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A stable trace formula III. Proof of the main theorems

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Introduction

This paper is the last of three articles designed to stabilize the trace formula. Our goal is to stabilize the global trace formula for a general connected group, subject to a condition on the fundamental lemma that has been established in some special cases. In the first article [I], we laid out the foundations of the process. We also stated a series of local and global theorems, which together amount to a stabilization of each of the terms in the trace formula. In the second paper [II], we established a key reduction in the proof of one of the global theorems. In this paper, we shall complete the proof of the theorems. We shall combine the global reduction of [II] with the expansions that were established in Section 10 of [I].

We refer the reader to the introduction of [I] for a general discussion of the problem of stabilization. The introduction of [II] contains further discussion of the trace formula, with emphasis on the "elliptic" coefficients $a^G_{\mathrm{ell}}(\dot\gamma_S)$. These objects are basic ingredients of the geometric side of the trace formula.

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However, it is really the dual "discrete" coefficients $a^G_{\mathrm{disc}}(\dot{\pi})$ that are the ultimate objects of study. These coefficients are basic ingredients of the spectral side of the trace formula. Any relationship among them can be regarded, at least in theory, as a reciprocity law for the arithmetic data that is encoded in automorphic representations.

The relationships among the coefficients $a^G_{\mathrm{disc}}(\dot{\pi})$ are given by Global Theorem 2. This theorem was stated in [I, §7], together with a companion, Global Theorem 2′, which more closely describes the relevant coefficients in the trace formula. The proof of Global Theorem 2 is indirect. It will be a consequence of a parallel set of theorems for all the other terms in the trace formula, together with the trace formula itself.

Let G be a connected reductive group over a number field F. For simplicity, we can assume for the introduction that the derived group G_{der} is simply connected. Let V be a finite set of valuations of F that contains the set of places at which G ramifies. The trace formula is the identity obtained from two different expansions of a certain linear form

$$I(f), \qquad f \in \mathcal{H}(G,V),$$

on the Hecke algebra of $G(F_V)$. The geometric expansion

$$I(f) = \sum_M |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M,V)} a^M(\gamma) I_M(\gamma, f) \tag{1}$$

is a linear combination of distributions parametrized by conjugacy classes γ in Levi subgroups $M(F_V)$. The spectral expansion

$$I(f) = \sum_M |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V)} a^M(\pi) I_M(\pi, f) d\pi \tag{2}$$

is a continuous linear combination of distributions parametrized by representations π of Levi subgroups $M(F_V)$. (We have written (2) slightly incorrectly, in order to emphasize its symmetry with (1). The right-hand side of (2) really represents a double integral over $\{(M,\Pi)\}$ that is known at present only to converge conditionally.) Local Theorems 1′ and 2′ were stated in [I, §6], and apply to the distributions $I_M(\gamma, f)$ and $I_M(\pi, f)$. Global Theorems 1′ and 2′, stated in [I, §7], apply to the coefficients $a^M(\gamma)$ and $a^M(\pi)$.

Each of the theorems consists of two parts (a) and (b). Parts (b) are particular to the case that G is quasisplit, and apply to "stable" analogues of the various terms in the trace formula. Our use of the word "stable" here (and in [I] and [II]) is actually slightly premature. It anticipates the assertions (b), which say essentially that the "stable" variants of the terms do indeed give rise to stable distributions. It is these assertions, together with the corresponding pair of expansions obtained from (1) and (2), that yield a stable trace formula.

Parts (a) of the theorems apply to "endoscopic" analogues of the terms in the trace formula. They assert that the endoscopic terms, *a priori* linear combinations of stable terms attached to endoscopic groups, actually reduce to the original terms. These assertions may be combined with the corresponding endoscopic expansions obtained from (1) and (2). They yield a decomposition of the original trace formula into stable trace formulas for the endoscopic groups of G.

Various reductions in the proofs of the theorems were carried out in [I] and [II] (and other papers) by methods that are not directly related to the trace formula. The rest of the argument requires a direct comparison of trace formulas. We are assuming at this point that G satisfies the condition [I, Assumption 5.2] on the fundamental lemma. For the assertions (a), we shall compare the expansions (1) and (2) with the endoscopic expansions established in [I, §10]. The aim is to show that (1) and (2) are equal to their endoscopic counterparts for any function f. For the assertions (b), we shall study the "stable" expansions established in [I, §10]. The aim here is to show that the expansions both vanish for any function f whose stable orbital integrals vanish. The assertions (a) and (b) of Global Theorem 2 will be established in Section 9, at the very end of the process. They will be a consequence of a term by term cancellation of the complementary components in the relevant trace formulas.

Many of the techniques of this paper are extensions of those in Chapter 2 of [AC]. In particular, Sections 2–5 here correspond quite closely to Sections 2.13–2.16 of [AC]. As in [AC], we shall establish the theorems by a double induction argument, based on integers

$$d_{\mathrm{der}} = \dim(G_{\mathrm{der}})$$

and

$$r_{\mathrm{der}} = \dim(A_M \cap G_{\mathrm{der}}),$$

for a fixed Levi subgroup M of G. In Section 1, we shall summarize what remains to be proved of the theorems. We shall then state formally the induction hypotheses on which the argument rests.

In Section 2, we shall apply the induction hypotheses to the endoscopic and stable expansions of [I, §10]. This will allow us to remove a number of inessential terms from the comparison. Among the most difficult of the remaining terms will be the distributions that originate with weighted orbital integrals. We shall begin their study in Section 3. In particular, we shall apply the technique of cancellation of singularities, introduced in the special case of division algebras by Langlands in 1984, in two lectures at the Institute for Advanced Study. The technique allows us to transfer the terms in question from the geometric side to the spectral side, by means of an application of the trace formula for M. The cancellation of singularities comes in showing that for suitable $v \in V$ and $f_v \in \mathcal{H}\Big(G(F_v)\Big)$, a certain difference of functions

$$\gamma_v \longrightarrow I_M^{\mathcal{E}}(\gamma_v, f_v) - I_M(\gamma_v, f_v), \qquad \gamma_v \in \Gamma_{G\text{-reg}}\Big(M(F_v)\Big),$$

can be expressed as an invariant orbital integral on $M(F_v)$. In Section 4, we shall make use of another technique, which comes from the Paley-Wiener theorem for real groups. We shall apply a weak estimate for the growth of spectral terms under the action on f of an archimedean multiplier α. This serves as a substitute for the lack of absolute convergence of the spectral side of the trace formula. In particular, it allows us to isolate terms that are discrete in the spectral variable. The results of Section 4 do come with certain restrictions on f. However, we will be able to remove the most serious of these restrictions in Section 5 by a standard comparison of distributions on a lattice.

The second half of the paper begins in Section 6 with a digression. In this section, we shall extend our results to the local trace formula. The aim is to complete the process initiated in [A10] of stabilizing the local trace formula. In particular, we shall see how such a stabilization is a natural consequence of the theorems we are trying to prove. The local trace formula has also to be applied in its own right. We shall use it to establish an unprepossessing identity (Lemma 6.5) that will be critical for our proof of Local Theorem 1. Local Theorem 1 actually implies all of the local theorems, according to reductions from other papers. We shall prove it in Sections 7 and 8. Following a familiar line of argument, we can represent the local group to which the theorem applies as a completion of a global group. We will then make use of the global arguments of Sections 2–5. By choosing appropriate functions in the given expansions, we will be able to establish assertion (a) of Local Theorem 1 in Section 7, and to reduce assertion (b) to a property of weak approximation. We will prove the approximation property in Section 8, while at the same time taking the opportunity to fill a minor gap at the end of the argument in [AC, §2.17].

We shall establish the global theorems in Section 9. With the proof of Local Theorem 1 in hand, we will see that the expansions of Sections 2–5 reduce immediately to two pairs of simple identities. The first pair leads directly to a proof of Global Theorem 1 on the coefficients $a_{\mathrm{ell}}^G(\dot{\gamma}_S)$. The second pair of identities applies to the dual coefficients $a_{\mathrm{disc}}^G(\dot{\pi})$. It leads directly to a proof of Global Theorem 2.

In the last section, we shall summarize some of the conclusions of the paper. In particular, we shall review in more precise terms the stablization process for both the global and local trace formulas. The reader might find it useful to read this section before going on with the main part of the paper.

1. The induction hypotheses

Our goal is to prove the general theorems stated in [I, §6,7]. This will yield both a stable trace formula, and a decomposition of the ordinary trace formula into stable trace formulas for endoscopic groups. Various reductions of the proof have been carried out in other papers, by methods that are generally independent of the trace formula. The rest of the proof will have to be established by an induction argument that depends intrinsically on the trace formula. In this section, we shall recall what remains to be proved. We shall then state the formal induction hypotheses that will be in force throughout the paper.

We shall follow the notation of the papers [I] and [II]. We will recall a few of the basic ideas in a moment. For the most part, however, we shall have to assume that the reader is familiar with the various definitions and constructions of these papers.

Throughout the present paper, F will be a local or global field of characteristic 0. The theorems apply to a K-group G over F that satisfies Assumption 5.2 of [I]. In particular,

$$G = \coprod_{\beta} G_{\beta}, \qquad \beta \in \pi_0(G),$$

is a disjoint union of connected reductive groups over F, equipped with some extra structure [A10, §2], [I, §4]. The disconnected K-group G is a convenient device for treating trace formulas of several connected groups at the same time. Any connected group G_1 is a component of an (essentially) unique K-group G [I, §4], and most of the basic objects that can be attached to G_1 extend to G in an obvious manner.

The study of endoscopy for G depends on a quasisplit inner twist ψ: $G \to G^*$ [A10, §1,2]. Recall that ψ is a compatible family of inner twists

$$\psi_\beta : \ G_\beta \longrightarrow G^*, \qquad \beta \in \pi_0(G),$$

from the components of G to a connected quasisplit group G^* over F. Unless otherwise stated, ψ will be assumed to be fixed. We also assume implicitly that if M is a given Levi sub(K-)group of G, then ψ restricts to an inner twist from M to a Levi subgroup M^* of G^*.

It is convenient to fix central data (Z, ζ) for G. We define the *center* of G to be a diagonalizable group $Z(G)$ over F, together with a compatible family of embeddings $Z(G) \subset G_\beta$ that identify $Z(G)$ with the center $Z(G_\beta)$ of any component G_β. The first object Z is an induced torus over F that is contained in $Z(G)$. The second object ζ is a character on either $Z(F)$ or $Z(\mathbb{A})/Z(F)$, according to whether F is local or global. The pair (Z, ζ) obviously determines a corresponding pair of central data (Z^*, ζ^*) for the connected group G^*.

Central data are needed for the application of induction arguments to endoscopic groups. Suppose that $G' \in \mathcal{E}_{\mathrm{ell}}(G)$ represents an elliptic endoscopic datum $(G', \mathcal{G}', s', \xi')$ for G over F [I, §4]. We assume implicitly that G' has been equipped with the auxiliary data $(\widetilde{G}', \widetilde{\xi}')$ required for transfer [A7, §2]. Then $\widetilde{G}' \to G'$ is a central extension of G' by an induced torus $\widetilde{C}'$ over F, while $\widetilde{\xi}'\colon \mathcal{G}' \to {}^L\widetilde{G}'$ is an L-embedding. The preimage $\widetilde{Z}'$ of Z in $\widetilde{G}'$ is an induced central torus over F. The constructions of [LS, (4.4)] provide a character $\widetilde{\eta}'$ on either $\widetilde{Z}'(F)$ on $\widetilde{Z}'(\mathbb{A})/\widetilde{Z}'(F)$, according to whether F is local or global. We write $\widetilde{\zeta}'$ for the product of $\widetilde{\eta}'$ with the pullback of ζ from Z to $\widetilde{Z}'$. The pair $(\widetilde{Z}', \widetilde{\zeta}')$ then serves as central data for the connected quasisplit group $\widetilde{G}'$. (The notation from [I] and [II] used here is slightly at odds with that of [A7] and [A10].)

The trace formula applies to the case of a global field, and to a finite set of valuations V of F that contains $V_{\mathrm{ram}}(G, \zeta)$. We recall that $V_{\mathrm{ram}}(G, \zeta)$ denotes the set of places at which G, Z or ζ are ramified. As a global K-group, G comes with a local product structure. This provides a product

$$G_V = \prod_{v \in V} G_v = \prod_v \Big(\coprod_{\beta_v} G_{v,\beta_v} \Big) = \coprod_{\beta_V} G_{V,\beta_V}$$

of local K-groups G_v over F_v, and a corresponding product

$$G_V(F_V) = \prod_{v \in V} G_v(F_v) = \prod_v \Big(\coprod_{\beta_v} G_{v,\beta_v}(F_v) \Big) = \coprod_{\beta_V} G_{V,\beta_V}(F_V)$$

of sets of F_v-valued points. Following the practice in [I] and [II], we shall generally avoid using separate notation for the latter. In other words, G_v will be allowed to stand for both a local K-group, and its set of F_v- valued points. The central data (Z, ζ) for G yield central data

$$(Z_V, \zeta_V) = \Big(\prod_v Z_v, \prod_v \zeta_v \Big) = \coprod_{\beta_V} (Z_{V,\beta_V}, \zeta_{V,\beta_V})$$

for G_V, with respect to which we can form the ζ_V^{-1}-equivariant Hecke space

$$\mathcal{H}(G_V, \zeta_V) = \coprod_{\beta_V} \mathcal{H}(G_{V,\beta_V}, \zeta_{V,\beta_V}).$$

The terms in the trace formula are linear forms in a function f in $\mathcal{H}(G_V, \zeta_V)$, which depend only on the restriction of f to the subset

$$G_V^Z = \Big\{ x \in G_V : \ H_G(x) \in \mathfrak{a}_Z \Big\}$$

of G_V. They can therefore be regarded as linear forms on the Hecke space

$$\mathcal{H}(G, V, \zeta) = \mathcal{H}(G_V^Z, \zeta_V) = \coprod_{\beta_V} \mathcal{H}(G_{V,\beta_V}^Z, \zeta_{V,\beta_V}).$$

We recall that some of the terms depend also on a choice of hyperspecial maximal compact subgroup

$$K^V = \prod_{v \notin V} K_v$$

of the restricted direct product

$$G^V(\mathbb{A}^V) = \prod_{v \notin V} G_v.$$

In the introduction, we referred to Local Theorems $1'$ and $2'$ and Global Theorems $1'$ and $2'$. These are the four theorems stated in [I, §6,7] that are directly related to the four kinds of terms in the trace formula. We shall investigate them by comparing the trace formula with the endoscopic and stable expansions in [I, §10]. In the end, however, it will not be these theorems that we prove directly. We shall focus instead on the complementary theorems, stated also in [I, §6,7]. The complementary theorems imply the four theorems in question, but they are in some sense more elementary.

Local Theorems 1 and 2 were stated in [I, §6], in parallel with Local Theorems $1'$ and $2'$. They apply to the more elementary situation of a local field. However, as we noted in [I, Propositions 6.1 and 6.3], they can each be shown to imply their less elementary counterparts. In the paper [A11], it will be established that Local Theorem 1 implies Local Theorem $1'$. In the paper [A12], it will be shown that Local Theorem 2 implies Local Theorem $2'$, and also that Local Theorem 1 implies Local Theorem 2. A proof of Local Theorem 1 would therefore suffice to establish all the theorems stated in [I, §6]. Since it represents the fundamental local result, we ought to recall the formal statement of this theorem from [I, §6].

Local Theorem 1. *Suppose that F is local, and that M is a Levi subgroup of G.*

(a) *If G is arbitrary,*

$$I_M^{\mathcal{E}}(\gamma, f) = I_M(\gamma, f), \qquad \gamma \in \Gamma_{G\text{-reg,ell}}(M, \zeta),\ f \in \mathcal{H}(G, \zeta).$$

(b) *Suppose that G is quasisplit, and that δ' belongs to the set $\Delta_{G\text{-reg,ell}}(\widetilde{M}', \widetilde{\zeta}')$, for some $M' \in \mathcal{E}_{\mathrm{ell}}(M)$. Then the linear form*

$$f \longrightarrow S_M^G(M', \delta', f), \qquad f \in \mathcal{H}(G, \zeta),$$

vanishes unless $M' = M^*$, in which case it is stable.

The notation here is, naturally, that of [I]. For example, $\Gamma_{G\text{-reg,ell}}(M, \zeta)$ stands for the subset of elements in $\Gamma(M, \zeta)$ of strongly G-regular, elliptic support in $M(F)$, while $\Gamma(M, \zeta)$ itself is a fixed basis of the space $\mathcal{D}(M, \zeta)$ of distributions on $M(F)$ introduced in [I, §1]. Similarly, $\Delta_{G\text{-reg,ell}}(\widetilde{M}', \widetilde{\zeta}')$

stands for the subset of elements in $\Delta(\widetilde{M}', \widetilde{\zeta}')$ of strongly G-regular, elliptic support in $\widetilde{M}'(F)$, while $\Delta(\widetilde{M}', \widetilde{\zeta}')$ is a fixed basis of the subspace $S\mathcal{D}(\widetilde{M}', \widetilde{\zeta}')$ of stable distributions in $\mathcal{D}(\widetilde{M}', \widetilde{\zeta}')$. We recall that G is defined to be quasisplit if it has a connected component G_β that is quasisplit. In this case, the Levi sub(K-)group M is also quasisplit, and there is a bijection $\delta \to \delta^*$ from $\Delta(M, \zeta)$ onto $\Delta(M^*, \zeta^*)$. The linear forms $I_M^{\mathcal{E}}(\gamma, f)$ and $S_M^G(M', \delta', f)$ are defined in [I, §6], by a construction that relies on the solution [Sh], [W] of the Langlands-Shelstad transfer conjecture. For p-adic F, this in turn depends on the Lie algebra variant of the fundamental lemma that is part of [I, Assumption 5.2]. If G is quasisplit (which is the only circumstance in which $S_M^G(M', \delta', f)$ is defined), the notation

$$S_M^G(\delta, f) = S_M^G(M^*, \delta^*, f), \qquad\qquad \delta \in \Delta_{G\text{-reg,ell}}(M, \zeta),$$

of [A10] and [I] is useful in treating the case that $M' = M^*$.

If $M = G$, there is nothing to prove. The assertions of the theorem in this case follow immediately from the definitions in [I, §6]. In the case of archimedean F, we shall prove the general theorem in [A13], by purely local means. We can therefore concentrate on the case that F is p-adic and $M \neq G$. We shall prove Local Theorem 1 under these conditions in Section 8. (One can also apply the global methods of this paper to the case of archimedean F, as in [AC]. However, some of the local results of [A13] would still be required in order to extend the cancellation of singularities in §3 to this case.)

Global Theorems 1 and 2 were stated in [I, §7], in parallel with Global Theorems 1′ and 2′. They apply to the basic building blocks from which the global coefficients in the trace formula are constructed. According to Corollary 10.4 of [I], Global Theorem 1 implies Global Theorem 1′, while by Corollary 10.8 of [I], Global Theorem 2 implies Global Theorem 2′. It would therefore be sufficient to establish the more fundamental pair of global theorems. We recall their formal statements, in terms of the objects constructed in [I, §7].

Global Theorem 1. *Suppose that F is global, and that S is a large finite set of valuations that contains $V_{\mathrm{ram}}(G, \zeta)$.*

(a) *If G is arbitrary,*

$$a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S) = a_{\mathrm{ell}}^{G}(\dot{\gamma}_S),$$

for any admissible element $\dot{\gamma}_S$ in $\Gamma_{\mathrm{ell}}^{\mathcal{E}}(G, S, \zeta)$.

(b) *If G is quasisplit, $b_{\mathrm{ell}}^G(\dot{\delta}_S)$ vanishes for any admissible element $\dot{\delta}_S$ in the complement of $\Delta_{\mathrm{ell}}(G, S, \zeta)$ in $\Delta_{\mathrm{ell}}^{\mathcal{E}}(G, S, \zeta)$.*

Global Theorem 2. *Suppose that F is global, and that $t \geq 0$.*

(a) *If G is arbitrary,*

$$a^{G,\mathcal{E}}_{\mathrm{disc}}(\dot{\pi}) = a^{G}_{\mathrm{disc}}(\dot{\pi}),$$

for any element $\dot{\pi}$ in $\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$.

(b) *If G is quasisplit, $b^{G}_{\mathrm{ell}}(\dot{\phi})$ vanishes for any $\dot{\phi}$ in the complement of $\Phi_{t,\mathrm{disc}}(G,\zeta)$ in $\Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$.*

The notation $\dot{\gamma}_S$, $\dot{\delta}_S$, $\dot{\pi}$ and $\dot{\phi}$ from [I] was meant to emphasize the essential global role of the objects in question. The first two elements are attached to G_S, while the last two are attached to $G(\mathbb{A})$. The objects they index in each case are basic constituents of the global coefficients for G_V, for any V with

$$V_{\mathrm{ram}}(G,\zeta) \subset V \subset S,$$

that actually occur in the relevant trace formulas. The domains $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$, $\Pi_{t,\mathrm{disc}}(G,\zeta)$, etc., were defined in [I, §2,3,7], while the objects they parametrize were constructed in [I, §7]. The notion of an admissible element in Global Theorem 1 is taken from [I, §1]. We shall establish Global Theorems 1 and 2 in Section 9, as the last step in our induction argument.

We come now to the formal induction hypotheses. The argument will be one of double induction on a pair of integers d_{der} and r_{der}, with

$$0 < r_{\mathrm{der}} < d_{\mathrm{der}}. \tag{1.1}$$

These integers are to remain fixed until we complete the argument at the end of Section 9. The hypotheses will be stated in terms of these integers, the derived multiple group

$$G_{\mathrm{der}} = \coprod_{\beta} G_{\beta,\mathrm{der}},$$

and the split component

$$A_{M\cap G_{\mathrm{der}}} = A_M \cap G_{\mathrm{der}}$$

of the Levi subgroup of G_{der} corresponding to M.

Local Theorem 1 applies to a local field F, a local K-group G over F that satisfies Assumption 5.2(2) of [1], and a Levi subgroup M of G. We assume inductively that this theorem holds if

$$\dim(G_{\mathrm{der}}) < d_{\mathrm{der}}, \qquad (F \text{ local}), \tag{1.2}$$

and also if

$$\dim(G_{\mathrm{der}}) = d_{\mathrm{der}}, \text{ and } \dim(A_M \cap G_{\mathrm{der}}) < r_{\mathrm{der}}, \qquad (F \text{ local}). \tag{1.3}$$

We are taking for granted the proof of the theorem for archimedean F [A13]. We have therefore to carry the hypotheses only for p-adic F, in which case G is just a connected reductive group. Global Theorems 1 and 2 apply to a global field F, and a global K-group G over F that satisfies Assumption 5.2(1) of [I]. We assume that these theorems hold if

$$\dim(G_{\mathrm{der}}) < d_{\mathrm{der}}, \qquad (F \text{ global}). \tag{1.4}$$

In both the local and global cases, we also assume that if G is *not* quasisplit, and

$$\dim(G_{\mathrm{der}}) = d_{\mathrm{der}}, \qquad (F \text{ local or global}), \tag{1.5}$$

the relevant theorems hold for the quasisplit inner K-form of G. We have thus taken on four induction hypotheses, which are represented by the four conditions (1.2)–(1.5). The induction hypotheses imply that the remaining theorems also hold. According to the results cited above, any of the theorems stated in [I, §6,7] are actually valid under any of the relevant conditions (1.2)–(1.5).

2. Application to endoscopic and stable expansions

We now begin the induction argument that will culminate in Section 9 with the proof of the global theorems. We have fixed the integers d_{der} and r_{der} in (1.1). In this section, we shall apply the induction hypotheses (1.2)–(1.5) to the terms in the main expansions of [I, §10]. The conclusions we reach will then be refined over the ensuing three sections. For all of this discussion, F will be global.

We fix the global field F. We also fix a global K-group G over F that satisfies Assumption 5.2(1) of [I], such that

$$\dim(G_{\mathrm{der}}) = d_{\mathrm{der}}.$$

Given G, we choose a corresponding pair of central data (Z, ζ). We then fix a finite set V of valuations of F that contains $V_{\mathrm{ram}}(G, \zeta)$. As we apply the induction hypotheses over the next few sections, we shall establish a series of identities that occur in pairs (a) and (b), and approximate what is required for the main theorems. The identities (b) apply to the case that G is quasisplit, and often to functions $f \in \mathcal{H}(G_V, \zeta_V)$ such that $f^G = 0$. We call such functions *unstable*, and we write $\mathcal{H}^{\mathrm{uns}}(G_V, \zeta_V)$ for the subspace of unstable functions in $\mathcal{H}(G_V, \zeta_V)$. It is clear that $\mathcal{H}^{\mathrm{uns}}(G_V, \zeta_V)$ can be defined by imposing a condition at any of the places v in V. It is the subspace of $\mathcal{H}(G_V, \zeta_V)$ spanned by functions $f = \prod_v f_v$ such that for some $v \in V$, f_v belongs to the local subspace

$$\mathcal{H}^{\mathrm{uns}}(G_v, \zeta_v) = \Big\{ f_v \in \mathcal{H}(G_v, \zeta_v) : \ f_v^G = 0 \Big\}$$

of unstable functions.

Our first step will be to apply the global descent theorem of [II], in the form taken by [II, Prop. 2.1] and its corollaries. Since the induction hypotheses (1.4) and (1.5) include the conditions imposed after the statement of Theorem 1.1 of [II], these results are valid for G. Let f be a fixed function in $\mathcal{H}(G_V, \zeta_V)$. Given f, we take S to be a large finite set of valuations of F containing V. To be precise, we require that S be such that the product of the support of f with the hyperspecial maximal compact subgroup K^V of $G^V(\mathbb{A}^V)$ is an S-admissible subset of $G(\mathbb{A})$, in the sense of [I, §1]. In [I, §8], we defined the linear form

$$I_{\mathrm{ell}}(f, S) = I_{\mathrm{ell}}(\dot{f}_S), \qquad \dot{f}_S = f \times u_S^V.$$

We also defined endoscopic and stable analogues $I^{\mathcal{E}}_{\mathrm{ell}}(f, S)$ and $S^G_{\mathrm{ell}}(f, S)$ of $I_{\mathrm{ell}}(f, S)$. The role of the results in [II] will be to reduce the study of these objects to that of distributions supported on unipotent classes.

Let us use the subscript *unip* to denote the unipotent variant of any object with the subscript *ell*. Thus, $\Gamma_{\mathrm{unip}}(G, V, \zeta)$ denotes the subset of classes in $\Gamma_{\mathrm{ell}}(G, V, \zeta)$ whose semisimple parts are trivial. Applying this convention to the "elliptic" objects of [I, §8], we obtain linear forms

$$I_{\mathrm{unip}}(f, S) = \sum_{\alpha \in \Gamma_{\mathrm{unip}}(G,V,\zeta)} a^G_{\mathrm{unip}}(\alpha, S) f_G(\alpha), \tag{2.1}$$

with coefficients

$$a^G_{\mathrm{unip}}(\alpha, S) = \sum_{k \in \mathcal{K}^V_{\mathrm{unip}}(\overline{G}, S)} a^G_{\mathrm{ell}}(\alpha \times k) r_G(k), \qquad \alpha \in \Gamma_{\mathrm{unip}}(G, V, \zeta).$$

We also obtain endoscopic and stable analogues $I^{\mathcal{E}}_{\mathrm{unip}}(f, S)$ and $S^G_{\mathrm{unip}}(f, S)$ of $I_{\mathrm{unip}}(f, S)$. These are defined inductively by the usual formula

$$I^{\mathcal{E}}_{\mathrm{unip}}(f, S) = \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G,S)} \iota(G, G') \widehat{S}^{\widetilde{G}'}_{\mathrm{unip}}(f', S) + \varepsilon(G) S^G_{\mathrm{unip}}(f, S),$$

with the requirement that $I^{\mathcal{E}}_{\mathrm{unip}}(f, S) = I_{\mathrm{unip}}(f, S)$ in case G is quasisplit. The natural variant of [I, Lemma 7.2] provides expansions

$$I^{\mathcal{E}}_{\mathrm{unip}}(f, S) = \sum_{\alpha \in \Gamma^{\mathcal{E}}_{\mathrm{unip}}(G,V,\zeta)} a^{G,\mathcal{E}}_{\mathrm{unip}}(\alpha, S) f_G(\alpha) \tag{2.2}$$

and

$$S^G_{\mathrm{unip}}(f, S) = \sum_{\beta \in \Delta^{\mathcal{E}}_{\mathrm{unip}}(G,V,\zeta)} b^G_{\mathrm{unip}}(\beta, S) f^{\mathcal{E}}_G(\beta), \tag{2.3}$$

with coefficients

$$a^{G,\mathcal{E}}_{\mathrm{unip}}(\alpha, S) = \sum_{k \in \mathcal{K}^{V,\mathcal{E}}_{\mathrm{unip}}(\overline{G}, S)} a^{G,\mathcal{E}}_{\mathrm{ell}}(\alpha \times k) r_G(k), \qquad \alpha \in \Gamma^{\mathcal{E}}_{\mathrm{unip}}(G, V, \zeta),$$

and

$$b^G_{\mathrm{unip}}(\beta,S)=\sum_{\ell\in\mathcal{L}^{V,\mathcal{E}}_{\mathrm{unip}}(\overline{G},S)} b^G_{\mathrm{ell}}(\beta\times\ell)r_G(\ell),\qquad \beta\in\Delta^{\mathcal{E}}_{\mathrm{unip}}(G,V,\zeta).$$

(See [I, (8.4)–(8.9)].)

The global descent theorem of [II] allows us to restrict our study of the "elliptic" coefficients to the special case in which the arguments have semisimple part that is central. Recall that the center of G is a diagonalizable group $Z(G)$ over F, together with a family of embeddings $Z(G)\subset G_\beta$. Let us write $Z(G)_{V,\mathfrak{o}}$ for the subgroup of elements z in $Z(G,F)$ such that for every $v\notin V$, the element z_v is bounded in $Z(G,F_v)$, which is to say that its image in G_v lies in the compact subgroup K_v. The group $Z(G)_{V,\mathfrak{o}}$ then acts discontinuously on G_V. Its quotient

$$Z(\overline{G})_{V,\mathfrak{o}}=Z(G)_{V,\mathfrak{o}}Z_V/Z_V$$

in turn acts discontinuously on $\overline{G}_V=G_V/Z_V$. If z belongs to $Z(\overline{G})_{V,\mathfrak{o}}$, and $f_z(x)=f(zx)$, we set

$$I_{z,\mathrm{unip}}(f,S)=I_{\mathrm{unip}}(f_z,S),$$

$$I^{\mathcal{E}}_{z,\mathrm{unip}}(f,S)=I^{\mathcal{E}}_{\mathrm{unip}}(f_z,S),$$

and

$$S^G_{z,\mathrm{unip}}(f,S)=S^G_{\mathrm{unip}}(f_z,S).$$

Lemma 2.1. (a) *In general*,

$$I^{\mathcal{E}}_{\mathrm{ell}}(f,S)-I_{\mathrm{ell}}(f,S)=\sum_{z\in Z(\overline{G})_{V,\mathfrak{o}}}\Big(I^{\mathcal{E}}_{z,\mathrm{unip}}(f,S)-I_{z,\mathrm{unip}}(f,S)\Big).$$

(b) *If G is quasisplit and f is unstable*,

$$S^G_{\mathrm{ell}}(f,S)=\sum_{z\in Z(\overline{G})_{V,\mathfrak{o}}} S^G_{z,\mathrm{unip}}(f,S).$$

Proof. Consider the expression in (a). It follows from the expansions [I, (8.5), (8.8)] that

$$I^{\mathcal{E}}_{\mathrm{ell}}(f,S)-I_{\mathrm{ell}}(f,S)=\sum_{\gamma\in\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G,V,\zeta)}\Big(a^{G,\mathcal{E}}_{\mathrm{ell}}(\gamma,S)-a^G_{\mathrm{ell}}(\gamma,S)\Big)f_G(\gamma).$$

The coefficients can in turn be expanded as

$$a^{G,\mathcal{E}}_{\mathrm{ell}}(\gamma,S)-a^G_{\mathrm{ell}}(\gamma,S)=\sum_{k\in\mathcal{K}^{V,\mathcal{E}}_{\mathrm{ell}}(\overline{G},S)}\Big(a^{G,\mathcal{E}}_{\mathrm{ell}}(\gamma\times k)-a^G_{\mathrm{ell}}(\gamma\times k)\Big)r_G(k),$$

by [I, (8.4), (8.6)]. Proposition 2.1(a) of [II] asserts that $a_{\mathrm{ell}}^{G,\mathcal{E}}(\gamma \times k)$ equals $a_{\mathrm{ell}}^{G}(\gamma \times k)$, whenever the semisimple part of $\gamma \times k$ is not central in G. It follows that if the semisimple part of γ is not central in G, $a_{\mathrm{ell}}^{G,\mathcal{E}}(\gamma, S)$ equals $a_{\mathrm{ell}}^{G}(\gamma, S)$. If the semisimple part of γ is central in G, γ has a Jordan decomposition that can be written

$$\gamma = z\alpha, \qquad z \in Z(\overline{G})_{V,\mathfrak{o}},\ \alpha \in \Gamma_{\mathrm{unip}}^{\mathcal{E}}(G, V, \zeta).$$

The trivial case of the general descent formula [II, Cor. 2.2(a)] then implies that

$$a_{\mathrm{ell}}^{G,\mathcal{E}}(\gamma, S) - a_{\mathrm{ell}}^{G}(\gamma, S) = a_{\mathrm{unip}}^{G,\mathcal{E}}(\alpha, S) - a_{\mathrm{unip}}^{G}(\alpha, S).$$

The formula (a) follows.

To deal with (b), we write

$$S_{\mathrm{ell}}^{G}(f, S) = \sum_{\delta \in \Delta_{\mathrm{ell}}^{\mathcal{E}}(G,V,\zeta)} b_{\mathrm{ell}}^{G}(\delta, S) f_{G}^{\mathcal{E}}(\delta),$$

and

$$b_{\mathrm{ell}}^{G}(\delta, S) = \sum_{\ell \in \mathcal{L}_{\mathrm{ell}}^{V,\mathcal{E}}(\overline{G},S)} b_{\mathrm{ell}}^{G}(\delta \times \ell) r_{G}(\ell),$$

according to [I, (8.9), (8.7)]. Since f is unstable, $f_{G}^{\mathcal{E}}(\delta)$ vanishes on the subset $\Delta_{\mathrm{ell}}(G, V, \zeta)$ of $\Delta_{\mathrm{ell}}^{\mathcal{E}}(G, V, \zeta)$. On the other hand, if δ lies in the complement of $\Delta_{\mathrm{ell}}(G, V, \zeta)$, and the semisimple part of δ is not central in G, Proposition 2.1(b) of [II] implies that $b_{\mathrm{ell}}^{G}(\delta, S) = 0$. If the semisimple part of δ is central in G, δ has a Jordan decomposition

$$\delta = z\beta, \qquad z \in Z(\overline{G})_{V,\mathfrak{o}},\ \alpha \in \Delta_{\mathrm{unip}}^{\mathcal{E}}(G, V, \zeta).$$

The simplest case of the descent formula [II, Cor. 2.2(b)] then implies that

$$b_{\mathrm{ell}}^{G}(\gamma, S) = b_{\mathrm{unip}}^{G}(\alpha, S).$$

The formula (b) follows. □

We have relied on our global induction hypotheses in making use of the descent formulas of [II]. The next stage of the argument depends on both the local and global induction hypotheses. We are going to study the expressions

$$I_{\mathrm{par}}(f) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M(\gamma) I_M(\gamma, f),$$

$$I_{\mathrm{par}}^{\mathcal{E}}(f) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathcal{E}}(M,V,\zeta)} a^{M,\mathcal{E}}(\gamma) I_M^{\mathcal{E}}(\gamma, f),$$

and

$$S_{\mathrm{par}}^{G}(f) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M, M') \cdot \sum_{\delta' \in \Delta(\widetilde{M}', V, \widetilde{\zeta}')} b^{\widetilde{M}'}(\delta') S_M^G(M', \delta', f),$$

that comprise the three geometric expansions in [I, §2,10]. However, we shall first study the complementary terms in the corresponding trace formulas. These include constituents of the three spectral expansions from [I, §3,10]. We shall show how to eliminate all the terms in the spectral expansions except for the discrete parts $I_{t,\mathrm{disc}}(f)$, $I^{\mathcal{E}}_{t,\mathrm{disc}}(f)$ and $S^G_{t,\mathrm{disc}}(f)$. As in [I, §3], the nonnegative real numbers t that parametrize these distributions are obtained from the imaginary parts of archimedean infinitesimal characters.

Proposition 2.2(a). (a) *In general*,

(2.4)
$$I^{\mathcal{E}}_{\mathrm{par}}(f)-I_{\mathrm{par}}(f)=\sum_t\Big(I^{\mathcal{E}}_{t,\mathrm{disc}}(f)-I_{t,\mathrm{disc}}(f)\Big)-\sum_z\Big(I^{\mathcal{E}}_{z,\mathrm{unip}}(f,S)-I_{z,\mathrm{unip}}(f,S)\Big).$$

(b) *If G is quasisplit and f is unstable*,

$$S^G_{\mathrm{par}}(f)=\sum_t S^G_{t,\mathrm{disc}}(f)-\sum_z S^G_{z,\mathrm{unip}}(f,S). \tag{2.5}$$

The sums over t in (a) *and* (b) *satisfy the global multiplier estimate* [I, (3.3)], *and in particular*, *converge absolutely.*

Proof. We begin with the assertion (a). By the geometric expansions [I, Prop. 2.2 and Th. 10.1(a)], we can write

$$I^{\mathcal{E}}_{\mathrm{par}}(f)-I_{\mathrm{par}}(f)=\Big(I^{\mathcal{E}}(f)-I(f)\Big)-\Big(I^{\mathcal{E}}_{\mathrm{orb}}(f)-I_{\mathrm{orb}}(f)\Big),$$

in the notation of [I]. Now

$$I^{\mathcal{E}}_{\mathrm{orb}}(f)-I_{\mathrm{orb}}(f)=\sum_{\gamma\in\Gamma^{\mathcal{E}}(G,V,\zeta)}\Big(a^{G,\mathcal{E}}(\gamma)-a^G(\gamma)\Big)f_G(\gamma),$$

by the definition [I, (2.11)] and the formula [I, Lemma 7.2(a)]. If we apply the global induction hypothesis (1.4) to the terms in the expansions [I, (2.8), (10.10)], we see that

$$a^{G,\mathcal{E}}(\gamma)-a^G(\gamma)=a^{G,\mathcal{E}}_{\mathrm{ell}}(\gamma,S)-a^G_{\mathrm{ell}}(\gamma,S).$$

It follows from [I, (8.5), (8.8)] that

$$I^{\mathcal{E}}_{\mathrm{orb}}(f)-I_{\mathrm{orb}}(f)=I^{\mathcal{E}}_{\mathrm{ell}}(f,S)-I_{\mathrm{ell}}(f,S).$$

Combining this with Lemma 2.1, we see that

$$I^{\mathcal{E}}_{\mathrm{par}}(f)-I_{\mathrm{par}}(f)=\Big(I^{\mathcal{E}}(f)-I(f)\Big)-\sum_z\Big(I^{\mathcal{E}}_{z,\mathrm{unip}}(f)-I_{z,\mathrm{unip}}(f)\Big).$$

The second step is to apply the spectral expansions for $I^{\mathcal{E}}(f)$ and $I(f)$. It follows from Propositions 3.1 and 10.5 of [I] that

$$I^{\mathcal{E}}(f) - I(f) = \sum_t \Big(I_t^{\mathcal{E}}(f) - I_t(f) \Big),$$

where the sums over t satisfy the global multiplier estimate [I, (3.3)]. We have to show that the summands reduce to the corresponding summands in (2.4).

By Proposition 3.3 and Theorem 10.6 of [I], we can write $I_t^{\mathcal{E}}(f) - I_t(f)$ as the sum of a distribution

$$I_{t,\mathrm{unit}}^{\mathcal{E}}(f) - I_{t,\mathrm{unit}}(f)$$

defined in [I, §3,7], and an expression

$$\sum_{M \in \mathcal{L}^0} |W_0^M| |W_0^G|^{-1} \int_{\Pi_t^{\mathcal{E}}(M,V,\zeta)} \Big(a^{M,\mathcal{E}}(\pi) I_M^{\mathcal{E}}(\pi, f) - a^M(\pi) I_M(\pi, f) \Big) d\pi.$$

Consider the terms in the expansion. The indices M are by definition proper Levi subgroups of G. For any such M, the global induction hypothesis (1.4) implies that $a^{M,\mathcal{E}}(\pi)$ equals $a^M(\pi)$. Local Theorem $2'$ would also tell us that the distributions $I_M^{\mathcal{E}}(\pi, f)$ and $I_M(\pi, f)$ are equal. At this point, we do not know that the theorem holds for arbitrary π. In the case at hand, however, π belongs to $\Pi_{\mathrm{unit}}^{\mathcal{E}}(M, V, \zeta)$, and therefore has unitary central character. In this case, the identity follows from the study of these distributions in terms of their geometric counterparts [A12], and the local induction hypothesis (1.2). (For special cases of this argument, the reader can consult the proof of Lemma 5.2 of [A2] and the discussion at the end of Section 10 of [AC].) The terms in the expansion therefore vanish. The remaining distribution has its own expansion

$$I_{t,\mathrm{unit}}^{\mathcal{E}}(f) - I_{t,\mathrm{unit}}(f) = \int_{\Pi_t^{\mathcal{E}}(G,V,\zeta)} \Big(a^{G,\mathcal{E}}(\pi) - a^G(\pi) \Big) f_G(\pi) d\pi,$$

according to [I, (3.16) and Lemma 7.3(a)]. Applying the global induction hypothesis (1.4) to the terms in the expansions [I, (3.12), (10.21)], we deduce that

$$a^{G,\mathcal{E}}(\pi) - a^G(\pi) = a_{\mathrm{disc}}^{G,\mathcal{E}}(\pi) - a_{\mathrm{disc}}^G(\pi).$$

It follows from [I, (8.13), (8.16)] that

$$I_{t,\mathrm{unit}}^{\mathcal{E}}(f) - I_{t,\mathrm{unit}}(f) = I_{t,\mathrm{disc}}^{\mathcal{E}}(f) - I_{t,\mathrm{disc}}(f).$$

This gives the reduction we wanted. Summing over t, we conclude that

$$I^{\mathcal{E}}(f) - I(f) = \sum_t \Big(I_{t,\mathrm{disc}}^{\mathcal{E}}(f) - I_{t,\mathrm{disc}}(f) \Big),$$

and that the identity of (a) is valid.

The argument in (b) is similar. Assume that G is quasisplit, and that f is unstable. The geometric expansion [I, Th. 10.1(b)] asserts that

$$S^G_{\mathrm{par}}(f) = S^G(f) - S^G_{\mathrm{orb}}(f),$$

in the notation of [I]. Now, $S^G_{\mathrm{orb}}(f)$ has a simple expansion

$$S^G_{\mathrm{orb}}(f) = \sum_{\delta\in\Delta^{\mathcal{E}}(G,V,\zeta)} b^G(\delta) f^{\mathcal{E}}_G(\delta),$$

according to [I, Lemma 7.2(b)]. Since f is unstable, the function $f^{\mathcal{E}}_G$ vanishes on the subset $\Delta(G,V,\zeta)$ of $\Delta^{\mathcal{E}}(G,V,\zeta)$. It follows from [I, Prop. 10.3(b) and (8.9)] that

$$S^G_{\mathrm{orb}}(f) = \sum_{\delta\in\Delta^{\mathcal{E}}(G,V,\zeta)} b^G_{\mathrm{ell}}(\delta,S) f^{\mathcal{E}}_G(\delta) = S^G_{\mathrm{ell}}(f,S).$$

Combining this with Lemma 2.1, we see that

$$S^G_{\mathrm{par}}(f) = S^G(f) - \sum_z S^G_{z,\mathrm{unip}}(f).$$

The second step again is to apply the appropriate spectral expansion. It follows from [I, Prop. 10.5] that

$$S^G(f) = \sum_t S^G_t(f),$$

where the sums over t satisfy the global multiplier estimate [I, (3.3)]. For a given t, Theorem 10.6 of [I] expresses $S^G_t(f)$ as the sum of a distribution $S^G_{t,\mathrm{unit}}$ defined in [I, §7], and an expansion in terms of distributions

$$S^G_M(M',\phi',f), \qquad M\in\mathcal{L}^0,\ M'\in\mathcal{E}_{\mathrm{ell}}(M,V),\ \phi'\in\Phi_t(\widetilde{M}',V,\widetilde{\zeta}').$$

Local Theorem $2'$ would tell us that the distribution $S^G_M(M',\phi')$ vanishes if $M'\neq M$, and is stable if $M'=M$. Since f is unstable, $S^G_M(M',\phi',f)$ ought then to vanish for any M'. Given that the element $\phi'\in\Phi_t(\widetilde{M}',V,\widetilde{\zeta}')$ at hand has unitary central character, this again follows from the study of the distributions in terms of their geometric counterparts [A12], and the local induction hypothesis (1.2), even though we have not yet established the theorem in general. The terms in the expansion therefore vanish. The remaining distribution has its own expansion

$$S^G_{t,\mathrm{unit}}(f) = \int_{\Phi^{\mathcal{E}}_t(G,V,\zeta)} b^G(\phi) f^{\mathcal{E}}_G(\phi)\,d\phi,$$

provided by [I, Lemma 7.3(b)]. We can then deduce that

$$S^G_{t,\mathrm{unit}}(f) = \sum_{\phi\in\Phi^{\mathcal{E}}_{t,\mathrm{unit}}(G,V,\zeta)} b^G_{\mathrm{disc}}(\phi) f^{\mathcal{E}}_G(\phi) = S^G_{t,\mathrm{disc}}(f),$$

from [I, Prop. 10.7(b) and (8.17)], and the fact that f is unstable. Summing over t, we conclude that

$$S^G(f) = \sum_t I_{t,\mathrm{disc}}(f).$$

The identity in (b) follows. □

We shall now study the expressions on the left-hand sides of (2.4) and (2.5). If M belongs to $\mathcal{L}^0$, the global induction hypothesis (1.4) implies that the coefficients $a^{M,\mathcal{E}}(\gamma)$ and $a^M(\gamma)$ are equal. We can therefore write the left-hand side of (2.4) as

$$I^{\mathcal{E}}_{\mathrm{par}}(f) - I_{\mathrm{par}}(f) = \sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma)\Big(I^{\mathcal{E}}_M(\gamma,f) - I_M(\gamma,f)\Big).$$

There are splitting formulas for $I^{\mathcal{E}}_M(\gamma,f)$ and $I_M(\gamma,f)$ that decompose these distributions into individual contributions at each place v in V [A10, (4.6), (6.2)], [A11]. The decompositions are entirely parallel. It follows from the induction hypothesis (1.2) that any of the cross terms in the two expansions cancel. To describe the remaining terms, we may as well assume that $f = \prod\limits_v f_v$. In particular,

$$f = f_v f^v, \qquad\qquad f^v = \prod_{w\neq v} f_w,$$

for any v. The left-hand side of (2.4) then reduces to

(2.6)
$$\sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{v\in V}\sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma)\Big(I^{\mathcal{E}}_M(\gamma_v,f_v) - I_M(\gamma_v,f_v)\Big) f^v_M(\gamma^v),$$

where $\gamma = \gamma_v\gamma^v$ is the decomposition of γ relative to the product $G_V = G_v G_V^v$. Similarly, there are splitting formulas [A10, (6.3), (6.3′)], [A11] for the distributions $S^G_M(M',\delta',f)$ that occur in the expansion of the left-hand side $S^G_{\mathrm{par}}(f)$ of (2.5). Applying the local induction hypothesis (1.2), one sees that $S^G_{\mathrm{par}}(f)$ equals

$$\sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M,M') \cdot \sum_{v\in V}\sum_{\delta'\in\Delta(\widetilde{M}',V,\widetilde{\zeta}')} b^{\widetilde{M}'}(\delta') S^G_M(M'_v,\delta'_v,f_v)(f^v)^{M'}\Big((\delta')^v\Big), \tag{2.7}$$

for any function $f = \Pi f_v$ such that $f^G = 0$, and for the decomposition $\delta' = \delta'_v(\delta')^v$ of δ'.

