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Pair correlation densities of inhomogeneous quadratic forms

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Abstract

Under explicit diophantine conditions on $(\alpha,\beta)\in\mathbb{R}^2$, we prove that the local two-point correlations of the sequence given by the values $(m-\alpha)^2+(n-\beta)^2$, with $(m,n)\in\mathbb{Z}^2$, are those of a Poisson process. This partly confirms a conjecture of Berry and Tabor [2] on spectral statistics of quantized integrable systems, and also establishes a particular case of the quantitative version of the Oppenheim conjecture for inhomogeneous quadratic forms of signature (2,2). The proof uses theta sums and Ratner's classification of measures invariant under unipotent flows.

1. Introduction

1.1. Let us denote by $0\le\lambda_1\le\lambda_2\le\cdots\to\infty$ the infinite sequence given by the values of

$$(m-\alpha)^2+(n-\beta)^2$$

at lattice points $(m,n)\in\mathbb{Z}^2$, for fixed $\alpha,\beta\in[0,1]$. In a numerical experiment, Cheng and Lebowitz [3] found that, for generic α,β, the local statistical measures of the *deterministic* sequence λ_j appear to be those of independent *random* variables from a Poisson process.

1.2. This numerical observation supports a conjecture of Berry and Tabor [2] in the context of quantum chaos, according to which the local eigenvalue statistics of generic quantized integrable systems are Poissonian. In the case discussed here, the λ_j may be viewed (up to a factor $4\pi^2$) as the eigenvalues of the Laplacian

$$-\Delta=-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}$$

with quasi-periodicity conditions

$$\varphi(x+k,y+l)=\mathrm{e}^{-2\pi\mathrm{i}(\alpha k+\beta l)}\varphi(x,y),\quad k,l\in\mathbb{Z}.$$

The corresponding classical dynamical system is the geodesic flow on the unit tangent bundle of the flat torus $\mathbb{T}^2$.

1.3. The asymptotic density of the sequence of λ_j is π, according to the well known formula for the number of lattice points in a large, shifted circle:

$$\#\{j : \lambda_j \leq \lambda\} = \#\{(m,n) \in \mathbb{Z}^2 : (m-\alpha)^2 + (n-\beta)^2 \leq \lambda\} \sim \pi\lambda$$

for $\lambda \to \infty$. The rate of convergence is discussed in detail by Kendall [11].

1.4. More generally, suppose we have a sequence $\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ of mean density D, i.e.,

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \#\{j : \lambda_j \leq \lambda\} = D.$$

For a given interval $[a,b] \subset \mathbb{R}$, the *pair correlation function* is then defined as

$$R_2[a,b](\lambda) = \frac{1}{D\lambda} \#\{j \neq k : \lambda_j \leq \lambda,\ \lambda_k \leq \lambda,\ a \leq \lambda_j - \lambda_k \leq b\}.$$

The following result is classical.

1.5. Theorem. *If the λ_j come from a Poisson process with mean density D,*

$$\lim_{\lambda\to\infty} R_2[a,b](\lambda) = D(b-a)$$

almost surely.

1.6. We will assume throughout most of the paper that $\alpha, \beta, 1$ are linearly independent over $\mathbb{Q}$. This makes sure that there are no systematic degeneracies in the sequence, which would contradict the independence we wish to establish. The symmetries leading to those degeneracies can, however, be removed without much difficulty. This will be illustrated in Appendix A.

1.7. We shall need a mild diophantine condition on α. An irrational number $\alpha \in \mathbb{R}$ is called *diophantine* if there exist constants $\kappa, C > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^\kappa}$$

for all $p, q \in \mathbb{Z}$. The smallest possible value of κ is $\kappa = 2$ [26]. We will say α is of *type* κ.

1.8. Theorem. *Suppose $\alpha, \beta, 1$ are linearly independent over $\mathbb{Q}$, and assume α is diophantine. Then*

$$\lim_{\lambda\to\infty} R_2[a,b](\lambda) = \pi(b-a).$$

This proves the Berry-Tabor conjecture for the spectral two-point correlations of the Laplacian in 1.2.

It is well known that almost all α (in the measure-theoretic sense) are diophantine [26]. We therefore have the following corollary.

1.9. Corollary. *Let α, β be independent uniformly distributed random variables in* $[0,1]$. *Then*

$$\lim_{\lambda\to\infty} R_2[a,b](\lambda) = \pi(b-a)$$

almost surely.

1.10. *Remark.* In [4], Cheng, Lebowitz and Major proved convergence of the expectation value[1]

$$\lim_{\lambda\to\infty} \mathbb{E}R_2[a,b](\lambda) = \pi(b-a),$$

that is, on average over α, β.

1.11. *Remark.* Notice that Theorem 1.8 is much stronger than the corollary. It provides explicit examples of "random" deterministic sequences that satisfy the pair correlation conjecture. An admissible choice is for instance $\alpha = \sqrt{2}$, $\beta = \sqrt{3}$ [26].

1.12. The statement of Theorem 1.8 does not hold for any rational α, β, where the pair correlation function is unbounded (see Appendix A.10 for details). This can be used to show that for generic (α, β) (in the topological sense) the pair correlation function does not converge to a uniform density:

1.13. Theorem. *For any $a > 0$, there exists a set $C \subset \mathbb{T}^2$ of second Baire category, for which the following holds.*[2]

(i) *For $(\alpha, \beta) \in C$, there exist arbitrarily large λ such that*

$$R_2[-a,a](\lambda) \geq \frac{\log\lambda}{\log\log\log\lambda}.$$

(ii) *For $(\alpha, \beta) \in C$, there exists an infinite sequence $L_1 < L_2 < \cdots \to \infty$ such that*

$$\lim_{j\to\infty} R_2[-a,a](L_j) = 2\pi a.$$

In the above, $\log\log\log\lambda$ may be replaced by any slowly increasing positive function $\nu(\lambda) \leq \log\log\log\lambda$ with $\nu(\lambda) \to \infty$ $(\lambda \to \infty)$.

1.14. The above results can be extended to the pair correlation densities of forms $(m_1-\alpha_1)^2 + \ldots + (m_k-\alpha_k)^2$ in more than two variables; see [16] for details.

[1] They consider a slightly different statistic, the number of lattice points in a random circular strip of fixed area. The variance of this distribution is very closely related to our pair correlation function.

[2] A set of first Baire category is a countable union of nowhere dense sets. Sets of second category are all those sets which are not of first category.

1.15. *A brief review.* After its formulation in 1977, Sarnak [25] was the first to prove the Berry-Tabor conjecture for the pair correlation of almost all positive definite binary quadratic forms

$$\alpha m^2 + \beta mn + \gamma n^2, \quad m, n \in \mathbb{Z}$$

("almost all" in the measure-theoretic sense). These values represent the eigenvalues of the Laplacian on a flat torus. His proof uses averaging techniques to reduce the pair correlation problem to estimating the number of solutions of systems of diophantine equations. The almost-everywhere result then follows from a variant of the Borel-Cantelli argument. For further related examples of sequences whose pair correlation function converges to the uniform density almost everywhere in parameter space, see [20], [22], [30], [31], [34]. Results on higher correlations have been obtained recently in [21], [23], [32].

Eskin, Margulis and Mozes [8] have recently given explicit diophantine conditions under which the pair correlation function of the above binary quadratic forms is Poisson. Their approach uses ergodic-theoretic methods based on Ratner's classification of measures invariant under unipotent flows. This will also be the key ingredient in our proof for the inhomogeneous set-up. New in the approach presented here is the application of theta sums [13], [14], [15].

The pair correlation problem for binary quadratic forms may be viewed as a special case of the quantitative version of the Oppenheim conjecture for forms of signature (2,2), which is particularly difficult [7].

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2. The plan

2.1. The plan is first to smooth the pair correlation function, i.e., to consider

$$R_2(\psi_1, \psi_2, h, \lambda) = \frac{1}{\pi\lambda} \sum_{j,k} \psi_1\left(\frac{\lambda_j}{\lambda}\right) \psi_2\left(\frac{\lambda_k}{\lambda}\right) \hat{h}(\lambda_j - \lambda_k).$$

Here $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}_+)$ are real-valued, and $\mathcal{S}(\mathbb{R}_+)$ denotes the Schwartz class of infinitely differentiable functions of the half line $\mathbb{R}_+$ (including the origin), which, as well as their derivatives, decrease rapidly at $+\infty$. It is helpful to think of ψ_1, ψ_2 as smoothed characteristic functions, i.e., positive and with compact support. Note that $\hat{h}$ is the Fourier transform of a compactly supported function $h \in \mathrm{C}(\mathbb{R})$, defined by

$$\hat{h}(s) = \int_{\mathbb{R}} h(u) e(\tfrac{1}{2}us)\, du,$$

with the shorthand $e(z) := \mathrm{e}^{2\pi \mathrm{i} z}$.

We will prove the following (Section 8).

2.2. THEOREM. *Let $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}_+)$ be real-valued, and $h \in \mathrm{C}(\mathbb{R})$ with compact support. Suppose $\alpha, \beta, 1$ are linearly independent over $\mathbb{Q}$, and assume α is diophantine. Then*

$$\lim_{\lambda\to\infty} R_2(\psi_1,\psi_2,h,\lambda) = \Big\{ \hat{h}(0) + \pi \int_{\mathbb{R}} \hat{h}(s)\, ds \Big\} \int_0^\infty \psi_1(r)\psi_2(r)\, dr.$$

The first term comes straight from the terms $j = k$; the second one is the more interesting.

Theorem 2.2 implies Theorem 1.8 by a standard approximation argument (Section 8).

2.3. Using the Fourier transform we may write

$$R_2(\psi_1,\psi_2,h,\lambda) = \frac{1}{\pi\lambda} \int_{\mathbb{R}} \Big(\sum_j \psi_1\Big(\frac{\lambda_j}{\lambda}\Big) e(\tfrac{1}{2}\lambda_j u)\Big) \overline{\Big(\sum_j \psi_2\Big(\frac{\lambda_j}{\lambda}\Big) e(\tfrac{1}{2}\lambda_j u)\Big)} h(u)\, du.$$

We will show that the inner sums can be viewed as a theta sum (see 4.14 for details)

$$\theta_\psi(u,\lambda) = \frac{1}{\sqrt{\lambda}} \sum_j \psi\left(\frac{\lambda_j}{\lambda}\right) e(\tfrac{1}{2}\lambda_j u)$$

living on a certain manifold Σ of finite volume (Sections 3 and 4). The integration in

$$R_2(\psi_1,\psi_2,h,\lambda) = \frac{1}{\pi} \int_{\mathbb{R}} \theta_{\psi_1}(u,\lambda) \overline{\theta_{\psi_2}(u,\lambda)} h(u)\, du$$

will then be identified with an orbit of a unipotent flow on Σ, which becomes equidistributed as $\lambda \to \infty$. The equidistribution follows from Ratner's classification of measures invariant under the unipotent flow (Section 5). A crucial subtlety is that Σ is noncompact, and that the theta sum is unbounded on this noncompact space. This requires careful estimates which guarantee that no positive mass of the above integral over a small arc of the orbit escapes to infinity (Section 6).

The only exception is a small neighbourhood of $u = 0$, where in fact a positive mass escapes to infinity, giving a contribution

$$2\pi^2 h(0) \int_0^\infty \psi_1(r)\psi_2(r)\, dr = \pi^2 \int_{\mathbb{R}} \hat{h}(s)\, ds \int_0^\infty \psi_1(r)\psi_2(r)\, dr,$$

which is the second term in Theorem 2.2.

The remaining part of the orbit becomes equidistributed under the above diophantine conditions, which yields

$$\frac{1}{\mu(\Sigma)} \int_\Sigma \theta_{\psi_1} \overline{\theta_{\psi_2}} d\mu \int_{\mathbb{R}} h(u)\, du,$$

where μ is the invariant measure (Section 7). The first integral can be calculated quite easily (Section 8). It is

$$\frac{1}{\mu(\Sigma)} \int_\Sigma \theta_{\psi_1} \overline{\theta_{\psi_2}} d\mu \int_{\mathbb{R}} h(u)\, du = \pi \int_0^\infty \psi_1(r)\psi_2(r)\, dr \int_{\mathbb{R}} h(u)\, du,$$

which finally yields

$$\pi \hat{h}(0) \int_0^\infty \psi_1(r)\psi_2(r)\, dr,$$

the first term in Theorem 2.2.

The proof of Theorem 1.13, which provides a set of counterexamples to the convergence to uniform density, is given in Section 9.

