbers satisfying the hypotheses of the above proposition. Note that, for an extremal real number ξ and a corresponding sequence $(\mathbf{x}_k)_{k\ge 1}$, Proposition 9.2 of [6] shows that there exists a constant $c_6 > 0$ such that

$$\{x_{k,0}\xi^3\} \ge c_6\|\mathbf{x}_k\|^{-1/\gamma^3}$$

for any sufficiently large k.

We also mention the following direct consequence of Corollary 5.2 of [6]:

PROPOSITION 2.3. *Let ξ be an extremal real number and let $(\mathbf{x}_k)_{k\ge 1}$ be as in Proposition 2.1. Then there exists an integer $k_0 \ge 1$ and a 2×2 matrix M with integral coefficients such that, viewing each $\mathbf{x}_k$ as a symmetric matrix, the point $\mathbf{x}_{k+2}$ is a rational multiple of $\mathbf{x}_{k+1}M\mathbf{x}_k$ when $k \ge k_0$ is odd, and a rational multiple of $\mathbf{x}_{k+1}{}^tM\mathbf{x}_k$ when $k \ge k_0$ is even.*

Proof. Corollary 5.2 together with formula (2.2) of [6] show that there exists an integer $k_0 \geq 1$ such that $\mathbf{x}_{k+2}$ is a rational multiple of $\mathbf{x}_{k+1}\mathbf{x}_{k-1}^{-1}\mathbf{x}_{k+1}$ for all $k > k_0$. If S is a 2×2 matrix such that $\mathbf{x}_{k+1}$ is a rational multiple of $\mathbf{x}_k S \mathbf{x}_{k-1}$ for some $k > k_0$, this implies that $\mathbf{x}_{k+2}$ is a rational multiple of $\mathbf{x}_k S \mathbf{x}_{k+1}$ and thus, by taking transpose, that $\mathbf{x}_{k+2}$ is a rational multiple of $\mathbf{x}_{k+1} {}^t S \mathbf{x}_k$. The conclusion then follows by induction on k, upon choosing M so that the required property holds for $k = k_0$. □

Note that, in the case where all points $\mathbf{x}_k$ have determinant 1, one may assume that $M \in \mathrm{GL}_2(\mathbf{Z})$ in the above proposition and the conclusion then becomes $\mathbf{x}_{k+2} = \pm \mathbf{x}_{k+1} S \mathbf{x}_k$ where S is either M or ${}^tM$ depending on the parity of $k \geq k_0$. This motivates the following definition:

Definition 2.4. Let $M \in \mathrm{GL}_2(\mathbf{Z})$ be a nonsymmetric matrix. We denote by $\mathcal{E}(M)$ the set of extremal real numbers ξ with the following property. There exists a sequence of points $(\mathbf{x}_k)_{k\geq 1}$ in $\mathbf{Z}^3$ satisfying the conditions of Proposition 2.1 which, viewed as symmetric matrices, belong to $\mathrm{GL}_2(\mathbf{Z})$ and satisfy the recurrence relation

$$\mathbf{x}_{k+2} = \mathbf{x}_{k+1} S \mathbf{x}_k, \quad (k \geq 1), \quad \text{where} \quad S = \begin{cases} M & \text{if } k \text{ is odd,} \\ {}^tM & \text{if } k \text{ is even.} \end{cases}$$

Examples of extremal real numbers are the Fibonacci continued fractions $\xi_{a,b}$ (see [5] and [6, §6]) where a and b denote distinct positive integers. They are defined as the real numbers

$$\xi_{a,b} = [0, a, b, a, a, b, \dots] = 1/(a + 1/(b + \cdots))$$

whose sequence of partial quotients begins with 0 followed by the elements of the Fibonacci word on $\{a, b\}$, the infinite word $abaab\cdots$ starting with a which is a fixed point of the substitution $a \mapsto ab$ and $b \mapsto a$. Corollary 6.3 of [6] then shows that such a number $\xi_{a,b}$ belongs to $\mathcal{E}(M)$ with

$$M = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} ab+1 & a \\ b & 1 \end{pmatrix}. \tag{2.2}$$

We conclude this section with the following result.