We have not yet used the induction hypothesis (1.3) that depends on the integer r_{der}. In order to apply it, we have to fix a Levi subgroup $M \in \mathcal{L}$ such that

$$\dim(A_M \cap G_{\mathrm{der}}) = r_{\mathrm{der}}.$$

Since r_{der} is positive, M actually lies in the subset $\mathcal{L}^0$ of proper Levi subgroups. The pair (G, M) will remain fixed until the end of Section 5.

If v belongs to V, M determines an element M_v in the set $\mathcal{L}_v^0 \subset \mathcal{L}_v$ of (equivalence classes of) proper Levi subgroups of G_v that contain a fixed minimal Levi subgroup of G_v. The real vector space

$$\mathfrak{a}_{M_v} = \mathrm{Hom}\Big(X(M)_{F_v}, \mathbb{R}\Big)$$

then maps onto the corresponding space $\mathfrak{a}_M$ for M. As usual, we write $\mathfrak{a}_{M_v}^{G_v}$ for the kernel in $\mathfrak{a}_{M_v}$ of the projection of $\mathfrak{a}_{M_v}$ onto $\mathfrak{a}_{G_v}$. We shall also write $V_{\mathrm{fin}}(G, M)$ for the set of p-adic valuations v in V such that

$$\dim(\mathfrak{a}_{M_v}^{G_v}) = \dim(\mathfrak{a}_M^G).$$

This condition implies that the canonical map from $\mathfrak{a}_{M_v}^{G_v}$ to $\mathfrak{a}_M^G$ is an isomorphism.

If v is any place in V, we shall say that a function $f_v \in \mathcal{H}(G_v, \zeta_v)$ is M-*cuspidal* if $f_{v,L_v} = 0$ for any element $L_v \in \mathcal{L}_v$ that does not contain a G_v-conjugate of M_v. Let $\mathcal{H}_M(G_V, \zeta_V)$ denote the subspace of $\mathcal{H}(G_V, \zeta_V)$ spanned by functions $f = \prod_v f_v$ such that f_v is M-cuspidal at two places v in V. In the case that G is quasisplit, we also set

$$\mathcal{H}_M^{\mathrm{uns}}(G_V, \zeta_V) = \mathcal{H}_M(G_V, \zeta_V) \cap \mathcal{H}^{\mathrm{uns}}(G_V, \zeta_V).$$

We write $W(M)$ for the Weyl group of (G, M) [A10, §1]. As in the case of connected reductive groups, $W(M)$ is a finite group that acts on $\mathcal{L}$.

Lemma 2.3. (a) *If G is arbitrary, $I_{\mathrm{par}}^{\mathcal{E}}(f) - I_{\mathrm{par}}(f)$ equals*

$$|W(M)|^{-1} \sum_{v \in V_{\mathrm{fin}}(G,M)} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M(\gamma)\Big(I_M^{\mathcal{E}}(\gamma_v, f_v) - I_M(\gamma_v, f_v)\Big) f_M^v(\gamma^v), \tag{2.8}$$

for any function $f = \prod_v f_v$ in $\mathcal{H}_M(G_V, \zeta_V)$.

(b) *If G is quasisplit, $S_{\mathrm{par}}^G(f)$ equals*

$$|W(M)|^{-1} \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M, M') \cdot \sum_{v \in V_{\mathrm{fin}}(G,M)} \sum_{\delta' \in \Delta(\widetilde{M}', V, \widetilde{\zeta}')} b^{\widetilde{M}'}(\delta') S_M^G(M_v', \delta_v', f_v)(f^v)^{M'}\Big((\delta')^v\Big), \tag{2.9}$$

for any function $f = \prod_v f_v$ in $\mathcal{H}_M^{\mathrm{uns}}(G_V, \zeta_V)$.

Proof. To establish (a), we write the expression (2.6) as

$$\sum_L |W(L)|^{-1} \sum_{v\in V} \sum_{\gamma\in\Gamma(L,V,\zeta)} a^L(\gamma)\Big(I_L^{\mathcal{E}}(\gamma_v,f_v)-I_L(\gamma_v,f_v)\Big)f_L^v(\gamma^v),$$

where L is summed over a set of representatives of W_0^G-orbits in $\mathcal{L}^0$. This is possible because the factors on the right depend only on the W_0^G-orbit of L, and the stabilizer of L in W_0^G equals $W_0^L W(L)$. If L does not contain a conjugate of M, our condition on f implies that $f_L^v(\gamma^v)=0$ for any v. The corresponding summand therefore vanishes. If L does contain a conjugate of M, but is not actually equal to such a conjugate, we have

$$\dim(A_L\cap G_{\mathrm{der}}) < \dim(A_M\cap G_{\mathrm{der}}) = r_{\mathrm{der}}.$$

In this case, the induction hypothesis (1.3) implies that $I_L^{\mathcal{E}}(\gamma_v,f_v)$ equals $I_L(\gamma_v,f_v)$, for any v. The corresponding summand again vanishes. This leaves only the element L that represents the orbit of M. The earlier expression (2.6) for $I_{\mathrm{par}}^{\mathcal{E}}(f)-I_{\mathrm{par}}(f)$ therefore reduces to

$$|W(M)|^{-1} \sum_{v\in V} \sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma)\Big(I_M^{\mathcal{E}}(\gamma_v,f_v)-I_M(\gamma_v,f_v)\Big)f_M^v(\gamma^v).$$

This is the same as the given expression (2.8), except that v is summed over V instead of the subset $V_{\mathrm{fin}}(G,M)$ of V.

Suppose that v belongs to the complement of $V_{\mathrm{fin}}(G,M)$ in V. If v is archimedean, $I_M^{\mathcal{E}}(\gamma_v,f_v)$ equals $I_M(\gamma_v,f_v)$, by [A13] and [A11]. If v is p-adic, the map from $\mathfrak{a}_{M_v}^{G_v}$ to $\mathfrak{a}_M^G$ has a nontrivial kernel. In this case, the descent formulas [A10, (4.5), (7.2)] (and their analogues [A11] for singular elements) provide an expansion

$$\begin{aligned} &I_M^{\mathcal{E}}(\gamma_v,f_v)-I_M(\gamma_v,f_v)\\ &\qquad = \sum_{L_v\in\mathcal{L}_v(M_v)} d_{M_v}^G(M,L_v)\Big(\widehat{I}_{M_v}^{L_v,\mathcal{E}}(\gamma_v,f_{v,L_v})-\widehat{I}_{M_v}^{L_v}(\gamma_v,f_{v,L_v})\Big), \end{aligned}$$

in which the coefficients $d_{M_v}^G(M,L_v)$ vanish unless L_v is a proper Levi subgroup of G_v. But if L_v is proper, our local induction hypothesis (1.2) tells us that $\widehat{I}_{M_v}^{L_v,\mathcal{E}}(\gamma_v,f_{v,L_v})$ equals $\widehat{I}_{M_v}^{L_v}(\gamma_v,f_{v,L_v})$. The summand for v in the expression above therefore vanishes in either case. We conclude that $I_{\mathrm{par}}^{\mathcal{E}}(f)-I_{\mathrm{par}}(f)$ equals (2.8), as required.

The proof of (b) is similar. We first write the expression (2.7) as

$$\begin{aligned} &\sum_L |W(L)|^{-1} \sum_{L'\in\mathcal{E}_{\mathrm{ell}}(L,V)} \iota(L,L')\\ &\qquad\cdot \sum_{v\in V} \sum_{\delta'\in\Delta(\widetilde{L}',V,\widetilde{\zeta}')} b^{\widetilde{L}'}(\delta')S_L^G(L'_v,\delta'_v,f_v)(f^v)^{L'}\Big((\delta')^v\Big), \end{aligned}$$

where L is summed over a set of representatives of W_0^G-orbits in $\mathcal{L}^0$. If L does not contain a conjugate of M,

$$(f^v)^{L'}\Big((\delta')^v\Big) = (f_L^v)^{L'}\Big((\delta')^v\Big) = 0, \qquad v \in V,$$

so the corresponding summand vanishes. If L strictly contains a conjugate of M, our induction hypothesis (1.3) implies that the distribution $S_L^G(L'_v, \delta'_v, f_v)$ vanishes if $L' \neq L$, and is stable if $L' = L$. Since the function f is unstable, the product

$$S_L^G(L'_v, \delta'_v, f_v)(f^v)^{L'}\Big((\delta')^v\Big), \qquad v \in V,$$

vanishes for any L', v and δ'. The corresponding summand again vanishes. The earlier expression (2.7) for $S_{\mathrm{par}}^G(f)$ therefore reduces to

$$|W(M)|^{-1} \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M, M') \sum_{v \in V} \cdot \sum_{\delta' \in \Delta(\widetilde{M}', V, \widetilde{\zeta}')} b^{\widetilde{M}'}(\delta') S_M^G(M'_v, \delta'_v, f_v)(f^v)^{M'}\Big((\delta')^v\Big).$$

This is the same as the required expression (2.9), except that v is summed over V instead of the subset $V_{\mathrm{fin}}(G, M)$. But if v belongs to the complement of $V_{\mathrm{fin}}(G, M)$ in V, the condition that f be unstable again allows us to deduce that the products

$$S_M^G(M'_v, \delta'_v, f_v)(f^v)^{M'}\Big((\delta')^v\Big)$$

all vanish. If v is archimedean, this follows from [A13] and [A11]. If v is p-adic, it is a simple consequence of the descent formulas [A10, (7.3), (7.3′)] (and their analogues [A11] for singular elements), and the local induction hypothesis (1.2). The summand corresponding to v therefore vanishes. We conclude that $S_{\mathrm{par}}^G(f)$ equals (2.9), as required. □

We remark that if M' and v are as in (2.9), the local endoscopic datum M'_v for M_v need not be elliptic. However, in this case, [A10, Lemma 7.1(b′)] (together with our induction hypotheses) implies that

$$S_M^G(M'_v, \delta'_v, f_v) = 0.$$

It follows that v could actually be summed over the subset

$$\begin{aligned} V_{\mathrm{fin}}(G, M') &= \Big\{ v \in V_{\mathrm{fin}}(G, M) : \ \mathfrak{a}_{M'_v} = \mathfrak{a}_{M_v} \Big\} \\ &= \Big\{ v \in V_{\mathrm{fin}} : \ \dim(\mathfrak{a}_{M'_v}^{G_v}) = \dim(\mathfrak{a}_M^G) \Big\} \end{aligned}$$

of $V_{\mathrm{fin}}(G, M)$ in (2.9).

3. Cancellation of p-adic singularities

To proceed further, we require more information about the linear forms in f_v that occur in (2.8) and (2.9). We shall extend the method of cancellation of singularities that was applied to the general linear group in [AC, §2.14]. In this paper, we need consider only the p-adic form of the theory, since the problems for archimedean places will be treated by local means in [A13] and [A11].

As in the last section, G is a fixed K-group over the global field F, with a fixed Levi subgroup M. Suppose that v belongs to the set V_{fin} of p-adic valuations in V. Then G_v is a connected reductive group over the field F_v. We shall define two subspaces of the Hecke algebra $\mathcal{H}(G_v, \zeta_v)$.

Let $\mathcal{H}(G_v, \zeta_v)^{00}$ be the subspace of functions in $\mathcal{H}(G_v, \zeta_v)$ whose strongly regular orbital integrals vanish near the center of G. Equivalently, $\mathcal{H}(G_v, \zeta_v)^{00}$ is the null space in $\mathcal{H}(G_v, \zeta_v)$ of the family of orbital integrals

$$f_v \longrightarrow f_{v,G}(z_v\alpha_v), \qquad f_v \in \mathcal{H}(G_v, \zeta_v),$$

in which z_v ranges over the center

$$Z(\overline{G}_v) = Z(G, F_v)/Z(F_v)$$

of $\overline{G}_v = G_v/Z_v$, and α_v ranges over $\Gamma_{\mathrm{unip}}(G_v, \zeta_v)$. For the latter description, we could equally well have replaced $\Gamma_{\mathrm{unip}}(G_v, \zeta_v)$ by the abstract set $R_{\mathrm{unip}}(G_v, \zeta_v)$ introduced in [A11]. This set is a second basis of the space of distributions spanned by the unipotent orbital integrals, which has the advantage of behaving well under induction. More precisely, $R_{\mathrm{unip}}(G_v, \zeta_v)$ is the disjoint union of the set $R_{\mathrm{unip,ell}}(G_v, \zeta_v)$ of elliptic elements in $R_{\mathrm{unip}}(G_v, \zeta_v)$, together with the subset

$$R_{\mathrm{unip,par}}(G_v, \zeta_v) = \left\{ \rho_v^{G_v} : \ \rho_v \in R_{\mathrm{unip,ell}}(L_v, \zeta_v), \ L_v \subsetneq G_v \right\}$$

of parabolic elements, induced from elliptic elements for proper parabolic subgroups of G_v. (See [A11].) We have reserved the symbol $\mathcal{H}(G_v, \zeta_v)^0$ to denote the larger subspace annihilated by just the parabolic elements. That is, $\mathcal{H}(G_v, \zeta_v)^0$ is the subspace of functions f_v in $\mathcal{H}(G_v, \zeta_v)$ such that

$$f_{v,G}(z_v\alpha_v) = 0, \qquad z_v \in Z(\overline{G}_v), \ \alpha_v \in R_{\mathrm{unip,par}}(G_v, \zeta_v).$$

Suppose now that v lies in our subset $V_{\mathrm{fin}}(G, M)$ of valuations v in V_{fin} such that $\mathfrak{a}_{M_v}^{G_v}$ maps isomorphically onto $\mathfrak{a}_M^G$. We shall define a map from $\mathcal{H}(G_v, \zeta_v)^0$ to another space, which represents an obstruction to the assertion of Local Theorem 1(a). In the case that G_v is quasisplit, we shall construct some further maps, one of which is defined on the space

$$\mathcal{H}^{\mathrm{uns}}(G_v, \zeta_v)^0 = \mathcal{H}^{\mathrm{uns}}(G_v, \zeta_v) \cap \mathcal{H}(G_v, \zeta_v)^0,$$

and represents an obstruction to the stability assertion of Local Theorem 1(b). The maps will take values in the function spaces $\mathcal{I}_{ac}(M_v, \zeta_v)$ and $S\mathcal{I}_{ac}(M_v, \zeta_v)$

introduced in earlier papers. (See for example [A1, §1].) We recall that $\mathcal{I}_{ac}(M_v, \zeta_v)$ and $S\mathcal{I}_{ac}(M_v, \zeta_v)$ are modest generalizations of the spaces $\mathcal{I}(M_v, \zeta_v)$ and $S\mathcal{I}(M_v, \zeta_v)$, necessitated by the fact that weighted characters have singularities in the complex domain. They are given by invariant and stable orbital integrals of functions in a space $\mathcal{H}_{ac}(M_v, \zeta_v)$. By definition, $\mathcal{H}_{ac}(M_v, \zeta_v)$ is the space of uniformly smooth, ζ_v^{-1}-equivariant functions f_v on M_v such that for any X_v in the group

$$\mathfrak{a}_{M,v} = \mathfrak{a}_{M_v,F_v} = H_{M_v}(M_v),$$

the restriction of f_v to the preimage of X_v in M_v has compact support. By uniformly smooth, we mean that the function f_v is bi-invariant under an open compact subgroup of G_v. An element in $\mathcal{I}_{ac}(M_v, \zeta_v)$ can be identified with a function on either of the sets $\Gamma(M_v, \zeta_v)$ or $R(M_v, \zeta_v)$ (by means of orbital integrals) or with a function on the product of $\Pi(M_v, \zeta_v)$ with $\mathfrak{a}_{M,v}/\mathfrak{a}_{Z,v}$ (by means of characters). Similarly, an element in $S\mathcal{I}_{ac}(M_v, \zeta_v)$ can be identified with a function on $\Delta(M_v, \zeta_v)$ (by means of stable orbital integrals) or with a function on the product of $\Phi(M_v, \zeta_v)$ with $\mathfrak{a}_{M,v}/\mathfrak{a}_{Z,v}$ (by means of "stable characters"). We emphasize that the sets $R(M_v, \zeta_v)$, $\Delta(M_v, \zeta_v)$ and $\Phi(M_v, \zeta_v)$ are all abstract bases of one sort or another. In particular, the general theory is not sufficiently refined for us to be able to identify the elements in $\Phi(M_v, \zeta_v)$ with stable characters in the usual sense.

The maps will actually take values in the appropriate subspace of cuspidal functions. We recall that a function in $\mathcal{I}_{ac}(M_v, \zeta_v)$ is *cuspidal* if it vanishes on any induced element

$$\gamma_v = \rho_v^{M_v}, \qquad \rho_v \in \Gamma(R_v, \zeta_v),$$

in $\Gamma(M_v, \zeta_v)$, where R_v is a proper Levi subgroup of M_v. Similarly, a function in $S\mathcal{I}_{ac}(M_v, \zeta_v)$ is *cuspidal* if it vanishes on any properly induced element

$$\delta_v = \sigma_v^{M_v}, \qquad \sigma_v \in \Delta(R_v, \zeta_v),$$

in $\Delta(M_v, \zeta_v)$.

Proposition 3.1. (a) *There is a map*

$$\varepsilon_M : \mathcal{H}(G_v, \zeta_v)^0 \longrightarrow \mathcal{I}_{ac}(M_v, \zeta_v),$$

which takes values in the subspace of cuspidal functions, such that

$$\varepsilon_M(f_v, \gamma_v) = I_M^{\mathcal{E}}(\gamma_v, f_v) - I_M(\gamma_v, f_v), \tag{3.1}$$

for any $f_v \in \mathcal{H}(G_v, \zeta_v)^0$ *and* $\gamma_v \in \Gamma(M_v, \zeta_v)$.

(b) *If* G_v *is quasisplit, there is a map*

$$\varepsilon^M = \varepsilon^{M^*} : \mathcal{H}^{\mathrm{uns}}(G_v, \zeta_v)^0 \longrightarrow S\mathcal{I}_{ac}(M_v, \zeta_v),$$

which takes values in the subspace of cuspidal functions, such that

$$\varepsilon^M(f_v,\delta_v) = S^G_M(\delta_v, f_v), \tag{3.2}$$

for any $f_v \in \mathcal{H}^{\mathrm{uns}}(G_v,\zeta_v)^0$ *and* $\delta_v \in \Delta(M_v,\zeta_v)$.

(b$'$) *If* G_v *is quasisplit and* M' *belongs to* $\mathcal{E}^0_{\mathrm{ell}}(M)$, *there is a map*

$$\varepsilon^{M'}:\ \mathcal{H}(G_v,\zeta_v)^0 \longrightarrow S\mathcal{I}_{ac}(\widetilde{M}'_v,\widetilde{\zeta}'_v),$$

which takes values in the subspace of cuspidal functions, such that

$$\varepsilon^{M'}(f_v,\delta'_v) = S^G_M(M'_v,\delta'_v,f_v), \tag{3.2$'$}$$

for any $f_v \in \mathcal{H}(G_v,\zeta_v)^0$ *and* $\delta'_v \in \Delta(\widetilde{M}'_v,\widetilde{\zeta}'_v)$.

Proof. The main point will be to establish that the assertions of the lemma hold locally around a singular point. To begin the proof of (a), we fix a function $f_v \in \mathcal{H}(G_v,\zeta_v)^0$. Consider a semisimple conjugacy class $c_v \in \Gamma_{ss}(\overline{M}_v)$ in $\overline{M}_v = M_v/Z_v$. We shall show that the right-hand side of (3.1) represents an invariant orbital integral of some function, for those strongly G-regular elements $\gamma_v \in \Gamma_{G\text{-reg}}(M_v,\zeta_v)$ in some neighbourhood of c_v. To do so, we shall use the results in [A11] on the comparison of germs of weighted orbital integrals.

According to the germ expansions for $I^{\mathcal{E}}_M(\gamma_v,f_v)$ and $I_M(\gamma_v,f_v)$ in [A11], the right-hand side of (3.1) equals

$$\sum_{L\in\mathcal{L}(M)}\ \sum_{\rho_v\in R_{d_v}(L_v,\zeta_v)} \Big(g^{L,\mathcal{E}}_M(\gamma_v,\rho_v) I^{\mathcal{E}}_L(\rho_v,f_v) - g^L_M(\gamma_v,\rho_v) I_L(\rho_v,f_v)\Big), \tag{3.3}$$

for any element $\gamma_v \in \Gamma_{G\text{-reg}}(M_v,\zeta_v)$ that is near c_v. Here, $d_v \in \Delta_{ss}(\overline{M}_v)$ is the stable conjugacy class of c_v, and $R_{d_v}(L_v,\zeta_v)$ denotes the set of elements in the basis $R(L_v,\zeta_v)$ whose semisimple part maps to the image of d_v in $\Delta_{ss}(\overline{L}_v)$. One might expect to be able to sum ρ_v over only the subset $R_{c_v}(L_v,\zeta_v)$ of elements in $R_{d_v}(L_v,\zeta_v)$ whose semisimple part maps to c_v. Indeed, $g^L_M(\gamma_v,\rho_v)$ vanishes by definition, unless ρ_v lies in $R_{c_v}(L_v,\zeta_v)$. Local Theorem 1 implies that the germs $g^{L,\mathcal{E}}_M$ and g^L_M are equal [A11], so we would expect $g^{L,\mathcal{E}}_M(\gamma_v,\rho_v)$ also to have this property. For the moment, we have to leave open the possibility that $g^{L,\mathcal{E}}_M$ represent a larger family of germs, but we shall soon rule this out.

We shall show that the summand with any $L \neq M$ in (3.3) vanishes. If L is distinct from G, the first local induction hypothesis (1.2) tells us that the distributions $I^{L,\mathcal{E}}_M(\gamma_v)$ and $I^L_M(\gamma_v)$ are equal. It follows from [A11] that the germs $g^{L,\mathcal{E}}_M(\gamma_v,\rho_v)$ and $g^L_M(\gamma_v,\rho_v)$ are also equal. In particular, the corresponding inner sum in (3.3) can be taken over the subset $R_{c_v}(L_v,\zeta_v)$ of $R_{d_v}(L_v,\zeta_v)$. If L is also distinct from M, the second local induction hypothesis (1.3) implies that $I^{\mathcal{E}}_L(\rho_v,f_v)$ equals $I_L(\rho_v,f_v)$. It follows that the summands in (3.3) with L

distinct from M and G all vanish. Consider next the summand with $L = G$. Then

$$I_G^{\mathcal{E}}(\rho_v, f) = I_G(\rho_v, f_v) = f_{v,G}(\rho_v).$$

Suppose first that c_v is not central in G_v. The descent formulas in [A11] provide parallel expansions for $g_M^{G,\mathcal{E}}(\gamma_v, \rho_v)$ and $g_M^G(\gamma_v, \rho_v)$ in terms of germs attached to the centralizer of c_v in G_v. The induction hypothesis (1.2) again implies that the germs are equal. In the remaining case that c_v is central in G_v, we have

$$R_{d_v}(G_v, \zeta_v) = R_{c_v}(G_v, \zeta_v) = \Big\{c_v\alpha_v :\ \alpha_v \in R_{\mathrm{unip}}(G_v, \zeta_v)\Big\}.$$

If α_v belongs to the subset $R_{\mathrm{unip,ell}}(G_v, \zeta_v)$ of $R_{\mathrm{unip}}(G_v, \zeta_v)$, the germs $g_M^{G,\mathcal{E}}(\gamma_v, c_v\alpha_v)$ and $g_M^G(\gamma_v, c_v\alpha_v)$ are equal. This is a simple consequence [A11] of the results of [A10, §10]. If α_v belongs to the complement $R_{\mathrm{unip,par}}(G_v, \zeta_v)$ of $R_{\mathrm{unip,ell}}(G_v, \zeta_v)$ in $R_{\mathrm{unip}}(G_v, \zeta_v)$, $f_{v,G}(c_v\alpha_v)$ equals 0, since f_v belongs to $\mathcal{H}(G_v, \zeta_v)^0$. In either case, the term in (3.3) corresponding to $\rho_v = c_v\alpha_v$ vanishes. This takes care of the summand with $L = G$.

We have shown that (3.3) reduces to the summand with $L = M$. We obtain

(3.4)
$$I_M^{\mathcal{E}}(\gamma_v, f_v) - I_M(\gamma_v, f_v) = \sum_{\rho_v \in R_{c_v}(M_v, \zeta_v)} g_M^M(\gamma_v, \rho_v)\Big(I_M^{\mathcal{E}}(\rho_v, f_v) - I_M(\rho_v, f_v)\Big),$$

for elements $\gamma_v \in \Gamma_{G\text{-reg}}(M_v, \zeta_v)$ that are close to c_v. Since $g_M^M(\gamma_v, \rho_v)$ is an ordinary Shalika germ, the right-hand side of (3.4) represents an invariant orbital integral in γ_v. We conclude that there exists a function $\varepsilon_M(f_v)$ in $\mathcal{I}(M_v, \zeta_v)$ such that (3.1) holds locally for any strongly G-regular element γ_v in some neighbourhood of c_v.

To establish the full assertion (a), we have to let c_v vary. The obvious technique to use is a partition of unity. However, something more is required, since we have to show that a function of noncompact support is uniformly smooth. We shall use constructions of [A1] and [A12] to represent $\varepsilon_M(f_v)$ in terms of some auxiliary functions in $\mathcal{I}_{ac}(M_v, \zeta_v)$.

Suppose that γ_v is any element in $\Gamma_{G\text{-reg}}(M_v, \zeta_v)$. Then we can write

$$I_M(\gamma_v, f_v) = {}^cI_M(\gamma_v, f_v) - \sum_{L \in \mathcal{L}^0(M)} \widehat{I}_M^L\Big(\gamma_v,\, {}^c\theta_L(f_v)\Big),$$

in the notation of [A1, Lemma 4.8]. One of the purposes of the paper [A12] is to establish endoscopic and stable versions of formulas such as this. The endoscopic form is

$$I_M^{\mathcal{E}}(\gamma_v, f_v) = {}^cI_M^{\mathcal{E}}(\gamma_v, f_v) - \sum_{L \in \mathcal{L}^0(M)} \widehat{I}_M^{L,\mathcal{E}}\Big(\gamma_v,\, {}^c\theta_L^{\mathcal{E}}(f_v)\Big),$$

where ${}^cI_M^{\mathcal{E}}(\gamma_v)$ and ${}^c\theta_L^{\mathcal{E}}$ are endoscopic analogues of, respectively, the supplementary linear form ${}^cI_M(\gamma_v)$ and the map ${}^c\theta_L$ from $\mathcal{H}_{ac}(G_v, \zeta_v)$ to $\mathcal{I}_{ac}(L_v, \zeta_v)$. Therefore, the difference

$$I_M^{\mathcal{E}}(\gamma_v, f_v) - I_M(\gamma_v, f_v)$$

can be expressed as

$$\Big({}^cI_M^{\mathcal{E}}(\gamma_v, f_v) - {}^cI_M(\gamma_v, f_v)\Big) - \sum_{L \in \mathcal{L}^0(M)} \Big(\widehat{I}_M^{L,\mathcal{E}}\Big(\gamma_v, {}^c\theta_L^{\mathcal{E}}(f_v)\Big) - \widehat{I}_M^L\Big(\gamma_v, {}^c\theta_L(f_v)\Big)\Big).$$

Suppose that $L \in \mathcal{L}^0(M)$. Since L is distinct from G, the induction hypothesis (1.2) tells us that $\widehat{I}_M^{L,\mathcal{E}}(\gamma_v) = \widehat{I}_M^L(\gamma_v)$. If L is also distinct from M, it follows from the induction hypothesis (1.3) and the results of [A12] that ${}^c\theta_L^{\mathcal{E}}(f_v) = {}^c\theta_L(f_v)$. Therefore the summands with $L \neq M$ in the last expression all vanish. We obtain

$$I_M^{\mathcal{E}}(\gamma_v, f_v) - I_M(\gamma_v, f_v) = {}^c\varepsilon_M(f_v, \gamma_v) - \Big({}^c\theta_M^{\mathcal{E}}(f_v, \gamma_v) - {}^c\theta_M(f_v, \gamma_v)\Big),$$

where

$${}^c\varepsilon_M(f_v, \gamma_v) = {}^cI_M^{\mathcal{E}}(\gamma_v, f_v) - {}^cI_M(\gamma_v, f_v).$$

We can of course restrict the variable γ_v to the strongly G-regular elements in some neighbourhood of c_v. Since the left-hand side of the last formula represents a function in $\mathcal{I}(M_v, \zeta_v)$ in such a neighbourhood, and since ${}^c\theta_M^{\mathcal{E}}(f_v, \gamma_v)$ and ${}^c\theta_M(f_v, \gamma_v)$ represent functions in $\mathcal{I}_{ac}(M_v, \zeta_v)$ for all γ_v, ${}^c\varepsilon_M(f_v, \gamma_v)$ must represent a function in $\mathcal{I}(M_v, \zeta_v)$, for all strongly G-regular elements γ_v near c_v. The advantage of the auxiliary function ${}^c\varepsilon_M(f_v, \gamma_v)$ is that it has bounded support in γ_v. This follows from [A1, Lemma 4.4] and its endoscopic analogue in [A12]. We can therefore use a finite partition of unity to construct a function ${}^c\varepsilon_M(f_v)$ in $\mathcal{I}(M_v, \gamma_v)$ whose value at any strongly G-regular element γ_v equals ${}^c\varepsilon_M(f_v, \gamma_v)$.

Having defined ${}^c\varepsilon_M(f_v)$, we set

$$\varepsilon_M(f_v) = {}^c\varepsilon_M(f_v) - \Big({}^c\theta_M^{\mathcal{E}}(f_v) - {}^c\theta_M(f_v)\Big).$$

Then $\varepsilon_M(f_v)$ is a function in $\mathcal{I}_{ac}(M_v, \zeta_v)$ such that (3.1) holds for every γ_v in $\Gamma_{G\text{-reg}}(M_v, \zeta_v)$. To show that (3.1) is valid for elements that are not strongly G-regular, we consider the ordinary Shalika germ expansion

$$\varepsilon_M(f_v, \gamma_v) = \sum_{\rho_v \in R_{c_v}(M_v, \zeta_v)} g_M^M(\gamma_v, \rho_v)\varepsilon_M(f_v, \rho_v)$$

of $\varepsilon_M(f_v)$, for $\gamma_v \in \Gamma_{G\text{-reg}}(M_v, \zeta_v)$ near c_v. The left-hand side of this expression equals the left-hand side of (3.4) by construction, and so the two right-hand sides must be equal. It follows from the linear independence of the germs

$$g_M^M(\gamma_v, \rho_v), \qquad \rho_v \in R_{c_v}(M_v, \zeta_v),$$

that

$$\varepsilon_M(f_v, \rho_v) = I_M^{\mathcal{E}}(\rho_v, f_v) - I_M(\rho_v, f_v), \qquad \rho_v \in R_{c_v}(M_v, \zeta_v).$$

This is equivalent to the identity

$$\varepsilon_M(f, \gamma_v) = I_M^{\mathcal{E}}(\gamma_v, f_v) - I_M(\gamma_v, f_v), \qquad \gamma_v \in \Gamma_{c_v}(M_v, \zeta_v),$$

since $\Gamma_{c_v}(M_v, \zeta_v)$ and $R_{c_v}(M_v, \zeta_v)$ represent bases of the same space. But the set $\Gamma(M_v, \zeta_v)$ is by definition a disjoint union of subsets $\Gamma_{c_v}(M_v, \zeta_v)$. We conclude that (3.1) holds in general.

The last step in the proof of (a) is to show that the function $\varepsilon_M(f_v)$ is cuspidal. Consider an element

$$\gamma_v = \rho_v^{M_v}, \qquad \rho_v \in \Gamma(R_v, \zeta_v),$$

induced from a proper Levi subgroup R_v of M_v. Applying the descent formulas [A10, (4.5), (7.2)] (or rather their generalizations [A11] to singular elements), we see that

$$\begin{aligned}\varepsilon_M(f_v, \gamma_v) &= I_M^{\mathcal{E}}(\gamma_v, f_v) - I_M(\gamma_v, f_v) \\ &= \sum_{L_v \in \mathcal{L}(M_v)} d_{R_v}^G(M, L_v)\Big(\widehat{I}_{R_v}^{L_v, \mathcal{E}}(\rho_v, f_{v,L_v}) - \widehat{I}_{R_v}^{L_v}(\rho_v, f_{v,L_v})\Big).\end{aligned}$$

The coefficient $d_{R_v}^G(M, L_v)$ is defined in [A10, §4], and actually equals the corresponding coefficient $d_{R_v}^{G_v}(M_v, L_v)$ in this case, since v belongs to $V_{\mathrm{fin}}(G, M)$. In any case, since R_v is proper in M_v, the coefficient vanishes unless L_v is a proper Levi subgroup of G_v. But if L_v is proper, the induction hypothesis (1.2) tells us that $\widehat{I}_{R_v}^{L_v, \mathcal{E}}(\rho_v, f_{v,L_v})$ equals $\widehat{I}_{R_v}^{L_v}(\rho_v, f_{v,L_v})$. The summand corresponding to L_v vanishes, so that $\varepsilon_M(f_v, \gamma_v) = 0$. Therefore $\varepsilon_M(f_v)$ is a cuspidal function in $\mathcal{I}_{ac}(M_v, \zeta_v)$.

The proofs of (b) and (b$'$) proceed along similar lines. Assume that G_v is quasisplit, and that f_v belongs to $\mathcal{H}(G_v, \zeta_v)^0$. We fix an endoscopic datum M' in $\mathcal{E}_{\mathrm{ell}}(M)$, and a semisimple stable conjugacy class d_v' in $\Delta_{ss}(\overline{M}_v') = \Delta_{ss}(\overline{\widetilde{M}}_v')$. We shall study $S_M^G(M_v', \delta_v', f_v)$, for strongly G-regular elements $\delta_v' \in \Delta_{G\text{-reg}}(\widetilde{M}_v', \widetilde{\zeta}_v')$ that are close to d_v'. In the special case that $M' = M^*$, we assume that f_v belongs to the subspace $\mathcal{H}^{\mathrm{uns}}(G_v, \zeta_v)^0$ of $\mathcal{H}(G_v, \zeta_v)^0$, and we write $d_v = d_v'$ and $\delta_v = \delta_v'$. In general, we take d_v to be the image of d_v' in $\Delta_{ss}(\overline{M}_v)$.

We shall apply the stable germ expansion of [A11]. According to this expansion, $S_M^G(M_v', \delta_v', f_v)$ equals the sum of

$$\sum_{\sigma_v \in \Delta_{d_v}^{\mathcal{E}}(G_v, \zeta_v)} h_M^G(M_v', \delta_v', \sigma_v) f_{v,G}^{\mathcal{E}}(\sigma_v) \tag{3.5}$$

and

$$\sum_{L\in\mathcal{L}^0(M)}\sum_{L'\in\mathcal{E}_{M'}(L)}\sum_{\sigma'_v\in\Delta_{d'_v}(\widetilde{L}'_v,\widetilde{\zeta}'_v)}\iota_{M'}(L,L')h^{\widetilde{L}'}_{\widetilde{M}'}(\delta'_v,\sigma'_v)S^G_L(L'_v,\sigma'_v,f_v), \tag{3.6}$$

for any element $\delta'_v \in \Delta_{G\text{-reg}}(\widetilde{M}'_v,\widetilde{\zeta}'_v)$ that is close to d'_v. Here $\Delta^{\mathcal{E}}_{d_v}(G_v,\zeta_v)$ denotes the set of elements in $\Delta^{\mathcal{E}}(G_v,\zeta_v)$ whose semisimple part maps to the image of d_v in $\Delta_{ss}(\overline{G}_v)$, and $\Delta_{d'_v}(\widetilde{L}'_v,\widetilde{\zeta}'_v)$ is a similarly defined subset of $\Delta(\widetilde{L}'_v,\widetilde{\zeta}'_v)$. The functions

$$\delta'_v \longrightarrow h^G_M(M'_v,\delta'_v,\sigma_v)$$

in (3.5) are the "stable" germs of [A11]. If $M'=M^*$ and $\delta'_v=\delta^*_v$, and if σ_v belongs to the subset $\Delta_{d_v}(G_v,\zeta_v)$ of $\Delta^{\mathcal{E}}_{d_v}(G_v,\zeta_v)$, we generally write

$$h^G_M(\delta^*_v,\sigma_v)=h^G_M(M'_v,\delta'_v,\sigma_v).$$

The germs $h^{\widetilde{L}'}_{\widetilde{M}'}(\delta'_v,\sigma'_v)$ in (3.6) follow this notation.

Consider the sum in (3.5). Suppose first that d_v is not central in G_v. The descent formula of [A11] then asserts that $h^G_M(M'_v,\delta'_v,\sigma_v)=0$, unless $M'=M^*$ and σ_v lies in the subset $\Delta_{d_v}(G_v,\zeta_v)$ of $\Delta^{\mathcal{E}}_{d_v}(G_v,\zeta_v)$. However this last condition implies that

$$f^{\mathcal{E}}_{v,G}(\sigma_v)=f^G_v(\sigma_v)=0,$$

since f_v is unstable. Therefore (3.5) vanishes in this case. In the remaining case that d_v is central in G_v, we have

$$\Delta^{\mathcal{E}}_{d_v}(G_v,\zeta_v)=\Big\{d_v\beta_v:\ \beta_v\in\Delta^{\mathcal{E}}_{\mathrm{unip}}(G_v,\zeta_v)\Big\}.$$

If β_v belongs to the subset $\Delta^{\mathcal{E}}_{\mathrm{unip,ell}}(G_v,\zeta_v)$ of elliptic elements in $\Delta^{\mathcal{E}}_{\mathrm{unip}}(G_v,\zeta_v)$, we apply the results on cuspidal functions in [A10, §10]. It is a simple consequence [A11] of these results that $h^G_M(M'_v,\delta'_v,d_v\beta_v)=0$, unless $M'=M^*$ and β_v lies in the subset $\Delta_{\mathrm{unip,ell}}(G_v,\zeta_v)$ of $\Delta^{\mathcal{E}}_{\mathrm{unip,ell}}(G_v,\zeta_v)$. But the last condition implies that

$$f^{\mathcal{E}}_{v,G}(d_v\beta_v)=f^G_v(d_v\beta_v)=0,$$

again because f_v is unstable. On the other hand, if β_v belongs to the complement $\Delta^{\mathcal{E}}_{\mathrm{unip,par}}(G_v,\zeta_v)$ of $\Delta^{\mathcal{E}}_{\mathrm{unip,ell}}(G_v,\zeta_v)$, $f^{\mathcal{E}}_{v,G}(d_v\beta_v)=0$, by virtue of the fact that f_v lies in $\mathcal{H}(G_v,\zeta_v)^0$. The sum (3.5) therefore vanishes in this case as well.

We have shown that $S^G_M(M'_v,\delta'_v,f_v)$ equals the expansion (3.6). Turning our attention to (3.6), we consider an outer summand in this expression corresponding to any $L\neq M$. If L' is an endoscopic datum for L that is distinct from L^*, $S^G_L(L'_v,\sigma'_v,f_v)=0$, by the induction assumption (1.3). This takes care of all the elements in the inner sum over $\mathcal{E}_{M'}(L)$, provided that $M'\neq M^*$.

If $M' = M^*$, the set $\mathcal{E}_{M'}(L)$ also contains L^*. In this case, however, the induction hypothesis (1.3) implies that the distributions

$$S_L^G(\sigma_v, f_v) = S_L^G(L_v^*, \sigma_v^*, f_v), \qquad \sigma_v = \sigma_v^* = \sigma_v',$$

are stable. Since f_v is unstable, the distributions vanish at f_v. It follows that the terms in (3.6) with $L \neq M$ vanish. We conclude that

$$S_M^G(M_v', \delta_v', f_v) = \sum_{\sigma_v' \in \Delta_{d_v'}(\widetilde{M}_v', \widetilde{\zeta}_v')} h_{\widetilde{M}'}^{\widetilde{M}'}(\delta_v', \sigma_v') S_M^G(M_v', \sigma_v', f_v), \tag{3.7}$$

for all $\delta_v' \in \Delta_{G\text{-reg}}(\widetilde{M}_v', \widetilde{\zeta}_v')$ that are close to d_v'. Since $h_{\widetilde{M}'}^{\widetilde{M}'}(\delta_v', \sigma_v')$ is a stabilized Shalika germ, the right-hand side of (3.7) represents a stable orbital integral in δ_v'. We conclude that there is a function $\varepsilon^{M'}(f_v)$ in $S\mathcal{I}(\widetilde{M}_v', \widetilde{\zeta}_v')$ such that (3.2) or (3.2$'$) holds (according to whether M' equals M^* or not) for every strongly G-regular element δ_v' that is close to d_v'.

To establish the full assertions (3.2) and (3.2$'$), we have to let d_v' vary. We again use the constructions of [A12]. Given $M' \in \mathcal{E}_{\text{ell}}(M)$ and $\delta_v' \in \Delta_{G\text{-reg}}(\widetilde{M}_v', \widetilde{\zeta}_v')$, we can express $S_M^G(M_v', \delta_v', f_v)$ as

$$^cS_M^G(M_v', \delta_v', f_v) - \sum_{L \in \mathcal{L}^0(M)} \sum_{L' \in \mathcal{E}_{M'}(L)} \widehat{S}_{\widetilde{M}'}^{\widetilde{L}'}\Big(\delta_v',\ {}^c\eta_L(L_v', f_v)\Big),$$

where ${}^cS_M^G(M_v', \delta_v')$ and ${}^c\eta_L(L_v')$ are "stable" analogues of ${}^cI_M(\gamma_v)$ and ${}^c\theta_L$ respectively. Suppose that $L \in \mathcal{L}^0(M)$ is distinct from M. It follows from the induction hypothesis (1.3) and [A12] that ${}^c\eta_L(L_v', f_v) = 0$ for any $L' \in \mathcal{E}(L)$ distinct from L^*, and that ${}^c\eta_L(L_v^*, f_v)$ depends only on f_v^G. But if L^* lies in $\mathcal{E}_{M'}(L)$, M' has to equal M^*, and $f_v^G = 0$ by assumption. The term corresponding to L therefore vanishes. We obtain

$$S_M^G(M_v', \delta_v', f_v) = {}^c\varepsilon^{M'}(f_v, \delta_v') - {}^c\eta_M(M_v', f_v, \delta_v'),$$

where

$${}^c\varepsilon^{M'}(f_v, \delta_v') = {}^cS_M^G(M_v', \delta_v', f_v).$$

We have already shown that $S_M^G(M_v', \delta_v', f_v)$ represents a function in $S\mathcal{I}(\widetilde{M}_v', \widetilde{\zeta}_v')$ for δ_v' near d_v'. Since ${}^c\eta_M(M_v', f_v, \delta_v')$ represents a function in $S\mathcal{I}_{ac}(\widetilde{M}_v', \widetilde{\zeta}_v')$ for all δ_v', we see that ${}^c\varepsilon^{M'}(f_v, \delta_v')$ represents a function in $S\mathcal{I}(\widetilde{M}_v', \widetilde{\zeta}_v')$ for δ_v' near d_v'. As in (a), the auxiliary function ${}^c\varepsilon^{M'}(f_v, \delta_v')$ has bounded support in δ_v' [A12]. We can therefore use a finite partition of unity to construct a function ${}^c\varepsilon^{M'}(f_v)$ in $S\mathcal{I}(\widetilde{M}_v', \widetilde{\zeta}_v')$ whose value at any strongly G-regular element δ_v' equals ${}^c\varepsilon^{M'}(f_v, \delta_v')$.

Having defined ${}^c\varepsilon^{M'}(f_v)$, we set

$$\varepsilon^{M'}(f_v) = {}^c\varepsilon^{M'}(f_v) - {}^c\eta_M(M_v', f_v).$$

Then $\varepsilon^{M'}(f_v)$ is a function in $S\mathcal{I}_{ac}(\widetilde{M}'_v, \widetilde{\zeta}'_v)$ such that the relevant identity (3.2) or (3.2′) holds for every δ'_v in $\Delta_{G\text{-reg}}(\widetilde{M}'_v, \widetilde{\zeta}'_v)$. To show that the identity holds for elements that are not strongly G-regular, we compare (3.7) with the Shalika germ expansion of $\varepsilon^{M'}(f_v)$ around any d'_v. It follows from the linear independence of the germs

$$h^{\widetilde{M}'}_{\widetilde{M}'}(\delta'_v, \sigma'_v), \qquad\qquad \sigma'_v \in \Delta_{d'_v}(\widetilde{M}'_v, \widetilde{\zeta}'_v),$$

that the required identity is valid for elements in $\Delta_{d'_v}(\widetilde{M}'_v, \widetilde{\zeta}'_v)$. It is therefore valid in general.

