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Stability and instability of the Cauchy horizon for the spherically symmetric Einstein-Maxwell-scalar field equations

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Abstract

This paper considers a trapped characteristic initial value problem for the spherically symmetric Einstein-Maxwell-scalar field equations. For an open set of initial data whose closure contains in particular Reissner-Nordström data, the future boundary of the maximal domain of development is found to be a light-like surface along which the curvature blows up, and yet the metric can be continuously extended beyond it. This result is related to the *strong cosmic censorship* conjecture of Roger Penrose.

1. Introduction

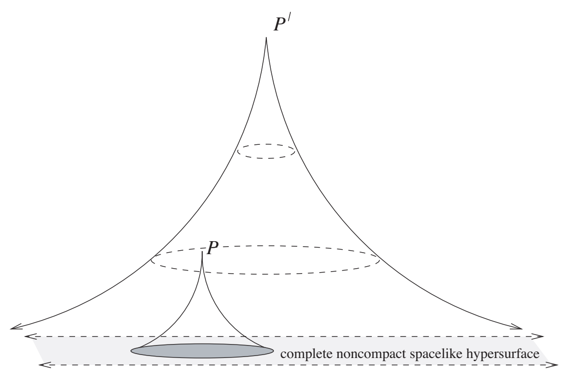
The principle of determinism in classical physics is expressed mathematically by the uniqueness of solutions to the initial value problem for certain equations of evolution. Indeed, in the context of the Einstein equations of general relativity, where the unknown is the very structure of space and time, uniqueness is equivalent on a fundamental level to the validity of this principle. The question of uniqueness may thus be termed the issue of the *predictability* of the equation.

The present paper explores the issue of predictability in general relativity. Since the work of Leray, it has been known that for the Einstein equations, contrary to common experience, uniqueness for the Cauchy problem in the large does *not* generally hold even within the class of smooth solutions. In other words, uniqueness may fail without any loss in regularity; such failure is thus a global phenomenon. The central question is whether this violation of predictability may occur in solutions representing actual physical processes. Physical phenomena and concepts related to the general theory of relativity, namely gravitational collapse, black holes, angular momentum, etc., must certainly come into play in the study of this problem. Unfortunately, the mathematical analysis of this exciting problem is very difficult, at present beyond reach for the vacuum Einstein equations in the physical dimension. Consequently, in this paper, I will resolve the issue of uniqueness in the context of a special, spherically symmetric initial value problem for a system of gravity coupled with matter, whose relation to the problem of gravitational collapse is well established in the physics literature. We will arrive at it here by reconciling the picture that emerges from the work of Demetrios Christodoulou [5]–the generic development of trapped regions and thus black holes–with the known unpredictability of the Kerr solutions in their corresponding black holes.

1.1. *Predictability for the Einstein equations and strong cosmic censorship.* To get a first glimpse of unpredictability, consider the Einstein equations in the vacuum,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0,$$

where the unknown is a Lorentzian metric $g_{\mu\nu}$ and the characteristic sets are its light cones. For any point P of spacetime, the hyperbolic nature of the equations determines the so-called past domain of influence of P, which in the present case of the vacuum equations is just its causal past $J^{-}(P)$. Uniqueness of the solution at P (modulo the diffeomorphism invariance) would follow from a domain of dependence argument. Such an argument requires, however, that $J^{-}(P)$ have compact intersection with the initial data; compare P and P' in the diagram below:



In what follows we shall encounter explicit solutions of the Einstein equations which contain points as in P' above, where the solution is regular and yet the compactness property essential to the domain of dependence argument fails. These solutions can then be easily seen to be nonunique as solutions to the initial value problem.[1]

[1]As this type of nonuniqueness is induced solely from the fact that the Einstein equations are quasilinear and the geometry of the characteristic set depends strongly on the unknown, it should be a feature of a broad class of partial differential equations.

It turns out that unpredictability of this nature occurs in particular in the most important family of special solutions of the Einstein equations, the so-called Kerr solutions. The current physical intuition for the final state of gravitational collapse of a star into a black hole derives from this family of solutions. One thus has to take seriously the possibility that nonuniqueness may be a general feature of gravitational collapse–in other words, that it *does* occur in actual physical processes. Penrose and Simpson [19] observed, however, that on the basis of a first-order calculation,[2] this scenario appeared to be unstable; this led Penrose to conjecture that, in the context of gravitational collapse, unpredictability is exceptional, i.e., for *generic* initial data in a certain class, the solution is unique. The conjecture goes by the name of *strong cosmic censorship.*

After the Einstein equations are coupled with equations for suitably chosen matter, and a regularity framework is set, strong cosmic censorship constitutes a purely mathematical question on the initial value problem, and thus provides an opportunity for the theory of partial differential equations to say something significant about fundamental physics. Unfortunately, all the difficulties of quasilinear hyperbolic equations with large data are present in this problem and make a general solution elusive at present. Nevertheless, this paper hopes to show that nonlinear analysis may still have something interesting to say at this time.

1.2. *Angular momentum in trapped regions and the formation of Cauchy horizons.* A formulation of the problem posed by strong cosmic censorship is sought which is analytically tractable yet still captures much of the essential physics. It turns out that the constraints induced by analysis are rather severe. Quasilinear hyperbolic equations become prohibitively difficult when the spatial dimension is greater than 1. Reducing the Einstein equations to a problem in $1+1$-dimensions in a way compatible with the physics of gravitational collapse leads necessarily to spherical symmetry.

The analytical study of the Einstein-scalar field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu},$$

$$g^{\mu\nu}(\partial_\mu\phi)_{;\nu} = 0,$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi,$$

[2]This calculation was in fact carried out in the context of a Reissner-Nordström background; see below.

under spherical symmetry[3] was introduced by Christodoulou in [10], where he discussed how this particular symmetry and scalar field matter impact on the gravitational collapse problem. (See also [7].) The equations reduce to the following system for a Lorentzian metric g and functions r and ϕ defined on a two-dimensional manifold Q:

$$K = \frac{1}{r^2}(1 - \partial^a r \partial_a r) + \partial^a \phi \partial_a \phi$$

$$\nabla_a \nabla_b r = \frac{1}{2r}(1 - \partial^c r \partial_c r) g_{ab} - r T_{ab}.$$

$$g^{ab} \nabla_a \nabla_b \phi + \frac{2}{r} \partial^a r \phi_a = 0.$$

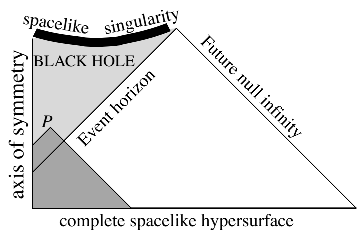
Here K denotes the Gauss curvature of g. Christodoulou's results of [5] are definitive: Gravitational collapse and the issue of predictability are completely understood in the context of the spherically symmetric Einstein-scalar field model. Nevertheless, that work leaves unanswered the question that motivated the formulation of strong cosmic censorship–the unpredictability of the Kerr solution.

Christodoulou was primarily interested in studying another phenomenon of gravitational collapse, the formation of black holes. The conjecture that in generic gravitational collapse, singularities are hidden behind black holes is known as *weak cosmic censorship*, even though strictly speaking it is not logically related to the issue of strong cosmic censorship (see [6]). Christodoulou proved this conjecture for the spherically symmetric Einstein-scalar field system. The key to his theorem is in fact the stronger result that, generically, so-called *trapped regions* form. In the 2-dimensional manifold Q, the trapped region is defined by the condition that the derivative of r in both forward characteristic directions is negative. A point $p \in Q$ in the trapped region corresponds to a *trapped surface* in the four-dimensional space-time manifold M.

Because of their global topological properties, in explicit solutions such as the Kerr solution, trapped surfaces must be present at all times. Christodoulou's solutions for the first time demonstrated that trapped regions–and thus black holes–can form *in evolution.* The geometry of black holes for the spherically symmetric Einstein-scalar field equations can be understood relatively easily; in particular these black holes always terminate in a spacelike singularity. Here is a depiction of the image of a conformal representation of

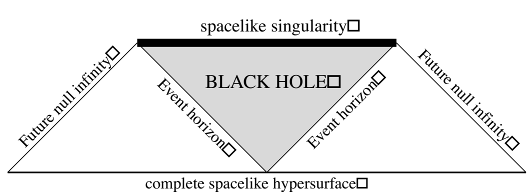
[3]Note that by Birkhoff's theorem, the vacuum equations under spherical symmetry admit only the Schwarzschild solutions.

the manifold Q into 2-dimensional Minkowski space:

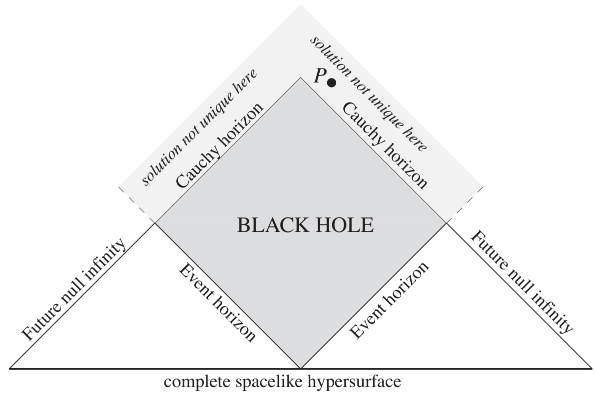


The causal structure of Q can be immediately read off, as characteristics correspond to straight lines at 45 and -45 degrees from the horizontal. Future null infinity and the singularity correspond to ideal points; they are not part of Q. The spacetime is future inextendible as a manifold with continuous Lorentzian metric (see §8), and the domain of dependence property is seen to hold for any point P in Q, as its past can never contain the intersection of the initial hypersurface with future null infinity. Thus, in this model, the theorem that trapped regions and thus black holes form generically yields immediately a proof of strong cosmic censorship.

The Kerr solutions constitute a two-parameter family parametrized by mass and angular momentum. These solutions indicate that the behavior of trapped regions exhibited by the spherically symmetric Einstein-scalar field equations is very special. Angular momentum is–in a certain sense–precisely a measure of spherical *asymmetry* of the metric. When the angular momentum parameter is set to zero in the Kerr solution, one obtains the so-called Schwarzschild solution. In this spherically symmetric solution, the trapped region, which coincides with the black hole, indeed terminates in a spacelike singularity, as in Christodoulou's solutions. Here again is a conformal representation of Q in the future of a complete spacelike hypersurface:



For every small nonzero value of the angular momentum, however, the future boundary of the black hole of the Kerr solution is a light-like surface beyond which the solution can be extended smoothly. To compare with the spherically symmetric case, a conformal representation of a 2-dimensional cross section, in the future of a complete-spacelike hypersurface, is depicted below:



This light-like surface is called a *Cauchy horizon*, as any Cauchy problem posed in its past is insufficient to uniquely determine the solution in its future. It thus signals the onset of unpredictability. (Note that the past of the point P in the figure above intersects the initial data in a noncompact set, i.e., it "contains" the point of intersection of the initial data set with future null infinity.)

It seems then that the (potential) driving force of unpredictability in gravitational collapse, after trapped surfaces have formed, is precisely the angular momentum invisible to the Einstein-scalar field model. A real first understanding of strong cosmic censorship in gravitational collapse must somehow come to terms with the possibility of the formation of Cauchy horizons generated by angular momentum.

1.3. *Maxwell's equations*: *charge as a substitute for angular momentum.* We are led to the Einstein-Maxwell-scalar field model:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2T_{\mu\nu} = 2(T^{\rm em}_{\mu\nu} + T^{sf}_{\mu\nu}) \tag{1}$$

$$F^{\mu\nu}_{;\nu} = 0, \tag{2}$$

$$F_{[\mu\nu,\rho]} = 0, \tag{3}$$

$$g^{\mu\nu}(\partial_\mu\phi)_{;\nu} = 0, \tag{4}$$

$$T^{\rm em}_{\mu\nu} = F_{\mu\lambda}F_{\nu\rho}g^{\lambda\rho} - \frac{1}{4}g_{\mu\nu}F_{\lambda\rho}F_{\sigma\tau}g^{\lambda\sigma}g^{\rho\tau},$$
$$T^{sf}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi,$$

in an effort to capture the physics of angular momentum in the trapped region, *while remaining in the realm of spherical symmetry.* The key observation is, in the words of John Wheeler, that charge is a "poor man's" angular momentum. It is well known that the trapped region of the (spherically symmetric)

Reissner-Nordström solution of the Einstein-Maxwell equations is similar to the Kerr solution's black hole, and in particular, also has as future boundary a Cauchy horizon leading to unpredictability for every small nonzero value of the charge parameter. In fact, the previous diagram of the 2-dimensional cross-section of the Kerr solution corresponds precisely to the manifold Q of group orbits of the Reissner-Nordström solution (see Section 3) in the past of the Cauchy horizon. Examining the nonlinear stability of the Reissner-Nordström Cauchy horizon will thus give insight to the predictability of general gravitational collapse.

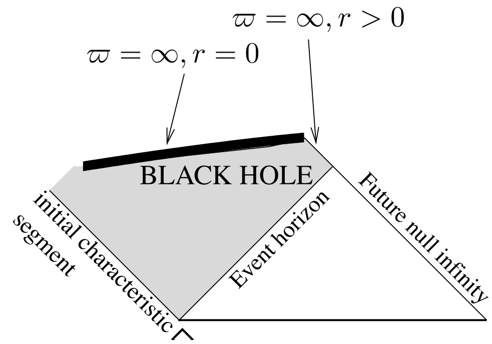
1.4. *Outline of the paper.* The spherically symmetric Einstein-Maxwell-scalar field system in null coordinates is derived in Section 2. In Section 3, the special Reissner-Nordström solution will be presented, and its important properties will be reviewed. The initial value problem to be considered in this work will be formulated in Section 4. The initial data will lie in the trapped region.

Section 5 will initiate the discussion on predictability for our initial value problem, in view of the simplifications in the conformal structure provided by spherical symmetry. There always exists a maximal region of spacetime, the so-called maximal domain of development, for which the initial value problem uniquely determines the solution. The conditions for predictability are then related to the behavior of the unique solution of the initial value problem on the boundary of this region.

In the following two sections, the analytical results necessary to settle the issue will be obtained. In Section 6, a theorem is proved which delimits the extent of the maximal domain of development of our initial data. This will be effected by proving that the function r, a parameter on the order of the metric itself, is stable in a neighborhood of the point at infinity of the event horizon. In Section 7, a theorem is proved which determines the behavior of ϖ, a parameter related directly to both the C^1 norm of the metric and its curvature, along the boundary of the maximal domain of development. In particular, for an open set of initial data, this parameter is found to blow up. This situation, illustrated in the figure on the next page,[4] is seen to be qualitatively different from both the Kerr picture and the picture of the solutions of Christodoulou.

Finally, Section 8 examines the implications of the stability and blow-up results on predictability and thus on strong cosmic censorship. In view of the opposite nature of the theorems established in Sections 6 and 7, different verdicts for cosmic censorship can be extracted, depending on the smoothness assumptions adopted in its formulation.

[4]The nature of the $r = 0$ "singular" boundary, when nonempty, is discussed in the appendix.



The analytical content of this paper is thus a combination of a stability theorem and a blow-up result for a system of quasilinear partial differential equations in one spatial and one temporal dimension. Not surprisingly, standard techniques like bootstrapping play an important role. However, as they evolve, both the matter and the gravitational field strength will become large, and so other methods will also have to come into play. It is well known (for instance from the work of Penrose [17]) that the Einstein equations have important monotonicity properties. This monotonicity is even stronger in the context of spherical symmetry, and plays an important role in the work of Christodoulou. The result of Section 6 hinges on a careful study of the geometry of the solutions, with arguments depending on monotonicity replacing bootstrap techniques in regions where the solution is large.

The strong cosmic censorship conjecture was formulated by Penrose based on a first order perturbation argument [19] which seemed to indicate that certain natural derivatives of any reasonable perturbation field blow up on the Reissner-Nordström-Cauchy horizon. This was termed the *blue-shift effect* (see [15]). It is not easy even to conjecture how this mechanism, assuming it is stable, affects the nonlinear theory. Israel and Poisson [18] first proposed the scenario expounded in Section 7, dubbing it "mass inflation", in the context of a related model which is simpler than the scalar field model considered here. In the context of the scalar field model, in order to produce this effect one needs to make some rough *a priori* assumptions on the metric on which the blue-shift effect is to operate. Because of the nonlinearity of the problem, and the large field strengths, it is difficult to justify such assumptions, even nonrigorously (see [1]).