LEMMA 2.5. *Assume that ξ belongs to $\mathcal{E}(M)$ for some nonsymmetric matrix*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}),$$

and let $(\mathbf{x}_k)_{k\geq 1}$ be as in Definition 2.4. Then, upon writing $\mathbf{x}_k = (x_{k,0}, x_{k,1}, x_{k,2})$, we have, for all $k \geq 2$,

(i) $\mathbf{x}_{k+2} = (a x_{k,0} + (b + c) x_{k,1} + d x_{k,2}) \mathbf{x}_{k+1} \pm \mathbf{x}_{k-1}$,

(ii) $x_{k,0} x_{k+1,2} - x_{k,2} x_{k+1,0} = \pm(a x_{k-1,0} - d x_{k-1,2}) \pm (b - c) x_{k-1,1}$.

Proof. For $k \geq 1$, we have

$$\mathbf{x}_{k+1} = \mathbf{x}_k S \mathbf{x}_{k-1} \quad \text{and} \quad \mathbf{x}_{k+2} = \mathbf{x}_{k+1} {}^t S \mathbf{x}_k$$

where S is M or ${}^tM$ according to whether k is even or odd, and so

$$\mathbf{x}_{k+2} = {}^t\mathbf{x}_{k+2} = \mathbf{x}_k S \mathbf{x}_{k+1} = (\mathbf{x}_k S)^2 \mathbf{x}_{k-1}.$$

Since Cayley-Hamilton's theorem gives

$$(\mathbf{x}_k S)^2 = \operatorname{trace}(\mathbf{x}_k S)\mathbf{x}_k S - \det(\mathbf{x}_k S) I,$$

we deduce

$$\mathbf{x}_{k+2} = \operatorname{trace}(\mathbf{x}_k S)\mathbf{x}_{k+1} - \det(\mathbf{x}_k S)\mathbf{x}_{k-1}$$

which proves (i). Finally, (ii) follows from the fact that the left-hand side of this inequality is the sum of the coefficients outside of the diagonal of the product

$$\mathbf{x}_k J \mathbf{x}_{k+1} \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and that, since $J\mathbf{x}_k J = \pm \mathbf{x}_k^{-1}$, we have

$$\mathbf{x}_k J \mathbf{x}_{k+1} = \pm J \mathbf{x}_k^{-1} \mathbf{x}_{k+1} = \pm J S \mathbf{x}_{k-1}.$$

3. A smaller class of real numbers

Although we expect that all extremal real numbers ξ satisfy a measure of approximation by algebraic integers of degree at most 3 which is close to that of Theorem 1.1, say with exponent $\gamma^2 + \varepsilon$ for any $\varepsilon > 0$, we could only prove in [6] that they satisfy a measure with exponent $\gamma^2 + 1$ (see [6, Th. 1.5]). Here we observe that the formulas of Lemma 2.5 show a particularly simple arithmetic for the elements ξ of $\mathcal{E}(M)$ when, in the notation of this lemma, the matrix M has $b = 1$, $c = -1$ and $d = 0$. Taking advantage of this, we will prove:

THEOREM 3.1. *Let a be a positive integer. Then, any element ξ of*

$$\mathcal{E}_a = \mathcal{E}\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$$

satisfies the measure of approximation of Theorem 1.1.

The proof of this result will be given in the next section. Below, we simply show that, for $a = 1$, the corresponding set of extremal real numbers is not empty.

PROPOSITION 3.2. *Let m be a positive integer. Then, the real number*

$$\eta = (m + 1 + \xi_{m,m+2})^{-1} = [0, m+1, m, m+2, m, m, m+2, \ldots]$$

belongs to the set $\mathcal{E}_1$.

Proof. We first note that, if a real number ξ belongs to $\mathcal{E}(M)$ for some nonsymmetric matrix $M \in \mathrm{GL}_2(\mathbf{Z})$ with corresponding sequence of symmetric matrices $(\mathbf{x}_k)_{k\geq 1}$, and if C is any element of $\mathrm{GL}_2(\mathbf{Z})$, then the real number η for which $(\eta, -1)$ is proportional to $(\xi, -1)C$ belongs to $\mathcal{E}({}^tCMC)$ with corresponding sequence $(C^{-1}\mathbf{x}_k{}^tC^{-1})_{k\geq 1}$. The conclusion then follows since $\xi_{m,m+2}$ belongs to $\mathcal{E}(M)$ where M is given by (2.2) with $a = m$ and $b = m+2$ and since

$$ {}^tCMC = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for} \quad C = \begin{pmatrix} 0 & -1 \\ -1 & m+1 \end{pmatrix}. $$

Remark. In fact, it can be shown that $\mathcal{E}_a$ is not empty for any integer $a \geq 1$. For example, consider the sequence of matrices $(\mathbf{x}_k)_{k\geq 1}$ defined recursively using the formula of Definition 2.4 with

$$ \mathbf{x}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} a^3+2a & a^3-a^2+2a-1 \\ a^3-a^2+2a-1 & a^3-2a^2+3a-2 \end{pmatrix} $$

and

$$ M = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}. $$

Then, using similar arguments as in [6, §6], it can be shown that $(\mathbf{x}_k)_{k\geq 1}$ is a sequence of symmetric matrices in $\mathrm{GL}_2(\mathbf{Z})$ which satisfies the four conditions of Proposition 2.1 for some real number ξ which therefore belongs to $\mathcal{E}_a$.