The final point to check is that the function $\varepsilon^{M'}(f_v)$ is cuspidal. Consider an element

$$\delta'_v = (\sigma'_v)^{\widetilde{M}'_v}, \qquad\qquad \sigma'_v \in \Delta(\widetilde{R}'_v, \widetilde{\zeta}'_v),$$

induced from a proper Levi subgroup R'_v of M'_v. We can represent R'_v as an element in $\mathcal{E}_{\mathrm{ell}}(R_v)$, for a proper Levi subgroup R_v of M_v that is uniquely determined up to conjugacy. We shall use the extension [A11] to singular elements of the relevant descent formula [A10, (7.3), (7.3′)], with the elements F_1, G_1, M_1, and R_1 of [A10, §7] taken to be F_v, G_v, M_v, and R_v, respectively. Suppose first that $M' = M^*$. Then $R_v = R'_v$, and $\delta_v = \delta'_v$ is induced from the element $\sigma_v = \sigma'_v$ in $\Delta(R_v, \zeta_v)$. The descent formula in this case is

$$S^G_M(\delta_v, f_v) = \sum_{L_v \in \mathcal{L}(R_v)} \mathrm{e}^G_{R_v}(M, L_v) \widehat{S}^{L_v}_{R_v}(\delta_v, f_v^{L_v}).$$

The coefficient $\mathrm{e}^G_{R_v}(M, L_v)$ vanishes by definition [A10, (6.1)], unless L_v is a proper Levi subgroup of G_v. But if L_v is proper, our induction hypothesis (1.2) implies that $S^{L_v}_{R_v}(S_v)$ is stable, and since f_v is assumed to be unstable in this case, $\widehat{S}^{L_v}_{R_v}(\delta_v, f_v^{L_v}) = 0$. The sum therefore vanishes. In the other case that $M' \neq M^*$, the descent formula is simply the identity

$$S^G_M(M'_v, \delta'_v, f_v) = 0.$$

(The hypotheses in [A10, §7] on which this identity rests are included in the induction hypothesis (1.2) and (1.3).) We have shown in both cases that

$$\varepsilon^{M'}(f_v, \delta'_v) = S^G_M(M'_v, \delta'_v, f_v) = 0.$$

Therefore $\varepsilon^{M'}(f_v)$ is a cuspidal function in $S\mathcal{I}_{ac}(\widetilde{M}'_v, \widetilde{\zeta}'_v)$. This completes the proof of Proposition 3.1. □

COROLLARY 3.2. *The mappings of Proposition* 3.1 *satisfy formulas*

$$\varepsilon_M(f_v) = {}^c\varepsilon_M(f_v) - \Big({}^c\theta^{\mathcal{E}}_M(f) - {}^c\theta_M(f_v)\Big) \tag{3.8}$$

and

$$\varepsilon^{M'}(f_v) = {}^c\varepsilon^{M'}(f_v) - {}^c\eta_M(M'_v, f_v), \tag{3.9}$$

in the notation of the proof. As above, f_v is any function in $\mathcal{H}(G_v, \zeta_v)^0$ that is unstable in the case that $M' = M^*$. $\square$

We now return to the discussion of Section 2. Recall that V is a finite set of valuations of F that contains $V_{\mathrm{ram}}(G, \zeta)$. Let $\mathcal{H}(G_V, \zeta_V)^0$ denote the subspace of $\mathcal{H}(G_V, \zeta_V)$ spanned by functions of the form

$$f = \prod_{v \in V} f_v, \qquad f_v \in \mathcal{H}(G_v, \zeta_v)^0. \tag{3.10}$$

For any such function, we can obviously write

$$f = f_v f^v, \qquad v \in V,\ f^v \in \mathcal{H}(G_V^v, \zeta_V^v)^0,$$

with the superscript as usual being used to denote an object associated to $V - \{v\}$. If G is quasisplit, we set

$$\mathcal{H}^{\mathrm{uns}}(G_V, \zeta_V)^0 = \mathcal{H}^{\mathrm{uns}}(G_V, \zeta_V) \cap \mathcal{H}(G_V, \zeta_V)^0.$$

Then $\mathcal{H}^{\mathrm{uns}}(G_V, \zeta_V)^0$ is spanned by functions of the form (3.10) such that $f_v^G = 0$ for some v.

We extend the maps of Lemma 3.1 to functions on G_V so that they are parallel to the expansions (2.8) and (2.9). Thus

$$\varepsilon_M :\ \mathcal{H}(G_V, \zeta_V)^0 \longrightarrow \mathcal{I}_{ac}(M_V, \zeta_V)$$

is a map such that

$$\varepsilon_M(f, \gamma) = \sum_{v \in V_{\mathrm{fin}}(G,M)} \varepsilon_M(f_v, \gamma_v) f_M^v(\gamma^v), \tag{3.11}$$

for any element $\gamma = \prod_v \gamma_v$ in $\Gamma(M, V, \zeta)$, and any function $f \in \mathcal{H}(G_V, \zeta_V)^0$ of the form (3.10). If G is quasisplit,

$$\varepsilon^M = \varepsilon^{M^*} :\ \mathcal{H}^{\mathrm{uns}}(G_V, \zeta_V)^0 \longrightarrow S\mathcal{I}_{ac}(M_V, \zeta_V)$$

is defined by

$$\varepsilon^M(f, \delta) = \sum_{v \in V_{\mathrm{fin}}(G,M)} \varepsilon^M(f_v, \delta_v) f^{v,M}(\delta^v), \tag{3.12}$$

for $\delta = \prod_v \delta_v$ in $\Delta(M, V, \zeta)$ and for $f \in \mathcal{H}^{\mathrm{uns}}(G_V, \zeta_V)^0$ of the form (3.10). (This is well defined, since if f_v does not belong to $\mathcal{H}^{\mathrm{uns}}(G_v, \zeta_v)^0$, $(f^v)^M = 0$, and the corresponding summand vanishes.) Finally, if G is quasisplit and $M' \in \mathcal{E}^0_{\mathrm{ell}}(M)$,

$$\varepsilon^{M'} :\ \mathcal{H}(G_V, \zeta_V)^0 \longrightarrow S\mathcal{I}_{ac}(\widetilde{M}'_V, \widetilde{\zeta}'_V)$$

is defined by

$$\varepsilon^{M'}(f,\delta') = \sum_{v\in V_{\mathrm{fin}}(G,M)} \varepsilon^{M'}(f_v,\delta'_v) f^{v,M'}\Big((\delta')^v\Big), \tag{3.12$'$}$$

for $\delta' = \prod_v \delta'_v$ in $\Delta(\widetilde{M}', V, \widetilde{\zeta}')$ and $f \in \mathcal{H}(G_V,\zeta_V)^0$ as in (3.10). The co-domains $\mathcal{I}_{ac}(M_V,\zeta_V)$, $S\mathcal{I}_{ac}(M_V,\zeta_V)$ and $S\mathcal{I}_{ac}(\widetilde{M}'_V,\widetilde{\zeta}'_V)$ are natural variants of the spaces discussed for v at the beginning of this section. For example, $\mathcal{I}_{ac}(M_V,\zeta_V)$ is defined by orbital integrals in terms of the space $\mathcal{H}_{ac}(M_V,\zeta_V)$ of uniformly smooth, ζ_V^{-1}-equivariant functions on M_V whose restrictions to each set

$$M_V^X = \{m \in M_V : \ H_M(m) = X\}, \qquad X \in \mathfrak{a}_M,$$

have compact support.

In Section 2, we also defined the space $\mathcal{H}_M(G_V,\zeta_V)$ of functions in $\mathcal{H}(G_V,\zeta_V)$ that are M-cuspidal at two places. Let $\mathcal{H}_M(G_V,\zeta_V)^0$ and $\mathcal{H}_M^{\mathrm{uns}}(G_V,\zeta_V)^0$ denote the intersections of $\mathcal{H}_M(G_V,\zeta_V)$ with $\mathcal{H}(G_V,\zeta_V)^0$ and $\mathcal{H}^{\mathrm{uns}}(G_V,\zeta_V)^0$ respectively. The following result is a corollary of Lemma 2.3 and Proposition 3.1.

Corollary 3.3. (a) *If G is arbitrary,*

$$I^{\mathcal{E}}_{\mathrm{par}}(f) - I_{\mathrm{par}}(f) = |W(M)|^{-1}\widehat{I}^M\Big(\varepsilon_M(f)\Big), \tag{3.13}$$

for any function f in $\mathcal{H}_M(G_V,\zeta_V)^0$.

(b) *If G is quasisplit,*

$$S^G_{\mathrm{par}}(f) = |W(M)|^{-1} \sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M,M')\widehat{S}^{\widetilde{M}'}\Big(\varepsilon^{M'}(f)\Big), \tag{3.14}$$

for any function f in $\mathcal{H}_M^{\mathrm{uns}}(G_V,\zeta_V)^0$.

Proof. To establish (a), we combine the expansion (2.8) of Lemma 2.3 with the definition (3.1) and (3.11) of $\varepsilon_M(f)$. We obtain

$$I^{\mathcal{E}}_{\mathrm{par}}(f) - I_{\mathrm{par}}(f) = |W(M)|^{-1} \sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma)\varepsilon_M(f,\gamma).$$

We are assuming that f is M-cuspidal at two places. It then follows from Corollary 3.2 that the function $\varepsilon_M(f)$ in $\mathcal{I}_{ac}(M,V,\zeta)$ is actually cuspidal at two places. This means that the geometric side of the trace formula for M, formulated as in [I, Prop. 2.2], simplifies at the function $\varepsilon_M(f)$ [A2, Th. 7.1(b)].

We obtain
$$\widehat{I}^M\Big(\varepsilon_M(f)\Big) = \sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma)\varepsilon_M(f,\gamma).$$

The identity (3.13) follows.

To deal with (b), we begin with the stable expansion (2.9) of Lemma 2.3. Combining this with the definitions (3.2), (3.2′), (3.12) and (3.12′), we obtain

$$S^G_{\mathrm{par}}(f) = |W(M)|^{-1} \sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M,M') \sum_{\delta'\in\Delta(\widetilde{M}',V,\widetilde{\zeta}')} b^{\widetilde{M}'}(\delta')\varepsilon^{\widetilde{M}'}(f,\delta').$$

Corollary 3.2 implies that the function $\varepsilon^{M'}(f)$ in $S\mathcal{I}_{ac}(\widetilde{M}'_V,\widetilde{\zeta}'_V)$ is cuspidal at two places. This means that the geometric side of the stable trace formula for $\widetilde{M}'$ simplifies at the function $\varepsilon^{M'}(f)$. There is no direct reference to such a simplified formula, but it is easily derived from the general stable expansion in [I, Th. 10.1(b)]. Indeed, we can write

$$\begin{aligned}&\widehat{S}^{\widetilde{M}'}\Big(\varepsilon^{M'}(f)\Big) - \widehat{S}^{\widetilde{M}'}_{\mathrm{orb}}\Big(\varepsilon^{M'}(f)\Big)\\ &\qquad = \sum_{\widetilde{R}'\in(\mathcal{L}^{\widetilde{M}'})^0} |W_0^{\widetilde{R}'}||W_0^{\widetilde{M}'}|^{-1} \sum_{\delta'\in\Delta(\widetilde{R}',V,\widetilde{\zeta}')} b^{\widetilde{R}'}(\delta')\widehat{S}^{\widetilde{M}'}_{\widetilde{R}'}\Big(\delta',\varepsilon^{M'}(f)\Big),\end{aligned}$$

by [I, (10.5)]. From the local induction hypothesis (1.2) and the splitting formula for $\widehat{S}^{\widetilde{M}'}_{\widetilde{R}'}\Big(\delta',\varepsilon^{M'}(f)\Big)$, we deduce that this expression equals zero. Combining the global induction hypothesis (1.4) with the expansion [I, Lemma 7.2(b)] for $\widehat{S}^{\widetilde{M}'}_{\mathrm{orb}}\Big(\varepsilon^{M'}(f)\Big)$, we then obtain the simplified expansion

$$\widehat{S}^{\widetilde{M}'}\Big(\varepsilon^{M'}(f)\Big) = \sum_{\delta'\in\Delta(\widetilde{M}',V,\widetilde{\zeta}')} b^{\widetilde{M}'}(\delta')\varepsilon^{\widetilde{M}'}(f,\delta').$$

The identity (3.14) follows. □

4. Separation by infinitesimal character

Proposition 2.2 and Corollary 3.3 are the main results so far. Together, they provide a pair of identities (a) and (b) that will be objects of study for the rest of the paper. We shall now apply the theory of archimedean multipliers, following the argument in [AC, §2.15]. We shall replace the function

$$f = \bigoplus_{\beta\in\pi_0(G)} f_\beta, \qquad f_\beta\in\mathcal{H}(G_{V,\beta_V},\zeta_{V,\beta_V}),$$

by its transform $f_\alpha = \bigoplus_\beta f_{\beta,\alpha}$ under a multiplier $\alpha\in C_c^\infty(\mathfrak{h}^Z)^{W_\infty}$. We will then study the resulting identities in terms of the function $\widehat{\alpha}(\nu)$. Our goal is to show that each side of the identity in question vanishes, by the comparison of a distribution that is discrete in ν with one that is continuous. This is a crucial step that goes back to the comparison of distributions in [L3, §11] by Langlands.

We are following notation at the beginning of [I, §3]. In particular, $\mathfrak{h}$ is a split Cartan subalgebra of a split form of the real group $G_{V_\infty,\beta_{V_\infty}}$, for any component $\beta \in \pi_0(G)$, and W_∞ is the corresponding Weyl group. Any element α in the space $\mathcal{E}(\mathfrak{h})^{W_\infty}$ of compactly supported, W_∞-invariant distributions on $\mathfrak{h}$ determines an endomorphism

$$f = \bigoplus_\beta f_\beta \longrightarrow f_\alpha = \bigoplus_\beta f_{\beta,\alpha},$$

of $\mathcal{H}(G_V, \zeta_V)$. We shall take α to be in the subspace $C_c^\infty(\mathfrak{h}^Z)^{W_\infty}$ of $\mathcal{E}(\mathfrak{h})^{W_\infty}$. As in [I, §3], $\mathfrak{h}^Z$ denotes the subspace of points in $\mathfrak{h}$ whose projection onto $\mathfrak{a}_G$ lies in $\mathfrak{a}_Z$ (relative to the canonical embedding $\mathfrak{a}_Z \subset \mathfrak{a}_G$). Then $\widehat{\alpha}(\nu)$ is a W_∞-invariant Paley-Wiener function on the complex dual space $\mathfrak{h}^*_{\mathbb{C}}/\mathfrak{a}^*_{G,Z,\mathbb{C}}$, where $\mathfrak{a}^*_{G,Z}$ is the annihilator of $\mathfrak{a}_Z$ in $\mathfrak{a}^*_G$. We recall that it is the W_∞-orbits in $\mathfrak{h}^*_{\mathbb{C}}/\mathfrak{a}^*_{G,Z,\mathbb{C}}$ that parametrize archimedean infinitesimal characters ν_π of elements π in the set

$$\Pi(G_V^Z, \zeta_V) = \coprod_{\beta_V} \Pi(G^Z_{V,\beta_V}, \zeta_{V,\beta_V})$$

of irreducible representations of (components of) G_V^Z. We recall also that there is a natural subset $\mathfrak{h}^*_u$ of $\mathfrak{h}^*_{\mathbb{C}}/i\mathfrak{a}^*_{G,Z}$, which embeds into $\mathfrak{h}^*_{\mathbb{C}}/\mathfrak{a}^*_{G,Z,\mathbb{C}}$, and whose W_∞-orbits contain the infinitesimal characters of all unitary representations. (See [A2, p. 536] and [A7, p. 558].)

It will sometimes be convenient to index "discrete" distributions in the trace formula by an archimedean infinitesimal character ν, rather than the norm $t = \|\mathrm{Im}(\nu)\|$. For this purpose, ν stands for an element in the set $\mathfrak{h}^*_u/W_\infty$ of W_∞-orbits in $\mathfrak{h}^*_u$. We recall that $\|\cdot\|$ is the restriction to $\mathfrak{h}^*_u$ of the Hermitian norm on $\mathfrak{h}^*_{\mathbb{C}}/\mathfrak{a}^*_{G,Z,\mathbb{C}}$ that is dual to a fixed, W_∞-invariant Euclidean inner product on $\mathfrak{h}^Z$. We shall use a double subscript (ν, disc) to denote the contribution of ν to any object that has been indexed by (t, disc). For example, if $t = \|\mathrm{Im}(\nu)\|$, $\Pi_{\nu,\mathrm{disc}}(G, V, \zeta)$ denotes the set of representations in $\Pi_{t,\mathrm{disc}}(G, V, \zeta)$ whose archimedean infinitesimal character equals ν. It is empty unless the projection of ν onto $\mathfrak{a}^*_{Z,\mathbb{C}}$ coincides with the differential of the archimedean infinitesimal character of ζ. Moreover,

$$I_{\nu,\mathrm{disc}}(f) = \sum_{\pi \in \Pi_{\nu,\mathrm{disc}}(G,V,\zeta)} a^G_{\mathrm{disc}}(\pi) f_G(\pi)$$

and

$$I_{t,\mathrm{disc}}(f) = \sum_{\{\nu : \|\mathrm{Im}(\nu)\| = t\}} I_{\nu,\mathrm{disc}}(f).$$

Following [I, §3], and earlier papers [A2], [AC] and [A7], we write

$$\mathfrak{h}_u^*(r) = \mathfrak{h}_u^*(r,0) = \Big\{\nu \in \mathfrak{h}_u^* :\ \|\mathrm{Re}(\nu)\| \leq r\Big\}$$

and

$$\mathfrak{h}_u^*(r,T) = \Big\{\nu \in \mathfrak{h}_u^*(r) :\ \|\mathrm{Im}(\nu)\| \geq T\Big\},$$

for any nonnegative numbers r and T.

LEMMA 4.1. *For any function $f \in \mathcal{H}(G_V, \zeta_V)$, r can be chosen so that for every $\alpha \in C_c^\infty(\mathfrak{h}^Z)^{W_\infty}$, the distributions $I_{\nu,\mathrm{disc}}(f_\alpha)$, $I^{\mathcal{E}}_{\nu,\mathrm{disc}}(f_\alpha)$ and $S^G_{\nu,\mathrm{disc}}(f_\alpha)$ vanish if ν does not belong to $\mathfrak{h}_u^*(r)/W_\infty$.*

Proof. As α varies, the functions f_α are uniformly K_∞-finite, where $K_\infty = \coprod_\beta K_{\infty,\beta_\infty}$ is a union of maximal compact subgroups. It follows from Harish-Chandra's Plancherel formula that the tempered characters

$$f_{\alpha,G}(\pi), \qquad \pi \in \Pi_{\mathrm{temp}}(G_V, \zeta_V),$$

are supported on representations π whose archimedean infinitesimal characters have bounded real part. The same property for the larger family of unitary characters is then easy to establish from the Langlands classification. In particular, we can find an r such that the function $I_{\nu,\mathrm{disc}}(f_\alpha)$ of ν is supported on $\mathfrak{h}_u^*(r)/W_\infty$. The corresponding assertions for the functions $I^{\mathcal{E}}_{\nu,\mathrm{disc}}(f_\alpha)$ and $S^G_{\nu,\mathrm{disc}}(f_\alpha)$ then follow from their inductive definitions in terms of $I_{\nu,\mathrm{disc}}(f_\alpha)$, and standard properties of archimedean transfer factors. □

We now recall the spaces $\mathcal{H}_M(G_V,\zeta_V)^0$ and $\mathcal{H}^{\mathrm{uns}}_M(G_V,\zeta_V)^0$, defined near the end of the last section. Let us write $\mathcal{H}_M(G_V,\zeta_V)^{00}$ and $\mathcal{H}_M(G_V,\zeta_V)^{00}$ for the subspaces spanned by functions $f = \prod_v f_v$ in $\mathcal{H}_M(G_V,\zeta_V)^0$ and $\mathcal{H}^{\mathrm{uns}}_M(G_V,\zeta_V)^0$ respectively, such that for some $v \in V_{\mathrm{fin}}$, f_v belongs to the space $\mathcal{H}(G_v,\zeta_v)^{00}$ defined at the beginning of the last section.

PROPOSITION 4.2. (a) *If G is arbitrary,*

$$I^{\mathcal{E}}_{\nu,\mathrm{disc}}(f) - I_{\nu,\mathrm{disc}}(f) = 0, \tag{4.1}$$

for any ν and any $f \in \mathcal{H}_M(G_V,\zeta_V)^{00}$.

(b) *If G is quasisplit,*

$$S^G_{\nu,\mathrm{disc}}(f) = 0, \tag{4.2}$$

for any ν and any $f \in \mathcal{H}^{\mathrm{uns}}_M(G_V,\zeta_V)^{00}$.

Proof. The proposition is a general analogue of the results in [AC, §2.15] for GL(n). (See also [A7, Lemmas 8.1 and 9.1].) To prove it, we shall combine the global multiplier estimate [I, (3.3)] with Proposition 2.2 and Corollary 3.3. This will allow us to express the left-hand sides of (4.1) and (4.2) each as the value of a certain limit. We shall then show that the two limits vanish.

To deal with (a), we fix the function f in $\mathcal{H}_M(G_V, \zeta_V)^{00}$. Suppose that $\alpha \in C_c^\infty(\mathfrak{h}^Z)^{W_\infty}$ is a multiplier. The function f_α certainly lies in $\mathcal{H}(G_V, \zeta_V)$. It therefore satisfies the identity (2.4) of Proposition 2.2. In fact, f_α belongs to $\mathcal{H}_M(G_V, \zeta_V)^{00}$, since the conditions that define this subspace of $\mathcal{H}(G_V, \zeta_V)$ are not affected by multipliers. We can therefore apply Corollary 3.3(a) to the value taken at f_α by the linear form on the left-hand side of (2.4). Moreover, the sum

$$\sum_z \Big(I^{\mathcal{E}}_{z,\mathrm{unip}}(f_\alpha, S) - I_{z,\mathrm{unip}}(f_\alpha, S) \Big)$$

obtained from the right-hand side of (2.4) is equal to zero, since the functions $f_{\alpha,z,G}$ vanish on $\Gamma^{\mathcal{E}}_{\mathrm{unip}}(G, V, \zeta)$. The identity reduces to

$$|W(M)|^{-1} \widehat{I}^M\Big(\varepsilon_M(f_\alpha)\Big) = \sum_t \Big(I^{\mathcal{E}}_{t,\mathrm{disc}}(f_\alpha) - I_{t,\mathrm{disc}}(f_\alpha) \Big).$$

For the linear form on the left-hand side, we note that since W_∞ contains the corresponding Weyl group attached to M, α determines a multiplier for M_V. It acts on the spaces $\mathcal{H}(M_V, \zeta_V)$ and $\mathcal{H}_{ac}(M_V, \zeta_V)$, and as explained on p. 530 of [A2], also on $\mathcal{I}_{ac}(M_V, \zeta_V)$. The various definitions tell us that the function $\varepsilon_M(f_\alpha)$ equals $\varepsilon_M(f)_\alpha$. Since this function is cuspidal at some place, we can expand the linear form on the left-hand side as

$$\widehat{I}^M\Big(\varepsilon_M(f)_\alpha\Big) = \sum_t \widehat{I}^M_{t,\mathrm{disc}}\Big(\varepsilon_M(f)_\alpha\Big),$$

by the simple version [A2, Lemma 7.1(a)] of the spectral expansion. The distribution $\widehat{I}^M_{t,\mathrm{disc}}$ depends (through t) on a Euclidean norm on the analogue $\mathfrak{h}^{M,Z}$ for M of the space $\mathfrak{h}^Z = \mathfrak{h}^{G,Z}$. We assume that this is the restriction of the Euclidean norm we fixed on $\mathfrak{h}^Z$.

We have obtained an identity

$$|W(M)|^{-1} \sum_t \widehat{I}^M_{t,\mathrm{disc}}\Big(\varepsilon_M(f)_\alpha\Big) = \sum_t \Big(I^{\mathcal{E}}_{t,\mathrm{disc}}(f_\alpha) - I_{t,\mathrm{disc}}(f_\alpha) \Big)$$

between two absolutely convergent sums over t. The right-hand sum satisfies the global multiplier estimate [I, (3.3)]. It happens that the left-hand sum also satisfies this stronger estimate, but the justification requires further comment.

The spectral expansion for functions in the standard Hecke algebra does satisfy the required estimate. This follows from Proposition 3.1 of [I], which is in turn a direct consequence of the proof of Lemma 6.3 of [A2]. However, the function $\varepsilon_M(f)$ belongs to $\mathcal{I}_{ac}(M_V, \zeta_V)$ rather than $\mathcal{I}(M_V, \zeta_V)$. Moreover, the multiplier α is supported on the space $\mathfrak{h}^Z = \mathfrak{h}^{G,Z}$, rather than the subspace $\mathfrak{h}^{M,Z}$ attached to M. With only these general conditions on $\varepsilon_M(f)$ and α, the estimate of [A2, Lemma 6.3] would actually fail. The estimate was carried out for the special case of what were called *moderate* functions. In the present context, the moderate functions form a space that lies between $\mathcal{I}(M_V, \zeta_V)$ and $\mathcal{I}_{ac}(M_V, \zeta_V)$. They are defined as on p. 531 of [A2] by a mild support condition, and a similarly mild growth condition. We shall prove that $\varepsilon_M(f)$ is a moderate function in $\mathcal{I}_{ac}(M_V, \zeta_V)$, in order to show that the left-hand side of the identity does satisfy the desired estimate.

To establish that $\varepsilon_M(f)$ is moderate, it will be enough to verify that for $v \in V_{\text{fin}}(G, M)$ and $f_v \in \mathcal{H}(G_v, \zeta_v)^0$, the function $\varepsilon_M(f_v)$ satisfies the local form of the two conditions on p. 531 of [A2]. The fact that $\varepsilon_M(f_v)$ is cuspidal means that the support condition is vacuous. To check the growth condition, we recall that

$$\varepsilon_M(f_v) = {}^c\varepsilon_M(f_v) - \Big({}^c\theta^{\mathcal{E}}_M(f_v) - {}^c\theta_M(f_v)\Big),$$

by Corollary 3.2. As we saw in the proof of Proposition 3.1, the function ${}^c\varepsilon_M(f_v)$ actually belongs to $\mathcal{I}(M_v, \zeta_v)$. It is therefore compactly supported on $\Gamma(M_v, \zeta_v)$. The functions ${}^c\theta^{\mathcal{E}}_M(f_v)$ and ${}^c\theta_M(f_v)$ belong to the larger space $\mathcal{I}_{ac}(M_v, \zeta_v)$. However, according to the assertion in [A1, Lemma 5.2] for ${}^c\theta_M(f_v)$, and its analogue [A12] for ${}^c\theta^{\mathcal{E}}_M(f_v)$, the two functions have a property of rapid decrease. More precisely, as functions on the product of $\Pi_{\text{temp}}(M_v, \zeta_v)$ with $\mathfrak{a}_{M,v}/\mathfrak{a}_{Z,v}$, ${}^c\theta_M(f_v)$ and ${}^c\theta^{\mathcal{E}}_M(f_v)$ are both rapidly decreasing in the second variable. Therefore $\varepsilon_M(f_v)$ has the same property. The required condition of moderate growth pertains to $\varepsilon_M(f_v)$ as a function on $\Gamma(M_v, \zeta_v)$. However, since $\varepsilon_M(f_v)$ is cuspidal, the condition is an obvious consequence of what we have just established. Therefore $\varepsilon_M(f_v)$ satisfies both conditions. It follows from (3.11) that $\varepsilon_M(f)$ is a moderate function in $\mathcal{I}_{ac}(M_V, \zeta_V)$. Once we know that $\varepsilon_M(f)$ is moderate, the relevant part of the proof of Lemma 6.3 of [A2] tells us that the spectral expansion of $\widehat{I}^M\Big(\varepsilon_M(f)_\alpha\Big)$ satisfies the global multiplier estimate. But the spectral expansion of $\widehat{I}^M\Big(\varepsilon_M(f)_\alpha\Big)$ is just the sum on the left-hand side of the identity we have been considering. Therefore, the left-hand side does satisfy the global multiplier estimate.

We have established an identity

$$\sum_{t \geq 0} \Big(I^{\mathcal{E}}_{t,\text{disc}}(f_\alpha) - I_{t,\text{disc}}(f_\alpha) - |W(M)|^{-1}\widehat{I}^M_{t,\text{disc}}\Big(\varepsilon_M(f)_\alpha\Big)\Big) = 0,$$

in which the sum over t satisfies the global multiplier estimate [I, (3.3)]. The estimate itself depends on the choice of a positive number T. Before applying it, we recall that

$$I^{\mathcal{E}}_{t,\text{disc}}(f_\alpha) - I_{t,\text{disc}}(f) = \sum_{\nu} \Big(I^{\mathcal{E}}_{\nu,\text{disc}}(f_\alpha) - I_{\nu,\text{disc}}(f)\Big),$$

where ν is summed over orbits $\nu \in \mathfrak{h}_u^*/W_\infty$ with $\|\mathrm{Im}(\nu)\| = t$. In fact, by Lemma 4.1, we can restrict ν to the orbits in $\mathfrak{h}_u^*(r)/W_\infty$, for some $r > 0$ that is independent of t and α. We can therefore express the sum

$$\sum_{\nu} \Big(I^{\mathcal{E}}_{\nu,\mathrm{disc}}(f_\alpha) - I_{\nu,\mathrm{disc}}(f_\alpha)\Big), \tag{4.3}$$

taken over the orbits $\nu \in \mathfrak{h}_u^*(r)/W_\infty$ with $\|\mathrm{Im}(\nu)\| \leq T$, as a difference between

$$|W(M)|^{-1} \sum_{t \leq T} \widehat{I}^M_{t,\mathrm{disc}}\Big(\varepsilon_M(f)_\alpha\Big) \tag{4.4}$$

and the expression obtained from the left-hand side of the last identity by restricting the sum to those t with $t > T$. It is to the last expression that we apply the global multiplier estimate. The resulting conclusion is that we may choose r, together with positive constants C and k, with the property that for any $N > 0$, $\alpha \in C_N^\infty(\mathfrak{h}^Z)^{W_\infty}$ and T, the difference between (4.3) and (4.4) has absolute value that is bounded by

$$C\mathrm{e}^{kN} \sup_{\nu \in \mathfrak{h}_u^*(r,T)} |\widehat{\alpha}(\nu)|.$$

(See [I, (3.3)].) We note that for any given T, the sums in (4.3) and (4.4) can be taken over finite sets that are independent of α.

Let $\nu_1 \in \mathfrak{h}_u^*(r)/W_\infty$ be a fixed infinitesimal character. According to [AC, Lemma 2.15.2], we can find a function $\alpha_1 \in C_c^\infty(\mathfrak{h}^Z)^{W_\infty}$ such that $\widehat{\alpha}_1$ maps $\mathfrak{h}_u^*(r)$ to the unit interval, and such that the inverse image of 1 under $\widehat{\alpha}_1$ is simply the set of points in the W_∞-orbit of ν_1. We fix α_1, and then choose $N_1 > 0$ such that α_1 belongs to $C_{N_1}^\infty(\mathfrak{h}^Z)^{W_\infty}$. Assuming that r and k have been chosen as above, we can find a positive number T such that

$$|\widehat{\alpha}_1(\nu)| \leq \mathrm{e}^{-2kN_1},$$

for all points ν in the set $\mathfrak{h}_u^*(r,T)$. This is possible because $\widehat{\alpha}_1$ is rapidly decreasing on the vertical strips. For each positive integer m, let α_m be the convolution of α_1 with itself m times. Then α_m belongs to $C_{mN_1}^\infty(\mathfrak{h}^Z)^{W_\infty}$, and

$$\widehat{\alpha}_m(\nu) = \Big(\widehat{\alpha}_1(\nu)\Big)^m.$$

Taking $\alpha = \alpha_m$ in the estimate above, we see that (4.3) equals the sum of (4.4) and an expression whose absolute value is bounded by

$$C\mathrm{e}^{-kN_1 m}.$$

It follows that the difference between (4.3) and (4.4) approaches zero as m approaches infinity.

Suppose that ν is any point in $\mathfrak{h}^*_u(r)/W_\infty$. Then

$$\begin{aligned} I^{\mathcal{E}}_{\nu,\mathrm{disc}}(f_{\alpha_m}) - I_{\nu,\mathrm{disc}}(f_{\alpha_m}) &= \sum_{\pi\in\Pi^{\mathcal{E}}_{\nu,\mathrm{disc}}(G,V,\zeta)} \Big(a^{G,\mathcal{E}}_{\mathrm{disc}}(\pi) - a^G_{\mathrm{disc}}(\pi)\Big) f_{\alpha_m,G}(\pi) \\ &= \sum_{\pi} \Big(a^{G,\mathcal{E}}_{\mathrm{disc}}(\pi) - a^G_{\mathrm{disc}}(\pi)\Big) f_G(\pi)\widehat{\alpha}_m(\nu) \\ &= \Big(\widehat{\alpha}_1(\nu)\Big)^m \Big(I^{\mathcal{E}}_{\nu,\mathrm{disc}}(f) - I_{\nu,\mathrm{disc}}(f)\Big). \end{aligned}$$

This equals $I^{\mathcal{E}}_{\nu_1,\mathrm{disc}}(f) - I_{\nu_1,\mathrm{disc}}(f)$ if $\nu = \nu_1$, and otherwise approaches zero as m approaches infinity. Since there are only finitely many nonzero terms in (4.3), we conclude that the value of (4.3) at $\alpha = \alpha_m$ approaches the difference

$$I^{\mathcal{E}}_{\nu_1,\mathrm{disc}}(f) - I_{\nu_1,\mathrm{disc}}(f)$$

as m approaches infinity. This difference therefore equals the corresponding limit

$$\lim_{m\to\infty} \Big(|W(M)|^{-1} \sum_{t\le T} \widehat{I}^M_{t,\mathrm{disc}}\Big(\varepsilon_M(f)_{\alpha_m}\Big)\Big) \tag{4.5}$$

of (4.4). We have reduced the proof of (a) to showing that the limit (4.5) is zero.

To deal with (b), we assume that G is quasisplit, and that f belongs to $\mathcal{H}^{\mathrm{uns}}_M(G_V,\zeta_V)^{00}$. We shall retrace the steps in the argument for (a) above, making modifications as necessary. If $\alpha \in C^\infty_c(\mathfrak{h}^Z)^{W_\infty}$ is a multiplier, the function f_α remains unstable. This follows from Shelstad's characterization of stability at the archimedean places in terms of tempered L-packets [Sh]. (This point is not essential to the argument, since we could have insisted at the outset that f be unstable at some finite place.) In particular, f_α satisfies the identity (2.5) of Proposition 2.2. The function f_α actually belongs to $\mathcal{H}^{\mathrm{uns}}_M(G_V,\zeta_V)^{00}$, since the conditions that define this subspace of $\mathcal{H}^{\mathrm{uns}}(G_V,\zeta_V)$ are not changed by multipliers. We can therefore apply Corollary 3.3(b) to the value at f_α of linear form on the left-hand side of (2.5). Moreover, the sum

$$\sum_z S^G_{z,\mathrm{unip}}(f_\alpha)$$

obtained from the right-hand side of (2.5) is equal to 0. The identity becomes

$$|W(M)|^{-1} \sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M,M')\widehat{S}^{\widetilde{M}'}\Big(\varepsilon^{M'}(f_\alpha)\Big) = \sum_t S^G_{t,\mathrm{disc}}(f_\alpha).$$

For the linear forms on the left-hand side, we recall that for any M', there is a multiplier α' for $\widetilde{M}'_V$ such that

$$\widehat{\alpha}'(\nu') = \widehat{\alpha}(\nu), \qquad \nu \in \mathfrak{h}^*_{\mathbb{C}}/\mathfrak{a}^*_{G,Z,\mathbb{C}}.$$

Here

$$\nu \longrightarrow \nu' = \nu + d\widetilde{\eta}'_\infty$$

is the affine linear embedding of $\mathfrak{h}^*_{\mathbb{C}}$ into the corresponding space $(\widetilde{\mathfrak{h}}')^*_{\mathbb{C}} \supset \mathfrak{h}^*_{\mathbb{C}}$ attached to $\widetilde{M}'$. We also write

$$t = \|\mathrm{Im}(\nu)\| \longrightarrow t' = \|\mathrm{Im}(\nu')\| = t + \|\mathrm{Im}(d\widetilde{\eta}'_\infty)\|$$

for the associated change of norms. (See [A7, p. 561] and [I, §7].) The action of α' on $\mathcal{I}_{ac}(\widetilde{M}'_V, \widetilde{\zeta}'_V)$ is uniquely determined by the given condition, even though α' itself is not.) The correspondence $\alpha \to \alpha'$ is compatible with the archimedean transfer map, from which it follows that $\varepsilon^{M'}(f_\alpha) = \varepsilon^{M'}(f)_{\alpha'}$. We can therefore expand the linear forms on the left-hand side as

$$\widehat{S}^{\widetilde{M}'}\Big(\varepsilon^{M'}(f_\alpha)\Big) = \sum_t \widehat{S}^{\widetilde{M}'}_{t'}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big),$$

by [I, Prop. 10.5].

We have obtained an identity

$$|W(M)|^{-1} \sum_t \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M, M') \widehat{S}^{\widetilde{M}'}_{t'}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big) = \sum_t S^G_{t,\mathrm{disc}}(f_\alpha),$$

between two absolutely convergent sums over t. The right-hand sum satisfies the global multiplier estimate [I, (3.3)]. We would like to show that the left sum over t satisfies this stronger estimate, and also that the linear forms $\widehat{S}^{\widetilde{M}'}_{t'}$ can be replaced by their "discrete" analogues.

The stronger estimate would follow from the proof of [I, Prop. 10.5] and [A2, Lemma 6.3], if it could be shown that for any M', the function $\varepsilon^{M'}(f)$ in $S\mathcal{I}_{ac}(\widetilde{M}'_V, \widetilde{\zeta}'_V)$ was moderate. This amounts to showing that $\varepsilon^{M'}(f)$ satisfies the analogues of the weak support and growth conditions on p. 531 of [A2]. As in (a), it is enough to verify that for any $v \in V_{\mathrm{fin}}(G, M)$, the function $\varepsilon^{M'}(f_v)$ satisfies the relevant form of these conditions. The fact that $\varepsilon^{M'}(f_v)$ is cuspidal means that the support condition is vacuous. The growth condition is a consequence of the identity

$$\varepsilon^{M'}(f_v) = {}^c\varepsilon^{M'}(f_v) - {}^c\eta_M(M'_v, f_v)$$

of Corollary 3.2. For as we saw in the proof of Proposition 3.1, the function ${}^c\varepsilon^{M'}(f_v)$ belongs to $S\mathcal{I}(\widetilde{M}'_v, \widetilde{\zeta}'_v)$, and therefore has compact support on $\Delta(\widetilde{M}'_v, \widetilde{\zeta}'_v)$. By the analogue [A12] of [A1, Lemma 5.2], the function ${}^c\eta_M(M'_v, f_v)$ is rapidly decreasing (in the sense of [A2, Lemma 5.2]). It follows easily that ${}^c\eta_M(M'_v, f_v)$ satisfies the relevant growth condition, at least on the elliptic elements on which $\varepsilon^{M'}(f_v)$ is supported. The same condition therefore holds for the original function $\varepsilon^{M'}(f_v)$. Recalling the definitions (3.12) and (3.12′), we conclude that $\varepsilon^{M'}(f)$ is a moderate function in $S\mathcal{I}_{ac}(\widetilde{M}'_V, \widetilde{\zeta}'_V)$. Therefore the sum

$$\sum_t \widehat{S}^{\widetilde{M}'}_{t'}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big)$$

does satisfy the global multiplier estimate.

For any t, the general spectral expansion [I, (10.18)] for $\widehat{S}^{\widetilde{M}'}_{t'}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big)$ is easily seen to simplify. Guided by the proof of [A2, Th. 7.1(a)], one applies the splitting formula in [A12] to the terms

$$\widehat{S}^{\widetilde{M}'}_{\widetilde{L}'}\Big(\phi', \varepsilon^{M'}(f)_{\alpha'}\Big), \qquad \widetilde{L}' \in (\mathcal{L}^{\widetilde{M}'})^0,\ \phi' \in \Pi_{t'}(\widetilde{L}', V, \widetilde{\zeta}'),$$

in this expansion. Since $\varepsilon^{M'}(f)_{\alpha'}$ is actually cuspidal at two places, the local induction hypothesis (1.2) implies immediately that these terms all vanish. It follows from [I, (10.18)] that

$$\widehat{S}^{\widetilde{M}'}_{t'}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big) = \widehat{S}^{\widetilde{M}'}_{t',\mathrm{unit}}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big).$$

Recall that Lemma 7.3(b) of [I] provides an expansion for $\widehat{S}^{\widetilde{M}'}_{t',\mathrm{unit}}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big)$, as well as an expansion for the more elementary linear form $\widehat{S}^{\widetilde{M}'}_{t',\mathrm{disc}}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big)$. One can compare the coefficients of the two expansions by means of the formula (10.22) of Proposition 10.7(b) of [I]. If we combine this formula with the global induction hypothesis (1.4), and the fact that $\varepsilon^{M'}(f)_{\alpha'}$ is cuspidal at some place, we find that

$$\widehat{S}^{\widetilde{M}'}_{t',\mathrm{unit}}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big) = \widehat{S}^{\widetilde{M}'}_{t',\mathrm{disc}}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big).$$

The right-hand side here represents a simple version of the stable spectral expansion of the original linear form $\widehat{S}^{\widetilde{M}'}_{t'}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big)$. This is the second point we wanted to check.

We have established that

$$\sum_{t\geq 0}\bigg(S^G_{t,\mathrm{disc}}(f_\alpha) - |W(M)|^{-1}\sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M,V)}\iota(M,M')\widehat{S}^{\widetilde{M}'}_{t',\mathrm{disc}}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big)\bigg) = 0,$$

where the sum over t satisfies the global multiplier estimate [I, (3.3)]. The rest of the argument is very similar to the discussion above for (a). By Lemma 4.1, we can write

$$S^G_{t,\mathrm{disc}}(f_\alpha) = \sum_\nu S^G_{\nu,\mathrm{disc}}(f_\alpha),$$

where ν is summed over the orbits in a set $\mathfrak{h}^*(r)/W_\infty$ with $\|\mathrm{Im}(\nu)\| = t$. We fix an orbit $\nu_1 \in \mathfrak{h}^*_u(r)/W_\infty$, and then choose a corresponding function $\alpha_1 \in C^\infty_{N_1}(\mathfrak{h}^Z)^{W_\infty}$ as above. Following the discussion of (a), we deduce that the linear form

$$S^G_{\nu_1,\mathrm{disc}}(f)$$

may be represented by the limit

$$\lim_{m\to\infty}\bigg(|W(M)|^{-1}\sum_{t\leq T}\sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M,V)}\iota(M,M')\widehat{S}^{\widetilde{M}'}_{t',\mathrm{disc}}\Big(\varepsilon^{M'}(f)_{\alpha'_m}\Big)\bigg), \tag{4.6}$$

for some $t > 0$. This reduces the proof of (b) to showing that the limit (4.6) is zero.

To deal with the limits (4.5) and (4.6), we first write

$$\widehat{I}^M_{t,\mathrm{disc}}\Big(\varepsilon_M(f)_{\alpha_m}\Big) = \sum_{\nu} \widehat{I}^M_{\nu,\mathrm{disc}}\Big(\varepsilon_M(f)_{\alpha_m}\Big)$$

and

$$\widehat{S}^{\widetilde{M}'}_{t',\mathrm{disc}}\Big(\varepsilon^{M'}(f)_{\alpha'_m}\Big) = \sum_{\nu} \widehat{S}^{\widetilde{M}'}_{\nu',\mathrm{disc}}\Big(\varepsilon^{M'}(f)_{\alpha'_m}\Big),$$

where ν is summed in each case over the infinitesimal characters for M_V with $\|\mathrm{Im}(\nu)\| = t$.