3. Schrödinger and Shale-Weil representation

3.1. Let ω be the standard symplectic form on $\mathbb{R}^{2k}$, i.e.,

$$\omega(\boldsymbol{\xi}, \boldsymbol{\xi}') = \mathbf{x} \cdot \mathbf{y}' - \mathbf{y} \cdot \mathbf{x}',$$

where

$$\boldsymbol{\xi} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, \qquad \boldsymbol{\xi}' = \begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix}, \qquad \mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathbb{R}^k.$$

The *Heisenberg group* $\mathbb{H}(\mathbb{R}^k)$ is then defined as the set $\mathbb{R}^{2k} \times \mathbb{R}$ with multiplication law [12]

$$(\boldsymbol{\xi}, t)(\boldsymbol{\xi}', t') = (\boldsymbol{\xi} + \boldsymbol{\xi}', t + t' + \tfrac{1}{2}\omega(\boldsymbol{\xi}, \boldsymbol{\xi}')).$$

Note that we have the decomposition

$$\left(\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, t \right) = \left(\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}, 0 \right) \left(\begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}, 0 \right) (\mathbf{0}, t - \tfrac{1}{2} \mathbf{x} \cdot \mathbf{y}).$$

3.2. The *Schrödinger representation* of $\mathbb{H}(\mathbb{R}^k)$ on $f \in \mathrm{L}^2(\mathbb{R}^k)$ is given by (cf. [12, p. 15])

$$\begin{aligned}
\left[W\left(\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}, 0 \right) f \right](\mathbf{w}) &= e(\mathbf{x} \cdot \mathbf{w})\, f(\mathbf{w}), && \text{with } \mathbf{x}, \mathbf{w} \in \mathbb{R}^k, \\
\left[W\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}, 0 \right) f \right](\mathbf{w}) &= f(\mathbf{w} - \mathbf{y}), && \text{with } \mathbf{y}, \mathbf{w} \in \mathbb{R}^k, \\
W(\mathbf{0}, t) &= e(t)\, \mathrm{id}, && \text{with } t \in \mathbb{R}.
\end{aligned}$$

Therefore for a general element $(\boldsymbol{\xi},t)$ in $\mathbb{H}(\mathbb{R}^k)$

$$\left[W\left(\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix},t\right)f\right](\mathbf{w}) = e(t-\tfrac{1}{2}\mathbf{x}\cdot\mathbf{y})\; e(\mathbf{x}\cdot\mathbf{w})\; f(\mathbf{w}-\mathbf{y}).$$

3.3. For every element M in the symplectic group $\mathrm{Sp}(k,\mathbb{R})$ of $\mathbb{R}^{2k}$, we can define a new representation W_M of $\mathbb{H}(\mathbb{R}^k)$ by

$$W_M(\boldsymbol{\xi},t) = W(M\boldsymbol{\xi},t).$$

All such representations are irreducible and, by the Stone-von Neumann theorem, unitarily equivalent (see [12] for details). That is, for each $M\in\mathrm{Sp}(k,\mathbb{R})$ there exists a unitary operator $R(M)$ such that

$$R(M)\; W(\boldsymbol{\xi},t)\; R(M)^{-1} = W(M\boldsymbol{\xi},t).$$

The $R(M)$ is determined up to a unitary phase factor and defines the projective *Shale-Weil representation* of the symplectic group. *Projective* means that

$$R(MM') = c(M,M')R(M)R(M')$$

with cocycle $c(M,M')\in\mathbb{C}$, $|c(M,M')|=1$, but $c(M,M')\neq 1$ in general.

3.4. For our present purpose it suffices to consider the group $\mathrm{SL}(2,\mathbb{R})$ which is embedded in $\mathrm{Sp}(k,\mathbb{R})$ by

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} \mapsto \begin{pmatrix} a\,1_k & b\,1_k\\ c\,1_k & d\,1_k\end{pmatrix}$$

where 1_k is the $k\times k$ unit matrix.

The action of $M\in\mathrm{SL}(2,\mathbb{R})$ on $\boldsymbol{\xi}\in\mathbb{R}^{2k}$ is then given by

$$M\boldsymbol{\xi} = \begin{pmatrix} a\mathbf{x}+b\mathbf{y}\\ c\mathbf{x}+d\mathbf{y}\end{pmatrix}, \quad \text{with } M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}, \quad \boldsymbol{\xi}=\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}.$$

3.5. For $M\in\mathrm{SL}(2,\mathbb{R})\hookrightarrow\mathrm{Sp}(k,\mathbb{R})$ we have the explicit representations (see [12, p. 61f]).

$$[R(M)f](\mathbf{w}) = \begin{cases} |a|^{k/2} e(\tfrac{1}{2}\|\mathbf{w}\|^2 ab) f(a\mathbf{w}) & (c=0)\\[2ex] |c|^{-k/2}\displaystyle\int_{\mathbb{R}^k} e\left[\frac{\tfrac{1}{2}(a\|\mathbf{w}\|^2+d\|\mathbf{w}'\|^2)-\mathbf{w}\cdot\mathbf{w}'}{c}\right] f(\mathbf{w}')\,d\mathbf{w}' & (c\neq 0).\end{cases}$$

Here $\|\cdot\|$ denotes the euclidean norm in $\mathbb{R}^k$,

$$\|\mathbf{x}\| = \sqrt{x_1^2+\cdots+x_k^2}.$$

3.6. If

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \quad M_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),$$

with $M_1 M_2 = M_3$, the corresponding cocycle is

$$c(M_1, M_2) = \mathrm{e}^{-\mathrm{i}\pi k \,\mathrm{sign}(c_1 c_2 c_3)/4},$$

where

$$\mathrm{sign}(x) = \begin{cases} -1 & (x < 0) \\ 0 & (x = 0) \\ 1 & (x > 0). \end{cases}$$

3.7. In the special case when

$$M_1 = \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} \cos\phi_2 & -\sin\phi_2 \\ \sin\phi_2 & \cos\phi_2 \end{pmatrix},$$

we find

$$c(M_1, M_2) = \mathrm{e}^{-\mathrm{i}\pi k(\sigma_{\phi_1} + \sigma_{\phi_2} - \sigma_{\phi_1+\phi_2})/4}$$

where

$$\sigma_\phi = \begin{cases} 2\nu & \text{if } \phi = \nu\pi, \\ 2\nu + 1 & \text{if } \nu\pi < \phi < (\nu+1)\pi. \end{cases}$$

3.8. Every $M \in \mathrm{SL}(2, \mathbb{R})$ admits the unique Iwasawa decomposition

$$M = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = (\tau, \phi),$$

where $\tau = u + \mathrm{i}v \in \mathfrak{H}$, $\phi \in [0, 2\pi)$. This parametrization leads to the well known action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathfrak{H} \times [0, 2\pi)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, \phi) = (\frac{a\tau + b}{c\tau + d}, \phi + \arg(c\tau + d) \bmod 2\pi).$$

We will sometimes use the convenient notation $(M\tau, \phi_M) := M(\tau, \phi)$ and $u_M := \mathrm{Re}(M\tau)$, $v_M := \mathrm{Im}(M\tau)$.

3.9. The (projective) Shale-Weil representation of $\mathrm{SL}(2, \mathbb{R})$ reads in these coordinates

$$[R(\tau, \phi) f](\mathbf{w}) = [R(\tau, 0) R(\mathrm{i}, \phi) f](\mathbf{w}) = v^{k/4} e(\tfrac{1}{2}\|\mathbf{w}\|^2 u) [R(\mathrm{i}, \phi) f](v^{1/2}\mathbf{w})$$

and

$$[R(\mathrm{i},\phi)f](\mathbf{w})$$
$$= \begin{cases} f(\mathbf{w}) & (\phi = 0 \bmod 2\pi) \\ f(-\mathbf{w}) & (\phi = \pi \bmod 2\pi) \\ |\sin\phi|^{-k/2} \displaystyle\int_{\mathbb{R}^k} e\left[\frac{\frac{1}{2}(\|\mathbf{w}\|^2 + \|\mathbf{w}'\|^2)\cos\phi - \mathbf{w}\cdot\mathbf{w}'}{\sin\phi}\right] & f(\mathbf{w}')\,d\mathbf{w}' \\ & (\phi \neq 0 \bmod \pi). \end{cases}$$

Note that $R(\mathrm{i},\pi/2) = \mathcal{F}$ is the Fourier transform.

3.10. For Schwartz functions $f \in \mathcal{S}(\mathbb{R}^k)$,

$$\lim_{\phi\to 0\pm} |\sin\phi|^{-k/2} \int_{\mathbb{R}^k} e\left[\frac{\frac{1}{2}(\|\mathbf{w}\|^2 + \|\mathbf{w}'\|^2)\cos\phi - \mathbf{w}\cdot\mathbf{w}'}{\sin\phi}\right] f(\mathbf{w}')\,d\mathbf{w}'$$
$$= \mathrm{e}^{\pm \mathrm{i}\pi k\pi/4} f(\mathbf{w}),$$

and hence this projective representation is in general discontinuous at $\phi = \nu\pi$, $\nu \in \mathbb{Z}$. This can be overcome by setting

$$\tilde{R}(\tau,\phi) = \mathrm{e}^{-\mathrm{i}\pi k\sigma_\phi/4} R(\tau,\phi).$$

In fact, $\tilde{R}$ corresponds to a unitary representation of the double cover of $\mathrm{SL}(2,\mathbb{R})$ [12]. This means in particular that (compare 3.7)

$$\tilde{R}(\mathrm{i},\phi)\tilde{R}(\mathrm{i},\phi') = \tilde{R}(\mathrm{i},\phi+\phi'),$$

where $\phi \in [0,4\pi)$ parametrizes the double cover of $\mathrm{SO}(2) \subset \mathrm{SL}(2,\mathbb{R})$.

4. Theta sums

4.1. The *Jacobi group* is defined as the semidirect product [1]

$$\mathrm{Sp}(k,\mathbb{R}) \ltimes \mathbb{H}(\mathbb{R}^k)$$

with multiplication law

$$(M;\boldsymbol{\xi},t)(M';\boldsymbol{\xi}',t') = (MM';\boldsymbol{\xi}+M\boldsymbol{\xi}',t+t'+\tfrac{1}{2}\omega(\boldsymbol{\xi},M\boldsymbol{\xi}')).$$

This definition is motivated by the fact that, since

$$R(M)W(\boldsymbol{\xi}',t') = W(M\boldsymbol{\xi}',t')R(M),$$

(recall 3.3) we have

$$\begin{aligned} &W(\boldsymbol{\xi},t)R(M)\;W(\boldsymbol{\xi}',t')R(M') \\ &\quad= \quad W(\boldsymbol{\xi},t)W(M\boldsymbol{\xi}',t')\;R(M)R(M') \\ &\quad= \quad c(M,M')^{-1}\;W(\boldsymbol{\xi}+M\boldsymbol{\xi}',t+t'+\tfrac{1}{2}\omega(\boldsymbol{\xi},M\boldsymbol{\xi}'))\;R(MM'). \end{aligned}$$

Hence

$$R(M;\boldsymbol{\xi},t)=W(\boldsymbol{\xi},t)R(M)$$

defines a projective representation of the Jacobi group, with cocycle $c(M,M')$ as above, the so-called *Schrödinger-Weil representation* [1].

Let us also put

$$\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)=W(\boldsymbol{\xi},t)\tilde{R}(\tau,\phi).$$

4.2. *Jacobi's theta sum.* We define Jacobi's theta sum for $f\in\mathcal{S}(\mathbb{R}^k)$ by

$$\Theta_f(\tau,\phi;\boldsymbol{\xi},t)=\sum_{\mathbf{m}\in\mathbb{Z}^k}[\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f](\mathbf{m}).$$

More explicitly, for $\tau=u+\mathrm{i}v$, $\boldsymbol{\xi}=\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}$,

$$\Theta_f(\tau,\phi;\boldsymbol{\xi},t)=v^{k/4}e(t-\tfrac12\mathbf{x}\cdot\mathbf{y})\sum_{\mathbf{m}\in\mathbb{Z}^k}f_\phi((\mathbf{m}-\mathbf{y})v^{1/2})e(\tfrac12\|\mathbf{m}-\mathbf{y}\|^2u+\mathbf{m}\cdot\mathbf{x}),$$

where

$$f_\phi=\tilde{R}(\mathrm{i},\phi)f.$$

It is easily seen that if $f\in\mathcal{S}(\mathbb{R}^k)$ then $f_\phi\in\mathcal{S}(\mathbb{R}^k)$ for ϕ fixed, and thus also $\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f\in\mathcal{S}(\mathbb{R}^k)$ for fixed $(\tau,\phi;\boldsymbol{\xi},t)$. This guarantees rapid convergence of the above series. We have the following uniform bound.

4.3. Lemma. *Let $f_\phi=\tilde{R}(\mathrm{i},\phi)f$, with $f\in\mathcal{S}(\mathbb{R}^k)$. Then, for any $R>1$, there is a constant c_R such that for all $\mathbf{w}\in\mathbb{R}^k$, $\phi\in\mathbb{R}$,*

$$|f_\phi(\mathbf{w})|\leq c_R(1+\|\mathbf{w}\|)^{-R}.$$

Proof. Since $f\in\mathcal{S}(\mathbb{R}^k)$, we can use repeated integration by parts to show that

$$\left||\sin\phi|^{-k/2}\int_{\mathbb{R}^k}e\left[\frac{\frac12(\|\mathbf{w}\|^2+\|\mathbf{w}'\|^2)\cos\phi-\mathbf{w}\cdot\mathbf{w}'}{\sin\phi}\right]f(\mathbf{w}')\,d\mathbf{w}'\right|\leq c_R'(1+\|\mathbf{w}\|)^{-R}$$

uniformly for all $\phi\notin(\nu\pi-\frac{1}{100},\nu\pi+\frac{1}{100})$, $\nu\in\mathbb{Z}$. That is,

$$|f_\phi(\mathbf{w})|\leq c_R'(1+\|\mathbf{w}\|)^{-R}$$

in the above range.

Furthermore $f_{\pi/2}$ is up to a phase factor $\mathrm{e}^{\mathrm{i}\pi k}$ the Fourier transform of f and therefore of Schwartz class as well. Again, after integration by parts,

$$\left||\sin\phi|^{-k/2}\int_{\mathbb{R}^k}e\left[\frac{\frac12(\|\mathbf{w}\|^2+\|\mathbf{w}'\|^2)\cos\phi-\mathbf{w}\cdot\mathbf{w}'}{\sin\phi}\right]f_{\pi/2}(\mathbf{w}')\,d\mathbf{w}'\right|$$
$$\leq c_R''(1+\|\mathbf{w}\|)^{-R}$$

for all $\phi \notin (\nu\pi - \frac{1}{100}, \nu\pi + \frac{1}{100})$, $\nu \in \mathbb{Z}$. This means

$$|f_{\phi+\pi/2}(\mathbf{w})| \le c''_R(1 + \|\mathbf{w}\|)^{-R}$$

in the above range, or, by replacement of $\phi \mapsto \phi - \pi/2$,

$$|f_\phi(\mathbf{w})| \le c''_R(1 + \|\mathbf{w}\|)^{-R},$$

for all $\phi \notin (\nu\pi + \frac{1}{2}\pi - \frac{1}{100}, \nu\pi + \frac{1}{2}\pi + \frac{1}{100})$, $\nu \in \mathbb{Z}$.