This difficulty is circumvented here with the help of a simple and very general monotonicity property of the solutions to the spherically symmetric wave equation (Proposition 5), which was unexpected as it is peculiar to trapped regions, i.e., it has no counterpart in more familiar metrics like Minkowski space, or the regular regions where most of the analysis of Christodoulou was carried out. In combination with the monotonicity properties discovered earlier, the new one provides a powerful tool which, under the assumption that the mass does *not* blow up, yields precisely the kind of control on the metric that is necessary for the blue shift mechanism to operate. This leads–by contradiction!–to the "mass inflation" scenario of Israel and Poisson.

The blue shift mechanism discovered by Penrose is crucial for the understanding of cosmic censorship in gravitational collapse, as it provides the initial impetus for fields to become large. Beyond that point, however, perturbation techniques, based on linearization, lose their effectiveness. I hope that this paper will demonstrate, if only in the context of this restricted model, that the proper setting for investigating the physical and analytical mechanisms regulating nonpredictability is provided by the theory of nonlinear partial differential equations.

2. The Einstein-Maxwell-scalar field equations under spherical symmetry

In this section we derive the Einstein-Maxwell-scalar field equations under the assumption of spherical symmetry.

For general information about the Einstein equations with matter see for instance [15]. The assumption of spherical symmetry on the metric, discussed in [7], is the statement that SO(3) acts on the spacetime by isometry. We furthermore assume that the Lie derivatives of the electromagnetic field $F_{\mu\nu}$ and the scalar field ϕ vanish in directions tangent to the group orbits.

Recall that the SO(3) action induces a 1+1-dimensional Lorentzian metric g_{ab} (with respect to local coordinates x^a) on the quotient manifold (possibly with boundary) Q, and the metric $g_{\mu\nu}$ and energy momentum tensor $T_{\mu\nu}$ take the form

$$\begin{aligned} g &= g_{ab}dx^a dx^b + r^2(x)\gamma_{AB}(y)dy^A dy^B, \\ T &= T_{ab}dx^a dx^b + r^2(x)S(x)\gamma_{AB}(y)dy^A dy^B, \end{aligned}$$

where y^A are local coordinates on the unit two-sphere and $\gamma_{AB}dy^A dy^B$ denotes its standard metric. The Einstein equations (1) reduce to the following system for r and a Lorentzian metric g_{ab} on Q:

$$K = \frac{1}{r^2}(1-\partial^a r\partial_a r) + (\mathrm{tr}T - 2S), \tag{5}$$

$$\nabla_a\nabla_b r = \frac{1}{2r}(1-\partial^c r\partial_c r)g_{ab} - r(T_{ab} - g_{ab}\mathrm{tr}T). \tag{6}$$

Here, K is the Gauss curvature of g_{ab}.

We would like to supplement equations (5) and (6) with additional equations on Q determining the evolution of the electromagnetic and scalar fields, in order to form a closed system. It turns out that, under spherical symmetry, the electromagnetic field decouples, and its contribution to the energy-momentum tensor is computable in terms of r.

To see this, first note that the requirement of spherical symmetry and the topology of S^2 together imply that $F_{aB} = 0$; also, F_{AB}, on each sphere, must equal a constant multiple of the volume form. Maxwell's equations then yield

$$F_{AB;a} = 0, \tag{7}$$

and this in turn implies that the above constant is independent of the radius of the spheres. Since the initial data described in the next section will satisfy

$$F_{AB} = 0, \tag{8}$$

by integration of (7) it follows that (8) holds identically. In the derivation of the equations, we will then assume (8) for convenience. This corresponds to the natural physical assumption that there is no magnetic charge.

It now follows that the electromagnetic contribution to the energy-momentum tensor is given by

$$T^{\mathrm{em}}_{ab} = g_{ab}\frac{1}{4}g^{cd}g^{st}F_{cs}F_{dt}. \tag{9}$$

Moreover, Maxwell's equation (2) implies that

$$F^{ab}_{;e} = -2r^{-1}\partial_e r F^{ab}. \tag{10}$$

Thus, we can compute

$$\begin{aligned}(g^{bd}g^{ac}F_{ab}F_{cd})_{;e} &= (g_{bd}g_{ac}F^{ab}F^{cd})_{;e}\\ &= g_{bd}g_{ac}F^{ab}_{;e}F^{cd} + g_{bd}g_{ac}F^{ab}F^{cd}_{;e}\\ &= -4r^{-1}\partial_e r g^{bd}g^{ac}F_{ab}F_{cd},\end{aligned}$$

which integrated gives

$$g^{bd}g^{ac}F_{ab}F_{cd} = -\frac{2e^2}{r^4},$$

where e^2 is a positive constant. We have obtained

$$T^{\mathrm{em}}_{ab} = -\frac{e^2}{2r^4}g_{ab}, \tag{11}$$

$$S^{\mathrm{em}} = \frac{e^2}{2r^4}, \tag{12}$$

and

$$\mathrm{tr}T^{\mathrm{em}} = g^{ab}T^{\mathrm{em}}_{ab} = -\frac{e^2}{r^4}. \tag{13}$$

The Maxwell equations are indeed decoupled, as their contribution to the energy-momentum tensor is computable in terms of r and the constant e. This constant is called the *charge*. We will thus no longer consider equations (2) and (3), as it is not the behavior of the electromagnetic field *per se* that is of interest, but rather its effect on the metric.

In view of the above calculations, the equations (5) and (6) for the metric reduce to

$$K = \frac{1}{r^2}(1 - \partial^a r \partial_a r) + \partial^a \phi \partial_a \phi, \tag{14}$$

$$\nabla_a \nabla_b r = \frac{1}{2r}(1 - \partial^c r \partial_c r) g_{ab} - r(\frac{e^2}{2r^4} g_{ab} + T^{sf}_{ab}), \tag{15}$$

and the wave equation (4) (see [10]) reduces to

$$g^{ab} \nabla_a \nabla_b \phi + \frac{2}{r} \partial^a r \partial_a \phi = 0. \tag{16}$$

We recall from [7] that the so-called mass function m, defined by

$$1 - \frac{2m}{r} = \partial^a r \partial_a r, \tag{17}$$

enjoys important positivity properties[5], which follow from the mass equation

$$\partial_a m = r^2 (T_{ab} - g_{ab} \mathrm{tr} T) \partial^b r. \tag{18}$$

In view of the above computations, we have

$$\partial_a m = r^2 (T^{sf}_{ab}) \partial^b r + \frac{e^2}{2r^2} \partial_a r.$$

Defining ϖ now by

$$\varpi = m + \frac{e^2}{2r},$$

we see from (11) that

$$\partial_a \varpi = r^2 (T^{sf}_{ab}) \partial^b r. \tag{19}$$

This is identical to the equation satisfied by the mass m in the Einstein-scalar field case considered in [10]. In particular, we will see that ϖ inherits the special monotonicity properties of m from that case.

Of course, the system (14)–(16) is not well-posed in the traditional sense, because of the general covariance of the equations. One can arrive at a well-posed system only after fixing the coordinates in terms of the metric. Since we will be considering an initial value problem where the initial data will be prescribed on two characteristic segments, emanating from a single point, it

[5]The proofs in [7] assumed the existence of a center of symmetry in the spacetime, which is not present in our case. For spacetimes evolving from a double characteristic initial value problem, one may substitute this assumption with an appropriate assumption on the metric on the initial characteristic segments. This assumption will hold in our problem, and thus in what follows we will refer freely to the results of [7].

is natural to introduce so-called null coordinates u and v, normalized on the initial segments. The metric in such coordinates takes the form

$$g = 2g_{uv}dudv = -\Omega^2 dudv. \tag{20}$$

The equations thus constitute a second order system for Ω, r, and ϕ.

To exploit the method of characteristics, we would like to recast the above system as a first order system. Introduce $\lambda = \partial_v r$, $\nu = \partial_u r$, $\theta = r\partial_v\phi$ and $\zeta = r\partial_u\phi$. From (17) we compute that

$$-\Omega^2 = \frac{4\partial_v r \partial_u r}{1 - \frac{2\varpi}{r} + \frac{e^2}{r^2}} = \frac{4\lambda\nu}{1-\mu}, \tag{21}$$

where we recall from [5] the notation $\mu = \frac{2m}{r}$. We thus can eliminate Ω in favor of ϖ. (Compare with [8].) It then follows that the metric and scalar field are completely described by $(r, \lambda, \nu, \varpi, \theta, \zeta)$, whose evolution in an arbitrary null coordinate system under the spherically symmetric Einstein-Maxwell-scalar field equations is governed by

$$\partial_u r = \nu, \tag{22}$$

$$\partial_v r = \lambda, \tag{23}$$

$$\partial_u \lambda = \lambda\left(-\frac{2\nu}{1-\mu}\frac{1}{r^2}\left(\frac{e^2}{r} - \varpi\right)\right), \tag{24}$$

$$\partial_v \nu = \nu\left(-\frac{2\lambda}{1-\mu}\frac{1}{r^2}\left(\frac{e^2}{r} - \varpi\right)\right), \tag{25}$$

$$\partial_u \varpi = \frac{1}{2}(1-\mu)\left(\frac{\zeta}{\nu}\right)^2 \nu, \tag{26}$$

$$\partial_v \varpi = \frac{1}{2}(1-\mu)\left(\frac{\theta}{\lambda}\right)^2 \lambda, \tag{27}$$

$$\partial_u \theta = -\frac{\zeta\lambda}{r}, \tag{28}$$

$$\partial_v \zeta = -\frac{\theta\nu}{r}. \tag{29}$$

3. The Reissner-Nordström solution

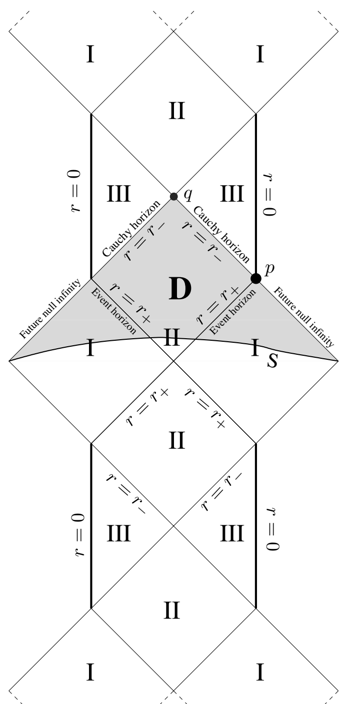
It turns out that any solution of the equations (22)–(29) with θ and ζ vanishing identically is isometric to a piece of the so-called Reissner-Nordström solution. This section outlines the most important properties of the Reissner-

Nordström solution and in particular how its nonpredictability arises. The nonpredictability of this solution will motivate the formulation of our initial value problem, in the next section.

Equations (26) and (27) and the vanishing of θ and ζ imply that ϖ is constant. The two constants e, ϖ determine a unique spherically symmetric, simply connected, maximally extended analytic Reissner-Nordström solution. Only the case $0 < e < \varpi$ will be considered here.

In view of the discussion of the introduction, the issue of predictability can be understood provided we know the conformal structure and can identify complete initial data. These aspects of the solution will be described in what follows. The reader can refer to [15] for explicit formulas for the metric in various coordinate patches.

It turns out that we can map conformally the spacetime Q of group orbits onto a domain of $1+1$-dimensional Minkowski space. Such a representation is depicted below:



The boundary of the domain is not included in Q, which is by definition open. This boundary is a convenient representation of ideal points, either singular (the part labelled $r = 0$) or "at infinity". We will not discuss the significance of future null infinity here, except to note that its intersection with the curve S is indeed "at infinity", in the sense that the total length of S in either of the I regions is infinite. The curve S thus corresponds in the 4-dimensional spacetime to a complete hypersurface with two asymptotically flat ends.

Since S is complete, uniqueness in the small holds for the initial value problem with data S, and uniqueness in the large is thus a reasonable question to ask. Yet as in the Kerr solution described in the introduction, the domain of dependence property fails outside the shaded area D. The region D corresponds to the *maximal domain of development* of the initial data. (See §5.)

Furthermore, it can be explicitly shown that the Reissner-Nordström solution, with its initial data on S, is indeed nonunique beyond the Cauchy horizon, as a solution of the initial value problem for the Einstein-Maxwell-scalar field equations. One can construct in fact an infinite family of smooth solutions extending D by first prescribing an arbitrary scalar field vanishing to infinite order on what will be two conjugate null curves, emanating to the future from the point q, and applying an appropriate local well-posedness argument. It is in this sense that the future boundary of D is a *Cauchy horizon*.

The infinite tower of regions I, II, and III indicates exactly how strange extensions beyond the Cauchy horizon can be. For the Kerr solution, there is an even more bizarre maximally analytic extension, containing closed time-like curves in the region beyond the Cauchy horizon.

Complete spacelike hypersurfaces with asymptotically flat ends satisfying the constraint equations for the spherically symmetric Einstein-Maxwell-scalar field system with nonzero charge will have topology at least as complicated as the Reissner-Nordström solution. Moreover, they will always contain a trapped surface. These global properties of solutions of this system render them totally inappropriate for studying the collapse of regular regions and the formation of trapped regions. In view of the discussion in the introduction, it is thus only in a neighborhood of the point p (from which the Cauchy horizon emanates) that the behavior of the Reissner-Nordström solution has implications on the collapse picture.

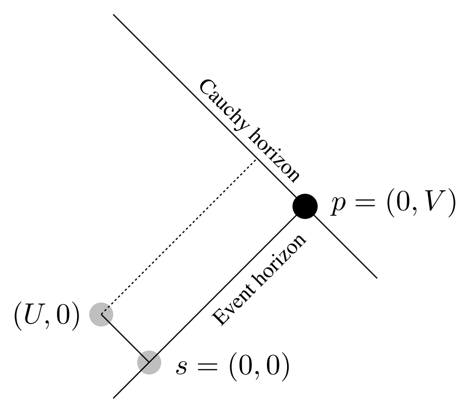
We will restrict our attention to a neighborhood of p. Let it be emphasized again that p is not included in the spacetime, as it corresponds to the point at infinity on the event horizon. The interior of region II to the future of the event horizon is trapped, *i.e.,* λ and ν are negative on it. The next section will formulate a trapped initial value problem for which the stability of the Cauchy horizon will be examined.

4. The initial value problem

A characteristic initial value problem, in an appropriate function class, will be formulated in this section. Its study, in Sections 6 and 7, will lead to the resolution of the question of predictability.

It will be convenient to retain Reissner-Nordström data on its event horizon and prescribe, along a conjugate ray, arbitrary matching data, finite in an appropriate norm. This formulation sidesteps the important question, currently open, of determining the behavior of scalar field matter on the event horizon in the vicinity of p, when these data arise in turn from complete spacelike initial data where ϕ is nonconstant in the domain of outer communications. By contrast, the data described below can easily be seen to arise from a complete spacelike hypersurface where ϕ vanishes in the domain of outer communications. Such data are the simplest ones for which the arguments in [2] [3] [18], in the context of the linearized problem, apply, and thus provide a natural starting point for studying the problem in the nonlinear setting. In fact, the method of this paper applies to a much wider class of initial data to be considered in a forthcoming paper.

We proceed to describe how initial data for $(r, \lambda, \nu, \varpi, \theta, \zeta)$ will be prescribed on two null line segments, which will define the $u = 0$ and $v = 0$ axes of our coordinate system.



On $u = 0$, the initial data will be determined from the Reissner-Nordström solution (to be denoted with the subscript RN) corresponding to the fixed parameters $0 < e < \varpi_0$. Choose a point s on the event horizon of a right-I region (strictly to the future of the point of intersection of the right-I and the corresponding left-I region) and parametrize the $u = 0$ line segment by $0 \le v \le V$ with $s = (0, 0)$ and $p = (0, V)$, and parametrization determined by the condition

$$\int_0^v \frac{\lambda_{\mathrm{RN}}}{1 - \mu_{\mathrm{RN}}}(0, v')dv' = r_+ \log \frac{V}{V - v}, \tag{30}$$

where r_+ is the larger root of $1 - \mu_{\mathrm{RN}} = 0$. With respect to these coordinates, set

$$(r, \lambda, \varpi, \theta, \zeta)|_{u=0} = (r_+, 0, \varpi_0, 0, 0).$$

Since λ and $1-\mu$ both vanish identically on the event horizon, the condition (30) needs some explanation: The equation

$$\partial_u\left(\frac{\lambda}{1-\mu}\right)=\left(\frac{\lambda}{1-\mu}\right)\frac{1}{r}\left(\frac{\zeta}{\nu}\right)^2\nu \tag{31}$$

implies that $\frac{\lambda_{\mathrm{RN}}}{1-\mu_{\mathrm{RN}}}$ is constant in u. The integral of (30) is thus equal to an integral along a parallel outgoing light ray segment contained in the interior of the right-I region of the Reissner-Nordström solution, which can be computed to be a positive function of the right endpoint, monotonically increasing to infinity as the endpoint tends to infinity. The choice (30) is thus valid.