LEMMA 4.3. (a) *If ν is any infinitesimal character for M_V, there is a Schwartz function*

$$\lambda \longrightarrow \varepsilon_{M,\nu}(f,\lambda), \qquad \lambda \in i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z},$$

such that for any $\alpha \in C^\infty_c(\mathfrak{h}^Z)^{W_\infty}$,

$$\widehat{I}^M_{\nu,\mathrm{disc}}\Big(\varepsilon_M(f)_\alpha\Big) = \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} \varepsilon_{M,\nu}(f,\lambda)\widehat{\alpha}(\nu+\lambda)d\lambda.$$

(b) *Suppose that G is quasisplit. Then for any infinitesimal character ν for M_V and any $M' \in \mathcal{E}_{\mathrm{ell}}(M,V)$, there is a Schwartz function*

$$\lambda \longrightarrow \varepsilon^{M'}_{\nu'}(f,\lambda), \qquad \lambda \in i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z},$$

such that for any $\alpha \in C^\infty_c(\mathfrak{h}^Z)^{W_\infty}$,

$$\widehat{S}^{\widetilde{M}'}_{\nu',\mathrm{disc}}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big) = \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} \varepsilon^{M'}_{\nu'}(f,\lambda)\widehat{\alpha}(\nu+\lambda)d\lambda.$$

Proof. Consider part (a). Any element ϕ in $\mathcal{I}_{ac}(M_V,\zeta_V)$ can be regarded as a function on the product of $\Pi(M_V,\zeta_V)$ with $\mathfrak{a}_M/\mathfrak{a}_Z$. For example, if ϕ is the image of a function $h \in \mathcal{H}_{ac}(M_V,\zeta_V)$, the value of ϕ is defined by an integral

$$\phi(\pi,X) = \int_{M^X_V/Z_V} h(m)\Theta_\pi(m)dm, \qquad (\pi,X) \in \Pi(M_V,\zeta_V)\times\mathfrak{a}_M/\mathfrak{a}_Z,$$

over a compact domain, in which Θ_π is the character of π, and

$$M^X_V = \Big\{m \in M_V :\ H_M(m) + \mathfrak{a}_Z = X\Big\}.$$

Now the given element $\alpha \in C^\infty_c(\mathfrak{h}^Z)^{W_\infty}$ is to be regarded as a multiplier for M_V. It transforms any function ϕ in $\mathcal{I}_{ac}(M_V,\zeta_V)$ to the function

$$\phi_\alpha(\pi,X) = \int_{\mathfrak{a}^{G,Z}_M/\mathfrak{a}_Z} \phi(\pi,X-Y)\alpha_M(\pi,Y)dY,$$

where $\mathfrak{a}_M^{G,Z}$ is the subspace of elements in $\mathfrak{a}_M$ whose image in $\mathfrak{a}_G$ lies in $\mathfrak{a}_Z$, and

$$\alpha_M(\pi, Y) = \int_{i\mathfrak{a}_{M,Z}^*/i\mathfrak{a}_{G,Z}^*} \widehat{\alpha}(\nu_\pi + \lambda)\mathrm{e}^{-\lambda(Y)}d\lambda, \qquad Y \in \mathfrak{a}_M^{G,Z}.$$

This follows easily from the fact that $i\mathfrak{a}_{M,Z}^*/i\mathfrak{a}_{G,Z}^*$ is isomorphic to the dual group of $\mathfrak{a}_M^{G,Z}/\mathfrak{a}_Z$. (See [A2, (6.1)].) We are most interested in the case that π is unitary, and the element $X \in \mathfrak{a}_M/\mathfrak{a}_Z$ is trivial. Then M_V^X equals the set we have denoted by M_V^Z. In order to match earlier notation, we generally reserve the symbol π for the restriction of the representation to this subset of M_V. Then π may be identified with an orbit $\{\pi_\lambda\}$ of $i\mathfrak{a}_{M,Z}^*$ in $\Pi_{\mathrm{unit}}(M_V, \zeta_V)$, or if one prefers, the representation in that orbit whose infinitesimal character has minimal norm. We shall usually suppress the element $X = 0$ from the notation in this case, and write

$$\phi(\pi) = \phi(\pi_\lambda, 0).$$

We use these remarks to express the value of $\widehat{I}_{\nu,\mathrm{disc}}^M$ at $\varepsilon_M(f)_\alpha$. We obtain a sum

$$\widehat{I}_{\nu,\mathrm{disc}}^M\Big(\varepsilon_M(f)_\alpha\Big) = \sum_{\pi\in\Pi_{\nu,\mathrm{disc}}(M,V,\zeta)} a_{\mathrm{disc}}^M(\pi)\varepsilon_M(f)_\alpha(\pi),$$

which can be taken over a finite set that is independent of α, and in which

$$\varepsilon_M(f)_\alpha(\pi) = \int_{\mathfrak{a}_M^{G,Z}/\mathfrak{a}_Z} \varepsilon_M(f, \pi, -Y)\alpha_M(\pi, Y)dY.$$

It would be enough to show that for any π in $\Pi_{\nu,\mathrm{disc}}(M, V, \zeta)$, the function

$$X \longrightarrow \varepsilon_M(f, \pi, X), \qquad X \in \mathfrak{a}_M^{G,Z}/\mathfrak{a}_Z,$$

is rapidly decreasing. For assertion (a) would then follow, with

$$\varepsilon_{M,\nu}(f, \lambda) = \sum_{\pi\in\Pi_{\nu,\mathrm{disc}}(M,V,\zeta)} a_{\mathrm{disc}}^M(\pi) \int_{\mathfrak{a}_M^{G,Z}/\mathfrak{a}_Z} \varepsilon_M(f, \pi, X)\mathrm{e}^{\lambda(X)}dX,$$

from an interchange of integrals in the last formula. As in the special case proved in [AC, Lemma 2.15.3], we shall combine the cuspidal property of the map ε_M with the fact that π is unitary. (The proof of [AC, Lemma 2.15.3] is a little hard to decipher, because of an unfortunate typographical error in the assertion of an earlier result [AC, Cor. 2.14.2]. The symbol $\Pi^+\Big(M(F_S)\Big)$ in the second sentence of that assertion should actually be $\Pi^{+}_{\mathrm{temp}}\Big(M(F_S)\Big)$. The earlier result was meant to serve as the special case of [AC, Lemma 2.15.3] in which π is tempered.)

We can assume that f is a product of $\prod\limits_v f_v$, as in (3.10). It is then not hard to see from the definition (3.11) that $\varepsilon_M(f,\pi,X)$ equals

$$\sum_{v\in V_{\mathrm{fin}}(G,M)}\int_{\mathfrak{a}^X_{M,V}/\mathfrak{a}_{Z,V}} f^v_M(\pi^v,X^v_V)\varepsilon_M(f_v,\pi_v,X_v)dX_V,$$

where $\pi=\pi^v\otimes\pi_v$, and $\mathfrak{a}^X_{M,V}$ is the set of vectors $X_V=X^v_V\oplus X_v$ in

$$\mathfrak{a}_{M,V}=\bigoplus_{w\in V}\mathfrak{a}_{M,w}$$

whose projection onto $\mathfrak{a}_M$ equals X. For any v,

$$X^v_V\longrightarrow f^v_M(\pi^v,X^v_V)=\prod_{w\in V^v}f_{M_w}(\pi_w,X_w),$$

is a smooth function of compact support on the quotient of $\mathfrak{a}_{M,V^v}$ by $\mathfrak{a}_{Z,V^v}$. The growth of $\varepsilon_M(f,\pi,X)$ is therefore reflected entirely in the growth of the functions $\varepsilon_M(f_v,\pi_v,X_v)$. It would be enough to show that for any $v\in V_{\mathrm{fin}}(G,M)$ and $\pi_v\in\Pi_{\mathrm{unit}}(M_v,\zeta_v)$,

$$X_v\longrightarrow\varepsilon_M(f_v,\pi_v,X_v)$$

is a rapidly decreasing function on the quotient of the lattice $\mathfrak{a}_{M,v}$ by $\mathfrak{a}_{Z,v}$.

To exploit the fact that $\varepsilon_M(f_v)$ is a cuspidal function, we expand $\varepsilon_M(f_v,\pi_v,X_v)$ in terms of the basis $T(M_v)$ of (possibly) nontempered virtual characters discussed in [A12], among other places. (The notation here differs slightly from that of the earlier papers [A5] and [A7], where the elements in $T(M_v)$ were taken to be tempered. For example, if G is an inner form of $\mathrm{GL}(n)$, $T(M_v)$ now represents the basis of standard characters used in the proof of [AC, Lemma 2.15.3].) We obtain a finite linear combination

$$\varepsilon_M(f_v,\pi_v,X_v)=\sum_{\tau_v\in T(M_v,\zeta_v)}\delta(\pi_v,\tau_v)\varepsilon_M(f_v,\tau_v,X_v),$$

where $T(M_v,\zeta_v)$ denotes the subset of elements in $T(M_v)$ with Z_v-central character equal to ζ_v. Recall that $T(M_v,\zeta_v)$ contains the subset $T_{\mathrm{ell}}(M_v,\zeta_v)$ of elliptic elements. If τ_v belongs to the complement of $T_{\mathrm{ell}}(M_v,\zeta_v)$ in $T(M_v,\zeta_v)$, τ_v is properly induced. In this case

$$\varepsilon_M(f_v,\tau_v,X_v)=0,$$

since $\varepsilon_M(f_v)$ is cuspidal. We can therefore expand $\varepsilon_M(f_v,\pi_v,X_v)$ as a finite linear combination

$$\varepsilon_M(f_v,\pi_v,X_v)=\sum_{\tau_v\in T_{\mathrm{ell}}(M_v,\zeta_v)}\delta(\pi_v,\tau_v)\varepsilon_M(f_v,\tau_v,X_v),$$

for complex numbers $\delta(\pi_v, \tau_v)$. The original representation π_v for M_v is unitary, and in particular, has unitary central character. We can therefore restrict the last sum to the set of elements in $T_{\mathrm{ell}}(M_v, \zeta_v)$ with unitary central character. But the set of elements in $T_{\mathrm{ell}}(M_v, \zeta_v)$ with unitary central character is precisely the subset $T_{\mathrm{temp,ell}}(M_v, \zeta_v)$ of tempered elements. It is therefore sufficient to prove that for any element τ_v in $T_{\mathrm{temp,ell}}(M_v, \zeta_v)$, the function

$$X_v \longrightarrow \varepsilon_M(f_v, \tau_v, X_v), \qquad X_v \in \mathfrak{a}_{M,v}/\mathfrak{a}_{Z,v},$$

is rapidly decreasing.

For the case of tempered τ_v, we have only to refer back to our proof that $\varepsilon_M(f_v)$ is moderate. Indeed, we can write

$$\varepsilon_M(f_v, \tau_v, X_v) = {}^c\varepsilon_M(f_v, \tau_v, X_v) - \Big({}^c\theta^{\mathcal{E}}_M(f_v, \tau_v, X_v) - {}^c\theta_M(f_v, \tau_v, X_v)\Big),$$

by Corollary 3.2. As in the proof of Proposition 3.1, ${}^c\varepsilon_M(f_v)$ belongs to $\mathcal{I}(M_v, \zeta_v)$. In particular, the function

$$X_v \longrightarrow {}^c\varepsilon_M(f_v, \tau_v, X_v), \qquad X_v \in \mathfrak{a}_{M,v}/\mathfrak{a}_{Z,v},$$

is actually of compact support. Moreover, [A1, Lemma 5.2] and its endoscopic analogue [A12] imply that the function

$$X_v \longrightarrow {}^c\theta^{\mathcal{E}}_M(f_v, \tau_v, X_v) - {}^c\theta_M(f_v, \tau_v, X_v), \qquad X_v \in \mathfrak{a}_{M,v}/\mathfrak{a}_{Z,v},$$

is rapidly decreasing. It follows that the original function $\varepsilon_M(f_v, \tau_v, X_v)$ is rapidly decreasing in X_v, as required. This completes the proof of assertion (a). Observe that the argument depends in an essential way on the original representations being unitary. For it would otherwise be necessary to contend with nonunitary twists of elements τ_v in $T_{\mathrm{temp,ell}}(M_v, \zeta_v)$, and since

$$\varepsilon_M(f_v, \tau_{v,\lambda}, X_v) = \mathrm{e}^{\lambda(X_v)}\varepsilon_M(f_v, \tau_v, X_v), \qquad \lambda \in \mathfrak{a}^*_{M,Z},$$

the functions in question could then have exponential growth.

For the second half (b) of the lemma, we fix the endoscopic datum $M' \in \mathcal{E}_{\mathrm{ell}}(M, V)$. We can regard any element in $S\mathcal{I}(\widetilde{M}'_V, \widetilde{\zeta}'_V)$ as a function on the product of $\Phi(\widetilde{M}'_V, \widetilde{\zeta}'_V)$ with $\mathfrak{a}_{\widetilde{M}'}/\mathfrak{a}_{\widetilde{Z}'}$. Since M' is elliptic, there is a canonical isomorphism $X' \to X$ from $\mathfrak{a}_{\widetilde{M}'}/\mathfrak{a}_{\widetilde{Z}'}$ onto the space $\mathfrak{a}_M/\mathfrak{a}_Z$. We can therefore take the second variable of a function in $S\mathcal{I}(\widetilde{M}'_V, \widetilde{\zeta}'_V)$ to lie in $\mathfrak{a}_M/\mathfrak{a}_Z$. If $\mathfrak{a}^{G,Z}_{\widetilde{M}'}$ denotes the subspace of elements in $\mathfrak{a}_{\widetilde{M}'}$ whose projection onto $\mathfrak{a}_G$ lies in $\mathfrak{a}_Z$, $X' \to X$ restricts to an isomorphism from $\mathfrak{a}^{G,Z}_{\widetilde{M}'}/\mathfrak{a}_{\widetilde{Z}'}$ onto $\mathfrak{a}^{G,Z}_M/\mathfrak{a}_Z$. This is dual to an isomorphism $\lambda \to \lambda'$ from $i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}$ onto $i\mathfrak{a}^*_{\widetilde{M}',\widetilde{Z}'}/i\mathfrak{a}^*_{\widetilde{G}',\widetilde{Z}'}$ with the property that $(\nu + \lambda)' = \nu' + \lambda'$.

The role of $\Pi_{\nu,\mathrm{disc}}(M, V, \zeta)$ in the proof of (a) is taken by the set $\Phi_{\nu',\mathrm{disc}}(\widetilde{M}', V, \widetilde{\zeta}')$ attached to M'. The elements in this set belong to $\Phi(\widetilde{M}'_V, \widetilde{\zeta}'_V)$, and have unitary central character. When they occur as the first component of a point in the domain of a function in $S\mathcal{I}_{ac}(\widetilde{M}'_V, \widetilde{\zeta}'_V)$, we suppress the second component from the notation if it is equal to zero. Then

$$S^{M'}_{\nu',\mathrm{disc}}\Big(\varepsilon^{M'}(f)_{\alpha'}\Big) = \sum_{\phi' \in \Phi_{\nu',\mathrm{disc}}(\widetilde{M}', V, \widetilde{\zeta}')} b^{\widetilde{M}'}_{\mathrm{disc}}(\phi')\varepsilon^{M'}(f)_{\alpha'}(\phi').$$

One verifies that the distributions on the right can be expanded as

$$\varepsilon^{M'}(f)_{\alpha'}(\phi') = \int_{\mathfrak{a}^{G,Z}_M/\mathfrak{a}_Z} \varepsilon^{M'}(f, \phi', -Y')\alpha_{M'}(\phi', Y')dY,$$

where

$$\alpha_{M'}(\phi', Y') = \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} \widehat{\alpha}'(\nu'+\lambda')\mathrm{e}^{-\lambda'(Y')}d\lambda = \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} \widehat{\alpha}(\nu+\lambda)\mathrm{e}^{-\lambda(Y)}d\lambda.$$

It would be enough to show that for any ϕ' in $\Phi_{\nu',\mathrm{disc}}(\widetilde{M}', V, \widetilde{\zeta}')$, the function

$$X \longrightarrow \varepsilon^{M'}(f, \phi', X'), \qquad X \in \mathfrak{a}^{G,Z}_M/\mathfrak{a}_Z,$$

is rapidly decreasing. For assertion (b) would then follow, with

$$\varepsilon^{M'}_{\nu'}(f, \lambda) = \sum_{\phi' \in \Phi_{\nu',\mathrm{disc}}(\widetilde{M}', V, \widetilde{\zeta}')} b^{\widetilde{M}'}_{\mathrm{disc}}(\phi') \int_{\mathfrak{a}^{G,Z}_M/\mathfrak{a}_Z} \varepsilon^{M'}(f, \phi', Y')\mathrm{e}^{\lambda(Y)}dY,$$

from an interchange of integrals.

We can assume that f equals a product $\prod_v f_v$, as in (3.10). We write $\phi' = \bigotimes_v \phi'_v$, for elements $\phi'_v \in \Phi(\widetilde{M}'_v, \widetilde{\zeta}'_v)$ with unitary central character, and then apply the definitions (3.12) and (3.12′) of $\varepsilon^{M'}(f_v)$. We see without difficulty that it would suffice to prove that for any $v \in V_{\mathrm{fin}}(G, M)$, the function

$$X_v \longrightarrow \varepsilon^{M'}(f_v, \phi'_v, X'_v), \qquad X_v \in \mathfrak{a}_{M,v}/\mathfrak{a}_{Z,v},$$

is rapidly decreasing. The function $f_v \in \mathcal{H}(G_v, \zeta_v)^0$ is of course fixed. In the special case that $M' = M^*$, we can assume that it lies in the subspace $\mathcal{H}^{\mathrm{uns}}(G_v, \zeta_v)^0$ of $\mathcal{H}(G_v, \zeta_v)^0$, since there can be at most one nonzero term on the right-hand side of (3.12).

By construction [A7], [A12], the set $\Phi(\widetilde{M}'_v, \widetilde{\zeta}'_v)$ of abstract stable characters is a union of the subset $\Phi_{\mathrm{ell}}(\widetilde{M}'_v, \widetilde{\zeta}'_v)$ of elliptic elements with the subset of elements induced from proper Levi subgroups of $\widetilde{M}'_v$. If $\widetilde{\phi}'_v$ is properly induced,

$$\varepsilon^{M'}(f_v, \phi'_v, X'_v) = 0,$$

since $\varepsilon^{M'}(f_v)$ is cuspidal. We may therefore assume that ϕ'_v is elliptic. But the set of elements in $\Phi_{\mathrm{ell}}(\widetilde{M}'_v, \widetilde{\zeta}'_v)$ with unitary central character is the subset $\Phi_{\mathrm{temp,ell}}(\widetilde{M}'_v, \widetilde{\zeta}'_v)$ of tempered elements. It is therefore sufficient to prove that for any element $\phi'_v \in \Phi_{\mathrm{temp,ell}}(\widetilde{M}'_v, \widetilde{\zeta}'_v)$, the function

$$X_v \longrightarrow \varepsilon^{M'}(f_v, \phi'_v, X'_v), \qquad X_v \in \mathfrak{a}_{M,v}/\mathfrak{a}_{Z,v},$$

is rapidly decreasing.

For the case of tempered ϕ'_v, we write

$$\varepsilon^{M'}(f_v, \phi'_v, X'_v) = {}^c\varepsilon^{M'}(f_v, \phi'_v, X'_v) - {}^c\eta_M(M'_v, f_v, \phi'_v, X'_v),$$

by Corollary 3.2. We can then argue as in the earlier proof that $\varepsilon^{M'}(f_v)$ is moderate. Since ${}^c\varepsilon^{M'}(f_v)$ belongs to $S\mathcal{I}(\widetilde{M}'_v, \widetilde{\zeta}'_v)$, as we observed in the proof of Proposition 3.1, the function

$$X_v \longrightarrow {}^c\varepsilon^{M'}(f_v, \phi'_v, X'_v), \qquad X_v \in \mathfrak{a}_{M,v}/\mathfrak{a}_{Z,v},$$

actually has compact support. Moreover, the stable analogue [A12] of [A1, Lemma 5.2] implies that the function

$$X_v \longrightarrow {}^c\eta_M(M'_v, f_v, \phi'_v, X'_v), \qquad X \in \mathfrak{a}_{M,v}/\mathfrak{a}_{Z,v},$$

is rapidly decreasing. Therefore the original function $\varepsilon^{M'}(f_v, \phi'_v, X'_v)$ is also rapidly decreasing, as required. This completes the proof of the remaining assertion (b) Lemma 4.3. □

With Lemma 4.3 in hand, we can now finish the proof of Proposition 4.2. We have to show that the limits (4.5) and (4.6) are both zero. According to the lemma, we can write the first limit (4.5) as

$$\lim_{m\to\infty}\Big(|W(M)|^{-1}\sum_{\nu}\int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} \varepsilon_{M,\nu}(f,\lambda)\widehat{\alpha}_m(\nu+\lambda)d\lambda\Big),$$

where ν is summed over the infinitesimal characters for M_V with $\|\mathrm{Im}(\nu)\| \le T$. The sum may be taken over a finite subset of $\mathfrak{h}^*_u(r)$, which is independent of m, and which represents a set of Weyl orbits in $\mathfrak{h}^*_u(r)/i\mathfrak{a}^*_{M,Z}$. Moreover, the integral converges absolutely uniformly in m. If $\lambda+\nu$ lies outside the W_∞-orbit of ν_1, we see that

$$\lim_{m\to\infty}\Big(\widehat{\alpha}_m(\nu+\lambda)\Big) = \lim_{m\to\infty}\Big(\widehat{\alpha}_1(\nu+\lambda)^m\Big) = 0,$$

since $0 \le \widehat{\alpha}(\nu+\lambda) < 1$. We conclude that the limit (4.5) vanishes. The treatment of the second limit (4.6) is identical. By Lemma 4.3, it equals

$$\lim_{m\to\infty}\Big(|W(M)|^{-1}\sum_{\nu}\sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M,V)}\iota(M,M')\int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} \varepsilon^{M'}_{\nu'}(f,\lambda)\widehat{\alpha}_m(\nu+\lambda)d\lambda\Big),$$

where ν is summed over the set of infinitesimal characters of M_V with $\|\mathrm{Im}(\nu)\| \le T$. Using the same arguments, we deduce that this limit also vanishes. We have shown that the required limits vanish, and therefore that the identities (4.1) and (4.2) hold, with $\nu = \nu_1$. The proof of Proposition 4.2 is complete. □

5. Elimination of restrictions on f

The next step will be to remove the local restrictions on f. We shall show that the identities of Proposition 4.2 remain valid without the constraints on the p-adic unipotent orbital integrals. This section can be regarded as a general (untwisted) analogue of the special case in [AC, §2.16] of inner forms of $\mathrm{GL}(n)$. As in the earlier special case, we shall relax the constraints one p-adic place at a time.

Let $v \in V_{\mathrm{fin}}$ be a fixed p-adic valuation. We have at our disposal three sets $\Pi(G_v)$, $T(G_v)$, and $\Phi^{\mathcal{E}}(G_v)$, consisting of virtual characters that are respectively irreducible, standard and endoscopic. The sets represent three different bases of the complex vector space of virtual characters on the connected p-adic group G_v. Likewise, we have three subsets $\Pi_{\mathrm{temp}}(G_v)$, $T_{\mathrm{temp}}(G_v)$ and $\Phi^{\mathcal{E}}_{\mathrm{temp}}(G_v)$, which represent three separate bases of the space of tempered virtual characters on G_v. It will be best to work with the latter two pairs of bases, since they behave well under induction. We shall of course also restrict ourselves to the subbases of elements that have central character on Z_v equal to ζ_v.

We shall consider a fixed connected component Ω_v in either of the two sets $T_{\mathrm{temp}}(G_v, \zeta_v)$ or $\Phi^{\mathcal{E}}_{\mathrm{temp}}(G_v, \zeta_v)$. Then Ω_v is a quotient of a compact torus under the action of some finite group. As such, it acquires a measure $d\omega$ from the Haar measure on the torus. Given Ω_v, we write $\Omega_{v,\mathbb{C}}$ for the complexified connected component in the associated set $T(G_v, \zeta_v)$ or $\Phi^{\mathcal{E}}(G_v, \zeta_v)$. The next lemma will be stated in terms of a space $\mathcal{H}(\Omega_v)$, which we define to be the subspace of functions $f_v \in \mathcal{H}(G_v, \zeta_v)$ such that the associated function $f_{v,G}(\tau_v)$ or $f^{\mathcal{E}}_{v,G}(\phi_v)$ (on either $T_{\mathrm{temp}}(G_v, \zeta_v)$ or $\Phi^{\mathcal{E}}_{\mathrm{temp}}(G_v, \zeta_v)$) is supported on Ω_v. At the beginning of Section 3, we defined two subspaces of $\mathcal{H}(G_v, \zeta_v)$ by imposing constraints on the unipotent orbital integrals. These provide corresponding subspaces

$$\mathcal{H}(\Omega_v)^0 \quad = \quad \mathcal{H}(\Omega_v) \cap \mathcal{H}(G_v, \zeta_v)^0$$

and

$$\mathcal{H}(\Omega_v)^{00} \quad = \quad \mathcal{H}(\Omega_v) \cap \mathcal{H}(G_v, \zeta_v)^{00}$$

of $\mathcal{H}(\Omega_v)$. We shall say that Ω_v is *elliptic* or *parabolic* according to whether the functions in $\mathcal{H}(\Omega_v)$ are cuspidal or not. Then Ω_v is parabolic if and only if it is induced from an elliptic component Ω_{L_v} (in either $T_{\mathrm{temp,ell}}(L_v, \zeta_v)$ or $\Phi^{\mathcal{E}}_{\mathrm{temp,ell}}(L_v, \zeta_v)$) attached to a proper Levi subgroup L_v of G_v over F_v.

LEMMA 5.1. (a) *Suppose that Ω_v is a parabolic component in $T_{\mathrm{temp}}(G_v, \zeta_v)$, and that f^v is a function in $\mathcal{H}(G^v_V, \zeta^v_V)$ such that the identity* (4.1) *holds for any function $f = f^v f_v$, with $f_v \in \mathcal{H}(\Omega_v)^{00}$. Then* (4.1) *also holds for any $f = f^v f_v$, with $f_v \in \mathcal{H}(\Omega_v)$.*

(b) *Suppose that G is quasisplit, that Ω_v is a parabolic component in $\Phi^{\mathcal{E}}_{\mathrm{temp}}(G_v, \zeta_v)$, and that $f^v \in \mathcal{H}(G^v_V, \zeta^v_V)$ is such that the identity* (4.2) *holds for any function $f = f^v f_v$, with $f_v \in \mathcal{H}(\Omega_v)^{00}$. Then* (4.2) *holds for any $f = f^v f_v$, with $f_v \in \mathcal{H}(\Omega_v)$.*

Proof. The basic idea is quite simple, and is familiar from the special cases in [AC] and [A7]. To treat the general case here, we have to deal with the usual minor technical complications. In particular, we need to account for the split component of the center of G_v, or rather, its quotient by the split component of Z_v. As in the last section, we shall work with the vector spaces $\mathfrak{a}_{G_v}$ and $\mathfrak{a}_{Z_v}$, and the canonical lattices that they contain. Suppose that X_v is a point in the quotient

$$\mathfrak{a}_{G,v}/\mathfrak{a}_{Z,v} = H_{G_v}(G_v)/H_{Z_v}(Z_v)$$

of these lattices. Let $\mathcal{H}(G_v^{X_v}, \zeta_v)$ be the subspace of functions in $\mathcal{H}(G_v, \zeta_v)$ that are supported on the subset

$$G_v^{X_v} = \Big\{ x \in G_v : \ H_{G_v}(x) + \mathfrak{a}_{Z,v} = X_v \Big\}$$

of G_v. We can then define the intersections

$$\mathcal{H}(\Omega_v, X_v) = \mathcal{H}(\Omega_v) \cap \mathcal{H}(G_v^{X_v}, \zeta_v)$$

and

$$\mathcal{H}(\Omega_v, X_v)^{00} = \mathcal{H}(\Omega_v)^{00} \cap \mathcal{H}(\Omega_v, X_v).$$

Any function in $\mathcal{H}(\Omega_v)$ is obviously a finite sum of functions in the various spaces $\mathcal{H}(\Omega_v, X_v)$. It is therefore enough to establish the assertions (a) and (b) for functions f_v in $\mathcal{H}(\Omega_v, X_v)$, for a fixed element X_v.

To deal with (a), we consider the pairs of elements

$$i = (z_v, \alpha_v), \qquad z_v \in Z(\overline{G}_v), \ \alpha_v \in R_{\mathrm{unip}}(G_v, \zeta_v),$$

that parametrize Shalika germs near the center of G_v. Any such i gives rise to the linear form

$$J_i(f_v) = f_{v,G}(z_v \alpha_v), \qquad f_v \in \mathcal{H}(G_v, \zeta_v),$$

on $\mathcal{H}(G_v, \zeta_v)$. Having fixed Ω_v and X_v, we let $I(\Omega_v, X_v)$ be a fixed maximal set of pairs $\{i\}$ for which the restrictions to $\mathcal{H}(\Omega_v, X_v)$ of the linear forms $\{J_i\}$ are linearly independent. By the trivial (abelian) case of the Howe conjecture, $I(\Omega_v, X_v)$ is a finite set. (The set is actually empty unless $Z(\overline{G}_v)$ intersects the group $\overline{G}_v^{X_v} = G_v^{X_v}/Z_v$.) The subspace $\mathcal{H}(\Omega_v, X_v)^{00}$ of $\mathcal{H}(\Omega_v, X_v)$ equals the intersection

$$\Big\{ f_v \in \mathcal{H}(\Omega_v, X_v) : \ J_i(f_v) = 0, \ i \in I(\Omega_v, X_v) \Big\}$$

of the kernels of linear forms in this finite set. We fix functions

$$\Big\{ f_v^j \in \mathcal{H}(\Omega_v, X_v) : \ j \in I(\Omega_v, X_v) \Big\}$$

with the property that

$$J_i(f_v^j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

for any i and j in $I(\Omega_v, X_v)$. The map

$$f_v \longrightarrow f_v^{00} = f_v - \sum_i J_i(f_v) f_v^i$$

is then a projection from $\mathcal{H}(\Omega_v, X_v)$ onto $\mathcal{H}(\Omega_v, X_v)^{00}$.

Consider the distribution on the left-hand side of (4.1). It has an expansion

$$I_{\nu,\mathrm{disc}}^{\mathcal{E}}(f) - I_{\nu,\mathrm{disc}}(f) = \sum_{\pi \in \Pi_{\nu,\mathrm{disc}}^{\mathcal{E}}(G,V,\zeta)} \Big(a_{\mathrm{disc}}^{G,\mathcal{E}}(\pi) - a_{\mathrm{disc}}^{G}(\pi) \Big) f_G(\pi),$$

for any function $f \in \mathcal{H}(G_V, \zeta_V)$. We take $f = f^v f_v$, and then consider the distribution as a linear form in f_v. As such, it has a further expansion associated to the basis $T(G_v, \zeta_v)$. To see this explicitly, we write

$$f_{v,G}(\pi_v) = \sum_{\tau_v \in T(G_v, \zeta_v)} \delta(\pi_v, \tau_v) f_{v,G}(\tau_v),$$

for coefficients $\delta(\pi_v, \tau_v)$ attached to any representation $\pi_v \in \Pi(G_v, \zeta_v)$. We assume that f_v belongs to the subspace $\mathcal{H}(G_v^{X_v}, \zeta_v)$ of $\mathcal{H}(G_v, \zeta_v)$. Following our general conventions, we write $\mathfrak{a}_{G_v,Z_v}^*$ for the annihilator in $\mathfrak{a}_{G_v}^*$ of the subspace $\mathfrak{a}_{Z_v} \subset \mathfrak{a}_{G_v}$. Then f_v has an equivariance property

$$f_{v,G}(\tau_{v,\lambda_v}) = \mathrm{e}^{\lambda_v(X_v)} f_{v,G}(\tau_v), \qquad \lambda_v \in i\mathfrak{a}_{G_v,Z_v}^*,$$

with respect to the action of $i\mathfrak{a}_{G_v,Z_v}^*$ on $T(G_v, \zeta_v)$. Let $\overline{T}(G_v, \zeta_v)$ denote the space of orbits of $i\mathfrak{a}_{G_v,Z_v}^*$ in $T(G_v, \zeta_v)$. The expansion becomes

$$(5.1) \qquad I_{\nu,\mathrm{disc}}^{\mathcal{E}}(f) - I_{\nu,\mathrm{disc}}(f) = \sum_{\tau_v \in \overline{T}(G_v, \zeta_v)} \alpha(\tau_v) f_{v,G}(\tau_v),$$

where the coefficient

$$\alpha(\tau_v) = \alpha_{\nu,\mathrm{disc}}(f^v, \tau_v)$$

equals

$$\sum_{\pi} \bigg(\Big(a_{\mathrm{disc}}^{G,\mathcal{E}}(\pi) - a_{\mathrm{disc}}^{G}(\pi) \Big) f_G^v(\pi^v) \sum_{\lambda_v \in i\mathfrak{a}_{G_v,Z_v}^*} \delta(\pi_v, \tau_{v,\lambda_v}) \mathrm{e}^{\lambda_v(X_v)} \bigg).$$

Since

$$\alpha(\tau_{v,\lambda_v}) = \mathrm{e}^{-\lambda_v(X_v)} \alpha(\tau_v),$$

the summand in (5.1) does indeed depend only on the image of τ_v in $\overline{T}(G_v, \zeta_v)$. Observe also that while the last sum is over representations $\pi \in \Pi^{\mathcal{E}}_{\nu,\mathrm{disc}}(G, V, \zeta)$ for the subset G^Z_V of G_V, the factors in a given summand depend on a choice of a representative $\pi^v \otimes \pi_v$ of π in $\Pi_{\mathrm{unit}}(G_V, \zeta_V)$. However, the product of these factors depends only on π. Finally, we note that for fixed f^v, the sum over τ_v in (5.1) can be taken over a finite set that depends only on the support of $f_{v,G}$ (as a function on $T(G_v, \zeta_v)$).

Suppose that f^v is as in (a), and that f_v belongs to $\mathcal{H}(\Omega_v, X_v)$. We shall write $\overline{\Omega}_v$ and $\overline{\Omega}_{v,\mathbb{C}}$ for the spaces of orbits of $i\mathfrak{a}^*_{G_v,Z_v}$ in Ω_v and $\Omega_{v,\mathbb{C}}$ respectively. By assumption, the linear form

$$\sum_{\omega \in \overline{\Omega}_{v,\mathbb{C}}} \alpha(\omega) f_{v,G}(\omega)$$

on the right-hand side of (5.1) vanishes if f_v lies in the subspace $\mathcal{H}(\Omega_v, X_v)^{00}$ of $\mathcal{H}(\Omega_v, X_v)$. Therefore

$$\sum_{\omega \in \overline{\Omega}_{v,\mathbb{C}}} \alpha(\omega) f^{00}_{v,G}(\omega) = 0.$$

It follows from the definition of f^{00}_v that

$$\sum_{\omega \in \overline{\Omega}_{v,\mathbb{C}}} \alpha(\omega) f_{v,G}(\omega) = \sum_{i \in I(\Omega_v, X_v)} a^i J_i(f_v),$$

where

$$a^i = \sum_{\omega \in \overline{\Omega}_{v,\mathbb{C}}} \alpha(\omega) f^i_{v,G}(\omega).$$

The function $\alpha(\omega)$ on the left-hand side of this identity is supported on finitely many elements in $\overline{\Omega}_{v,\mathbb{C}}$. To deal with the right-hand side, we recall that the Fourier transform of any p-adic orbital integral, as a distribution on $T_{\mathrm{temp}}(G_v, \zeta_v)$, is a smooth function. This is a special case of [A6, Th. 4.1]. (See Remark 4(c) on p. 185 of [A6], as well as Corollary 9.1 of that paper.) Therefore

$$J_i(f_v) = \int_{\overline{\Omega}_v} A_i(\omega) f_{v,G}(\omega) d\omega, \qquad f_v \in \mathcal{H}(\Omega_v, X_v),$$

where each $A_i(\omega)$ is a smooth function on Ω_v such that

$$A_i(\omega_{\lambda_v}) = \mathrm{e}^{-\lambda_v(X_v)} A_i(\omega), \qquad \lambda_v \in i\mathfrak{a}^*_{G_v,Z_v}.$$

Setting

$$A(\omega) = \sum_i a^i A_i(\omega),$$

we conclude that

$$\sum_{\omega \in \overline{\Omega}_{v,\mathbb{C}}} \alpha(\omega) f_{v,G}(\omega) = \int_{\overline{\Omega}_v} A(\omega) f_{v,G}(\omega) d\omega, \qquad f_v \in \mathcal{H}(\Omega_v, X_v). \tag{5.2}$$

We shall complete the proof of (a) by showing that the discrete distribution on the left-hand side of (5.2) can be compatible with the continuous distribution on the right-hand side only if both sides equal zero. This is a p-adic variant of the comparison we applied to archimedean multipliers in the last section. The arguments are essentially those of [AC, p. 191] and [A7, p. 567].

By assumption, the component Ω_v is parabolic. It is therefore induced from an elliptic component

$$\Omega_{L_v} \subset T_{\mathrm{temp,ell}}(L_v, \zeta_v)$$

attached to a proper Levi subgroup L_v of G_v over F_v. We can then identify Ω_v with the set of orbits $\Omega_{L_v}/W(\Omega_{L_v})$, where $W(\Omega_{L_v})$ is the stabilizer of Ω_{L_v} in the Weyl group $W(L_v)$ of (G_v, A_{L_v}). Now the real vector space $i\mathfrak{a}^*_{L_v,Z_v}$ acts transitively on the elliptic component Ω_{L_v}. Let $i\Lambda^*_v$ be the stabilizer in $i\mathfrak{a}^*_{L_v,Z_v}$ of any point ω_0 in Ω_{L_v}. Then $i\Lambda^*_v$ is a lattice in $i\mathfrak{a}^*_{L_v,Z_v}$. For any choice of base point ω_0, we can identify Ω_{L_v} with the compact torus

$$i\mathfrak{a}^*_v = i\mathfrak{a}^*_{L_v,Z_v}/i\Lambda^*_v.$$

The smoothness condition for the function $A(\omega)$ in (5.2) pertains of course to the co-ordinates defined by the torus, and as we noted earlier, the measure $d\omega$ is induced from a Haar measure on the torus.

It is a simple consequence of the trace Paley-Wiener theorem [BDK] that the image of $\mathcal{H}(\Omega_v, X_v)$ under the map

$$f_v \longrightarrow f_{v,G}(\omega), \qquad \omega \in \Omega_v,$$

is the $i\mathfrak{a}^*_{G_v,Z_v}$-equivariant Paley-Wiener space on Ω_v. In other words, the image can be identified with the space of $W(\Omega_{L_v})$-invariant functions ϕ_v on Ω_{L_v}, which coincide with finite Fourier series on the torus $i\mathfrak{a}^*_v$, and which satisfy the condition

$$\phi_v(\omega_{\lambda_v}) = \phi_v(\omega)\mathrm{e}^{\lambda_v(X_v)}, \qquad \omega \in \Omega_{L_v},\ \lambda_v \in i\mathfrak{a}^*_{G_v,Z_v}.$$

We can obviously identify each side of (5.2) with a linear form in ϕ_v. We obtain

(5.3)
$$\sum_{\omega\in\overline{\Omega}_{L_v,\mathbb{C}}} |W(\Omega_{L_v},\omega)||W(\Omega_{L_v})|^{-1}\alpha(\omega)\phi_v(\omega) = \int_{\overline{\Omega}_{L_v}} |W(\Omega_{L_v})|^{-1}A(\omega)\phi_v(\omega)d\omega,$$

where $\overline{\Omega}_{L_v,\mathbb{C}}$ and $\overline{\Omega}_{L_v}$ are the quotients of $\Omega_{L_v,\mathbb{C}}$ and Ω_{L_v} by $i\mathfrak{a}^*_{G_v,Z_v}$, and $W(\Omega_{L_v},\omega)$ is the stabilizer of ω in $W(\Omega_{L_v})$. This identity holds for any function ϕ_v that lies in the $i\mathfrak{a}^*_{G_v,Z_v}$-equivariant Paley-Wiener space on Ω_{L_v}, and is symmetric under $W(\Omega_{L_v})$. But as equivariant functions on $\Omega_{L_v,\mathbb{C}}$ and Ω_{L_v} respectively, both $\alpha(\omega)$ and $A(\omega)$ are also symmetric under $W(\Omega_{L_v})$. It follows that (5.3) actually holds for any ϕ_v in the full equivariant Paley-Wiener space on Ω_{L_v}.

To exploit (5.3), we identify Ω_{L_v} with $i\mathfrak{a}_v^*$ by choosing a base point ω_0. Then ϕ_v ranges over the space of finite, $i\mathfrak{a}_{G_v,Z_v}^*$-equivariant Fourier series on the torus $i\mathfrak{a}_v^*$. We shall consider the Fourier transform of each side of (5.3) as a distribution on the dual group

$$\Lambda_v = \mathrm{Hom}(\Lambda_v^*, \mathbb{Z}).$$

Let $\Lambda_v^{X_v} \subset \Lambda_v$ be the preimage of X_v under the canonical map from Λ_v to $\mathfrak{a}_{G,v}/\mathfrak{a}_{Z,v}$. Then $\Lambda_v^{X_v}$ is an affine sublattice of Λ_v, on which the kernel Λ_v^0 of the map acts simply transitively. The image of the space of test functions ϕ_v under Fourier transform is the space of functions of finite support on $\Lambda_v^{X_v}$. The Fourier transform of the distribution $\alpha(\omega)$ on the left-hand side of (5.3) is a finite linear combination of (nonunitary) exponential functions on $\Lambda_v^{X_v}$ (relative to the action of Λ_v^0). The Fourier transform of the distribution $A(\omega)$ on the right-hand side of (5.3) is a rapidly decreasing function on $\Lambda_v^{X_v}$. The resulting two distributions on $\Lambda_v^{X_v}$ are incompatible. They can be equal only if they are both simultaneously equal to zero. It follows that the left-hand side of (5.3) vanishes, and hence, that the left-hand side of (5.2) also vanishes.

We have established that

$$\sum_{\omega \in \overline{\Omega}_{v,\mathbb{C}}} \alpha(\omega) f_{v,G}(\omega) = 0, \qquad f_v \in \mathcal{H}(\Omega_v, X_v).$$

It follows from (5.1) that

$$I_{\nu,\mathrm{disc}}^{\mathcal{E}}(f) - I_{\nu,\mathrm{disc}}(f) = \sum_{\tau \in \overline{T}(G_v,\zeta_v)} \alpha(\tau_v) f_{v,G}(\tau_v) = 0,$$

for any function $f = f^v f_v$ with $f_v \in \mathcal{H}(\Omega_v, X_v)$. The same formula therefore holds for any such function with $f_v \in \mathcal{H}(\Omega_v)$. This completes the proof of part (a) of Lemma 4.1.

We need only add a few remarks in the case of part (b), since the proof is essentially the same. In this case, G is quasisplit, and Ω_v is a parabolic connected component in $\Phi_{\mathrm{temp}}^{\mathcal{E}}(G_v, \zeta_v)$. Here, we take $I(\Omega_v, X_v)$ to be a maximal set of pairs

$$i = (z_v, \beta_v), \qquad z_v \in Z(\overline{G}_v),\ \beta_v \in \Delta_{\mathrm{unip}}^{\mathcal{E}}(G_v, \zeta_v),$$

such that the endoscopic orbital integrals

$$J_i(f_v) = f_{v,G}^{\mathcal{E}}(z_v \beta_v), \qquad f_v \in \mathcal{H}(\Omega_v, \zeta_v),$$

on $\mathcal{H}(\Omega_v, X_v)$ are linearly independent. We then define a projection

$$f_v \longrightarrow f_v^{00} = f_v - \sum_i J_i(f_v) f_v^i,$$

from $\mathcal{H}(\Omega_v, X_v)$ onto $\mathcal{H}(\Omega_v, X_v)^{00}$, as in (a).

The distribution on the left-hand side of (4.2) has an expansion

$$S^G_{\nu,\mathrm{disc}}(f) = \sum_{\phi\in\Phi^{\mathcal{E}}_{\nu,\mathrm{disc}}(G,V,\zeta)} b^G_{\mathrm{disc}}(\phi) f^{\mathcal{E}}_G(\phi).$$

Assume that $f = f^v f_v$, where f_v belongs to $\mathcal{H}(G_v^{X_v},\zeta_v)$. We can then consider the distribution as a linear form in f_v. We see that

$$S^G_{\nu,\mathrm{disc}}(f) = \sum_{\phi_v\in\overline{\Phi}^{\mathcal{E}}(G_v,\zeta_v)} \beta(\phi_v) f^{\mathcal{E}}_{v,G}(\phi_v), \tag{5.4}$$

where the set $\overline{\Phi}^{\mathcal{E}}(G_v,\zeta_v) = \Phi^{\mathcal{E}}(G_v^{Z_v},\zeta_v)$ equals the space of orbits of $i\mathfrak{a}^*_{G_v,Z_v}$ in $\Phi^{\mathcal{E}}(G_v,\zeta_v)$, and the coefficient

$$\beta(\phi_v) = \beta_{\nu,\mathrm{disc}}(f^v,\phi_v)$$

is defined as a double sum

$$\sum_{\phi}\sum_{\lambda_v} b^G_{\mathrm{disc}}(\phi) f^{v,\mathcal{E}}_G(\phi^v)$$

over elements $\phi\in\Phi^{\mathcal{E}}_{\nu,\mathrm{disc}}(G,V,\zeta)$ and $\lambda_v\in i\mathfrak{a}^*_{G_v,Z_v}$ such that ϕ has a representative $\phi^v\otimes\phi_{v,\lambda_v}$ in $\Phi(G_v,\zeta_v)$. Suppose that f^v is as in (b), and that f_v belongs to $\mathcal{H}(\Omega_v,\zeta_v)$. Combining (5.4) with the projection $f_v\to f_v^{00}$, we obtain an identity

$$\sum_{\omega\in\overline{\Omega}_{v,\mathbb{C}}} \beta(\omega) f^{\mathcal{E}}_{v,G}(\omega) = \sum_{i\in I(\Omega_v,X_v)} b^i J_i(f_v),$$

where

$$b^i = \sum_{\omega\in\overline{\Omega}_{v,\mathbb{C}}} \beta(\omega)(f^i_{v,G})^{\mathcal{E}}(\omega).$$

Since the endoscopic orbital integrals $J_i(f_v)$ are finite linear combinations of invariant orbital integrals, their Fourier transforms are also given by smooth functions. Therefore

$$J_i(f_v) = \int_{\overline{\Omega}_v} B_i(\omega) f^{\mathcal{E}}_{v,G}(\omega)d\omega, \qquad f_v\in\mathcal{H}(\Omega_v,X_v),$$

where each $B_i(\omega)$ is a smooth function on Ω_v such that

$$B_i(\omega_{\lambda_v}) = \mathrm{e}^{-\lambda_v(X_v)}B_i(\omega).$$

It follows that

$$\sum_{\omega\in\overline{\Omega}_{v,\mathbb{C}}} \beta(\omega) f^{\mathcal{E}}_{v,G}(\omega) = \int_{\overline{\Omega}_v} \beta(\omega) f^{\mathcal{E}}_{v,G}(\omega)d\omega, \qquad f_v\in\mathcal{H}(\Omega_v,X_v), \tag{5.5}$$

where

$$B(\omega) = \sum_i \beta^i B_i(\omega).$$

The rest of the proof of (b) is the same as that of (a). We identify Ω_v with a set of orbits $\Omega_{L_v}/W(\Omega_{L_v})$, for an elliptic component

$$\Omega_{L_v} \subset \Phi^{\mathcal{E}}_{\mathrm{temp,ell}}(L_v, \zeta_v)$$

attached to a proper Levi subgroup $L_v \subset G_v$. For any base point $\omega_0 \in \Omega_{L_v}$, we identify Ω_{L_v} in turn with a compact torus

$$i\mathfrak{a}^*_v = i\mathfrak{a}^*_{L_v,Z_v}/i\Lambda^*_v.$$

The proof of (b) is then established by transforming (5.5) into an identity between distributions on a corresponding affine lattice $\Lambda_v^{X_v}$. □

Corollary 5.2. (a) *The identity* (4.1) *of Proposition* 4.2(a) *holds for any function* f *in* $\mathcal{H}_M(G_V, \zeta_V)$.