Clearly for each $\phi \in \mathbb{R}$ at least one of the bounds applies; we put $c_R = \max\{c'_R, c''_R\}$. □

4.4. The following transformation formulas are crucial for our further investigations:

Jacobi 1.

$$\Theta_f\left(-\frac{1}{\tau}, \phi + \arg\tau; \begin{pmatrix} -\mathbf{y} \\ \mathbf{x} \end{pmatrix}, t\right) = \mathrm{e}^{-\mathrm{i}\pi k/4}\Theta_f\left(\tau, \phi; \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, t\right).$$

Proof. The Poisson summation formula states that for any $f \in \mathcal{S}(\mathbb{R}^k)$,

$$\sum_{\mathbf{m}\in\mathbb{Z}^k} [\mathcal{F}f](\mathbf{m}) = \sum_{\mathbf{m}\in\mathbb{Z}^k} f(\mathbf{m})$$

where $\mathcal{F}$ is the Fourier transform. Because

$$\mathcal{F} = R(\mathrm{i}, \pi/2) = R(S), \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and secondly $\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t)f \in \mathcal{S}(\mathbb{R}^k)$ for fixed $(\tau, \phi; \boldsymbol{\xi}, t)$, the Poisson summation formula yields

$$\sum_{\mathbf{m}\in\mathbb{Z}^k} [R(S)\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t)f](\mathbf{m}) = \sum_{\mathbf{m}\in\mathbb{Z}^k} [\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t)f](\mathbf{m}).$$

We have

$$R(S)\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t) = R(S)W(\boldsymbol{\xi}, t)\tilde{R}(\tau, 0)\tilde{R}(\mathrm{i}, \phi) = W(S\boldsymbol{\xi}, t)R(S)R(\tau, 0)\tilde{R}(\mathrm{i}, \phi);$$

furthermore

$$R(S)R(\tau, 0) = R\left(-\frac{1}{\tau}, \arg\tau\right) = R\left(-\frac{1}{\tau}, 0\right)R(\mathrm{i}, \arg\tau),$$

since $(\tau, 0)$ and $(-\frac{1}{\tau}, 0)$ are upper triangular matrices, and hence the corresponding cocycles are trivial, i.e., equal to 1 (recall 3.6). Finally, since $0 < \arg\tau < \pi$ for $\tau \in \mathfrak{H}$,

$$R(\mathrm{i}, \arg\tau)\tilde{R}(\mathrm{i}, \phi) = \mathrm{e}^{\mathrm{i}\pi k/4}\tilde{R}(\mathrm{i}, \arg\tau)\tilde{R}(\mathrm{i}, \phi) = \mathrm{e}^{\mathrm{i}\pi k/4}\tilde{R}(\mathrm{i}, \phi + \arg\tau).$$

Collecting all terms, we find

$$R(S)\tilde{R}(\tau, \phi; \boldsymbol{\xi}, t) = \mathrm{e}^{\mathrm{i}\pi k/4}\tilde{R}\left(-\frac{1}{\tau}, \phi + \arg\tau; S\boldsymbol{\xi}, t\right),$$

and hence

$$\sum_{\mathbf{m}\in\mathbb{Z}^k}\left[\tilde{R}\left(-\frac{1}{\tau},\phi+\arg\tau;S\boldsymbol{\xi},t\right)f\right](\mathbf{m})=\mathrm{e}^{-\mathrm{i}\pi k/4}\sum_{\mathbf{m}\in\mathbb{Z}^k}[\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f](\mathbf{m}),$$

which proves the claim. □

Jacobi 2.

$$\Theta_f\left(\tau+1,\phi;\begin{pmatrix}\mathbf{s}\\ \mathbf{0}\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix},t+\tfrac{1}{2}\mathbf{s}\cdot\mathbf{y}\right)=\Theta_f\left(\tau,\phi;\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix},t\right),$$

with

$$\mathbf{s}={}^{\mathrm{t}}(\tfrac{1}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2})\in\mathbb{R}^k.$$

Proof. Clearly for any $f\in\mathcal{S}(\mathbb{R}^k)$

$$\sum_{\mathbf{m}\in\mathbb{Z}^k}\left[\tilde{R}\left(\mathrm{i}+1,0;\begin{pmatrix}\mathbf{s}\\ \mathbf{0}\end{pmatrix},0\right)f\right](\mathbf{m})=\sum_{\mathbf{m}\in\mathbb{Z}^k}f(\mathbf{m}),$$

and hence also (replace f with $\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f$)

$$\sum_{\mathbf{m}\in\mathbb{Z}^k}\left[\tilde{R}\left(\mathrm{i}+1,0;\begin{pmatrix}\mathbf{s}\\ \mathbf{0}\end{pmatrix},0\right)\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f\right](\mathbf{m})=\sum_{\mathbf{m}\in\mathbb{Z}^k}[\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f](\mathbf{m}).$$

We conclude by noticing

$$\begin{aligned}&\tilde{R}\left(\mathrm{i}+1,0;\begin{pmatrix}\mathbf{s}\\ \mathbf{0}\end{pmatrix},0\right)\tilde{R}\left(\tau,\phi;\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix},t\right)\\&\qquad=\tilde{R}\left(\tau+1,\phi;\begin{pmatrix}\mathbf{s}\\ \mathbf{0}\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix},t+\tfrac{1}{2}\mathbf{s}\cdot\mathbf{y}\right),\end{aligned}$$

where we have used that $c((\mathrm{i},0),(\tau,\phi))=1$ since $(\mathrm{i},0)$ is an upper triangular matrix; cf. 3.6. □

Jacobi 3.

$$\Theta_f\left(\tau,\phi;\begin{pmatrix}\mathbf{k}\\ \mathbf{l}\end{pmatrix}+\boldsymbol{\xi},r+t+\tfrac{1}{2}\omega\left(\begin{pmatrix}\mathbf{k}\\ \mathbf{l}\end{pmatrix},\boldsymbol{\xi}\right)\right)=(-1)^{\mathbf{k}\cdot\mathbf{l}}\,\Theta_f(\tau,\phi;\boldsymbol{\xi},t)$$

for any $\mathbf{k},\mathbf{l}\in\mathbb{Z}^k$, $r\in\mathbb{Z}$.

Proof. By virtue of 3.2 we have for all f

$$\sum_{\mathbf{m}\in\mathbb{Z}^k}\left[W\left(\begin{pmatrix}\mathbf{k}\\ \mathbf{l}\end{pmatrix},r\right)f\right](\mathbf{m})=e(-\tfrac{1}{2}\mathbf{k}\cdot\mathbf{l})\sum_{\mathbf{m}\in\mathbb{Z}^k}f(\mathbf{m}),$$

and therefore, replacing f with $W(\boldsymbol{\xi},t)\tilde{R}(\tau,\phi)f$,

$$\sum_{\mathbf{m}\in\mathbb{Z}^k}\left[W\left(\begin{pmatrix}\mathbf{k}\\ \mathbf{l}\end{pmatrix},r\right)W(\boldsymbol{\xi},t)\tilde{R}(\tau,\phi)f\right](\mathbf{m}) \\ = e(-\tfrac{1}{2}\mathbf{k}\cdot\mathbf{l})\sum_{\mathbf{m}\in\mathbb{Z}^k}[W(\boldsymbol{\xi},t)\tilde{R}(\tau,\phi)f](\mathbf{m}),$$

which gives the desired result. □

4.5. In what follows, we shall only need to consider products of theta sums of the form

$$\Theta_f(\tau,\phi;\boldsymbol{\xi},t)\overline{\Theta_g(\tau,\phi;\boldsymbol{\xi},t)},$$

where $f,g\in\mathcal{S}(\mathbb{R}^k)$. Clearly such combinations do not depend on the t-variable. Let us therefore define the semi-direct product group

$$G^k=\mathrm{SL}(2,\mathbb{R})\ltimes\mathbb{R}^{2k}$$

with multiplication law

$$(M;\boldsymbol{\xi})(M';\boldsymbol{\xi}')=(MM';\boldsymbol{\xi}+M\boldsymbol{\xi}'),$$

and put

$$\Theta_f(\tau,\phi;\boldsymbol{\xi})=v^{k/4}\sum_{\mathbf{m}\in\mathbb{Z}^k}f_\phi((\mathbf{m}-\mathbf{y})v^{1/2})e(\tfrac{1}{2}\|\mathbf{m}-\mathbf{y}\|^2u+\mathbf{m}\cdot\mathbf{x}).$$

By virtue of Lemma 4.3 and the Iwasawa parametrization 3.8, $\Theta_f\overline{\Theta_g}$ is a continuous $\mathbb{C}$-valued function on G^k.

4.6. A short calculation yields that the set

$$\Gamma^k=\left\{\left(\begin{pmatrix}a&b\\c&d\end{pmatrix};\begin{pmatrix}ab\mathbf{s}\\cd\mathbf{s}\end{pmatrix}+\mathbf{m}\right):\ \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z}),\ \mathbf{m}\in\mathbb{Z}^{2k}\right\}\subset G^k,$$

with $\mathbf{s}={}^{\mathrm{t}}(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})\in\mathbb{R}^k$, is closed under multiplication and inversion, and therefore forms a subgroup of G^k. Note also that the subgroup

$$N=\{1\}\ltimes\mathbb{Z}^{2k}$$

is normal in Γ^k.

4.7. Lemma. *Γ^k is generated by the elements*

$$\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix};\mathbf{0}\right),\quad\left(\begin{pmatrix}1&1\\0&1\end{pmatrix};\begin{pmatrix}\mathbf{s}\\ \mathbf{0}\end{pmatrix}\right),\quad\left(\begin{pmatrix}1&0\\0&1\end{pmatrix};\mathbf{m}\right),\quad\mathbf{m}\in\mathbb{Z}^{2k}.$$

Proof. The map

$$\mathrm{SL}(2,\mathbb{Z}) \to N\backslash\Gamma^k, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \begin{pmatrix} ab\mathbf{s} \\ cd\mathbf{s} \end{pmatrix} + \mathbb{Z}^{2k} \right)$$

defines a group isomorphism. The matrices $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ generate $\mathrm{SL}(2,\mathbb{Z})$, hence the lemma. □

4.8. PROPOSITION. *The left action of the group Γ^k on G^k is properly discontinuous. A fundamental domain of Γ^k in G^k is given by*

$$\mathcal{F}_{\Gamma^k} = \mathcal{F}_{\mathrm{SL}(2,\mathbb{Z})} \times \{\phi \in [0,\pi)\} \times \{\boldsymbol{\xi} \in [-\tfrac{1}{2}, \tfrac{1}{2})^{2k}\}.$$

where $\mathcal{F}_{\mathrm{SL}(2,\mathbb{Z})}$ is the fundamental domain in $\mathfrak{H}$ of the modular group $\mathrm{SL}(2,\mathbb{Z})$, given by $\{\tau \in \mathfrak{H} : u \in [-\frac{1}{2}, \frac{1}{2}), |\tau| > 1\}$.

Proof. As mentioned before, the matrices $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ generate $\mathrm{SL}(2,\mathbb{Z})$, which explains $\mathcal{F}_{\mathrm{SL}(2,\mathbb{Z})}$. Note furthermore that $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ generates the shift $\phi \mapsto \phi + \pi$. □

4.9. PROPOSITION. *For $f, g \in \mathcal{S}(\mathbb{R}^k)$, $\Theta_f(\tau,\phi;\boldsymbol{\xi})\overline{\Theta_g(\tau,\phi;\boldsymbol{\xi})}$ is invariant under the left action of Γ^k.*

Proof. This follows directly from Jacobi 1–3, since the left action of the generators from 4.7 is

$$\left(\tau,\phi; \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}\right) \mapsto \left(-\frac{1}{\tau}, \phi + \arg\tau; \begin{pmatrix} -\mathbf{y} \\ \mathbf{x} \end{pmatrix}\right),$$

$$(\tau,\phi;\boldsymbol{\xi}) \mapsto \left(\tau+1, \phi; \begin{pmatrix} \mathbf{s} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}\right),$$

and

$$(\tau,\phi;\boldsymbol{\xi}) \mapsto (\tau,\phi;\boldsymbol{\xi}+\mathbf{m}),$$

respectively. □

We find the following uniform estimate.

4.10 PROPOSITION. *Let $f, g \in \mathcal{S}(\mathbb{R}^k)$. For any $R > 1$,*

$$\Theta_f\left(\tau,\phi; \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}\right) \overline{\Theta_g\left(\tau,\phi; \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}\right)} = v^{k/2} \sum_{\mathbf{m}\in\mathbb{Z}^k} f_\phi((\mathbf{m}-\mathbf{y})v^{1/2}) \overline{g_\phi((\mathbf{m}-\mathbf{y})v^{1/2})} + O_R(v^{-R})$$

uniformly for all $(\tau,\phi;\boldsymbol{\xi}) \in G^k$ *with* $v > \frac{1}{2}$. *In addition,*

$$\Theta_f\left(\tau,\phi;\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\right)\overline{\Theta_g\left(\tau,\phi;\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\right)}$$
$$= v^{k/2} f_\phi((\mathbf{n}-\mathbf{y})v^{1/2})\overline{g_\phi((\mathbf{n}-\mathbf{y})v^{1/2})} + O_R(v^{-R}),$$

uniformly for all $(\tau,\phi;\boldsymbol{\xi}) \in G^k$ *with* $v > \frac{1}{2}$, $\mathbf{y} \in \mathbf{n} + [-\frac{1}{2},\frac{1}{2}]^k$ *and* $\mathbf{n} \in \mathbb{Z}^k$.