The $v=0$ line segment will be parametrized by $0\le u\le U$, so that $\nu(u,0)=-1$, and we will prescribe an arbitrary decreasing function $\frac{\lambda}{1-\mu}(u,0)$ with u derivative vanishing at $(0,0)$. In particular, integrating (25) yields that on $0\times[0,V)$, ν equals ν_{RN} with respect to the coordinates introduced above. By (31), the u derivative of $\frac{\lambda}{1-\mu}$ will then determine ζ (up to a sign), since $r-r_+=u$. Equation (26) will determine ϖ, and thus $1-\mu$ and λ will be determined. Equation (28) then determines θ. In particular we have

$$\frac{\zeta}{\nu}\to 0 \text{ as } u\to 0. \tag{32}$$

We note that the two quantities $\frac{\zeta}{\nu}$ and $\frac{\theta}{\lambda}$ which appear naturally in $\partial_r\varpi$ satisfy the equations

$$\partial_u\frac{\theta}{\lambda}=-\frac{\zeta}{\nu}\frac{\nu}{r}-\frac{\theta}{\lambda}\left(-\frac{2\nu}{1-\mu}\frac{1}{r^2}\left(\frac{e^2}{r}-\varpi\right)\right), \tag{33}$$

$$\partial_v\frac{\zeta}{\nu}=-\frac{\theta}{\lambda}\frac{\lambda}{r}-\frac{\zeta}{\nu}\left(-\frac{2\lambda}{1-\mu}\frac{1}{r^2}\left(\frac{e^2}{r}-\varpi\right)\right), \tag{34}$$

which at times will be more convenient to work with than (28), (29).

The above parametrizations for u and v have been chosen to be symmetric in the sense that

$$\int_u^U\frac{\nu_{\mathrm{RN}}}{1-\mu_{\mathrm{RN}}}(u',0)du'=\int_{r_+-U}^{r_+-u}\frac{r^2dr}{(r-r_-)(r-r_+)}\sim\log u. \tag{35}$$

Here the notation $A\sim B$ signifies that $A<CB$ and $B<CA$ for some fixed constant C. When restricted to smaller U, (35) will also hold with ν_{RN} and $1-\mu_{\mathrm{RN}}$ replaced by the ν and $1-\mu$ of our initial data. This follows from (32), (26), and the relation

$$\partial_u(1-\mu)=\frac{-1}{r}\left(\frac{\zeta}{\nu}\right)^2\nu(1-\mu)-\frac{2\nu}{r^2}\left(\frac{e^2}{r}-\varpi\right). \tag{36}$$

Indeed, (36) and (32) imply that

$$\alpha_+<-\partial_u(1-\mu)(u,0)\le\alpha_++\varepsilon$$

for $\varepsilon = \varepsilon(U) \to 0$ as $U \to 0$, where

$$\alpha_+ = -\frac{2}{r_+^2}\left(\frac{e^2}{r_+} - \varpi_0\right) = \frac{r_+ - r_-}{r_+^2},$$

and thus

$$\frac{1}{(\alpha_+ + \varepsilon)u} \leq \frac{\nu}{1-\mu}(u, 0) < \frac{1}{\alpha_+ u}. \tag{37}$$

In particular, $1-\mu < 0$ on the interval $((0, U], 0)$, and this interval is contained in the trapped region (see [7]; this can also easily be seen to follow from (31)).

The set of all locally C^1 functions $(r, \lambda, \nu, \varpi)$ and locally C^0 functions (θ, ζ) on the null segments which can be constructed in the above way will define the class R_0. Membership in class R_0 will be the most basic assumption on initial data. We will usually need to consider initial data that satisfy the additional restriction

$$\sup_{\substack{v=0 \\ 0\leq u\leq U}} \left|\frac{\zeta}{\nu}\right| \frac{1}{u^s} \to 0 \tag{38}$$

for some $s > 0$. These will be dubbed R_1-initial data. The statements defining R_0 and R_1 can be interpreted as conditions of regularity of the scalar field across the event horizon as measured with respect to the natural parameter r.

Let it be emphasized once again that, despite the finite choice of coordinates for v, the initial data are in a very definite sense complete in the v direction. The question of predictability is thus reasonable to ask, although one has to be careful to disentangle the trivial considerations which arise from the fact that the data are incomplete in the u direction. A precise framework for examining this issue will be developed in the next section.

5. The maximal domain of development

For the initial value problem in general relativity, strong cosmic censorship is typically formulated in terms of the extendibility of the maximal domain of development. (See §8.) This extendibility can be thought of as depending on the "boundary" behavior of the solution in this domain, a concept not so easy to define. The reader should refer to [13] for definitions valid in general, and a nice discussion of the relevant concepts. Since conformal structure is locally trivial in $1 + 1$ dimensions, these issues are markedly simpler for the spherically symmetric equations, and in particular the notion of boundary for the maximal domain of development can be properly defined without recourse to complicated constructions.

We begin by mentioning that the notions of causal past, future, etc., can be formulated *a priori* in terms of our null coordinates. We define first

$$\mathbf{D}(U) = \{(u,v)|0 < u < U, 0 \leq v < V\}\,,$$

$$\overline{\mathbf{D}(U)} = \{(u,v)|0 < u < U, 0 \leq v \leq V\}\,.$$

The *causal past* of a set $\mathbf{S} \subset \mathbf{D}(U)$, denoted by $J^-(\mathbf{S})$, is then simply

$$J^-(\mathbf{S}) = \bigcup_{(u,v)\in\mathbf{S}} J^-((u,v)) = \bigcup_{(u,v)\in\mathbf{S}} \{(u',v')|0 < u' \leq u, 0 < v' \leq v\}.$$

Replacing $\leq$ in the above equation by $<$ defines the so-called *chronological past* $I^-(\mathbf{S})$. Similarly, one can define causal and chronological future $J^+(\mathbf{S})$ and $I^+(\mathbf{S})$, and thereby, in a standard way, the domain of influence and domain of dependence of an achronal set $\mathbf{S}$.

Given $(u,v) \in \mathbf{D}(U)$, a solution of the initial value problem with initial data $(\hat{r}, \hat{\lambda}, \hat{\nu}, \hat{\varpi}, \hat{\theta}, \hat{\zeta})$ of class R_0, defined on the initial null segments, are locally C^1 functions $(r, \lambda, \nu, \varpi)$ and C^0 functions (θ, ζ) defined in $I^-(u,v)$ that satisfy the equations (22)–(29), and the initial conditions

$$(r, \lambda, \nu, \varpi, \theta, \zeta)|_{\text{Initial}} = (\hat{r}, \hat{\lambda}, \hat{\nu}, \hat{\varpi}, \hat{\theta}, \hat{\zeta}).$$

Introducing the notation

$$|\psi|^k_{(u,v)} = |\psi|_{C^k(I^-(u,v))},$$

we define the norm

$$|(r, \lambda, \nu, \varpi, \theta, \zeta)|_{(u,v)} = \max\{|r^{-1}|^1_{(u,v)}, |\lambda|^1_{(u,v)}, |\nu|^1_{(u,v)}, |\varpi|^1_{(u,v)}, |\theta|^0_{(u,v)}, |\zeta|^0_{(u,v)}\}.$$

Set theoretic arguments, a local existence theorem, and the domain of dependence theorem for the function space defined by the above norm guarantee the existence of a unique solution to the initial value problem in a nonempty open set

$$\mathbf{E}(U) \subset \mathbf{D}(U),$$

uniquely determined by the properties

1. $\mathbf{E}(U)$ is a past set, i.e. $J^-(\mathbf{E}(U)) \subset \mathbf{E}(U)$, and

2. For each $(u,v) \in \partial\overline{\mathbf{E}(U)} \cap \mathbf{D}(U)$, we have

$$|(r, \lambda, \nu, \varpi, \theta, \zeta)|_{(u,v)} = \infty.$$

Here $\overline{\mathbf{E}(U)}$ denotes the closure of $\mathbf{E}(U)$ in $\overline{\mathbf{D}(U)}$. $\mathbf{E}(U)$ is the so-called *maximal domain of development* of our initial data set. We will refer to $\partial\overline{\mathbf{E}(U)}$ as the *boundary* of the maximal domain of development; it is clearly nonempty.

It turns out that for $(u, v) \in \partial\overline{\mathbf{E}(U)} \cap \mathbf{D}(U)$, we have in fact that $r(u, v) = 0$ and $\varpi(u, v) = \infty$. The proof of this is deferred to the appendix. It implies in particular that an *a priori* lower bound for $0 < c < r(u, v)$ induces $(u, v) \in \mathbf{E}(U)$. This fact will be used in the sequel without mention.

Of course, the other part of the boundary of the maximal domain of development, i.e., $\partial\overline{\mathbf{E}(U)} \setminus \mathbf{D}(U)$, if nonempty, potentially causes problems for predictability. It is not immediately clear, however, whether this set should be considered in the first place a boundary or whether it represents ideal points at infinity. (Compare with future null infinity of the Reissner-Nordström of the diagram of Section 3.) The latter scenario is excluded by the following:

PROPOSITION 1. *Let $(r, \lambda, \nu, \varpi, \theta, \zeta)$ be a solution of the equations with R_0-initial data. Then all C^1 timelike curves in $\mathbf{E}(U)$ have finite length.*

The proof here will actually only show that almost all C^1 time-like curves are of finite length. In the process, we will introduce some of the fundamental inequalities for the analysis of our equations. The reader can recover the full result of the proposition from the estimates for ν in Section 6.

For the slighter weaker result then, by virtue of the co-area formula, it suffices to bound the double integral $\int\int_{\mathbf{X}} g_{uv} du dv$, where

$$\mathbf{X} = \mathbf{E}(U)/((0, u) \times [V - v, V)),$$

in terms of a finite constant depending on u and v.

We note first, from the results of [7], that it follows immediately, for R_0 data, that $\mathbf{E}(U)$ is trapped, i.e.,

$$\nu < 0, \tag{39}$$

$$\lambda < 0, \tag{40}$$

and $1 - \mu < 0$. The reader unfamiliar with the results of [7] may derive these inequalities directly from the equations. From $1 - \mu < 0$ it follows that $r = 0$ implies $\varpi = \infty$, and thus the norm $|\varpi|_{(u,v)}$ blows up. Sequences of points (u_i, v_i) for which $r(u_i, v_i) \to 0$ must then approach the boundary. We thus have the inequality

$$r > 0. \tag{41}$$

In fact, by equations (22), (23), the above inequalities (39), (40) can be rewritten

$$\partial_u r < 0, \qquad \partial_v r < 0.$$

This in particular implies that the r function can be extended to the boundary, and sequences (u_i, v_i) as above correspond to points (u, v) on the boundary with $r(u, v) = 0$.

This immediately derives, from (30) and (31), the bound

$$\frac{\lambda}{1-\mu}(u,v) \leq \frac{\lambda}{1-\mu}(0,v) = \frac{r_+}{V-v}, \tag{42}$$

and from (37) and

$$\partial_v\left(\frac{\nu}{1-\mu}\right) = \left(\frac{\nu}{1-\mu}\right)\frac{1}{r}\left(\frac{\theta}{\lambda}\right)^2\lambda, \tag{43}$$

the bound

$$\frac{\nu}{1-\mu}(u,v) < \frac{\nu}{1-\mu}(u,0) < \frac{1}{\alpha_+ u}, \tag{44}$$

for all (u,v).

To bound now the double integral in $\mathbf{X}$, it certainly suffices to establish bounds

$$\int_0^V -g_{uv}(u,v)dv < \frac{C}{u}, \tag{45}$$

with $u > 0$, and

$$\int_0^U -g_{uv}(u,v)du < \frac{C}{V-v}. \tag{46}$$

Recall from (20) and (21) that

$$\int_0^V -g_{uv}(u,v)dv = \int_0^V \Omega^2 dv = -\int_0^V \frac{2\lambda\nu}{1-\mu}.$$

By the bounds (44) and (41) it follows that

$$-\int_0^V \frac{\lambda\nu}{1-\mu} < -\frac{1}{\alpha_+ u}\int_0^V \lambda dv < \frac{1}{\alpha_+ u} r(u,0),$$

which yields (45). The estimate (46) follows similarly by applying (42). □

It should be noted that bounds of the form (44) and (42) are a general property of spherically symmetric trapped regions, independent of the choice of matter model (in regular regions, one has only the bound (42); see [7]). Their applicability is severely restricted, however, by the fact that the bounds become degenerate near $u = 0$ or $v = V$. Of course, it is precisely this degeneracy that is responsible for the so-called blue-shift effect discussed in the introduction. On the other hand, degeneracy renders the task of controlling the solution–in its domain of existence–much more difficult. For example, integrating the equation (25) using the bound (42) or (24) using (44) in the hopes of obtaining a lower bound on r near the Cauchy horizon is fruitless.[6] It turns out that to

[6]These bounds are however useful for the issue of local existence.

exploit to the maximum extent the *control* provided by (44) and (42), one must consider various regions separately, taking advantage either of their shape or of the signs they determine. This will be one of the main themes of the next section.

6. Stability of the area radius

In this section, it will be shown that, after restricting to sufficiently small U, the maximal domain of development of R_1 data coincides with the maximal domain of development for the Reissner-Nordström solution, so that its boundary will be the Reissner-Nordström Cauchy horizon. Moreover, the behavior of r along the Cauchy horizon will approach its Reissner-Nordström value as the point at infinity on the event horizon is approached. The precise result is contained in the following:

THEOREM 1. *Let $(r, \lambda, \nu, \varpi, \theta, \zeta)$ be a solution of the equations with R_1-initial data. For sufficiently small U,*

$$\mathbf{E}(U) = \mathbf{D}(U)$$

and $r > k$ in $\mathbf{D}(U)$, for some positive $k > 0$. Moreover, r extends to a continuous function along the Cauchy horizon with

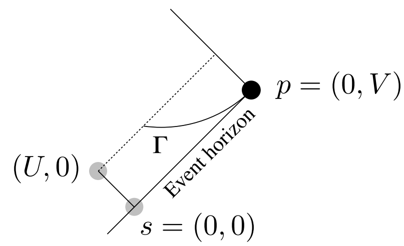
$$\lim_{u \to 0} r(u, V) = r_{-}.$$

We discussed at the end of the previous section the fact that the bounds (42) and (44), when substituted in (25) and (24), are in themselves insufficient to provide the desired global control of r. These bounds were obtained by integrating (24) and (25) in absolute value. It is clear that to obtain a better bound, one must understand the signs of the right-hand sides, or what is equivalent, the sign of the quantity

$$\frac{e^2}{r} - \varpi. \tag{47}$$

On the initial segments, this quantity is negative, bounded strictly away from zero. This is the unfavorable sign from the point of view of controlling r. One may at first hope that the region where (47) is negative could be controlled *a priori* in such a way as to control all the dangerous contributions in (25). That such an attempt is fruitless can be seen from consideration of the Reissner-Nordström solution:

In the Reissner-Nordström solution, the quantity (47) indeed monotonically increases on every line of constant u, approaching the positive ("good") constant $\frac{e^2}{r_-} - \varpi_0$, on the Cauchy horizon. In particular, there is a spacelike curve Γ terminating at $p = (0, V)$ such that $\frac{e^2}{r} - \varpi$ is negative in its past, positive in its future, and vanishes on it.



The behavior of ν on Γ, however, is already bad: $-\nu \sim u^{-1}$. All that can be obtained then is $-\nu < u^{-1}$ in the future of Γ. Integrating this bound in the future of Γ is clearly insufficient to retrieve the desired lower bound on r.

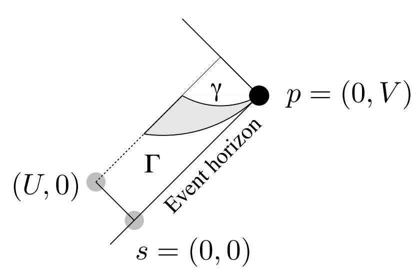
What ensures the boundedness of r from below for the Reissner-Nordström solution is the favorable contribution to ν, given by the sign of $\frac{e^2}{r} - \varpi$ in (25) in some region to the future of Γ. It would seem then that to control r in our case we would need to be able to extract a quantitative estimate of this contribution, but unfortunately, as will be shown in Section 7, one cannot expect that the Reissner-Nordström behavior of the sign of (47) will persist up to the Cauchy horizon. For if r is bounded below by a positive number, and $\varpi \to \infty$, the quantity (47) will become negative, and thus contribute again unfavorably to ν in (25).

It seems then that the proof of Theorem 1 must incorporate:

1. The existence of a definite region of favorable contribution from which we can extract a good bound for ν from (24).

2. A way of extending the bound obtained on ν in the future of this region which does not depend on the sign of $\frac{e^2}{r} - \varpi$.