(b) *If* G *is quasisplit, the identity* (4.2) *of Proposition* 4.2(b) *holds for any function* f *in* $\mathcal{H}^{\mathrm{uns}}_M(G_v, \zeta_v)$.

Proof. Proposition 4.2(a) applies to any function in $\mathcal{H}_M(G_V, \zeta_V)^{00}$. We have to show that it remains valid for functions in the larger space $\mathcal{H}_M(G_V, \zeta_V)$. Now $\mathcal{H}_M(G_V, \zeta_V)^{00}$ is spanned by functions $f = \prod_{v \in V} f_v$ that satisfy the following conditions.

(i) For each place $v \in V_{\mathrm{fin}}(G, M)$, f_v belongs to $\mathcal{H}(G_v, \zeta_v)^0$.

(ii) At some place $v \in V_{\mathrm{fin}}$, f_v belongs to $\mathcal{H}(G_v, \zeta_v)^{00}$.

(iii) At two places $v \in V$, f_v is M-cuspidal.

The larger space $\mathcal{H}_M(G_V, \zeta_V)$ is spanned by functions $f = \Pi f_v$ that satisfy only condition (iii). Notice that if f_v is M-cuspidal, the restriction of the function $f_{v,G}$ to any connected component of $T_{\mathrm{temp}}(G_v, \zeta_v)$ is also M-cuspidal. We can therefore span $\mathcal{H}_M(G_V, \zeta_V)$ by functions that satisfy (iii), and the following support condition.

(s_a) For each place $v \in V_{\mathrm{fin}}$, there is a connected component Ω_v of $T_{\mathrm{temp}}(G_v, \zeta_v)$ such that f_v belongs to $\mathcal{H}(\Omega_v)$.

We consider a function $f = \prod_v f_v$ that satisfies (iii) and (s_a), for fixed components Ω_v in the sets $T_{\mathrm{temp}}(G_v, \zeta_v)$. To establish part (a) of the corollary, it is enough to show that any such f satisfies the identity (4.1). We are free to enlarge the set V if we choose. In particular, the left-hand side of (4.1) remains unchanged if V is replaced by a larger set $V_1 = V \cup \{v_1\}$, and f is replaced by a function $f_1 = f u_{v_1}$. The local component u_{v_1} here stands for the unit in an unramified Hecke algebra at v_1. It lies in $\mathcal{H}(\Omega_{v_1}, \zeta_{v_1})$, where

ζ_{v_1} is an unramified character on Z_{v_1}, and Ω_{v_1} is the parabolic component of unramified representations in $T_{\mathrm{temp}}(G_{v_1}, \zeta_{v_1})$. We may therefore assume that the set

$$V_{\mathrm{par}} = V_{\mathrm{par}}(f) = \Big\{ v \in V_{\mathrm{fin}} : \ \Omega_v \text{ is parabolic} \Big\}$$

is nonempty.

Suppose that f satisfies the extra constraint that f_v belongs to $\mathcal{H}(G_v, \zeta_v)^{00}$, for each $v \in V_{\mathrm{par}}$. Then f satisfies the condition (ii). If v lies in the complement of V_{par} in V_{fin}, Ω_v is elliptic, and $\mathcal{H}(G_v, \zeta_v)$ equals $\mathcal{H}(\Omega_v, \zeta_v)^0$ by definition. Therefore f satisfies condition (i) as well as (ii) and (iii). In other words, f belongs to $\mathcal{H}_M(G_V, \zeta_V)^{00}$, and consequently satisfies the identity (4.1). To remove the extra constraints, we apply Lemma 5.1(a) to each of the places $v \in V_{\mathrm{par}}$. We thereby deduce that the identity remains in force without the requirement that f_v lie in $\mathcal{H}(\Omega_v, \zeta_v)^{00}$. This establishes that (4.1) holds for any f that satisfies (iii) and (s_a), which in turn yields the required assertion of part (a).

The same argument applies to part (b), except that $\mathcal{H}_M^{\mathrm{uns}}(G_V, \zeta_V)^{00}$ and $\mathcal{H}_M^{\mathrm{uns}}(G_V, \zeta_V)$ play the roles of $\mathcal{H}_M(G_V, \zeta_V)^{00}$ and $\mathcal{H}_M(G_V, \zeta_V)$. The space $\mathcal{H}_M^{\mathrm{uns}}(G_V, \zeta_V)^{00}$ is spanned by functions $f = \prod\limits_{v \in V} f_v$ that satisfy conditions (i)–(iii), and also the following supplementary condition.

(iv) At some place $v \in V$, $f_v^G = 0$.

The larger space $\mathcal{H}_M^{\mathrm{uns}}(G_v, \zeta_v)$ is spanned by functions $f = \Pi f_v$ that satisfy only conditions (iii) and (iv). Observe that if f_v is either M-cuspidal or unstable, the restriction of the function $f_{v,G}^{\mathcal{E}}$ to any connected component of $\Phi_{\mathrm{temp}}^{\mathcal{E}}(G_v, \zeta_v)$ has the same property. We can therefore span $\mathcal{H}_M^{\mathrm{uns}}(G_V, \zeta_V)$ by functions that satisfy (iii), (iv), and the following support condition.

(s_b) For each place $v \in V_{\mathrm{fin}}$, there is a connected component Ω_v of $\Phi_{\mathrm{temp}}^{\mathcal{E}}(G_v, \zeta_v)$ such that f_v belongs to $\mathcal{H}(\Omega_v)$.

We can then derive the assertion of part (b) as above. $\square$

6. Local trace formulas

Our concern has been the global trace formula, and the stabilization of its various terms. However, there will come a point in the next section when we have to apply the local trace formula. In the present section, we shall lay the groundwork for this. In particular, we shall take up the study begun in [A10, §9] of how to stabilize the local trace formula.

For the next three sections, F will be a local field. We take G to be a reductive K-group over F, which for the moment is arbitrary. In this context, Z stands for a central induced torus in G over F, and ζ is a character on $Z(F)$. Before we discuss stabilization, we have first to reformulate the invariant local trace formula of [A5] so that it is compatible with the canonically normalized weighted characters of [A8]. As might be expected from the global constructions in [I, §2–3], the result will be a little simpler than the formula of [A5, §4] that depends on a noncanonical choice of normalizing factors.

We temporarily adopt notation from [A10, §8–9], in which $V = \{v_1, v_2\}$ is reduced to the role of an index set of order two. Then $F_V = F \times F$, $G_V = G(F_V) = G(F) \times G(F)$, and $\zeta_V = \zeta \times \zeta^{-1}$, while

$$f = f_1 \times \overline{f}_2, \qquad f_i \in \mathcal{C}(G, \zeta) = \mathcal{C}\Big(G(F), \zeta\Big),$$

is a function in the Schwartz space $\mathcal{C}(G_V, \zeta_V)$. The geometric side of the local trace formula will be the linear form

$$I(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{G\text{-reg,ell}}(M,V,\zeta)} I_M(\gamma, f) d\gamma, \tag{6.1}$$

defined [A10, (9.2)] in terms of the invariant distributions $I_M(\gamma, f)$ in [A10, §4]. We have written $\Gamma_{G\text{-reg,ell}}(M, V, \zeta)$ for the subset of strongly G-regular, elliptic elements in the basis $\Gamma(M, \zeta) = \Gamma\Big(M(F), \zeta\Big)$, identified with its diagonal image

$$\Big\{(\gamma, \gamma) : \ \gamma \in \Gamma_{G\text{-reg,ell}}(M, \zeta)\Big\}$$

in $\Gamma(M_V, \zeta_V)$. (The set $\Gamma_{G\text{-reg,ell}}(M, V, \zeta)$ is bijective with the family $\Gamma_{G\text{-reg,ell}}(\overline{M})$ of strongly G-regular, elliptic conjugacy classes in $\overline{M}(F) = M(F)/Z(F)$ used to state [A10, (9.2)].) The spectral side will be a distribution

$$I_{\text{disc}}(f) = \int_{T_{\text{disc}}(G,V,\zeta)} i^G(\tau) f_G(\tau) d\tau \tag{6.2}$$

that is essentially discrete. Here we are following notation of [A5, §3] (with obvious modifications for the character ζ). We have written $T_{\text{disc}}(G, V, \zeta)$ for the diagonal image

$$\Big\{(\tau, \tau^\vee) : \ \tau \in T_{\text{disc}}(G, \zeta)\Big\}$$

in $T_{\text{temp}}(G_V, \zeta_V)$ of the subset $T_{\text{disc}}(G, \zeta)$ of $T_{\text{temp}}(G, \zeta) = T_{\text{temp}}\Big(G(F), \zeta\Big)$ defined as on p. 96 of [A5]. As on the geometric side, we do not generally distinguish between the element τ attached to $G(F)$ and the corresponding pair $(\tau, \tau^\vee)$ associated to G_V. Thus, $i^G(\tau)$ is the function [A5, (3.2)] on $T_{\text{disc}}(G, \zeta)$, and $d\tau$ is a measure on $T_{\text{disc}}(G, \zeta)$ defined as in [A5, (3.5)] (with $i\mathfrak{a}^*_{G,Z}$ playing the role of $i\mathfrak{a}^*_G$), while

$$f_G(\tau) = (f_1)_G(\tau)(\overline{f}_2)_G(\tau^\vee) = f_{1,G}(\tau)\overline{f_{2,G}(\tau)}.$$

PROPOSITION 6.1. $I(f) = I_{\mathrm{disc}}(f)$.

Proof. We can afford to be brief, since the proof is similar to that of [A5, Th. 4.2]. The discussion of [A5, §4] applies only to a function in the Hecke space, but it extends easily to the Schwartz space by the arguments of [A6, §5]. We note in passing that while the formula of [A5, §4] is close to the assertion of this proposition, the invariant local trace formula established in the paper [A6, §5] is of a rather different nature. The latter was designed to prove the qualitative theorems in [A6, §4] for distributions on $G(F)$, rather than for the comparison of distributions on different groups.

The starting point is the noninvariant trace formula, which consists of two different expansions of a noninvariant linear form $J(f)$. As formulated in [A5, Prop. 4.1], the expansions are

$$J(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{G\text{-reg,ell}}(M,V,\zeta)} J_M(\gamma, f)d\gamma \tag{6.3}$$

and

$$J(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{T_{\mathrm{disc}}(M,V,\zeta)} i^M(\tau)J_M(\tau, f)d\tau. \tag{6.4}$$

The point here is that the distribution $J_M(\tau, f)$, defined for example as in [A5, §4], actually equals a canonically normalized weighted character. To put it another way,

$$J_M(\tau, f) = \phi_M(\tau, f) = \phi_M(f_1 \times \overline{f}_2, \tau \times \tau^\vee),$$

where

$$\phi_M : \mathcal{C}(G_V, \zeta_V) \longrightarrow \mathcal{I}(M_V, \zeta_V)$$

is the mapping of [A8, §3] and [A10, §4]. This property is not hard to establish from the definitions just cited. Since we have already proved the analogous global property in [I, §3], we shall leave the details to the reader. (The property is closely related to the analyticity assertions of [A4, Lemma 12.1] and [A8, Prop. 2.3]. Unnormalized weighted characters are generally only meromorphic.)

Following the usual recipe, we define invariant linear forms

$$I^L : \mathcal{C}(L_V, \zeta_V) \longrightarrow \mathbb{C}, \qquad L \in \mathcal{L},$$

inductively by setting

$$J(f) = \sum_{L \in \mathcal{L}} |W_0^L||W_0^G|^{-1}(-1)^{\dim(A_L/A_G)} \widehat{I}^L\Big(\phi_L(f)\Big).$$

It follows by induction from (6.3) and the definition

$$J_M(\gamma, f) = \sum_{L \in \mathcal{L}(M)} \widehat{I}_M^L\Big(\gamma, \phi_L(f)\Big)$$

that $I^G(f)$ is equal to the expansion (6.1) for $I(f)$. On the other hand, if we define invariant linear forms

$$I_M^L(\tau) : \mathcal{C}(L_V, \zeta_V) \longrightarrow \mathbb{C}, \qquad L \in \mathcal{L},$$

inductively by setting

$$J_M(\tau, f) = \sum_{L \in \mathcal{L}(M)} \widehat{I}_M^L\Big(\tau, \phi_L(f)\Big),$$

then

$$I_M^G(\tau, f) = \begin{cases} f_G(\tau), & \text{if } M = G, \\ 0, & \text{if } M \neq G. \end{cases}$$

It follows by induction from (6.4) that $I^G(f)$ is also equal to the expansion (6.2) for $I_{\mathrm{disc}}(f)$. We have shown that $I(f)$ equals $I_{\mathrm{disc}}(f)$, as required. □

The proposition asserts that the expansions on the right-hand sides of (6.1) and (6.2) are equal. This is the invariant local trace formula we were seeking. It differs from the earlier formula in [A5, Th. 4.2] as follows. On the geometric side, the invariant distributions $I_M(\gamma, f)$ in (6.1) are defined in terms of the weighted characters of [A8], while their counterparts in [A5, (4.10)] use the weighted characters on p. 101 of [A5]. On the spectral side, the distribution $I_{\mathrm{disc}}(f)$ in (6.2) is essentially discrete in the variable τ, while its counterpart [A5, (4.11)] contains continuous terms that arise from normalizing factors for intertwining operators.

We turn now to the question of stabilizing the terms in (6.1) and (6.2). Suppose that G' is an endoscopic datum for G. Following the convention in [A10, §9], we shall identify G' with the diagonal endoscopic datum

$$G'_V = G' \times \overline{G}'$$

for $G_V = G \times G$. We recall that if G' represents the datum $(G', \mathcal{G}', s', \xi')$, then $\overline{G}'$ represents the adjoint datum $\Big(G', \mathcal{G}', (s')^{-1}, \xi'\Big)$. Recall that the Langlands-Shelstad transfer factor attached to (G, G') depends on a choice of auxiliary data $\widetilde{G}' \to G'$ and $\widetilde{\xi}'$: $\mathcal{G}' \to {}^L\widetilde{G}'$ for G'. We would like to choose compatible auxiliary data for $\overline{G}'$. Since $\overline{G}'$ equals G' as a quasisplit group, we can set $\widetilde{\overline{G}}' = \widetilde{G}'$. The choice of L-embedding

$$\widetilde{\overline{\xi}}' : \overline{\mathcal{G}}' = \mathcal{G}' \longrightarrow {}^L\widetilde{\overline{G}}' = {}^L\widetilde{G}'$$

is then dictated by the following lemma, which was suggested to me by Kottwitz.

Lemma 6.2. *Given $\widetilde{\xi}'$, there exists $\widetilde{\overline{\xi}}'$ so that the relative transfer factor for $(G, \overline{G}')$ is the inverse of the relative transfer factor for (G, G').*

Proof. This is Lemma 8.1 of [A10], which was stated essentially without proof. However, there is one point that ought to be verified in detail. In fact, the description of $\widetilde{\overline{\xi}}'$ given on p. 258 of [A10] is not correct, since the map $\widetilde{\overline{\xi}}'$ defined there is not an embedding of the required type. (Its restriction to the subgroup $\widehat{G}' = \widehat{\overline{G}}'$ is not the identity embedding of this group into $\widehat{\widetilde{G}}' = \widehat{\widetilde{\overline{G}}}'$.) We need to see how the choice of $\widetilde{\overline{\xi}}'$ is forced on us by the transfer factor for (G, G').

We may assume that the group $\widetilde{G}' = \widetilde{\overline{G}}'$ equals G' [LS, (4.4)]. Then $\widetilde{\xi}'$ is simply an L-isomorphism, which we use to identify $\mathcal{G}'$ with ${}^L G'$. We can then treat ξ' as an L-embedding of ${}^L G'$ into ${}^L G$. With this assumption, the relative transfer factor for (G, G') is defined as a product of four terms [LS, (3.2)–(3.5)]. The relative transfer factor for $(G, \overline{G}')$ is defined by a similar product, except that the element s' in the factors Δ_I and Δ_1 [LS, (3.2), (3.4)] has to be replaced by its inverse $(s')^{-1}$. If $\{\chi_\alpha\}$ are the χ-data for G' on which the other two factors Δ_{II} and Δ_2 [LS, (3.3), (3.5)] depend, we are free to take $\{\chi_\alpha^{-1}\}$ to be the χ-data for $\overline{G}'$. It is then clear from the definitions [LS, (3.2)–(3.4)] that the three factors $\overline{\Delta}_1$, $\overline{\Delta}_{\mathrm{II}}$ and $\overline{\Delta}_1$ for $\overline{G}'$ are all inverses of the corresponding factors Δ_{I}, Δ_{II} and Δ_1 for G'. The remaining factor

$$\overline{\Delta}_2 = \overline{\Delta}_2(\delta', \gamma), \qquad \delta' \in \Delta_{G\text{-reg}}(G'),\ \gamma \in \Gamma_{\mathrm{reg}}(G),$$

for $\overline{G}'$ is absolute, in the sense that it can be defined without reference to a base point. It is also the only factor that depends on the choice of $\widetilde{\overline{\xi}}'$. We are now treating $\widetilde{\overline{\xi}}'$ as an L-automorphism of ${}^L G'$. The choice of $\widetilde{\overline{\xi}}'$ is therefore equivalent to that of an L-embedding

$$\overline{\xi}' = \xi' \circ (\widetilde{\overline{\xi}}')^{-1} :\ {}^L G' \longrightarrow {}^L G,$$

which coincides with ξ' on $\widehat{G}'$. We can assume that the restriction of $\widetilde{\overline{\xi}}'$ to the Weil group W_F of F preserves a Γ-splitting of $\widehat{G}'$. Then $\overline{\xi}'$ is of the form

$$\overline{\xi}'(g' \times w) = z'(w)\xi'(g' \times w), \qquad g' \in \widehat{G}',\ w \in W_F,$$

where z' is a 1-cocycle from W_F to the center $Z(\widehat{G}')$ of $\widehat{G}'$. We have to show that z' can be chosen so that $\overline{\Delta}_2(\delta', \gamma)$ equals the inverse of the corresponding factor $\Delta_2(\delta', \gamma)$ for G'. This is the point that is not immediately obvious from the definitions.

The assertion is not in itself hard to verify, but it does require a recapitulation of the various objects [LS, (2.5), (2.6), (3.5)] that go into the construction of $\Delta_2(\delta', \gamma)$. The χ-data are attached to the maximal torus T in G^* (the underlying quasisplit inner form of G) that is the image of the centralizer of (a representative of) δ' under a fixed admissible embedding. Their role is to provide two L-embeddings ξ'_T: ${}^L T \to {}^L G'$ and ξ_T: ${}^L T \to {}^L G$. The factor Δ_2 is defined in terms of these embeddings by the local Langlands correspondence on $T(F)$. To conserve notation, we assume that the restrictions of ξ'_T, ξ' and ξ_T to the relevant dual groups are simply the trivial injections of embedded subgroups $\widehat{T} \subset \widehat{G}'$, $\widehat{G}' \subset \widehat{G}$ and $\widehat{T} \subset \widehat{G}$. The factor is then defined by

$$\Delta_2(\delta', \gamma) = \langle a, \delta \rangle,$$

where δ is the image of δ' in $T(F)$, and a is the 1-cocycle from W_F to $\widehat{T}$ defined by

$$\xi' \circ \xi'_T = a\xi_T.$$

Our task is to choose $\overline{\xi}'$ so that the corresponding cocycle $\overline{a}$ for $\overline{G}'$ maps to the image of a^{-1} in $H^1(W_F, \widehat{T})$.

The value of the L-embedding ξ_T at an element $w \in W_F$ is given by a product

$$\xi_T(w) = r(w)n(w).$$

Here r: $W_F \to \widehat{T}$ is the 1-chain defined in [LS, (2.5)] in terms of the χ-data $\{\chi_\alpha\}$ and a fixed gauge on the roots of $(\widehat{G}, \widehat{T})$, while

$$n(w) = n\Big(\omega_T(\sigma)\Big) \times w$$

is the element in ${}^LG$ defined in [LS, (2.1)] in terms of a fixed Γ_F-splitting

$$\Big(\widehat{T}, \widehat{B}, \{X_\alpha : \ \alpha \in \Delta(\widehat{B}, \widehat{T})\}\Big)$$

for $(\widehat{G}, \widehat{T})$. We recall that σ is the image of w in the Galois group $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$, while $\omega_T(\sigma)$ is the element in the Weyl group Ω of $(\widehat{G}, \widehat{T})$ defined by the action of Γ_F on T, and $n\Big(\omega_T(\sigma)\Big)$ is a representative of $\omega_T(\sigma)$ in the normalizer $\widehat{N}$ of $\widehat{T}$ in $\widehat{G}$. The value at w of the second embedding ξ'_T is given by a corresponding product

$$\xi'_T(w) = r'(w)n'(w).$$

In this product, $n'(\omega)$ is defined in terms of a fixed splitting

$$\Big(\widehat{T}, \widehat{B}', \{X_\beta : \ \beta \in \Delta(\widehat{B}', \widehat{T})\}\Big)$$

for $(\widehat{G}', \widehat{T})$ such that $\widehat{B}' = \widehat{G}' \cap \widehat{B}$ [LS, (3.1)]. The two elements $n(w)$ and $\xi'\Big(n'(w)\Big)$ in ${}^LG$ have the same action by conjugation on $\widehat{T}$. Their quotient

$$b(w) = \xi'\Big(n'(w)\Big)n(w)^{-1}$$

therefore lies in $\widehat{T}$. Since the quotient

$$c(w) = r'(w)r(w)^{-1}$$

also lies in $\widehat{T}$, we obtain a decomposition

$$a(w) = b(w)c(w).$$

We must compare this with the corresponding decomposition for $\overline{a}(w)$. Now $\overline{a}(w)$ is defined by replacing ξ' with $\overline{\xi}' = z'\xi'$, for a 1-cocycle $z' \in Z^1\Big(W_F, Z(\widehat{G}')\Big)$ to be chosen, and by replacing the χ-data $\{\chi_\alpha\}$ by $\{\chi_\alpha^{-1}\}$. This has the effect of replacing $b(w)$ by $z'(w)b(w)$, and $c(w)$ by $c(w)^{-1}$, as one sees easily from the construction in [LS, (2.5)]. It follows that

$$a(w)\overline{a}(w) = \Big(b(w)c(w)\Big)\Big(z'(w)b(w)c(w)^{-1}\Big) = z'(w)b(w)^2.$$

It would be enough to show that the 1-cocycle $b(w)^2$ in $Z^1(W_F, \widehat{T})$ maps into the image of $H^1\Big(W_F, Z(\widehat{G}')\Big)$ in $H^1(W_F, \widehat{T})$. For we could then take $z'(w)$ to be any element in $Z^1\Big(W_F, Z(\widehat{G}')\Big)$ whose image in $H^1(W_F, \widehat{T})$ equals that of $b(w)^{-2}$. This would in turn yield a formula

$$\overline{\Delta}_2(\delta', \gamma)\Delta_2(\delta', \gamma) = \langle \overline{a}a, \delta\rangle = \langle z'b^2, \delta\rangle = 1$$

that gives the desired relation between the two factors.

The map θ: $w \to \omega_T(\sigma) \times w$ is a homomorphism from W_F to the group ${}^L\Omega = \Omega \rtimes W_F$. The map $w \to n(w)$ is the composition of this homomorphism with a function

$$\nu : \ \omega \times w \longrightarrow n(\omega) \times w$$

from ${}^L\Omega$ to the group ${}^LN = \widehat{N} \rtimes W_F$, where W_F acts on $\widehat{N}$ by means of the fixed Γ_F-splitting of $(\widehat{G}, \widehat{T})$. To define $n(\omega)$, Langlands and Shelstad first set

$$n(\omega_\alpha) = n(\alpha) = \exp(X_\alpha)\exp(-X_{-\alpha})\exp(X_\alpha),$$

for any simple root α, and for the root vectors X_α and $X_{-\alpha}$ given by the splitting. Following [Sp], they then define

$$n(\omega) = n(\alpha_1)\cdots n(\alpha_n),$$

for an arbitrary element $\omega \in \Omega$ with reduced decomposition $\omega = \omega_{\alpha_1}\cdots\omega_{\alpha_n}$ into simple reflections. There are of course similar maps θ': $W_F \to {}^L\Omega'$ and ν': ${}^L\Omega' \to {}^LN'$ for G'. We therefore have a diagram

$$\begin{tikzcd} & {}^L\Omega' \arrow[r, "\nu'"] \arrow[d, "\xi'"] & {}^LN' \arrow[d, "\xi'"] \\ W_F \arrow[ur, "\theta'"] \arrow[dr, "\theta"'] & & \\ & {}^L\Omega \arrow[r, "\nu"] & {}^LN \end{tikzcd}$$

with vertical arrows obtained from the L-embedding ξ': ${}^LG' \to {}^LG$. The square is not generally commutative. However, the obstruction

$$\beta(\theta') = \xi'\Big(\nu'(\theta')\Big)\nu\Big(\xi'(\theta')\Big)^{-1}, \qquad \theta' \in {}^L\Omega',$$

does belong to $\widehat{T}$. Since

$$b(w) = \beta\Big(\theta'(w)\Big), \qquad w \in W_F,$$

it would obviously be enough to arrange things so that for any $\theta' \in {}^L\Omega'$, $\beta(\theta')^2$ lies in $Z(\widehat{G}')$.

The map ν depends on our fixed splitting $\Big(\widehat{T}, \widehat{B}, \{X_\alpha\}\Big)$ for $(\widehat{G}, \widehat{T})$. We shall expand the set $\{X_\alpha\}$ into a complete family of root vectors $\{X_\beta\}$, where β runs over the set $\Phi(\widehat{G}, \widehat{T})$ of all roots of $(\widehat{G}, \widehat{T})$. We claim that this can be done in such a way that if

$$\gamma = \theta\beta, \qquad \beta, \gamma \in \Phi(\widehat{G}, \widehat{T}),\ \theta \in {}^L\Omega,$$

then

$$\mathrm{Ad}\Big(\nu(\theta)\Big)X_\beta = \mathrm{Ad}(u)X_\gamma,$$

for some element $u \in \widehat{T}$ with $u^2 = 1$. It is clearly enough to show that the condition holds if $\beta = \gamma$, and $X_\beta = X_\gamma$ is any associated root vector. In the special case that $\beta = \gamma$ is simple, the condition follows (with $u = 1$) from [Sp, Prop. 11.2.11]. If $\beta = \gamma$ is arbitrary, we choose $\omega \in \Omega$ so that $\alpha = \omega\beta$ is simple. Then $(\omega^{-1}\theta\omega)\alpha = \alpha$, and

$$\nu(\omega^{-1}\theta\omega)X_\alpha = X_\alpha.$$

Since Lemma 2.1.A of [LS] implies that

$$\nu(\theta)\nu(\omega) = u\nu(\omega)\nu(\omega^{-1}\theta\omega),$$

for some element $u \in \widehat{T}$ with $u^2 = 1$, the condition holds in this case as well. The claim follows. Having chosen the family $\{X_\beta\}$, we take

$$\Big(\widehat{T}, \widehat{B}', \{X_\beta :\ \beta \in \Delta(\widehat{B}', \widehat{T})\}\Big), \qquad \widehat{B}' = \widehat{G}' \cap \widehat{B},$$

to be the splitting for $(\widehat{G}', \widehat{T})$. To ensure that this is a Γ_F-splitting, we might have to replace ξ' by an L-embedding whose restriction to W_F differs from that of ξ' by some $\widehat{T}$-conjugate. Such a change serves only to multiply a by a 1-coboundary from W_F to $\widehat{T}$, and therefore has no effect on the image of a in $H^1(W_F, \widehat{T})$.

We can now complete the argument. Suppose that θ' is an element in ${}^L\Omega'$. Then

$$\beta(\theta')\nu(\theta) = \xi'(n'),$$

where $\theta = \xi'(\theta')$ and $n' = \nu'(\theta')$. Assume first that θ' belongs to the subgroup W_F of ${}^L\Omega'$. Let γ be a simple root of $(\widehat{G}', \widehat{T})$, and set $\beta = \theta^{-1}\gamma$. Then $\mathrm{Ad}\Big(\xi'(n')\Big)X_\beta$ equals X_γ, since n' preserves the splitting for $(\widehat{G}', \widehat{T})$ and ξ' is a homomorphism. Therefore

$$X_\gamma = \mathrm{Ad}\Big(\xi'(n')\Big)X_\beta = \mathrm{Ad}\Big(\beta(\theta')\Big)\mathrm{Ad}\Big(\nu(\theta)\Big)X_\beta = \mathrm{Ad}\Big(\beta(\theta')u\Big)X_\gamma,$$

for an element $u \in \widehat{T}$ with $u^2 = 1$. It follows that $\gamma\Big(\beta(\theta')^2\Big)$ equals 1 for any simple root γ of $(\widehat{G}', \widehat{T})$. We conclude that $\beta(\theta')^2$ belongs to $Z(\widehat{G}')$, as required. Assume now that θ' belongs to the subgroup Ω' of ${}^L\Omega'$. Then $\xi'(n') = n' = n'(\theta')$, and $\nu(\theta) = \nu(\theta') = n(\theta')$, since ξ' restricts to the trivial embedding of $\widehat{G}'$. Consider the special case that $\theta' = \omega_\beta$, for a simple root β of $(\widehat{G}', \widehat{T})$. In this case, we choose $\omega \in \Omega$ so that $\alpha = \omega\beta$ is a simple root for $(\widehat{G}, \widehat{T})$. It then follows from [LS, Lemma 2.1.A] and the definitions above that

$$\begin{aligned} n'(\theta') = n(\beta) &= \exp(X_\beta)\exp(-X_{-\beta})\exp(X_\beta) \\ &= \mathrm{Int}(u)\mathrm{Int}\Big(n(\omega)\Big)^{-1}\Big(\exp(X_\alpha)\exp(-X_{-\alpha})\exp\, X_\alpha\Big) \\ &= \mathrm{Int}(u)\Big(n(\omega)^{-1}n(\alpha)n(\omega)\Big) \\ &= \mathrm{Int}(u)\mathrm{Int}(u')n(\omega^{-1}\omega_\alpha\omega) = (uu')\omega(uu')^{-1}n(\theta), \end{aligned}$$

for elements $u, u' \in \widehat{T}$ of square 1. Therefore $\beta(\theta')$ equals the product $uu'\omega(uu')^{-1}$, an element whose square is also equal to 1. Finally, if θ' is an arbitrary element in Ω', we write θ' as a product $\omega_{\beta_1}\cdots\omega_{\beta_n}$ of simple reflections in Ω'. It then follows from [LS, Lemma 2.1.A], and what we have just proved, that

$$\begin{aligned} n'(\theta') &= n'(\omega_{\beta_1})\cdots n'(\omega_{\beta_n}) \\ &= u_1 n(\omega_{\beta_1})u_2 n(\omega_{\beta_2})\cdots u_n n(\omega_{\beta_n}) \\ &= un(\omega_{\beta_1})\cdots n(\omega_{\beta_n}) = uu'n(\theta), \end{aligned}$$

where $u_1, \dots, u_n$, u and u' are all elements in $\widehat{T}$ of square 1. Therefore $\beta(\theta')$ equals uu', an element whose square is again equal to 1. We have now only to recall that ${}^L\Omega'$ is a semidirect product of the two subgroups Ω' and W_F. We conclude that $\beta(\theta')^2$ lies in $Z(\widehat{G}')$ for any element θ' in ${}^L\Omega'$. This is what we set out to prove. As explained above, the embedding

$$\overline{\xi}'(w) = \beta\Big(\theta'(w)\Big)^{-2}\xi'(w), \qquad w \in W_F,$$

provides a factor $\overline{\Delta}_2(\delta', \gamma)$ that is the inverse of $\Delta_2(\delta', \gamma)$. □

We return to the setting of Proposition 6.1, in which $f \in \mathcal{C}(G_V, \zeta_V)$ is a function of the form $f_1 \times \overline{f}_2$. Lemma 6.2 applies directly to the transfer

$$f \longrightarrow f' = f^{G'_V} = f_1^{G'} \times \overline{f}_2^{\overline{G}'}$$

of f to an endoscopic group $G'_V = G' \times \overline{G}'$. According to the discussion on p. 269 of [A10], which is based on the assertion of Lemma 6.2, f' is equal to the product

$$f'_1 \times \overline{f'_2} = f_1^{G'} \times \overline{f_2^{G'}}.$$

In particular, f' is a function in $S\mathcal{I}(\widetilde{G}'_V, \widetilde{\zeta}'_V)$, where $\widetilde{\zeta}'_V = \widetilde{\zeta}' \times (\widetilde{\zeta}')^{-1}$. The transfer mappings were used in [A10, §9] to construct supplementary linear forms $I^{\mathcal{E}}(f)$ and $S^G(f)$ from $I(f)$. They are defined by the familiar formula

$$I^{\mathcal{E}}(f) = \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G)} \iota(G, G')\widehat{S}'(f') + \varepsilon(G)S^G(f),$$

in which the linear forms $\widehat{S}' = \widehat{S}^{\widetilde{G}'}$ on $S\mathcal{I}(\widetilde{G}'_V, \widetilde{\zeta}'_V)$ are determined inductively by the further requirement that $I^{\mathcal{E}}(f) = I(f)$ in case G is quasisplit. We recall that

$$\iota(G, G') = |\mathrm{Out}_{G'}(G)|^{-1}|Z(\widehat{G}')^{\Gamma}/Z(\widehat{G}^{\Gamma})|^{-1},$$

and that $\varepsilon(G)$ equals 1 or 0, according to whether or not G is quasisplit. One of the main results of [A10] was Theorem 9.1. This theorem provides geometric expansions

$$I^{\mathcal{E}}(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{G\text{-reg,ell}}(M,V,\zeta)} I^{\mathcal{E}}_M(\gamma, f)d\gamma \tag{6.5}$$

and

$$\begin{aligned} S^G(f) \quad = \quad & \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M)} \iota(M, M') \\ & \cdot \int_{\Delta_{G\text{-reg,ell}}(\widetilde{M}',V,\widetilde{\zeta}')} n(\delta')^{-1} S^G_M(M', \delta', f)d\delta', \end{aligned} \tag{6.6}$$

the latter valid for G quasisplit, that are reminiscent of the global geometric expansions of [I, Prop. 10.1].

In [A10, §10], we also stabilized a special case of the spectral side, in which the function f_1 was cuspidal. (The results were used in the cancellation of p-adic singularities in §3.) The formal aspects of the process work in general, being no different from the construction above. For any function $f = f_1 \times \overline{f}_2$, we set

$$I^{\mathcal{E}}_{\mathrm{disc}}(f) = \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G)} \iota(G, G')\widehat{S}'_{\mathrm{disc}}(f') + \varepsilon(G)S^G_{\mathrm{disc}}(f),$$

for linear forms $\widehat{S}'_{\mathrm{disc}} = \widehat{S}^{\widetilde{G}'}_{\mathrm{disc}}$ on $S\mathcal{I}(\widetilde{G}'_V, \widetilde{\zeta}'_V)$ that are defined inductively by the condition that $I^{\mathcal{E}}_{\mathrm{disc}}(f) = I_{\mathrm{disc}}(f)$ in case G is quasisplit. The linear form S^G_{disc} is defined as usual only when G is quasisplit. It follows inductively from Proposition 6.1 and the two sets of definitions that

$$I^{\mathcal{E}}(f) = I^{\mathcal{E}}_{\mathrm{disc}}(f) \tag{6.7}$$

in general, and that

$$S^G(f) = S^G_{\mathrm{disc}}(f), \tag{6.8}$$

in case G is quasisplit.

The linear forms $I^{\mathcal{E}}_{\mathrm{disc}}(f)$ and $S^{G}_{\mathrm{disc}}(f)$ have expansions that are parallel to (6.2). To state them, we have first to define the relevant coefficients by local analogues of the global definitions [I, (7.7), (7.8)]. If τ belongs to $T_{\mathrm{temp}}(G_V, \zeta_V)$, we set

$$i^{G,\mathcal{E}}(\tau) = \sum_{G'}\sum_{\phi'} \iota(G,G') s^{\widetilde{G}'}(\phi')\Delta_G(\phi',\tau) + \varepsilon(G)\sum_{\phi} s^G(\phi)\Delta_G(\phi,\tau), \tag{6.9}$$

with G', ϕ' and ϕ summed over the sets $\mathcal{E}^0_{\mathrm{ell}}(G)$, $\Phi_{\mathrm{temp}}(\widetilde{G}'_V, \widetilde{\zeta}'_V)$ and $\Phi^{\mathcal{E}}_{\mathrm{temp}}(G_V, \zeta_V)$, respectively, and with coefficients $s^{\widetilde{G}'}(\phi')$ defined inductively by the requirement that

$$i^{G,\mathcal{E}}(\tau) = i^G(\tau), \tag{6.10}$$

in the case that G is quasisplit. It is understood that $i^G(\tau)$ is defined to be 0 for any τ in the complement of $T_{\mathrm{disc}}(G,V,\zeta)$ in $T_{\mathrm{temp}}(G_V,\zeta_V)$. Like the original coefficients $i^G(\tau)$, both $i^{G,\mathcal{E}}(\tau)$ and $s^G(\phi)$ are supported on sets that are discrete modulo the diagonal action of $i\mathfrak{a}^*_{G,Z}$. Following the general prescription in [I, §7], we can define $i\mathfrak{a}^*_{G,Z}$-discrete subsets $T^{\mathcal{E}}_{\mathrm{disc}}(G,V,\zeta) \supset T_{\mathrm{disc}}(G,V,\zeta)$ and $\Phi^{\mathcal{E}}_{\mathrm{disc}}(G,V,\zeta)$ of $T_{\mathrm{temp}}(G_V,\zeta_V)$ and $\Phi^{\mathcal{E}}_{\mathrm{temp}}(G_V,\zeta_V)$, respectively, which contain the support of the respective coefficients $i^{G,\mathcal{E}}(\tau)$ and $s^G(\phi)$. The sums over ϕ' and ϕ in (6.9) may then be restricted to the subsets

$$\Phi_{\mathrm{disc}}(\widetilde{G}', V, \widetilde{\zeta}') = \Phi^{\mathcal{E}}_{\mathrm{disc}}(\widetilde{G}', V, \widetilde{\zeta}') \cap \Phi_{\mathrm{temp}}(\widetilde{G}'_V, \widetilde{\zeta}'_V)$$

and $\Phi^{\mathcal{E}}_{\mathrm{disc}}(G,V,\zeta)$ of $\Phi_{\mathrm{temp}}(\widetilde{G}'_V, \widetilde{\zeta}'_V)$ and $\Phi^{\mathcal{E}}_{\mathrm{temp}}(G_V,\zeta_V)$ respectively. We note that the Haar measure on $i\mathfrak{a}^*_{G,Z}$ determines natural measures $d\tau$, $d\phi'$, and $d\phi$ on the respective spaces $T^{\mathcal{E}}_{\mathrm{disc}}(G,V,\zeta)$, $\Phi_{\mathrm{disc}}(\widetilde{G}', V, \widetilde{\zeta}')$ and $\Phi^{\mathcal{E}}_{\mathrm{disc}}(G,V,\zeta)$.

PROPOSITION 6.3. (a) *If G is arbitrary*,

$$I^{\mathcal{E}}_{\mathrm{disc}}(f) = \int_{T^{\mathcal{E}}_{\mathrm{disc}}(G,V,\zeta)} i^{G,\mathcal{E}}(\tau) f_G(\tau) d\tau. \tag{6.11}$$

(b) *If G is quasisplit*,

$$S^G_{\mathrm{disc}}(f) = \int_{\Phi^{\mathcal{E}}_{\mathrm{disc}}(G,V,\zeta)} s^G(\phi) f^{\mathcal{E}}_G(\phi) d\phi. \tag{6.12}$$

Proof. The assertions of the proposition have the same form as those of Lemmas 7.2 and 7.3 of [I]. The proofs are similar. □

COROLLARY 6.4. (a) *Assume that Local Theorem* 1(a) *holds for G and its Levi subgroups. Then*

$$i^{G,\mathcal{E}}(\tau) = i^G(\tau), \qquad \tau \in T^{\mathcal{E}}_{\mathrm{disc}}(G,V,\zeta).$$

(b) *Assume that G is quasisplit, and that Local Theorem* 1(b) *holds for G and its Levi subgroups. Then the coefficient $s^G(\phi)$ vanishes on the complement of $\Phi_{\mathrm{disc}}(G,V,\zeta)$ in $\Phi^{\mathcal{E}}_{\mathrm{disc}}(G,V,\zeta)$.*

Proof. Consider part (a). We first combine the assertion of Local Theorem 1(a) with the splitting formulas [A10, (4.6), (6.2)] for the product $f = f_1 \times \overline{f}_2$. We obtain

$$I^{\mathcal{E}}_M(\gamma, f) = I_M(\gamma, f), \qquad \gamma \in \Gamma_{G\text{-reg,ell}}(M,V,\zeta),$$

in the usual way. It follows from the expansions (6.1) and (6.5) that $I^{\mathcal{E}}(f) = I(f)$. Therefore

$$I^{\mathcal{E}}_{\mathrm{disc}}(f) = I^{\mathcal{E}}(f) = I(f) = I_{\mathrm{disc}}(f).$$

The identity between the coefficients $i^{G,\mathcal{E}}(\tau)$ and $i^G(\tau)$ then follows from a comparison of the expansions (6.2) and (6.11) for $I_{\mathrm{disc}}(f)$ and $I^{\mathcal{E}}_{\mathrm{disc}}(f)$. The proof of (b) is similar. □

Remarks. 1. Part (a) of Corollary 6.4 is equivalent to the assertion that $I^{\mathcal{E}}_{\mathrm{disc}}(f) = I_{\mathrm{disc}}(f)$. Part (b) is equivalent to the assertion that the distribution $S^G(f)$ is stable. This second assertion is of course required to complete the inductive definition of $I^{\mathcal{E}}_{\mathrm{disc}}(f)$.

2. If F is archimedean, Corollary 6.4 could be established directly from Langlands's parametrization of tempered representations [L2], the character identities of Shelstad [Sh], and local analogues of the results in [A3].

We now return to the induction hypothesis of Section 1, with fixed integers d_{der} and r_{der}. Since our intention is to apply the local trace formula to the proof of Local Theorem 1, and since the archimedean case is treated in [A13], we take F to be a p-adic field. The K-group G is then just a connected reductive group over F. We assume that (G,F) satisfies Assumption 5.2(2) of [I], and that $\dim(G_{\mathrm{der}}) = d_{\mathrm{der}}$. Given G, we fix a Levi subgroup M with $\dim(A_M \cap G_{\mathrm{der}}) = r_{\mathrm{der}}$. For simplicity, we shall assume that G_{der} is simply connected, and that the central induced torus Z in G is trivial. If G' is any endoscopic group for G, we can then take the central extension $\widetilde{G}'$ to be G' itself.

We recall that $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$ denotes the set of isomorphism classes of pairs (M',δ'), where M' is an elliptic endoscopic datum for M, and δ' belongs to the set $\Delta_{G\text{-reg,ell}}(M')$ of G-regular, elliptic stable conjugacy classes in $M'(F)$. Then $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$ can be identified with the quotient of

$$\Big\{(M',\delta') : \ M' \in \mathcal{E}_{\mathrm{ell}}(M), \ \delta' \in \Delta_{G\text{-reg,ell}}(M')\Big\},$$

under the action of the finite group $\mathrm{Out}_M(M')$ on $\Delta_{G\text{-reg,ell}}(M')$. If δ is the image in $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$ of a pair (M',δ'), we shall write $\overline{\delta}$ and δ^{-1} for the images of the respective pairs $(\overline{M}',\delta')$ and $\big(M',(\delta')^{-1}\big)$. We recall also that $\Delta_{G\text{-reg,ell}}(M)$ can be identified with a subset of $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$. It will be convenient to set

$$\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M) = \begin{cases} \Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M), & \text{if } G \text{ is not quasisplit,} \\ \Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M) - \Delta_{G\text{-reg,ell}}(M), & \text{if } G \text{ is quasisplit.} \end{cases}$$

If G is quasisplit and δ represents a pair (M',δ'), we consider the linear form

$$\varepsilon(f_*,\delta) = S^G_M(M',\delta',f_*), \qquad f_* \in \mathcal{H}(G).$$

Local Theorem 1(b) asserts that this linear form vanishes if δ lies in $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M)$. The local trace formula allows a modest step in this direction.

Lemma 6.5. *Suppose that G is quasisplit, and that*

$$\varepsilon(f_*,\delta) = \varepsilon(\delta) f^{\mathcal{E}}_{*,M}(\delta), \qquad \delta \in \Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M),\ f_* \in \mathcal{H}(G), \tag{6.13}$$

for a smooth function $\varepsilon(\delta)$ on the complement $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M)$ of $\Delta_{G\text{-reg,ell}}(M)$. Then

$$\varepsilon(\delta) + \varepsilon(\delta^{-1}) = 0. \tag{6.14}$$

Remark. The linear forms $\varepsilon(f_*,\delta)$ and $f^{\mathcal{E}}_{*,M}(\delta)$ depend on implicit choices of (absolute) transfer factors. However, the effects of these choices on the two sides of (6.13) are easily seen to cancel. It follows that $\varepsilon(\delta)$ is independent of the choices, and depends on δ only as a class in $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M)$. (See [A11, §4].)

