Annals of Mathematics, **158** (2003), 953–976

A C^2-smooth counterexample to the Hamiltonian Seifert conjecture in $\mathbb{R}^4$

By Viktor L. Ginzburg and Başak Z. Gürel*

Abstract

We construct a proper C^2-smooth function on $\mathbb{R}^4$ such that its Hamiltonian flow has no periodic orbits on at least one regular level set. This result can be viewed as a C^2-smooth counterexample to the Hamiltonian Seifert conjecture in dimension four.

1. Introduction

The "Hamiltonian Seifert conjecture" is the question whether or not there exists a proper function on $\mathbb{R}^{2n}$ whose Hamiltonian flow has no periodic orbits on at least one regular level set. We construct a C^2-smooth function on $\mathbb{R}^4$ with such a level set. Following the tradition of [Gi4], [He1], [He2], [Ke], [KuG], [KuGK], [KuK1], [KuK2], [Sc], we can call this result a C^2-smooth counterexample to the Hamiltonian Seifert conjecture in dimension four. We emphasize that in this example the Hamiltonian vector field is C^1-smooth while the function is C^2.

In dimensions greater than six, C^∞-smooth counterexamples to the Hamiltonian Seifert conjecture were constructed by one of the authors, [Gi1], and simultaneously by M. Herman, [He1], [He2]. In dimension six, a $C^{2+\alpha}$-smooth counterexample was found by M. Herman, [He1], [He2]. This smoothness constraint was later relaxed to C^∞ in [Gi2]. A very simple and elegant construction of a new C^∞-smooth counterexample in dimensions greater than four was recently discovered by E. Kerman, [Ke]. The flow in Kerman's example has dynamics different from the ones in [Gi1], [Gi2], [He1], [He2]. We refer the reader to [Gi3], [Gi4] for a detailed discussion of the Hamiltonian Seifert conjecture. The reader interested in the results concerning the original Seifert conjecture settled by K. Kuperberg, [KuGK], [KuK1], should consult [KuK2], [KuK3]. Here we only mention that a C^1-smooth counterexample to the Seifert conjecture on S^3 was constructed by P. Schweitzer, [Sc]. Later, the smooth-

*This work was partially supported by the NSF and by the faculty research funds of the University of California, Santa Cruz.

ness in this example was improved to C^2 by J. Harrison, [Ha]. A C^1-smooth volume-preserving counterexample on S^3 was found by G. Kuperberg, [KuG]. The ideas from both P. Schweitzer's and G. Kuperberg's constructions play an important role in this paper.

An essential difference of the Hamiltonian case from the general one is manifested by the almost existence theorem, [HZ1], [HZ2], [St], which asserts that almost all regular levels of a proper Hamiltonian have periodic orbits (see Remark 2.3). In other words, regular levels without periodic orbits are exceptional in the sense of measure theory.

The existence of a C^2-counterexample to the Hamiltonian Seifert conjecture in dimension four was announced by the authors in [GG], where a proof was also outlined. Here we give a detailed construction of this counterexample.

Acknowledgments. The authors are deeply grateful to Helmut Hofer, Anatole Katok, Ely Kerman, Krystyna Kuperberg, Mark Levi, Debra Lewis, Rafael de la Llave, Eric Matsui, and Maria Schonbek for useful discussions and suggestions.

2. Main results

Recall that *characteristics* on a hypersurface M in a symplectic manifold (W, η) are, by definition, the (unparametrized) integral curves of the field of directions $\ker(\eta|_M)$.

Let $\mathbb{R}^{2n}$ be equipped with its standard symplectic structure.

THEOREM 2.1. *There exists a C^2-smooth embedding $S^3 \hookrightarrow \mathbb{R}^4$ which has no closed characteristics. This embedding can be chosen C^0-close and C^2-isotopic to an ellipsoid.*

As an immediate consequence we obtain

THEOREM 2.2. *There exists a proper C^2-function $F: \mathbb{R}^4 \to \mathbb{R}$ such that the level $\{F = 1\}$ is regular and the Hamiltonian flow of F has no periodic orbits on $\{F = 1\}$. In addition, F can be chosen so that this level is C^0-close and C^2-isotopic to an ellipsoid.*

Remark 2.3. Regular levels of F without periodic orbits are exceptional in the sense that the set of corresponding values of F has zero measure. This is a consequence of the almost existence theorem, [HZ1], [HZ2], [St], which guarantees that for a C^2-smooth (and probably even C^1-smooth) function, periodic orbits exist on a full measure subset of the set of regular values. In particular, since all values of F near $F = 1$ are regular, almost all levels of F near this level carry periodic orbits.

Remark 2.4. It is quite likely that our construction gives an embedding $S^3 \hookrightarrow \mathbb{R}^4$ without closed characteristics, which is $C^{2+\alpha}$-*smooth.*

Remark 2.5. Similarly to its higher-dimensional counterparts, [Gi1], [Gi2], Theorem 2.1 extends to other symplectic manifolds as follows. Let (W, η) be a four-dimensional symplectic manifold and let $i: M \hookrightarrow W$ be a C^∞-smooth embedding such that $i^*\eta$ has only a finite number of closed characteristics. Then there exists a C^2-smooth embedding $i': M \hookrightarrow W$, which is C^0-close and isotopic to i, such that $i'^*\eta$ has no closed characteristics.

The rest of the paper is devoted to the proof of Theorem 2.1. The idea of the proof is to adjust Schweitzer's construction, [Sc], of an aperiodic C^1-flow on S^3 to make it embeddable into $\mathbb{R}^4$ as a Hamiltonian flow. This is done by introducing a Hamiltonian version of Schweitzer's plug. More specifically, the flow on Schweitzer's plug is defined as the Hamiltonian flow of a certain multi-valued function K which we use to find a symplectic embedding of the plug (see Proposition 3.2 and Remark 3.4). The existence of such a function K depends heavily on the choice of a Denjoy vector field in Schweitzer's plug. In fact, the Denjoy vector field is required to be essentially as smooth as a Denjoy vector field can be (see Remark 6.2). Implicitly, the idea to define the flow on Schweitzer's plug using the Hamilton equation goes back to G. Kuperberg's paper [KuG].

As of this moment we do not know if G. Kuperberg's flow can be embedded into $\mathbb{R}^4$. The two constructions differ in an essential way. The Denjoy flow and the function K in G. Kuperberg's example are required to have properties very different from the ones we need. As a consequence, our method to embed the plug into $\mathbb{R}^4$ does not apply to G. Kuperberg's plug. (For example, one technical but essential discrepancy between the methods is as follows. In G. Kuperberg's construction, it is important to take a rotation number which cannot be too rapidly approximated by rationals, while the Denjoy map is not required to be smoother than just C^1. On the other hand, in our construction the value of a rotation number is irrelevant, but the smoothness of the Denjoy map plays a crucial role.)

The proof is organized as follows. In Section 3 we describe the symplectic embedding of Schweitzer's flow assuming the existence of the plug with required properties. In Sections 4 and 5 we derive the existence of such a flow on the plug from the fact (Lemma 5.2) that there exists a "sufficiently smooth" Denjoy flow on T^2. Finally, this "sufficiently smooth" Denjoy flow is constructed in Section 6.

3. Proof of Theorem 2.1: The symplectic embedding

Let us first fix the notation. Throughout this paper σ denotes the standard symplectic form on $\mathbb{R}^{2m}$ or the pull-back of this form to $\mathbb{R}^{2m+1}$ by the projection $\mathbb{R}^{2m+1} \to \mathbb{R}^{2m}$ along the first coordinate; I^{2m} stands for a cube in $\mathbb{R}^{2m}$ whose edges are parallel to the coordinate axes. The product $[a, b] \times I^{2m}$ is always assumed to be embedded into $\mathbb{R}^{2m+1}$ (henceforth, the standard embedding) so that the interval $[a, b]$ is parallel to the first coordinate. We refer to the direction along the first coordinate t (time) in $\mathbb{R}^{2m+1}$ (or $[a, b]$ in $[a, b] \times I^{2m}$) as the vertical direction.

All maps whose smoothness is not specified are C^∞-smooth.

Theorem 2.1 follows, as do similar theorems in dimensions greater than four, from the existence of a symplectic plug. The definitions of a plug vary considerably (see [Gi1], [Ke], [KuG]), and here we use the one more suitable for our purposes.

A *C^k-smooth symplectic plug* in dimension $2n$ is a C^k-embedding J of $P = [a, b] \times I^{2n-2}$ into $P \times \mathbb{R} \subset \mathbb{R}^{2n}$ such that the following conditions hold:

P1. *The boundary condition*: The embedding J is the identity embedding of P into $\mathbb{R}^{2n-1}$ near the boundary ∂P. Thus the characteristics of $J^*\sigma$ are parallel to the vertical direction near ∂P.

P2. *Aperiodicity*: The characteristic foliation of $J^*\sigma$ is aperiodic, i.e., $J^*\sigma$ has no closed characteristics.

P3. *Trapped trajectories*: There is a characteristic of $J^*\sigma$ beginning on $\{a\} \times I^{2n-2}$ that never exits the plug. Such a characteristic is said to be trapped in P.

P4. *The embedding* J is C^0-close to the standard embedding and C^k-isotopic to it.

P5. *Matched ends or the entrance-exit condition*: If two points (a, x), the "entrance", and (b, y), the "exit", are on the same characteristic, then $x = y$. In other words, for a characteristic that meets both the bottom and the top of the plug, its top end lies exactly above the bottom end.

