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Branched polymers and dimensional reduction

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Abstract

We establish an exact relation between self-avoiding branched polymers in $D+2$ continuum dimensions and the hard-core continuum gas at negative activity in D dimensions. We review conjectures and results on critical exponents for $D+2 = 2, 3, 4$ and show that they are corollaries of our result. We explain the connection (first proposed by Parisi and Sourlas) between branched polymers in $D+2$ dimensions and the Yang-Lee edge singularity in D dimensions.

1. Introduction

A branched polymer is usually defined [Sla99] to be a finite subset $\{y_1, \ldots, y_N\}$ of the lattice $\mathbb{Z}^{D+2}$ together with a tree graph whose vertices are $\{y_1, \ldots, y_N\}$ and whose edges $\{y_i, y_j\}$ are such that $|y_i - y_j| = 1$ so that points in an edge of the tree graph are necessarily nearest neighbors. A tree graph is a connected graph without loops. Since the points y_i are distinct, branched polymers are *self-avoiding*. Figure 1 shows a branched polymer with $N = 9$ vertices on a two-dimensional lattice.

Critical exponents may be defined by considering statistical ensembles of branched polymers. Define two branched polymers to be equivalent when one is a lattice translate of the other, and let c_N be the number of equivalence classes of branched polymers with N vertices.

For example, $c_1, c_2, c_3 = 1, 2, 6$, respectively, in $\mathbb{Z}^2$. Some authors prefer to consider the number of branched polymers that contain the origin. This is Nc_N, since there are N representatives of each class which contain the origin.

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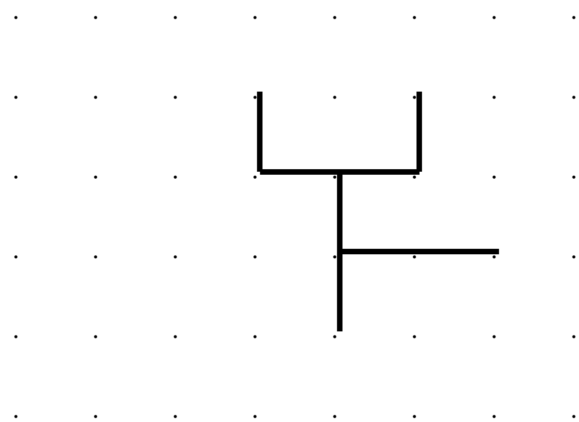

Figure 1.

One expects that c_N has an asymptotic law of the form

$$c_N \sim N^{-\theta} \mathrm{z}_c^{-N}, \tag{1.1}$$

in the sense that $\lim_{N\to\infty} -\frac{1}{\ln N} \ln[c_N \mathrm{z}_c^N] = \theta$. The *critical exponent* θ is conjectured to be *universal*, meaning that (unlike z_c) it should be independent of the local structure of the lattice. For example, it should be the same on a triangular lattice, or in the continuum model to be considered in this paper.

In 1981 Parisi and Sourlas [PS81] conjectured exact values of θ and other critical exponents for self-avoiding branched polymers in $D+2$ dimensions by relating them to the Yang-Lee singularity of an Ising model in D dimensions. Various authors [Dha83], [LF95], [PF99] have also argued that the exponents of the Yang-Lee singularity are related in simple ways to exponents for the hard-core gas at the negative value of activity which is the closest singularity to the origin in the pressure. In this paper we consider these models in the continuum and show that there is an exact relation between the hard-core gas in D dimensions and branched polymers in $D+2$ dimensions. We prove that the Mayer expansion for the pressure of the hard-core gas is exactly equal to the generating function for branched polymers.

Following [Frö86], we rewrite c_N in a way that motivates the continuum model we will study in this paper. Let T be an abstract tree graph on N vertices labeled $1, \ldots, N$ and let $y = (y_1, \ldots, y_N)$ be a sequence of distinct points in $\mathbb{Z}^{D+2}$. We say y *embeds* T if $y_{ij} := y_i - y_j$ has length one for all *edges* $\{i, j\}$ in the tree T. This condition holds for y if and only if it holds for any translate $y' = (y_1 + u, \ldots, y_N + u)$. Therefore it is a condition on the class $[y]$ of sequences equivalent to y under translation. Then

$$c_N = \frac{1}{N!} \sum_{T,[y]} 1_{y \text{ embeds } T}. \tag{1.2}$$

Proof. $\sum_T 1_{y \text{ embeds } T}$ is a symmetric function of $y_1, \ldots, y_N$ because a permutation π of $\{1, \ldots, N\}$ induces a permutation of tree graphs in the range of the sum. Therefore, in the right-hand side of the claim, we can drop the $\frac{1}{N!}$ and sum over representatives $(y_1, \ldots, y_N)$ of $[y]$ whose points are in lexicographic order. Then the vertices in the abstract tree T may be replaced by points according to $i \leftrightarrow y_i$ and the claim follows. □

We describe the two systems to be related by dimensional reduction now.

The hard-core gas. Suppose we have "particles" at positions $x_1, \ldots, x_N$ in a rectangle $\Lambda \subset \mathbb{R}^D$. Let $x_{ij} = x_i - x_j$ and define the *Hard-Core Constraint*:

$$J(\{1, \ldots, N\}, \mathbf{x}) = \begin{cases} 1 \text{ if all } |x_{ij}| \geq 1 \\ 0 \text{ otherwise} \end{cases} . \tag{1.3}$$

By definition, the *Partition Function for the Hard-Core Gas* is the following power series in z:

$$Z_{\mathrm{HC}}(\mathrm{z}) = \sum_{N \geq 0} \frac{\mathrm{z}^N}{N!} \int (d^D x)^N \, J(\{1, \ldots, N\}, \mathbf{x}), \tag{1.4}$$

where each x_i is integrated over Λ. For $D = 0$, Λ is an abstract one-point space and the integrals can be omitted. Then, the hard-core constraint eliminates all terms with $N > 1$ and the partition function reduces to $1 + \mathrm{z}$.

Branched polymers in the continuum. A *branched polymer* is a tree graph T on vertices $\{1, \ldots, N\}$ together with an embedding into $\mathbb{R}^{D+2}$, i.e. positions $y_i \in \mathbb{R}^{D+2}$ for each $i = 1, \ldots, N$, such that

(1) If $ij \in T$ then $|y_{ij}| = 1$;

(2) If $ij \notin T$ then $|y_{ij}| \geq 1$.

Define the weight $W(T)$ of a tree by

$$W(T) := \int \prod_{ij \in T} \underbrace{d\Omega(y_{ij})}_{\substack{\text{surface measure} \\ \text{on unit ball}}} \prod_{ij \notin T} \mathbb{1}_{\{|y_{ij}| \geq 1\}}, \tag{1.5}$$

where the integral is over $\mathbb{R}^{[D+2]N}/\mathbb{R}^{D+2}$, or, more concretely, $y_1 = 0$. If $N = 1$, $W(T) := 1$. The *generating function* for branched polymers is

$$Z_{\mathrm{BP}}(\mathrm{z}) = \sum_{N=1}^{\infty} \frac{\mathrm{z}^N}{N!} \sum_{T \text{ on } \{1, \ldots, N\}} W(T). \tag{1.6}$$

Our main theorem is

THEOREM 1.1. *For all* z *such that the right-hand side converges absolutely, the thermodynamic limit exists and satisfies*

$$\lim_{\Lambda \nearrow \mathbb{R}^D} \frac{1}{|\Lambda|} \log Z_{\mathrm{HC}}(\mathrm{z}) = -2\pi Z_{\mathrm{BP}} \left(-\frac{\mathrm{z}}{2\pi}\right). \tag{1.7}$$

Here lim *is omitted when* $D = 0$.