Proof. The lemma will be a simple consequence of the local trace formula, in the form of a local analogue of Lemma 2.3(b). Let $\mathcal{H}^{\text{uns}}_M(G,V)$ be the subspace of $\mathcal{H}(G,V)$ spanned by functions $f = f_1 \times \overline{f}_2$ such that both f_1 and f_2 are M-cuspidal, and such that either $f_1^G = 0$ or $f_2^G = 0$. If f belongs to this space, the expression (6.6) simplifies. Arguing as in the proof of Lemma 2.3(b), we see that $S^G(f)$ equals

$$|W(M)|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{M'\in\mathcal{E}_{\text{ell}}(M)} \iota(M,M')$$
$$\cdot \int_{\Delta_{G\text{-reg,ell}}(M')} n(\delta')^{-1}\Big(S^G_M(M',\delta',f_1)\overline{f}_2^{\overline{M}'}(\delta') + S^G_M(\overline{M}',\delta',\overline{f}_2)f_1^{M'}(\delta')\Big)d\delta'.$$

If δ is the image of (M',δ') in $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$, the last integrand equals

$$n(\delta')^{-1}\Big(\varepsilon(f_1,\delta)\overline{f}^{\mathcal{E}}_{2,M}(\overline{\delta}) + \varepsilon(\overline{f}_2,\overline{\delta})f^{\mathcal{E}}_{1,M}(\delta)\Big).$$

According to the definitions [A7, §1,3],

$$\begin{aligned}\iota(M,M')n(\delta')^{-1} &= |\mathrm{Out}_M(M')|^{-1}|Z(\widehat{M}')^\Gamma/Z(\widehat{M})^\Gamma|^{-1}|(\widehat{T}')^\Gamma/Z(\widehat{M}')^\Gamma\Big)^{-1}\\ &= |\mathrm{Out}_M(M')|^{-1}n(\delta)^{-1},\end{aligned}$$

where $T' = M_{\delta'}$, and

$$n(\delta) = |(\widehat{T}')^\Gamma/Z(\widehat{M})^\Gamma|.$$

Setting

$$c_M = |W(M)|^{-1}(-1)^{\dim(A_M/A_G)},$$

and noting that $\mathrm{Out}_M(M')$ acts freely on $\Delta_{G\text{-reg,ell}}(M')$, we see that $S^G(f)$ equals

$$c_M \int_{\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)} n(\delta)^{-1}\Big(\varepsilon(f_1,\delta)\overline{f}^{\mathcal{E}}_{2,M}(\overline{\delta}) + \varepsilon(\overline{f}_2,\overline{\delta})f^{\mathcal{E}}_{1,M}(\delta)\Big)d\delta.$$

Suppose that, in addition to the conditions above, both f_1 and f_2 are unstable. Then $f^{\mathcal{E}}_{1,M}$ and $\overline{f}^{\mathcal{E}}_{2,M}$ are both supported on the subset $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M)$ of $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$. Our expression for $S^G(f)$ reduces to

$$c_M \int_{\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M)} n(\delta)^{-1}\Big(\varepsilon(\delta) + \varepsilon(\overline{\delta})\Big)f^{\mathcal{E}}_{1,M}(\delta)\overline{f}^{\mathcal{E}}_{2,M}(\overline{\delta})d\delta. \tag{6.15}$$

Since $S^G(f)$ equals $S^G_{\mathrm{disc}}(f)$, this in turn equals the expansion

$$\int_{\Phi^{\mathcal{E}}_{\mathrm{disc}}(G,V)} s^G(\phi)f^{\mathcal{E}}_G(\phi)d\phi \tag{6.16}$$

for $S^G_{\mathrm{disc}}(f)$ given by (6.12).

It is not hard to show that the equality between (6.15) and (6.16) forces each expression to vanish. The argument is similar to that of Section 5, except simpler, since the linear forms in (6.15) and (6.16) are tempered. We shall give a brief sketch. Let f_2 be fixed, and consider (6.15) and (6.16) as linear forms on the space of functions

$$\phi_1 \longrightarrow f^{\mathcal{E}}_{1,G}(\phi_1), \qquad \phi_1 \in \Phi^{\mathcal{E}}_{\mathrm{temp}}(G),\ f_1 \in \mathcal{H}^{\mathrm{uns}}_M(G).$$

The distribution corresponding to (6.15) can be identified with a smooth function on the image of the space

$$\Phi^{\mathcal{E},0}_{\mathrm{temp,ell}}(M) = \Phi^{\mathcal{E}}_{\mathrm{temp,ell}}(M) - \Phi_{\mathrm{temp,ell}}(M)$$

in $\Phi^{\mathcal{E}}_{\mathrm{temp}}(G)$. We note that $\Phi^{\mathcal{E},0}_{\mathrm{temp,ell}}(M)$ is a disjoint union of compact tori, of dimension equal to that of A_M. The distribution attached to (6.16), on the other hand, is supported on a finite union of $i\mathfrak{a}^*_G$-orbits in $\Phi^{\mathcal{E}}_{\mathrm{temp}}(G)$. The two distributions are incompatible. Applying the usual comparison argument, we see without difficulty that each distribution equals zero. Therefore, the expressions (6.15) and (6.16) both vanish.

We have established that (6.15) vanishes for any function $f = f_1 \times \overline{f}_2$, with $f_i \in \mathcal{H}^{\mathrm{uns}}_M(G)$. The Weyl group $W(M)$ of (G, A_M) operates freely on $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M)$, the domain of integration in (6.15), and the integrand in (6.15) is invariant under this action. If α is any $W(M)$-invariant function on the Paley-Wiener space on $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M)$, we can choose f so that the function

$$f^{\mathcal{E}}_M(\delta) = f^{\mathcal{E}}_{1,M}(\delta)\overline{f}^{\mathcal{E}}_{2,M}(\overline{\delta})$$

equals $\alpha(\delta)$. It follows that the coefficients of $f^{\mathcal{E}}_M(\delta)$ in (6.15) vanish. In particular,

$$\varepsilon(\delta) + \varepsilon(\overline{\delta}) = 0.$$

The last step will be to show that $\varepsilon(\overline{\delta})$ equals $\varepsilon(\delta^{-1})$. To this end, we consider the opposition involution θ_0 of G. By definition, θ_0 is the unique automorphism of G that preserves a given F-splitting, and maps any strongly regular element x to a conjugate of x^{-1}. It follows easily from the definition that θ_0 commutes with any automorphism of G that preserves the splitting. Since G is quasisplit, this implies that θ_0 is defined over F. We reserve the symbol θ for the $G(F)$-conjugate

$$\theta = \mathrm{Int}(w_M) \circ \theta_0, \qquad w_M \in G(F),$$

of θ_0 that maps M to itself, and restricts to the opposition involution of M. Then θ is also an involution of G that is defined over F.

As in the discussion preceding [A10, Lemma 3.1], θ determines an involution θ' on the set of pairs (M', δ'), and an involution θ on $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$. From the symmetry condition of [A10, Lemma 3.1], we obtain

$$\varepsilon(\theta f_*, \theta\delta) = S^G_M(\theta' M', \theta'\delta', \theta f_*) = S^G_M(M', \delta', f_*) = \varepsilon(f_*, \delta),$$

for any $f_* \in \mathcal{H}(G)$, where $\theta f_* = f_* \circ \theta^{-1}$. Since

$$(\theta f_*)^{\mathcal{E}}_M(\theta\delta) = f^{\mathcal{E}}_{*,M}(\delta),$$

by similar considerations, we see that

$$\varepsilon(\delta) = \varepsilon(\theta\delta), \qquad \delta \in \Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M).$$

Now the dual of θ restricts to the opposition involution of $\widehat{M}$, which restricts in turn to the opposition involution of $\widehat{M}'$. It follows from the definitions that $\theta'(M', \delta')$ equals $\bigl(\overline{M}', (\delta')^{-1}\bigr)$, and therefore that $\theta\delta = (\overline{\delta})^{-1}$. We conclude that

$$\varepsilon(\overline{\delta}) = \varepsilon\Bigl((\theta\delta)^{-1}\Bigr) = \varepsilon\Bigl(\theta(\delta^{-1})\Bigr) = \varepsilon(\delta^{-1}).$$

The formula (6.1) follows. □

For later use, we recall that elements in $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$ can be represented in slightly different form. Suppose that $\delta \in \Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$ is the image of a pair (M', δ'). Suppose also that $T' \to T^*$ is an admissible embedding [LS, (1.3)] of the torus $T' = M'_{\delta'}$ into the quasisplit inner form M^* of M, and that $\delta^* \in T^*(F)$ is the corresponding image of δ'. The stable conjugacy class of δ^* in $M^*(F)$ (which we can also denote by δ^*) then depends only on δ. The endoscopic datum M' also gives a second piece of information. It provides an element s'_M in $\widehat{T}'$, which can be pulled back under the dual mapping $\widehat{T}^* \to \widehat{T}'$ to an element κ^* in the group

$$\mathcal{K}(M^*_{\delta^*}) = \mathcal{K}(T^*) = \pi_0\Big((\widehat{T}^*)^{\Gamma}/Z(\widehat{M})^{\Gamma}\Big), \qquad \Gamma = \mathrm{Gal}(\overline{F}/F).$$

We have thus a correspondence

$$(M', \delta') \longrightarrow (\delta^*, \kappa^*).$$

If $M^*_{G\text{-reg,ell}}(F)$ denotes the set of G-regular, elliptic elements in $M^*(F)$, we write $D^{\mathcal{E}}_{G\text{-reg,ell}}(M)$ for the quotient of the set

$$\Big\{(\delta^*, \kappa^*) : \ \delta^* \in M^*_{G\text{-reg,ell}}(F), \ \kappa^* \in \mathcal{K}(M^*_{\delta^*})\Big\}$$

defined by stable conjugacy in $M^*(F)$. The correspondence $(M', \delta') \to (\delta^*, \kappa^*)$ then determines a canonical bijection

$$\delta \longrightarrow d, \qquad \delta \in \Delta^{\mathcal{E}}_{G\text{-reg,ell}}(F),$$

from $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$ onto $D^{\mathcal{E}}_{G\text{-reg,ell}}(M)$.

The bijection $\delta \to d$ was part of the proof of [A7, Lemma 2.2] and [A10, Lemma 2.3]. We shall use it in Section 7 in conjunction with a fixed elliptic torus $T \subset M$ over F. Any such T can be mapped to a maximal torus $T^* \subset M^*$ over F by the inverse of an admissible isomorphism i: $T^* \to T$ [K2, §9]. Since i is unique up to stable conjugacy, we can thereby identify any stable conjugacy class in $M^*_{G\text{-reg,ell}}(F)$ that intersects $T^*(F)$ with an orbit in $T_{G\text{-reg}}(F)$ under the rational Weyl group $W_F(M,T)$ of (M,T). The quotient of the set

$$\Big\{(t, \kappa) : \ t \in T_{G\text{-reg}}(F), \ \kappa \in \mathcal{K}(T)\Big\}$$

by $W_F(M,T)$ represents in this way a subset of $D^{\mathcal{E}}_{G\text{-reg,ell}}(M)$. If t belongs to $T_{G\text{-reg}}(F)$, let $\mathcal{F}(t)$ be the set of elements in $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$ whose image in $D^{\mathcal{E}}_{G\text{-reg,ell}}(M)$ can be represented by a pair of the form (t, κ). There is then a canonical bijection

$$\delta \longrightarrow \kappa(\delta), \qquad \delta \in \mathcal{F}(t),$$

from $\mathcal{F}(t)$ onto $\mathcal{K}(T)$. One observes that an element $\delta \in \mathcal{F}(t)$ belongs to the subset $\Delta_{G\text{-reg,ell}}(M)$ of $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$ if and only if $\kappa(\delta) = 1$. Moreover, for any such δ, $\overline{\delta}$ is the element in $\mathcal{F}(t)$ with $\kappa(\overline{\delta}) = \kappa(\delta)^{-1}$, and δ^{-1} is the element in $\mathcal{F}(t^{-1})$ with $\kappa(\delta^{-1}) = \kappa(\delta)$.

7. Local Theorem 1

We have reached the critical stage of our extended induction argument. We recall that the induction hypotheses were stated formally at the end of Section 1, in terms of two fixed positive integers d_{der} and r_{der}. In the next two sections, we shall prove Local Theorem 1. This will take care of the part of the induction argument that depends on r_{der}. We shall establish Global Theorems 1 and 2 in Section 9, thereby completing the induction argument.

The setting will be that of the latter part of Section 6. Then G is a connected reductive group over the p-adic local field F that satisfies Assumption 5.2(2) of [I], with $\dim(G_{\mathrm{der}}) = d_{\mathrm{der}}$. Furthermore, M is a fixed Levi subgroup of G with $\dim(A_M \cap G_{\mathrm{der}}) = r_{\mathrm{der}}$. Since r_{der} is positive, M is proper in G. We have finished our discussion of the local trace formula. We can therefore allow f to stand for a function on $G(F)$, as in the statement of Local Theorem 1, rather than on $G(F) \times G(F)$ (as in the last section). Our goal is to prove Local Theorem 1 for G.

The discussion will be simpler if we do not have to deal with central data.

LEMMA 7.1. *Assume that Local Theorem* 1 *is valid under the restriction that* G_{der} *is simply connected and* $Z = 1$. *Then it is also valid without this restriction.*

Proof. The proof is similar to that of Proposition 2.1 of [II]. It is actually simpler, since we are working in a local context, with elements whose centralizers are connected. We shall therefore be brief.

The first step is to reduce Local Theorem 1 to the case that G_{der} is simply connected. Given G and M, let $\widetilde{G}$ be a z-extension of G [K1, §1], and let $\widetilde{M}$ be the preimage of M in $\widetilde{G}$. Then $\widetilde{G}$ is a central extension of G by an induced torus $\widetilde{C}$ over F such that $\widetilde{G}_{\mathrm{der}}$ is simply connected, and $\widetilde{M}$ is a Levi subgroup of $\widetilde{G}$. The pair $(\widetilde{G}, \widetilde{M})$ then satisfies the conditions we imposed on (G, M) above. We write $\widetilde{Z}$ for the preimage of Z in $\widetilde{G}$, and $\widetilde{\zeta}$ for the pullback of ζ to $\widetilde{Z}(F)$. We have to check that if Local Theorem 1 holds for $\widetilde{G}$, $\widetilde{M}$, $\widetilde{Z}$, and $\widetilde{\zeta}$, then it is also valid for G, M, Z and ζ.

Recall [K1, §1] that $G(F) \cong \widetilde{G}(F)/\widetilde{C}(F)$. We can therefore identify functions (or distributions) on $G(F)$ with functions (or distributions) on $\widetilde{G}(F)$ that are invariant under translation by $\widetilde{C}(F)$. In particular, there is a canonical isomorphism $f \to \widetilde{f}$ from $\mathcal{H}(G, \zeta)$ onto $\mathcal{H}(\widetilde{G}, \widetilde{\zeta})$. We can also assume that the fixed bases $\Gamma_{G\text{-reg,ell}}(\widetilde{M}, \widetilde{\zeta})$, $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(\widetilde{M}, \widetilde{\zeta})$, etc., of $\widetilde{\zeta}$-equivariant distributions for $\widetilde{G}$ are the images of the corresponding bases $\Gamma_{G\text{-reg,ell}}(M, \zeta)$, $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M, \zeta)$, etc., for M under the canonical maps $\gamma \to \widetilde{\gamma}$, $\delta \to \widetilde{\delta}$, etc., of distributions. It follows from the definitions that

$$I_M(\gamma, f) = I_{\widetilde{M}}(\widetilde{\gamma}, \widetilde{f}), \qquad \gamma \in \Gamma_{G\text{-reg,ell}}(M, \zeta),\ f \in \mathcal{H}(G, \zeta).$$

The endoscopic and stable analogues of these distributions satisfy similar formulas. As in the proof of [II, Prop. 2.1], we obtain identities

$$I^{\mathcal{E}}_M(\gamma, f) = I^{\mathcal{E}}_{\widetilde{M}}(\widetilde{\gamma}, \widetilde{f})$$

and also

$$S^G_M(M', \delta', f) = S^{\widetilde{G}}_{\widetilde{M}}(\widetilde{M}', \widetilde{\delta}', \widetilde{f}), \qquad M' \in \mathcal{E}_{\mathrm{ell}}(M),\ \delta' \in \Delta_{G\text{-reg,ell}}(\widetilde{M}', \widetilde{\zeta}'),$$

in the case that G is quasisplit. (The last identity is really a tautology, since we can take $\delta' = \widetilde{\delta}'$.) It follows from these formulas that the assertions of Local Theorem 1 are valid for G, M, Z, and ζ if they hold for $\widetilde{G}$, $\widetilde{M}$, $\widetilde{Z}$ and $\widetilde{\zeta}$.

The second step is to reduce Local Theorem 1 to the case that Z is trivial. Given G, M, Z and ζ, we define a projection

$$f \longrightarrow f^{\zeta} = \int_{Z(F)} \zeta(z) f_z dz, \qquad f \in \mathcal{H}(G),$$

from $\mathcal{H}(G)$ onto $\mathcal{H}(G, \zeta)$, where $f_z(x) = f(zx)$ for any $x \in G(F)$. We have to compare linear forms on $\mathcal{H}(G)$ at a given function f with the values of corresponding linear forms on $\mathcal{H}(G, \zeta)$ at f^{ζ}.

Suppose that γ_ζ belongs to the fixed basis $\Gamma_{G\text{-reg,ell}}(M, \zeta)$ of ζ-equivariant distributions on $M(F)$ [I, §1], and that $\gamma \in \Gamma_{G\text{-reg,ell}}(M)$ is a conjugacy class that maps to γ_ζ. We can then compare the orbital integral on $\mathcal{H}(G, \zeta)$ at γ_ζ with the orbital integral on $\mathcal{H}(G)$ at γ. The relation is

$$f^{\zeta}_G(\gamma_\zeta) = (\gamma/\gamma_\zeta)^{-1} \int_{Z(F)} f_{z,G}(\gamma) \zeta(z) dz, \qquad f \in \mathcal{H}(G),$$

where (γ/γ_ζ) is the ratio of the given invariant measure on γ with the signed measure that comes with γ_ζ. A similar relation holds for weighted orbital integrals, and the associated invariant distributions. It follows directly from the definitions that

$$I_M(\gamma_\zeta, f^{\zeta}) = (\gamma/\gamma_\zeta)^{-1} \int_{Z(F)} I_M(\gamma, f_z) \zeta(z) dz. \tag{7.1}$$

If we combine this with the discussion at the end of [I, §4] and the definitions in [I, §6], we see that

$$I^{\mathcal{E}}_M(\gamma_\zeta, f^{\zeta}) = (\gamma/\gamma_\zeta)^{-1} \int_{Z(F)} I_M(\gamma, f_z) \zeta(z) dz. \tag{7.2}$$

In the case that G is quasisplit, we also obtain

$$S^G_M(M', \delta'_\zeta, f^{\zeta}) = (\delta'/\delta'_\zeta)^{-1} \int_{Z(F)} S^G_M(M', \delta', f_z) \zeta(z) dz, \tag{7.3}$$

for $M \in \mathcal{E}_{\mathrm{ell}}(M)$ and $\delta'_\zeta \in \Delta_{G\text{-reg,ell}}(\widetilde{M}', \widetilde{\zeta}')$, and for an element $\delta' \in \Delta_{G\text{-reg,ell}}(\widetilde{M}', \widetilde{\eta}')$ that maps to δ'_ζ. The ratio (δ'/δ'_ζ) is defined in the obvious way [I, (1.6)]. The general assertions of Local Theorem 1 apply to the distributions on the left-hand sides of (7.1)–(7.3). The corresponding assertions for the case that Z is trivial apply to the distributions on the right-hand sides. It follows from these formulas that Local Theorem 1 hold for arbitrary (Z, ζ) if it is valid in the case that Z is trivial. This gives the second reduction, and completes the proof of the lemma. $\square$

We have reduced the proof of Local Theorem 1 for G, M, Z and ζ to the case that G_{der} is simply connected and Z is trivial. We assume from now on that these conditions hold. In particular, the role of the general basis $\Gamma_{G\text{-reg,ell}}(M, \zeta)$ can be taken simply by the family $\Gamma_{G\text{-reg,ell}}(M)$ of strongly G-regular, elliptic conjugacy classes in $M(F)$. Moreover, if M' belongs to $\mathcal{E}_{\mathrm{ell}}(M)$, we can replace the basis $\Delta_{G\text{-reg,ell}}(\widetilde{M}', \widetilde{\zeta}')$ by the corresponding family $\Delta_{G\text{-reg,ell}}(M')$ of stable conjugacy classes in $M'(F)$. This is because the derived group of M is also simply connected, so there exists an admissible embedding ${}^L M' \to {}^L M$ of L-groups [L4]. Elliptic conjugacy classes of course meet elliptic maximal tori. It will be convenient to let T denote an arbitrary, but fixed elliptic maximal torus in M. We will then work with those classes that have representatives in $T_{G\text{-reg}}(F)$. This was the setting at the end of Section 6.

The proof of Local Theorem 1 will be global. We shall use all the global information we accumulated over the first half of the paper. The local objects F, G, M and T have been fixed. They are assumed implicitly to have been equipped with a quasisplit inner twist

$$\psi : \ (G, M) \longrightarrow (G^*, M^*),$$

by which we mean an M^*-inner class of isomorphisms from (G, M) to a quasisplit pair (G^*, M^*). We are also going to fix a suitable finite Galois extension E of F, over which G, M and T split. Given E, we propose to choose global objects corresponding to the components of the local datum (F, E, G, M, T, ψ). We shall denote these by the same symbols, but augmented as in [A7, §7–9] by a dot on top. Thus, $(\dot{F}, \dot{E}, \dot{G}, \dot{M}, \dot{T}, \dot{\psi})$ stands for the following set of objects: a finite Galois extension $\dot{F} \subset \dot{E}$ of number fields, a trio of connected reductive groups

$$\dot{T} \subset \dot{M} \subset \dot{G}$$

over $\dot{F}$ that split over $\dot{E}$, with $\dot{M}$ being a Levi subgroup in $\dot{G}$ and $\dot{T}$ an elliptic maximal torus in $\dot{M}$, and a quasisplit inner twist

$$\dot{\psi} : \ (\dot{G}, \dot{M}) \longrightarrow (\dot{G}^*, \dot{M}^*).$$

If v is any valuation of $\dot{F}$ that lies in the set $V_{\mathrm{fin}}(\dot{G}, \dot{M})$, and for which $\dot{E}_v$ is a field, the completion $(\dot{F}_v, \dot{E}_v, \dot{G}_v, \dot{M}_v, \dot{T}_v, \dot{\psi}_v)$ is a local datum of the kind we started with. In particular, it makes sense to speak of an isomorphism from

(F, E, G, M, T, ψ) to such a completion. Any such isomorphism would of course map the Galois group $\Gamma = \mathrm{Gal}(E/F)$ isomorphically onto the decomposition group $\dot{\Gamma}_v = \mathrm{Gal}(\dot{E}_v/\dot{F}_v)$ of $\dot{\Gamma} = \mathrm{Gal}(\dot{E}/\dot{F})$.

LEMMA 7.2. *We can choose E and $(\dot{F}, \dot{E}, \dot{G}, \dot{M}, \dot{T}, \dot{\psi})$, together with isomorphisms*

$$\phi_u : (F, E, G, M, T, \psi) \longrightarrow (\dot{F}_u, \dot{E}_u, \dot{G}_u, \dot{M}_u, \dot{T}_u, \dot{\psi}_u), \qquad u \in U, \tag{7.4}$$

for a finite set U of p-adic valuations $\{u\}$ of $\dot{F}$ such that $\dot{E}_u$ is a field, with the following properties.

(i) *$(\dot{G}, \dot{F})$ satisfies Assumption 5.2(1) of [I].*

(ii) *If G is quasisplit over F, $\dot{G}$ is quasisplit over $\dot{F}$.*

(iii) *For any valuation $v \notin U$, $\dot{G}_v$ is quasisplit over $\dot{F}_v$.*

(iv) *$|U| \geq 3$.*

(v) *There is a place $v \notin U$ such that $\dot{E}_v$ is a field.*

Proof. The lemma is a simple exercise in the approximation of local data by global data. A less elaborate version, with some details omitted, was given in [A7, pp. 576–577]. In the discussion here, we shall make use of [I, Lemma 5.3], which asserts that the global form (1) and the local form (2) of Assumption 5.2 of [I] both remain valid under inner twists of the group, and under finite extensions of the ground field.

The local pair (G, F) satisfies Assumption 5.2(2) of [I]. This implies that (G^*, F) is isomorphic to a completion $(\dot{G}^*_u, \dot{F}_u)$, for a quasisplit global pair $(\dot{G}^*, \dot{F})$ that satisfies Assumption 5.2(1). Let $\dot{E}$ be a finite Galois extension of $\dot{F}$ such that $\dot{G}^*$ splits over $\dot{E}$, and such that T splits over the completion of $\dot{E}$ defined by the valuation u of $\dot{F}$. Replacing $\dot{F}$ by the fixed field of a decomposition group in $\mathrm{Gal}(\dot{E}/\dot{F})$ over u, we can assume that $E = \dot{E}_u$ is a field. Then E is a finite Galois extension of F over which G^* and T split. Moreover, the associated Galois groups satisfy

$$\mathrm{Gal}(E/F) \cong \mathrm{Gal}(\dot{E}_u/\dot{F}_u) \cong \mathrm{Gal}(\dot{E}/\dot{F}).$$

It follows easily that there is a Levi subgroup $\dot{M}^*$ of $\dot{G}^*$ over $\dot{F}$, and an isomorphism

$$\phi^*_u : (F, E, G^*, M^*) \longrightarrow (\dot{F}_u, \dot{E}_u, \dot{G}^*_u, \dot{M}^*_u).$$

The local field $E \supset F$ will be the required extension. However, we shall still have to modify the global fields $\dot{E} \supset \dot{F}$ in order to accommodate the extra conditions.

Since M^* is quasisplit over F, the torus $T \subset M$ transfers to M^*. More precisely, we can find a maximal torus $T^* \subset M^*$ over F, together with an isomorphism i: $T^* \to T$ over F that is admissible in the sense of [K2, §9], which is to say that i is M-conjugate to the restriction of ψ^{-1} to T^*. Let $\dot{M}^*_u(T)$ be the set of elements in $\dot{M}^*_u = \dot{M}^*(\dot{F}_u)$ that are $\dot{M}^*_u$-conjugate to $\phi^*_u\Big(T^*(F)\Big)$. The set of strongly G-regular points in $\dot{M}^*_u(T)$ is open in $\dot{M}^*_u$, and intersects any open neighbourhood of 1 in $\dot{M}^*_u$ in a nonempty open set. Since the closure of $\dot{M}^*(\dot{F})$ in $\dot{M}^*_u$ contains an open neighbourhood of 1 [KR, Lemma 1(a)], $\dot{M}^*(\dot{F})$ intersects the set of strongly G-regular points in $\dot{M}^*_u(T)$. Let $\dot{T}^*$ be the centralizer in $\dot{M}^*$ of any point in this intersection. Then $\dot{T}^*$ is a maximal torus in $\dot{M}^*$ over $\dot{F}$ that is $\dot{M}^*_u$-conjugate to $\phi^*_u(T^*)$. Replacing ϕ^*_u with an M^*_u-conjugate, we can assume that ϕ^*_u takes T^* to $\dot{T}^*$.

The torus $\dot{T}^*$ need not split over $\dot{E}$. However, it does split over the completion $\dot{E}_u$. We can therefore find a finite Galois extension $\dot{E}'$ of $\dot{F}$ over which $\dot{T}$ splits, and which embeds in $\dot{E}_u$. Replacing $\dot{E}'$ by the composite $\dot{E}'\dot{E}$, if necessary, we can also assume that $\dot{E}'$ contains $\dot{E}$. If u' is the valuation in $\dot{E}'$, obtained from an embedding of $\dot{E}'$ into $\dot{E}_u$, the decomposition group for $\dot{E}'/\dot{F}$ at u' is a subgroup of $\mathrm{Gal}(\dot{E}'/\dot{F})$ that is isomorphic to $\mathrm{Gal}(E/F)$. Let $\dot{F}' \subset \dot{E}'$ be the fixed field of this subgroup. The associated valuation u' on $\dot{F}'$ has the property that $\dot{E}'_{u'}$ is a field such that $\dot{E}'_{u'}/\dot{F}'_{u'}$ is isomorphic to E/F. Replacing $\dot{F} \subset \dot{E}$ by $\dot{F}' \subset \dot{E}'$, if necessary, we can assume that $\dot{T}$ does split over $\dot{E}$.

We have constructed quasisplit global objects, and an isomorphism

$$\phi^*_u: \ (F, E, G^*, M^*, T^*) \longrightarrow (\dot{F}_u, \dot{E}_u, \dot{G}^*_u, \dot{M}^*_u, \dot{T}^*_u). \tag{7.5}$$

It is easy to modify the construction so that there are several such isomorphisms. Let $\dot{F}''$ be a large finite extension of $\dot{F}$ in which u splits completely, and let $\dot{E}''$ be a composite of $\dot{E}$ with $\dot{F}''$. If u'' is any valuation on $\dot{F}''$ over u, $\dot{E}''_{u''}$ is a field such that $\dot{E}''_{u''}/\dot{F}''_{u''}$ is isomorphic to E/F. Replacing $\dot{F} \subset \dot{E}$ by $\dot{F}''/\dot{E}''$, if necessary, we can assume that there are isomorphisms (7.5) for each u in an arbitrarily large finite set U^+ of p-adic valuations on $\dot{F}$.

The local inner twist

$$\psi: \ (G, M) \longrightarrow (G^*, M^*)$$

determines an element α_G in the image of $H^1(F, M^* \cap G^*_{\mathrm{ad}})$ in $H^1(F, G^*_{\mathrm{ad}})$. Recall that there is a canonical bijection from $H^1(F, G^*_{\mathrm{ad}})$ onto the finite abelian group $\pi_0\Big(Z(\widehat{G}_{\mathrm{sc}})^\Gamma\Big)^*$ [K3, Th. 1.2]. Let n_G be the order of the image of α_G in $\pi_0\Big(Z(\widehat{G}_{\mathrm{sc}})^\Gamma\Big)^*$. We take U to be any proper subset of U^+, with $|U| \geq 3$, such that n_G divides $|U|$. The element

$$\bigoplus_{u \in U} \phi^*_u(\alpha_G)$$

then lies in the kernel of the composition of maps

$$\bigoplus_{u\in U} H^1(\dot F_u, \dot G^*_{u,\mathrm{ad}}) \xrightarrow{\sim} \bigoplus_{u\in U} \pi_0\Big(Z(\widehat G^*_{\mathrm{sc}})^{\dot\Gamma_u}\Big)^* \longrightarrow \pi_0\Big(Z(\widehat G^*_{\mathrm{sc}})^{\Gamma}\Big)^*.$$

According to [K3, Th. 2.2, Cor. 2.5], we can build a global inner form of $\dot G^*$ from the local inner forms of $\{\dot G^*_u:\ u\in U\}$ associated to the classes $\{\phi^*_u(\alpha_G)\}$. More precisely, taking [A10, Lemma 2.1] and [I, Lemma 4.1] into consideration, we see that we can find a global inner twist

$$\dot\psi:\ (\dot G, \dot M) \longrightarrow (\dot G^*, \dot M^*),$$

where $\dot G$ is a reductive group over $\dot F$ with Levi subgroup $\dot M$, together with isomorphisms

$$\phi_u:\ (F, E, G, M) \longrightarrow (\dot F_u, \dot E_u, \dot G_u, \dot M_u), \qquad u\in U,$$

such that each map $\phi^*_u \circ \psi$ is $\dot M^*_u$-conjugate to $\dot\psi_u \circ \phi_u$. It is clear that $\dot G$ is quasisplit over $\dot F$ if G is quasisplit over F, and that $\dot G_v$ is quasisplit over $\dot F_v$ in general, for each $v \notin U$.

The last point is to transfer the maximal torus $\dot T^*$ of $\dot M^*$ to a maximal torus $\dot T$ of $\dot M$. For each $v\in U$, the map

$$i_v = \phi_v \circ i \circ (\phi^*_v)^{-1}:\ \dot T^*_v \longrightarrow \dot M_v$$

is admissible, and in particular, is defined over $\dot F_v$. We may as well also fix admissible maps i_v: $\dot T^*_v \to \dot M_v$ for the valuations v in the complement of U, subject to the natural conditions [K2, (9.2.1)] at the unramified places. Since $\dot M_v$ is quasisplit for each such v, this is possible. We seek an admissible global embedding

$$j:\ \dot T^* \longrightarrow \dot M$$

over $\dot F$ that is $\dot M_v$-conjugate to i_v for each v. There is a general obstruction to the existence of such an embedding, which is defined in [K2, §9] as an element in the dual of the finite abelian group $\mathcal K(\dot T^*)$. The group $\mathcal K(\dot T^*)$, taken relative to $\dot M^*$, is defined to be the subgroup of elements in $\pi_0\Big(\big(\widehat T^*/Z(\widehat M^*)\big)^{\dot\Gamma}\Big)$ whose image in $H^1\big(\dot F, Z(\widehat M^*)\big)$ is locally trivial. If v belongs to U^+, it follows from the fact that $\dot E_v$ is a field that

$$\mathcal K(\dot T^*) = \pi_0\Big((\widehat T^*_v)^{\dot\Gamma_v}/Z(\widehat M^*_v)^{\dot\Gamma_v}\Big) = \mathcal K(\dot T^*_v).$$

The local group $\mathcal K(\dot T^*_v)^*$ acts simply transitively on the set of $\dot M_v$-conjugacy classes of admissible embeddings i_v. We are certainly free to modify i_v at any v outside U. Replacing i_v by its image under the appropriate element in $\mathcal K(\dot T^*_v)^*$, for some v in the complement of U in U^+, we can assume that the global obstruction vanishes. We then obtain a global embedding j that maps $\dot{T}^*$ to a maximal torus $\dot{T}$ in $\dot{M}$. The torus $\dot{T}$ over $\dot{F}$ provides the last component of the global datum $(\dot{F}, \dot{E}, \dot{G}, \dot{M}, \dot{T}, \dot{\psi})$. Replacing ϕ_u by some $\dot{M}_u$-conjugate, for each $u \in U$, we can assume that ϕ_u maps T to $\dot{T}_u$. These maps become the required isomorphisms (7.4). □

We fix the various objects provided by the lemma. We also fix a place $u_0 \in U$, and use the isomorphism ϕ_{u_0} to identify the local datum (F, E, G, M, T, ψ) with the completion of $(\dot{F}, \dot{E}, \dot{G}, \dot{M}, \dot{T}, \dot{\psi})$ at u_0. Let V be a finite set of valuations that contains U, and also all the ramified places for $\dot{E}$, $\dot{G}$ and $\dot{T}$. If f is a given function in $\mathcal{H}(G)$, we choose a function

$$\dot{f} = \prod_{v \in V} \dot{f}_v$$

in $\mathcal{H}(G_V)$ such that $\dot{f}_{u_0} = f$. Since $|U| \geq 3$, we can fix two other places u_1 and u_2 in U. We assume that if v equals u_1 or u_2, the function $\dot{f}_v$ is supported on the open subset of elements in $\dot{G}_v$ that are stably conjugate to points in $\dot{T}_{G\text{-reg}}(\dot{F}_v)$. Then $\dot{f}$ belongs to the space $\mathcal{H}_{\dot{M}}(\dot{G}_V)$, which was introduced in the context of global K-groups in Section 2. The connected group $\dot{G}$ is a component of an (essentially) unique global K-group [I, §4], which by [I, Lemma 5.3] and the proof of the last lemma we may assume satisfies Assumption 5.2 of [I]. We can regard $\dot{f}$ as a function on the K-group that is supported on $\dot{G}_V$. The various global results of Sections 2–5 therefore makes sense for $\dot{f}$. We shall apply them to our study of the relevant linear forms in f.

As always, we have to separate the discussion into the two parts (a) and (b). Recall that we are trying to prove Local Theorem 1 for (G, M). The assertion (a) of the theorem is trivial if G is quasisplit, while assertion (b) applies only to this case. We may as well then treat (a) and (b) as two disjoint cases, corresponding respectively to whether G is not, or is, quasisplit over F. This corresponds in turn to whether $\dot{G}$ is not, or is, quasisplit over $\dot{F}$.

To deal with (a), we shall apply the formula (2.4) of Proposition 2.2(a). We first recall that our function $\dot{f} \in \mathcal{H}_M(\dot{G}_V)$ satisfies an identity

$$I^{\mathcal{E}}_{\nu,\mathrm{disc}}(\dot{f}) - I_{\nu,\mathrm{disc}}(\dot{f}) = 0, \qquad \nu \in i\mathfrak{h}^*_u/W_\infty,$$

by Corollary 5.2(a). This implies that the term

$$\sum_t \Big(I^{\mathcal{E}}_{t,\mathrm{disc}}(\dot{f}) - I_{t,\mathrm{disc}}(\dot{f}) \Big) = \sum_\nu \Big(I^{\mathcal{E}}_{\nu,\mathrm{disc}}(\dot{f}) - I_{\nu,\mathrm{disc}}(\dot{f}) \Big)$$

on the right-hand side of (2.4) vanishes. We also note that $\dot{f}$ vanishes on an invariant neighbourhood of the center of $\dot{G}_V$, since the corresponding property holds for $\dot{f}_{u_1}$ and $\dot{f}_{u_2}$. Therefore the other term

$$\sum_z \Big(I^{\mathcal{E}}_{z,\mathrm{unip}}(\dot{f}, S) - I_{t,\mathrm{unip}}(\dot{f}, S) \Big)$$

on the right-hand side of (2.4) vanishes as well. It follows that the left-hand side

$$I^{\mathcal{E}}_{\mathrm{par}}(\dot{f}) - I_{\mathrm{par}}(\dot{f})$$

of (2.4) equals zero. Applying the expansion (2.8) of Lemma 2.3(a) for this linear form, we see that

$$\sum_{v\in V_{\mathrm{fin}}(\dot{G},\dot{M})}\ \sum_{\dot{\gamma}\in\Gamma(\dot{M},V)} a^{\dot{M}}(\dot{\gamma})\Big(I^{\mathcal{E}}_{\dot{M}}(\dot{\gamma}_v,\dot{f}_v) - I_{\dot{M}}(\dot{\gamma}_v,\dot{f}_v)\Big)\dot{f}^v_{\dot{M}}(\dot{\gamma}^v) = 0. \tag{7.6}$$

The left-hand side of (7.6) can be identified with the expansion [I, (2.11)] of the linear form $I^{\dot{M}}_{\mathrm{orb}}(\dot{h})$, for some function $\dot{h}\in\mathcal{H}(\dot{M}_V)$. Since $\dot{M}$ is a proper Levi subgroup, our induction hypotheses imply that

$$I^{\dot{M}}_{\mathrm{orb}}(\dot{h}) = I^{\dot{M},\mathcal{E}}_{\mathrm{orb}}(\dot{h}) = \sum_{\dot{M}'\in\mathcal{E}(\dot{M},V)} \iota(\dot{M},\dot{M}')\widehat{S}^{\dot{M}'}_{\mathrm{orb}}(\dot{h}').$$

Given the expansion for $\widehat{S}^{\dot{M}'}_{\mathrm{orb}}(\dot{h}')$ in [I, Lemma 7.2(b)], together with the induction hypothesis (1.4) that the function $b^{\dot{M}'}(\dot{\delta}')$ is supported on the subset $\Delta(\dot{M}',V)$ of $\Delta^{\mathcal{E}}(\dot{M}',V)$, we can then rewrite the left-hand side of (7.6) as an expansion in terms of $\dot{M}'$ and $\dot{\delta}'$. We conclude that

(7.7)

$$\sum_{\dot{M}'\in\mathcal{E}_{\mathrm{ell}}(\dot{M},V)} \iota(\dot{M},\dot{M}') \sum_{v\in V_{\mathrm{fin}}(\dot{G},\dot{M})}\ \sum_{\dot{\delta}'\in\Delta(\dot{M}',V)} b^{\dot{M}'}(\dot{\delta}')\varepsilon_{\dot{M}}(\dot{f}_v,\dot{\delta}'_v)(\dot{f}^v)^{\dot{M}'}\Big((\dot{\delta}')^v\Big) = 0,$$

where

$$\varepsilon_{\dot{M}}(\dot{f}_v,\dot{\delta}'_v) = \sum_{\dot{\gamma}_v\in\Gamma(\dot{M}_v)} \Delta(\dot{\delta}'_v,\dot{\gamma}_v)\Big(I^{\mathcal{E}}_{\dot{M}}(\dot{\gamma}_v,\dot{f}_v) - I_{\dot{M}}(\dot{\gamma}_v,\dot{f}_v)\Big),$$

for any element $\dot{\delta}'_v$ in $\Delta(\dot{M}'_v)$. (We cannot actually claim that the function

$$\dot{\gamma} \longrightarrow \sum_v \Big(I^{\mathcal{E}}_{\dot{M}}(\dot{\gamma}_v,\dot{f}_v) - I_{\dot{M}}(\dot{\gamma}_v,\dot{f}_v)\Big)\dot{f}^v_{\dot{M}}(\dot{\gamma}^v), \qquad \dot{\gamma}\in\Gamma_{G\text{-reg}}(\dot{M}_V),$$

belongs to $\mathcal{I}(\dot{M}_V)$, since $\dot{f}$ is not required to lie in $\mathcal{H}^0(G_V)$. However, the conditions at u_1 and u_2 allow us to truncate the function near the singular set without changing the value of the left-hand side of (7.6). Alternatively, one can simply note that the proof of [I, Lemma 7.2] is formal, and does not require that the underlying function lie in the Hecke space.)

For the second case (b), in which $\dot{G}$ is quasisplit, we have to impose the extra condition that $\dot{f}_v$ be unstable for some v. The function $\dot{f}$ then lies in $\mathcal{H}^{\mathrm{uns}}_{\dot{M}}(\dot{G}_V)$. In this case, we apply the formula (2.5) of Proposition 2.2(b). It follows from Corollary 5.2(b) that the term

$$\sum_t S^{\dot{G}}_{t,\mathrm{disc}}(\dot{f}) = \sum_\nu S^{\dot{G}}_{\nu,\mathrm{disc}}(\dot{f})$$

on the right-hand side of (2.5) vanishes. Since $\dot{f}$ vanishes on an invariant neighbourhood of the center of $\dot{G}_V$, the other term

$$\sum_z S^{\dot{G}}_{z,\mathrm{unip}}(\dot{f}, S)$$

on the right-hand side of (2.5) also vanishes. Therefore the left-hand side

$$S^{\dot{G}}_{\mathrm{par}}(\dot{f})$$

of (2.5) equals zero. Applying the expansion (2.9) of Lemma 2.3(b) for this linear form, we see that

(7.8)

$$\sum_{\dot{M}'\in\mathcal{E}_{\mathrm{ell}}(\dot{M},V)} \iota(\dot{M},\dot{M}') \sum_{v\in V_{\mathrm{fin}}(\dot{G},\dot{M})} \sum_{\dot{\delta}'\in\Delta(\dot{M}',V)} b^{\dot{M}'}(\dot{\delta}')\varepsilon^{\dot{M}'}(\dot{f}_v,\dot{\delta}'_v)(\dot{f}^v)^{\dot{M}'}\Big((\dot{\delta}')^v\Big) = 0,$$

where

$$\varepsilon^{\dot{M}'}(\dot{f}_v,\dot{\delta}'_v) = S^{\dot{G}}_{\dot{M}}(\dot{M}'_v,\dot{\delta}'_v,\dot{f}_v).$$

The formulas (7.7) and (7.8), corresponding to the two cases (a) and (b), are almost identical. We shall analyze them together. Suppose that in addition to the conditions we have already imposed, the function $\dot{f}$ is admissible in the sense of [I, §1]. The summands in (7.7) and (7.8) are then supported on classes $\dot{\delta}'$ that are admissible. This means that we can take $S = V$ in the expansion [I, (10.11)] for $b^{\dot{M}'}(\dot{\delta}')$. It is then a consequence of the definitions that the right-hand side of [I, (10.11)] vanishes. Therefore, the global coefficient $b^{\dot{M}'}(\dot{\delta}')$ in (7.7) and (7.8) reduces to the more elementary "elliptic" coefficient $b^{\dot{M}'}_{\mathrm{ell}}(\dot{\delta}')$. We shall apply the global descent formula [II, Cor. 2.2(b)] to this latter coefficient.