THEOREM 3.1. *In dimension four, there exists a C^2-smooth symplectic plug.*

Proof of Theorem 2.1. Theorem 2.1 readily follows from Theorem 3.1. Consider an irrational ellipsoid in $\mathbb{R}^4$ and pick two little balls each of which is centered at a point on a closed characteristic on the ellipsoid. Intersections of these balls with the ellipsoid can be viewed symplectically as open subsets in $\mathbb{R}^3$. By scaling the plug we can assume that $[a, b] \times I^2$ can be embedded into each of these open balls so that the closed characteristic on an ellipsoid matches a trapped trajectory in the plug. Now we perturb the ellipsoid by means of the embedding J within each of these open subsets. The resulting embedding has no closed characteristics, C^0-close to the ellipsoid and C^2-isotopic to it. $\square$

Proof of Theorem 3.1. First observe that it suffices to construct a semi-plug, i.e., a "plug" satisfying only the conditions (P1)–(P4). Indeed, a plug can then be obtained by combining two symmetric semi-plugs. More precisely, suppose that a semi-plug with embedding J_- has been constructed. Without loss of generality we may assume that $[a,b] = [-1,0]$. Define a semi-plug on $[0,1] \times I^2$ with embedding J_+ by setting $J_+(t,x) = RJ_-(-t,x)$, $t \in [0,1]$ and $x \in I^2$, where R is the reflection of $\mathbb{R}^4$ in $\mathbb{R}^3$. Combined together, these semi-plugs give rise to a plug on $[-1,1] \times I^2$.

We will construct a semi-plug by perturbing the standard embedding of $[a,b]\times I^2$ on a subset $M \subset [a,b]\times I^2$. This subset is diffeomorphic to $[-1,1]\times\Sigma$, where Σ is a punctured torus.

It is more convenient to perform this perturbation using slightly different "coordinates" on a neighborhood of M. More specifically, we will first consider an embedding of M into another four-dimensional symplectic manifold (W, σ_W) such that the pull-back of σ_W is still $\sigma|_M$. Then we C^0-perturb this embedding so that the characteristic vector field of the new pull-back will have properties similar to those of Schweitzer's plug. By the symplectic neighborhood theorem, a neighborhood of M in W is symplectomorphic to that of M in $\mathbb{R}^4$. This will allow us to turn the embedding $M \hookrightarrow W$ into the required embedding $J\colon M \hookrightarrow \mathbb{R}^4$. (See the diagrams (3.1) and (3.2) below.)

To construct the perturbed embedding $M \hookrightarrow W$, we first embed M into $[-1,1] \times T^2$ by puncturing the torus in a suitable way. Then we find a map $j\colon [-1,1]\times T^2 \to W$ such that the characteristic vector field of $j^*\sigma_W$ is aperiodic and has trapped trajectories.

The embedding j is constructed as follows. Let (x,y) be coordinates on T^2. Consider the product $W = (-2,2) \times S^1 \times T^2$ with coordinates (t,x,u,y) and symplectic form $\sigma_W = dt\wedge dx + du\wedge dy$. The map j is a C^0-small perturbation of

$$j_0\colon [-1,1] \times T^2 \to W; \quad j_0(t,x,y) = (t,x,x,y).$$

Note that $j_0(t,x,y) = (t,x,K_0,y)$, where $K_0(t,x,y) = x$. To define j, let us replace K_0 by a mapping $K\colon [-1,1] \times T^2 \to S^1$ to be specified later on. In other words, set

$$j\colon [-1,1] \times T^2 \to (-2,2) \times S^1 \times T^2, \quad \text{where} \quad j(t,x,y) = (t,x,K,y).$$

It is clear that j is an embedding. (An explanation of the origin of j is given in Remark 3.4.) The pull-back $j^*\sigma_W$ is the form

$$j^*\sigma_W = dt \wedge dx + (\partial_x K) dx \wedge dy + (\partial_t K) dt \wedge dy$$

with characteristic vector field

$$v = (\partial_x K)\partial_t - (\partial_t K)\partial_x + \partial_y.$$

To ensure that (P1)–(P4) hold we need to impose some requirements on K.

To specify these requirements, consider a Denjoy vector field $\partial_y + h\partial_x$ on T^2. This vector field should satisfy certain additional conditions which will be detailed in Section 6. Denote by $\mathfrak{D}$ the Denjoy continuum for this field. (Recall that $\mathfrak{D}$ is the closure of a trajectory of the Denjoy vector field; see Section 6.1 for the precise definition.)[1] Pick a point (x_0, y_0) in the complement of $\mathfrak{D}$. Fix a small, disjoint from $\mathfrak{D}$, neighborhood V_0 of (x_0, y_0). Consider the tubular neighborhood of the line $(t, x_0, y_0 + t)$ in $[-1, 1] \times T^2$ of the form $\{(t, x, y + t) \mid (x, y) \in V_0,\ t \in [-1, 1]\}$. Fix also a small neighborhood of the boundary $\partial([-1, 1] \times T^2)$ and denote by N the union of these neighborhoods.

PROPOSITION 3.2. *There exists a C^2-smooth mapping $K\colon [-1,1]\times T^2 \to S^1$ such that*

- K1. *v is equal to the Denjoy vector field (i.e., $\partial_x K = 0$ and $\partial_t K = -h$) at every point of $\{0\} \times \mathfrak{D}$;*
- K2. *the t-component of v is positive (i.e., $\partial_x K > 0$) on the complement of $\{0\} \times \mathfrak{D}$;*
- K3. *K is C^0-close[2] to the map $K_0\colon (t, x, y) \mapsto x$;*
- K4. *$K = K_0$ on N.*

Let us defer the proof of the proposition to Section 4 and finish the proof of Theorem 3.1. From now on we assume that K is as in Proposition 3.2.

By (K1) and (K2), v has a trapped trajectory and is aperiodic. Indeed, by (K1), $\{0\} \times \mathfrak{D}$ is invariant under the flow of v and on this set the flow is a Denjoy flow. By (K2), the vertical component of v is nonzero unless the point is in $\{0\} \times \mathfrak{D}$. This implies that periodic orbits can only occur within $\{0\} \times \mathfrak{D}$. Since the Denjoy flow is aperiodic, so is the entire flow of v. Furthermore, it is easy to see that since $\{0\} \times \mathfrak{D}$ is invariant, there must be a trapped trajectory. Furthermore, $v = \partial_t + \partial_y$ on N by (K4).

Now we are in a position to define J. Let Σ be the torus T^2 punctured at (x_0, y_0). To be more accurate, Σ is obtained by deleting a neighborhood of (x_0, y_0), contained in V_0. There exists a symplectic bridge immersion of $(\Sigma, dx \wedge dy)$ into some cube I^2 with the standard symplectic structure. Hence, there exists an embedding

$$M = [-1, 1] \times \Sigma \hookrightarrow [a, b] \times I^2 \subset \mathbb{R}^3 \subset \mathbb{R}^4$$

such that the pull back of σ is $dx \wedge dy$. Henceforth, we identify M with its image in $\mathbb{R}^4$.

---

[1]We also refer the reader to [HS], [KH], [Sc] for a discussion of Denjoy maps and vector fields.

[2]More specifically, for any $\varepsilon > 0$ there exists K satisfying (K1)–(K2) and (K4) such that $\| K - K_0 \| < \varepsilon$. The required value of ε is determined by the size of the neighborhood U in the symplectic neighborhood theorem; see below.

On the other hand, we can embed M into $[-1,1]\times T^2$ by means of

$$\varphi\colon M=[-1,1]\times\Sigma\to[-1,1]\times T^2;\quad \varphi(t,x,y)=(t,x,y+t).$$

Then $\varphi_*\partial_t=\partial_t+\partial_y$ and $(j_0\varphi)^*\sigma_W=dx\wedge dy$. The argument, similar to the proof of the symplectic neighborhood theorem, [McDS, Lemma 3.14], shows (see [Gi1, Section 4] for details) that a "neighborhood" of M in $\mathbb{R}^4$ is symplectomorphic to a "neighborhood" U of $j_0\varphi(M)$ in W. More precisely, for a small $\delta>0$, there exists a symplectomorphism

$$\psi\colon M\times(-\delta,\delta)\to U\subset W$$

extending $j_0\varphi$, i.e., such that $\psi|_M=j_0\varphi$. These maps form the following diagram:

$$\begin{array}{ccccc} M & \hookrightarrow & (-\delta,\delta)\times M & \subset & \mathbb{R}^4\\ \| & & \downarrow\psi & & \\ M & \xrightarrow{j_0\varphi} & U & \subset & W\end{array}\tag{3.1}$$

By (K3), j is C^0-close to j_0. Furthermore, $j=j_0$ on N by (K4). Hence, j can be assumed to take values in U (see Remark 3.3).

Finally, set

$$J=\psi^{-1}j\varphi$$

on M. In other words, J is defined by the diagram:

$$\begin{array}{ccccc} M & \xrightarrow{J} & (-\delta,\delta)\times M & \subset & \mathbb{R}^4\\ \| & & \downarrow\psi & & \\ M & \xrightarrow{j\varphi} & U & \subset & W\end{array}\tag{3.2}$$

Then $(J^*\sigma)|_M=(j\varphi)^*\sigma_W$. To finish the definition of J, we extend it as the standard embedding to $[a,b]\times I^2\smallsetminus M$.