The expansion of the left-hand side as a power series in z is known [Rue69] to be convergent for |z| small. Theorem 1.1 shows that the radius of convergence of both sides is the same, as the coefficients are identical at every order. Nothing is known in general about the maximal domain of analyticity of the left-hand side (the pressure of the hard-core gas), but it is presumably larger than the disk of convergence of the right-hand side.

Consequences for critical exponents. For $D = 0, 1$ the left-hand side can be computed exactly, and so we obtain exact formulas for the weights of polymers of size N in dimension $d = D + 2 = 2, 3$:

COROLLARY 1.2.

$$\frac{1}{N!} \sum_{T \text{ on } \{1,\dots,N\}} W(T) = \begin{cases} N^{-1}(2\pi)^{N-1} & \text{if } d = 2 \\ \frac{N^{N-1}}{N!}(2\pi)^{N-1} & \text{if } d = 3 \end{cases}. \tag{1.8}$$

Proof. For $D = 0$ the left-hand side of (1.8) is $\log(1 + \mathrm{z})$, and so

$$Z_{\mathrm{BP}}(\mathrm{z}) = -\frac{1}{2\pi} \log(1 - 2\pi \mathrm{z}) = \sum_{N=1}^{\infty} \frac{1}{2\pi N} (2\pi \mathrm{z})^N, \tag{1.9}$$

which leads to the $d = 2$ result. For $D = 1$, the pressure

$$\lim_{\Lambda \nearrow \mathbb{R}^D} |\Lambda|^{-1} \log Z_{\mathrm{HC}}(\mathrm{z})$$

of the hard-core gas is also computable (see [HH63], for example). It is the largest solution to $xe^x = \mathrm{z}$ for $\mathrm{z} > \tilde{\mathrm{z}}_c := -e^{-1}$, and thus $2\pi Z_{\mathrm{BP}}\left(-\frac{\mathrm{z}}{2\pi}\right) = T(-\mathrm{z})$. Here $T(\mathrm{z}) = -\mathrm{Lambert}W(-\mathrm{z})$ is the tree function, whose N^{th} derivative at 0 is N^{N-1} (see [CGHJK]). Hence,

$$Z_{\mathrm{BP}}(\mathrm{z}) = \frac{1}{2\pi} T(2\pi \mathrm{z}) = \sum_{N=1}^{\infty} \frac{N^{N-1}}{2\pi N!} (2\pi \mathrm{z})^N. \tag{1.10}$$

□

One can check directly from the definition above that the volume of the set of configurations available to dimers and trimers is indeed π, $4\pi^2/3$, respectively, in $d = 2$ and $2\pi, 6\pi^2$, respectively, in $d = 3$. For larger values of N, Corollary 1.2 describes a new set of geometric-combinatoric identities for disks in the plane and for balls in $\mathbb{R}^3$.

From Corollary 1.2 we see immediately that the critical activity z_c for branched polymers in dimension $d = 2$ is exactly $\frac{1}{2\pi}$, and that $\theta = 1$. For $d = 3$, Stirling's formula may be used to generate large N asymptotics:

$$\frac{1}{N!} \sum_{T \text{ on } \{1,\ldots,N\}} W(T) = (2\pi)^{N-\frac{1}{2}} e^{-(N+1)} N^{-\frac{3}{2}} (1 + O(N^{-1})). \tag{1.11}$$

Hence $\mathrm{z}_c = \frac{e}{2\pi}$ and $\theta = \frac{3}{2}$.

For $D = 2$, the pressure of a gas of hard disks is not known, but if we assume the singularity at negative activity is in the same universality class as that of Baxter's model of hard hexagons on a lattice [Bax82], then the pressure has a leading singularity of the form $(\mathrm{z} - \tilde{\mathrm{z}}_c)^{2-\alpha_{\mathrm{HC}}}$ with $\alpha_{\mathrm{HC}} = \frac{7}{6}$ [Dha83], [BL87]. We may define another critical exponent γ_{BP} from the leading singularity of $Z_{\mathrm{BP}}(\mathrm{z})$:

(1.12)
$$\left(\mathrm{z}\frac{d}{d\mathrm{z}}\right)^2 Z_{\mathrm{BP}}(\mathrm{z}) \sim (\mathrm{z} - \mathrm{z}_c)^{-\gamma_{\mathrm{BP}}}, \text{or equivalently } Z_{\mathrm{BP}}(\mathrm{z}) \sim (\mathrm{z} - \mathrm{z}_c)^{2-\gamma_{\mathrm{BP}}}.$$

Theorem 1.1 implies that the singularity of the pressure of the hard-core gas and the singularity of Z_{BP} are the same, so that

$$\gamma_{\mathrm{BP}} = \alpha_{\mathrm{HC}}. \tag{1.13}$$

Hence we expect that $\gamma_{\mathrm{BP}} = \frac{7}{6}$ in dimension $d = 4$. In general, if the exponent θ is well-defined, then it equals $3 - \gamma_{\mathrm{BP}}$ by an Abelian theorem. Thus θ should equal $\frac{11}{6}$ in $d = 4$.

These values for $\theta(d)$ for $d = 2, 3, 4$ agree with the Parisi-Sourlas relation

$$\theta(d) = \sigma(d-2) + 2 \tag{1.14}$$

[PS81] when known or conjectured values of the Yang-Lee edge exponent $\sigma(D)$ are assumed [Dha83], [Car85] (see Section 2). Of course, the exponents are expected to be universal, so one should find the same values for other models of branched polymers (e.g., lattice trees) and also for animals.

A Generalization: *Soft polymers and the soft-core gas.* We define

$$Z_v(\mathrm{z}) = \sum_{N \geq 0} \frac{\mathrm{z}^N}{N!} \int (d^D x)^N \prod_{1 \leq i < j \leq N} e^{-v(|x_{ij}|^2)}, \tag{1.15}$$

where $x_i \in \Lambda \subset \mathbb{R}^D$ and $v(r^2)$ is a differentiable, rapidly decaying, spherically symmetric two-particle potential. The inverse temperature, β, has been included in v. With $w(x) \equiv v(|x|^2)$, let us assume $\hat{w}(k) > 0$ for a repulsive interaction. Then there is a corresponding branched polymer model in $D+2$ dimensions with

$$W_v(T) := \int \prod_{ij\in T} \left[-2v'(|y_{ij}|^2) d^{D+2} y_{ij}\right] \prod_{1\le i<j\le N} e^{-v(|y_{ij}|^2)}. \tag{1.16}$$