Proof. Suppose $\mathbf{y} \in \mathbf{n} + [-\frac{1}{2},\frac{1}{2}]^k$ for an arbitrary integer $\mathbf{n} \in \mathbb{Z}^k$.

By virtue of Lemma 4.3 we have for any $T > 1$

$$|f_\phi((\mathbf{m}-\mathbf{y})v^{1/2})| \le c_T(1 + \|\mathbf{m}-\mathbf{y}\| v^{1/2})^{-T} = O_T(\|\mathbf{m}-\mathbf{n}\|^{-T}\, v^{-T/2}),$$

which holds uniformly for $v > \frac{1}{2}$, $\phi \in \mathbb{R}$ and $\mathbf{y} \in \mathbf{n} + [-\frac{1}{2},\frac{1}{2}]^k$, if $\mathbf{m} \neq \mathbf{n}$.

Likewise for g_ϕ,

$$|g_\phi((\tilde{\mathbf{m}}-\mathbf{y})v^{1/2})| \le \tilde{c}_T(1 + \|\tilde{\mathbf{m}}-\mathbf{y}\| v^{1/2})^{-T} = O_T(\|\tilde{\mathbf{m}}-\mathbf{n}\|^{-T}\, v^{-T/2}),$$

again uniformly for $v > \frac{1}{2}$, $\phi \in \mathbb{R}$ and $\mathbf{y} \in \mathbf{n} + [-\frac{1}{2},\frac{1}{2}]^k$, if $\tilde{\mathbf{m}} \neq \mathbf{n}$.

Hence the leading order contributions come from terms with $\tilde{\mathbf{m}} = \mathbf{m}$, the sum of all other terms contributes $O_T(v^{-T/2})$. □

The following lemmas will be useful later on.

4.11 Lemma. *The subgroup*

$$\Gamma_\theta \ltimes \mathbb{Z}^{2k},$$

where

$$\Gamma_\theta = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) :\ ab \equiv cd \equiv 0 \bmod 2\right\}$$

denotes the theta group, is of index three in Γ^k.

Proof. It is well known [9] that Γ_θ is of index three in $\mathrm{SL}(2,\mathbb{Z})$ and

$$\mathrm{SL}(2,\mathbb{Z}) = \bigcup_{j=0}^{2} \Gamma_\theta \begin{pmatrix} 0 & -1\\ 1 & 1\end{pmatrix}^j .$$

By virtue of the group isomorphism employed in the proof of Lemma 4.7, we infer that

$$\Gamma^k = \bigcup_{j=0}^{2} (\Gamma_\theta \ltimes \mathbb{Z}^{2k}) \left(\begin{pmatrix} 0 & -1\\ 1 & 1\end{pmatrix}; \begin{pmatrix}\mathbf{0}\\ \mathbf{s}\end{pmatrix}\right)^j .$$ □

4.12 Lemma. Γ^k *is of finite index in* $\mathrm{SL}(2,\mathbb{Z}) \ltimes (\frac{1}{2}\mathbb{Z})^{2k}$.

Proof. The subgroup $\Gamma_\theta \ltimes \mathbb{Z}^{2k} \subset \Gamma^k$ is of finite index in $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$ and thus also in $\mathrm{SL}(2,\mathbb{Z}) \ltimes (\frac{1}{2}\mathbb{Z})^{2k}$. □

4.13. *Remark.* Note that

$$\mathrm{SL}(2,\mathbb{Z}) \ltimes (\tfrac{1}{2}\mathbb{Z})^{2k} = \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}; \mathbf{0} \right) (\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}) \left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; \mathbf{0} \right),$$

i.e., $\mathrm{SL}(2,\mathbb{Z}) \ltimes (\frac{1}{2}\mathbb{Z})^{2k}$ is isomorphic to $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$.

4.14. In this paper, we will be interested in the case of quadratic forms in two variables, i.e., $k = 2$. The corresponding theta sum (defined for general k in 4.5) reads then

$$\begin{aligned} \Theta_f(\tau, \phi; \boldsymbol{\xi}) &= v^{1/2} \sum_{(m,n)\in\mathbb{Z}^2} f_\phi((m-y_1)v^{1/2}, (n-y_2)v^{1/2}) \\ &\quad \times e(\tfrac{1}{2}(m-y_1)^2 u + \tfrac{1}{2}(n-y_2)^2 u + m x_1 + n x_2), \end{aligned}$$

where $\boldsymbol{\xi} = {}^{\mathrm{t}}(x_1, x_2, y_1, y_2) \in \mathbb{R}^4$. This theta sum is related to the one introduced in Section 2 by

$$\theta_{\psi_1}(u,\lambda)\overline{\theta_{\psi_2}(u,\lambda)} = \Theta_f(\tau,\phi;\boldsymbol{\xi})\overline{\Theta_g(\tau,\phi;\boldsymbol{\xi})}$$

with

$$\tau = u + \mathrm{i}\frac{1}{\lambda}, \qquad \phi = 0, \qquad \boldsymbol{\xi} = {}^{\mathrm{t}}(0,0,\alpha,\beta),$$

and

$$f(w_1,w_2) = \psi_1(w_1^2 + w_2^2), \qquad g(w_1,w_2) = \psi_2(w_1^2 + w_2^2).$$

Recall that $f_\phi|_{\phi=0} = f$ and likewise $g_\phi|_{\phi=0} = g$.

The crucial advantage in dealing with Θ_f rather than the original θ_ψ is that the extra set of variables allows us to realize Θ_f as a function on a finite-volume manifold and to employ ergodic-theoretic techniques.

5. Unipotent flows

5.1. Put

$$\Psi_0^t = \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}; \mathbf{0} \right).$$

For $t \in \mathbb{R}$, Ψ_0^t generates a unipotent one-parameter-subgroup of G^k, denoted by $\Psi_0^{\mathbb{R}}$. For any lattice Γ in G^k, we now define the flow $\Psi^t : \Gamma\backslash G^k \to \Gamma\backslash G^k$ by right translation by Ψ_0^t,

$$\Psi^t(g) := g\Psi_0^t.$$

Hence for $g = (M; \boldsymbol{\xi})$ we have

$$\Psi^t(g) = \left(M \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}; \boldsymbol{\xi} \right).$$

When projected onto $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$, this flow becomes the classical horocycle flow.

5.2. Similarly,

$$\Phi_0^t = \left(\begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} ; \mathbf{0} \right),$$

generates a one-parameter-subgroup of G^k. The flow $\Phi^t : \Gamma\backslash G^k \to \Gamma\backslash G^k$ defined by

$$\Phi^t(g) := g\Phi_0^t,$$

represents a lift of the classical geodesic flow on $\Gamma\backslash \operatorname{SL}(2,\mathbb{R})$.

5.3. We are interested in averages of the form

$$\int F(u + \mathrm{i}v, 0; \boldsymbol{\xi})\, h(u)\, du$$

where F is a continuous function $\Gamma\backslash G^k \to \mathbb{R}$, and h is a continuous probability density with compact support. Setting $g_0 = (\mathrm{i}, 0; \boldsymbol{\xi})$, and $v = e^{-t}$, we may write the above integral as

$$\rho_t(F) = \int F(g_0 \Psi_0^u \Phi_0^t)\, h(u)\, du = \int F \circ \Phi^t \circ \Psi^u(g_0)\, h(u)\, du,$$

which may therefore be interpreted as the average along an orbit of the unipotent flow Ψ^u, which is translated by Φ^t. Since $\rho_t(1) = 1$, ρ_t defines a probability measure on $\Gamma\backslash G^k$.

5.4. Proposition. *Let Γ be a subgroup of $\operatorname{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$ of finite index. Then the family of probability measures $\{\rho_t : t \geq 0\}$ is relatively compact, i.e., every sequence of measures contains a subsequence which converges weakly to a probability measure on $\Gamma\backslash G^k$.*

Proof. Consider the function

$$X_R(\tau) = \sum_{\gamma \in \{\Gamma_\infty \cup (-1)\Gamma_\infty\}\backslash \operatorname{SL}(2,\mathbb{Z})} \chi_R(\operatorname{Im}(\gamma\tau)),$$

where χ_R is the characteristic function of the open interval (R, ∞), and

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\} \subset \operatorname{SL}(2,\mathbb{Z}).$$

For $u + \mathrm{i}v \in \mathcal{F}_{\operatorname{SL}(2,\mathbb{Z})}$, we thus have

$$X_R(u + \mathrm{i}v) = \begin{cases} 1 & (v > R) \\ 0 & (v \leq R). \end{cases}$$

Because Γ is a finite index subgroup of $\operatorname{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$, X_R represents the characteristic function of a set in $\Gamma\backslash G^k$, whose complement is compact.

By construction, the function X_R is independent of ϕ and $\boldsymbol{\xi}$; we can therefore apply the equidistribution theorem for arcs of long closed horocycles on $\Gamma\backslash\mathfrak{H}$ (see, e.g., [10] and [15, Cor. 5.2]), which yields for $g_0 = (\mathrm{i}, 0; \boldsymbol{\xi})$,

$$\lim_{t\to\infty} \rho_t(X_R) = \lim_{v\to 0} \int X_R(u+\mathrm{i}v)\, h(u)\, du = \frac{1}{\mu(\mathcal{F}_{\mathrm{SL}(2,\mathbb{Z})})} \int_{\mathcal{F}_{\mathrm{SL}(2,\mathbb{Z})}} X_R(u+\mathrm{i}v)\, \frac{du\, dv}{v^2}.$$

Now

$$\int_{\mathcal{F}_{\mathrm{SL}(2,\mathbb{Z})}} X_R(u+\mathrm{i}v)\, \frac{du\, dv}{v^2} = \int_R^\infty \frac{dv}{v^2} = R^{-1}.$$

Hence, given any $\varepsilon > 0$, we find some $R > 1$ such that

$$\sup_{t\geq 0} \rho_t(X_R) \leq \varepsilon.$$

The family of ρ_t is therefore tight, and the proposition follows from the Helly-Prokhorov theorem [28]. $\square$

5.5. Proposition. *If ν is a weak limit of a subsequence of the probability measures ρ_t with $t \to \infty$, then ν is invariant under the action of $\Psi^{\mathbb{R}}$, i.e., $\nu \circ \Psi^{\mathbb{R}} = \nu$.*

Proof. Suppose $\{\rho_{t_i} : i \in \mathbb{N}\}$ is a convergent subsequence with weak limit ν. That is, for any bounded continuous function F, we have

$$\lim_{i\to\infty} \rho_{t_i}(F) = \nu(F).$$

For any fixed $s \in \mathbb{R}$, we find

$$\begin{aligned} \rho_t(F\circ\Psi^s) &= \int F(g_0 \Psi_0^u \Phi_0^t \Psi_0^s)\, h(u)\, du = \int F(g_0 \Psi_0^{u+s\exp(-t)} \Phi_0^t)\, h(u)\, du \\ &= \int F(g_0 \Psi_0^u \Phi_0^t)\, h(u - s\exp(-t))\, du. \end{aligned}$$

Furthermore

$$\begin{aligned} \Big|\rho_t(F\circ\Psi^s) - \rho_t(F)\Big| &= \Big|\int F(g_0 \Psi_0^u \Phi_0^t)\, \Big[h(u - s\exp(-t)) - h(u)\Big]\, du\Big| \\ &\leq (\sup|F|) \int \Big|h(u - s\exp(-t)) - h(u)\Big|\, du. \end{aligned}$$

Hence, given any $\varepsilon > 0$, we find a T such that

$$|\rho_t(F\circ\Psi^s) - \rho_t(F)| < \varepsilon$$

for all $t > T$. Because the function $\tilde{F} = F \circ \Psi^s$ (s is fixed) is bounded continuous, the limit

$$\lim_{i\to\infty} \rho_{t_i}(F\circ\Psi^s) = \nu(F\circ\Psi^s)$$

exists, and we know from the above inequality that

$$|\nu(F \circ \Psi^s) - \nu(F)| \leq \varepsilon$$

for any $\varepsilon > 0$. Therefore $\nu(F \circ \Psi^s) = \nu(F)$. □

5.6. Ratner [18], [19] gives a classification of all ergodic $\Psi^{\mathbb{R}}$-invariant measures on $\Gamma\backslash G^k$. We will now investigate which of these measures are possible limits of the sequence $\{\rho_t\}$. The answer will be unique, translates of orbits of $\Psi^{\mathbb{R}}$ become equidistributed.

5.7. THEOREM. *Let Γ be a subgroup of $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$ of finite index. Fix some point*

$$g_0 = \left(\mathrm{i}, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}\right) \in \Gamma\backslash G^k$$

such that the components of the vector $({}^{\mathrm{t}}\mathbf{y}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$. Let h be a continuous probability density $\mathbb{R} \to \mathbb{R}_+$ with compact support. Then, for any bounded continuous function F on $\Gamma\backslash G^k$,

$$\lim_{t\to\infty} \int_{\mathbb{R}} F \circ \Phi^t \circ \Psi^u(g_0)\, h(u)\, du = \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\, d\mu$$

where μ is the Haar measure of G^k.

This theorem is a special case of Shah's more general Theorem 1.4 in [27] on the equidistribution of translates of unipotent orbits. Because of the simple structure of the Lie groups studied here, the proof of Theorem 5.7 is less involved than in the general context.