Step 1 is a question of stability. The region of favorable contribution will be of the form $I^+(\Gamma) \cap I^-(\gamma)$,



where Γ is a curve corresponding to the Reissner-Nordström Γ above, to be specified in Proposition 2, and γ is defined by a relation

$$\gamma = \left\{(u,v) \mid u^Q = V - v\right\},$$

for some $Q = Q(s)$ to be chosen later. (This s will depend on the initial data; recall the definition of R_1-data.) We must derive sufficient information on the behavior of the solution in this region to extract the necessary favorable contribution. This will require a combination of a lot of bootstrapping, with careful *a priori* understanding of the geometry of the region.

Step 2 will require bounds independent of the size of the data. We will see that although it is impossible to control (47) independently of ϖ, it is possible to control the quotient

$$\frac{\frac{e^2}{r} - \varpi}{1-\mu},$$

from above, independently of ϖ. This control depends crucially on the global monotonicity properties of ϖ and r.

We are now ready to begin the proof of Theorem 1. Step 1, as outlined above, is achieved by three stability propositions, the most basic of which is

PROPOSITION 2. *Let $(r, \lambda, \nu, \varpi, \theta, \zeta)$ be a solution to the equations for R_0-initial data. For sufficiently small U, there exists a spacelike curve $\Gamma \subset \mathbf{E}(U)$, terminating at $p = (0, V)$, such that, for $(u', v') \in \Gamma$,*

$$\left(\frac{e^2}{r} - \varpi\right)(u', v') = 0 \tag{48}$$

where

$$I^-(\Gamma) \subset \mathbf{G} = \left\{(u,v) \,\middle|\, \left(\frac{e^2}{r} - \varpi\right)(u,v) \leq 0\right\}, \tag{49}$$

with $\overline{I^-(\Gamma)}$ containing in particular $(0,U) \times 0$. Moreover, as $u' \to 0$,

$$\varpi(u') \to \varpi_0, \tag{50}$$

$$r(u') \to r_0, \tag{51}$$

on Γ, where $r_0 = \frac{e^2}{\varpi_0}$.

From the equations (33) and (34) we deduce that in the region $\mathbf{G}$

$$\partial_u \left|\frac{\theta}{\lambda}\right| \leq -\left|\frac{\zeta}{\nu}\right| \frac{\nu}{r} \tag{52}$$

and

$$\partial_v \left|\frac{\zeta}{\nu}\right| \leq -\left|\frac{\theta}{\lambda}\right| \frac{\lambda}{r}. \tag{53}$$

We seek bounds on $\frac{\theta}{\lambda}$ and $\frac{\zeta}{\nu}$ at any fixed point $(\tilde{u},\tilde{v})$ such that

$$J^-(\tilde{u},\tilde{v}) \subset \mathbf{G}. \tag{54}$$

Assume (as a bootstrap assumption) a bound

$$c < r, \tag{55}$$

for some $c > 0$ to be determined later.

Integrating the inequality (52) along the $v = \tilde{v}$ edge of $J^-(\tilde{u},\tilde{v})$ gives

$$\left|\frac{\theta}{\lambda}\right|(\tilde{u},\tilde{v}) \le \int_0^{\tilde{u}} -\left|\frac{\zeta}{\nu}\right| \frac{\nu}{r}(u,\tilde{v})du.$$

Thus,

$$\left|\frac{\theta}{\lambda}\right|(\tilde{u},\tilde{v}) \le \left(\sup_{J^-(\tilde{u},\tilde{v})}\left|\frac{\zeta}{\nu}\right|\right) \int_0^{\tilde{u}} -\frac{\nu}{r}(u,\tilde{v})du.$$

This then implies

$$\left|\frac{\theta}{\lambda}\right|(\tilde{u},\tilde{v}) \le \left(\sup_{J^-(\tilde{u},\tilde{v})}\left|\frac{\zeta}{\nu}\right|\right) (\log r_+ - \log r(\tilde{u},\tilde{v})),$$

and thus,

$$\left|\frac{\theta}{\lambda}\right|(\tilde{u},\tilde{v}) \le C \sup_{J^-(\tilde{u},\tilde{v})}\left|\frac{\zeta}{\nu}\right|.$$

Since this remains true if $(\tilde{u},\tilde{v})$ is replaced by any point $(\hat{u},\hat{v}) \in J^-(\tilde{u},\tilde{v})$, we have

$$\sup_{J^-(\hat{u},\hat{v})}\left|\frac{\theta}{\lambda}\right| \le C \sup_{J^-(\hat{u},\hat{v})}\left|\frac{\zeta}{\nu}\right|. \tag{56}$$

Now integrating the inequality (53) along the $u = \hat{u}$ edge of $J^-(\hat{u},\hat{v})$ gives

$$\left|\frac{\zeta}{\nu}\right|(\hat{u},\hat{v}) \le \int_0^{\hat{v}} -\left|\frac{\theta}{\lambda}\right| \frac{\lambda}{r}(\hat{u},v)dv + \left|\frac{\zeta}{\nu}\right|(\hat{u},0).$$

Thus,

$$\left|\frac{\zeta}{\nu}\right|(\hat{u},\hat{v}) \le \int_0^{\hat{v}} \left(\sup_{J^-(\hat{u},v)}\left|\frac{\theta}{\lambda}\right|\right)\left(-\frac{\lambda}{r}\right)(\hat{u},v)dv + \left|\frac{\zeta}{\nu}\right|(\hat{u},0).$$

In fact,

$$\sup_{J^-(\hat{u},\hat{v})}\left|\frac{\zeta}{\nu}\right| \le \sup_{(u,v)\in J^-(\hat{u},\hat{v})}\left\{\int_0^{v} \left(\sup_{J^-(\hat{u},v')}\left|\frac{\theta}{\lambda}\right|\right)\left(-\frac{\lambda}{r}\right)(u,v')dv'\right\} + \sup_{\substack{v=0\\0\le u\le U}}\left|\frac{\zeta}{\nu}\right|.$$

Since the integrand is positive, r is nonincreasing in u, and $|\lambda|$ is nondecreasing in u, and by virtue of (24) and the hypothesis (54), we can bound the first supremum term on the right-hand side by the corresponding integral along the segment $\hat{u} \times [0, \hat{v}]$. That is,

$$\sup_{J^-(\hat{u},\hat{v})} \left|\frac{\zeta}{\nu}\right| \leq \int_0^{\hat{v}} \left(\sup_{J^-(\hat{u},v')} \left|\frac{\theta}{\lambda}\right| \right) \left(-\frac{\lambda}{r}\right) (\hat{u}, v')dv' + \sup_{\substack{v=0 \\ 0\leq u\leq U}} \left|\frac{\zeta}{\nu}\right|.$$

Thus, substituting this into (56) yields

$$\sup_{J^-(\hat{u},\hat{v})} \left|\frac{\theta}{\lambda}\right| \leq C \int_0^{\hat{v}} \left(\sup_{J^-(\hat{u},v')} \left|\frac{\theta}{\lambda}\right| \right) \left(-\frac{\lambda}{r}\right) (\hat{u}, v')dv' + \sup_{\substack{v=0 \\ 0\leq u\leq U}} \left|\frac{\zeta}{\nu}\right|. \tag{57}$$

Now, applying Gronwall's lemma, we obtain that

$$\left|\frac{\theta}{\lambda}\right|(\tilde{u}, \tilde{v}) \leq C \sup_{\substack{v=0 \\ 0\leq u\leq U}} \left|\frac{\zeta}{\nu}\right|, \tag{58}$$

and thus also

$$\left|\frac{\zeta}{\nu}\right|(\tilde{u}, \tilde{v}) \leq C \sup_{\substack{v=0 \\ 0\leq u\leq U}} \left|\frac{\zeta}{\nu}\right|, \tag{59}$$

for some constant C.

We will thus obtain bounds of the type (58), (59), as long as we can retrieve our assumption (55) for some choice of c.

For convenience, we introduce a further bootstrap assumption,

$$\varpi < 2\varpi_0. \tag{60}$$

Integrating equation (26), and taking account of the bounds (55), (60), and (59), we obtain

$$\varpi - \varpi_0 \leq C \left(\sup_{\substack{v=0 \\ 0\leq u\leq U}} \left|\frac{\zeta}{\nu}\right| \right)^2. \tag{61}$$

Since by (30),

$$\sup_{\substack{v=0 \\ 0\leq u\leq U}} \left|\frac{\zeta}{\nu}\right| \to 0 \tag{62}$$

as $U \to 0$, we can choose U sufficiently small so that $\varpi - \varpi_0 < \varpi_0$. In particular, this retrieves (60). But now, since $1 - \mu < 0$ in $\mathbf{D}(U)$, the relation

$$1 - \mu = 1 - \frac{2\varpi}{r} + \frac{e^2}{r^2}$$

yields the lower bound

$$\frac{e^2}{4\varpi_0} < r.$$

In particular, if $c < \frac{e^2}{4\varpi_0}$ we achieve (55) with $c' > c$ replacing c. All the bootstrap assumptions have been improved, as desired, so we indeed obtain–after a standard continuity argument–(58) and (59). Recalling the initial data, we have, for any fixed ε, the bounds

$$\left|\frac{\theta}{\lambda}\right|(u,v) < \varepsilon, \tag{63}$$

$$\left|\frac{\zeta}{\nu}\right|(u,v) < \varepsilon, \tag{64}$$

after appropriately restricting U.

Consider now the region $\mathbf{X}$ defined by

$$\mathbf{X} = \left\{(u,v)\,|\,J^{-}(u,v) \subset \mathbf{G}\right\}. \tag{65}$$

By local well posedness, and the fact that after restricting U, we have $\frac{e^2}{r} - \varpi < -C < 0$ on the initial axes $u = 0$ and $v = 0$, it follows that $\mathbf{X}$ is nonempty. Moreover, $\mathbf{X}$ is a past set, so that $\partial\mathbf{X}$ is an achronal Lipshitz curve, terminating at p.

We first exclude the case where this curve has a component on the Reissner-Nordström Cauchy horizon, i.e., the case where after sufficient restriction of U,

$$\partial\mathbf{X} = \{v = V\}.$$

If this is the case, it is clear that after restricting to even smaller U, we have (63), (64) and a lower bound on r, throughout $\mathbf{D}(U)$. The sign of (25) and the equation (43) yield, for fixed $u > 0$, a bound

$$(1-\mu)(u,v) < (1-\mu)(u,0) < -C < 0 \tag{66}$$

on $u \times [0, V]$. Integrating (31) from 0 to u, using the bound for r^{-1} and $\left|\frac{\zeta}{\nu}\right|$, we obtain

$$\frac{\lambda}{1-\mu}(u,v) \sim \frac{\lambda}{1-\mu}(0,v) = \frac{r_+}{V-v}.$$

This and our bound (66) imply that $\int_0^V \lambda(u,v)dv = -\infty$, which contradicts our bound on r, in view of (23).

It now follows that the set

$$\mathbf{Y} = \partial\mathbf{X} \cap \left\{(u,v)\left|\left(\frac{e^2}{r} - \varpi\right)(u,v) = 0\right.\right\}$$

is nonempty, and that $\partial\mathbf{X} \setminus \mathbf{Y}$ consists of null segments emanating from points of $\mathbf{Y}$. Our bounds (63) and (64) imply, however, that on $\partial\mathbf{X}$,

$$\partial_v\left(\frac{e^2}{r} - \varpi\right) = \frac{-e^2}{r^2}\lambda - \frac{1}{2}(1-\mu)\left(\frac{\theta}{\lambda}\right)^2\lambda > (-\lambda)(Ce^2 - C'\varepsilon) > 0 \tag{67}$$

and similarly

$$\partial_u \left(\frac{e^2}{r} - \varpi\right) > \frac{-e^2}{r^2}\nu - \frac{1}{2}(1-\mu)\left(\frac{\zeta}{\nu}\right)^2 \nu > 0, \tag{68}$$

after sufficient restriction of U. It follows that

$$\frac{e^2}{r} - \varpi > 0$$

on $\partial\mathbf{X} \setminus \mathbf{Y}$, and thus this set must be empty, i.e.,

$$\mathbf{Y} = \partial\mathbf{X}. \tag{69}$$

Defining now $\Gamma = \partial\mathbf{X}$, we see that the relation (49) is immediate, (48) follows from (69), and the inequalities (67) and (68) yield that Γ is a spacelike curve. The bounds (63) and (64) then give (50), and this, in conjunction with (48), gives (51). $\square$

Since we expect to use to our advantage in (24) and (25) the potential size of $\int \frac{\nu}{1-\mu} du$ and $\int \frac{\lambda}{1-\mu} dv$, special care has to be taken where $\frac{e^2}{r} - \varpi \approx 0$. We will thus need the following refinement of Proposition 2:

Proposition 3. *Let $(r, \lambda, \nu, \varpi, \theta, \zeta)$ be a solution to the equations for R_0-initial data. For sufficiently small $E > 0$ and $U = U(E)$, there exists for each $-\alpha_+ \frac{r_+^2}{2} < \eta \leq E$ a spacelike curve $\Gamma_\eta \subset \mathbf{E}(U)$, terminating at $p = (0, V)$, such that, for $(u', v') \in \Gamma_\eta$,*

$$\left(\frac{e^2}{r} - \varpi\right)(u', v') = \eta, \tag{70}$$

and

$$I^-(\Gamma_\eta) \subset \left\{ (u, v) \,\middle|\, \left(\frac{e^2}{r} - \varpi\right)(u, v) \leq \eta \right\}. \tag{71}$$

Moreover, as $u' \to 0$,

$$\varpi(u') \to \varpi_0, \tag{72}$$

$$r(u') \to r_\eta, \tag{73}$$

on Γ, where $r_\eta = \frac{e^2}{\varpi_0 + \eta}$.

We first consider the case of negative η. Defining Γ_η by

$$\left\{ \left(\frac{e^2}{r} - \varpi\right) = \eta \right\} \cap I^-(\Gamma),$$

we have, from the bounds derived in Proposition 2, that, after restricting to sufficiently small U, independent of η, Γ_η is a spacelike curve satisfying all the required conditions.

For the case of positive η, consider the region

$$\left\{0 \leq \frac{e^2}{r} - \varpi_0 \leq 3E\right\} \cap I^+(\Gamma). \tag{74}$$

For sufficiently small E, the relation

$$1 - \mu_0 = \frac{1}{r_0^2}((r_0 - e)(r_0 + e)),$$

together with the fact that $r_0 - e$ is bounded away from 0, and that $\varpi \geq \varpi_0$, yield a bound $|1 - \mu| > c > 0$ in the region defined by (74).

The locus of points satisfying $\frac{e^2}{r} - \varpi_0 = 3E$ is either empty or is an achronal curve terminating at p, since it is defined by the condition $r =$ constant. In particular, this implies that for any point (u', v') in the region defined by (74), the second inequality of (74) holds on $J^-(u', v')$. In fact, $J^-(u', v') \cap I^+(\Gamma)$ is contained in the region (74). This can be seen immediately: Since Γ is spacelike there exists a unique $v'' < v'$ such that $(u', v'') \in \Gamma$. The first inequality of (74) now follows since $r(u', v'') \geq r(u', v')$, $\left(\frac{e^2}{r} - \varpi\right)(u', v'') = 0$, and $\varpi(u, v'') \geq \varpi_0$.

We are now set to bound $\frac{\theta}{\lambda}$ and $\frac{\zeta}{\nu}$ in the region defined by (74). Integrating (33) yields

$$\left|\frac{\theta}{\lambda}\right|(u', v') \leq \left(\int_{(u'', v')}^{(u', v')} \left|\frac{\zeta}{\nu}\right| \frac{(-\nu)}{r} du + \left|\frac{\theta}{\lambda}\right|(u'', v')\right) e^{\int_{(u'', v')}^{(u', v')} \frac{2\nu}{1-\mu} \frac{1}{r^2}\left(\frac{e^2}{r} - \varpi\right) du}, \tag{75}$$

where (u'', v') is defined as the unique point on the line $v = v'$ such that $(u'', v') \in \Gamma$. The existence and uniqueness of such a point follow because Γ is spacelike and its past contains $(0, U) \times 0$.