The characteristic vector field of $J^*\sigma$ is ∂_t in the complement of M and $(\varphi^{-1})_*v$ on M. Since $(\varphi^{-1})_*v=\partial_t$ near ∂M, these vector fields match smoothly at ∂M. It is clear that (P1) is satisfied. Since v has a trapped trajectory and is aperiodic, the same is true for $(\varphi^{-1})_*v$; i.e., the conditions (P2) and (P3) are met. The condition (P4) is easy to verify. Hence, J is indeed a semi-plug. □

Remark 3.3. The following argument shows in more detail why j can be assumed to take values in U. Let us slightly shrink M by enlarging the puncture in T^2 and shortening the interval $[-1,1]$. Denote the resulting manifold with corners by M'. The shrinking is made so that $\partial M'\subset N$ and hence $M\smallsetminus M'\subset N$. It follows that U contains a genuine neighborhood U' of $j_0\varphi(M')$. Thus, if K is sufficiently C^0-close to K_0, we have $j(\varphi(M'))\subset U'$. On $j_0\varphi(M\smallsetminus M')$, we have $K=K_0$ by (K5) and hence $j=j_0$. Therefore, $j(\varphi(M))\subset U$.

Remark 3.4. The definition of the embedding j can be explained as follows. Let us view the annulus $[-1,1]\times S^1$ with symplectic form $dt\wedge dx$ as a symplectic manifold and the product $[-1,1]\times T^2$ as the extended phase space with the y-coordinate as the time-variable. Then we can regard K as a (multi-valued) time-dependent Hamiltonian on $[-1,1]\times S^1$. The embeddings j_0 and j identify the coordinates t, x, and y on $[-1,1]\times T^2$ with those on W. Hence, we can view W as the further extended time-energy phase space with the cyclic energy-coordinate u. Then j is the graph of the time-dependent Hamiltonian K in the extended time-energy phase space W. Now it is clear that v is just the Hamiltonian vector field of K.

Remark 3.5. In the proof of Proposition 3.2 we will not require the Denjoy continuum $\mathfrak{D}$ to have zero measure. As a consequence, the union of characteristics entirely contained in the semi-plug can have Hausdorff dimension two because this set is the image of $\mathfrak{D}$ by a C^2-smooth embedding.

4. Proof of Proposition 3.2

Recall that $\partial_y + h\partial_x$ is a Denjoy vector field on T^2 whose choice will be discussed later on and $\mathfrak{D}$ is the Denjoy continuum for this field. Recall also that V_0 is a small, disjoint from $\mathfrak{D}$, neighborhood of (x_0, y_0). Fix a slightly larger neighborhood V_1 of (x_0, y_0) which contains the closure of V_0 and is still disjoint from $\mathfrak{D}$. Let $\varepsilon > 0$ be sufficiently small.

Proposition 3.2 is an immediate consequence of the following

PROPOSITION 4.1. *There exists a C^2-smooth mapping $K\colon[-\varepsilon,\varepsilon]\times T^2\to S^1$ which satisfies* (K1)–(K3) *and the requirement*

K4′. *$K = K_0$ for all t and (x, y) in the fixed neighborhood V_1 of (x_0, y_0).*

Proof of Proposition 3.2. Let K be as in Proposition 4.1. We extend this function to $[-1,1]\times T^2$ as the linear combination $\phi(t)K(t,x,y)+(1-\phi(t))x$, where ϕ is a bump function equal to 1 for t close to 0 and vanishing for t near $\pm\varepsilon$. Note that this linear combination is well defined, as an element of the short arc connecting $K(t,x,y)$ and x, due to (K3). Clearly, the linear combination satisfies (K1)–(K3). If the range of t, for which $\phi(t) = 1$, is sufficiently small it also satisfies (K4). □

Proof of Proposition 4.1.

Step 1: *The extension of h to* $[-1,1]\times T^2$. Our first goal is to extend h from T^2 to $H\colon[-1,1]\times T^2\to\mathbb{R}$ smoothly and so that $\partial_x H - \partial_x h$ is of order one in t.

LEMMA 4.2. *Assume that α is sufficiently close to* 1 *and h is $C^{1+\alpha}$. Then there exists a C^1-function $H: [-1, 1] \times T^2 \to \mathbb{R}$ such that*

H1. $H(0, x, y) = h(x, y)$;

H2. $\partial_x H(t, x, y) = \partial_x h(x, y) + o(t)$ *uniformly in* (x, y);

H3. *the function* $\int_0^t H(\tau, x, y)\, d\tau$ *is* C^2 *in* (t, x, y).

At this moment only the assertion of Lemma 4.2 is essential and we defer its proof to Section 5.

Remark 4.3. Since H is only C^1-smooth, the condition (H2) does not hold automatically. However, as is easy to see from the proof of the lemma, one can find an extension H such that $\partial_x H(t, x, y) = \partial_x h(x, y) + o(t^k)$ for any given k and (H3) still holds, provided that α is sufficiently close 1 (in fact, $k/(k + 1) < \alpha < 1$).

Step 2: *The definition of K.* From now on we fix the extension H, but allow the interval $[-\varepsilon, \varepsilon]$, on which it is considered, to vary. We will construct the function K of the form

$$K(t, x, y) = \int_0^t [-H(\tau, x, y) + f(x, y)\tau]\, d\tau + A(x, y), \tag{4.1}$$

where the "constant" of integration A and the correction function f are chosen so as to make (K1)–(K3) and (K4′) hold. Note that A is actually a function $T^2 \to S^1$, whereas H and f are real valued functions. The main difficulty in the proof below comes from the combination of the conditions (K1) and (K2).

Step 3: *The auxiliary functions A and f.* Let us now specify the requirements the functions A and f have to meet.

LEMMA 4.4. *There exist a C^2-function $A: T^2 \to S^1$ and C^∞-function $f: T^2 \to \mathbb{R}$ satisfying the following conditions*:

A1. $\partial_x A \geq \eta(\partial_x h)^2$ *for some constant $\eta > 0$ and $\partial_x A$ vanishes exactly on the Denjoy set* $\mathfrak{D}$;

A2. *there exists an open set $U \subset T^2$, containing $\mathfrak{D}$, such that $U \cap V_1 = \emptyset$ and*

$$\partial_x A|_{T^2 \setminus U} \geq \mathit{const} > 0, \tag{4.2}$$

$$\partial_x f|_U \geq 4\eta^{-1} + 2; \tag{4.3}$$

A3. *A is C^0-close to $(x, y) \mapsto x$*;

A4. $A(x, y) = x$ *for* $(x, y) \in V_1$.

This lemma will also be proved in Section 5.

Remark 4.5. More specifically, the condition (A3) means that for fixed h and V_1 one can find A arbitrarily C^0-close to $(x, y) \mapsto x$ and satisfying other requirements of the lemma.

Step 4: *The properties of* K. Let us now prove that K given by (4.1), i.e.,

$$K = -\int_0^t H\,d\tau + \frac{t^2}{2} f + A,$$

satisfies the requirements of Proposition 4.1, provided that $\varepsilon > 0$ is small enough and A and f are as in Lemma 4.4. The function K is C^2. Indeed, the first term is C^2 by (H3). By Lemma 4.4, the next term is C^∞ and the last term, A, is C^2.

Condition (K1) is obvious: $\partial_t K|_{t=0} = -H|_{t=0} = -h$ by (H1) and $\partial_x K|_{\{0\}\times\mathfrak{D}} = \partial_x A|_{\mathfrak{D}} = 0$ by (A1).

Let us now turn to (K2). We will first show that

$$\partial_x K = -\int_0^t \partial_x H\,d\tau + \frac{t^2}{2}\partial_x f + \partial_x A \geq 0 \tag{4.4}$$

and then prove that the equality occurs only on $\{0\} \times \mathfrak{D}$.

Assume first that $(x, y) \in U$. By (H2) and (4.3), we have

$$\partial_x K \geq \partial_x A - t\partial_x h + 2\eta^{-1}t^2 + \Big(t^2 + o(t^2)\Big).$$

Obviously,

$$t^2 + o(t^2) \geq 0 \tag{4.5}$$

for $(x, y) \in U$ and all $t \in [-\varepsilon, \varepsilon]$, provided that $\varepsilon > 0$ is small. Hence, to verify (4.4), it suffices to show that

$$\partial_x A - t\partial_x h + 2\eta^{-1}t^2 \geq 0. \tag{4.6}$$

By (A1), this follows from

$$\eta(\partial_x h)^2 - t\partial_x h + 2\eta^{-1}t^2 \geq 0.$$

Here all the terms are nonnegative except, maybe, $-t\partial_x h$. Hence, it suffices to prove that at least one of the following two inequalities holds:

$$\eta(\partial_x h)^2 - t\partial_x h \geq 0, \tag{4.7}$$

$$-t\partial_x h + 2\eta^{-1}t^2 \geq 0. \tag{4.8}$$

Inequality (4.7) holds if (but not only if)

$$|t| \leq \eta|\partial_x h| \tag{4.9}$$

and (4.8) holds if (but not only if)

$$|t| \geq \frac{\eta|\partial_x h|}{2}. \tag{4.10}$$

Clearly, at least one of the inequalities (4.9) and (4.10) holds. This proves (4.4) for $(x, y) \in U$.

Assume now that $(x, y) \in T^2 \smallsetminus U$. Then, by (A2) or, more specifically, by (4.2),

$$\partial_x K = \partial_x A + O(t) > const + O(t) > 0,$$

when $\varepsilon > 0$ is small. Thus (K2) holds for $(x, y) \in (T^2 \smallsetminus U)$.

To finish the proof of (K2) we need to show that for $(x, y) \in U$ the equality in (4.4) implies that $t = 0$ and $(x, y) \in \mathfrak{D}$. Thus, assume that $(x, y) \in U$ and $\partial_x K(t, x, y) = 0$. Then (4.5) and (4.6) must become equalities. The equality (4.5) is possible only when $t = 0$. Setting $t = 0$ in the equality (4.6), we conclude that $\partial_x A(x, y) = 0$ and hence $(x, y) \in \mathfrak{D}$ by (A1).