5.8. Before we begin with the proof of Theorem 5.7, we consider the special test function

$$F_\delta(M; \boldsymbol{\xi}) = \sum_{\gamma \in \mathrm{SL}(2,\mathbb{Z})} f_\delta(\gamma M)\, \eta_D(\gamma \boldsymbol{\xi}),$$

with (in the Iwasawa parametrization 3.8)

$$f_\delta(M) = f_\delta(\tau, \phi) = \chi_1(u + v \cot \phi)\, \chi_2(v^{-1/2} \cos \phi)\, \chi_3(v^{-1/2} \sin \phi)$$

where χ_j $(j = 1, 2, 3)$ is the characteristic function of the interval $[s_j, s_j + \delta_j]$. We assume in the following that s_j ranges over the fixed compact interval I_j, and that I_3 is furthermore properly contained in $\mathbb{R}^+$, i.e., $s_3 \geq \overline{s}$ for some constant $\overline{s} > 0$. Clearly f_δ has compact support in $\mathrm{SL}(2,\mathbb{R})$. The function $\eta_D : \mathbb{T}^{2k} \to \mathbb{R}$ is the characteristic function of a domain D in $\mathbb{T}^{2k}$ with smooth boundary.

Clearly, F_δ may be viewed as a function on $\Gamma\backslash G^k$, for Γ is a subgroup of $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$.

5.9. LEMMA. *Suppose the components of the vector* $({}^{\mathrm{t}}\mathbf{y}, 1) \in \mathbb{R}^{k+1}$ *are linearly independent over* $\mathbb{Q}$. *Then, given intervals* I_1, I_2, I_3 *as above, there exists a constant* $C > 0$ *such that, for any domain* $D \subset \mathbb{T}^{2k}$ *with smooth boundary,* $\delta_1, \delta_2, \delta_3 > 0$ *(sufficiently small) and* $s_1 \in I_1, s_2 \in I_2, s_3 \in I_3$,

$$\limsup_{v \to 0} \int_{\mathbb{R}} F_\delta \left(u + \mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix} \right) h(u)\, du \le C\delta_1\delta_2(s_3 + \delta_3) \int_{\mathbb{T}^{2k}} \eta_D(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

The constant C *may depend on the choice of* $h, \mathbf{y}, I_1, I_2, I_3$.

5.10. *Proof.*

5.10.1. Given any $\varepsilon > 0$ and any domain $D \subset \mathbb{T}^{2k}$ with smooth boundary, we can cover D by a large but finite number of nonoverlapping cubes $C_j \subset \mathbb{T}^{2k}$, in such a way that

$$\eta_D \le \sum_j \eta_{C_j}, \qquad \int_{\mathbb{T}^{2k}} \left(\sum_j \eta_{C_j} - \eta_D \right) d\boldsymbol{\xi} < \varepsilon.$$

We may therefore assume without loss of generality that $\eta_D(\boldsymbol{\xi})$ is the characteristic function of an arbitrary cube in $\mathbb{T}^{2k}$, i.e., $\eta_D(\boldsymbol{\xi}) = \eta_1(\mathbf{x})\eta_2(\mathbf{y})$, where η_1, η_2 are characteristic functions of arbitrary cubes in $\mathbb{T}^k$.

5.10.2. We recall that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$,

$$F_\delta \left(u + \mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix} \right) = \sum_\gamma f_\delta \left(\frac{a(u + \mathrm{i}v) + b}{c(u + \mathrm{i}v) + d}, \arg(c\tau + d) \right) \eta_1(b\mathbf{y})\, \eta_2(d\mathbf{y}).$$

In particular (with $\phi = 0$),

$$v_\gamma^{-1/2} \cos\phi_\gamma = v^{-1/2}(cu + d), \qquad v_\gamma^{-1/2} \sin\phi_\gamma = v^{1/2}c,$$

$$u_\gamma = \mathrm{Re}\, \frac{a(u + \mathrm{i}v) + b}{c(u + \mathrm{i}v) + d} = \frac{a}{c} - \frac{1}{c} \frac{cu + d}{|c\tau + d|^2} = \frac{a}{c} - v_\gamma \cot\phi_\gamma.$$

One then finds that

$$F_\delta \left(u + \mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix} \right) = \sum_\gamma \chi_1(\frac{a}{c})\, \chi_2(v^{-1/2}(cu + d))\, \chi_3(cv^{1/2})\, \eta_1(b\mathbf{y})\, \eta_2(d\mathbf{y}),$$

which, after being integrated against $h(u)du$, yields

$$v \sum_\gamma \chi_1(\frac{a}{c})\, \chi_3(cv^{1/2})\, \eta_1(b\mathbf{y})\, \eta_2(d\mathbf{y}) \int \chi_2(cv^{1/2}t)\, h(vt - \frac{d}{c})\, dt.$$

The compactness of the support of h implies that $\frac{d}{c} = vt + O(1)$, and hence

$|d| \ll |s_2+\delta_2|v^{1/2}+|c|$, i.e., $|d| \ll |c|$ for v small. Therefore

$$\int F_\delta\left(u+\mathrm{i}v,0;\begin{pmatrix}\mathbf{0}\\ \mathbf{y}\end{pmatrix}\right)h(u)du$$
$$\ll \delta_2\, v^{1/2} \sum_{\substack{\gamma \\ |d|\le A|c|}} \frac{1}{|c|}\chi_1\left(\frac{a}{c}\right)\,\chi_3(cv^{1/2})\,\eta_1(b\mathbf{y})\,\eta_2(d\mathbf{y}),$$

where $A>0$ and the implied constant depend only on h, if v is small enough.

5.10.3. There are only finitely many terms with $d=0$, which thus give a total contribution of order $v^{1/2}$; we will thus assume in the following $d\neq 0$. Likewise, if $b=0$, we have $ad=1$ and $c\in\mathbb{Z}$. This leads to a contribution of order $v^{1/2}|\log v|$, which tends to zero in the limit $v\to 0$.

The solutions of the equation $ad-bc=1$ with $b,d\neq 0$ can be obtained in the following way. Take nonzero coprime integers $b,d\in\mathbb{Z}$, $\gcd(b,d)=1$, and suppose a_0,c_0 solves $a_0d-bc_0=1$. (Such a solution can always be found.) All other solutions must then be of the form $a=a_0+mb$, $c=c_0+md$ with $m\in\mathbb{Z}$. We may assume without loss of generality that $0\le c_0\le |d|-1$. So, for v sufficiently small,

$$\int F_\delta\left(u+\mathrm{i}v,0;\begin{pmatrix}\mathbf{0}\\ \mathbf{y}\end{pmatrix}\right)h(u)du$$
$$\ll \quad \delta_2\, v^{1/2} \sum_{\substack{b,d,m\in\mathbb{Z}\\ 0<|d|\le A|c_0+md|}} \frac{1}{|c_0+md|}\chi_1\left(\frac{b}{d}+\frac{1}{(c_0+md)d}\right)$$
$$\times\chi_3((c_0+md)v^{1/2})\,\eta_1(b\mathbf{y})\,\eta_2(d\mathbf{y})+O_{\delta,\eta}(v^{1/2}\log v),$$

where $a_0=a_0(b,d)$ and $c_0=c_0(b,d)$ are chosen as above. We have dropped the restriction that $\gcd(b,d)=1$.

For terms with $|m|>1$, we obtain upper bounds by observing

$$\frac{1}{|c_0+md|}\le\frac{1}{(|m|-1)|d|},$$

and replacing the restriction imposed by χ_3 with the condition $(|m|-1)|d|\le v^{-1/2}(s_3+\delta_3)$. For terms with $m=0,\pm1$, we have

$$\frac{1}{|c_0+md|}\le\frac{A}{|d|}$$

and we replace the restriction corresponding to χ_3 with $|d|\le Av^{-1/2}(s_3+\delta_3)$, since $|d|\le A|c_0+md|$.

The restriction coming from χ_1 means for $d>0$

$$s_1d-\frac{1}{c_0+md}\le b\le (s_1+\delta_1)d-\frac{1}{c_0+md},$$

which we extend to

$$s_1 d - \frac{A}{|d|} \le b \le (s_1+\delta_1)d + \frac{A}{|d|},$$

and for $d<0$,

$$(s_1+\delta_1)d - \frac{1}{c_0+md} \le b \le s_1 d - \frac{1}{c_0+md},$$

which we extend to

$$(s_1+\delta_1)d - \frac{A}{|d|} \le b \le s_1 d + \frac{A}{|d|}.$$

We thus have (with $n=|m|-1$ for $|m|>1$, and $n=1$ for $m=0,\pm 1$)

$$\int F_\delta\left(u+\mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}\right) h(u)du \\ \ll \delta_2\, v^{1/2} \sum_{b,d,n\in\mathbb{Z}} \frac{1}{|nd|} \eta_1(b\mathbf{y})\, \eta_2(d\mathbf{y}) + O_{\delta,\eta}(v^{1/2}\log v),$$

with the summation restricted to

$$s_1|d| - \frac{A}{|d|} \le \pm b \le (s_1+\delta_1)|d| + \frac{A}{|d|}, \quad n|d| \le \max(A,1) v^{-1/2}(s_3+\delta_3), \quad n>0.$$

5.10.4. Since the components of $({}^{\mathrm{t}}\mathbf{y}, 1)$ are linearly independent over $\mathbb{Q}$, Weyl's equidistribution theorem ([33, Satz 4]) implies that

$$\sum_{s_1|d| - \frac{A}{|d|} \le \pm b \le (s_1+\delta_1)|d| + \frac{A}{|d|}} \eta_1(b\mathbf{y}) \ll |d|\delta_1 \int_{\mathbb{T}^k} \eta_1(\mathbf{x}) d\mathbf{x},$$

uniformly for $|d| > v^{-1/4}$ large enough. For $|d| \le v^{-1/4}$ we use the trivial bound

$$\sum_{s_1|d| - \frac{A}{|d|} \le \pm b \le (s_1+\delta_1)|d| + \frac{A}{|d|}} \eta_1(b\mathbf{y}) = O_{\delta,\eta}(v^{-1/4}),$$

for small enough v. Therefore

$$\int F_\delta\left(u+\mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}\right) h(u)du \\ \ll \delta_1\delta_2\, v^{1/2} \sum_{\substack{n>0 \\ |d| \ll n^{-1} v^{-1/2}(s_3+\delta_3)}} \frac{1}{n} \eta_2(d\mathbf{y}) \int_{\mathbb{T}^k} \eta_1(\mathbf{x}) d\mathbf{x} + O_{\delta,\eta}(v^{1/4}(\log v)^2),$$

where the last term includes all contributions from terms with $|d| \le v^{-1/4}$.

5.10.5. We split the remaining sum over n into terms with $0 < n < v^{-1/4}$ and terms with $n \ge v^{-1/4}$. In the first case we have, for $v \to 0$,

$$n v^{1/2} \sum_{0<|d| \ll n^{-1} v^{-1/2}(s_3+\delta_3)} \eta_2(d\mathbf{y}) \ll (s_3+\delta_3) \int_{\mathbb{T}^k} \eta_2(\mathbf{x}) d\mathbf{x}$$

by Weyl's equidistribution theorem. For $n \geq v^{-1/4}$ one simply uses the trivial bound

$$\sum_{0<|d|\ll n^{-1}v^{-1/2}(s_3+\delta_3)} \eta_2(d\mathbf{y}) \ll n^{-1}v^{-1/2}(s_3+\delta_3).$$

5.10.6. We conclude

$$\limsup_{v\to 0} \int F_\delta \left(u+\mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix} \right) h(u)du$$

$$\ll \delta_1\delta_2(s_3+\delta_3) \int_{\mathbb{T}^k} \eta_1(\mathbf{x})d\mathbf{x} \limsup_{v\to 0} \left[\sum_{n<v^{-1/4}} n^{-2} \int_{\mathbb{T}^k} \eta_2(\mathbf{x})d\mathbf{x} + \sum_{n\geq v^{-1/4}} n^{-2} \right].$$

Since $\lim_{v\to 0} \sum_{n<v^{-1/4}} n^{-2} = \frac{\pi^2}{6} < \infty$ and $\lim_{v\to 0} \sum_{n\geq v^{-1/4}} n^{-2} = 0$, the lemma is proved. □

5.11. *Proof of Theorem* 5.7.

5.11.1. By Propositions 5.4 and 5.5, we find a convergent subsequence of ρ_{t_i} with weak limit ν invariant under $\Psi^{\mathbb{R}}$. Hence for any bounded continuous function F on $\Gamma\backslash G^k$,

$$\lim_{i\to\infty} \rho_{t_i}(F) = \nu(F).$$

5.11.2. Following [17], we denote by $\mathcal{H}$ the collection of all closed connected subgroups H of G^k such that $\Gamma\cap H$ is a lattice in H and the subgroup, which is generated by all unipotent one-parameter subgroups of G^k contained in H, acts ergodically from the right on $\Gamma\backslash\Gamma H$ with respect to the H-invariant probability measure. This collection is countable ([18, Th. 1.1]), and we call $\mathcal{H}^* \subset \mathcal{H}$ the set containing one representative of each Γ-conjugacy class.

Because $\mathrm{SL}(2,\mathbb{R})\ltimes\{\mathbf{0}\}$ and $\{1\}\ltimes\mathbb{R}^{2k}$ are each generated by unipotent one-parameter subgroups, so is G^k, which of course acts ergodically (with respect to Haar measure μ) from the right on $\Gamma\backslash G^k$, and so $G^k \in \mathcal{H}$.