By the above bounds, and since an *a priori* upper bound on ϖ is not necessary in view of the sign of $\frac{e^2}{r} - \varpi$ in (75), we obtain

$$\left|\frac{\theta}{\lambda}\right|(u', v') \leq C\left(\int_{(u'', v')}^{(u', v')} \left|\frac{\zeta}{\nu}\right| \frac{(-\nu)}{r} du + \left|\frac{\theta}{\lambda}\right|(u'', v')\right),$$

and similarly,

$$\left|\frac{\zeta}{\nu}\right|(u', v') \leq C\left(\int_{(u', v'')}^{(u', v')} \left|\frac{\theta}{\lambda}\right| \frac{(-\lambda)}{r} dv + \left|\frac{\zeta}{\nu}\right|(u', v'')\right),$$

where $(u, v''(u))$ is defined as the unique point on the line of constant u such that $(u, v''(u)) \in \Gamma$. The two inequalities above now yield

$$\begin{aligned}\left|\frac{\theta}{\lambda}\right|(u',v') \quad \leq \quad & C^2\int_{(u'',v')}^{(u',v')}\int_{(u,v'')}^{(u,v')}\left|\frac{\theta}{\lambda}\right|\frac{(-\lambda)}{r}dv\frac{(-\nu)}{r}du \\ & + C^2\int_{u''}^{u'}\left|\frac{\zeta}{\nu}\right|(u,v''(u))\frac{(-\nu)}{r}du \\ & + C\left|\frac{\theta}{\lambda}\right|(u'',v').\end{aligned}$$

Since $\int \lambda dv$ and $\int \nu du$ can be bounded in absolute value by CE, we obtain that

$$\left|\frac{\theta}{\lambda}\right|(u',v') \leq C'E \sup_{J^-(u',v')\cap I^+(\Gamma)}\left|\frac{\theta}{\lambda}\right| + D, \tag{76}$$

where D bounds the supremum of $\left|\frac{\theta}{\lambda}\right|$ and $\left|\frac{\zeta}{\nu}\right|$ on Γ, and thus can be made arbitrarily small by suitably restricting U. It follows that

$$\sup_{J^-(u',v')\cap I^+(\Gamma)}\left|\frac{\theta}{\lambda}\right| \leq CE \sup_{J^-(u',v')\cap I^+(\Gamma)}\left|\frac{\theta}{\lambda}\right| + D, \tag{77}$$

whence, for $E < (2C)^{-1}$,

$$\sup\left|\frac{\theta}{\lambda}\right| \leq \frac{D}{1-CE}, \tag{78}$$

which can be made arbitrarily small by appropriately restricting U.

In view of (78), (27) is a linear differential inequality for ϖ that induces the bound

$$|\varpi - \varpi_0| < E, \tag{79}$$

after restricting U. In particular we have an upper bound for $|1-\mu|$. Applying again (78), and the above bounds on r, we deduce from (67) that

$$\partial_v\left(\frac{e^2}{r} - \varpi\right) > 0. \tag{80}$$

Thus, in the region defined by (74),

$$0 \leq \frac{e^2}{r} - \varpi \leq 3E.$$

This can be slightly refined. First, note that every line of constant u must leave the region defined by (74), as the value of v is increased. For otherwise, the above bounds would imply that the solution exists up to the Reissner-Nordström Cauchy horizon, with lower bounds on r, $\left|\frac{\theta}{\lambda}\right|$, $\left|\frac{\zeta}{\nu}\right|$ and a lower bound $|1-\mu| > C'$ in the future of Γ. In that case,

$$\int_0^u \frac{1}{r}\left(\frac{\zeta}{\nu}\right)^2(-\nu)(\tilde{u},v)d\tilde{u} < \tilde{C}.$$

Then applying (31), for $(u,v'')\in\Gamma$ we would obtain

$$\begin{aligned}\int_{(u,v'')}^{(u,V)}-\lambda(u,\tilde v)d\tilde v &> C'^{-1}\int_{(u,v'')}^{(u,V)}\frac{\lambda}{1-\mu}(u,\tilde v)d\tilde v\\ &> (C'\tilde C)^{-1}\int_{(0,v'')}^{(0,V)}\frac{\lambda}{1-\mu}(u,\tilde v)d\tilde v=\infty,\end{aligned}$$

which contradicts our bound on r, by (23).

It follows now from our bounds in the region (74) that for every $0<\eta\le E$, and every u, there is a unique point $(u,v(u))$ with the property that

$$\frac{e^2}{r}(u,v)-\varpi_0=\eta+E.$$

Moreover, as the locus of points satisfying the above equality is a curve of constant r, it is clear that since λ and ν are negative, the set of such points forms a spacelike curve $\Gamma_{\eta+E,0}$. It follows that on $\Gamma_{\eta+E,0}$ we have

$$\eta\le\frac{e^2}{r}-\varpi.$$

In particular, this means that we can choose for each u a unique (by (80)) point in the region $I^+(\Gamma)\cap I^-(\Gamma_{\eta+E,0})$ such that $\frac{e^2}{r}-\varpi=\eta$, and it is clear, again from our bounds, that this set of points forms a spacelike curve Γ_η with the desired properties. □

In the region $I^-(\Gamma_E)$, good estimates for the matter were derived relatively easily, and this allowed us to prove that the Reissner-Nordström solution is in some sense stable. Such estimates are more difficult in $I^+(\Gamma_E)$. As explained at the beginning of the proof, however, we can establish stability at least on some subregion of $I^+(\Gamma_E)$. The geometry of this region must be precisely understood *a priori*. The first step is to understand the geometry of Γ_E itself, i.e., its position in (u,v) coordinates, and the behavior of λ and ν. By fixing any curve Γ_{η^*} with $\eta^*<0$, chosen from the foliation of $I^-(\Gamma_E)$ by the Γ_η produced in Proposition 3, and considering separately its past and future, one may derive bounds on the desired behavior of Γ_E. Choosing η^* smaller induces better bounds but requires more restriction on U. The optimal bounds reflect the proper combination of these choices, as explained in the following:

LEMMA 1. *For $\delta>0$ and $(u,v)\in I^+(\Gamma_{-\alpha_+r_+^2/2+\delta})\cap I^-(\Gamma_E)$,*

$$|\lambda|(u,v)\sim\left(\eta+\frac{\alpha_+r_+^2}{2}\right)\frac{1}{V-v},|\nu|(u,v)\sim\left(\eta+\frac{\alpha_+r_+^2}{2}\right)\frac{1}{u},\tag{81}$$

where η is defined by $(u,v)\in\Gamma_\eta$, and

$$\left(\frac{V}{V-v}\right)^{\alpha_+ r_+ - \varepsilon(\delta,U)} < C(\delta^{-1})\alpha_+ \frac{1}{u} < \left(\frac{V}{V-v}\right)^{\alpha_+ r_+ + \varepsilon(\delta,U)}, \tag{82}$$

for some constant ε, where $\varepsilon \to 0$ as $(\delta, U) \to (0,0)$.

To prove the two relations (81), we first note that by integrating (31) and (43) and applying the bounds in $I^-(\Gamma_E)$ derived previously for r^{-1}, $\frac{\theta}{\lambda}$, $\frac{\zeta}{\nu}$, we deduce that

$$\frac{\lambda}{1-\mu}(u,v) \sim \frac{\lambda}{1-\mu}(0,v) \sim \frac{1}{V-v}, \frac{\nu}{1-\mu}(u,v) \sim \frac{\nu}{1-\mu}(u,0) \sim \frac{1}{u}$$

in this region. Thus to show (81) it suffices to bound $|1-\mu|$ away from 0. For $\eta \geq 0$, we have $(u,v) \in I^+(\Gamma) \cap I^-(\Gamma_E)$, and we recall that a lower bound for $|1-\mu|$ in this region has already been established in the context of the proof of Proposition 3. For $(u,v) \in \Gamma_\eta$ with $\eta < 0$, we first note that $r_+ - r(u,v)$ is bounded from below by a constant multiple of $\eta + \frac{\alpha_+ r_+^2}{2}$. This is so because the upper bounds on $\left|\frac{\theta}{\lambda}\right|$ and $|1-\mu|$ imply that $\partial_r \varpi(\cdot, v)$ can be made arbitrarily small by restricting U, and thus, in particular, it is bounded. We now fix some $\eta_0 < 0$, and consider separately the cases $\eta \leq \eta_0$ and $\eta > \eta_0$. For $\eta \leq \eta_0$, since (36) implies

$$\partial_u(1-\mu) \leq -\frac{2\nu}{r^2}\left(\frac{e^2}{r} - \varpi\right), \tag{83}$$

and $\left(\frac{e^2}{r} - \varpi\right) \leq \eta_0$ in $I^-(\Gamma_{\eta_0})$, we obtain the desired lower bound on $|1-\mu|$ by integrating (83) on the segment from $(0,v)$ to (u,v). As for $\eta > \eta_0$, since the Γ_η are spacelike, there exists a point $(u', v) \in \eta_0$, with $u' < u$, and by (83) it suffices to obtain the bound at that point. We just appeal then to the previous case $\eta \leq \eta_0$.

We now turn to establishing (82). The first step is to derive the inequalities (82) just for $(u,v) \in \Gamma_{-\alpha_+ r_+^2/2+\delta}$. The idea is to compare the bounds on ν given by (81) with what one obtains by integrating (25). Since we are able to control all the terms in (25), integration will yield bounds for the size of the domain.

Integrating (25) on the segment $u \times [0,v]$, where $(u,v) \in \Gamma_{-\alpha_+ r_+^2/2+\delta}$, recalling the initial condition $\nu(u,0) = -1$, and substituting from (81), we obtain

$$C(\delta^{-1})\frac{1}{u} \sim -\nu(u,v) = \exp \int_0^v -\frac{2}{r^2}\left(\frac{e^2}{r} - \varpi\right)\frac{\lambda}{1-\mu}(u,v')dv'. \tag{84}$$

For $(u',v') \in J^-(u,v)$, we can bound the terms appearing on the right-hand side of (84) by

$$-\alpha_+ - \varepsilon(\delta, U) < \frac{2}{r^2}\left(\frac{e^2}{r} - \varpi\right)(u',v') < -\alpha_+ + \varepsilon(\delta, U),$$

and also,

$$(1-\varepsilon(\delta,U))\frac{r_+}{(V-v)} < \frac{\lambda}{1-\mu} \leq \frac{r_+}{(V-v)},$$

in view of our bounds on $\left|\frac{\zeta}{\nu}\right|$. Thus

$$\begin{aligned}(1-\varepsilon)(\alpha_+-\varepsilon)r_+\log\frac{V}{(V-v)} &= (1-\varepsilon)\int_0^v (\alpha_+-\varepsilon)r_+\frac{1}{V-v'}dv' \\ &\leq \int_0^v -\frac{2}{r^2}\left(\frac{e^2}{r}-\varpi\right)\frac{\lambda}{1-\mu}(u,v')dv' \\ &\leq \int_0^v (\alpha_++\varepsilon)r_+\frac{1}{V-v'}dv' \\ &= (\alpha_++\varepsilon)r_+\log\frac{V}{(V-v)},\end{aligned}$$

and (82) for $(u,v)\in\Gamma_{-\alpha_+r_+^2/2+\delta}$ follows from exponentiating the above inequality and then combining the result with (84), after redefinition of ε.

The next step is to show that (82) actually holds in fact for all (u',v') satisfying

$$(u',v')\in I^+(\Gamma_{-\alpha_+r_+^2/2+\delta})\cap I^-(\Gamma_E), \tag{85}$$

thus completing the proof of the lemma. The argument is similar to that of the first step, but the bounds are here easier. Since $\Gamma_{-\alpha_+r_+^2/2+\delta}$ is spacelike we can define u and v to be the unique solutions of the equations $(u,v')\in\Gamma_{-\alpha_+r_+^2/2+\delta}$, $(u',v)\in\Gamma_{-\alpha_+r_+^2/2+\delta}$, respectively. We can now bound $\frac{u}{u'}$ and $\frac{V-v'}{V-v}$ above and below by constants depending on δ^{-1} by integrating the relations (81), upon recalling (22), (23), our bounds on r, and our lower bound $|1-\mu|>c(\delta)$. It is now clear that these bounds together with (82) on $\Gamma_{-\alpha_+r_+^2/2+\delta}$ imply that (82) holds for all (u',v') satisfying (85). □

Our next proposition will require the stronger assumption (38) which defines R_1-initial data.

By the estimates (58), (59), and (78) of the two preceding propositions, it should be clear that (38) holds in fact on Γ_E, and also with $\frac{\theta}{\lambda}$ replacing $\frac{\zeta}{\nu}$.

Given s from (38), we will define a curve

$$\gamma=\left\{(u,v)\mid u^Q=V-v\right\}, \tag{86}$$

for some $Q=Q(s)>(\alpha_+r_+)^{-1}$. It turns out that if $Q(s)-(\alpha_+r_+)^{-1}$ is sufficiently small then (38) implies that $\frac{\theta}{\lambda},\frac{\zeta}{\nu}$ are bounded in $I^-(\gamma)$. These bounds then, together with Lemma 1, will allow us to derive bounds for the behavior of ν on γ, and thus also for the behavior of $1-\mu$. Proving the above statements is the content of:

PROPOSITION 4. *Let $(r, \lambda, \nu, \varpi, \theta, \zeta)$ be a solution of the equations for R_1-initial data. There exists a $Q(s)$, where s is as in the definition of R_1-data, and a $\tau > 0$, such that, after restriction of U, γ defined by* (86) *satisfies $\gamma \subset \mathbf{E}(U)$, and for $(u, v) \in \gamma$,*

$$-\nu < \frac{1}{u^{1-\tau}}(u, v).$$

Moreover, as $u \to 0$,

$$(1 - \mu)(u, v) \to 0,$$

$$r(u, v) \to r_-.$$

After restricting U, Lemma 1 implies that $\gamma \subset I^+(\Gamma_E)$. The proposition will be proved by establishing a number of bounds in $I^+(\Gamma_E) \cap I^-(\gamma)$. By a continuity argument, since Γ_E and γ are spacelike, it suffices to show that these bounds are implied by certain bootstrap assumptions, chosen so as to hold on Γ_E, providing that these assumptions can then be improved.

In what follows (u, v) will denote a fixed point in $I^-(\gamma)$. Our first bootstrap assumption is

$$(\varpi - \varpi_0)(u, v) < \varepsilon. \tag{87}$$

It follows, of course, that

$$(\varpi - \varpi_0)(u', v') < \varepsilon,$$

for $(u', v') \in J^-(u, v)$, and thus, for sufficiently small ε, we have in fact

$$B > \frac{2}{r^2}\left(\frac{e^2}{r} - \varpi\right) > \tilde{E} > 0 \tag{88}$$

in $J^-(u, v) \cap I^+(\Gamma_E)$. Implicit in the above is in particular the bound

$$0 < c < r. \tag{89}$$

In view of our assumption (87), this follows immediately from

$$\varpi - \frac{e^2}{2r} = m \geq 0.$$

Our other bootstrap assumption is

$$\left|\frac{\zeta}{\nu}\right|(u', v') < u'^{\sigma}, \tag{90}$$

for some $\sigma > 0$ which we will specify later on.

We note that (88) implies that $\partial_v(-\nu) < 0$, and so from Lemma 1 we deduce that

$$-\nu(u', v') \leq C\frac{1}{u'}.$$

Defining now the points (u'', v) and (u, v'') to be the unique (since γ and Γ_E are spacelike) points on the line segments of constant u, v, respectively, lying also on Γ_E, we have

$$\int_{u''}^{u'} \left(\frac{\zeta}{\nu}\right)^2 (-\nu)(\tilde{u}, v')d\tilde{u} < \varepsilon'(U) \tag{91}$$

where $\varepsilon' \to 0$ as $U \to 0$, and

$$\int_{u''}^{u'} \left|\frac{\zeta}{\nu}\right| (-\nu)(\tilde{u}, v')d\tilde{u} < Cu'^{\sigma}. \tag{92}$$

The bounds (88) and (44) together with Lemma 1 imply

$$\begin{aligned} &\exp\left[\int_{(u'',v')}^{(u',v')} \frac{\nu}{1-\mu}\frac{2}{r^2}\left(\frac{e^2}{r} - \varpi\right)(\tilde{u}, v')d\tilde{u}\right] \\ &\qquad < \quad \tilde{C}\left[\frac{u'}{(V-v')^{(\alpha_+ r_+ +\delta)}}\right]^{B(\alpha_+)^{-1}}, \end{aligned} \tag{93}$$

while (91) and (42) imply that

$$\left(1 - \frac{\varepsilon'}{c}\right)\frac{r_+}{V - v'} < \frac{\lambda}{1-\mu}(u', v') \leq \frac{r_+}{V - v'} \tag{94}$$

and thus

$$\begin{aligned} &\tilde{C}^{-1}\left[\frac{u'^{(\alpha_+ r_+ -\delta)^{-1}}}{V - v'}\right]^{\tilde{E}(1-\varepsilon'/c)} \\ &\qquad < \quad \exp\left[\int_{(u',v'')}^{(u',v')} \frac{\lambda}{1-\mu}\frac{2}{r^2}\left(\frac{e^2}{r} - \varpi\right)(u', \tilde{v})d\tilde{v}\right] \\ &\qquad < \quad \tilde{C}\left[\frac{u'^{(\alpha_+ r_+ +\delta)^{-1}}}{V - v'}\right]^{Br_+}. \end{aligned} \tag{95}$$

Integrating (33) and applying (93) yield

$$\begin{aligned} \left|\frac{\theta}{\lambda}\right|(u', v') \quad &\leq \quad \tilde{C}\left[\frac{u'}{(V-v')^{(\alpha_+ r_+ +\delta)}}\right]^{B(\alpha_+)^{-1}} \\ &\cdot\left\{\int_{(u'',v')}^{(u',v')} \left|\frac{\zeta}{\nu}\right| \frac{-\nu}{r}(\tilde{u}, v')d\tilde{u} + \left|\frac{\theta}{\lambda}\right|(u'', v')\right\}. \end{aligned} \tag{96}$$

Fixing $\tau > 0$, through appropriate choice of δ_0 and $Q_0 > (\alpha_+ r_+)^{-1}$, we can ensure that

$$\left[\frac{u'}{(V-v')^{(\alpha_+ r_+ +\delta)}}\right]^{B\alpha_+^{-1}} \leq u'^{-\tau}$$

for all $\delta \le \delta_0$, $(r_+\alpha_+)^{-1} < Q \le Q_0$. This can be seen as follows: Defining v by $(u', v) \in \gamma$, note that $V - v' \ge V - v = u'^Q$ yields the bound

$$\begin{aligned}\left[\frac{u'}{(V-v')^{(\alpha_+ r_+ + \delta)}}\right]^{B(\alpha_+)^{-1}} &\le \left[\frac{u'}{u'^{Q(\alpha_+ r_+ + \delta)}}\right]^{B(\alpha_+)^{-1}} \\ &= u'^{-B(\alpha_+)^{-1}[Q(\alpha_+ r_+ + \delta)-1]}.\end{aligned}$$

Thus we must simply choose δ_0 and Q_0 such that

$$B(\alpha_+)^{-1}[Q_0(\alpha_+ r_+ + \delta_0) - 1] \le \tau.$$

In particular, we can make such choices for $\tau = \sigma/2$.