The condition (K3) follows from (A3). Indeed, if $\varepsilon > 0$ is small, K is C^0-close to A which, in turn, is C^0-close to K_0 by (A3).

The condition (K4$'$) need not be satisfied for K. By (A3), on $[-\varepsilon, \varepsilon] \times T^2$ the function K is C^0-close to K_0, provided that $\varepsilon > 0$ is small. Moreover, by (A4), the function $\partial_x K$ is C^0-close to 1 and $\partial_y K$ is C^0-close to 0 on a neighborhood of $[-\varepsilon, \varepsilon] \times \mathrm{closure}(V_1)$, for small $\varepsilon > 0$. Now it is easy to see that (taking a smaller $\varepsilon > 0$, if necessary) we can modify K near and on $[-\varepsilon, \varepsilon] \times V_1$ so as to keep (K1)–(K3) and make the new function satisfy (K4$'$). Indeed, let $\phi: T^2 \to [0, 1]$ be a bump function equal to 1 on V_1 and 0 outside of a small neighborhood of V_1. Then the linear combination $x\phi + (1 - \phi)K$ still satisfies (K1)–(K3) and also (K4$'$) if $\varepsilon > 0$ is small enough. Note that this linear combination is well defined due to (A3). □

5. Proofs of Lemmas 4.2 and 4.4

5.1. *Proof of Lemma* 4.2. For $t \neq 0$ and $(x, y) \in T^2$, set $x_\pm = x \pm t^s/2$ and $y_\pm = y \pm t^s/2$ and define

$$H(t, x, y) = \frac{1}{t^{2s}} \int_{y_-}^{y_+} \int_{x_-}^{x_+} h(\xi, \zeta)\, d\xi d\zeta,$$

where s is an even positive integer to be specified later. Also, let $H(0, x, y) = h(x, y)$. In other words, $H(t, x, y)$ is obtained by averaging h over the square with side t^s, centered at (x, y).[3]

Condition (H2): First note that H is obviously differentiable in x and y for every t. Furthermore, it is easy to see that H satisfies (H2), i.e., $\partial_x H = \partial_x h + o(t)$, provided that

$$s\alpha > 1. \tag{5.1}$$

[3] This extension of h by averaging is somewhat similar to the one from [KuG].

Indeed, since h is continuous,

$$\partial_x H(t,x,y) = \frac{1}{t^{2s}} \int_{y_-}^{y_+} \Big(h(x_+,\zeta) - h(x_-,\zeta) \Big)\, d\zeta.$$

By the mean value theorem, $h(x_+,\zeta) - h(x_-,\zeta) = t^s \partial_x h(x_0,\zeta)$, where x_0 is some point in $[x_-,x_+]$, depending on ζ. Since the distance between (x,y) and (x_0,ζ) does not exceed $t^s/\sqrt{2}$ and $\partial_x h$ is α-Hölder, we have $|\partial_x h(x_0,\zeta) - \partial_x h(x,y)| \leq \mathit{const} \cdot (t^s)^\alpha$ with *const* independent of (x,y). Hence,

$$\begin{aligned} |\partial_x H(t,x,y) - \partial_x h(x,y)| &\leq \frac{1}{t^s} \int_{y_-}^{y_+} \Big| \partial_x h(x_0,\zeta) - \partial_x h(x,y) \Big|\, d\zeta \\ &\leq \mathit{const} \cdot t^{s\alpha}, \end{aligned}$$

where *const* is independent of (x,y). This proves (H2), provided that (5.1) holds.

C^1-*smoothness of* H: We will show that H is C^1, provided that h is C^1 and $s > 1$. (Note that (5.1) implies that $s > 1$.) The proof essentially amounts to repeated applications of the mean value theorem. However, for the sake of completeness, we give a detailed argument below.

It is clear that $\partial_x H$ is continuous in all variables for $t \neq 0$ and continuous in x and y for all t. Its continuity in t at $t = 0$ follows from (H2). Moreover, it is easy to see that $\partial_x H \to \partial_x h$ as $t \to 0$ even if h is just C^1. The same reasoning applies to $\partial_y H$.

It remains to show that $\partial_t H$ exists and is continuous. Again this is obvious for $t \neq 0$. Using the fact that h is C^1, one can easily check that

$$|H(t,x,y) - h(x,y)| \leq \mathit{const} \cdot t^s.$$

This immediately implies that $\partial_t H\, |_{t=0} = 0$ when $s > 1$. Thus, to establish the continuity of $\partial_t H$ at $t = 0$, we need to prove that $\partial_t H \to 0$ uniformly in (x,y) as $t \to 0$. A straightforward calculation shows that

$$\partial_t H(t,x,y) = \frac{s}{2t^{s+1}} \int_{y_-}^{y_+} \mathcal{J}_y(\zeta)\, d\zeta + \frac{s}{2t^{s+1}} \int_{x_-}^{x_+} \mathcal{J}_x(\xi)\, d\xi, \tag{5.2}$$

where

$$\mathcal{J}_y(\zeta) = h(x_+,\zeta) + h(x_-,\zeta) - \frac{2}{t^s} \int_{x_-}^{x_+} h(\xi,\zeta)\, d\xi$$

and, similarly,

$$\mathcal{J}_x(\xi) = h(\xi,y_+) + h(\xi,y_-) - \frac{2}{t^s} \int_{y_-}^{y_+} h(\xi,\zeta)\, d\zeta.$$

By the mean value theorem, we have

$$\begin{aligned}|\mathcal{J}_y(\zeta)| &= |h(x_+,\zeta)+h(x_-,\zeta)-2h(x_0,\zeta)| \\ &= |\partial_x h(x_2,\zeta)(x_+-x_0)-\partial_x h(x_1,\zeta)(x_0-x_-)| \\ &\le \Big(|\partial_x h(x_2,\zeta)|+|\partial_x h(x_1,\zeta)|\Big)\cdot t^s \\ &\le const\cdot t^s,\end{aligned}$$

where x_0, x_1, and x_2 are some points in $[x_-,x_+]$, depending on ζ, and the constant can be taken independent of (x,y).) As a consequence, the first term in (5.2) (whose absolute value is bounded by $const\cdot t^{s-1}$) goes to zero as $t\to 0$ if $s>1$. A similar argument shows that the second term also goes to zero. Therefore, $\partial_t H\to 0$ as $t\to 0$ uniformly in (x,y), and hence $\partial_t H$ is continuous.

Condition (H3): Let us now prove (H3), i.e., that

$$F=\int_0^t H\,d\tau$$

is C^2, under some additional constraints on s and α.

It is clear that F is C^1. Moreover, the continuity of $\partial_x H$ and $\partial_y H$ implies that $\partial_x\partial_t F=\partial_x H=\partial_t\partial_x F$ and $\partial_y\partial_t F=\partial_y H=\partial_t\partial_y F$. Thus these partial derivatives exist and are continuous.

Furthermore, we claim that the second order partial derivatives of F in x and y also exist and are continuous, provided that

$$s(1-\alpha)<1. \tag{5.3}$$

Let us examine, for example, $\partial_x\partial_y F$. Note that $F|_{t=0}=0$, and hence $\partial_x\partial_y F|_{t=0}=0$. Thus we may assume that $t\neq 0$. Clearly,

$$\partial_x\partial_y F=\partial_x\int_0^t G(\tau,x,y)\,d\tau, \tag{5.4}$$

where

$$G(\tau,x,y)=\frac{1}{\tau^{2s}}\int_{x_-}^{x_+}(h(\xi,y_+)-h(\xi,y_-))\,d\xi.$$

We claim that in (5.4) the integration in τ and ∂_x can be interchanged. Indeed,

$$\begin{aligned}\Big|\partial_x G(\tau,x,y)\Big| &= \frac{1}{\tau^{2s}}\Big|(h(x_+,y_+)-h(x_-,y_+))-(h(x_+,y_-)-h(x_-,y_-))\Big| \\ &= \frac{1}{\tau^s}\Big|\partial_x h(x_2,y_+)-\partial_x h(x_1,y_-)\Big|.\end{aligned}$$

Here x_1 and x_2 are some points whose distance to x does not exceed $\tau^s/2$ and the second equality follows from the mean value theorem. Using the fact that $\partial_x h$ is α-Hölder, we obtain

$$|\partial_x G(\tau,x,y)|\le\frac{const}{\tau^s}(\tau^s)^\alpha=\frac{const}{\tau^{s(1-\alpha)}},$$

where the constant is independent of x and y. As a consequence, if $s(1-\alpha) < 1$, i.e., (5.3) holds, the integral $\int_0^t \partial_x G(\tau, x, y)\, d\tau$ converges absolutely and uniformly in (x, y). Thus it follows from (5.3) that the derivative $\partial_x \partial_y F$ exists and

$$\partial_x \partial_y F = \int_0^t \partial_x G(\tau, x, y)\, d\tau. \tag{5.5}$$

In addition, this implies that

$$\partial_x \partial_y F \to 0 \quad \text{as } t \to 0 \text{ uniformly in } x \text{ and } y. \tag{5.6}$$

Let us prove now that $\partial_x \partial_y F$ is continuous. The above analysis shows that this derivative is everywhere continuous in t and in (x, y) at $t = 0$. Hence, we only need to verify its continuity in (x, y) at $t \neq 0$. For $t \neq 0$, the integral (5.5) can be broken up into two parts: the integral over $[0, \delta]$ and the integral over $[\delta, t]$. By (5.6), the first part can be made arbitrarily small uniformly in (x, y) by choosing $\delta > 0$ small. The second part is obviously continuous in (x, y). This implies that $\partial_x \partial_y F$ is continuous in (x, y).