Note that by assumption, $v'(r^2)$ is rapidly decaying, so the monomers are stuck together along the branches of a tree. The polymers are softly self-avoiding, with the same weighting factor as for the soft-core gas, albeit in two more dimensions. Defining, as before,

$$Z_{\mathrm{BP},v} = \sum_{N\ge 1} \frac{\mathrm{z}^N}{N!} \sum_{T \text{ on } \{1,\dots,N\}} W(T), \tag{1.17}$$

we will prove:

Theorem 1.3. *For all* z *such that the right-hand side converges absolutely*,

$$\lim_{\Lambda\nearrow\mathbb{R}^D} \frac{1}{|\Lambda|} \log Z_v(\mathrm{z}) = -2\pi Z_{\mathrm{BP},v}\left(-\frac{\mathrm{z}}{2\pi}\right). \tag{1.18}$$

Note that by the sine-Gordon transformation [KUH63], [Frö76]

$$Z_v(\mathrm{z}) = \int \exp\left(\int dx\, \hat{\mathrm{z}} e^{i\varphi(x)}\right) d\mu_w(\varphi), \tag{1.19}$$

where $d\mu_w$ is the Gaussian measure with covariance w, and $\hat{\mathrm{z}} := \mathrm{z}e^{v(0)/2}$. Thus Theorem 1.3 gives an identity relating certain branched polymer models and $-\hat{\mathrm{z}}e^{i\varphi}$ field theories. As discussed in Section 2, an expansion of $-\hat{\mathrm{z}}e^{i\varphi}$ about the critical point reveals an $i\varphi^3$ term (along with higher order terms), so we have a direct connection between branched polymers and the field theory of the Yang-Lee edge.

Green's function relations and exponents. Green's functions are defined through functional derivatives as follows. In the definition (1.4) of the hard-core partition function Z_{HC} each dx_j is replaced by $dx_j \exp(h(x_j))$ where $h(x)$ is a continuous function on Λ. Let $h = \alpha h_1 + \beta h_2$. Then there exists a measure $G_{\mathrm{HC},\Lambda}(dx_1, dx_2; \mathrm{z})$ on $\Lambda \times \Lambda$ such that

$$\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\bigg|_{\alpha=\beta=0} \log Z_{\mathrm{HC}} = \int G_{\mathrm{HC},\Lambda}(dx_1, dx_2; \mathrm{z}) h_1(x_1) h_2(x_2). \tag{1.20}$$

This measure is called a density-density correlation or 2-point Green's function because $G_{\mathrm{HC},\Lambda}(d\tilde{x}_1, d\tilde{x}_2; \mathrm{z})$ equals the correlation of $\rho(d\tilde{x}_1)$ with $\rho(d\tilde{x}_2)$ where $\rho(d\tilde{x}) = \sum \delta_{x_j}(d\tilde{x})$ is a random measure interpreted as the empirical particle density at $\tilde{x}$ of the random hard-core configuration $\{x_1, \dots, x_N\}$. (The underlying probability distribution on hard-core configurations is known as the

Grand Canonical Ensemble; $Z_{\mathrm{HC}}(\mathrm{z})$ is its normalizing constant, cf. (1.4).) For z in the interior of the domain of convergence of the power series Z_{BP}, term by term differentiation is legitimate and the weak limit as the volume $\Lambda \nearrow \mathbb{R}^D$ of $G_{\mathrm{HC},\Lambda}(dx_1, dx_2; \mathrm{z})$ exists. It is a translation-invariant measure which we write as $G_{\mathrm{HC}}(dx; \mathrm{z})\, dx_1$, where $x = x_2 - x_1$. These claims are easy consequences of our identities but we omit the details since they are known [Rue69].

For branched polymers we define $\hat{W}(T)$ by changing the definition (1.5) of the weight $W(T)$ by (i) including an extra Lebesgue integration over $y_1 = (x_1, z_1) \in \hat{\Lambda}$, where $\hat{\Lambda}$ is a rectangle in $\mathbb{R}^{D+2}$, and (ii) inserting $\prod_j \exp(h(y_j))$ under the integral. Then $\hat{Z}_{\mathrm{BP}}$ is defined by replacing $W(T)$ by $\hat{W}(T)$ in (1.6). We define the finite-volume branched polymer Green's function as a measure by taking derivatives at zero with respect to α and β when $h = \alpha h_1 + \beta h_2$. The derivatives can be taken term by term and the infinite volume limit as $\hat{\Lambda} \to \mathbb{R}^{D+2}$ is easily verified to be

(1.21)
$$G_{\mathrm{BP}}(d\tilde{y}_1, d\tilde{y}_2; \mathrm{z}) := \sum_{N=1}^{\infty} \frac{\mathrm{z}^N}{N!} \sum_{T \text{ on } \{1,\ldots,N\}} \int_{(\mathbb{R}^{D+2})^N} \prod_{ij \in T} d\Omega(y_{ij}) \rho(d\tilde{y}_1) \rho(d\tilde{y}_2),$$

where $\rho(d\tilde{y}) = \sum \delta_{y_j}(d\tilde{y})$. This can be written as $G_{\mathrm{BP}}(d\tilde{y}; \mathrm{z}) d\tilde{y}_1$ where $\tilde{y} = \tilde{y}_2 - \tilde{y}_1$.

THEOREM 1.4. *If* z *is in the interior of the domain of convergence of* Z_{BP}, *then for all continuous compactly supported functions* f *of* $x \in \mathbb{R}^D$,

$$\int_{\mathbb{R}^D} f(x) G_{\mathrm{HC}}(dx; \mathrm{z}) = -2\pi \int_{\mathbb{R}^{D+2}} f(x) G_{\mathrm{BP}} \left(dy; -\frac{\mathrm{z}}{2\pi}\right), \tag{1.22}$$

where $y = (x, z) \in \mathbb{R}^{D+2}$.

In effect, G_{HC} can be obtained by integrating G_{BP} over the two extra dimensions. Note that $G_{\mathrm{BP}}(dy; \mathrm{z})$ is invariant under rotations of y. Therefore, we can define a distribution $G_{\mathrm{BP}}(t; \mathrm{z})$ on functions with compact support in $\mathbb{R}^+$ by $\int f(t) G_{\mathrm{BP}}(t; \mathrm{z}) dt = \int G_{\mathrm{BP}}(dy; \mathrm{z}) f(|y|^2)$. $G_{\mathrm{HC}}(t; \mathrm{z})$ is defined analogously. Then Theorem 1.4 implies that, in dimension $D \geq 1$,

$$G_{\mathrm{BP}} \left(t; -\frac{\mathrm{z}}{2\pi}\right) = \frac{1}{2\pi^2} \frac{d}{dt} G_{\mathrm{HC}}(t; \mathrm{z}), \tag{1.23}$$

where the derivative is a weak derivative. A similar theorem holds for Green's functions associated with soft polymers and the soft-core gas.