Substituting the above bound and the bootstrap assumption (92) into (96) and recalling the definition of R_1-data, it follows that if σ is chosen less than s, we have

$$\left|\frac{\theta}{\lambda}\right|(u', v') \le u'^{\tau}. \tag{97}$$

We have absorbed here the constant $\tilde{C}$ by restricting U.

Integrating the equation (34) and applying (95), we obtain

$$\begin{aligned}\left|\frac{\zeta}{\nu}\right|(u', v') \le{}& \left[\frac{u'^{(\alpha_+ r_+ + \delta)^{-1}}}{(V-v')}\right]^{Br_+} \\ &\cdot\left\{\int_{v''}^{v'}\left[\frac{u'^{(\alpha_+ r_+ - \delta)^{-1}}}{V-\tilde{v}}\right]^{-\tilde{E}(1-\varepsilon'/c)}\left|\frac{\theta}{\lambda}\right|\frac{(-\lambda)}{r}(u', \tilde{v})d\tilde{v}\right. \\ &\left.\quad + \left|\frac{\zeta}{\nu}\right|(u', v'')\right\}.\end{aligned} \tag{98}$$

For $(u, v') \in \gamma$, we have

$$\left[\frac{u'^{(\alpha_+ r_+ + \delta)^{-1}}}{(V-v')}\right]^{Br_+} \le \left[\frac{u^{(\alpha_+ r_+ + \delta)^{-1}}}{(V-v')}\right]^{Br_+} = (V-v')^{-[1-Q^{-1}(\alpha_+ r_+ + \delta)^{-1}]Br_+},$$

and thus, given $\tilde{\tau} > 0$, choosing δ_1, Q_1 such that

$$[1 - Q_1^{-1}(\alpha_+ r_+ + \delta_1)^{-1}]Br_+ \le \tilde{\tau},$$

gives

$$\left[\frac{u'^{(\alpha_+ r_+ + \delta)^{-1}}}{(V-v')}\right]^{Br_+} \le (V-v')^{-\tilde{\tau}}$$

for all $\delta \le \delta_1$, $(\alpha_+ r_+)^{-1} < Q \le Q_1$. From (97) we obtain

$$\left|\frac{\theta}{\lambda}\right|(u', \tilde{v}) \le u'^{\tau} \le u^{\tau} = (V-v')^{\tau Q^{-1}}$$

after appropriately restricting Q, and (88) implies that $\partial_u(-\lambda) < 0$, and thus

$$-\lambda(u,\tilde{v}) < C\frac{1}{V-\tilde{v}}, \tag{99}$$

by Lemma 1. We compute

$$\begin{aligned}\int_{v''}^{v'} \left[\frac{u'^{(\alpha_+r_+-\delta)^{-1}}}{V-\tilde{v}}\right]^{-\tilde{E}(1-\varepsilon'/c)} & \left|\frac{\theta}{\lambda}\right| \frac{(-\lambda)}{r}(u',\tilde{v})d\tilde{v} \\ &\leq C\left[\frac{u'^{(\alpha_+r_+-\delta)^{-1}}}{V-v'}\right]^{-\tilde{E}(1-\varepsilon'/c)} (V-v')^{\tau Q^{-1}} \\ &\leq (V-v')^{\tilde{E}/2},\end{aligned}$$

where here we must restrict U, and choose Q_2, δ_2 such that

$$\tilde{E}\left(1-\frac{\varepsilon'}{c}\right)[1-Q^{-1}(\alpha_+r_+-\delta)^{-1}] \geq \frac{\tilde{E}}{2}$$

for $\delta \leq \delta_2$, $(\alpha_+r_+)^{-1} < Q \leq Q_2$. Again restricting U, we obtain

$$\left|\frac{\zeta}{\nu}\right|(u',v'') \leq u'^s = (V-v')^{Q^{-1}s},$$

and thus, from the above,

$$\begin{aligned}\left|\frac{\zeta}{\nu}\right|(u',v') &\leq (V-v')^{-\tilde{\tau}+Q^{-1}s} + (V-v')^{-\tilde{\tau}+\tilde{E}/2} \\ &= u^{-\tilde{\tau}Q+s} + u^{-\tilde{\tau}Q+Q\tilde{E}/2} \\ &\leq u'^{-\tilde{\tau}Q+s} + u'^{-\tilde{\tau}Q+Q\tilde{E}/2},\end{aligned}$$

which improves our bootstrap assumption (90) after restricting U, provided that $\sigma, \tilde{\tau}$ are chosen such that $-\tilde{\tau}Q+s > \sigma$ and $-\tilde{\tau}Q+Q\tilde{E}/2 > \sigma$. The bound (87) is now easily improved by applying (91). This concludes the bootstrap argument.

We now proceed to obtain bounds for ν on γ. Let (u,v) denote a point on γ. As before (u,v'') will be on Γ_E. From (95) and (25) it follows that

$$-\nu(u,v) \leq -\tilde{C}\nu(u,v'')\left[\frac{u^{(\alpha_+r_+-\delta)^{-1}}}{V-v}\right]^{-\tilde{E}(1-\varepsilon'/c)}$$

and thus, for appropriate choice of constants and restriction of U, we deduce from Lemma 1 that

$$-\nu < \frac{1}{u^{1-\tau}}, \tag{100}$$

for some $0 < \tau < 1$.

In analogy with (91), we certainly have

$$\int_{v''}^{v} \left(\frac{\theta}{\lambda}\right)^2 (-\lambda)(u', v')dv' < \varepsilon'(U), \tag{101}$$

and thus,

$$\frac{\nu}{1-\mu}(u, v) > C^{-1}\frac{1}{u}. \tag{102}$$

The bound (100) along with (102) now show that $(1-\mu)(u, v) \to 0$ as $u \to 0$. Since (101) implies that $\varpi(u, v) \to \varpi_0$, it follows that $r(u, v)$ must approach a root of

$$r^2 - 2\varpi_0 r + e^2 = 0,$$

and since r is bounded above away from r_+ on Γ, that root must be r_-, and the proposition is proven. □

This completes Step 1 of the proof of Theorem 1. The second part of the proof amounts basically to showing that (100) carries over to the future of γ.

It is convenient to argue by contradiction. Assume that there exists a spacelike curve $r = r_- - k$ terminating at p for some k. (This is certainly incompatible with the statement of the theorem.)

We now proceed to bound ν in the region

$$\mathbf{S} = I^+(\gamma) \cap \{r \geq r_- - k\}.$$

We partition this region into three sets $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$ where

$$\begin{aligned} \mathbf{S}_1 &= \left\{\frac{\varpi}{r} > C\right\} \cap \mathbf{S}, \\ \mathbf{S}_2 &= \left\{\frac{e^2}{r} - \varpi > 0\right\} \cap \mathbf{S}, \\ \mathbf{S}_3 &= (\mathbf{S}_1 \cup \mathbf{S}_2)^c \cap \mathbf{S}, \end{aligned}$$

and C must be chosen large enough, in a way which will become apparent below.

We will show that we have a bound

$$-\nu < \frac{\tilde{C}}{u^{1-\tau}}. \tag{103}$$

Since such a bound holds on γ by Proposition 4, and also $\partial_v(-\nu) < 0$ on $\mathbf{S}_2$, it follows that it suffices to prove that the bound (103) holds for each point (u', v') of $\mathbf{S}_1 \cup \mathbf{S}_3$ provided that it holds for a corresponding point (u', v'') defined uniquely by the two relations

$$\left(\frac{e^2}{r} - \varpi\right)(u', v'') = 0, \tag{104}$$

and

$$\left(\frac{e^2}{r} - \varpi\right)(u', v) \leq 0, \tag{105}$$

for $v'' \leq v \leq v'$.

In $\mathbf{S}_1$, we can bound

$$\frac{1}{r^2}\frac{\frac{e^2}{r} - \varpi}{1-\mu} = \frac{1}{r}\frac{\frac{e^2}{r^2} - \frac{\varpi}{r}}{1 - 2\frac{\varpi}{r} + \frac{e^2}{r^2}}$$

by C and the lower bound on r provided that the former is big enough in comparison to the latter. If $(u', v') \in \mathbf{S}_3$, we note that $(1-\mu)(u', v')$ is bounded from above by $(1-\mu)(u', v'')$. (See (36).) In view of (104), $(1-\mu)(u', v'')$ is seen to be bounded away from zero by writing

$$1 - \mu(u', v'') = \frac{1}{r(u', v'')^2}(r(u', v'') - e)(r(u', v'') + e)$$

and recalling that $r < e - \varepsilon$ in $I^+(\Gamma)$. This then bounds

$$\frac{1}{r^2}\frac{\frac{e^2}{r} - \varpi}{1-\mu}$$

in absolute value, since on $\mathbf{S}_3$, ϖ is bounded. From the above and the bound on $\int -\lambda dv$ given by our bounds on r, a bound (103) throughout the region $I^+(\gamma) \cap \{r \geq r_- - k\}$ follows from (25).

In view of Proposition 4, we can restrict U so that r is bounded below by $r_- - \frac{k}{2}$ on γ. By our hypothesis that the curve $r = r_- - k$ terminates at p, we now have

$$(r_- - \frac{k}{2}) - (r_- - k) < \int_{(u',v')}^{(u'',v')} -\nu dv$$

for $(u', v') \in \gamma$, where (u'', v') is defined uniquely by $(u'', v') \in \{r = r_- - k\}$, and thus from (103),

$$\frac{k}{2} < C(u''^{\tau} - u'^{\tau}). \tag{106}$$

Since $(u''^{\tau} - u'^{\tau}) \to 0$, as $v' \to V$, we obtain a contradiction.

It follows then that in $\mathbf{D}(U)$, we have $r > r_- - k$ with $k \to 0$ as $U \to 0$. The solution thus exists up to the Cauchy horizon, and by our bounds on ν, it follows that r is in fact continuous with $\lim_{u\to 0} r(u, V) = r_-$. The theorem is proved. $\square$

7. Blow-up of the mass

In the previous section, it was shown that there exists a nontrivial class of initial data such that for small enough U, the solution exists all the way up to the Cauchy horizon. We are interested in determining the behavior of the mass parameter ϖ of the solutions as the Cauchy horizon is approached. In this section, we will construct a large set of initial data for which the mass parameter blows up.

We note that since $\zeta = \theta = 0$ at the origin of our coordinate system, the condition that $\zeta > 0$ on $v = 0$ in a sufficiently small neighborhood of the origin minus the origin itself defines an open set of initial data. This of course implies that θ is positive on that neighborhood.

We define a new class of data R_2 by the open conditions

$$\theta, \zeta > 0 \text{ on } (0, U] \times 0, \tag{107}$$

and

$$\zeta > cu^s \text{ on some interval } (0, u') \subset (0, U), \tag{108}$$

for some $s < s_0$, where $s_0 > 0$ is a constant depending on the fraction $\frac{r_-}{r_+}$, satisfying $s_0 \to \infty$ as $\frac{r_-}{r_+} \to 0$. We will only consider the case

$$\frac{r_-}{r_+} < \frac{1}{\sqrt{2}}. \tag{109}$$

A more careful analysis based on the first order perturbation theory [3] could perhaps entirely remove the condition (108) if a condition similar to (109) is satisfied. Note that (109), for fixed ϖ_0, is a smallness assumption on e.

The condition (107) implies a monotonicity property for the scalar field, which will play a fundamental role in this section:

Proposition 5. *Let ζ, θ be a solution of the equations* (28), (29) *in a region $\mathbf{R}$ satisfying $\nu \le 0$, $\lambda \le 0$, with ν and λ in C^1_{loc}, and $r > 0$. If $\zeta \ge 0$, $\theta \ge 0$ on an achronal curve K, then $\zeta \ge 0$, $\theta \ge 0$ and*

$$\partial_v \zeta \ge 0, \qquad \partial_u \theta \ge 0 \tag{110}$$

in $\mathbf{R} \cap d(K)$ where $d(K)$ denotes the domain of dependence of K.

Notice first that if K were an achronal curve for which the *strict* inequalities

$$\theta, \zeta > c > 0 \tag{111}$$

held, then it would follow that

$$\theta, \zeta \ge 0 \tag{112}$$

in $d(K)$. For if not, there would exist a point (u,v) such that $\theta(u,v)=0$ or $\zeta(u,v)=0$, but $\theta,\zeta>0$ in $d(K)\cap J^-(u,v)$. Integration of the equations (28) and (29) would then yield a contradiction.

We can now extend (112) to the case where the inequalities (111) are not strict by noting the following continuity principle:

Given a fixed metric satisfying the conditions of the theorem, the initial value problem for θ and ζ is well posed on all compact subsets of K, for instance in the C^0 norm. In particular, we can approximate θ and ζ on K in this norm by a sequence θ_i, ζ_i satisfying the strict inequalities (111). The inequality (112) is satisfied for all θ_i, ζ_i, and by continuous dependence on the data, it is thus satisfied for θ, ζ.

The inequality (112), our assumptions on ν, λ, and the equations (28) and (29) now imply that $\partial_u\theta\geq 0$ and $\partial_v\zeta\geq 0$, as desired. □

In particular, we can apply the above proposition to our solutions with K equal to our initial null segments to yield the inequalities (110) in $\mathbf{D}(U)$. We see, applying again (28) and (29) and our knowledge of λ and ν that the inequalities (110) are strict in this case.

The monotonicity established above can now be applied to the mass difference. This gives the following:

Proposition 6. *Let $(r,\lambda,\nu,\varpi,\theta,\zeta)$ be a solution of the equations satisfying* (107). *Then*

$$\varpi(u'',v'')-\varpi(u',v'')\geq\varpi(u'',v')-\varpi(u',v') \tag{113}$$

for $u''>u'$ and $v''>v'$, and similarly, with the roles of u and v interchanged.

We will prove (113); the analogous result with u, v interchanged follows immediately. We can write (26) as

$$\partial_u\varpi=\frac{1}{2}\zeta^2\frac{1-\mu}{\nu}. \tag{114}$$

Now, by Proposition 5, it follows that $\partial_v(\zeta^2)\geq 0$. Also, it is clear from (44) that

$$\partial_v\left(\frac{1-\mu}{\nu}\right)\geq 0.$$

Thus,

$$\partial_u\varpi(u,v'')\geq\partial_u\varpi(u,v').$$

Integrating the above inequality from u' to u'', gives the result of the proposition. □

We are now ready to prove the main result of this section:

THEOREM 2. *Let $(r, \lambda, \nu, \varpi, \theta, \zeta)$ be a solution of the equations with R_2-initial data. Then $\varpi = \infty$ on the Cauchy horizon.*

As explained in the introduction, this result is motivated by what is essentially a first-order perturbation theory calculation. For the considered class of initial data, we solve the linear system for $\theta_{\rm lin}$, $\zeta_{\rm lin}$ on the fixed Reissner-Nordström background, and compute that for $v < V$, $u > 0$,

$$\int_v^V \left(\frac{\theta^2_{\rm lin}}{|\lambda_{\rm RN}|} \right)(u, v')dv' = \infty. \tag{115}$$

To first order, then, the back-effect on ϖ is computed by integrating (27). Writing $1 - \mu = 1 - \frac{2\varpi}{r} + \frac{e^2}{r^2}$, and integrating (27) as a linear equation in ϖ we have that ϖ blows up at the Cauchy horizon.