4. Proof of Theorem 3.1

We fix a positive integer a, a real number $\xi \in \mathcal{E}_a$, and a corresponding sequence of points $(\mathbf{x}_k)_{k\geq 1}$ of $\mathbf{Z}^3$ as in Definition 2.4. For simplicity, we also define

$$ X_k = \|\mathbf{x}_k\| \quad \text{and} \quad \delta_k = \{x_{k,2}\xi\}, \quad (k \geq 1). $$

The constant c_3 being as in Proposition 2.1, we first note that

$$ \begin{aligned} (4.1) \quad \{x_{k,0}\xi\} &\leq |x_{k,0}\xi - x_{k,1}| \leq c_3 X_k^{-1}, \\ \{x_{k,1}\xi\} &\leq |x_{k,1}\xi - x_{k,0}\xi^2| + |x_{k,0}\xi^2 - x_{k,2}| \leq (|\xi|+1)c_3 X_k^{-1}. \end{aligned} $$

For $k \geq 2$, the recurrence formula of Lemma 2.5 (i) implies

$$ x_{k+2,2} = a x_{k,0} x_{k+1,2} \pm x_{k-1,2} \tag{4.2} $$

and Lemma 2.5 (ii) gives

$$ x_{k,0}x_{k+1,2} = x_{k,2}x_{k+1,0} \pm a x_{k-1,0} \pm 2x_{k-1,1}. $$

Using (4.1), the latter relation leads to the estimate

$$ \{x_{k,0}x_{k+1,2}\xi\} \leq X_k\{x_{k+1,0}\xi\} + a\{x_{k-1,0}\xi\} + 2\{x_{k-1,1}\xi\} \leq c_7 X_{k-1}^{-1} $$

for some constant $c_7 > 0$ (since $X_k X_{k+1}^{-1} \le c_3^{2+\gamma} X_{k-1}^{-1}$ by virtue of Proposition 2.1 (i)). Combining this with (4.2), we deduce

$$|\delta_{k+2} - \delta_{k-1}| \le a\{x_{k,0}x_{k+1,2}\xi\} \le ac_7 X_{k-1}^{-1}.$$

Since the sequence $(X_k)_{k\ge 1}$ grows at least geometrically, this in turn implies that, for any pair of integers j and k which are congruent modulo 3 with $j \ge k \ge 1$, we have

$$|\delta_j - \delta_k| \le c_8 X_k^{-1}$$

with some other constant $c_8 > 0$. Since

$$|\{x_{k,0}\xi^3\} - \delta_k| \le |x_{k,0}\xi^3 - x_{k,2}\xi| \le c_3|\xi|X_k^{-1}, \quad (k \ge 1),$$

we conclude that, for $i = 1, 2, 3$, the limit

$$\theta_i = \lim_{j\to\infty} \{x_{i+3j,0}\xi^3\} = \lim_{j\to\infty} \delta_{i+3j}$$

exists and that

$$|\theta_i - \{x_{k,0}\xi^3\}| \le (c_8 + c_3|\xi|)X_k^{-1}$$

for $k \equiv i \bmod 3$. Since, for all sufficiently large k, Proposition 9.2 of [6] gives

$$\{x_{k,0}\xi^3\} \ge c_9 X_k^{-1/\gamma^3}$$

with a constant $c_9 > 0$, these numbers θ_i are nonzero. Thus the sequence $\Big(\{x_{k,0}\xi^3\}\Big)_{k\ge 1}$ has (at most three) nonzero accumulation points and therefore is bounded below by some positive constant, say for $k \ge k_0$, to exclude the finitely many indices k where $x_{k,0} = 0$. Applying Proposition 2.2 to the subsequence $(\mathbf{x}_k)_{k\ge k_0}$, we conclude that ξ has the approximation property stated in Theorem 1.1.

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