Other partial derivatives of F in x and y can be dealt with in a similar fashion. Hence, to ensure that (H2) and (H3) hold, it suffices to have s and α satisfy (5.1) and (5.3) simultaneously. Obviously, for every α sufficiently close to 1, there exists an even positive integer s satisfying these inequalities. This completes the proof of the lemma.

Remark 5.1. The assumption that s is a positive even integer can be dropped if we replace t^s by $|t|^s$ in the definition of H. Then (5.1) and (5.3) have a solution $s > 0$ if and only if $1/2 < \alpha < 1$.

5.2. *Proof of Lemma* 4.4.

Step 1: *The function A.* The following lemma, which we will prove in Section 6, plays a crucial role in the construction of A.

LEMMA 5.2. *There exists a Denjoy vector field* $\partial_y + h\partial_x$ *which is* $C^{1+\alpha}$ *for all* $\alpha \in (0, 1)$ *and such that*

D1. $\partial_x h$ *vanishes on* $\mathfrak{D}$;

D2. $\int_0^x (\partial_x h(\xi, y))^2\, d\xi$ *is* C^2 *in* (x, y).

Assuming Lemma 5.2, let us continue the proof of Lemma 4.4. The essence of the requirements on A is that A should be C^2, and the derivative $\partial_x A$ should vanish on $\mathfrak{D}$ and be bounded from below by $\eta(\partial_x h)^2$. If $\eta(\partial_x h)^2$ were not sufficiently smooth, these conditions would be hard to satisfy. However, since $\int_0^x \eta(\partial_x h)^2\, d\xi$ is C^2 by Lemma 5.2, we may simply take $\eta(\partial_x h)^2$ as $\partial_x A$ with some additional correction terms. These extra terms are needed to make A into a function $T^2 \to S^1$ meeting other requirements of Lemma 4.4.

Let us now outline the construction of A omitting some details to be filled in at the concluding part of the proof (Step 3). Pick a smooth C^1-small nonnegative function $a: T^2 \to \mathbb{R}$ which vanishes exactly on $\mathfrak{D}$. There exists a smooth nonnegative function $b: T^2 \to \mathbb{R}$ which vanishes on $\mathfrak{D}$ (but not only on $\mathfrak{D}$) and such that

$$(5.7) \qquad (x,y) \mapsto \int_0^x b(\xi, y)\, d\xi \quad \text{is } C^0\text{-close to} \quad (x,y) \mapsto x.$$

Pick a small $\eta_1 > 0$ and set[4]

$$A(x,y) = \frac{\int_0^x \left[\eta_1 (\partial_x h)^2 + a + b\right]\, d\xi}{\int_0^1 \left[\eta_1 (\partial_x h)^2 + a + b\right]\, d\xi}.$$

This is a function $T^2 \to S^1$. Indeed, $\int_0^1 \partial_x A\, dx = 1$ for any y and $A(x,0) = A(x,1)$ for any x by the definition of A. By (D2) and since a and b are smooth, A is C^2. By taking a and $\eta_1 > 0$ small, we can ensure that A is C^0-close to $(x,y) \mapsto x$, i.e., the requirement (A3) is met. Also, A obviously satisfies (A1) for some $\eta > 0$.

One can construct b in such a way that $b \geq$ *const* on the complement of some neighborhood U of $\mathfrak{D}$, which implies (4.2), and so that $b|_{V_1} = 1$. Then on V_1, the function A is C^1-close to x, provided that $\eta_1 > 0$ is small and a is C^1-small. Now it is easy to alter A on and near V_1 so that (A4) is satisfied (i.e., $A(x,y) = x$ on V_1) and the conditions (A1) and (4.2) still hold.

Step 2: *The function f.* First note that it suffices to construct a function f such that $\partial_x f|_U \geq$ *const*. To define such a function f, we pick a smooth function f' such that $f'|_U \geq$ *const* and such that the mean value of $x \mapsto f'(x,y)$ is zero for every y. (This is possible if U is sufficiently small.) Then $f(x,y) = \int_0^x f'(\xi, y)\, d\xi$ satisfies (4.3).

Step 3: *The detailed construction of the neighborhood U and the functions b and f.* Let us cover T^2 by two open overlapping cylinders $C_1 = S^1 \times I_1$ and $C_2 = S^1 \times I_2$, where I_1 and I_2 are two arcs covering the circle S^1 with coordinate y.

First we describe b and f on C_1. For the sake of brevity let us denote C_1 by C and I_1 by I. Without loss of generality we may assume that $0 \in I$ and $V_1 \subset C$. The Denjoy flow gives rise to a C^1-diffeomorphism $\varphi: C \to S^1 \times I$ which sends $\mathfrak{D} \cap C$ to a cylindrical set, i.e., $\varphi(\mathfrak{D} \cap C) = \mathfrak{D}_0 \times I$, where $\mathfrak{D}_0 = \mathfrak{D} \cap \{y = 0\}$. It is easy to see that $\mathfrak{D}_0$ can be covered by a finite collection of disjoint arbitrarily short open intervals $\Gamma_1, \ldots, \Gamma_k$. Then $\varphi(\mathfrak{D} \cap C)$ is covered by stripes $\Gamma_i \times I$ and thus $\mathfrak{D} \cap C$ is covered by the skewed stripes $\varphi^{-1}(\Gamma_i \times I)$.

[4]Throughout the proof we identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$.

The intersection of $\varphi^{-1}(\Gamma_i \times I)$ with $S^1 \times \{y\}$ is an arc whose end-points are C^1-functions of y. For each stripe, let us approximate these functions by C^∞-functions. If the approximations are accurate enough, the new end-point functions still bound nonoverlapping skewed open stripes in C which cover $\mathfrak{D} \cap C$. Denote these stripes by $L_1, \dots, L_k$ and set $U_1 = \cup L_i$.

Note that the end-points of $L_i \cap (S^1 \times \{y\})$ are smooth functions of y and that all stripes L_i can be made arbitrarily narrow by taking short intervals Γ_i. In addition, we can always make U_1 disjoint from V_1.

It is not hard to see that there exists a C^∞-function b on C which is identically zero on U_1 and such that (5.7) holds. Indeed, on $S^1 \times \{y\}$ we take a smooth function which is equal to zero on all arcs $L_i \cap (S^1 \times \{y\})$ and which has high bumps in between these arcs. Since the arcs are short, b can be chosen to satisfy (5.7). This function can obviously be made smooth in y because the end-points of the arcs are smooth. In addition, it is easy to see that we can take b to be equal to 1 on V_1.

The function f is defined in a similar fashion. For example, we can take f' equal to 1 on U_1 and, for each y, use the complement of $U_1 \cap (S^1 \times \{y\})$ in $S^1 \times \{y\}$ to make sure that f' has zero mean. Then, as we have pointed out above, we set $f(x, y) = \int_0^x f'(\xi, y)\, d\xi$.

For the second cylinder C_2 the argument is similar. We obtain the function b on T^2 from its counterparts b_1 on C_1 and b_2 on C_2 by pasting b_1 and b_2 on $C_1 \cap C_2$ using cut-off functions in y. The construction of f is finished in a similar way. It is clear that there exists a small neighborhood U of $\mathfrak{D}$ (contained in $U_1 \cup U_2$) such that $b|_U = 0$ and $\partial_x f|_U = 1$. This is the required neighborhood U. (Note that in general we cannot take $U = U_1 \cup U_2$.) The proof of the lemma is completed.

6. Proof of Lemma 5.2

The proof of Lemma 5.2 is based on the existence of a $C^{1+\alpha}$ Denjoy diffeomorphism Φ such that $(\Phi' - 1)^2$ is C^1. Therefore, we first outline the construction of such a Denjoy diffeomorphism, and then proceed with the proof of the lemma.

6.1. *Construction of the Denjoy diffeomorphism.*

LEMMA 6.1. *There exists a Denjoy diffeomorphism Φ which is $C^{1+\alpha}$ for all $\alpha \in (0, 1)$ and such that $(\Phi' - 1)^2$ is C^1.*

Proof of Lemma 6.1. We prove Lemma 6.1 in two steps. First, we define the required Denjoy diffeomorphism Φ and show that Φ' is α-Hölder for every $\alpha \in (0, 1)$; then we prove that $(\Phi' - 1)^2$ is C^1.

Step 1: *Definition of* Φ. In the construction of Φ we closely follow the general description of Denjoy maps in [KH, §12.2]. Pick $\beta \in (0, 1)$ and let

$$l_n := k_\beta(|n|+2)^{-1}(\log(|n|+2))^{-1/\beta} \tag{6.1}$$

be the length of the interval I_n inserted into S^1 to "blow up" an orbit, a_n, of an irrational rotation. Here k_β is a constant depending on β chosen so that $\sum_{n\in\mathbb{Z}} l_n < 1$. We emphasize that this choice of l_n is essential in order to make the series $\sum_{n\in\mathbb{Z}} l_n$ converge very slowly which, in turn, results in a small Denjoy continuum, $S^1 \smallsetminus \bigcup_{n\in\mathbb{Z}} \mathit{Int}(I_n)$. This slow convergence is the main factor which ensures that $(\Phi'-1)^2$ is C^1 and the second assertion (D2) of Lemma 5.2 holds, as will become clear later on.