Let

$$\begin{aligned} N(H) &= \{g \in G^k : \Psi_0^{\mathbb{R}} \subset g^{-1}Hg\}, \\ S(H) &= \bigcup_{H'\in\mathcal{H},\, H'\subset H,\, H'\neq H} N(H'), \end{aligned}$$

and

$$T_H = \pi(N(H)\backslash S(H)),$$

where π is the natural quotient map $G^k \to \Gamma\backslash G^k$. We denote by ν_H the restriction of ν on T_H. Then, for any $g \in N(H)\backslash S(H)$, the group $g^{-1}Hg$ is the smallest closed subgroup of G^k which contains $\Psi_0^{\mathbb{R}}$ and whose orbit through $\pi(g)$ is closed in $\Gamma\backslash G^k$ (cf. [17, Lemma 2.4]).

For all Borel measurable subsets $\mathcal{A} \subset \Gamma\backslash G^k$, the $\Psi^{\mathbb{R}}$-invariant measure ν admits the decomposition (see [17, Th. 2.2])

$$\nu(\mathcal{A}) = \sum_{H\in\mathcal{H}^*} \nu_H(\mathcal{A}).$$

Furthermore (see [17] for details), for any $\Psi^{\mathbb{R}}$-ergodic component ι of ν_H, with ι a probability measure, there exists a $g \in N(H)$ such that ι is the unique $g^{-1}Hg$-right-invariant probability measure on the closed orbit $\Gamma\backslash\Gamma Hg$. In particular, if $\nu(\pi(S(G^k))) = 0$, then $\nu = \mu$ (up to normalization).

5.11.3. Let us suppose first that there is at least one $H \in \mathcal{H}$ with $\nu_H \neq 0$, whose projection onto the $\mathrm{SL}(2,\mathbb{R})$-component is a closed connected subgroup L of $\mathrm{SL}(2,\mathbb{R})$ with $L \neq \mathrm{SL}(2,\mathbb{R})$ (compare Appendix B). Let Λ be the projection of Γ onto its $\mathrm{SL}(2,\mathbb{R})$-component. Since $\Gamma \cap H$ is a lattice in H, $\Lambda \cap L$ is a lattice in L. We can therefore construct a bounded continuous function $F(\tau,\phi;\boldsymbol{\xi}) = F(\tau,\phi)$ such that

$$\int F d\nu \neq \frac{1}{\mu(\Gamma\backslash G^k)} \int F d\mu.$$

With F independent of $\boldsymbol{\xi}$, we apply the equidistribution theorem for long arcs of closed horocycles [10], [15], which yields

$$\lim_{t\to\infty} \rho_t(F) = \frac{1}{\mu(\Gamma\backslash G^k)} \int F d\mu.$$

For the above subsequence (5.11.1) we find, however,

$$\lim_{i\to\infty} \rho_{t_i}(F) = \int F d\nu,$$

which leads to a contradiction. We shall therefore assume in the following that $L = \mathrm{SL}(2,\mathbb{R})$.

5.11.4. The most general form of a closed connected subgroup H, for which $L = \mathrm{SL}(2,\mathbb{R})$ and which contains a conjugate of $\Psi_0^{\mathbb{R}}$, is (see Appendix B)

$$H = (1;\boldsymbol{\xi}_0)H_0(1;-\boldsymbol{\xi}_0), \quad H_0 = \mathrm{SL}(2,\mathbb{R}) \ltimes \Omega,$$

where Ω is a closed connected subgroup of $\mathbb{R}^{2k}$ (i.e., Ω is a closed linear subspace of $\mathbb{R}^{2k}$), which is invariant under the action of $\mathrm{SL}(2,\mathbb{R})$. Since $\mathrm{SL}(2,\mathbb{R}) \ltimes \{\mathbf{0}\}$ and $\{1\} \ltimes \Omega$ are generated by unipotent one-parameter subgroups, the same holds for H_0 and hence for H. The right action of H on $\Gamma\backslash\Gamma H$ is obviously ergodic with respect to the (unique) H-invariant probability measure ι, and therefore $H \in \mathcal{H}$.

5.11.5. Let us consider the orbit

$$\Gamma\backslash\Gamma H g = \Gamma\backslash\Gamma(1;\boldsymbol{\xi}_0)H_0\tilde{g}$$

with $g \in N(H)$ and thus $\tilde{g} = (1;-\boldsymbol{\xi}_0)g \in N(H_0)$. Note that

$$\left(1;\begin{pmatrix}\mathbf{a}\\ \mathbf{0}\end{pmatrix}\right)\ \Psi_0^t\ \left(1;\begin{pmatrix}\mathbf{a}\\ \mathbf{0}\end{pmatrix}\right)^{-1} = \Psi_0^t \in H_0$$

for all $t \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^k$, and

$$\left(1;\begin{pmatrix}\mathbf{0}\\ \mathbf{b}\end{pmatrix}\right)\ \Psi_0^t\ \left(1;\begin{pmatrix}\mathbf{0}\\ \mathbf{b}\end{pmatrix}\right)^{-1} = \left(1;\begin{pmatrix}-t\mathbf{b}\\ \mathbf{0}\end{pmatrix}\right)\Psi_0^t.$$

The right-hand side is an element of H_0 for all $t \in \mathbb{R}$ if and only if

$$\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix};\mathbf{0}\right)\left(1;\begin{pmatrix}-t\mathbf{b}\\ \mathbf{0}\end{pmatrix}\right)\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix};\mathbf{0}\right) = \left(1;\begin{pmatrix}\mathbf{0}\\ t\mathbf{b}\end{pmatrix}\right) \in H_0.$$

We therefore have the explicit representation

$$N(H_0) = H_0\ \left\{\left(1;\begin{pmatrix}\mathbf{a}\\ \mathbf{0}\end{pmatrix}\right) :\ \mathbf{a} \in \mathbb{R}^k\right\}.$$

5.11.6. Let us suppose in the following that $\nu_H \neq 0$ for some $H \neq G^k$, i.e., $\Omega \neq \mathbb{R}^{2k}$. We denote by $B_k(r)$ the open ball $\{\mathbf{x} \in \mathbb{R}^k : \|\mathbf{x}\| < r\}$. Then, for any $r > 0$, we define

$$\begin{aligned}\Sigma(r) &= \Gamma(1;\boldsymbol{\xi}_0)H_0\left\{\left(1;\begin{pmatrix}\mathbf{a}\\ \mathbf{0}\end{pmatrix}\right) :\ \mathbf{a} \in B_k(r)\right\}\\ &= \left\{\left(M;\boldsymbol{\xi} + M\begin{pmatrix}\mathbf{a}\\ \mathbf{0}\end{pmatrix}\right) :\ M \in \mathrm{SL}(2,\mathbb{R}),\ \boldsymbol{\xi} \in \widetilde{\Omega},\ \mathbf{a} \in B_k(r)\right\},\end{aligned}$$

where $\widetilde{\Omega} = \Gamma(\boldsymbol{\xi}_0 + \Omega)$ is a closed subset in $\mathbb{R}^{2k}$. We fix r large enough so that the restriction of ν_H on $\Gamma\backslash\Sigma(r)$ is nonzero.

5.11.7. Let us discuss the structure of $\widetilde{\Omega}$ in more detail: Since Γ is of finite index in $\Gamma' = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$ we see that $\Gamma \cap H$ is of finite index in $\Gamma' \cap H$. Furthermore $\Gamma \cap H$ is a lattice in H, and so $\Gamma' \cap H$ is a lattice in H. Then clearly $(1;-\boldsymbol{\xi}_0)\Gamma'(1;\boldsymbol{\xi}_0) \cap H_0$ must be a lattice in H_0. With

$$(1;-\boldsymbol{\xi}_0)\Gamma'(1;\boldsymbol{\xi}_0) = \{(M;(M-1)\boldsymbol{\xi}_0 + \mathbf{m}) :\ M \in \mathrm{SL}(2,\mathbb{Z}),\ \mathbf{m} \in \mathbb{Z}^{2k}\},$$

the lattice property in turn implies that $(M-1)\boldsymbol{\xi}_0 \in \Omega + \mathbb{Z}^{2k}$ for all M in a finite index subgroup $\Lambda \subset \mathrm{SL}(2,\mathbb{Z})$. The orbit $\mathrm{SL}(2,\mathbb{Z})\boldsymbol{\xi}_0/(\Omega + \mathbb{Z}^{2k})$ is therefore finite in $\mathbb{R}^{2k}/(\Omega + \mathbb{Z}^{2k})$; we denote by $\{\boldsymbol{\xi}_0^{(1)}, \boldsymbol{\xi}_0^{(2)}, \ldots, \boldsymbol{\xi}_0^{(J)}\}$ a finite set of representatives. With this, we conclude

$$\Gamma'(\boldsymbol{\xi}_0 + \Omega) = \bigcup_{j=1}^{J} \boldsymbol{\xi}_0^{(j)} + \Omega + \mathbb{Z}^{2k}.$$

The fact that $(1;-\boldsymbol{\xi}_0)\Gamma'(1;\boldsymbol{\xi}_0)\cap H_0$ is a lattice in H_0 implies also that $\mathbb{Z}^{2k}\cap\Omega$ is a euclidean lattice in Ω. Hence there is a compact fundamental domain $\mathcal{F}_{\mathbb{Z}^{2k}\cap\Omega}\subset\Omega$. We may therefore write

$$\Gamma'(\boldsymbol{\xi}_0+\Omega)=\bigcup_{j=1}^{J}\boldsymbol{\xi}_0^{(j)}+\mathcal{F}_{\mathbb{Z}^{2k}\cap\Omega}+\mathbb{Z}^{2k}.$$

Note that $\mathcal{F}_{\mathbb{Z}^{2k}\cap\Omega}$ is also compact in $\mathbb{R}^{2k}$, since Ω is closed.

We conclude by observing that $\Gamma'(\boldsymbol{\xi}_0+\Omega)$ is, of course, a finite covering of $\widetilde{\Omega}$, because Γ has finite index in Γ'.

5.11.8. Consider the subset $\Sigma_\delta(r)$ of $\Sigma(r)$, given by

$$\Sigma_\delta(r)=\Gamma\left\{\left(M;\boldsymbol{\xi}+M\begin{pmatrix}\mathbf{a}\\ \mathbf{0}\end{pmatrix}\right):\ M\in\mathcal{D}_\delta,\ \boldsymbol{\xi}\in\widetilde{\Omega},\ \mathbf{a}\in B_k(r)\right\},$$

where $\mathcal{D}_\delta$ is an open subset of $\mathrm{SL}(2,\mathbb{R})$ specified below.

In the Iwasawa parametrization 3.8

$$M=\begin{pmatrix} uv^{-1/2}\sin\phi+v^{1/2}\cos\phi & uv^{-1/2}\cos\phi-v^{1/2}\sin\phi\\ v^{-1/2}\sin\phi & v^{-1/2}\cos\phi\end{pmatrix},$$

we have

$$\begin{aligned}\Sigma_\delta(r)\quad=\quad\Gamma\Bigg\{\left(\tau,\phi;\boldsymbol{\xi}+\begin{pmatrix}(u+v\cot\phi)\mathbf{a}\\ \mathbf{a}\end{pmatrix}\right):&\\ (\tau,\phi)\in\mathcal{D}_\delta,\ \boldsymbol{\xi}\in\widetilde{\Omega},\ \mathbf{a}\in B_k(rv^{-1/2}\sin\phi)&\Bigg\},\end{aligned}$$

where $\mathcal{D}_\delta$ is now chosen to be the open set of elements $(\tau,\phi)\in\mathrm{SL}(2,\mathbb{R})$ subject to the restrictions

$$0<u+v\cot\phi<\delta,\quad -1<v^{-1/2}\cos\phi<1,\quad 1<v^{-1/2}\sin\phi<2.$$

For the set

$$\Pi_\delta(r)=\Gamma\left\{\left(\tau,\phi;\boldsymbol{\xi}+\begin{pmatrix}(u+v\cot\phi)\mathbf{a}\\ \mathbf{a}\end{pmatrix}\right):\ (\tau,\phi)\in\mathcal{D}_\delta,\ \boldsymbol{\xi}\in\widetilde{\Omega},\ \mathbf{a}\in B_k(r)\right\},$$

we find $\Pi_\delta(r)\subset\Sigma_\delta(r)\subset\Pi_\delta(2r)$. Let us finally define

$$\begin{aligned}&\widehat{\Pi}_{\varepsilon,\delta}(r)\\ &\quad=\Gamma\left\{\left(\tau,\phi;\boldsymbol{\zeta}+\boldsymbol{\xi}+\begin{pmatrix}\mathbf{0}\\ \mathbf{a}\end{pmatrix}\right):(\tau,\phi)\in\mathcal{D}_\delta,\ \boldsymbol{\zeta}\in B_{2k}(\varepsilon),\ \boldsymbol{\xi}\in\widetilde{\Omega},\ \mathbf{a}\in B_k(r)\right\},\end{aligned}$$

where $B_{2k}(\varepsilon)\subset\mathbb{R}^{2k}$ is the open ball of radius ε about the origin. Thus, $\widehat{\Pi}_{\varepsilon,\delta}(r)$ is a full dimensional (but thin) open set, which contains $\Pi_\delta(r)$ if $\delta>0$ is chosen small enough. That is, for any $\varepsilon>0$ there is a $\delta>0$ such that

$$\Sigma_\delta(r)\subset\widehat{\Pi}_{\varepsilon,\delta}(2r).$$

5.11.9. The characteristic function of $\widehat{\Pi} := \widehat{\Pi}_{\varepsilon,\delta}(2r)$ therefore satisfies $\chi_{\widehat{\Pi}}(\tau, \phi; \boldsymbol{\xi}) = 1$ for all $(\tau, \phi; \boldsymbol{\xi}) \in \Sigma_\delta(r)$. Hence, and because $\nu_H|_{\Gamma\backslash\Sigma(r)} \neq 0$, there is a constant $c_\flat > 0$ which is independent of δ and ε, such that, for all $\varepsilon > 0$, $\delta > 0$ sufficiently small,

$$\nu_H(\Gamma\backslash\widehat{\Pi}) = \int \chi_{\widehat{\Pi}}\, d\nu_H \geq c_\flat \int_{\substack{0<u+v\cot\phi<\delta \\ -1<v^{-1/2}\cos\phi<1 \\ 1<v^{-1/2}\sin\phi<2}} \frac{du\, dv\, d\phi}{v^2} = 4c_\flat\, \delta$$

and so

$$\nu(\Gamma\backslash\widehat{\Pi}) \geq \nu_H(\Gamma\backslash\widehat{\Pi}) \geq 4c_\flat\, \delta.$$