For $t > 1$, which is twice the hard-core radius, $G_{\mathrm{HC}}(t; \mathrm{z})$ and $G_{\mathrm{BP}}(t; \mathrm{z})$ are functions, so one may define correlation exponents ν and η from the asymptotic form of Green's functions as $\mathrm{z} \nearrow \mathrm{z}_c$. The correlation length $\xi_{\mathrm{HC}}(\mathrm{z})$ is defined from the rate of decay of G_{HC}:

$$\xi_{\mathrm{HC}}(\mathrm{z})^{-1} := \lim_{x \to \infty} -\frac{1}{x} \log |G_{\mathrm{HC}}(x^2; \mathrm{z})|. \tag{1.24}$$

if the limit exists. Then the correlation length exponent $\nu_{\rm HC}$ is defined if $\xi_{\rm HC}(\mathrm{z}) \sim (\mathrm{z}-\tilde{\mathrm{z}}_c)^{-\nu_{\rm HC}}$ as $\mathrm{z} \searrow \tilde{\mathrm{z}}_c := -2\pi \mathrm{z}_c$. One can then send $x \to \infty$ and $\mathrm{z} \searrow \tilde{\mathrm{z}}_c$ while keeping $\hat{x} := x/\xi(\mathrm{z})$ fixed. If there is a number $\eta_{\rm HC}$ such that the scaling function

$$K_{\rm HC}(\hat{x}) := \lim_{x\to\infty, \mathrm{z}\searrow \tilde{\mathrm{z}}_c} x^{D-2+\eta_{\rm HC}} G_{\rm HC}(x^2;\mathrm{z}) \tag{1.25}$$

is defined and nonzero (at least for $\hat{x} > 0$), then $\eta_{\rm HC}$ is called the anomalous dimension. Similar definitions can be applied in the case of branched polymers when one considers the behavior of $G_{\rm BP}(y^2;\mathrm{z})$ as $\mathrm{z} \nearrow \mathrm{z}_c$ (D is replaced with $d = D+2$ in (1.25)). Then (1.23) implies that for $D \geq 1$,

$$\xi_{\rm BP}(\mathrm{z}) = \xi_{\rm HC}\left(-\frac{\mathrm{z}}{2\pi}\right), \tag{1.26}$$

$$\nu_{\rm BP} = \nu_{\rm HC}, \tag{1.27}$$

$$\eta_{\rm BP} = \eta_{\rm HC}, \tag{1.28}$$

$$K_{\rm BP}(\hat{x}) = \frac{1}{4\pi^2}\left[\hat{x}K'_{\rm HC}(\hat{x}) - (D-2+\eta_{\rm HC})K_{\rm HC}(\hat{x})\right], \tag{1.29}$$

when the hard-core quantities are defined.

In conclusion, we see from (1.13), (1.27), (1.28) that the exponents $\gamma_{\rm BP}$, $\nu_{\rm BP}$, $\eta_{\rm BP}$ are equal to their hard-core counterparts $\alpha_{\rm HC}$, $\nu_{\rm HC}$, $\eta_{\rm HC}$ in two fewer dimensions. If the relation $D\nu_{\rm HC} = 2 - \alpha_{\rm HC}$ holds for $D \leq 6$ (hyperscaling conjecture) then a dimensionally reduced form of hyperscaling will hold for branched polymers (cf. [PS81]):

$$(d-2)\nu_{\rm BP} = 2 - \gamma_{\rm BP}. \tag{1.30}$$

For $D = 1$ one has $\alpha_{\rm HC} = \frac{3}{2}$, $\eta_{\rm HC} = -1$, $K_{\rm HC}(\hat{x}) = -4\hat{x}^{-2}e^{-\hat{x}}$ [BI03, eq. 1.19]. Thus our results prove that the branched polymer model $Z_{\rm BP}(\mathrm{z})$ has exponents $\gamma_{\rm BP} = \frac{3}{2}$, $\nu_{\rm BP} = \frac{1}{2}$, $\eta_{\rm BP} = -1$, and scaling function

$$K_{\rm BP}(\hat{x}) = \frac{1}{\pi^2 \hat{x}} e^{-\hat{x}} \tag{1.31}$$

in three dimensions. The form of (1.31) was conjectured by Miller [Mil91], under the assumption that a relation like (1.23) holds between branched polymers in $d = 3$ and the one-dimensional Ising model near the Yang-Lee edge (see §2).

For $D = 2$, the conjectured value of $\alpha_{\rm HC}$ is $\frac{7}{6}$, as mentioned above. Hyperscaling and Fisher's relation $\alpha_{\rm HC} = \nu_{\rm HC}(2-\eta_{\rm HC})$ then lead to conjectures $\nu_{\rm HC} = \frac{5}{12}$, $\eta_{\rm HC} = -\frac{4}{5}$. Assuming these are correct, the results above imply the same values for branched polymers in $d = 4$.

In high dimensions ($d > 8$) it has been proved that $\gamma_{\rm BP} = \frac{1}{2}$, $\nu_{\rm BP} = \frac{1}{4}$, $\eta_{\rm BP} = 0$ (at least for spread-out lattice models) [HS90], [HS92], [HvS03]. While our results do not apply to lattice models, they give a strong indication that the corresponding hard-core exponents have the same (mean-field) values for $D > 6$.

2. Background and relation to earlier work

In this section we consider theoretical physics issues raised by our results.

Three classes of models are relevant to this discussion. *Branched polymers* and *repulsive gases* were defined in Section 1. We also consider the *Yang-Lee edge* $h_\sigma(T)$, defined for the Ising model above the critical temperature as the first occurrence of Lee-Yang zeroes [YL52] on the imaginary magnetic field axis. The density of zeroes is expected to exhibit a power-law singularity $g(h) \sim |h - h_\sigma(T)|^\sigma$ for $|\mathrm{Im}\, h| > |\mathrm{Im}\, h_\sigma(T)|$ [KG71]. This should lead to a branch cut in the magnetization, a singular part of the same form, and a free-energy singularity of the form $(h - h_\sigma(T))^{\sigma+1}$. In zero and one dimensions, the Ising model in a field is solvable and one obtains $\sigma(0) = -1$, $\sigma(1) = -\frac{1}{2}$ [Fis80]. Above six dimensions, a mean-field model of this critical point should give the correct value of σ. Take the standard interaction potential

$$V(\varphi) = \frac{1}{2} r\varphi^2 + u\varphi^4 + h\varphi, \tag{2.1}$$

and let h move down the imaginary axis. The point φ_h where $V'(\varphi_h) = 0$ moves up from the origin, and when h reaches the Yang-Lee edge $h_\sigma(r, u)$, one finds a critical point with $V'(\varphi_{h_c}) = V''(\varphi_{h_c}) = 0$. One can easily see that $|\varphi_h - \varphi_{h_c}| \sim |h - h_c|^{1/2}$, which means that $\sigma = \frac{1}{2}$ in mean field theory. Note that the expansion of $V(\varphi + \varphi_{h_c})$ then begins with a φ^3 term with purely imaginary coefficient.

The repulsive-core singularity and the Yang-Lee edge. The singularity in the pressure found for repulsive lattice and continuum gases at negative activity is known as the *repulsive-core singularity.* Theorem 1.1 relates this singularity to the branched polymer critical point. Poland [Pol84] first proposed that the exponent characterizing the singularity should be universal, depending only on the dimension. Baram and Luban [BL87] extended the class of models to include nonspherical particles and soft-core repulsions. The connection with the Yang-Lee edge goes back to two articles: Cardy [Car82] related the Yang-Lee edge in D dimensions to directed animals in $D + 1$ dimensions, and Dhar [Dha83] related directed animals in $D+1$ dimensions to hard-core lattice gases in D dimensions. Another indirect link arises from the hard hexagon model which, as explained above, has a free-energy singularity of the form $(\mathrm{z}_c - \mathrm{z})^{2-\alpha_{\mathrm{HC}}}$ with $\alpha_{\mathrm{HC}} = \frac{7}{6}$. Equating $2 - \alpha_{\mathrm{HC}}$ with $\sigma + 1$ leads to the value $\sigma(2) = -\frac{1}{6}$, which is consistent with the conformal field theory value for the Yang-Lee edge exponent σ [Car85].