To construct a Denjoy diffeomorphism Φ, it suffices to define the derivative Φ', since Φ is then obtained by integration. Let $\varphi\colon[0,1]\to\mathbb{R}$ be a bump function satisfying $\int_0^1\varphi(x)\,dx = 1$. Define the smooth function

$$\varphi_n(x) := c_n\,\varphi\Big((x-a_n)/l_n\Big)$$

on the interval $I_n = [a_n, a_n+l_n]$, where $c_n = (l_n - l_{n+1})/l_n$, and note that $\int_{I_n}\varphi_n(x)\,dx = c_n\,l_n$. Finally, let

$$\Phi'(x) = \left\{\begin{array}{ll} 1 & \text{for } x\in S^1\smallsetminus\bigcup_{n\in\mathbb{Z}} I_n, \\ 1+\varphi_n(x) & \text{for } x\in I_n \end{array}\right\}.$$

This completes the construction of Φ. It is well known, [KH], and easy to see that Φ is $C^{1+\alpha}$ for any $\alpha\in(0,1)$. (Moreover, one can show that

$$|\Phi'(x)-\Phi'(x_0)| \le \mathit{const}|x-x_0|\Big|\log|x-x_0|\Big|^{1/\beta}$$

for any x and x_0 in S^1.) For what follows, we only need that Φ is $C^{1+\alpha}$ for some $\alpha\in(1/2,1)$ and also some estimates on the C^1-norm of $(\Phi'-1)|_{I_n}$ which result from (6.1).

Let us now list some properties of c_n and l_n to be used later on:

First, we note that, as is true for any Denjoy map,

$$c_n \to 0 \quad\text{as}\quad n\to\infty.$$

In fact, $c_n = O(1/n)$.

Furthermore, (6.1) guarantees[5] that

$$\frac{c_n^2}{l_n} \to 0 \quad\text{as}\quad n\to\infty. \tag{6.2}$$

Indeed,

$$\frac{|c_n|}{l_n} = \frac{|l_n - l_{n+1}|}{l_n^2} = O\bigg(\Big(\log\,(|n|+2)\Big)^{1/\beta}\bigg),$$

[5]This is the main point in the proof where the specific choice of l_n made above is essential.

as can be seen by expanding the left-hand side in $|n|^a \Big(\log(|n|+2)\Big)^b$ for $a \le 0$, and $b \ge 0$. Thus,

$$\frac{c_n^2}{l_n} = l_n \frac{c_n^2}{l_n^2} = \frac{O\Big(\Big(\log(|n|+2)\Big)^{1/\beta}\Big)}{(|n|+2)} \to 0 \quad \text{as} \quad n \to \infty.$$

We finish this discussion by establishing the following estimates to be used in the rest of the proof:[6]

$$\|(\Phi'-1)\,|_{I_n}\| = O(|c_n|) \to 0 \tag{6.3}$$

$$\Big\|\partial_x (\Phi'-1)\,|_{I_n}\Big\| = O(|c_n|/l_n) \tag{6.4}$$

$$\Big\|\partial_x (\Phi'-1)^2\,|_{I_n}\Big\| = O(c_n^2/l_n) \to 0. \tag{6.5}$$

To prove these estimates, we first recall that $(\Phi'-1)|_{I_n} = \varphi_n$. Then, since $\|\varphi_n\| = |c_n| \cdot \|\varphi\|$, we have (6.3). The second estimate, (6.4), is proved as follows:

$$\|\partial_x (\Phi'-1)\,|_{I_n}\| = \|\partial_x \varphi_n\| \le \|\varphi'\| \frac{|c_n|}{l_n}.$$

Finally, (6.5) is a consequence of the first two estimates and (6.2).

Step 2: *Proof that* $(\Phi'-1)^2$ *is* C^1. Let $\mathfrak{D}_0 := S^1 \smallsetminus \bigcup_{n\in\mathbb{Z}} Int(I_n)$ denote the Denjoy continuum. Observe that, since on each I_n the function φ_n is smooth, $\Phi'-1$ as well as $(\Phi'-1)^2$ are also smooth on I_n. Hence, we need to prove that for $x_0 \in \mathfrak{D}_0$, $(\Phi'-1)^2$ is differentiable, its derivative at x_0 is zero, and $\partial_x (\Phi'-1)^2(x) \to 0$ as $x \to x_0$.

Recall that $\Phi'-1$ is α-Hölder continuous with $\alpha > 1/2$ and $(\Phi'-1)^2 \equiv 0$ on $\mathfrak{D}_0$. It readily follows that $(\Phi'-1)^2$ is differentiable, and its derivative is zero on $\mathfrak{D}_0$.

To finish the proof, it remains to show that $\partial_x (\Phi'-1)^2(x) \to 0$ as $x \to x_0 \in \mathfrak{D}_0$. Let x_k be a sequence in $S^1 \smallsetminus \mathfrak{D}_0$ converging to x_0. Since $\mathfrak{D}_0$ is nowhere dense, there exists a sequence of intervals, I_{n_k}, such that $x_k \in I_{n_k}$ for $k \in \mathbb{N}$. Then, by (6.5), $\Big|\partial_x (\Phi'-1)^2\,(x_k)\Big| \to 0$ as $k \to \infty$, and this, together with $\partial_x (\Phi'-1)^2(x_0) = 0$, proves the assertion. □

Remark 6.2. The Denjoy map defined by (6.1) is essentially as smooth as a Denjoy map can be made, up to functions growing slower than logarithms, e.g., iterations of logarithms. The next significant improvement in smoothness would be to have $\log \Phi'$ of bounded variation or satisfying the Zygmund condition which is impossible; see [HS], [JS], [KH].

[6]Throughout the rest of the proof $\|\ \|$ denotes the sup-norm on I_n.

Now we are in a position to prove Lemma 5.2 which asserts: *There exists a Denjoy vector field* $\partial_y + h\partial_x$ *which is* $C^{1+\alpha}$ *for all* $\alpha \in (0,1)$ *and such that*

D1. $\partial_x h$ *vanishes on* $\mathfrak{D}$;

D2. $\int_0^x (\partial_x h(\xi, y))^2 \, d\xi$ *is* C^2 *in* (x, y).

To prove this lemma we show that the Denjoy vector field $\partial_y + h\partial_x$ on T^2 for Φ described above satisfies (D1) and (D2). The proof of (D1) is straightforward and based on the explicit formula for h. The proof of (D2) is divided into two parts. In the first part (Section 6.3), we show that $(\partial_x h)^2$ is C^1, which obviously means that $\partial_x \int_0^x (\partial_x h(\xi, y))^2 \, d\xi$ is C^1. In the second part (Section 6.4), we show that $\partial_y \int_0^x (\partial_x h(\xi, y))^2 \, d\xi$ is C^1. These two results imply (D2).

6.2. *Explicit formula for* h *and the proof of* (D1). First we give explicit formulas for h and $\partial_x h$, and fix notation. Let

$$\Phi_y(x) = (1 - \delta(y))\, x + \delta(y)\, \Phi(x), \tag{6.6}$$

where $\delta\colon [0,1] \to [0,1]$ is a nonnegative, increasing, smooth function which is 0 for y close to 0 and 1 for y close to 1. Then the x-component h of a Denjoy vector field can be expressed as

$$h(x, y) = \left(\partial_y \Phi_y \circ \Phi_y^{\mathrm{inv}}\right)(x),$$

where the function $\Phi_y^{\mathrm{inv}}(x)$ is the inverse of $\Phi_y(x)$ in the x-variable.

Analyzing the smoothness of these functions, we first observe that $\Phi_y(x)$ is clearly $C^{1+\alpha}$ in (x, y). Moreover, $\Phi_y(x)$ is C^∞ for $x \notin \mathfrak{D}_0$. Furthermore, $\Phi_y^{\mathrm{inv}}(x)$ is also $C^{1+\alpha}$ in (x, y). To see this note that by the implicit function theorem $\Phi_y^{\mathrm{inv}}(x)$ is C^1 in (x, y) and

$$\partial_y \Phi_y^{\mathrm{inv}}(x) = -\frac{\partial_y \Phi_y\left(\Phi_y^{\mathrm{inv}}(x)\right)}{\partial_x \Phi_y\left(\Phi_y^{\mathrm{inv}}(x)\right)},$$

where the denominator is bounded away from zero. Now it readily follows that $\partial_y \Phi_y^{\mathrm{inv}}(x)$ is C^α in (x, y), for the numerator is $C^{1+\alpha}$ and the denominator is C^α. A similar argument shows that $\partial_x \Phi_y^{\mathrm{inv}}(x)$ is C^α in (x, y).

As a consequence, h is $C^{1+\alpha}$, and hence

$$\partial_x h(x, y) = \delta'(y)\left(\Phi' \circ \Phi_y^{\mathrm{inv}}(x) - 1\right) \partial_x \Phi_y^{\mathrm{inv}}(x)$$

is C^α.

Finally, for a fixed $y \in [0,1]$, with the notation from Section 5.2, let $\mathfrak{D}_y := \mathfrak{D} \cap \{y\}$. Thus, $\mathfrak{D}_y = \Phi_y(\mathfrak{D}_0)$.

Proof of (D1). Let $(x, y) \in \mathfrak{D}$, i.e., $x \in \mathfrak{D}_y = \Phi_y(\mathfrak{D}_0)$. Thus, $\Phi_y^{\mathrm{inv}}(x) \in \mathfrak{D}_0$. Since $\Phi' - 1 \equiv 0$ on $\mathfrak{D}_0$, we conclude that $\partial_x h(x, y) = 0$; i.e., $\partial_x h$ vanishes on $\mathfrak{D}$.