In view of the equations (28) and (29) for θ, ζ, and the behavior of λ, ν in $I^+(\Gamma)$, it is not unreasonable to hope that in the nonlinear problem, θ is at least as large as $\theta_{\rm lin}$. It turns out that condition (108) implies that the full extent of the linear theory is not necessary to obtain (115), but rather, simple ordinary differential equation estimates, along with monotonicity, suffice. We will see that this carries over to the nonlinear case.

The quantity λ, however, poses a difficulty. It can easily be seen that (essentially) $|\lambda| > |\lambda_{\rm RN}|$, and thus $|\lambda^{-1}| < |\lambda^{-1}_{\rm RN}|$. It is not immediately clear whether the gain in θ is sufficient to overcome the loss in $|\lambda^{-1}|$, so as to be able to show that (115) is still true in the nonlinear case.

It follows from the above discussion that the calculation (115), as it stands, is useful for the nonlinear theory only in the case where

$$\lambda \sim \lambda_{\rm RN}. \tag{116}$$

In Step 1 of the proof, we show that the assumption

$$\lim_{u \to 0} \varpi(u, V) = \varpi_0 \tag{117}$$

implies essentially (116), and thus allows the "first-order" argument–as we have described it above–to go through, yielding infinite mass, and thus a contradiction. The fact that (117) is sufficient to imply essentially (116) depends heavily on the monotonicity of Proposition 5, and in addition, on the monotonicity of λ in $I^+(\Gamma)$ that will follow from (117).

This will reduce the problem to the case

$$\lim_{u \to 0} \varpi(u, V) > \varpi_0. \tag{118}$$

Step 2 will show how this inequality immediately implies that the mass blows up. The argument depends entirely on the monotonicity of Proposition 6: The crucial fact is that since there exists a spacelike curve Γ terminating at $(0, V)$ along which $\varpi(u)|_\Gamma \to \varpi_0$ as $u \to 0$, the assumption (118) implies that a mass difference bounded below away from 0 between Γ and the Cauchy horizon persists as $u \to 0$. The monotonicity of Proposition 6 then yields infinite mass difference along any curve of constant $u > 0$.

We are now prepared to carry out the proof. Assume first (117). We hope to derive a contradiction (Step 1).

It is clear (for instance from our construction of Γ_E), that in the case of (117), we have in fact that

$$\frac{e^2}{r} - \varpi > 0$$

in $I^+(\Gamma)$, after restricting U appropriately. This then implies the important inequality

$$\partial_u\left(-\frac{\theta}{\lambda}\right) \geq 0 \tag{119}$$

in $I^+(\Gamma)$.

It turns out that (119) allows us to control the behavior of $\frac{\nu}{1-\mu}$. First note that for each $u' < U$, denoting $(u', v'(u'))$ to be the unique point satisfying $(u', v') \in \gamma$, we certainly have

$$\int_{(u',v')}^{(u',V)} \left(\frac{\theta}{\lambda}\right)^2 (-\lambda) dv < \infty. \tag{120}$$

For otherwise, integrating (27), we obtain

$$\begin{aligned}\varpi(u', v) = &\Bigg\{\varpi(u', v') \\ &- \int_{v'}^{v} \frac{1}{2}\left(\frac{\theta}{\lambda}\right)^2 (-\lambda)\left(1 + \frac{e^2}{r^2}\right) \exp\left[-\int_{(u',v')}^{(u',\hat{v})} \frac{1}{r}\left(\frac{\theta}{\lambda}\right)^2 (-\lambda) d\tilde{v}\right] d\hat{v}\Bigg\} \\ &\cdot \exp\left[\int_{(u',v')}^{(u',v)} \frac{1}{r}\left(\frac{\theta}{\lambda}\right)^2 (-\lambda) d\tilde{v}\right],\end{aligned}$$

and by the bound $r \leq r_- + \varepsilon$ in $I^+(\gamma)$, after restricting U, and noting the inequality $\varpi(u', v') \geq \varpi_0$, we can bound the integral of the second term by

$$\varepsilon' \varpi(u', v') \int_{v'}^{v} \frac{1}{r}\left(\frac{\theta}{\lambda}\right)^2 (-\lambda) \exp\left[-\int_{(u',v')}^{(u',\hat{v})} \frac{1}{r}\left(\frac{\theta}{\lambda}\right)^2 (-\lambda) d\tilde{v}\right] d\hat{v} < \varepsilon' \varpi(u', v').$$

If U is restricted so that $\varepsilon' < 1$, this gives that $\varpi \to \infty$, contradicting (117).

The inequality (120) now implies that as $v_1, v_2 \to V$, $v_2 > v_1$,

$$\int_{(u',v_1)}^{(u',v_2)} \left(\frac{\theta}{\lambda}\right)^2 (-\lambda) dv \to 0.$$

By (119) it follows that for $u \leq u'$,

$$\int_{(u,v_1)}^{(u,v_2)} \left(\frac{\theta}{\lambda}\right)^2 (-\lambda) dv \leq \int_{(u',v_1)}^{(u',v_2)} \left(\frac{\theta}{\lambda}\right)^2 (-\lambda) dv.$$

Since γ is a spacelike curve terminating at p, in particular $u' \to 0$ implies $v' \to V$, it is clear that

$$\lim_{U\to 0} \sup_{0<u'<U} \int_{(u',v')}^{(u',V)} \left(\frac{\theta}{\lambda}\right)^2 (-\lambda) dv = 0. \tag{121}$$

The bound (121) and the known properties of $\frac{\nu}{1-\mu}$ on γ now clearly imply that

$$\frac{\nu}{1-\mu}(u,v) \geq \frac{1}{(\alpha_+ + \varepsilon)u}, \tag{122}$$

for some constant ε satisfying $\varepsilon \to 0$ as $U \to 0$.

We now proceed to see how (122) also determines the behavior of λ along any line $u = u_0$ as $v \to V$. First, note that from our assumption (117) and the result of Theorem 1, we have

$$\left|\frac{2}{r^2}\left(\frac{e^2}{r} - \varpi\right) - \alpha_-\right| < \varepsilon(U) \tag{123}$$

in $I^+(\gamma)$, where $\varepsilon(U) \to 0$ as $U \to 0$, and α_- is defined by

$$\alpha_- = \frac{2}{r_-^2}\left(\frac{e^2}{r_-} - \varpi_0\right) = \frac{r_+ - r_-}{r_-^2}.$$

In view of (123) and (122), we have determined bounds in $J^+(\gamma)$ for all the factors in (24).

On γ itself, Lemma 1 and the fact that $-\lambda$ is nonincreasing in u on $J^+(\Gamma) \cap J^-(\gamma)$ yields a bound

$$-\lambda(u,v) < \frac{C}{V-v}.$$

By virtue of the above bound, it follows immediately from integration of (24) that

$$\begin{aligned} -\lambda(u,v) &\leq (V-v)^{Q^{-1}\alpha_-\alpha_+^{-1}-\varepsilon-1} u^{-(\alpha_-\alpha_+^{-1}+\varepsilon)} \\ &< (V-v)^{r_+\alpha_- -\varepsilon-1} u^{-\alpha_-\alpha_+^{-1}-\varepsilon} \end{aligned} \tag{124}$$

in $I^+(\gamma)$, where ε is some generic constant with $\varepsilon \to 0$ as $U \to 0$. The dependence $\varepsilon(U)$ includes of course a choice of $Q(U)$ such that $Q - (\alpha_+ r_+)^{-1} \to 0$ as $U \to 0$. Thus, assuming only (117), we have obtained that λ behaves essentially like λ_{RN}.

On the other hand, we can quickly obtain rough[7] lower bounds for θ in two stages.

[7]By not deriving sharp bounds, we lose in accuracy *vis à vis* the predictions of first order perturbation theory.

First, we determine lower bounds for the behavior of θ on $\Gamma_{\alpha_+ r_+^2/2+\delta}$, through integration of (28). This in turn requires lower bounds for $-\lambda$ in $I^-(\Gamma_{\alpha_+ r_+^2/2+\delta})$. Writing for instance

$$\lambda = \frac{\lambda}{1-\mu}\frac{1-\mu}{\nu}\nu,$$

and recalling our bounds on the first two factors on the right-hand side, as well as

$$-\nu \geq (V-v)^{-\alpha_+ r_+ + \delta},$$

which follows from Lemma 1, we obtain

$$-\lambda \geq (V-v)^{-\alpha_+ r_+ -1+\delta} u.$$

Substituting this relation in (28), and noting that after restricting U, we have by (108) that $\zeta(u,0) > cu^s$, and thus by Proposition 5, $\zeta(u,v) > cu^s$, we obtain

$$\theta(u,v) \geq (V-v)^{-\alpha_+ r_+ -1+\delta+(s+2)(\alpha_+ r_+ +\delta)}, \tag{125}$$

for $(u,v) \in \Gamma_{\alpha_+ r_+^2/2+\delta}$. Now, we simply note that (125) must hold in $I^-(\Gamma_{\alpha_+ r_+^2/2+\delta})$ by virtue of Proposition 5.

Computing

$$\int_{(u,v)}^{(u,V)} \left(\frac{\theta}{\lambda}\right)^2 (-\lambda) dv,$$

using the above bounds (125) and (124), after suitable choice of δ and U, it is clear that this integral is infinite for s satisfying

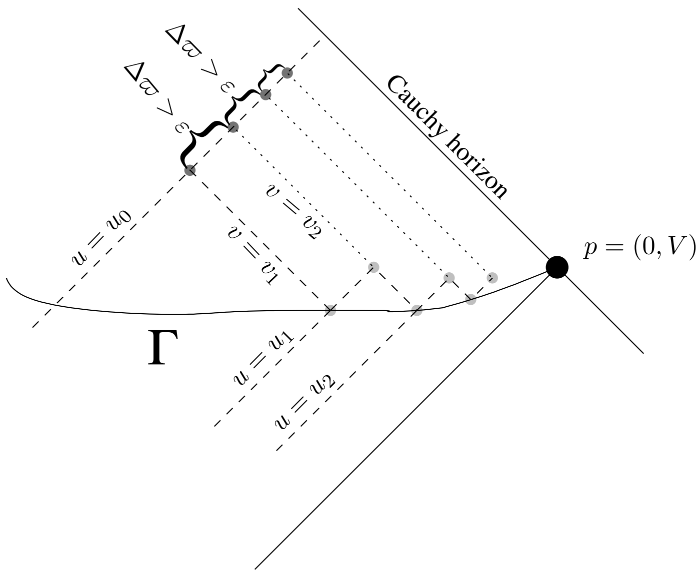
$$s < \frac{1}{2}\left(\frac{r_+}{r_-}\right)^2 - 1, \tag{126}$$

and this contradicts (121). Note that (109) implies that solutions to (126) exist, and that we can choose $s \to \infty$ as $\frac{r_-}{r_+} \to 0$. This finishes the case (117).

We have thus reduced the problem to the case where (118) holds (Step 2).

Define $\varpi^* = \inf_{v=V} \varpi$. By (118), $\varpi^* > \varpi_0$. Thus, given ε small, we can choose a U such that $\varpi|_\Gamma < \varpi_0 + \varepsilon$, where $\varpi_0 + 3\varepsilon < \varpi^*$.

The argument will be similar to the one that obtained (121), except that here we will not argue by contradiction. Two sequences of points (u_i, v_i), $(u_i, \tilde{v}_i)$ will be constructed as follows: For some $(u_i, v_i) \in \Gamma$, let $(u_i, \tilde{v}_i)$ be the point where $\varpi(u_i, \tilde{v}_i) = \varpi_0 + 2\varepsilon$. Define now $v_{i+1} = \tilde{v}_i$ and choose u_{i+1} so that $(u_{i+1}, v_{i+1}) \in \Gamma$. The original (u_1, v_1) can be chosen arbitrarily on Γ. We note that this procedure produces an infinite sequence of points, and of course $v_{i+1} > v_i$.



Now choose a $u_0 > u_1$ and consider the points (u_0, v_i) and $(u_0, \tilde{v}_i)$. From (113), it is clear that $\varpi(u_0, \tilde{v}_i) - \varpi(u_0, v_i) > \varepsilon$. Thus

$$\varpi(u_0, v_N) \geq \varpi(u_0, v_1) + \sum_{i=1}^{N} \varpi(u_0, \tilde{v}_i) - \varpi(u_0, v_i) > N\varepsilon.$$

It follows that as $N \to \infty$, $\varpi(u_0, v_N) \to \infty$, i.e. $\varpi = \infty$ on the Cauchy horizon. $\square$

8. Predictability and strong cosmic censorship

The notion of predictability is intimately related to the required regularity for solutions to the Einstein equations. For a careful general discussion of some aspects of these issues, the reader may consult references [12], [13] and [6]. This section will present two alternative notions of predictability that have appeared in the literature, one for relatively high, the other for relatively low regularity, and discuss the status of each in the context of the initial value problem of this paper and in the light of the results of Sections 6 and 7.

As strong cosmic censorship is the conjecture that "generic" initial data in the class of physically reasonable data is predictable, it turns out that the two different formulations of predictability will lead to different "verdicts" for the conjecture. The reader may be concerned that the class of initial data considered here does not seem broad enough to properly address the problem of "genericity". In particular, as explained in the beginning of Section 4,

R_1 data do *not* in fact arise from sufficiently general, physically appropriate, spherically symmetric, spacelike, complete initial data.[8] Nonetheless, as discussed in the introduction, from the point of view of the arguments based on linearization (see [3], [19]) which originally motivated the formulation of strong cosmic censorship, R_1 data can be thought of as a first attempt at a sufficiently general class of data, and R_2 data can be thought of as "generic" in this class. It is in this informal sense that we will refer in this section to our result as "supporting" or "not supporting" the strong cosmic censorship conjecture.

The two formulations of predictability to be presented in this section are applicable to the initial value problem for any Einstein-matter system for which there is a well-defined notion of *maximal domain of development*. In the class of solutions considered here for the equations of this paper, such a notion was defined in Section 5. For another definite example, one can consider smooth initial data for the vacuum Einstein equations, and the maximal domain of development constructed by Choquet-Bruhat and Geroch [4]. In general, one might then ask whether the verdict on predictability may be changed by the introduction of an alternative notion of maximal domain of development.[9] For the initial value problem considered in this paper, however, it is clear that any notion of maximal domain of development must agree with $(\mathbf{D}(U), g)$, as any extension, in the sense of the definition below, would fail to be globally hyperbolic.

To formulate predictability we need the following notion of local future extension:

Definition 1. A $(3+1)$-dimensional Lorentzian manifold $(\tilde{M}, \tilde{g})$ is a *local future extension* of a $(3+1)$-dimensional Lorentzian manifold (M, g) if

$$(M, g) \subset (\tilde{M}, \tilde{g}), \tag{127}$$

$$I^-(M') \setminus M' \subset M, \tag{128}$$

where $M' = \tilde{M} \setminus M$. If in addition $(\tilde{M}, \tilde{g})$ satisfies

$$M \neq \tilde{M}, \tag{129}$$

then the extension is called *nontrivial*.

We are now ready to present our two formulations of predictability. The most common notion is set by:

[8] And in fact, as explained at the end of Section 3, the Einstein-Maxwell-scalar field system is in a sense inappropriate for studying the problem with such data.

[9] This is what happens, for instance, in equations for which solutions from general data evolve shocks. Thus one has to be very careful in interpreting the formulations for general matter.

Formulation 1. A complete initial value set in general relativity is said to be *predictable* if no nontrivial extension (M, g) of its maximal domain of dependence $\mathbf{D}$ exists, where g is a metric with continuous curvature.[10]

This formulation captures uniqueness in any class of solutions that are sufficiently regular to satisfy the Einstein equations in the classical sense.

In our problem, the initial value set is not complete in the u direction, and thus one must reformulate the condition (129) for nontriviality of future extensions, replacing it for instance by the requirement that $\tilde{M}$ not contain limits of sequences of points on the spheres whose u coordinates tend to U. From Theorem 2, one obtains that for R_2 initial data any (nontrivial) future extension $\tilde{M}$ of $\mathbf{D}$ contains the limit of a sequence of points x_i for which $\varpi(x_i) \to \infty$. This statement can be made without the restriction on smallness of U, by virtue of the appendix. By equations (14) and (17), it follows that $\tilde{M}$ contains the limit of a sequence of points for which the scalar curvature diverges, and since $1 - \mu = \partial^a r \partial_a r$, the C^1 norm of the metric at such a sequence, with respect to any coordinate chart, also diverges. Thus, for such extensions not only the curvature is discontinuous but even the metric cannot be C^1. This proves in particular that according to Formulation 1, R_2 data are indeed predictable. Thus, in this view, the result of Theorem 2 certainly supports strong cosmic censorship, even though the geometry of the trapped region is qualitatively different, for instance, from the solutions of Christodoulou [5].