More recently, Lai and Fisher [LF95] and Park and Fisher [PF99] assembled additional evidence for the proposition that the hard-core repulsive singularity is of the Yang-Lee class. In the latter article, a model with hard cores and additional attractive and repulsive terms was translated into field theory by means of a sine-Gordon transformation. When the repulsive terms dominate, a saddle point analysis leads to the $i\varphi^3$ field theory. We can simplify this picture by considering an interaction potential $w(x-y)$ with $\hat{w}(k) > 0$, $\int d^D k\, \hat{w}(k) < \infty$. Then the sine-Gordon transformation (1.19) leads to an interaction $-\hat{\mathrm{z}}e^{i\varphi}$, where φ is a Gaussian field with covariance w and $\hat{\mathrm{z}} := \mathrm{z}e^{w(0)/2}$. In a mean-field analysis, φ is assumed to be constant, and with $r = (\hat{w}(0))^{-1}$ we obtain a potential

$$V(\varphi) = -\hat{\mathrm{z}}e^{i\varphi} + \frac{1}{2} r\varphi^2. \tag{2.2}$$

If we put $\varphi = ix$, the saddle-point equation is

$$\frac{\hat{\mathrm{z}}}{r} = xe^x, \tag{2.3}$$

which has two solutions for $-e^{-1} < \frac{\hat{\mathrm{z}}}{r} < 0$. When $\hat{\mathrm{z}} = \hat{\mathrm{z}}_c = -\frac{r}{e}$, the two critical points coincide at $\varphi_{\hat{\mathrm{z}}_c}$ such that $V'(\varphi_{\hat{\mathrm{z}}_c}) = V''(\varphi_{\hat{\mathrm{z}}_c}) = 0$. Expanding about this point gives an $i\varphi^3$ field theory, plus higher-order terms. Complex interactions play an essential role here, since for real models, stability considerations prevent one from finding a critical theory by causing two critical points to coincide—normally at least three are needed, as for φ^4 theory. Observe that for $\hat{\mathrm{z}} - \hat{\mathrm{z}}_c$ small and positive, the critical point $\varphi_{\hat{\mathrm{z}}}$ satisfies $\varphi_{\hat{\mathrm{z}}} - \varphi_{\hat{\mathrm{z}}_c} \sim (\hat{\mathrm{z}} - \hat{\mathrm{z}}_c)^{\frac{1}{2}}$. Hence this sine-Gordon form of the Yang-Lee edge theory also has $\sigma = \frac{1}{2}$ in mean field theory.

Branched polymers and the Yang-Lee edge. In [PS81], Parisi and Sourlas connected branched polymers in d dimensions with the Yang-Lee edge in $d-2$ dimensions (see also Shapir's field theory representation of lattice branched polymers [Sha83], [Sha85], and [Frö86]). Working with the $n \to 0$ limit of a $\boldsymbol{\varphi}^3$ model, the leading diagrams are the same as those of a φ^3 model in an imaginary random magnetic field. Dimensional reduction [PS79] relates this to the Yang-Lee edge interaction $i\varphi^3$ in two fewer dimensions. The free-energy singularities should coincide, so that $2 - \gamma_{\mathrm{BP}}(d) = \sigma(d-2) + 1$; therefore $\theta(d) = 3 - \gamma_{\mathrm{BP}}(d) = \sigma(d-2) + 2$. There are some potential flaws in this argument. First, a similar dimensional reduction argument for the Ising model in a random (real) magnetic field leads to value of 3 for the lower critical dimension [PS79], [KW81], in contradiction to the proof of long-range order in $d = 3$ [Imb84], [Imb85]. See [BD98], [PS02], [Fel02] for recent discussions of this issue. Second, nonsupersymmetric terms were discarded in the Parisi-Sourlas approach, also in Shapir's work. Though irrelevant in the renormalization group sense, such terms could interfere with dimensional reduction. Finding a more rigorous basis for dimensional reduction continues to be an important issue; for example Cardy's recent results on two-dimensional self-avoiding loops and vesicles [Car01] depend on a reduction of branched polymers to the zero-dimensional $i\varphi^3$ theory.

Our Theorems 1.1 and 1.3 provide an exact relationship between branched polymers and the repulsive-core singularity in two fewer dimensions. When combined with the solid connection between repulsive gases and the Yang-Lee edge, they leave little room to doubt the Parisi-Sourlas claims for branched polymers. In terms of exponents, we have $2 - \gamma_{\mathrm{BP}}(d) = 2 - \alpha_{\mathrm{HC}}(d-2) = \sigma(d-2) + 1$, and the Parisi-Sourlas relation (1.14) follows as above.

3. A fundamental theorem of calculus

The proof of Theorem 1.1 relies on an interpolation formula, Theorem 3.1 below. The idea is to decouple the particles of the hard-core gas by adding coordinates for two additional dimensions and then separating the particles in the new directions. Like the fundamental theorem of calculus, this interpolation formula has a boundary term–the weight J in (1.4)–and derivative terms, which involve tree graphs. The latter become independent as the particles are spread out in the extra dimensions (see Section 4). This leads to a formula for $\log Z_{\mathrm{HC}}(\mathrm{z})$ and our main results (see Section 5).

Suppose $f(\mathbf{t})$ is a smooth function of a collection $\mathbf{t} = (t_{ij}), (t_i)$ of variables

$$\underbrace{(t_{ij})_{1\le i<j\le N}}_{\text{bond variables}} \quad \text{and} \quad \underbrace{(t_i)_{1\le i\le N}}_{\text{vertex variables}} \;,$$

which is compactly supported in (t_i). A subset F of bonds $\{ij|1 \le i < j \le N\}$ is called a *graph* on *vertices* $\{1, \ldots, N\}$. A subset R of vertices is called a set of *roots*. *Forests* are graphs that have no loops. Note that the empty graph is a forest by this definition. The connected components of a forest are *trees*, provided we declare that the graph with no bonds and just one vertex is also a tree.

$f^{(F,R)}(\mathbf{t})$ denotes the derivative with respect to the variables t_{ij} with $ij \in F$ and t_i with $i \in R$. Let $z_1, \ldots, z_N$ be complex numbers, $z_{ij} = z_i - z_j$ and set

$$t_{ij} = |z_{ij}|^2, \quad t_i = |z_i|^2. \tag{3.1}$$

THEOREM 3.1 (Forest-Root Formula).

$$f(\mathbf{0}) = \sum_{(F,R)} \int_{\mathbb{C}^N} f^{(F,R)}(\mathbf{t}) \left(\frac{d^2 z}{-\pi}\right)^N, \tag{3.2}$$

where F, R is summed over all forests F and all sets R of roots constrained by the condition that each tree in F contains exactly one root from R. $d^2 z = du\, dv$ where $z = u + iv$.