6.3. *Proof of* (D2), *Part* I: $(\partial_x h)^2$ *is* C^1. Note that the existence and continuity of the partial derivatives of $(\partial_x h)^2$ are nontrivial only at the points of $\mathfrak{D}$.

First, observe that both of the partial derivatives $\partial_x (\partial_x h)^2$ and $\partial_y (\partial_x h)^2$ exist and vanish at $(x, y) \in \mathfrak{D}$. This follows immediately from the facts that $\partial_x h$ is α-Hölder continuous with $\alpha > 1/2$ and $\partial_x h$ vanishes on $\mathfrak{D}$ by (D1).

To examine the continuity of $\partial_x (\partial_x h)^2$ and $\partial_y (\partial_x h)^2$ we adopt new notation for $(\partial_x h)^2$. Fix $y \in [0, 1]$ and let $F_y : S^1 \to \mathbb{R}$ be the function defined by

$$F_y(\xi) = \Big(\Phi'(\xi) - 1\Big)^2 \frac{\Big(\delta'(y)\Big)^2}{\Big(1 + \delta(y)\left(\Phi'(\xi) - 1\right)\Big)^2}. \tag{6.7}$$

Then

$$(\partial_x h(x, y))^2 = F_y \circ \Phi_y^{\mathrm{inv}}(x). \tag{6.8}$$

It follows that F_y vanishes on $\mathfrak{D}_0$ for every y. The function F_y is clearly differentiable since $F_y(\xi) = (\partial_x h)^2 (\Phi_y(\xi), y)$, where $(\partial_x h)^2$ is differentiable as is shown above and $\Phi_y(\xi)$ is $C^{1+\alpha}$ as proved in Section 6.2. Furthermore, $\partial_\xi F_y$ and $\partial_y F_y$ are both zero on $\mathfrak{D}_0$, for the partial derivatives of $(\partial_x h)^2$ vanish on $\mathfrak{D}$.

To prove that $(\partial_x h)^2$ is C^1 in (x, y), it suffices to show that $F_y(\xi)$ is C^1 in (ξ, y). (Indeed, Φ_y^{inv} is $C^{1+\alpha}$ and (6.8) implies that $(\partial_x h)^2$ is C^1 if F_y is C^1.) Thus, it remains to prove that $\partial_\xi F_y$ and $\partial_y F_y$ are continuous.

Continuity of $\partial_y F_y(\xi)$. This follows immediately from (6.7) since δ is C^∞-smooth.

Continuity of $\partial_\xi F_y(\xi)$. First note that a straightforward calculation using (6.7) shows that

$$\partial_\xi F_y(\xi) = \Big(\delta'(y)\Big)^2 \frac{\Big(1 + \delta(y)(\Phi' - 1)\Big)\partial_\xi (\Phi' - 1)^2 - 2\,\delta(y)(\Phi' - 1)^2 \partial_\xi (\Phi' - 1)}{\Big(1 + \delta(y)(\Phi' - 1)\Big)^3}$$

on $S^1 \smallsetminus \mathfrak{D}_0 = \bigcup_{n \in \mathbb{Z}} \mathit{Int}(I_n)$, and, as discussed above, $\partial_\xi F_y(\xi) = 0$ on $\mathfrak{D}_0$. It follows immediately that $\partial_\xi F_y(\xi)$ is continuous in y for every ξ.

Clearly, $\partial_\xi F_y(\xi)$ is continuous in ξ on the complement of $\mathfrak{D}_0$ for every fixed y. Let us show the continuity at (ξ, y) with $\xi \in \mathfrak{D}_0$. Note that the denominator in the expression for $\partial_\xi F_y$ is bounded away from zero. Using the estimates (6.3), (6.4), and (6.5), it is easy to see that the asymptotic behavior of $\|\partial_\xi F_y|_{I_n}\|$ as $n \to \infty$ is determined by $\|\partial_\xi (\Phi' - 1)^2\|$; i.e.,

$$\left\|\partial_\xi F_y|_{I_n}\right\| = O(c_n^2/l_n) \to 0. \tag{6.9}$$

Arguing as in the proof of the fact that $(\Phi' - 1)^2$ is continuously differentiable (see Section 6.1), we conclude that $\partial_\xi F_y(\xi)$ is continuous.

This finishes the proof that F_y, and hence $(\partial_x h)^2$, is C^1.

6.4. *Proof of* (D2), *Part* II: $\partial_y \int_0^x (\partial_x h(\xi, y))^2 \, d\xi$ *is* C^1. First let us write the function $\int_0^x (\partial_x h(\xi, y))^2 \, d\xi$ in a form more convenient for our analysis. Setting $\xi = \Phi_y(\eta)$, we obtain

$$\begin{aligned} \int_0^x (\partial_x h(\xi, y))^2 \, d\xi &= \int_0^x F_y\left(\Phi_y^{\mathrm{inv}}(\xi)\right) d\xi \\ &= \int_{\Phi_y^{\mathrm{inv}}(0)}^{\Phi_y^{\mathrm{inv}}(x)} F_y(\eta)\, \partial_\eta \Phi_y(\eta)\, d\eta. \end{aligned}$$

Hence, define

$$G_y(u) := \int_0^u F_y(\eta)\, \partial_\eta \Phi_y(\eta)\, d\eta. \tag{6.10}$$

Then

$$\begin{aligned} \int_0^x (\partial_x h(\xi, y))^2 \, d\xi &= \int_{\Phi_y^{\mathrm{inv}}(0)}^{\Phi_y^{\mathrm{inv}}(x)} F_y(\eta)\, \partial_\eta \Phi_y(\eta)\, d\eta \\ &= \int_0^{\Phi_y^{\mathrm{inv}}(x)} F_y(\eta)\, \partial_\eta \Phi_y(\eta)\, d\eta - \int_0^{\Phi_y^{\mathrm{inv}}(0)} F_y(\eta)\, \partial_\eta \Phi_y(\eta)\, d\eta \\ &= G_y \circ \Phi_y^{\mathrm{inv}}(x) - G_y \circ \Phi_y^{\mathrm{inv}}(0). \end{aligned}$$

Thus, our goal is to prove that $\partial_y[G_y \circ \Phi_y^{\mathrm{inv}}(x)]$ is a C^1-function. We do this in two steps: first, we show that the function $G_y(u)$ is C^2 and then, using this result, we prove that $\partial_y[G_y \circ \Phi_y^{\mathrm{inv}}(x)]$ is a C^1-function.

6.4.1. *Step* 1: *Proof that* $G_y(u)$ *is* C^2. Let us show that both of the partial derivatives $\partial_u G_y(u)$ and $\partial_y G_y(u)$ are C^1.

Proof that $\partial_u G_y(u)$ *is* C^1. Let $\tilde{F}_y(u) := \partial_u G_y(u)$. Thus, by (6.10),

$$\tilde{F}_y(u) = F_y(u)\, \partial_u \Phi_y(u). \tag{6.11}$$

First, let us consider $\partial_y \partial_u G_y = \partial_y \tilde{F}_y$. For $u \in S^1 \smallsetminus \mathfrak{D}_0$, the function $\tilde{F}_y(u)$ is smooth. Hence, as long as $u \in S^1 \smallsetminus \mathfrak{D}_0$, the derivative $\partial_y \tilde{F}_y$ exists (and is continuous). For u_0 in $\mathfrak{D}_0$, the partial derivative $\partial_y \tilde{F}_y(u_0)$ exists and is zero. The reason is that $\tilde{F}_y(u_0) = 0$ for all $y \in [0, 1]$. Furthermore, $\partial_y \tilde{F}_y(u)$ is continuous in u and smooth in y, i.e., $\partial_y \tilde{F}_y(u)$ is infinitely differentiable in y

and every derivative is continuous in (u,y) as immediately follows from (6.6) and (6.7). This proves the continuity of $\partial_y \partial_u G_y$.

Let us now focus on the partial derivative $\partial_u^2 G_y = \partial_u \tilde{F}_y$. As before, this partial derivative obviously exists when $u \in S^1 \smallsetminus \mathfrak{D}_0$. Furthermore, we claim that $\partial_u \tilde{F}_y(u_0)$ exists and is zero for any $u_0 \in \mathfrak{D}_0$. To see this, recall that as proved in Section 6.3, $F_y(u_0) = 0$ and $\partial_u F_y(u_0) = 0$ for all $u_0 \in \mathfrak{D}_0$. Hence,

$$\begin{aligned}
\partial_u \tilde{F}_y(u_0) &= \partial_u \Big(F_y(u)\, \partial_u \Phi_y(u) \Big)\big|_{u=u_0} \\
&= \lim_{u \to u_0} \frac{F_y(u)\, \partial_u \Phi_y(u) - \overbrace{F_y(u_0)}^{0}\, \partial_u \Phi_y(u_0)}{u - u_0} \\
&= \lim_{u \to u_0} \frac{F_y(u) - F_y(u_0)}{u - u_0}\, \partial_u \Phi_y(u) \\
&= \underbrace{\partial_u F_y(u_0)}_{0}\, \partial_u \Phi_y(u_0) \\
&= 0.
\end{aligned}$$

To show that $\partial_u \tilde{F}_y$ is continuous, we first express $\partial_u \tilde{F}_y$ on each I_n as follows

$$\partial_u \tilde{F}_y(u) = \underbrace{\partial_u F_y(u)}_{O(c_n^2/l_n)}\, \underbrace{\partial_u \Phi_y(u)}_{O(1)} + \underbrace{F_y(u)}_{O(c_n^2)}\, \underbrace{\partial_u^2 \Phi_y(u)}_{O(|c_n|/l_n)},$$

where the braces indicate asymptotic behavior as $|n| \to \infty$. The estimate $\|\partial_u \tilde{F}\| = O(c_n^2/l_n)$ has been established above, see (6.9); the estimate $\|\partial_u \Phi_y\| = O(1)$ follows from the definition of Φ (see (6.6)) and (6.3); the estimate $\|F_y\| = O(c_n^2)$ is a consequence of the definition of F_y (i.e., (6.7)) and (6.3). Finally, $\|\partial_u^2 \Phi_y\| = O(|c_n|/l_n)$ results from (6.6) and (6.4).