Since $\Gamma\backslash\widehat{\Pi}$ is open, we have along the subsequence $t_1, t_2, \dots$ in 5.11.1 (Theorem 1, p. 311 in [28])

$$\liminf_{i\to\infty} \rho_{t_i}(\chi_{\widehat{\Pi}}) \geq \nu(\Gamma\backslash\widehat{\Pi}) \geq 4c_\flat\, \delta.$$

5.11.10. On the other hand,

$$\chi_{\widehat{\Pi}}(\tau, \phi; \boldsymbol{\xi}) \leq F_{\varepsilon,\delta}(\tau, \phi; \boldsymbol{\xi}) = \sum_{\gamma\in\mathrm{SL}(2,\mathbb{Z})} f_\delta(\gamma\tau, \phi_\gamma)\eta_\varepsilon(\gamma\boldsymbol{\xi}),$$

where (as in 5.8)

$$f_\delta(\tau, \phi) = \chi_1(u + v\cot\phi)\ \chi_2(v^{-1/2}\cos\phi)\ \chi_3(v^{-1/2}\sin\phi)$$

and χ_1, χ_2, χ_3 are the characteristic functions of the intervals $[0, \delta]$, $[-1, 1]$, $[1, 2]$, respectively. The function η_ε is the characteristic function of the set

$$\left\{ \left(\boldsymbol{\zeta} + \boldsymbol{\xi} + \begin{pmatrix} \mathbf{0} \\ \mathbf{a} \end{pmatrix} \right) :\ \boldsymbol{\zeta} \in B_{2k}(\varepsilon),\ \boldsymbol{\xi} \in \widetilde{\Omega},\ \mathbf{a} \in B_k(r) \right\} + \mathbb{Z}^{2k}.$$

By Lemma 5.9, there is a constant $c_\sharp > 0$ which is independent of δ and ε, such that

$$\limsup_{i\to\infty} \rho_{t_i}(\chi_{\widehat{\Pi}}) \leq c_\sharp \delta \int_{\mathbb{T}^{2k}} \eta_\varepsilon(\boldsymbol{\xi}) d\boldsymbol{\xi}$$

for all sufficiently small $\varepsilon, \delta > 0$.

We conclude that

$$4c_\flat \leq c_\sharp \int_{\mathbb{T}^{2k}} \eta_\varepsilon(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

This contradicts our assumption that $c_\flat > 0$, if we can show that the integral over η_ε tends to zero, as $\varepsilon \to 0$. We will check this by a dimension consideration.

5.11.11. To this end we need to show that, if $\Omega \neq \mathbb{R}^{2k}$, we have

$$\dim \left\{ \left(\boldsymbol{\xi} + \begin{pmatrix} \mathbf{0} \\ \mathbf{a} \end{pmatrix} \right) :\ \boldsymbol{\xi} \in \widetilde{\Omega},\ \mathbf{a} \in B_k(r) \right\} < 2k.$$

In view of 5.11.7 this holds if and only if the dimension of the linear space

$$V = \Omega + W, \qquad W = \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{a} \end{pmatrix} : \ \mathbf{a} \in \mathbb{R}^k \right\},$$

is strictly less than $2k$. Suppose $\dim V = \kappa$, and let $\mathbf{b}_1, \ldots, \mathbf{b}_\lambda$ form a basis of Ω. Then there exist vectors $\mathbf{b}_{\lambda+1}, \ldots, \mathbf{b}_\kappa \in W$ such that $\mathbf{b}_1, \ldots, \mathbf{b}_\kappa$ is a basis of V. Hence

$$V = \Omega \oplus U, \quad U = \operatorname{span}\{\mathbf{b}_{\lambda+1}, \ldots, \mathbf{b}_\kappa\}.$$

The linear subspace

$$U^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} U \subset \begin{pmatrix} \mathbb{R}^k \\ \mathbf{0} \end{pmatrix}$$

clearly satisfies $U \cap U^* = \{\mathbf{0}\}$, and also $U^* \cap \Omega = \{\mathbf{0}\}$ since $U \cap \Omega = \{\mathbf{0}\}$ and Ω is $\mathrm{SL}(2,\mathbb{R})$-invariant. Hence

$$V \oplus U^* \subset \mathbb{R}^{2k},$$

and so $\dim V = 2k$ implies $\dim U^* = \dim U = 0$, which occurs only if $\Omega = V$. Thus $\dim \Omega < 2k$ implies $\dim V < 2k$ and the claim is proved.

5.11.12. Therefore $\nu_H \neq 0$ if and only if $H = G^k$, and hence the only limit measure of converging subsequences is the normalized μ. The uniqueness of the limit measure implies finally that every subsequence converges [28]. □

6. Diophantine conditions

6.1. So far, all equidistribution results are valid only in the case of bounded test functions F. We will now extend these results to unbounded test functions F, which grow moderately in the cusps of $\Gamma\backslash G^k$. This will, however, only be possible under certain diophantine assumptions on $\mathbf{y}$.

6.2. To this end let us discuss the following model situation. Let $G = G^1$ and $\Gamma = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$. Define furthermore the subgroup

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\} \subset \mathrm{SL}(2,\mathbb{Z}),$$

and put

$$v_\gamma := \operatorname{Im}(\gamma\tau) = \frac{v}{|c\tau + d|^2}, \qquad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and

$$y_\gamma := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (\gamma\boldsymbol{\xi}) = cx + dy, \qquad \text{with } \gamma\boldsymbol{\xi} = \gamma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Let χ_R be the characteristic function of the interval $[R, \infty)$,

$$\chi_R(t) = \begin{cases} 1 & (t \geq R) \\ 0 & (t < R). \end{cases}$$

For any $f \in \mathrm{C}(\mathbb{R})$, which is rapidly decreasing at $\pm\infty$, and $\beta \in \mathbb{R}$, the function

$$F_R(\tau; \boldsymbol{\xi}) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \sum_{m \in \mathbb{Z}} f\Big((y_\gamma + m) v_\gamma^{1/2}\Big)\, v_\gamma^\beta\, \chi_R(v_\gamma)$$

is readily seen to be invariant under the action of Γ. If τ lies in the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ given by $\mathcal{F}_{\mathrm{SL}(2,\mathbb{Z})} = \{\tau \in \mathfrak{H} : u \in [-\frac{1}{2}, \frac{1}{2}), |\tau| > 1\}$, and if furthermore $R > 1$, then $F_R(\tau; \boldsymbol{\xi})$ clearly has the representation

$$F_R(\tau; \boldsymbol{\xi}) = \sum_{m \in \mathbb{Z}} \Big\{ f\Big((y + m) v^{1/2}\Big) + f\Big((-y + m) v^{1/2}\Big)\Big\} v^\beta \chi_R(v).$$

The sum over m is rapidly converging because f is rapidly decreasing at $\pm\infty$.

We note that F_R can alternatively be represented as

$$F_R(\tau; \boldsymbol{\xi}) = \sum_{(\gamma; \mathbf{n}) \in \widehat{\Gamma}_\infty \backslash \Gamma} f\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (\gamma \boldsymbol{\xi} + \mathbf{n}) v_\gamma^{1/2}\right)\, v_\gamma^\beta\, \chi_R(v_\gamma)$$

with the abelian subgroup

$$\widehat{\Gamma}_\infty = \left\{ \left(\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} n \\ 0 \end{pmatrix} \right) : m, n \in \mathbb{Z} \right\} \subset \Gamma.$$

6.3. We will assume from now on that $f \geq 0$. The L^1 norm of F_R over $\Gamma \backslash G$ is then

$$\mu(F_R) = \int_{\Gamma \backslash G} F_R(\tau; \boldsymbol{\xi})\, d\mu(\tau, \phi; \boldsymbol{\xi})$$

with Haar measure

$$d\mu(\tau, \phi; \boldsymbol{\xi}) = \frac{du\, dv\, d\phi\, dx\, dy}{v^2}.$$

Then

$$\mu(F_R) = \int_{\widehat{\Gamma}_\infty \backslash G} f\Big(y v^{1/2}\Big)\, v^\beta\, \chi_R(v)\, d\mu(\tau, \phi; \boldsymbol{\xi})$$

and so

$$\mu(F_R) = 2\pi \int_{\mathbb{R}} f(w) dw \int_R^\infty v^{\beta - 5/2} dv = 2\pi \frac{R^{-(3/2 - \beta)}}{3/2 - \beta} \int_{\mathbb{R}} f(w) dw$$

for $\beta < 3/2$, and $\mu(F_R) = \infty$ otherwise. Of special interest will be the case $\beta = 1$, for which

$$\mu(F_R) = 4\pi R^{-1/2} \int_{\mathbb{R}} f(w) dw.$$

6.4. There is a well known one-to-one correspondence between the coset $\Gamma_\infty\backslash\Gamma$ and the set

$$\{(0,1),(0,-1),(1,0),(-1,0)\}\cup\{(c,d)\in\mathbb{Z}^2 : c,d\neq 0,\ \gcd(c,d)=1\},$$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c,d).$$

We may therefore write

$$\begin{aligned} F_R(\tau;\boldsymbol{\xi}) = & \sum_{m\in\mathbb{Z}} \Big\{ f\Big((y+m)v^{1/2}\Big) + f\Big((-y+m)v^{1/2}\Big)\Big\} v^\beta \chi_R(v) \\ & + \sum_{m\in\mathbb{Z}} \left\{ f\left((x+m)\frac{v^{1/2}}{|\tau|}\right) + f\left((-x+m)\frac{v^{1/2}}{|\tau|}\right)\right\} \frac{v^\beta}{|\tau|^{2\beta}}\chi_R\left(\frac{v}{|\tau|^2}\right) \\ & + \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ c,d\neq 0}} \sum_{m\in\mathbb{Z}} f\left((cx+dy+m)\frac{v^{1/2}}{|c\tau+d|}\right) \frac{v^\beta}{|c\tau+d|^{2\beta}}\chi_R\left(\frac{v}{|c\tau+d|^2}\right). \end{aligned}$$

From here on, we will only consider the case $\beta=1$, and $\boldsymbol{\xi} = {}^{\mathrm{t}}(0,y)$.

6.5. Proposition. *Suppose $h\in \mathrm{C}(\mathbb{R})$ is positive and has compact support, and let y be diophantine of type κ. Then, for any $R>1$ and ε,ε' with $0<\varepsilon<1$ and $0<\varepsilon'<\frac{1}{\kappa-1}$,*

$$\limsup_{v\to 0} \int_{|u|>v^{1-\varepsilon}} F_R\left(u+\mathrm{i}v;\begin{pmatrix}0\\y\end{pmatrix}\right)\, h(u)\,du = O_{\varepsilon,\varepsilon'}(R^{-\varepsilon'/2}),$$

where y, f and h are fixed.

The proof of this proposition requires the following lemma.

6.6. Lemma. *Let α be diophantine of type κ, and $f\in\mathrm{C}(\mathbb{R})$ be rapidly decreasing at $\pm\infty$ and positive, $f\geq 0$. Then, for any fixed $A>1$ and $0<\varepsilon<\frac{1}{\kappa-1}$,*

$$\sum_{d=1}^{D}\sum_{m\in\mathbb{Z}} f\Big(T(d\alpha+m)\Big) \ll \begin{cases} T^{-A} & (D\leq T^\varepsilon) \\ 1 & (T^\varepsilon\leq D\leq T^{\frac{1}{\kappa-1}}) \\ D\,T^{-\frac{1}{\kappa-1}} & (D\geq T^{\frac{1}{\kappa-1}}), \end{cases}$$

uniformly for all $D,T>1$.

6.7. *Proof.*

6.7.1. Order $\alpha, 2\alpha, \ldots, D\alpha$ mod 1 in the unit interval $[0,1]$, and denote these numbers by $0 < \varphi_1 < \ldots < \varphi_D < 1$. Clearly $\varphi_{j+1} - \varphi_j = k_j\alpha$ mod 1 for some integer $k_j \in [-D, D]$; therefore, and because α is of type κ,

$$\varphi_{j+1} - \varphi_j \geq \frac{C}{|k_j|^{\kappa-1}} \geq \frac{C}{D^{\kappa-1}},$$

for some suitable constant $C > 0$. Hence in any interval of length ℓ there can be at most $O(D^{\kappa-1}\ell + 1)$ points.