This result is a generalization of Theorem 3.1 in [BW88]. That paper and this one rely on Lemma 6.1, an idea which is common to all of the papers [PS79], [PS80], [Lut83], [AB84]. The proof will be given in Sections 6 and 7. The assumption of compact support simplifies our discussion here. But having proved the theorem in this case it holds, when we take limits, for any function which decays to zero at infinity and whose first derivatives are continuous and integrable.

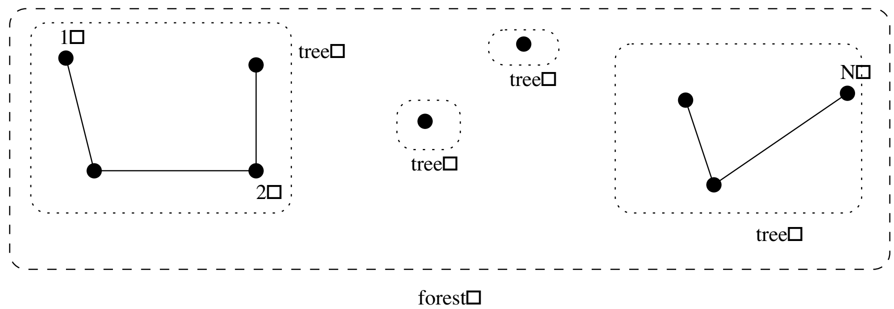

Figure 2. Example of a forest

4. A tree formula for connected parts

Let J be a function on finite subsets X of $\{1, 2, \ldots\}$. The *connected part* of J is a new function J_c on finite subsets, uniquely defined by solving

$$J(X) = \sum_{\{X_1,\ldots X_n\},\ \text{a partition of}\ X} J_c(X_1)\cdots J_c(X_n) \tag{4.1}$$

recursively, starting with $J_c(X) = J(X)$ if $|X| = 1$.

Corollary 4.1. *Let $J(X) = J(X, \mathbf{t})$ depend on auxiliary parameters $t_{ij} \geq 0$ for each unordered pair $\{i, j\} \subset X$, $i \neq j$. Assume that*

$$J(X \cup Y) = J(X)J(Y) \tag{4.2}$$

for disjoint sets X, Y whenever t_{ij} is sufficiently large for all $i \in X$, $j \in Y$, or vice versa. Then

$$J_c(X, \mathbf{0}) = \sum_{T\ \text{on}\ X} \left(-\frac{1}{\pi}\right)^{N-1} \int_{\mathbb{C}^N/\mathbb{C}} J^{(T)}(X, \mathbf{t}), \tag{4.3}$$

where $N = |X|$ denotes the number of vertices in X, and $J^{(T)}$ denotes the first partial derivatives with respect to each of the variables t_{ij} for ij in the tree graph T; cf. $f^{(F,R)}$ above. The integral is over $z_i \in \mathbb{C}$, $i = 1, \ldots, N$ with simultaneous translations $z_i \to z_i + c$ of all vertices factored out, and $t_{ij} = |z_i - z_j|^2$.

Remark. This result was first proved for two-body interactions in [BW88]. A simpler proof based on the Forest-Root formula will be given here for arbitrary $J(X)$.

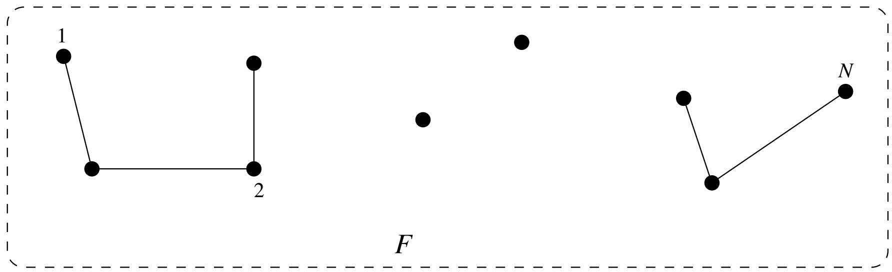

Figure 3. Partition on X defined by a forest F

Proof. Replace the labels $\{1, \ldots, N\}$ in the Forest-Root formula by the elements of X. Let g be a smooth, decreasing, compactly supported function with $g(0) = 1$. Apply the Forest-Root Formula (3.2) to

$$f\big((t_{ij}), (t_i)\big) = J(X, (t_{ij})) \prod_i g(\varepsilon t_i), \tag{4.4}$$

and let $\varepsilon > 0$ tend to zero. Then Corollary 4.1 is proved by the following considerations:

1. A forest F on a set of vertices X uniquely determines a partition of X, each subset being the vertices in one of the trees of F. Therefore,

   (4.5)
   $$\sum_F (\cdots) = \sum_{\{X_1,\ldots,X_n\},\text{ a partition of } X} \ \sum_{F,\text{ compatible with } \{X_1,\ldots,X_n\}} (\cdots).$$

2. Consider any tree T of F, and let r be its root. There is a factor $\varepsilon g'(\varepsilon t_r)$ from the root derivative at r. Each of the other factors $g(\varepsilon t_i)$ for $i \neq r$ can be replaced by $g(\varepsilon t_r)$ because hypothesis (4.2) makes any t_{ij}-derivative vanish for $t_{ij} \geq$ const. This forces all z_i, with i a vertex in T, to be equal to within $O(1)$, and all $g(\varepsilon t_i)$ to be equal to within $O(\varepsilon)$.

3. From the last item, and the sum over r in T (which comes from the sum over R), there arises a factor $(-\varepsilon N(T)/\pi) g'(\varepsilon t_r) g^{N(T)-1}(\varepsilon t_r)$ for each tree. (Here $N(T) = |T| + 1$ denotes the number of vertices in T.) This is a very "flat" probability density on $\mathbb{C}$. The trees are distributed in z-space according to the product of these probability densities.

4. As $\varepsilon \to 0$ the probability that any pair of trees is within distance $o(\varepsilon^{-1})$ tends to zero. Thus, except for a set of vanishingly small measure, $J(X, t)$ factors into a product of terms, one for each tree on an X_i.

5. In the limit $\varepsilon \to 0$, $\sum_R \int_{\mathbb{C}^N} f^{(F,R)}(d^2z/(-\pi))^X$ equals the product over trees $T \subset F$ of factors
$$I(T) := \Big(-\frac{1}{\pi}\Big)^{|T|} \int_{\mathbb{C}^{N(T)}/\mathbb{C}} J^{(T)}(X_T, \mathbf{t}), \tag{4.6}$$
where X_T is the set of vertices in T.