Now it is clear that $\|\partial_u \tilde{F}_y(u)_{I_n}\| = O(c_n^2/l_n)$. Since, by (6.2), $c_n^2/l_n \to 0$ as $|n| \to \infty$, $\partial_u \tilde{F}_y(u)$ can be shown to be continuous in a fashion similar to the cases discussed before.

This completes the proof of the fact that $\partial_u G_y(u) = \tilde{F}_y(u)$ is C^1.

Proof that $\partial_y G_y(u)$ is C^1. Note that, since $\partial_y \tilde{F}_y(\eta)$ is continuous in (η, y) and its domain is compact, the functions $\partial_y \tilde{F}_y$ converge uniformly to $\partial_y \tilde{F}_y|_{y=y_0}$ as $y \to y_0$ for any $y_0 \in [0,1]$. Thus, we have

$$\partial_y G_y(u) = \partial_y \int_0^u \tilde{F}_y(\eta)\, d\eta = \int_0^u \partial_y \tilde{F}_y(\eta)\, d\eta.$$

This implies that $\partial_u \partial_y G_y(u)$ is continuous, for $\partial_u \partial_y G_y(u) = \tilde{F}_y(u)$ is continuous (in fact, C^1). To show that $\partial_y^2 G_y$ is continuous we recall that $\tilde{F}_y(u)$ is infinitely differentiable in y and every derivative is continuous in (u,y). Hence, as above, the integration and differentiation can be interchanged, and

$$\partial_y^2 G_y(u) = \int_0^u \partial_y^2 \tilde{F}_y(\eta)\, d\eta$$

is continuous because the integrand is continuous. This completes Step 1.

6.4.2. *Step* 2: *Proof that* $\partial_y[G_y \circ \Phi_y^{\mathrm{inv}}(x)]$ *is* C^1. We first write this partial derivative explicitly as follows:

$$\partial_y\Big(G_y \circ \Phi_y^{\mathrm{inv}}(x)\Big) = \partial_u G_y\Big(\Phi_y^{\mathrm{inv}}(x)\Big)\,\partial_y \Phi_y^{\mathrm{inv}}(x) + \partial_y G_y\Big(\Phi_y^{\mathrm{inv}}(x)\Big).$$

The second term of the sum on the right-hand side is C^1 because it is the composition of C^1 and $C^{1+\alpha}$ functions. Thus, we focus on the first summand which is

$$\begin{aligned}\partial_u G_y\Big(\Phi_y^{\mathrm{inv}}(x)\Big)\,\partial_y \Phi_y^{\mathrm{inv}}(x) &= \tilde{F}_y\Big(\Phi_y^{\mathrm{inv}}(x)\Big)\,\partial_y \Phi_y^{\mathrm{inv}}(x)\\ &= F_y\Big(\Phi_y^{\mathrm{inv}}(x)\Big)\,\partial_u\Phi\Big(\Phi_y^{\mathrm{inv}}(x)\Big)\,\partial_y \Phi_y^{\mathrm{inv}}(x),\end{aligned}$$

where the last equality follows from (6.11). The product of the last two terms can be further simplified. Applying ∂_y to the identity $\Phi_y\Big(\Phi_y^{\mathrm{inv}}(x)\Big) \equiv x$, we obtain

$$\partial_u \Phi_y\Big(\Phi_y^{\mathrm{inv}}(x)\Big)\,\partial_y \Phi_y^{\mathrm{inv}}(x) + (\partial_y \Phi_y)\,\Big(\Phi_y^{\mathrm{inv}}(x)\Big) = 0,$$

and hence

$$\tilde{F}_y\Big(\Phi_y^{\mathrm{inv}}(x)\Big)\,\partial_y \Phi_y^{\mathrm{inv}}(x) = -[F_y\,\partial_y\Phi_y] \circ \Phi_y^{\mathrm{inv}}(x).$$

Recall that $F_y(u)$, $\partial_y\Phi_y(u)$ and $\Phi_y^{\mathrm{inv}}(x)$ are all C^1-functions. Hence, the left-hand side is also C^1.

This concludes Step 2 and hence the proof of the fact that

$$\partial_y \int_0^x (\partial_x h(\xi, y))^2\, d\xi$$

is C^1.

Remark 6.3. Note that the norms of $(\Phi' - 1)^2|_{I_n}$ and $\partial_u F_y(u)|_{I_n}$ and $\partial_u \tilde{F}_y(u)|_{I_n}$ converge to zero only as $O(c_n^2/l_n)$. (One can also show that the same is true for the ∂_x- and ∂_y-partial derivatives of $G_y \circ \Phi_y^{\mathrm{inv}}(x)$.) A faster rate of convergence, e.g., $O(|c_n|^3/l_n)$, would be likely to result in an "unacceptably" smooth Denjoy map and vector field.

UNIVERSITY OF CALIFORNIA, SANTA CRUZ, CA 95064, USA
E-mail address: ginzburg@math.ucsc.edu

UNIVERSITY OF CALIFORNIA, SANTA CRUZ, CA, USA
Current address: SUNY AT STONY BROOK, STONY BROOK, NY 11794, USA
E-mail address: basak@math.sunysb.edu

References

[Gi1] V. L. Ginzburg, An embedding $S^{2n-1} \to \mathbb{R}^{2n}$, $2n-1 \geq 7$, whose Hamiltonian flow has no periodic trajectories, *IMRN* (1995), no. 2, 83–98.

[Gi2] ———, A smooth counterexample to the Hamiltonian Seifert conjecture in $\mathbb{R}^6$, *IMRN* (1997), no. 13, 641–650.

[Gi3] ———, Hamiltonian dynamical systems without periodic orbits, in *Northern California Symplectic Geometry Seminar*, 35–48, *Amer. Math. Soc. Transl. Ser*. **196**, A. M. S., Providence, RI, 1999.

[Gi4] V. L. Ginzburg, The Hamiltonian Seifert conjecture: examples and open problems, in *Proc. of the Third European Congress of Mathematics* (Barcelona, 2000), *Progr. in Math*. **202** (2001), vol. II, pp. 547–555.

[GG] V. L. Ginzburg and B. Z. Gürel, On the construction of a C^2-counterexample to the Hamiltonian Seifert Conjecture in $\mathbb{R}^4$, *Electron. Res. Announc. Amer. Math. Soc*. **8** (2002), 1–10.

[Ha] J. Harrison, A C^2 counterexample to the Seifert conjecture, *Topology* **27** (1988), 249–278.

[He1] M.-R. Herman, Fax to Eliashberg, 1994.

[He2] ———, Examples of compact hypersurfaces in $\mathbb{R}^{2p}$, $2p \geq 6$, with no periodic orbits, in *Hamiltonian Systems with Three or More Degrees of Freedom* (C. Simo, ed.), *NATO Adv. Sci. Inst. Ser.* C, *Math. Phys. Sci*. **533**, Kluwer Acad. Publ., Dordrecht, 1999.

[HZ1] H. Hofer and E. Zehnder, Periodic solution on hypersurfaces and a result by C. Viterbo, *Invent. Math*. **90** (1987), 1–9.

[HZ2] ———, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser, Boston, 1994.

[HS] J. Hu and D. Sullivan, Topological conjugacy of circle diffeomorphisms, *Ergodic Theory Dynam. Systems* **17** (1997), 173–186.

[JS] M. Jakobson and G. Świątek, One-dimensional maps, in *Handbook of Dynamical Systems*, Vol. 1A, 599–664, North-Holland, Amsterdam, 2002.

[KH] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, *Encyc. of Mathematics and its Applications* **54**, Cambridge Univ. Press, Cambridge, 1995.

[Ke] E. Kerman, New smooth counterexamples to the Hamiltonian Seifert conjecture, *J. Symplectic Geom*. **1** (2002), 253–267.

[KuG] G. Kuperberg, A volume-preserving counterexample to the Seifert conjecture, *Comment. Math. Helv*. **71** (1996), 70–97.

[KuGK] G. Kuperberg and K. Kuperberg, Generalized counterexamples to the Seifert conjecture, *Ann. of Math*. **144** (1996), 239–268.

[KuK1] K. Kuperberg, A smooth counterexample to the Seifert conjecture in dimension three, *Ann. of Math*. **140** (1994), 723–732.

[KuK2] ———, Counterexamples to the Seifert conjecture, *Proc. Internat. Congress of Mathematicians* (Berlin, 1998), *Doc. Math*. (1998) Extra Vol. **II**, 831–840.

[KuK3] ———, Aperiodic dynamical systems, *Notices Amer. Math. Soc*. **46** (1999), 1035–1040.

[McDS] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, *Oxford Math. Monographs*, Oxford Univ. Press, New York, 1995.

[Sc] P. A. Schweitzer, Counterexamples to the Seifert conjecture and opening closed leaves of foliations, *Ann. of Math*. **100** (1970), 229–234.

[St] M. Struwe, Existence of periodic solutions of Hamiltonian systems on almost every energy surface, *Bol. Soc. Bras. Mat*. **20** (1990), 49–58.

(Received October 14, 2001)