Nevertheless, Formulation 1 of predictability seems unnatural from the point of view of both physics and the theory of partial differential equations. The blow-up of the curvature or even of the C^1 norm of the metric does not necessarily imply that an observer crossing the Cauchy horizon is destroyed, and thus indicates that under such circumstances, there should still be some notion of continuation of spacetime, which would necessarily fail to be globally hyperbolic. Moreover, as is well known, solutions to partial differential equations can often be defined in rough spaces, and such rough solutions may have physical meaning. (For recent work on rough solutions to the Einstein equations, see [16].) This motivates an alternative notion of predictability:

Formulation 2. An initial value set is said to be *predictable* if there is no nontrivial future extension (M, g) of its maximal domain of development where g is a continuous Lorentzian metric.[11]

It turns out that Theorem 1 rules out predictability, in the sense of Formulation 2. We have the following:

[10] Variations of this formulation add the requirement that the Einstein tensor satisfies the positive energy condition, and/or require only that the metric be $C^{1,1}$.

[11] This is the notion Christodoulou [6] uses in formulating strong cosmic censorship for the vacuum Einstein equations.

Proposition 7. *Let* $(r, \lambda, \nu, \varpi, \theta, \zeta)$ *be the unique solution of the equations with* R_1*-data in* $\mathbf{D}(U)$, *with the additional assumption* (107) *on the data for some* $U_1 \le U$. *Denote by* M *the spherically symmetric Lorentzian manifold whose spheres are the interior points of* $\mathbf{D}(U)$ *and whose metric* g *is determined by* r, λ, ν, *and* ϖ. *Then there exists a nontrivial future extension* $(\tilde{M}, \tilde{g})$ *of* (M, g), *where* $\tilde{g}$ *is a continuous Lorentzian metric.*

It will suffice to construct a $1+1$-dimensional Lorentzian manifold $(\tilde{Q}, \tilde{g})$ nontrivially extending the manifold (Q, g) represented by $(\mathbf{D}(U), g)$, and a continuous function $\tilde{r}$ on $\tilde{Q}$ bounded away from zero such that, on Q, $\tilde{r}$ coincides with r. An extension $\tilde{M}$ can then be defined by the metric $\tilde{g}_{\mu\nu} = \tilde{g}_{ab} + \tilde{r}^2\gamma_{AB}$.

The proof of Proposition 7 will require certain bounds on θ and ζ. Such bounds, as well as an implied bound for the scalar field ϕ, are obtained in the following:

Proposition 8. *Let* $(r, \lambda, \nu, \varpi, \theta, \zeta)$ *be a solution to the equations for* R_1 *data, with the additional assumption* (107). *The scalar field* ϕ *is uniformly bounded in* $\mathbf{D}(U)$ *and extends to a continuous function of* r *on the Cauchy horizon.*

Using the bounds of r derived before, and noting that ϕ is bounded on the initial segments, we can obtain a bound on ϕ by integrating either $\partial_v\phi = r^{-1}\theta$, or $\partial_u\phi = r^{-1}\zeta$. Consider first the set $I^-(\gamma)$ and recall the bound (90) for $\frac{\zeta}{\nu}$. This together with the bound

$$-\nu(u,v) < C\frac{1}{u}$$

implies that

$$\zeta < \frac{1}{u^{1-\sigma}}$$

for some positive σ. In particular, integrating this inequality on a segment of constant v, and using the bound on r, we obtain a bound for ϕ along γ.

Turning now to the set $I^+(\gamma)$, the bounds (97) and (99) imply that, along γ,

$$\theta < C\frac{1}{(V-v)^{1-\tau'}}. \tag{130}$$

We will show that a bound of this form for θ is preserved in $I^+(\gamma)$.

For $(u, \tilde{v}) \in \gamma$, $(\tilde{u}, v) \in \gamma$, noting the bound

$$\int_{\tilde{v}}^{V} -\nu(u,v)dv < Cu^{-1+\tau}(V-\tilde{v}) = Cu^{-1+\tau+Q},$$

and using (103), we deduce the inequalities

$$\zeta(u,v) \le C\frac{1}{u^{1-\sigma}} + Cu^{-1+\tau+Q}\sup_{v^*\in[\tilde{v},v]}\theta(u,v^*),$$

and

$$\theta(u,v) \leq C\frac{1}{(V-v)^{1-\tau'}} + C\int_{\tilde{u}}^{u} \sup_{I^+(\gamma)\cap J^-(u^*,v)} \zeta(-\lambda)(u^*,v)du^*$$

in $I^+(\gamma)$. Here $(\tilde{u},v)$ is on γ. Since $(V-v)^{1-\tau'} < u^{1-\sigma}$, after appropriate redefining of τ', we obtain

$$\begin{aligned}\theta(u,v) \quad \leq \quad & C(V-v)^{-1+\tau'} \\ & + C^2\int_{\tilde{u}}^{u} \sup_{v^*\in[\tilde{v},v]} \theta(u^*,v^*)u^{*-1+\tau+Q}(-\lambda)(u^*,v)du^*.\end{aligned}$$

Note that the bound

$$-\lambda(u,v) < \frac{1}{V-v}$$

in $I^+(\gamma)$ implies

$$\int_{\tilde{u}}^{u} -Cu^{*-1+\tau+(\alpha_+r_+)^{-1}+\varepsilon}\lambda(u^*,v)du^* < C(V-v)^{-1}\tilde{u}^{\tau+Q} < C.$$

Thus, taking the supremum and applying Gronwall's inequality yield a bound (130) on θ, as desired. Since $(V-v)^{-1+\tau'}$ is integrable, integration of (130) (recalling our bound for r) yields a bound for ϕ.

The scalar field ϕ can now be extended to the Cauchy horizon by monotonicity, since our additional assumption (107) implies that Proposition 5 holds and thus $\partial_v\phi$ is positive.

Moreover, since $\zeta(u,v)$ is nondecreasing in v, it tends, as $v\to V$, to a lower semi-continuous function $\zeta(u,V)$ defined on the Cauchy horizon, with range $(0,\infty]$. Integrating the inequalities (103), (130) and recalling our lower bound on r yield

$$\zeta < \frac{C}{u^{1-\tau}} \tag{131}$$

for some constant C and some $0<\tau<1$.

Since r is a continuous nonzero function of u, nonincreasing in v, we have by the monotone convergence theorem

$$\phi(u_2,v)-\phi(u_1,v) = \int_{u_1}^{u_2} r^{-1}\zeta(u,v)du \to_{v\to V} \int_{u_1}^{u_2} r^{-1}\zeta(u,V)du.$$

It then follows from (131) that ϕ is a continuous function of u.

To retrieve the last assertion of the proposition about ϕ as a function of r, all that is necessary is that r be continuously invertible as a function of u. This would follow immediately if ν is bounded away from zero on the Cauchy horizon. We consider separately the case in which the result of Theorem 2 holds and the case in which it does not.

In the first case, *i.e.* $\varpi = \infty$ on the Cauchy horizon, note that the spacelike curve $\tilde{\gamma}$ defined by

$$r = \frac{e^2}{2r_-\varpi_0}$$

has the property that $\partial_v(-\nu) > 0$ in $I^+(\tilde{\gamma})$. One can certainly bound ν uniformly away from zero on the piece of $\tilde{\gamma}$ with $u \in [u_1, u_2]$, where $0 < u_1 < u_2$. This yields that $0 < c < -\nu$ on $[u_1, u_2] \times V$ for $c = c(u_1)$.

Even when the result of Theorem 2 does not hold, the first step of its proof is still valid, whence it follows that $\varpi - \varpi_0 < \varepsilon$ after appropriately restricting U. From this, one can deduce the inequality

$$\int_v^V \frac{\lambda}{1-\mu}(u, v')dv' < \infty.$$

Otherwise, integrating (25) would yield in view of our control on ϖ that $\nu = 1 - \mu = 0$ on the Cauchy horizon, and thus r and ϖ would be constant, contradicting Proposition 6. Now since the above integral is monotone, as a function of u, and by the bound $\nu(u, v) > -c' > 0$ on any segment $v \times [u_1, u_2]$, with $0 < u_1 < u_2$, $v < V$, where $c' = c'(u_1, v)$, it follows again that $\nu(u, V) > -c > 0$, with $c = c(u_1)$.

This lower bound combined with the upper bound (103) yield

$$0 < -\nu(u, V) < Cu^{1-\tau}$$

on the Cauchy horizon, and this implies that ϕ is indeed a well-defined, continuous function of r. $\square$

We now return to the proof of Proposition 7. First note that fixing any interval $I = [u_1, U] \times V$ on the Cauchy horizon, with $u_1 > 0$, we have, in virtue of our bounds for ζ and ν derived above,

$$-\int_{u_1}^{u_2} \left(\frac{\zeta}{\nu}\right)^2 \nu(u, v)du < c \tag{132}$$

for all v, where $c = c(u_1)$. This will allow us to define a new system of null coordinates (u', v'), now anchored at the point at infinity p, related to (u, v) by a locally C^1 diffeomorphism in $\mathbf{D}(U)$. We show that, unlike the (u, v) coordinates, the (u', v') coordinates remain regular on the Cauchy horizon; they break down however, on the event horizon.

The origin $(0, 0)$ of the (u', v') coordinate system will correspond to the point $(0, V)$ in (u, v) coordinates. We normalize u' on the Cauchy horizon by the condition $\nu' = -1$. In view of our results on r, this is well defined. Here we recall that ν, λ, θ, and ζ are defined from r, ϕ by coordinate derivatives. Primes will thus indicate the corresponding quantities defined in terms of (u', v') coordinates. The point $(r(U, V) - r_-, 0)$ will thus represent (U, V).

To define the coordinate v', note that since $-\nu$ is bounded below and $\frac{\nu}{1-\mu}$ nonincreasing in v, it follows that $1-\mu(U,v) < -C < 0$. This then implies

$$\int_{v_0}^{V} \frac{\lambda}{1-\mu}(U,v)dv < C^{-1}\int \lambda dv < \tilde{C},$$

for all $v_0 \in [0,V]$. Thus, we can define a coordinate

$$v'(v) = -\int_{v}^{V} \frac{\lambda}{1-\mu}(U,v)dv,$$

because the integrand is positive. Then $\mathbf{D}(U)$ corresponds in the (u',v') coordinates to

$$\mathbf{D}(U) = (0, r(U,V)-r_-) \times (v'(0),0).$$

By account of all our estimates, $\nu'(u',v')$ is continuous and nonzero in $(0,r(U,V)-r_-) \times (v'(0),0]$. Moreover, $\frac{\lambda'}{1-\mu}(u',v')$ is a continuous function, bounded below by 1. This follows by integrating (31), in the primed coordinates, starting out from points on the line $U \times (v'(0),0)$, and using the bound (132), which holds in the new coordinates as well. Application of the monotone convergence theorem implies that $\frac{\lambda'}{1-\mu}$ extends continuously to $(0,r(U,V)-r_-) \times (v'(0),0]$, as a function of u, and our bounds on θ imply that this extension is also continuous in v.

We conclude that both $-\Omega'^2 = 4\frac{\lambda'\nu'}{1-\mu}$ and r are continuous functions in any $(u,U] \times (v'_1,0]$ satisfying $0 < c(u) < (\Omega'^2, r) < C$. Thus, $-\Omega'^2$ and r can be extended to continuous functions $-\tilde{\Omega}'^2$ and $\tilde{r}$, defined on

$$\tilde{K} = (0,U) \times (v'(0),V')$$

for some positive V', and bounded $0 < (\tilde{\Omega}'^2, \tilde{r}) < \tilde{C}$, so that the metric $\tilde{g} = -\tilde{\Omega}^2 du'dv'$ coincides with $g = -\Omega^2 dudv$ in $\mathbf{D}(U)$. An extension of g with the required properties has thus been constructed. □

Similar to the smooth extensions of the maximum domain of development of the Reissner-Nordström solution considered in Section 3, the above extension $\tilde{Q} \supset Q$ is global, i.e., inextendible curves in $\tilde{Q}$ which do not approach the trivial boundary $u = U$ must necessarily intersect $\tilde{Q} \setminus Q$.

From the above, it becomes clear that whether strong cosmic censorship holds hinges critically on the precise formulation of this principle. In the end, the reader may choose for himself which interpretation he prefers.

9. Appendix: The future boundary of trapped regions

This last section will be concerned with the behavior of the solution at the part of the boundary of the maximal domain of development that does *not* lie on the Reissner-Nordström Cauchy horizon, namely $\partial\overline{\mathbf{E}(U)} \cap \mathbf{D}(U)$.

After restricting U, we have found this boundary to be empty for R_1 data. The following proposition may provide some justification for our arbitrary restriction of U in the study of the issue of predictability.

PROPOSITION 9. *Let $(r, \lambda, \nu, \varpi, \theta, \zeta)$ be a solution of the equations with R_0 data. If $\partial\overline{\mathbf{E}(U)} \cap \mathbf{D}(U)$ is nonempty, then it is an achronal set terminating at a point on $[0, U) \times V$, and the functions r and ϖ can be extended continuously (the latter as a continuous function to $[0, \infty]$) to this set so that $r = 0$ and $\varpi = \infty$ hold identically.*

We proceed with the proof. Given a point (u, v), note that the signs of λ and ν imply that $r^{-1}(u, v)$ and $\varpi(u, v)$ bound r^{-1} and ϖ, respectively, in $J^-(u, v)$. Applying the above *a priori* bounds and (42) into (25) yield the bound

$$|\nu|(u', v') < \left(\frac{V}{V - v'}\right)^C, \tag{133}$$

for $(u', v') \in \mathbf{R}(u, v)$, where $C = C(r^{-1}(u, v), \varpi(u, v))$. We certainly have an upper bound

$$|1 - \mu| < C \tag{134}$$

which comes from writing

$$1 - \mu = 1 - \frac{2\varpi}{r} + \frac{e^2}{r^2}.$$

Thus, in view again of (42), it follows that we can also bound λ:

$$|\lambda| < \frac{C}{(V - v)}. \tag{135}$$

Assuming local well-posedness, in the sense that for fixed initial data and fixed v, there is a u' sufficiently small so that a solution exists in $J^-(u', v)$ with the norm of Section 5 bounded, the bounds for r^{-1} and ϖ, along with (44), (42), (133), (134), and (135) allow us to control, first, the C^0 norm of r, λ, ν, ϖ, θ, ζ, and then the norm of Section 5, in all of $J^-(u, v)$. Thus, standard techniques imply that $\partial\overline{\mathbf{E}(U)} \cap \mathbf{D}(U)$ is an achronal curve, such that if (u, v) is a point at which the curve is spacelike, either $r(u, v) = 0$ or $\varpi(u, v) = \infty$. Let us denote the set of the spacelike points of $\partial\overline{\mathbf{E}(U)} \cap \mathbf{D}(U)$ by $\mathbf{X}$. We note that if $\partial\overline{\mathbf{E}(U)} \cap \mathbf{D}(U)$ is nonempty, then so is $\mathbf{X}$.

We first show that in fact $r = 0$ and $\varpi = \infty$ identically on $\mathbf{X}$. The relation $1 - \mu \leq 0$ implies (actually, all that is needed is that m be positive, which is true in greater generality; see [7])

$$\frac{e^2}{2r} \leq \varpi. \tag{136}$$

Thus, for $(u, v) \in \mathbf{X}$, if $r = 0$, then $\varpi = \infty$, and we simply must exclude the case $r(u, v) > 0$, $\varpi(u, v) = \infty$. This will follow if we can bound $|\nu|$ and $|\lambda|$. Indeed, in that case θ and ζ may be bounded by integrating the linear system (28), (29). This in turn will induce ϖ to satisfy a linear differential inequality from (26), a solution of which cannot blow up in finite u-time. To obtain such bounds for ν and λ from integration of (25) and (24), we partition the domain of integration, as in Step 2 in the proof of Theorem 1, into the set where $\frac{\varpi}{r}$ is small and the set where $\frac{\varpi}{r}$ is large, and in the latter case, make use of the denominator $1 - \mu$.

We can finally extend the equalities $\varpi = \infty$, $r = 0$, from $\mathbf{X}$ to $\partial\overline{\mathbf{E}(U)} \cap \mathbf{D}(U)$ by noticing that r is nonincreasing on null segments, while ϖ is nondecreasing. $\square$

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