6. The sum over forests factors into independent sums over trees on each of the X_i. It follows that $\Sigma_{T \text{ on } X} I(T)$ solves the recursion (4.1); therefore it must be $J_c(X, \mathbf{0})$. □

5. Proof of the main results

We prove Theorem 1.1 (the relation between the hard-core gas and branched polymers) by applying the tree formula for the connected parts to the Mayer expansion:

Theorem 5.1 ([May40]). *The formal power series for the logarithm of the partition function is given by*
$$\log Z_{\mathrm{HC}}(\mathrm{z}) = \sum_{N\geq 1} \frac{\mathrm{z}^N}{N!} \int (d^D x)^N J_c(\{1, \ldots, N\}, \mathbf{x}). \tag{5.1}$$

Proof of Theorem 1.1. The hard-core constraint for particles with labels in X can be written as
$$J(X, \mathbf{x}) = \prod_{ij \in X} \mathbb{1}_{\{|x_{ij}|^2 \geq 1\}}. \tag{5.2}$$
Let
$$J(X, \mathbf{x}, \mathbf{t}) = \prod_{ij \in X} \mathbb{1}_{\{|x_{ij}|^2 + t_{ij} \geq 1\}}. \tag{5.3}$$
Replace each indicator function by a smooth approximation and apply Corollary 4.1, noting that
$$\mathbb{1}_{\{|x_{ij}|^2 + z_{ij}\bar{z}_{ij} \geq 1\}} \tag{5.4}$$
is a hard-core condition in $D+2$ dimensions, and each t_{ij}-derivative becomes $\frac{1}{2}$ surface measure when the smoothing approximation is removed by taking a limit outside the integrals. If we put $y_i = (x_i, z_i)$, a $(D+2)$-dimensional vector, then, by Theorem 5.1,
$$\log\ Z_{\mathrm{HC}}(\mathrm{z}) = \sum_{N\geq 1} \frac{\mathrm{z}^N}{N!} \sum_{T \text{ on } \{1,\ldots,N\}} \Big(-\frac{1}{\pi}\Big)^{N-1} \int dx_1 \prod_{ij\in T} \Big[\frac{1}{2}\, d\Omega(y_{ij})\Big] \prod_{ij \notin T} \mathbb{1}_{\{|y_{ij}| \geq 1\}}, \tag{5.5}$$

where the integral is over $(x_1,\dots,x_N)\in\Lambda^N$ and $(z_2,\dots,z_N)\in\mathbb{R}^{2[N-1]}$ and $z_1=0$. Consider the integrations over $y_2,\dots,y_N$. (i) By the monotone convergence theorem the infinite volume limit as $\Lambda\to\mathbb{R}^D$ exists for each term in the sum over N. (ii) By translation invariance the limit is independent of x_1 which is set equal to zero. (iii) Division by $|\Lambda|$ cancels the remaining dx_1 integration over Λ. (iv) By absolute convergence of the sum over N the infinite volume limit can also be exchanged with the sum over N. Theorem 1.1 is proved. □

A similar argument can be used to prove Theorem 1.3. It is necessary to relax the condition (4.2) in Corollary 4.1. The proof of Corollary 4.1 remains valid if J has a clustering property:

$$J(X\cup Y)\to J(X)J(Y) \text{ when all } t_{ij}\to\infty \text{ for } i\in X, j\in Y, \tag{5.6}$$

and if a similar statement holds with $(\frac{\partial}{\partial t})^F$applied. This is satisfied for

$$J(X,\mathbf{x},\mathbf{t})=\prod_{ij\in X}e^{-v(|x_{ij}|^2+t_{ij})}=\prod_{ij\in X}e^{-v(|y_{ij}|^2)}, \tag{5.7}$$

provided v, v' vanish at infinity. We further assume that $v'(|y|^2)$ is an integrable function of $y\in\mathbb{R}^{D+2}$. When evaluating $J^{(T)}$ in (4.3), the factors $-v'(|y_{ij}|^2)$ ensure convergence of (1.16). An extra factor of 2 has been inserted in (1.16) so that the combination $-\frac{\mathrm{z}}{2\pi}$ appears in (1.18). □

Proof of Theorem 1.4. If $\prod dx_j$ is replaced by $\prod\exp(h(x_j))dx_j$ in the definition (1.4) of Z_{HC} the proof of (5.5) generalizes to

$$\log Z_{\mathrm{HC}}(\mathrm{z}e^h)=\sum_{N\geq 1}\frac{\mathrm{z}^N}{N!}\sum_{T\text{ on }\{1,\dots,N\}}\left(-\frac{1}{\pi}\right)^{N-1}$$

$$\int dx_1\prod_{ij\in T}\left[\frac{1}{2}\,d\Omega(y_{ij})\right]\prod_{ij\notin T}\mathbb{1}_{\{|y_{ij}|\geq 1\}}\prod_j\exp(h(x_j)),$$

where the integral is over $(x_1,\dots,x_N)\in\Lambda^N$ and $(z_1,\dots,z_N)\in\mathbb{R}^{2N}/\mathbb{R}^2$. We differentiate with respect to α,β at zero with $h=\alpha h_1+\beta h_2$ and h_i compactly supported. The left-hand side becomes the finite-volume Green's function $G_{\mathrm{HC},\Lambda}(d\tilde{y}_1,d\tilde{y}_2;\mathrm{z})$ integrated against the test functions $h_1(\tilde{x}_1)$ and $h_2(\tilde{x}_2)$, and the right-hand side becomes

$$-2\pi\sum_{N=1}^{\infty}\frac{1}{N!}\left(-\frac{\mathrm{z}}{2\pi}\right)^N\sum_{T\text{ on }\{1,\dots,N\}}\int\prod_{ij\in T}d\Omega(y_{ij})\rho(h_1)\rho(h_2), \tag{5.8}$$

where $\rho(h)=\sum h(x_j)$ and the integral is over $x_j\in\Lambda$ for $j=1,\dots,N$ and $(z_1,\dots,z_N)\in\mathbb{R}^{2N}/\mathbb{R}^2$. The integration over $\mathbb{R}^{2N}/\mathbb{R}^2$ can be rewritten as an integral over $\mathbb{R}^{2N}$ by replacing $\rho(h_1)$ by $\sum h_1(x_j)\delta(z_j)$. The test functions localize at least one x_j in their support, so as in the proof of Theorem 1.1, the infinite volume limit $\Lambda \to \mathbb{R}^D$ exists by the monotone and dominated convergence theorems. The limit is easily verified to be

$$\int G_{\mathrm{BP}}(d\tilde{y}_1, d\tilde{y}_2; \mathrm{z}) h_1(\tilde{y}_1)\delta(\tilde{z}_1)h_2(\tilde{y}_2), \tag{5.9}$$

and this proves Theorem 1.4. □

The generalization of Theorem 1.4 to n-point functions is straightforward. On the branched polymer side of the identity all but one of the points are integrated over two extra dimensions. In Fourier space, this means that branched polymer Green's functions equate to hard-core Green's functions when all components of momenta for the extra dimensions are set to zero.

6. Proof of the Forest-Root formula

Define the differential forms

$$\tau_{ij} = z_{ij}\bar{z}_{ij} + dz_{ij}d\bar{z}_{ij}/(2\pi i), \tag{6.1}$$

$$\tau_i = z_i\bar{z}_i + dz_i d\bar{z}_i/(2\pi i). \tag{6.2}$$

Forms are multiplied by the wedge product. Suppose $g(t_1)$ is a smooth function on the real line. Then we define a new form by the Taylor series

$$g(\tau_1) = g(z_1\bar{z}_1) + g'(z_1\bar{z}_1)dz_1 d\bar{z}_1/(2\pi i), \tag{6.3}$$

which terminates after one term because all higher powers of $dz_1 d\bar{z}_1$ vanish. More generally, given any smooth function of the variables $(t_i), (t_{ij})$, we define $g(\tau)$ by the analogous multivariable Taylor expansion. By *definition*, integration over $\mathbb{C}^N$ of forms is zero on all forms of degree not equal to $2N$.