We can assume that the summands in (7.7) and (7.8) corresponding to a given $\dot{\delta}' \in \Delta(\dot{M}',V)$ are nonzero. It follows from the conditions on $\dot{f}_{u_1}$ and $\dot{f}_{u_2}$, and the global descent formula for $b^{\dot{M}'}_{\mathrm{ell}}(\dot{\delta}')$, that $\dot{\delta}'$ belongs to the subset $\Delta_{G\text{-reg,ell},V}(\dot{M}')$ of elements in $\Delta(\dot{M}',V)$ that lie in $\Delta_{G\text{-reg,ell}}(\dot{M}')$, and are V-admissible. (Recall that $\Delta_{G\text{-reg,ell}}(\dot{M}')$ denotes the set of strongly G-regular, elliptic stable conjugacy classes in $\dot{M}'(\dot{F})$, and can be identified with a subset of $\Delta(\dot{M}',V)$.) Since $\dot{\delta}'$ is strongly regular, the global descent formula is very simple. We obtain

$$b^{\dot{M}'}_{\mathrm{ell}}(\dot{\delta}') = j^{\dot{M}'}(V,\dot{\delta}')b^{\dot{T}'}_{\mathrm{ell}}(1) = \tau(\dot{M}')\tau(\dot{T}')^{-1}\tau(\dot{T}') = \tau(\dot{M}'),$$

where $\dot{T}'$ is the centralizer of $\dot{\delta}'$ in $\dot{M}'$. It follows from the formula [K2, Th. 8.3.1] for $\iota(\dot{M},\dot{M}')$ that

$$\iota(\dot{M},\dot{M}')b^{\dot{M}'}_{\mathrm{ell}}(\dot{\delta}') = \tau(\dot{M})\tau(\dot{M}')^{-1}|\mathrm{Out}_{\dot{M}}(\dot{M}')|^{-1}\tau(\dot{M}') = \tau(\dot{M})|\mathrm{Out}_{\dot{M}}(\dot{M}')|^{-1}.$$

The Tamagawa number $\tau(\dot{M})$ is nonzero, and is of course independent of $\dot{M}'$ and $\dot{\delta}'$. Moreover, the group $\mathrm{Out}_{\dot{M}}(\dot{M}')$ acts freely on the set of pairs

$$\Big\{(\dot{M}',\dot{\delta}'):\ \dot{M}'\in\mathcal{E}_{\mathrm{ell}}(\dot{M}),\ \dot{\delta}'\in\Delta_{G\text{-reg,ell}}(\dot{M}')\Big\}$$

that are relevant to $\dot{M}$. We write $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(\dot{M})$ for the quotient. We also write $\Delta^{\mathcal{E}}_{G\text{-reg,ell},V}(\dot{M})$ for the subset of orbits in $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(\dot{M})$ for which $\dot{M}'$ lies in $\mathcal{E}_{\mathrm{ell}}(\dot{M},V)$ and $\dot{\delta}'$ lies in $\Delta_{G\text{-reg,ell},V}(\dot{M}')$. The summands in (7.7) and (7.8) then depend only on the image $\dot{\delta}$ of $(\dot{M}',\dot{\delta}')$ in $\Delta^{\mathcal{E}}_{G\text{-reg,ell},V}(\dot{M})$. In order to combine the two cases (a) and (b), we set

$$\varepsilon(\dot{f}_v,\dot{\delta}_v)=\begin{cases}\varepsilon_{\dot{M}}(\dot{f}_v,\dot{\delta}'_v), & \text{if } G \text{ is not quasisplit,}\\[4pt] \varepsilon^{\dot{M}'}(\dot{f}_v,\dot{\delta}'_v), & \text{if } G \text{ is quasisplit.}\end{cases}$$

The equations (7.7) and (7.8) can then be written together in the form

$$\sum_{\dot{\delta}\in\Delta^{\mathcal{E}}_{G\text{-reg,ell},V}(\dot{M})}\ \sum_{v\in V_{\mathrm{fin}}(\dot{G},\dot{M})}\varepsilon(\dot{f}_v,\dot{\delta}_v)\dot{f}^{v,\mathcal{E}}_{\dot{M}}(\dot{\delta}^v)=0. \tag{7.9}$$

To see how to separate the terms in (7.9), we should view the indices $\dot{\delta}$ in terms of the global form of the set $D^{\mathcal{E}}_{G\text{-reg,ell}}(M)$ defined in Section 6. Let $D^{\mathcal{E}}_{G\text{-reg,ell}}(\dot{M})$ be the quotient of the set of $\dot{M}$-relevant pairs in

$$\Big\{(\dot{\delta}^*,\dot{\kappa}^*):\ \dot{\delta}^*\in\dot{M}^*_{G\text{-reg,ell}}(\dot{F}),\ \dot{\kappa}^*\in\mathcal{K}(\dot{M}^*_{\dot{\delta}^*})\Big\}$$

that is defined by stable conjugacy in $\dot{M}^*(\dot{F})$. The group $\dot{T}^*=\dot{M}^*_{\dot{\delta}^*}$ here is of course a maximal torus in $\dot{M}^*$ over $\dot{F}$, and the global group $\mathcal{K}(\dot{M}^*_{\dot{\delta}^*})=\mathcal{K}(\dot{T}^*)$ is defined in [K3, (4.6)]. As in the local case, there is a correspondence $(\dot{M}',\dot{\delta}')\to(\dot{\delta}^*,\dot{\kappa}^*)$ that yields a well defined bijective mapping $\dot{\delta}\to\dot{d}$ from $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(\dot{M})$ onto $D^{\mathcal{E}}_{G\text{-reg,ell}}(\dot{M})$. This mapping underlies some of the basic constructions of [L5]; it is also a special case of either [K3, Lemma 9.7] or [II, Prop. 3.1]. Now, suppose that $\dot{T}\subset\dot{M}$ is the elliptic torus provided by Lemma 7.2. The quotient of the set

$$\Big\{(\dot{t},\dot{\kappa}):\ \dot{t}\in\dot{T}_{G\text{-reg,ell}}(\dot{F}),\ \dot{\kappa}\in\mathcal{K}(\dot{T})\Big\}$$

by the rational Weyl group $W_{\dot{F}}(\dot{M},\dot{T})$ then represents a subset of $D^{\mathcal{E}}_{G\text{-reg,ell}}(\dot{M})$. We note that if v is any valuation such that $\dot{E}_v$ is a field, the definition [K3, (4.6)] reduces to

$$\mathcal{K}(\dot{T})=\pi_0\Big(\widehat{T}^{\dot{\Gamma}}/Z(\widehat{M})^{\dot{\Gamma}}\Big)=\pi_0\Big(\widehat{T}^{\dot{\Gamma}_v}/Z(\widehat{M})^{\dot{\Gamma}_v}\Big)=\mathcal{K}(\dot{T}_v).$$

Following notation at the end of Section 6, we set $\mathcal{F}(\dot{t})$ equal to the fibre in $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(\dot{M})$ of a given point $\dot{t}$ in $\dot{T}_{G\text{-reg}}(\dot{F})$, and we write $\dot{\delta}\to\kappa(\dot{\delta})$ for the canonical bijection from $\mathcal{F}(\dot{t})$ onto $\mathcal{K}(\dot{T})$. Our immediate concern will be the case that $\dot{t}$ lies in the subset $\dot{T}_{G\text{-reg},V}(\dot{F})$ of V-admissible elements in $\dot{T}_{G\text{-reg}}(\dot{F})$. The fibre $\mathcal{F}(\dot{t})$ will then be contained in $\Delta^{\mathcal{E}}_{G\text{-reg,ell},V}(\dot{M})$. We are going to isolate the contribution to (7.9) of those elements $\dot{\delta}$ in $\mathcal{F}(\dot{t})$.

The sum over $\dot{\delta}$ in (7.9) can be restricted to a finite set that depends only on the support of $\dot{f}$. (See for example [A2, §3].) Having once chosen a bound for the support of $\dot{f}$, which we take to be an admissible subset of $\dot{G}_V$, we can shrink any of the functions $\dot{f}_v$ without enlarging this finite set or affecting the admissibility of $\dot{f}$. We note that by [A10, Lemma 3.1], the summand in (7.9) depends only on the $W(\dot{M})$-orbit of $\dot{\delta}$, relative to the free action of the Weyl group $W(\dot{M})$ of $(\dot{G}, A_{\dot{M}})$ on $\Delta_{G\text{-reg,ell},V}(\dot{M})$. We can therefore regard (7.9) as a sum over a finite set of $W(\dot{M})$-orbits in $\Delta_{G\text{-reg,ell},V}(\dot{M})$. Let $\dot{t}$ be a fixed point in $\dot{T}_{G\text{-reg},V}(\dot{F})$. If v belongs to V, we shall write $\dot{t}_v^{\dot{G}}$ for the stable conjugacy class of $\dot{t}$ in $\dot{G}_v$. Having fixed $\dot{t}$, we consider the distribution

$$\dot{f}_v \longrightarrow \dot{f}^{\mathcal{E}}_{v,\dot{M}}(\dot{\delta}_v) = \sum_{\dot{\gamma}_v \in \Gamma(\dot{M}_v)} \Delta_{\dot{M}}(\dot{\delta}_v, \dot{\gamma}_v) \dot{f}_{v,\dot{M}}(\dot{\gamma}_v), \qquad \dot{f}_v \in \mathcal{H}(\dot{G}_v),$$

on $\dot{G}_v$ associated to an arbitrary element $\dot{\delta}$ in $\Delta^{\mathcal{E}}_{G\text{-reg,ell},V}(\dot{M})$. The support of this distribution equals $\dot{t}_v^{\dot{G}}$ if $w\dot{\delta}$ lies in $\mathcal{F}(\dot{t})$ for some $w \in W(\dot{M})$, and is otherwise disjoint from $\dot{t}_v^{\dot{G}}$. Now suppose that v equals one of our two places u_1 and u_2 in U. In this case, we assume that the function $\dot{f}_v$ is supported on a small neighbourhood of $\dot{t}_v^{\dot{G}}$. If $\dot{\delta}$ indexes a nonzero summand in (7.9), one of the terms

$$\dot{f}^{\mathcal{E}}_{u_i,\dot{M}}(\dot{\delta}_{u_i}),$$

must be nonzero, from which it follows that the $W(\dot{M})$-orbit of $\dot{\delta}$ meets $\mathcal{F}(\dot{t})$. The identity (7.9) therefore reduces to

$$\sum_{\dot{\delta} \in \mathcal{F}(\dot{t})} \sum_{v \in V_{\mathrm{fin}}(\dot{G},\dot{T})} \varepsilon(\dot{f}_v, \dot{\delta}_v) \dot{f}^{v,\mathcal{E}}_{\dot{M}}(\dot{\delta}^v) = 0. \tag{7.10}$$

We have replaced the set $V_{\mathrm{fin}}(\dot{G}, \dot{M})$ in (7.9) by the subset

$$V_{\mathrm{fin}}(\dot{G}, \dot{T}) = \Big\{ v \in V_{\mathrm{fin}} : \ \dim(\mathfrak{a}^{\dot{G}_v}_{\dot{T}_v}) = \dim(\mathfrak{a}^{\dot{G}}_{\dot{T}}) \Big\}$$

of places at which $\dot{T}_v$ is $\dot{M}_v$-elliptic, since the induction hypothesis (1.2) and the appropriate descent formula imply that $\varepsilon(\dot{f}_v, \dot{\delta}_v)$ vanishes if v lies in the complement of this subset.

We have imposed strict support constraints on the functions $\dot{f}_v$, when v equals u_1 or u_2. However, we are still free to specify the values taken by the functions

$$\dot{\gamma}_v \longrightarrow \dot{f}_{v,\dot{M}}(\dot{\gamma}_v), \qquad v \in \{u_1, u_2\},$$

on the $\dot{M}_v$-conjugacy classes in the $\dot{M}_v$-stable conjugacy class $\dot{t}_v^{\dot{M}}$. To see how to do this in a way that exploits the conditions of Lemma 7.2, we recall an elementary property of the local transfer factors. If $\dot{\gamma}_v$ and $\dot{\gamma}_v^0$ are $\dot{M}_v$-conjugacy classes in $\dot{t}_v^{\dot{M}}$, and $\dot{\delta}$ belongs to $\mathcal{F}(\dot{t})$, we have

$$\Delta_{\dot{M}}(\dot{\delta}_v, \dot{\gamma}_v) = \Big\langle \mathrm{inv}(\dot{\gamma}_v^0, \dot{\gamma}_v), \dot{\kappa}_v \Big\rangle \Delta_{\dot{M}}(\dot{\delta}_v, \dot{\gamma}_v^0),$$

where $\dot{\kappa}_v$ is the image of the element $\dot{\kappa} = \kappa(\dot{\delta})$ in $\mathcal{K}(\dot{T}_v)$, and $\mathrm{inv}(\dot{\gamma}_v^0, \dot{\gamma}_v)$ is the element in the set

$$\mathcal{E}(\dot{T}_v) \cong \mathcal{K}(\dot{T}_v)^*$$

that measures the difference between $\dot{\gamma}_v^0$ and $\dot{\gamma}_v$. Therefore

$$\dot{f}^{\mathcal{E}}_{v,\dot{M}}(\dot{\delta}_v) = \Delta_{\dot{M}}(\dot{\gamma}_v, \dot{\gamma}_v^0)\Big(\sum_{\dot{\gamma}_v} \Big\langle \mathrm{inv}(\dot{\gamma}_v^0, \dot{\gamma}_v), \dot{\kappa}_v \Big\rangle \dot{f}_{v,\dot{M}}(\dot{\gamma}_v)\Big).$$

We recall that

$$\mathcal{E}(\dot{T}_v) = \mathrm{Im}\Big(H^1(\dot{F}_v, \dot{T}_{\mathrm{sc},v}) \longrightarrow H^1(\dot{F}_v, \dot{T}_v)\Big),$$

where $\dot{T}_{\mathrm{sc}}$ here stands for the preimage of $\dot{T}$ in the simply connected cover of $\dot{M}_{\mathrm{der}}$. Since v is p-adic, $\mathcal{E}(\dot{T}_v)$ equals the set $\mathcal{D}(\dot{T}_v)$ [L4, p. 702] that, together with the base point $\dot{\gamma}_v^0$, parametrizes the $\dot{M}_v$-conjugacy classes in $\dot{t}_v^{\dot{M}}$. We note that it is immaterial whether the groups $\mathcal{E}(\dot{T}_v)$ and $\mathcal{K}(\dot{T}_v)$ are defined relative to $\dot{M}_v$ or $\dot{G}_v$. To put it another way, the set of $\dot{M}_v$-conjugacy classes $\{\dot{\gamma}_v\}$ in $\dot{t}_v^{\dot{M}}$ is bijective with the set of $\dot{G}_v$-conjugacy classes in $\dot{t}_v^{\dot{G}}$. It follows that the linear forms

$$\dot{\gamma}_v \longrightarrow \dot{f}_{v,\dot{M}}(\dot{\gamma}_v), \qquad \dot{f}_v \in \mathcal{H}(\dot{G}_v),$$

form a basis of the space of invariant distributions on $\dot{G}_v$ that are supported on $\dot{t}_v^{\dot{G}}$. We are assuming that v equals u_1 or u_2. Therefore $\mathcal{K}(\dot{T})$ is isomorphic to $\mathcal{K}(\dot{T}_v)$, and $\dot{\delta} \to \dot{\kappa}_v$ is a bijection from $\mathcal{F}(\dot{t})$ onto $\mathcal{K}(\dot{T}_v)$. Since $\mathcal{K}(\dot{T}_v)$ is dual to $\mathcal{E}(\dot{T}_v)$, we conclude that the linear forms

$$\dot{\delta} \longrightarrow \dot{f}^{\mathcal{E}}_{v,\dot{M}}(\dot{\delta}_v), \qquad \dot{\delta} \in \mathcal{F}(\dot{t}),\ \dot{f}_v \in \mathcal{H}(\dot{G}_v),$$

are also a basis of the space of invariant distributions on $\dot{G}_v$ that are supported on $\dot{t}_v^{\dot{G}}$.

Let $\dot{\sigma}$ be a fixed element in $\mathcal{F}(\dot{t})$, and set $\sigma = \dot{\sigma}_{u_0}$. Suppose that the original function $f = \dot{f}_{u_0}$ in $\mathcal{H}(G)$ is such that

$$f^{\mathcal{E}}_M(\sigma) = 0.$$

For each u in $\{u_1, u_2\}$, we fix $\dot{f}_u$ so that

$$\dot{f}^{\mathcal{E}}_{u,\dot{M}}(\dot{\delta}_u) = \begin{cases} 1, & \text{if } \dot{\delta} = \dot{\sigma}, \\ 0, & \text{otherwise,} \end{cases}$$

for any $\dot{\delta} \in \mathcal{F}(\dot{t})$. This is possible by the discussion above. If v lies in the complement of $\{u_0, u_1, u_2\}$ in V, we take $\dot{f}_v$ to be a function such that $\dot{f}^{\mathcal{E}}_{v,\dot{M}}(\dot{\sigma}_v) = 1$. The functions $\dot{f}_{u_1}$ and $\dot{f}_{u_2}$ are assumed to satisfy the earlier conditions, and $\dot{f} = \prod_{v \in V} \dot{f}_v$ is required to be admissible. If G is quasisplit, we can also assume that $\dot{f}^{\dot{G}}_{u_0} = f^G = 0$, if $\kappa(\dot{\sigma}) = 1$, and that $\dot{f}^{\dot{G}}_{u_1} = \dot{f}^{\dot{G}}_{u_2} = 0$, if $\kappa(\dot{\sigma}) \neq 1$. This is possible because for any $u \in U$, the linear forms

$$\dot{f}_u \longrightarrow \dot{f}^{\mathcal{E}}_{u,\dot{M}}(\dot{\delta}_u), \qquad \dot{\delta} \in \mathcal{F}(\dot{t}),\ \kappa(\dot{\delta}) \neq 1,$$

on $\mathcal{H}^{\mathrm{uns}}(\dot{G}_v)$ are linearly independent. The function $\dot{f}$ is then unstable. In all cases, $\dot{f}$ has the appropriate constraints, and therefore satisfies (7.10). The complement in V of any v of course contains one of the places u_1 or u_2. It follows that the terms in (7.10) with $\dot{\delta} \neq \dot{\sigma}$ all vanish. If $\dot{\delta} = \dot{\sigma}$, the terms with $v \neq u_0$ also vanish, while the term with $v = u_0$ equals a nonzero multiple of $\varepsilon(f, \sigma)$. The identity (7.10) therefore implies that $\varepsilon(f, \sigma) = 0$.

We have reached the conclusion that $\varepsilon(f, \sigma)$ vanishes for any function $f \in \mathcal{H}(G)$ such that $f^{\mathcal{E}}_M(\sigma) = 0$, and such that $f^G = 0$, in case G is quasisplit and $\kappa(\dot{\sigma}) = 1$. This relation applies to the point $\sigma = \dot{\sigma}_{u_0}$, for any $\dot{\sigma}$ in the fibre $\mathcal{F}(\dot{t})$, any element $\dot{t}$ in $\dot{T}_{G\text{-reg},V}(\dot{F})$, and any V that is large relative to the support of f. The set V at this point actually plays no role. For if $\dot{t}$ is any element in $\dot{T}_{G\text{-reg}}(\dot{F})$, we can always choose the finite set V such that $\dot{t}$ lies in $\dot{T}_{G\text{-reg},V}(\dot{F})$. The relation therefore holds, with the conditions on f, for any $\dot{\sigma}$ in $\mathcal{F}(\dot{t})$. Reformulated in terms of the next lemma, it will be the main step in our proof of Local Theorem 1.

LEMMA 7.3. (i) *There is a smooth function* $\varepsilon(\delta)$ *on the set* $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M)$ *defined in Section* 6 *such that*

$$\varepsilon(f, \delta) = \varepsilon(\delta) f^{\mathcal{E}}_M(\delta), \qquad f \in \mathcal{H}(G),\ \delta \in \Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M). \tag{7.11}$$

(ii) *If* G *is quasisplit and* δ *lies in* $\Delta_{G\text{-reg,ell}}(M)$, *the distribution*

$$\delta \longrightarrow \varepsilon(f, \delta), \qquad f \in \mathcal{H}(G),$$

is stable.

Proof. Since the original elliptic torus $T \subset M$ was arbitrary, it would be enough to treat points δ in $\mathcal{F}(t)$, for elements $t \in T_{G\text{-reg}}(F)$. Let κ be a fixed element in $\mathcal{K}(T)$. For any given t, we then take δ to be the point in $\mathcal{F}(t)$ with $\kappa(\delta) = \kappa$.

Suppose that δ lies in $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M)$. We first consider the special case that $t = \dot{t}_{u_0}$, for a rational element $\dot{t} \in T_{G\text{-reg}}(\dot{F})$. Then δ equals $\dot{\delta}_{u_0}$, for the element $\dot{\delta} \in \mathcal{F}(\dot{t})$ such that $\kappa(\dot{\delta})_{u_0}$ equals κ. The conditions on δ rule out the case that $\dot{G}$ is quasisplit and $\kappa(\dot{\delta}) = 1$. The relation $\varepsilon(f, \dot{\delta}_{u_0}) = 0$ is then valid for any function $f \in \mathcal{H}(G)$ with $f^{\mathcal{E}}_M(\dot{\delta}_{u_0}) = 0$. This relation in turn implies that there is a complex number $\varepsilon(\dot{\delta}_{u_0})$ such that

$$\varepsilon(f, \dot{\delta}_{u_0}) = \varepsilon(\dot{\delta}_{u_0}) f^{\mathcal{E}}_M(\dot{\delta}_{u_0}),$$

for any $f \in \mathcal{H}(G)$ at all. Now the functions $\varepsilon(f, \delta)$ and $f^{\mathcal{E}}_M(\delta)$ vary smoothly with t. Moreover, $\dot{T}(\dot{F})$ is dense in $T(F) = \dot{T}(\dot{F}_{u_0})$, since $\dot{F}_v$ is a field for some

$v \neq u_0$ [KR, Lemma 1(b)]. It follows that

$$\varepsilon(f, \delta) = \varepsilon(\delta) f_M^{\mathcal{E}}(\delta)$$

in general, for a function $\varepsilon(\delta)$ that varies smoothly with t. This gives the assertion (i).

For the assertion (ii), we assume that G is quasisplit and $\kappa = 1$. The element δ then lies in $\Delta_{G\text{-reg,ell}}(M)$. Let $f \in \mathcal{H}(G)$ be a function with $f^G = 0$. If t is of the form $\dot{t}_{u_0}$, δ is of the form $\dot{\delta}_{u_0}$, for the element $\dot{\delta} \in \mathcal{F}(\dot{t})$ with $\kappa(\dot{\delta}) = 1$. In this case, we have established that $\varepsilon(f, \delta) = 0$. Since $\dot{T}(\dot{F})$ is dense in $T(F)$, the equation $\varepsilon(f, \delta) = 0$ then holds in general. The assertion (ii) follows. □

We have established the local identity (7.11) for G by representing G as a completion $\dot{G}_{u_0}$ of the global group $\dot{G}$. A similar identity can be established for the other completions

$$\dot{G}_v, \qquad v \in V_{\text{fin}}(\dot{G}, \dot{M}),$$

of $\dot{G}$, by embedding any $\dot{G}_v$ in its own (possibly different) global group. We obtain

$$\varepsilon(\dot{f}_v, \dot{\delta}_v) = \varepsilon(\dot{\delta}_v) \dot{f}_{v,\dot{M}}^{\mathcal{E}}(\dot{\delta}_v), \qquad \dot{f}_v \in \mathcal{H}(\dot{G}_v),\ \dot{\delta}_v \in \Delta_{G\text{-reg,ell}}^{\mathcal{E},0}(\dot{M}_v),$$

for a smooth function $\varepsilon(\dot{\delta}_v)$ on $\Delta_{G\text{-reg,ell}}^{\mathcal{E},0}(\dot{M}_v)$.

Corollary 7.4. *Suppose that G is not quasisplit. Suppose also that $\dot{t}$ belongs to $\dot{T}_{G\text{-reg}}(\dot{F})$, and that $\dot{\delta}$ is an element in $\mathcal{F}(\dot{t})$. Then*

$$\sum_{u \in U} \varepsilon(\dot{\delta}_u) = 0. \tag{7.12}$$

Proof. If u belongs to U, $\dot{G}_u$ is not quasisplit. In this case, $\dot{\delta}_u$ lies in $\Delta_{G\text{-reg,ell}}^{\mathcal{E},0}(\dot{M}_u)$ by definition, and the function $\varepsilon(\dot{\delta}_u)$ is defined. If v is a valuation outside of U, $\dot{G}_v$ is quasisplit. In this case

$$\varepsilon(\dot{f}_v, \dot{\delta}_v) = \varepsilon_{\dot{M}}(\dot{f}_v, \dot{\delta}'_v) = \sum_{\dot{\gamma}_v} \Delta(\dot{\delta}'_v, \dot{\gamma}_v)\Big(I_{\dot{M}}^{\mathcal{E}}(\dot{\gamma}_v, \dot{f}_v) - I_{\dot{M}}(\dot{\gamma}_v, \dot{f}_v)\Big) = 0,$$

again by definition. The identity (7.10) becomes

$$\sum_{\dot{\sigma} \in \mathcal{F}(\dot{t})} \Big(\sum_{u \in U} \varepsilon(\dot{\sigma}_u)\Big) \dot{f}_{\dot{M}}^{\mathcal{E}}(\dot{\sigma}) = 0.$$

This formula holds for any large finite set $V \supset U$, and any admissible function

$\dot{f}$ in $\mathcal{H}(\dot{G}_V)$. We can clearly choose $\dot{f}$ so that

$$\dot{f}_M^{\mathcal{E}}(\dot{\sigma}) = \begin{cases} 1, & \text{if } \dot{\sigma} = \dot{\delta}, \\ 0, & \text{if } \dot{\sigma} \neq \dot{\delta}, \end{cases}$$

for any $\dot{\sigma} \in \mathcal{F}(\dot{t})$. The formula (7.12) follows. □

We are now in a position to prove part (a) of Local Theorem 1. This corresponds to the case that G is not quasisplit. The assertion is that if γ belongs to $\Gamma_{G\text{-reg,ell}}(M)$, the distribution

$$I_M^{\mathcal{E}}(\gamma, f) - I_M(\gamma, f), \qquad f \in \mathcal{H}(G),$$

vanishes. Recall that

$$\varepsilon_M(f, \delta') = \sum_{\gamma \in \Gamma_{G\text{-reg,ell}}(M)} \Delta(\delta', \gamma)\Big(I_M^{\mathcal{E}}(\gamma, f) - I_M(\gamma, f)\Big),$$

for any pair (M', δ') that represents a point δ in $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$. The last formula can be inverted by the adjoint relations [A7, Lemma 2.2] for transfer factors. It is therefore enough to prove that for any such δ, the distribution

$$\varepsilon_M(f, \delta') = \varepsilon(f, \delta), \qquad f \in \mathcal{H}(G),$$

vanishes.

Since G is not quasisplit, we will be able to apply the last corollary. Suppose that κ belongs to $\mathcal{K}(T)$. Then κ equals $\dot{\kappa}_{u_0}$, for a unique element $\dot{\kappa}$ in $\mathcal{K}(\dot{T})$. For each $u \in U$, we choose a point $\dot{t}_u$ in $\dot{T}_{G\text{-reg}}(\dot{F}_u)$, and we let $\dot{\delta}_u$ be the element in $\mathcal{F}(\dot{t}_u)$ such that $\kappa(\dot{t}_u)$ equals $\dot{\kappa}_u$. The group $\dot{T}(\dot{F})$ is dense in $\dot{T}(\dot{F}_U)$. This follows from [KR, Lemma 1(b)], and the condition (v) of Lemma 7.2 that $\dot{E}_v$ is a field for some $v \notin U$. We can therefore approximate the points $\{\dot{t}_u\}$ simultaneously by an element $\dot{t}$ in $\dot{T}_{G\text{-reg,ell}}(\dot{F})$, and the points $\{\dot{\delta}_u\}$ simultaneously by the element $\dot{\delta} \in \mathcal{F}(\dot{t})$ such that $\kappa(\dot{\delta})$ equals $\dot{\kappa}$. We now apply Corollary 7.4 to $\dot{\delta}$. Since the functions $\varepsilon(\dot{\delta}_u)$ are smooth, the identity (7.12) of Corollary 7.4 extends to the general family of points $\{\dot{\delta}_u\}$. The points were of course chosen independently of each other, so that (7.12) implies that each of the functions

$$\varepsilon(\dot{\kappa}_u) = \varepsilon(\dot{\delta}_u), \qquad u \in U,$$

is constant. Furthermore, from [A10, Lemma 3.1] and the existence of the isomorphisms (7.4), we see that for any $u \in U$, $\varepsilon(\dot{\kappa}_u)$ equals $\varepsilon(\kappa)$. The formula (7.12) then yields

$$\varepsilon(\kappa) = |U|^{-1} \sum_{u \in U} \varepsilon(\dot{\kappa}_u) = 0.$$

We conclude that

$$\varepsilon(\delta) = \varepsilon(\kappa) = 0, \qquad t \in T_{G\text{-reg}}(F),$$

for the element $\delta = \dot{\delta}_{u_0}$ in $\mathcal{F}(t)$ with $\kappa(\delta) = \kappa$. But the objects t, κ, and T were completely arbitrary. It follows that

$$\varepsilon(f,\delta) = \varepsilon(\delta) f_M^{\mathcal{E}}(\delta) = 0, \qquad f \in \mathcal{H}(G),$$

for any element δ in the set $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M) = \Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M)$. This completes the proof of part (a) of Local Theorem 1.

It remains to establish part (b) of Local Theorem 1. We are now in the case that G is quasisplit. There are actually two assertions. One is that if δ belongs to $\Delta_{G\text{-reg,ell}}(M)$, the distribution

$$f \longrightarrow S^G_M(\delta, f), \qquad f \in \mathcal{H}(G),$$

is stable. This has already been proved. Since

$$S^G_M(\delta,f) = \varepsilon^{M^*}(f,\delta) = \varepsilon(f,\delta),$$

the assertion is just part (ii) of Lemma 7.3. The other assertion is that if δ belongs to the complement $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M)$ of $\Delta_{G\text{-reg,ell}}(M)$, and is represented by a pair (M', δ'), the distribution

$$S^G_M(M',\delta',f) = \varepsilon^{M'}(f,\delta') = \varepsilon(f,\delta), \qquad f \in \mathcal{H}(G),$$

vanishes. This is more difficult. It requires a property of weak approximation on $\dot{T}$, whose proof we postpone until the next section. In the remaining part of this section, we shall formulate an analogue of Corollary 7.4, which will be used in conjunction with Lemma 6.5 to establish the approximation property.

Suppose that V is a finite set of valuations of $\dot{F}$ that contains U, and outside of which $\dot{G}$, $\dot{T}$ and $\dot{E}$ are unramified. We assume also that V contains the finite set $V_{\mathrm{fund}}(\dot{G})$ of Assumption 5.2(1) of [I], outside of which the generalized fundamental lemma is assumed to hold. Given V, we write $S(\dot{E},V)$ for the set of valuations $v \notin V$ that split completely in $\dot{E}$, and $W(\dot{E},V)$ for the complement of $S(\dot{E},V)$ in the set of all valuations of $\dot{F}$.

Corollary 7.5. *Suppose that G is quasisplit. Suppose also that $\dot{t}$ is a point in $\dot{T}_{G\text{-reg}}(\dot{F})$ such that $\dot{t}_v$ is bounded for every v in the complement of V in $W(\dot{E},V)$, and that $\dot{\delta}$ is an element in $\mathcal{F}(\dot{t})$ with $\kappa(\dot{\delta}) \neq 1$. Then $\varepsilon(\dot{\delta}_v)$ is defined for any v in $V_{\mathrm{fin}}(\dot{G},\dot{T})$, and*

$$\sum_{v \in V_{\mathrm{fin}}(\dot{G},\dot{T})} \varepsilon(\dot{\delta}_v) = 0. \tag{7.13}$$

Proof. If v belongs to $V_{\mathrm{fin}}(\dot{G},\dot{T})$, the map

$$\mathfrak{a}^{\dot{G}_v}_{\dot{T}_v} \longrightarrow \mathfrak{a}^{\dot{G}}_{\dot{T}}$$

is an isomorphism. It follows easily that the canonical map

$$\mathcal{K}(\dot{T}) = \pi_0\Big(\widehat{T}^{\dot{\Gamma}}/Z(\widehat{G})^{\dot{\Gamma}}\Big) \longrightarrow \mathcal{K}(\dot{T}_v) = \pi_0\Big(\widehat{T}^{\dot{\Gamma}_v}/Z(\widehat{G})^{\dot{\Gamma}_v}\Big),$$

which we are denoting by $\dot{\kappa} \to \dot{\kappa}_v$, is injective. Set $\dot{\kappa} = \kappa(\dot{\delta})$. Then $\dot{\kappa}_v \neq 1$. Since $\dot{\kappa}_v = \kappa(\dot{\delta}_v)$, the point $\dot{\delta}_v$ lies in $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(\dot{M}_v)$, and the function $\varepsilon(\dot{\delta}_v)$ is defined.

The required identity (7.13) would follow directly from (7.10), were it not for the fact that V has been chosen here independently of $\dot{\delta}$. Given $\dot{\delta}$, we choose a finite set V^+ of valuations containing V, such that $\dot{\delta}$ is V^+-admissible. We can then apply (7.10) to V^+. Isolating the element $\dot{\delta} \in \mathcal{F}(\dot{t})$ by an admissible function $\dot{f}^+ \in \mathcal{H}(\dot{G}_{V^+})$, as in the proof of Corollary 7.4, we see that

$$\sum_{v \in V^+_{\mathrm{fin}}(\dot{G},\dot{T})} \varepsilon(\dot{\delta}_v) = 0.$$

To establish (7.13), it would be enough to show that $\varepsilon(\dot{\delta}_v) = 0$ for every v in the complement of V in $V^+_{\mathrm{fin}}(\dot{G},\dot{T})$.

We first observe that $V^+_{\mathrm{fin}}(\dot{G},\dot{T})$ is contained in $W(\dot{E},V)$. Indeed, if v belongs to the complement $S(\dot{E},V)$ of $W(\dot{E},V)$, $\dot{T}_v$ is a split torus over $\dot{F}_v$. The group

$$\mathcal{K}(\dot{T}_v) = \pi_0\Big(\widehat{T}^{\dot{\Gamma}_v}/Z(\widehat{G})^{\dot{\Gamma}_v}\Big) = \pi_0\Big(\widehat{T}/Z(\widehat{G})\Big)$$

is then trivial, and $\dot{\kappa}_v = 1$. In particular, v cannot lie in $V^+_{\mathrm{fin}}(\dot{G},\dot{T})$.

Suppose that v lies in the complement of V in $V^+_{\mathrm{fin}}(\dot{G},\dot{T})$. Then v belongs to the complement of V in $W(\dot{E},V)$ and so the element $\dot{t}_v$ in $\dot{T}_v$ is bounded. This implies that $\dot{\delta}_v$ is bounded, as is the element $\dot{\delta}'_v$ attached to any pair $(\dot{M}',\dot{\delta}')$ that represents $\dot{\delta}$. Let $\dot{f}_v$ be the characteristic function of a hyperspecial maximal compact subgroup of $\dot{G}_v$. We shall apply the identity

$$\varepsilon(\dot{f}_v,\dot{\delta}_v) = \varepsilon(\dot{\delta}_v)\dot{f}^{\mathcal{E}}_{v,\dot{M}}(\dot{\delta}_v).$$

According to Assumption 5.2(1) of [I], the standard fundamental lemma is valid for $\dot{G}_v$. It asserts that the factor

$$\dot{f}^{\mathcal{E}}_{v,\dot{M}}(\dot{\delta}_v) = \dot{f}^{\dot{M}'}_v(\dot{\delta}'_v)$$

on the right-hand side of the identity equals $\dot{h}^{\dot{M}'}_v(\dot{\delta}'_v)$, where $\dot{h}_v$ is the characteristic function of a hyperspecial maximal compact subgroup of $\dot{M}'_v$. As a bounded, G-regular stable conjugacy class in $\dot{M}'_v$, $\dot{\delta}'_v$ intersects the support of $\dot{h}'_v$. It follows that the stable orbital integral $\dot{h}^{\dot{M}'}_v(\dot{\delta}'_v)$ is nonzero. The factor $\dot{f}^{\mathcal{E}}_{v,\dot{M}}(\dot{\delta}_v)$ is therefore nonzero. The generalized fundamental lemma is valid for $(\dot{G}_v,\dot{M}_v)$, again by Assumption 5.2(1) of [I]. It can be applied to the term

$$\varepsilon(\dot{f}_v,\dot{\delta}_v) = \varepsilon^{\dot{M}'}(\dot{f}_v,\dot{\delta}'_v) = S^G_M(\dot{M}'_v,\dot{\delta}'_v,\dot{f}_v)$$

on the left-hand side of the identity. The generalized fundamental lemma was actually formulated [I, Conjecture 5.1] in terms of the weighted orbital integrals $J_{\dot{M}}(\cdot,\dot{f}_v)$. However, one sees easily from the unramified local analogue of [A9, Th. 5] (which is actually a consequence of this theorem) that it is equivalent to the special case of Local Theorem 1(b) in which G is unramified, and f is a unit in the Hecke algebra. In other words, the generalized fundamental lemma implies that $S_M^G(\dot{M}'_v, \dot{\delta}'_v, \dot{f}_v)$ vanishes. The factor $\varepsilon(\dot{f}_v, \dot{\delta}_v)$ is therefore equal to zero. Putting the two pieces of information together, we conclude that $\varepsilon(\dot{\delta}_v) = 0$, as required. This completes the proof of the corollary. □

8. Weak approximation

In this section, we shall finish the proof of Local Theorem 1. We fix local data F, G, M, T and ψ as at the beginning of the last section. We can then make use of the Galois extension $E \supset F$ and global datum $(\dot{F}, \dot{E}, \dot{G}, \dot{M}, \dot{T}, \dot{\psi})$ provided by Lemma 7.2. We also fix the place $u_0 \in U$ as before, and use the isomorphism ϕ_{u_0} in (7.4) to identify (F, E, G, M, T, ψ) with the completion of $(\dot{F}, \dot{E}, \dot{G}, \dot{M}, \dot{T}, \dot{\psi})$ at u_0.

We assume that G is quasisplit over the local field F. The group $\dot{G}$ is then quasisplit over the global field $\dot{F}$. We can also assume that the inner twists ψ and $\dot{\psi}$ are each equal to 1. To complete the proof of Local Theorem 1, we have to show that if δ belongs to $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M)$, and is represented by a pair (M', δ'), the linear form

$$S_M^G(M', \delta', f) = \varepsilon^{M'}(f, \delta') = \varepsilon(f, \delta), \qquad f \in \mathcal{H}(G),$$

vanishes. This is equivalent to showing that the function

$$\varepsilon(\delta), \qquad \delta \in \Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M),$$

of Lemma 7.3 vanishes. We shall establish the result as a general property of any family of such functions that satisfy the global identity (7.13) of Corollary 7.5, and the local identity (6.14) of Lemma 6.5.

Suppose that V is a finite set of valuations of $\dot{F}$ that contains U. As usual, we assume that $\dot{G}$, $\dot{T}$ and $\dot{E}$ are unramified outside of V. We also assume that

$$\dot{T}(\mathbb{A}) = \dot{T}(\dot{F})\dot{T}(\dot{F}_V)\dot{R}^V, \tag{8.1}$$

where

$$\dot{R}^V = \prod_{v \notin V} \dot{R}_v$$

is the maximal compact subgroup of $\dot{T}(\mathbb{A}^V)$. As at the end of Section 7, we write $S(\dot{E}, V)$ for the set of valuations $v \notin V$ that split completely in $\dot{E}$, and $W(\dot{E}, V)$ for the complement of $S(\dot{E}, V)$ in the set of all valuations.

LEMMA 8.1. *Suppose that for each $v \in V$, $\mathrm{e}_v(\dot{t}_v)$ is a smooth function on $\dot{T}_{G\text{-reg}}(\dot{F}_v)$ that depends only on the isomorphism class of $(\dot{F}_v, \dot{E}_v, \dot{G}_v, \dot{M}_v, \dot{T}_v)$.*

Assume that

$$\sum_{v\in V} \mathrm{e}_v(\dot{t}_v) = 0, \tag{8.2}$$

for any element $\dot{t} \in T_{G\text{-reg}}(\dot{F})$ *such that for each* w *in the complement of* V *in* $W(\dot{E}, V)$, *the point* $\dot{t}_w$ *is bounded. Assume also that the function* $\mathrm{e} = \mathrm{e}_{u_0}$ *satisfies the formula*

$$\mathrm{e}(t) + \mathrm{e}(t^{-1}) = 0, \qquad t \in T_{G\text{-reg}}(F). \tag{8.3}$$

Then e *vanishes identically on* $T_{G\text{-reg}}(F)$.

Proof. Before we can exploit the identities (8.2) and (8.3), we have first to make some simple remarks relating to Langlands duality for tori. This discussion will be quite general. It applies to the case that $\dot{T}$ is any torus over $\dot{F}$, $\dot{E}$ is any finite Galois extension of $\dot{F}$ that splits $\dot{T}$, and V is any finite set of valuations of $\dot{F}$ that satisfies (8.1), and outside of which $\dot{E}$ is unramified.

Suppose that W is any set of valuations of $\dot{F}$ that contains V. Then

$$\dot{R}_W^V = \prod_{v\in W-V} \dot{R}_v$$

is a maximal compact subgroup of

$$\dot{T}(\mathbb{A}_W^V) = \Big\{\dot{t}_V \in \dot{T}(\mathbb{A}) : \ \dot{t}_v = 1 \text{ for any } v \in V \cup {}^cW\Big\}.$$

We shall write $\dot{T}_{V,W}$ for the closure in

$$\dot{T}_V = \prod_{v\in V} \dot{T}_v = \prod_{v\in V} \dot{T}(\dot{F}_v)$$

of the subgroup

$$\dot{T}_V \cap \dot{T}(\dot{F})\dot{R}_W^V \dot{T}(\mathbb{A}^W). \tag{8.4}$$

For example, we could take W to be the set of all valuations of $\dot{F}$. In this case, the group (8.4) is already closed. It is equal to the discrete subgroup

$$\dot{\Gamma}_V = \dot{T}_V \cap \dot{T}(\dot{F})\dot{R}^V$$

of $\dot{T}_V$. We can of course also take W to be the set $W(\dot{E}, V)$, in which case we shall write

$$\dot{T}_{V,\dot{E}} = \dot{T}_{V,W(\dot{E},V)}.$$

Observe that under the stated conditions of the lemma, the identity (8.2) is valid for any strongly $\dot{G}$-regular element $\dot{t}_V = \prod_{v\in V} \dot{t}_v$ in $\dot{T}_{V,\dot{E}}$. This is because for any $w \neq V$, an element $\dot{t}_w \in \dot{T}_w$ is bounded if and only if it lies in $\dot{R}_w$.

The global Weil group $W_{\dot{F}}$ acts on the dual torus $\widehat{T} = \widehat{\dot{T}}$. We shall consider subgroups of the continuous cohomology group $H^1(W_{\dot{F}}, \widehat{T})$. If W is any set of valuations of $\dot{F}$, we write $H^1(W_{\dot{F}}, \widehat{T})_W$ for the kernel of the map

$$H^1(W_{\dot{F}}, \widehat{T}) \longrightarrow \bigoplus_{v \notin W} H^1(W_{\dot{F}_v}, \widehat{T}).$$

The quotient

$$H^1(W_{\dot{F}}, \widehat{T})^W = H^1(W_{\dot{F}}, \widehat{T})/H^1(W_{\dot{F}}, \widehat{T})_W$$

then maps injectively into the direct sum over $v \notin W$ of the groups $H^1(W_{\dot{F}_v}, \widehat{T})$. If W is the empty set, for example, $H^1(W_{\dot{F}}, \widehat{T})_W$ is the group $H^1(W_{\dot{F}}, \widehat{T})_{\mathrm{lt}}$ of locally trivial classes in $H^1(W_{\dot{F}}, \widehat{T})$. According to the Langlands correspondence for tori [L1], the associated quotient $H^1(W_{\dot{F}}, \widehat{T})^{\mathrm{lt}}$ is dual to $\dot{T}(\mathbb{A})/\dot{T}(\dot{F})$. Let us write $H^1_V(\cdot, \cdot)$ for the subgroup of classes in a given cohomology group that are unramified at each place outside of V. Then $H^1_V(W_{\dot{F}}, \widehat{T})^{\mathrm{lt}}$ is dual to the group

$$\dot{T}(\mathbb{A})/\dot{T}(\dot{F})\dot{R}^V \;\cong\; \dot{T}_V/\dot{T}_V \cap \dot{T}(\dot{F})\dot{R}^V = \dot{T}_V/\dot{\Gamma}_V.$$

More generally, suppose that W is any set of valuations that contains V. We claim that the closed subgroup

$$H^1_V(W_{\dot{F}}, \widehat{T})^{\mathrm{lt}}_W = H^1_V(W_{\dot{F}}, \widehat{T})_W/H^1_V(\dot{W}_{\dot{F}}, \widehat{T})_{\mathrm{lt}}$$

of $H^1_V(W_{\dot{F}}, \widehat{T})^{\mathrm{lt}}$ is the annihilator of the closed subgroup $\dot{T}_{V,W}/\dot{\Gamma}_V$ of $\dot{T}_V/\dot{\Gamma}_V$. Indeed, the elements in $H^1_V(W_{\dot{F}}, \widehat{T})^{\mathrm{lt}}_W$ correspond to continuous characters on $\dot{T}(\mathbb{A})/\dot{T}(\dot{F})$ that are trivial on $\dot{R}^V_W\dot{T}(\mathbb{A}^W)$, from which it follows that $H^1_V(W_{\dot{F}}, \widehat{T})^{\mathrm{lt}}_W$ annihilates $\dot{T}_{V,W}/\dot{\Gamma}_V$. Conversely, since the embedding

$$\dot{T}_w \longrightarrow \dot{T}(\mathbb{A})/\dot{T}(\dot{F})$$

is dual to the restriction

$$H^1(W_{\dot{F}}, \widehat{T})^{\mathrm{lt}} \longrightarrow H^1(W_{\dot{F}_w}, \widehat{T}),$$

any element in $H^1_V(W_{\dot{F}}, \widehat{T})^{\mathrm{lt}}$ that annihilates $\dot{T}_{V,W}/\dot{\Gamma}_V$ belongs to $H^1_V(W_{\dot{F}}, \widehat{T})^{\mathrm{lt}}_W$. The claim follows. We conclude that the group $H^1_V(W_{\dot{F}}, \widehat{T})^{\mathrm{lt}}_W$ is dual to the quotient $\dot{T}_V/\dot{T}_{V,W}$. If $W = V$, the assertion is a special case of [KR, Lemma 1(a)]. We shall be concerned with the case that W equals the set $W(\dot{E}, V)$.

The action of $W_{\dot{F}}$ on $\widehat{T}$ factors through the quotient

$$\mathrm{Gal}(\dot{E}/\dot{F}) \cong W_{\dot{E}/\dot{F}}/W_{\dot{E}/\dot{E}} \cong W_{\dot{F}}/W_{\dot{E}}.$$

The inflation map embeds the relative cohomology group

$$H^1(\dot{E}/\dot{F}, \widehat{T}) = H^1\Big(\mathrm{Gal}(\dot{E}/\dot{F}), \widehat{T}\Big)$$

into $H^1(W_{\dot F}, \widehat T)$. We claim that $H^1(\dot E/\dot F, \widehat T)$ equals the subgroup

$$H^1(W_{\dot F}, \widehat T)_{\dot E} = H^1(W_{\dot F}, \widehat T)_{W(\dot E, V)}$$

of $H^1(W_{\dot F}, \widehat T)$. To see this, we first note that $H^1(\dot E/\dot F, \widehat T)$ is the kernel of the map

$$H^1(W_{\dot F}, \widehat T) \longrightarrow H^1(W_{\dot E}, \widehat T) = H^1(W_{\dot E/\dot E}, \widehat T), \tag{8.5}$$

and that $H^1(W_{\dot F}, \widehat T)_{\dot E}$ is the kernel of the map

$$H^1(W_{\dot F}, \widehat T) \longrightarrow \bigoplus_{w \in S(\dot E, V)} H^1(\dot F_w, \widehat T). \tag{8.6}$$

Let $S^\sim(\dot E, V)$ be the set of valuations of $\dot E$ that divide those valuations of $\dot F$ that lie in $S(\dot E, V)$. The composition of (8.5) with the map

$$H^1(W_{\dot E/\dot E}, \widehat T) \longrightarrow \bigoplus_{w^\sim \in S^\sim(\dot E, V)} H^1(\dot E_{w^\sim}, \widehat T)$$

is then equal to the composition of (8.6) with the map

$$\bigoplus_{w \in S(\dot E, V)} H^1(\dot F_w, \widehat T) \longrightarrow \bigoplus_{w^\sim \in S^\sim(\dot E, V)} H^1(\dot E_{w^\sim}, \widehat T).$$

The last two maps are both injective. This is obvious in the case of the second map. For the first map, it is a consequence of the analogue of the Tchebotarev density theorem for the idele class group $C_{\dot E} \cong W_{\dot E/\dot E}$ [Se, Th. 2, p. I-23], and the fact that $S^\sim(\dot E, V)$ is a set of valuations of $\dot E$ of positive density whose associated Frobenius elements map surjectively onto any finite quotient of $C_{\dot E}$. We have shown that the two groups $H^1(\dot E/\dot F, \widehat T)$ and $H^1(W_{\dot F}, \widehat T)_{\dot E}$ represent the kernel of the same map. They are therefore equal, as claimed. In particular, the elements in $H^1(W_{\dot F}, \widehat T)_{\dot E}$ are unramified outside of V, since the same is true of the elements in $H^1(\dot E/\dot F, \widehat T)$. We apply what we have just observed to the quotient of each group by the subgroup of locally trivial classes. We conclude that $H^1(\dot E/\dot F, \widehat T)^{\mathrm{lt}}$ equals the group

$$H^1_V(W_{\dot F}, \widehat T)^{\mathrm{lt}}_{\dot E} = H^1_V(W_{\dot F}, \widehat T)^{\mathrm{lt}}_{W(\dot E, V)}.$$

It then follows from the remarks of the previous paragraph that $H^1(\dot E/\dot F, \widehat T)^{\mathrm{lt}}$ can be identified with the group of characters of $\dot T_V/\dot\Gamma_V$ that are trivial on the closed subgroup $\dot T_{V,\dot E}/\dot\Gamma_V$. In other words, $H^1(\dot E/\dot F, \widehat T)^{\mathrm{lt}}$ is in duality with $\dot T_V/\dot T_{V,\dot E}$.