The following lemma exploits the supersymmetry of this setup to localize the evaluation of integrals on $\mathbb{C}^N$ to the origin. It will be proved in Section 7.

LEMMA 6.1 (supersymmetry and localization). *For f smooth and compactly supported*,

$$\int_{\mathbb{C}^N} f(\tau) = f(0). \tag{6.4}$$

Let G be any graph on vertices $\{1, 2, \ldots, N\}$. Define

$$(dzd\bar{z})^G = \prod_{ij\in G} dz_{ij}d\bar{z}_{ij}, \tag{6.5}$$

and analogously, for R any subset of vertices

$$(dzd\bar{z})^R = \prod_{i\in R} dz_i d\bar{z}_i. \tag{6.6}$$

The Taylor series that defines $f(\tau)$ can be written in the form

$$f(\tau) = \sum_{G,R} f^{(G,R)}(z\bar{z}) \left(\frac{dzd\bar{z}}{2\pi i}\right)^G \left(\frac{dzd\bar{z}}{2\pi i}\right)^R, \tag{6.7}$$

where G is summed over all graphs and R is summed over all subsets of vertices.

- $(dzd\bar{z})^G = 0$ if G contains a loop L because $\sum_{ij\in L} z_{ij} = 0$. Therefore G must be a forest.
- $(dzd\bar{z})^F(dzd\bar{z})^R$ has degree $2N$ if and only if R has the same number of vertices as there are trees in F. This is because a tree on m vertices has $m-1$ lines.
- Each tree contains exactly one vertex from R, because, if T is a tree which includes two vertices a, b from R then $(dzd\bar{z})^T(dzd\bar{z})^R = 0$ since $z_a - z_b$ is a sum of z_{ij} over ij in the path in T joining a to b.

By these considerations Theorem 3.1 is reduced to:

Lemma 6.2.

$$\left(\frac{dzd\bar{z}}{2\pi i}\right)^F \left(\frac{dzd\bar{z}}{2\pi i}\right)^R = \frac{d^2z_1}{-\pi} \cdots \frac{d^2z_N}{-\pi}. \tag{6.8}$$

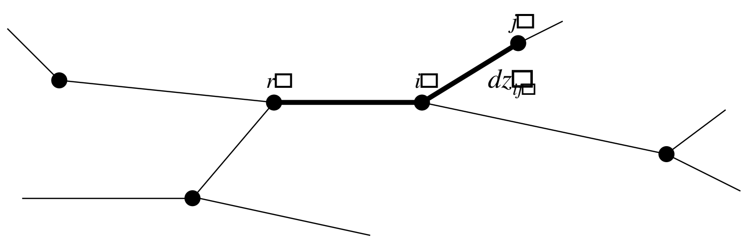

Figure 4. The unique path property

Proof. Suppose, by changing the labels if necessary, that vertices are labeled in such a way that as one traverses any path in F starting at a root r, the vertices one encounters have increasing labels. Thus, in the figure, $r < i < j$. Let $j = N$. Then z_{ij} may be replaced by z_j because

$$dz_j = \sum_{kl \text{ in path}} dz_{kl} + dz_{\text{root}}, \tag{6.9}$$

and $(dzd\bar{z})^F(dzd\bar{z})^R$ already contains dz_{root} and the other terms in the path. This argument may be repeated for j decreasing through $N-1, N-2, \ldots 1$. The lemma then follows from the fact that if $z = x+iy$, then $dzd\bar{z} = -2i\,dx\,dy$. $\square$

7. Equivariant flows and dimensional reduction

Proof of Lemma 6.1 (the supersymmetry/localization lemma). We prove the identity $\int_{\mathbb{C}^N} f(\tau) = f(0)$ first in the special case $N = 1$ so that $f(\tau) = f(\tau_1)$:

$$\int_{\mathbb{C}} f(\tau_1) = \underbrace{\int_{\mathbb{C}} f(z_1\bar{z}_1)}_{0 \text{ by definition}} + \int_{\mathbb{C}} f'(z_1\bar{z}_1)dz_1 d\bar{z}_1/(2\pi i) \tag{7.1}$$
$$= -2\int_0^\infty f'(r^2)rdr = f(0).$$

Note that this proof for $N = 1$ generalizes to the case where f depends only on vertex variables $\tau_1, \ldots, \tau_N$. The remaining argument is a reduction to this case borrowing ideas from the proof of the Duistermaat-Heckmann theorem in [AB84], as explained in [Wit92].

There is a flow on $\mathbb{C}^N$: $z_j \longmapsto e^{-2\pi i\theta} z_j$. Let V be the associated vector field and let i_V be the associated interior product which is an antiderivation on forms. By definition $i_V dz_i = -2\pi i z$ and $i_V d\bar{z}_i = 2\pi i\bar{z}$. Let $\mathcal{L}_V$ be the associated Lie derivative on forms:

$$\mathcal{L}_V dz_i = \frac{d}{d\theta}\bigg|_{\theta=0} d(e^{-2\pi i\theta} z_i) = -2\pi i dz_i, \qquad \mathcal{L}_V d\bar{z}_i = 2\pi i d\bar{z}_i. \tag{7.2}$$

It is a derivation on forms. Define the antiderivation $Q = d + i_V$ and note that $Q^2 = di_V + i_V d = \mathcal{L}_V$ by Cartan's formula for $\mathcal{L}_V$.

1. $\tau_{ij} = Qu_{ij}$ with $u_{ij} = z_{ij}d\bar{z}_{ij}/(2\pi i)$.
2. $Q\tau_{ij} = Q^2 u_{ij} = \mathcal{L}_V u_{ij} = 0$ because u_{ij} is invariant under the flow.
3. For any smooth function g, $Qg(\tau) = \sum g^{(ij)}(\tau)\, Q\tau_{ij} = 0$.
4. Fix any bond ij and define $f(\beta, \tau)$ by replacing τ_{ij} in $f(\tau)$ by $\beta\tau_{ij}$. Then
$$\frac{d}{d\beta} f(\beta, \tau) = d \text{ (some form)} + \text{(a form of degree} < 2N), \tag{7.3}$$
because the β-derivative is $f^{(ij)}(\beta, \tau)\tau_{ij}$ which equals
$$f^{(ij)}(\beta, \tau)Qu = Q(f^{(ij)}(\beta, \tau)u), \tag{7.4}$$
and $Q = d + i_V$ and i_V lowers the degree by one.
5. $\frac{d}{d\beta}\int_{\mathbb{C}^N} f(\beta, \tau) = 0$ because the integral annihilates the part of lower degree and also annihilates the part in the image of d by Stokes' theorem.

Using the assumption that f has compact support in each t_i, we can interchange the limit $\beta \to 0$ with the integral over $\mathbb{C}^N$. Thus every τ_{ij} in f can be deformed to 0 and Lemma 6.1 is reduced to the case where f is a function only of $(\tau_1, \tau_2, \ldots, \tau_N)$. □

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