At this point, we return to conditions of the lemma. In particular, we assume that $\dot T$ satisfies the conditions of the earlier Lemma 7.2. Following Section 7, we identify the global Galois group $\mathrm{Gal}(\dot E/\dot F)$ with the local Galois group $\mathrm{Gal}(E/F) = \mathrm{Gal}(\dot{E}_{u_0}/\dot{F}_{u_0})$ at the fixed place $u_0 \in U$. Since $\dot{E}_{u_0}$ is a field, the group $H^1(\dot{E}/\dot{F}, \widehat{T})_{\mathrm{lt}}$ is trivial. Therefore

$$H^1(E/F, \widehat{T}) = H^1(\dot{E}/\dot{F}, \widehat{T}) = H^1(\dot{E}/\dot{F}, T)^{\mathrm{lt}}.$$

For any place $v \in V$, we set

$$\dot{T}_{v,\dot{E}} = \dot{T}_v \cap \dot{T}_{V,\dot{E}} = \dot{T}(\dot{F}_v) \cap \dot{T}_{V,\dot{E}}.$$

According to the local Langlands correspondence for tori, the group $H^1(W_{\dot{F}_v}, \widehat{T})$ is dual to $\dot{T}_v$. Since $H^1(E/F, \widehat{T})$ represents the annihilator of $\dot{T}_{V,\dot{E}}/\dot{\Gamma}_V$ in the group of characters on $\dot{T}_V/\dot{\Gamma}_V$, $\dot{T}_{v,\dot{E}}$ is just the subgroup of $\dot{T}_v$ annihilated by the image of the composition

$$H^1(E/F, \widehat{T}) \longrightarrow H^1(\dot{E}_v/\dot{F}_v, \widehat{T}) \longrightarrow H^1(W_{\dot{F}_v}, \widehat{T}).$$

Consider the case that v belongs to the subset U of V. The restriction map of $H^1(E/F, \widehat{T})$ to $H^1(\dot{E}_u/\dot{F}_u, \widehat{T})$ is then an isomorphism, which identifies $H^1(E/F, \widehat{T})$ with the character group of $\dot{T}_u/\dot{T}_{u,\dot{E}}$. But $H^1(E/F, \widehat{T})$ has also been identified with the character group of $\dot{T}_V/\dot{T}_{V,\dot{E}}$. It follows that the canonical injection

$$\dot{T}_u/\dot{T}_{u,\dot{E}} \longrightarrow \dot{T}_V/\dot{T}_{V,\dot{E}} \tag{8.7}$$

is actually an isomorphism.

We are now ready to apply the identity (8.2). Suppose that v belongs to V, and that $\dot{t}_v$ is an element in $\dot{T}_{\dot{G}\text{-reg}}(\dot{F}_v)$. We can then find a $\dot{G}$-regular element $\dot{t}_V$ in $\dot{T}_{V,\dot{E}}$ whose image in $\dot{T}_v$ equals $\dot{t}_v$. To see this, we have only to choose a place $u \in U$ distinct from v, and then use the bijectivity of the map (8.7). Suppose that α is a point in $\dot{T}_{v,\dot{E}}$ such that the product $\dot{s}_v = \alpha\dot{t}_v$ is also strongly $\dot{G}$-regular. The element $\dot{s}_V = \alpha\dot{t}_V$ obviously remains in $\dot{T}_{V,\dot{E}}$, and has the same component as $\dot{t}_V$ at each place w in $V - \{v\}$. Applying the extension of the identity (8.2) to elements in $\dot{T}_{V,\dot{E}}$, we see that

$$\mathrm{e}_v(\dot{s}_v) - \mathrm{e}_v(\dot{t}_v) = \sum_{w \in V} \mathrm{e}_w(\dot{s}_w) - \sum_{w \in V} \mathrm{e}_w(\dot{t}_w) = 0.$$

The function e_v is therefore invariant under translation by $\dot{T}_{v,\dot{E}}$. In other words, it extends to a function on $\dot{T}_v/\dot{T}_{v,\dot{E}}$.

The last step will be to apply the identity (8.3). Suppose that x is the trivial coset $T(F)_{\dot{E}} = \dot{T}_{u_0,\dot{E}}$ in $T(F)/T(F)_{\dot{E}} = \dot{T}_{u_0}/\dot{T}_{u_0,\dot{E}}$. Then (8.3) yields

$$\mathrm{e}(x) = \tfrac{1}{2}\Big(\mathrm{e}(x) + \mathrm{e}(x)\Big) = \tfrac{1}{2}\Big(\mathrm{e}(x) + \mathrm{e}(x^{-1})\Big) = 0.$$

To deal with the other cosets, we choose two places u_1 and u_2 in U that are distinct from u_0. Then there are isomorphisms

$$(F, E, G, M, T) \longrightarrow (\dot{F}_{u_i}, \dot{E}_{u_i}, \dot{G}_{u_i}, \dot{M}_{u_i}, \dot{T}_{u_i}), \qquad i = 1, 2,$$

of local data. By assumption,

$$\mathrm{e}(x) = \mathrm{e}_{u_i}(\dot{x}_{u_i}), \qquad x \in T(F)/T(F)_{\dot{E}},$$

where $x \to \dot{x}_{u_i}$ denotes the isomorphism

$$T(F)/T(F)_{\dot{E}} \longrightarrow \dot{T}_{u_i}/\dot{T}_{u_i,\dot{E}}, \qquad i = 1, 2.$$

Suppose that x and y are points in $T(F)/T(F)_{\dot{E}}$. For each v in the complement of $\{u_0, u_1, u_2\}$ in V, choose a point $\dot{t}_v$ in $\dot{T}_{v,\dot{E}}$, and set

$$\dot{t}_V = xy \cdot \dot{x}_{u_1}^{-1} \cdot \dot{y}_{u_2}^{-1} \cdot \prod_v \dot{t}_v, \qquad v \notin \{u_0, u_1, u_2\}.$$

Letting the valuations u in (8.7) run over the set $\{u_0, u_1, u_2\}$, we see that $\dot{t}_V$ belongs to $\dot{T}_{V,\dot{E}}$. Set

$$\varepsilon_0 = \sum_v \mathrm{e}_v(\dot{t}_v), \qquad v \notin \{u_0, u_1, u_2\}.$$

It then follows from (8.3) and the extension of (8.2) to $\dot{T}_{V,\dot{E}}$ that

$$\begin{aligned} \mathrm{e}(xy) - \mathrm{e}(x) - \mathrm{e}(y) + \varepsilon_0 &= \mathrm{e}(xy) + \mathrm{e}(x^{-1}) + \mathrm{e}(y^{-1}) + \varepsilon_0 \\ &= \sum_{v \in V} \mathrm{e}_v(\dot{t}_v) = 0. \end{aligned}$$

Taking $x = y = 1$, we deduce that $\varepsilon_0 = 1$. Therefore

$$\mathrm{e}(xy) = \mathrm{e}(x) + \mathrm{e}(y),$$

for any points x and y in $T(F)/T(F)_{\dot{E}}$. In other words, e is a homomorphism from the finite group $T(F)/T(F)_{\dot{E}}$ to the additive group $\mathbb{C}$. Any such homomorphism must be trivial. It follows that the original function e on $T_{G\text{-reg}}(F)$ vanishes identically. □

We can now complete the proof of Local Theorem 1. Let κ be any element in $\mathcal{K}(T)$ with $\kappa \neq 1$, and let $\dot{\kappa}$ be the element in $\mathcal{K}(\dot{T})$ such that $\kappa = \dot{\kappa}_{u_0}$. Then $\dot{\kappa} \neq 1$. If v belongs to the subset $V_{\mathrm{fin}}(\dot{G}, \dot{T})$ of V, the element $\dot{\kappa}_v \in \mathcal{K}(\dot{T}_v)$ is also distinct from 1, as we saw at the beginning of the proof of Corollary 7.5. In this case we set

$$\mathrm{e}_v(\dot{t}_v) = \varepsilon(\dot{\delta}_v), \qquad \dot{t}_v \in \dot{T}_{G\text{-reg}}(\dot{T}_v),$$

where $\dot{\delta}_v$ is the element in $\mathcal{F}(\dot{t}_v)$ such that $\kappa(\dot{\delta}_v) = \dot{\kappa}_v$. If v lies in the complement of $V_{\mathrm{fin}}(\dot{G}, \dot{T})$ in V, we simply set $\mathrm{e}_v(\dot{t}_v) = 0$. The relations (8.2) and (8.3) then follow from Corollary 7.5 and Lemma 6.5, respectively. The last lemma asserts that $\mathrm{e}(t)$ vanishes identically on $T_{G\text{-reg}}(F)$. Therefore

$$\varepsilon(\delta) = 0, \qquad t \in T_{G\text{-reg}}(F),$$

where δ is the element in $\mathcal{F}(t)$ with $\kappa(\delta) = \kappa$. But any element δ in $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M)$ can be expressed in this form, for some choice of T, κ and t. In other words, $\varepsilon(\delta)$ vanishes identically on the set $\Delta^{\mathcal{E},0}_{G\text{-reg,ell}}(M)$. This is what we needed to show in order to establish the remaining assertion of Local Theorem 1.

We have shown that the assertions of Local Theorem 1 all are valid for G and M. This completes the part of the induction argument that depends on the integer

$$r_{\text{der}} = \dim(A_M \cap G_{\text{der}}).$$

Letting r_{der} vary, we conclude that Local Theorem 1 holds for any Levi subgroup M of G. The group G was fixed at the beginning of Section 7. The choice was subject only to Assumption 5.2(2) of [I], and the condition that $\dim(G_{\text{der}}) = d_{\text{der}}$. Therefore, as we noted in Section 1, all the local theorems stated in [I, §6] hold for any G with $\dim(G_{\text{der}}) = d_{\text{der}}$, so long as the relevant half of Assumption 5.2 of [I] is valid. Of course, this last assertion depends on the global induction assumption (1.4). To complete the induction argument, we must establish the global theorems for K-groups G with $\dim(G_{\text{der}}) = d_{\text{der}}$. We shall do so in Section 9.

The arguments that have lead to a proof of Local Theorem 1 generalize the techniques of Chapter 2 of [AC]. In particular, the discussion in Sections 7 and 8 here is loosely modeled on [AC, §2.17]. The analogue in [AC] of Local Theorem 1 is Theorem A(i), stated in [AC, §2.5]. There is actually a minor gap at the end of the proof of this result. The misstatement occurs near the top of p. 196 of [AC], with the sentence "But as long as k is large enough ...". For one cannot generally approximate elements in a local group by rational elements that are integral almost everywhere. The gap could be filled almost immediately with the local trace formula (and its Galois-twisted analogue) for $\mathrm{GL}(n)$. We shall resolve the problem instead by more elementary means. We shall establish a second lemma on weak approximation that is in fact simpler than the last one.

We may as well apply the "dot" notation above to the setting of [AC, §2.17]. Then $\dot{E}/\dot{F}$ is a cyclic extension of number fields. There are actually two cases to consider. If $\dot{E} = \dot{F}$, $\dot{G}$ is an inner form of the general linear group $\mathrm{GL}(n)$. If $\dot{E} \neq \dot{F}$, the problem falls into the general framework of twisted endoscopy. In this case, $\dot{G}$ is a component in a nonconnected reductive group $\dot{G}^+$ over $\dot{F}$ with $\dot{G}^0 = \mathrm{Res}_{\dot{E}/\dot{F}}\Big(\mathrm{GL}(n)\Big)$. In either case, $\dot{M}$ is a proper "Levi subset" of $\dot{G}$. Suppose that $V \supset V_{\text{ram}}(\dot{G})$ is a finite set of valuations outside of which $\dot{G}$ and $\dot{E}$ are unramified. The problem is to show that the smooth function

$$\mathrm{e}(\dot{\gamma}_V) = \varepsilon_{\dot{M}}(\dot{\gamma}_V), \qquad \dot{\gamma}_V \in \dot{M}_{G\text{-reg},V},$$

in [AC, (2.17.6)] vanishes. The formula (8.2) has an analogue here. It is the partial vanishing property

$$\mathrm{e}(\dot{\gamma}_V) = 0, \tag{8.8}$$

which applies to any $\dot{\gamma} \in \dot{M}_{G\text{-reg}}(\dot{F})$ such that $\dot{\gamma}_w$ is bounded for every w in the complement of V in the set $W(\dot{E}, V)$ defined as above. This property follows from [AC, (2.17.4), and Lemmas 2.4.2 and 2.4.3], as on p. 194–195 of [AC].

LEMMA 8.2. *Suppose that* $\mathrm{e}(\dot{\gamma}_V)$ *is any smooth function on* $M_{\dot{G}\text{-reg},V}$ *that vanishes under the conditions of* (8.8). *Then* $\mathrm{e}(\dot{\gamma}_V)$ *vanishes for any* $\dot{\gamma}_V$ *in* $\dot{M}_{G\text{-reg},V}$.

Proof. Suppose $\dot{E} = \dot{F}$. Then $W(\dot{E}, V)$ equals V, by definition. Since $\Gamma_{\dot{F}}$ acts trivially on $Z(\widehat{\dot{M}})$, $\dot{M}(\dot{F})$ is dense in $\dot{M}_V$ [KR, Lemma 1(b)]. The lemma then follows in this case from (8.8).

We can therefore assume that $\dot{E} \neq \dot{F}$. If W is any set of valuations of $\dot{F}$, let $W^{\sim}$ denote the set of valuations of $\dot{E}$ that divide valuations in W. We also write $\dot{G}^{\sim}$ for the general linear group of rank n over $\dot{E}$, and $\dot{M}^{\sim}$ for the Levi subgroup of $\dot{G}^{\sim}$ corresponding to $\dot{M}$. There is then a bijection $\dot{\gamma} \to \dot{\gamma}^{\sim}$ from $\dot{M}(\dot{F})$ onto $\dot{M}^{\sim}(\dot{E})$, and a compatible bijection $\dot{\gamma}_V \to \dot{\gamma}_{V^{\sim}}$ from $\dot{M}_{\dot{G}\text{-reg},V}$ onto $\dot{M}^{\sim}_{\dot{G}\text{-reg},V^{\sim}}$. It would be enough to show that the smooth function

$$\mathrm{e}^{\sim}(\dot{\gamma}_{V^{\sim}}) = \mathrm{e}(\dot{\gamma}_V), \qquad \dot{\gamma}_V \in \dot{M}_{\dot{G}\text{-reg},V},$$

on $\dot{M}^{\sim}_{\dot{G}\text{-reg},V^{\sim}}$ vanishes.

It follows from [KR, Lemma 1(b)] that $\dot{M}^{\sim}(\dot{E})$ is dense in $\dot{M}^{\sim}_{V^{\sim}}$. We may therefore assume that $\dot{\gamma}_{V^{\sim}}$ is the image of an element in $\dot{M}^{\sim}_{\dot{G}\text{-reg}}(\dot{E})$, and in particular that $\dot{\gamma}_{V^{\sim}}$ lies in $\dot{T}^{\sim}_{\dot{G}\text{-reg}}(\dot{E}_{V^{\sim}})$, for a maximal torus $\dot{T}^{\sim}$ in $\dot{M}^{\sim}$ over $\dot{E}$. Set

$$W^{\sim} = W^{\sim}(\dot{E}, V) = W(\dot{E}, V^{\sim}).$$

Following the notation of the proof of the last lemma, we write $\dot{T}^{\sim}_{V^{\sim},W^{\sim}}$ for the closure in $\dot{T}^{\sim}_{V^{\sim}} = \dot{T}^{\sim}(\dot{E}_{V^{\sim}})$ of the set of points $\dot{\gamma}^{\sim}$ in $\dot{T}^{\sim}(\dot{E})$ that are bounded at each valuation in the complement of $V^{\sim}$ in $W^{\sim}$. If $\dot{\gamma}^{\sim}$ is of this form, and is also $\dot{G}$-regular, the preimage $\dot{\gamma}$ of $\dot{\gamma}^{\sim}$ in $\dot{M}(\dot{F})$ satisfies (8.8). It follows that

$$\mathrm{e}^{\sim}(\dot{\gamma}_{V^{\sim}}) = 0,$$

for any $\dot{G}$-regular point $\dot{\gamma}_{V^{\sim}}$ in $\dot{T}^{\sim}_{V^{\sim},W^{\sim}}$. It would therefore be enough to show that $\dot{T}^{\sim}_{V^{\sim},W^{\sim}}$ equals $\dot{T}^{\sim}_{V^{\sim}}$. Replacing $V^{\sim}$ by a finite set $V_1^{\sim}$ that contains $V^{\sim}$, if necessary, we can assume that $\dot{T}^{\sim}$ is unramified outside of $V^{\sim}$. For if $\dot{\gamma}_{V_1^{\sim}}$ is any point in $\dot{T}^{\sim}_{V_1^{\sim}}$ that is bounded at each place in $W^{\sim} \cap (V_1^{\sim} - V^{\sim})$, and $\dot{\gamma}^{\sim}$ is a point in $\dot{T}^{\sim}(\dot{E})$ that is bounded at each place in $W^{\sim} - V_1^{\sim}$, and approximates $\dot{\gamma}_{V^{\sim}}$, then $\dot{\gamma}^{\sim}$ is bounded at each place in the set

$$W^{\sim} - V^{\sim} = (W^{\sim} - V_1^{\sim}) \cup \Big(W^{\sim} \cap (V_1^{\sim} - V^{\sim})\Big),$$

and approximates the component $\dot{\gamma}_{V^{\sim}}$ of $\dot{\gamma}_{V_1^{\sim}}$ in $\dot{T}^{\sim}_{V^{\sim}}$.

We shall again use Langlands duality for tori. As in the proof of the last lemma, the quotient $\dot{T}^{\sim}_{V^{\sim}}/\dot{T}^{\sim}_{V^{\sim},W^{\sim}}$ is dual to the group

$$H^1_{V^{\sim}}(W_{\dot{E}}, \widehat{\dot{T}}^{\sim})^{\mathrm{lt}}_{W^{\sim}} = H^1_{V^{\sim}}(W_{\dot{E}}, \widehat{\dot{T}}^{\sim})_{W^{\sim}}/H^1(W_{\dot{E}}, \widehat{\dot{T}})_{\mathrm{lt}}.$$

Recall that $H^1_{V^{\sim}}(W_{\dot{E}}, \widehat{\dot{T}}^{\sim})_{W^{\sim}}$ is the kernel of the map

$$H^1_{V^{\sim}}(W_{\dot{E}}, \widehat{\dot{T}}^{\sim}) \longrightarrow \bigoplus_{w^{\sim} \in S^{\sim}} H^1(\dot{E}_{w^{\sim}}, \widehat{\dot{T}}^{\sim}),$$

where $H^1_{V^{\sim}}(W_{\dot{E}}, \widehat{\dot{T}}^{\sim})$ denotes the subgroup of elements in $H^1(W_{\dot{E}}, \widehat{\dot{T}}^{\sim})$ that are unramified outside of $V^{\sim}$, and

$$S^{\sim} = S^{\sim}(\dot{E}, V) = S(\dot{E}, V^{\sim}).$$

We have only to show that any class in $H^1_{V^{\sim}}(W_{\dot{E}}, \widehat{\dot{T}}^{\sim})_{W^{\sim}}$ is locally trivial. Now $S^{\sim}$ represents a set of valuations on $\dot{E}$ of positive density. It follows from results on equidistribution [Se, Th. 2, p. I-23] that any class in $H^1_{V^{\sim}}(W_{\dot{E}}, \widehat{\dot{T}}^{\sim})_{W^{\sim}}$ is the inflation of a class in $H^1(\dot{E}^{\sim}/\dot{E}, \widehat{\dot{T}}^{\sim})$, for a Galois extension $\dot{E}^{\sim} \supset \dot{E}$ that splits $\dot{T}^{\sim}$, and is unramified outside of $V^{\sim}$. But $\dot{T}^{\sim}$ is a maximal torus in a general linear group. We can therefore assume by Shapiro's lemma that $\mathrm{Gal}(\dot{E}^{\sim}/\dot{E})$ acts trivially on the dual torus $\widehat{\dot{T}}^{\sim}$. Furthermore, any conjugacy class in $\mathrm{Gal}(\dot{E}^{\sim}/\dot{E})$ is the Frobenius class of some valuation in $S^{\sim}$. It follows that any element in $H^1(\dot{E}^{\sim}/\dot{E}, \widehat{\dot{T}}^{\sim})$ that is locally trivial at each place in $S^{\sim}$ is in fact trivial. The group $H^1_{V^{\sim}}(W_{\dot{E}}, \widehat{\dot{T}})_{W^{\sim}}$ is therefore actually zero. We conclude that $\dot{T}^{\sim}_{V^{\sim},W^{\sim}}$ equals $\dot{T}^{\sim}_{V^{\sim}}$, as required. □

9. Global Theorems 1 and 2

We are now at the final stage of our induction argument. Our task is to prove Global Theorems 1 and 2. This will take care of the part of the argument that depends on the remaining integer d_{der}.

We revert back to the setting of the first half of the paper, in which F is a global field. Then G is a global K-group over F that satisfies Assumption 5.2 of [I], such that $\dim(G_{\mathrm{der}}) = d_{\mathrm{der}}$. As usual, (Z, ζ) represents a pair of central data for G. Let V be a finite set of valuations of F that contains $V_{\mathrm{ram}}(G, \zeta)$. The local results completed in Section 8 imply that Local Theorems $1'$ and $2'$

of [I, §6] are valid for functions f in $\mathcal{H}(G_V, \zeta_V)$. The resulting simplification of the formulas established in Section 2–5 will lead directly to a proof of the global theorems.

Recall the linear forms $I_{\mathrm{par}}(f)$, $I^{\mathcal{E}}_{\mathrm{par}}(f)$ and $S^G_{\mathrm{par}}(f)$ introduced in Section 2. According to Local Theorem $1'$(a), we have

$$\begin{aligned}
&I^{\mathcal{E}}_{\mathrm{par}}(f) - I_{\mathrm{par}}(f)\\
&\qquad = \sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma)\Big(I^{\mathcal{E}}_M(\gamma,f) - I_M(\gamma,f)\Big) = 0,
\end{aligned}$$

for any f in $\mathcal{H}(G_V, \zeta_V)$. If G is quasisplit, the two assertions of Local Theorem $1'$(b) imply that

$$\begin{aligned}
S^G_{\mathrm{par}}(f) &= \sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M,M')\\
&\qquad \cdot \sum_{\delta'\in\Delta(\widetilde{M}',V,\widetilde{\zeta}')} b^{\widetilde{M}'}(\delta')S^G_M(\widetilde{M}',\delta',f)\\
&= \sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\delta^*\in\Delta(M^*,V,\zeta^*)} b^{M^*}(\delta^*)S^G_M(M^*,\delta^*,f) = 0,
\end{aligned}$$

for any function f in $\mathcal{H}^{\mathrm{uns}}(G_V,\zeta_V)$. The left-hand sides of the expressions (2.4) and (2.5) in Proposition 2.2 thus vanish. It remains only to consider the corresponding right-hand sides.

We have already finished the part of the general induction argument that applies to the integer r_{der}. The assertions of Corollary 5.2 therefore hold for any Levi subgroup M of G, and in particular, if M equals the minimal Levi subgroup M_0. In other words, the identity

$$I^{\mathcal{E}}_{\nu,\mathrm{disc}}(f) - I_{\nu,\mathrm{disc}}(f) = 0 \tag{9.1}$$

of Proposition 4.2(a) is valid for any f in the space $\mathcal{H}_{M_0}(G_V,\zeta_V) = \mathcal{H}(G_V,\zeta_V)$. Similarly, if G is quasisplit, the identity

$$S^G_{\nu,\mathrm{disc}}(f) = 0 \tag{9.2}$$

of Proposition 4.2(b) is valid for any f in the space $\mathcal{H}^{\mathrm{uns}}_{M_0}(G_V,\zeta_V) = \mathcal{H}^{\mathrm{uns}}(G_V,\zeta_V)$. In particular, the terms

$$I^{\mathcal{E}}_{t,\mathrm{disc}}(f) - I_{t,\mathrm{disc}}(f) = \sum_{\{\nu:\,\|\mathrm{Im}(\nu)\|=t\}} \Big(I^{\mathcal{E}}_{\nu,\mathrm{disc}}(f) - I_{\nu,\mathrm{disc}}(f)\Big), \qquad f\in\mathcal{H}(G_V,\zeta_V),$$

and

$$S^G_{t,\mathrm{disc}}(f) = \sum_{\{\nu:\,\|\mathrm{Im}\,\nu\|=t\}} S^G_{\nu,\mathrm{disc}}(f), \qquad f\in\mathcal{H}^{\mathrm{uns}}(G_V,\zeta_V),$$

on the right-hand sides of (2.4) and (2.5) both vanish. Having already observed that the left-hand sides of these formulas vanish, we conclude that the sum of the remaining terms on each right-hand side vanishes. In other words,

$$\sum_z \Big(I^{\mathcal{E}}_{z,\mathrm{unip}}(f,S) - I_{z,\mathrm{unip}}(f,S)\Big) = 0, \qquad f \in \mathcal{H}(G_V,\zeta_V), \tag{9.3}$$

and

$$\sum_z S^G_z(f,S) = 0, \qquad f \in \mathcal{H}^{\mathrm{uns}}(G_V,\zeta_V), \tag{9.4}$$

in the case that G is quasisplit.

We have two theorems to establish. The geometric Global Theorem 1 applies to any finite set of valuations $S \supset V_{\mathrm{ram}}(G,\zeta)$, and to elements $\dot{\gamma}_S \in \Gamma^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$ and $\dot{\delta}_S \in \Delta^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$ that are admissible in the sense of [I, §1]. According to [II, Prop. 2.1] (and the trivial case of [II, Cor. 2.2]), the global descent formulas of [II] reduce Global Theorem 1 to the case of unipotent elements. We can therefore assume that $\dot{\gamma}_S$ and $\dot{\delta}_S$ belong to the respective subsets $\Gamma^{\mathcal{E}}_{\mathrm{unip}}(G,S,\zeta)$ and $\Delta^{\mathcal{E}}_{\mathrm{unip}}(G,S,\zeta)$ of $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$ and $\Delta^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$. To deal with this case, we shall apply the formulas (9.3) and (9.4), with V equal to S, and $f = \dot{f}_S$ an admissible function in $\mathcal{H}(G_S,\zeta_S)$.

The formulas (2.1) and (2.2) provide expansions for the summands on the left-hand side of (9.3). We obtain

$$\begin{aligned}
&\sum_{z\in Z(\overline{G})_{S,\mathfrak{o}}} \ \sum_{\dot{\alpha}_S\in\Gamma^{\mathcal{E}}_{\mathrm{unip}}(G,S,\zeta)} \Big(a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\alpha}_S) - a^G_{\mathrm{ell}}(\dot{\alpha}_S)\Big)\dot{f}_{S,G}(z\dot{\alpha}_S)\\
&\qquad = \sum_z\sum_{\dot{\alpha}_S}\Big(a^{G,\mathcal{E}}_{\mathrm{unip}}(\dot{\alpha}_S,S) - a^G_{\mathrm{unip}}(\dot{\alpha}_S,S)\Big)\dot{f}_{S,z,G}(\dot{\alpha}_S)\\
&\qquad = \sum_z \Big(I^{\mathcal{E}}_{z,\mathrm{unip}}(\dot{f}_S,S) - I_{z,\mathrm{unip}}(\dot{f}_S,S)\Big) = 0,
\end{aligned}$$

since the identities $a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\alpha}_S) = a^{G,\mathcal{E}}_{\mathrm{unip}}(\dot{\alpha}_S,S)$ and $a^G_{\mathrm{ell}}(\dot{\alpha}_S) = a^G_{\mathrm{unip}}(\dot{\alpha}_S,S)$ are trivial consequences of the fact that $V = S$. But the linear forms

$$\dot{f}_S \longrightarrow \dot{f}_{S,G}(z\dot{\alpha}_S), \qquad z \in Z(\overline{G})_{S,\mathfrak{o}},\ \dot{\alpha}_S \in \Gamma^{\mathcal{E}}_{\mathrm{unip}}(G,S,\zeta),$$

on the subspace of admissible functions in $\mathcal{H}(G_S,\zeta_S)$ are linearly independent. We conclude that

$$a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\alpha}_S) - a^G_{\mathrm{ell}}(\dot{\alpha}_S) = 0,$$

for any element $\dot{\alpha}_S$ in $\Gamma^{\mathcal{E}}_{\mathrm{unip}}(G,S,\zeta)$. This completes the proof of part (a) of Global Theorem 1 for $\dot{\gamma}_S$ unipotent, and hence in general.

To deal with part (b) of Global Theorem 1, we take G to be quasisplit, and set

$$\Delta^{\mathcal{E},0}_{\mathrm{unip}}(G,S,\zeta) = \Delta^{\mathcal{E}}_{\mathrm{unip}}(G,S,\zeta) - \Delta_{\mathrm{unip}}(G,S,\zeta).$$

The formula (2.3) provides an expansion for the summands on the left-hand side of (9.4). Taking $\dot{f}_S$ to be unstable, we obtain

$$\begin{aligned}\sum_{z\in Z(\overline{G})_{S,\mathfrak{o}}}\ \sum_{\dot\beta_S\in\Delta^{\mathcal{E},0}_{\mathrm{unip}}(G,S,\zeta)} & b^G_{\mathrm{ell}}(\dot\beta_S)\dot{f}^{\mathcal{E}}_{S,G}(z\dot\beta_S)\\ &= \sum_z \sum_{\dot\beta_S\in\Delta^{\mathcal{E}}_{\mathrm{unip}}(G,S,\zeta)} b^G_{\mathrm{unip}}(\dot\beta_S,S)\dot{f}^{\mathcal{E}}_{S,z,G}(\dot\beta_S)\\ &= \sum_z S^G_z(\dot{f}_S,S)=0,\end{aligned}$$

since $\dot{f}^{\mathcal{E}}_{S,z,G}$ vanishes on $\Delta_{\mathrm{unip}}(G,S,\zeta)$. But the linear forms

$$\dot{f}_S \longrightarrow \dot{f}^{\mathcal{E}}_{S,G}(z\dot\beta_S), \qquad z\in Z(\overline{G})_{S,\mathfrak{o}},\ \dot\beta_S\in\Delta^{\mathcal{E},0}_{\mathrm{unip}}(G,S,\zeta),$$

on the subspace of admissible functions in $\mathcal{H}^{\mathrm{uns}}(G_S,\zeta_S)$ are linearly independent. We conclude that

$$b^G_{\mathrm{ell}}(\dot\beta_S)=0,$$

for any element $\dot\beta_S$ in the complement $\Delta^{\mathcal{E},0}_{\mathrm{unip}}(G,S,\zeta)$ of $\Delta_{\mathrm{unip}}(G,S,\zeta)$ in $\Delta^{\mathcal{E}}_{\mathrm{unip}}(G,S,\zeta)$. This completes the proof of part (b) of Global Theorem 1 for $\dot\delta_S$ unipotent, and hence in general.

The spectral Global Theorem 2 concerns adelic elements $\dot\pi\in\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$ and $\dot\phi\in\Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$. We shall apply the formulas (9.1) and (9.2), which pertain to functions $f\in\mathcal{H}(G_V,\zeta_V)$. Recall that for any f in $\mathcal{H}(G_V,\zeta_V)$, $\dot{f}=fu^V$ is a function in the adelic Hecke algebra $\mathcal{H}(G,\zeta)=\mathcal{H}\Big(G(\mathbb{A}),\zeta\Big)$. Conversely, any $\dot{f}\in\mathcal{H}(G,\zeta)$ can be obtained in this way from a function $f\in\mathcal{H}(G_V,\zeta_V)$, for some finite set $V\supset V_{\mathrm{ram}}(G,\zeta)$.

We combine (9.1) with the expansions in [I, (3.6)] and the first part of [I, Lemma 7.3(a)]. We obtain

$$\begin{aligned}\sum_{\dot\pi\in\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)} & \Big(a^{G,\mathcal{E}}_{\mathrm{disc}}(\dot\pi)-a^G_{\mathrm{disc}}(\dot\pi)\Big)\dot{f}_G(\dot\pi)\\ &= I^{\mathcal{E}}_{t,\mathrm{disc}}(\dot{f})-I_{t,\mathrm{disc}}(\dot{f})\\ &= I^{\mathcal{E}}_{t,\mathrm{disc}}(f)-I_{t,\mathrm{disc}}(f)\\ &= \sum_{\{\nu:\,\|\mathrm{Im}(\nu)\|=t\}}\Big(I^{\mathcal{E}}_{\nu,\mathrm{disc}}(f)-I_{\nu,\mathrm{disc}}(f)\Big)=0,\end{aligned}$$

for $\dot{f}$ and f related as above. But the linear forms

$$\dot{f}\longrightarrow \dot{f}_G(\dot\pi), \qquad \dot\pi\in\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta),$$

on $\mathcal{H}(G,\zeta)$ are linearly independent. We conclude that

$$a^{G,\mathcal{E}}_{\mathrm{disc}}(\dot{\pi}) - a^{G}_{\mathrm{disc}}(\dot{\pi}) = 0,$$

for any element $\dot{\pi}$ in $\Pi_{t,\mathrm{disc}}(G,\zeta)$. This is the required assertion of part (a) of Global Theorem 2.

To deal with part (b) of Global Theorem 2, we take G to be quasisplit, and we set

$$\Phi^{\mathcal{E},0}_{t,\mathrm{disc}}(G,\zeta) = \Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta) - \Phi_{t,\mathrm{disc}}(G,\zeta).$$

We combine (9.2) with the first expansion in [I, Lemma 7.3(b)]. Assume that $\dot{f}$ belongs to $\mathcal{H}^{\mathrm{uns}}(G,\zeta)$. Then $\dot{f} = fu^V$, for some V and some $f \in \mathcal{H}^{\mathrm{uns}}(G_V,\zeta_V)$. We obtain

$$\begin{aligned}\sum_{\dot{\phi}\in\Phi^{\mathcal{E},0}_{t,\mathrm{disc}}(G,\zeta)} b^G_{\mathrm{disc}}(\dot{\phi})\dot{f}^{\mathcal{E}}_G(\dot{\phi}) &= \sum_{\dot{\phi}\in\Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)} b^G_{\mathrm{disc}}(\dot{\phi})\dot{f}^{\mathcal{E}}_G(\dot{\phi}) = S^G_{t,\mathrm{disc}}(\dot{f})\\ &= S^G_{t,\mathrm{disc}}(f) = \sum_{\{\nu:\,\|\mathrm{Im}(\nu)\|=t\}} S^G_{t,\mathrm{disc}}(f) = 0,\end{aligned}$$

since $\dot{f}^{\mathcal{E}}_G$ vanishes on $\Phi_{t,\mathrm{disc}}(G,\zeta)$. But the linear forms

$$\dot{f} \longrightarrow \dot{f}^{\mathcal{E}}_G(\dot{\phi}), \qquad \dot{\phi}\in\Phi^{\mathcal{E},0}_{t,\mathrm{disc}}(G,\zeta),$$

on $\mathcal{H}^{\mathrm{uns}}(G,\zeta)$ are linearly independent. We conclude that

$$b^G_{\mathrm{disc}}(\dot{\phi}) = 0,$$

for any $\dot{\phi}$ in the complement $\Phi^{\mathcal{E},0}_{t,\mathrm{disc}}(G,\zeta)$ of $\Phi_{t,\mathrm{disc}}(G,\zeta)$ in $\Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$. This is the required assertion of part (b) of Global Theorem 2.

We have shown that the assertions of Global Theorems 1 and 2 are all valid for G. As we recalled in Section 1, this implies that all the global theorems in [I, §7] hold for G. With the proof of Global Theorems 1 and 2, we have completed the part of the induction argument that depends on the remaining integer d_{der}. We have thus finished the last step of an inductive proof that began formally in Section 1, but which has really been implicit in definitions and results from [I] and [II], and related papers.

10. Concluding remarks

We have solved the problems posed in Section 1. That is, we have proved Local Theorem 1 and Global Theorems 1 and 2. This completes the proof of the local theorems stated in [I, §6] and the global theorems stated in [I, §7].

The results are valid for any K-group that satisfies Assumption 5.2 of [I]. We recall once again that any connected reductive group G_1 is a component of an essentially unique K-group G. The theorems for G, taken as a whole, represent a slight generalization of the corresponding set of theorems for G_1.

Recall that Assumption 5.2 of [I] is the assertion that various forms of the fundamental lemma are valid. It has been established in a limited number of cases [I, §5]. For example, it holds if G equals $\mathrm{SL}(p)$, for p prime. The assumption includes the standard form of the fundamental lemma for both the group and its Lie algebra. I expect that the equivalence of the two must be known, for almost all places v, but I have not checked it myself. Granting this, Assumption 5.2 of [I] also holds if G is an inner K-form of $\mathrm{GSp}(4)$ or $\mathrm{SO}(5)$.

The theorems yield a stabilization of the trace formula. This amounts to the construction of a stable trace formula, and a decomposition of the ordinary trace formula into stable trace formulas for endoscopic groups. We shall conclude the paper with a brief recapitulation of the process.

Suppose that F is global, and that G is a K-group over F with central data (Z,ζ). Let f be a function in $\mathcal{H}(G,V,\zeta)$, where V is a finite set of valuations of F that contains $V_{\mathrm{ram}}(G,\zeta)$. The *ordinary trace formula* is the identity given by two different expansions:

$$I(f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma)I_M(\gamma,f) \tag{10.1}$$

and

$$\begin{aligned} I(f) &= \sum_t I_t(f) \\ &= \sum_t \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi_t(M,V,\zeta)} a^M(\pi)I_M(\pi,f)d\pi \end{aligned} \tag{10.2}$$

of a linear form $I(f)$ on $\mathcal{H}(G,V,\zeta)$. The *stable trace formula* applies to the case that G is quasisplit. It is the identity given by two different expansions:

$$S(f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta\in\Delta(M,V,\zeta)} b^M(\delta)S_M(\delta,f) \tag{10.3}$$

and

$$\begin{aligned} S(f) &= \sum_t S_t(f) \\ &= \sum_t \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Phi_t(M,V,\zeta)} b^M(\phi)S_M(\phi,f) \end{aligned} \tag{10.4}$$

of a stable linear form $S(f) = S^G(f)$ on $\mathcal{H}(G,V,\zeta)$. The theorems assert that the terms in these two expansions are in fact stable, and that the more complicated expansions in [I] reduce to the ones above. (See [I, Lemma 7.2(b), Lemma 7.3(b), (10.5) and (10.18)].) The actual stabilization can be described in terms of the *endoscopic trace formula*, *a priori* a third trace formula. It is the identity given by two different expansions

$$I^{\mathcal{E}}(f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma^{\mathcal{E}}(M,V,\zeta)} a^{M,\mathcal{E}}(\gamma)I_M^{\mathcal{E}}(\gamma,f) \tag{10.5}$$

and

$$(10.6)\quad I^{\mathcal{E}}(f) = \sum_t I_t^{\mathcal{E}}(f) = \sum_t \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi_t^{\mathcal{E}}(M,V,\zeta)} a^{M,\mathcal{E}}(\pi) I_M^{\mathcal{E}}(\pi,f)d\pi$$

of a third linear form $I^{\mathcal{E}}(f)$ on $\mathcal{H}(G,V,\zeta)$. The theorems assert that there is a term by term identification of these two expansions with the original ones.

The linear form $I(f)$ is defined explicitly by either of the two expansions (10.1) and (10.2). The other two linear forms are defined inductively in terms of $I(f)$ by setting

$$(10.7)\qquad I^{\mathcal{E}}(f) = \sum_{G'\in\mathcal{E}^0_{\mathrm{ell}}(G,V)} \iota(G,G')\widehat{S}'(f') + \varepsilon(G)S(f),$$

and also $I^{\mathcal{E}}(f) = I(f)$ in case G is quasisplit. Since the terms in (10.3) and (10.4) are stable, the linear form $S(f)$ is indeed stable. Since the terms in (10.5) and (10.6) are equal to the corresponding terms in (10.1) and (10.2), respectively, $I^{\mathcal{E}}(f)$ equals $I(f)$ in general. The definition (10.7) therefore reduces to the identity

$$(10.8)\qquad I(f) = \sum_{G'\in\mathcal{E}(G,V)} \iota(G,G')\widehat{S}'(f').$$

In particular, it represents a decomposition of the ordinary trace formula into stable trace formulas for endoscopic groups.

The reason for stabilizing the trace formula is to establish relationships among the spectral coefficients $a^G(\pi)$, $b^G(\phi)$ and $a^{G,\mathcal{E}}(\pi)$. These are of course the terms that concern automorphic representations. The relationships among them are given by Global Theorem 2. The proof of this theorem is indirect, being a consequence of the relationships established among the complementary terms, and of the trace formulas themselves. Having completed the process, one might be inclined to ignore the stable trace formula, and the relationships among the complementary terms. However, the general stable trace formula is likely to have other applications. For example, its analogue for function fields will surely be needed to extend the results of Lafforgue for $\mathrm{GL}(n)$.

The theorems also yield a stabilization of the local trace formula. Suppose that F is local, and that G is a local K-group over F with central data (Z,ζ). Let $f = f_1 \times \overline{f}_2$ be a function in $\mathcal{H}(G,V,\zeta)$, where $V = \{v_1, v_2\}$ as in Section 6. The *ordinary local trace formula* is the identity given by two different expansions

$$(10.9)\qquad I(f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{G\text{-reg,ell}}(M,V,\zeta)} I_M(\gamma,f)d\gamma$$

and

$$I_{\mathrm{disc}}(f) = \int_{T_{\mathrm{disc}}(G,V,\zeta)} i^G(\tau) f_G(\tau) d\tau \tag{10.10}$$

of a linear form $I(f) = I_{\mathrm{disc}}(f)$ on $\mathcal{H}(G, V, \zeta)$. The *stable local trace formula* applies to the case that G is quasisplit. It is the identity given by two different expansions

(10.11)

$$S(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Delta_{G\text{-reg,ell}}(M,V,\zeta)} n(\delta)^{-1} S_M(\delta, f) d\delta$$

and

$$S_{\mathrm{disc}}(f) = \int_{\Phi_{\mathrm{disc}}(G,V,\zeta)} s^G(\phi) f^G(\phi) d\phi \tag{10.12}$$

of a stable linear form

$$S(f) = S^G(f) = S^G_{\mathrm{disc}}(f) = S_{\mathrm{disc}}(f)$$

on $\mathcal{H}(G, V, \zeta)$. The theorems assert that the terms in these two expansions are in fact stable, and that the more complicated expressions (6.6) and (6.12) reduce to the ones above. (See [A10, (9.8)].) The actual stabilization can again be described in terms of what is *a priori* a third trace formula. The *endoscopic local trace formula* is the identity given by two different expansions

(10.13)

$$I^{\mathcal{E}}(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{G\text{-reg,ell}}(M,V,\zeta)} I^{\mathcal{E}}_M(\gamma, f) d\gamma$$

and

$$I^{\mathcal{E}}_{\mathrm{disc}}(f) = \int_{T^{\mathcal{E}}_{\mathrm{disc}}(G,V,\zeta)} \iota^{G,\mathcal{E}}(\tau) f_G(\tau) d\tau \tag{10.14}$$

of a third linear form $I^{\mathcal{E}}(f) = I^{\mathcal{E}}_{\mathrm{disc}}(f)$ on $\mathcal{H}(G, V, \zeta)$. The theorems assert that there is a term by term identification of these two expansions with the original ones.

The linear form $I(f) = I_{\mathrm{disc}}(f)$ is defined by the right-hand side of (10.9) or (10.10). The other two linear forms are defined inductively in terms of $I(f)$ by setting

$$I^{\mathcal{E}}(f) = \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G)} \iota(G, G') \widehat{S}'(f') + \varepsilon(G) S(f), \tag{10.15}$$

and also $I^{\mathcal{E}}(f) = I(f)$ in case G is quasisplit. Since the terms in (10.11) and (10.12) are stable, the linear form $S(f)$ is indeed stable. Since the terms in (10.13) and (10.14) are equal to the corresponding terms in (10.9) and (10.10), respectively, $I^{\mathcal{E}}(f)$ equals $I(f)$ in general. The definition (10.15) therefore reduces to the identity

$$I(f) = \sum_{G' \in \mathcal{E}(G)} \iota(G, G') \widehat{S}'(f'). \tag{10.16}$$

In particular, it represents a decomposition of the ordinary local trace formula into stable local trace formulas for endoscopic groups.

It is interesting to note that the stabilization of the local trace formula is almost completely parallel to that of the global trace formula. This seems remarkable, especially since the terms in the various local expansions stand for completely separate objects. The stable local trace formula is not so deep as its global counterpart. It has no direct bearing on automorphic representations, even though it was required at one point for the global stabilization. However, one could imagine direct applications of the stable local trace formula to questions in p-adic algebraic geometry.

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