6.7.2. As to the first bound, take $\chi_{[-R,R]}$ to be the characteristic function of the interval $[-R, R]$ with $R > 1$. Then

$$\sum_{d=1}^{D} \sum_{m\in\mathbb{Z}} \chi_{[-R,R]}\Big(T(d\alpha + m)\Big) = 0$$

for $\frac{CT}{D^{\kappa-1}} > R$, since $|d\alpha + m| \geq \frac{C}{d^{\kappa-1}} \geq \frac{C}{D^{\kappa-1}}$. The argument in 6.7.1 shows that

$$\sum_{d=1}^{D} \sum_{m\in\mathbb{Z}} \chi_{[-R,R]}\Big(T(d\alpha + m)\Big) = O(RD^{\kappa-1}T^{-1} + 1) = O(R)$$

for $D^{\kappa-1}T^{-1} \leq 1$; hence

$$\sum_{d=1}^{D} \sum_{m\in\mathbb{Z}} \chi_{[-R,R]}\Big(T(d\alpha + m)\Big) = \begin{cases} O(R) & \text{if } R \geq C\frac{T}{D^{\kappa-1}}, \\ 0 & \text{if } R < C\frac{T}{D^{\kappa-1}}, \end{cases}$$

in the range $D \leq T^{\frac{1}{\kappa-1}}$. Since f is rapidly decreasing, we have for any $B > 3$

$$f(t) \ll \sum_{R=1}^{\infty} R^{-B} \chi_{[-R,R]}(t),$$

and hence, when $D \leq T^{\frac{1}{\kappa-1}}$,

$$\sum_{d=1}^{D} \sum_{m\in\mathbb{Z}} f\Big(T(d\alpha + m)\Big) \ll \sum_{R \geq C\frac{T}{D^{\kappa-1}}}^{\infty} R^{-(B-1)} \ll \left(\frac{D^{\kappa-1}}{T}\right)^{B-2}$$

which proves the first bound in the range $D \leq T^{\varepsilon} \leq T^{\frac{1}{\kappa-1}}$, for $\varepsilon < \frac{1}{\kappa-1}$.

6.7.3. To prove the second and third relation, we follow [5, pp. 13–14]. Given any positive integer q (to be fixed later) divide the sum over d into blocks of the form

$$\sum_{d=b}^{b+q-1} \sum_{m\in\mathbb{Z}} f\Big(T(d\alpha + m)\Big) = \sum_{d=0}^{q-1} \sum_{m\in\mathbb{Z}} f\Big(T(b\alpha + d\alpha + m)\Big).$$

(The last block might contain less than q terms, but this is irrelevant since we are seeking an upper bound.) There are $O(\frac{D}{q}+1)$ such blocks. Take a rational approximation $\frac{p}{q}$ to α with $|\alpha-\frac{p}{q}|\leq q^{-2}$ and p,q coprime, then the above sum is

$$\sum_{d=0}^{q-1}\sum_{m\in\mathbb{Z}} f\left(T\left(b\alpha+\frac{dp+O(1)}{q}+m\right)\right).$$

Since dp runs through a full set of residues mod q, the above equals

$$\sum_{r=0}^{q-1}\sum_{m\in\mathbb{Z}} f\left(T\left(b\alpha+\frac{r+O(1)}{q}+m\right)\right)=\sum_{r\in\mathbb{Z}} f\left(\frac{T}{q}(qb\alpha+r+O(1))\right).$$

The term $qb\alpha$ may be replaced by the nearest integer $+O(1)$, and so

$$\sum_{r\in\mathbb{Z}} f\left(\frac{T}{q}(qb\alpha+r+O(1))\right)=\sum_{r\in\mathbb{Z}} f\left(\frac{T}{q}(r+O(1))\right)$$

which in turn is clearly bounded by $O(\frac{q}{T}+1)$ for f is rapidly decreasing. Therefore

$$\sum_{d=1}^{D}\sum_{m\in\mathbb{Z}} f\Big(T(d\alpha+m)\Big)\ll\left(\frac{D}{q}+1\right)\left(\frac{q}{T}+1\right).$$

6.7.4. By Dirichlet's theorem, we may take $\frac{p}{q}$ such that $q\leq T$ and $|\alpha-\frac{p}{q}|\leq q^{-1}T^{-1}$. Since α is of type κ, we have $Cq^{-\kappa}\leq|\alpha-\frac{p}{q}|$, so that

$$T^{\frac{1}{\kappa-1}}\ll q\leq T,$$

and finally

$$\sum_{d=1}^{D}\sum_{m\in\mathbb{Z}} f\Big(T(d\alpha+m)\Big)\ll\frac{D}{T^{\frac{1}{\kappa-1}}}+1. \qquad\square$$

6.8. *Proof of Proposition* 6.5.

6.8.1. Because we are only concerned with upper bounds, we may assume in the following without loss of generality that f is positive and even, i.e., $f\geq 0$, $f(-w)=f(w)$.

It follows from the expansion in 6.4 that, for $v<1$, the first term is absent, since $\chi_R(v)=0$ (recall: $R>1$); hence we are left with

$$\begin{aligned}F_R\left(\tau;\begin{pmatrix}0\\ y\end{pmatrix}\right)&=2\sum_{m\in\mathbb{Z}} f\left(m\frac{v^{1/2}}{|\tau|}\right)\frac{v}{|\tau|^2}\chi_R\left(\frac{v}{|\tau|^2}\right)\\ &+2\sum_{\substack{(c,d)\in\mathbb{Z}^2\\ \gcd(c,d)=1\\ c>0,d\neq 0}}\sum_{m\in\mathbb{Z}} f\left((dy+m)\frac{v^{1/2}}{|c\tau+d|}\right)\frac{v}{|c\tau+d|^2}\chi_R\left(\frac{v}{|c\tau+d|^2}\right).\end{aligned}$$

6.8.2. As to the first term in the above expansion, a simple change of variable $u = vt$ shows that

$$\begin{aligned}
&\int_{|u|>v^{1-\varepsilon}} 2\sum_{m\in\mathbb{Z}} f\left(m\frac{v^{1/2}}{|\tau|}\right)\frac{v}{|\tau|^2}\chi_R\left(\frac{v}{|\tau|^2}\right)h(u)\,du \\
&\qquad = 2\int_{|t|>v^{-\varepsilon}} \sum_{m\in\mathbb{Z}} f\left(\frac{m}{v^{1/2}(t^2+1)^{1/2}}\right)\frac{1}{t^2+1}\chi_R\left(\frac{1}{v(t^2+1)}\right)h(vt)\,dt \\
&\qquad \ll_{f,h} \int_{|t|>v^{-\varepsilon}} \frac{dt}{t^2+1},
\end{aligned}$$

since the sum over m is converging uniformly with respect to t and v due to the fact that $v(t^2+1) \le R^{-1} < 1$.

For $\varepsilon > 0$ the value of the above integral converges to zero as $v \to 0$.

6.8.3. We obtain an upper bound for the remaining terms, by dropping the condition $|u| > v^{1-\varepsilon}$ in the integral. We are thus led to estimate

$$S(v) = \sum_{\substack{(c,d)\in\mathbb{Z}^2\\ \gcd(c,d)=1\\ c>0,d\neq 0}} \sum_{m\in\mathbb{Z}} J(v,c,d,m)$$

with

$$J(v,c,d,m) = \int_{\mathbb{R}} f\left((dy+m)\frac{v^{1/2}}{|c\tau+d|}\right)\frac{v}{|c\tau+d|^2}\chi_R\left(\frac{v}{|c\tau+d|^2}\right)\, h(u)\,du.$$

We substitute $t = v^{-1}(u + \frac{d}{c})$ for u, yielding

$$\frac{1}{c^2}\int_{\mathbb{R}} f\left((dy+m)\frac{1}{\sqrt{c^2v(t^2+1)}}\right)\frac{1}{t^2+1}\chi_R\left(\frac{1}{c^2v(t^2+1)}\right)\, h(vt-\frac{d}{c})\,dt.$$

The range of integration is bounded by

$$R < \frac{1}{c^2v(t^2+1)}; \qquad \text{i.e.,} \quad |t| \ll \frac{1}{c\sqrt{vR}}.$$

This implies $|vt| \ll v^{1/2}c^{-1}R^{-1/2}$ is uniformly close to zero, and hence, because of the compact support of h, we find $|d| \le Mc$, for some constant $M > 0$ depending only on the support of h. Therefore

$$S(v) \ll \sum_{c=1}^{\infty} \sum_{0<|d|\le Mc} \sum_{m\in\mathbb{Z}} K(v,c,d,m),$$

with

$$K(v,c,d,m) = \frac{1}{c^2}\int_{\mathbb{R}} f\left((dy+m)\frac{1}{\sqrt{c^2v(t^2+1)}}\right)\frac{1}{t^2+1}\chi_R\left(\frac{1}{c^2v(t^2+1)}\right)\, dt.$$

6.8.4. In order to apply Lemma 6.6 with $D = Mc$, $T = (c^2 v(t^2+1))^{-1/2} > \sqrt{R} > 1$, we split the t-range of integration into the ranges

$$\begin{aligned}
(1): &\quad Mc \leq (c^2 v(t^2+1))^{-\varepsilon'/2} \\
(2): &\quad (c^2 v(t^2+1))^{-\varepsilon'/2} \leq Mc \leq (c^2 v(t^2+1))^{-\frac{1}{2(\kappa-1)}} \\
(3): &\quad Mc \geq (c^2 v(t^2+1))^{-\frac{1}{2(\kappa-1)}},
\end{aligned}$$

which correspond to

$$\begin{aligned}
(1): &\quad D \leq T^{\varepsilon'} \\
(2): &\quad T^{\varepsilon'} \leq D \leq T^{\frac{1}{\kappa-1}} \\
(3): &\quad D \geq T^{\frac{1}{\kappa-1}}.
\end{aligned}$$

Here, $\varepsilon' < \frac{1}{\kappa-1}$.

We denote the corresponding integrals by $K_1(v,c,d,m)$, $K_2(v,c,d,m)$ and $K_3(v,c,d,m)$, respectively.

6.8.5. Because $R^{-1/2} \geq T^{-1}$,

$$\begin{aligned}
\sum_{c>0} \sum_{0<|d|\leq Mc} \sum_{m\in\mathbb{Z}} K_1(v,c,d,m) &\ll R^{-A/2} \sum_{c>0} \frac{1}{c^2} \int_{(1)} \frac{1}{t^2+1} \chi_R\left(\frac{1}{c^2 v(t^2+1)}\right) dt \\
&\ll R^{-A/2} \sum_{c>0} \frac{1}{c^2} \int \frac{1}{t^2+1}\, dt \\
&\ll R^{-A/2}.
\end{aligned}$$

6.8.6. In order to obtain an upper bound, we can relax the second range $T^{\varepsilon'} \leq D \leq T^{\frac{1}{\kappa-1}}$ to $R^{\varepsilon'/2} \leq D$, since $R^{1/2} \leq T$. This yields

$$\sum_{c>0} \sum_{0<|d|\leq Mc} \sum_{m\in\mathbb{Z}} K_2(v,c,d,m) \ll \sum_{Mc \geq R^{\varepsilon'/2}} c^{-2} \int \frac{1}{t^2+1}\, dt \ll R^{-\varepsilon'/2}.$$

6.8.7. In the third range, we find (putting $\delta = \frac{1}{\kappa-1}$),

$$\begin{aligned}
&\sum_{c>0} \sum_{0<|d|\leq Mc} \sum_{m\in\mathbb{Z}} K_3(v,c,d,m) \\
&\quad \ll \sum_{c>0} \frac{1}{c^2} \int_{(3)} c^{1+\delta} v^{\delta/2} (t^2+1)^{\frac{\delta}{2}-1} \chi_R\left(\frac{1}{c^2 v(t^2+1)}\right) dt \\
&\quad \leq \sum_{c>0} c^{-1+\delta} v^{\delta/2} \int_{\mathbb{R}} (t^2+1)^{\frac{\delta}{2}-1} \chi_R\left(\frac{1}{c^2 v(t^2+1)}\right) dt \\
&\quad = v^{\delta/2} \int_{\mathbb{R}} \left\{ \sum_{c=1}^{\infty} c^{-1+\delta} \chi_R\left(\frac{1}{c^2 v(t^2+1)}\right) \right\} (t^2+1)^{\frac{\delta}{2}-1}\, dt.
\end{aligned}$$

For the inner sum there exist the upper bounds

$$\sum_{c=1}^{\infty} c^{-1+\delta}\chi_R\left(\frac{1}{c^2 v(t^2+1)}\right) \ll \int_0^{\infty} x^{-1+\delta}\chi_R\left(\frac{1}{x^2 v(t^2+1)}\right) dx$$
$$= [v(t^2+1)]^{-\delta/2}\int_0^{\infty} x^{-1+\delta}\chi_R\left(\frac{1}{x^2}\right) dx = [v(t^2+1)]^{-\delta/2}\left\{\frac{x^\delta}{\delta}\right\}_0^{R^{-1/2}},$$

and so

$$\sum_{c>0}\sum_{d\ll c}\sum_{m\in\mathbb{Z}} K_3(v,c,d,m) \ll \frac{R^{-\delta/2}}{\delta}\int_{\mathbb{R}} (t^2+1)^{-1}\,dt = \frac{\pi R^{-\delta/2}}{\delta}.$$

The proof of Proposition 6.5 is complete. □

7. Equidistribution and unbounded test functions

7.1. Let us define the characteristic function on $\Gamma\backslash G^k$ (cf. the proof of Proposition 5.4):

$$X_R(\tau) = \sum_{\gamma\in\{\Gamma_\infty\cup(-1)\Gamma_\infty\}\backslash\,\mathrm{SL}(2,\mathbb{Z})} \chi_R(v_\gamma),$$

where χ_R is the characteristic function of $[R,\infty)$.

7.2. We shall consider functions on $\Gamma\backslash G^k$, which grow moderately in the cusps. To be more precise, we will require that, for some fixed constant $L>1$, the function F *is dominated by* F_R; that is, for all sufficiently large $R>1$,

$$|F(\tau,\phi;\boldsymbol{\xi})|X_R(\tau) \le L + F_R(\tau;\boldsymbol{\xi})$$

uniformly for all $(\tau,\phi;\boldsymbol{\xi}) \in G^k$. The function $F_R(\tau;\boldsymbol{\xi})$ is now viewed as a function on G^k (rather than G^1 as in Section 6); that is, for

$$\boldsymbol{\xi} = {}^{\mathrm{t}}(x_1,\ldots,x_k,y_1,\ldots,y_k)$$

we put

$$F_R(\tau;\boldsymbol{\xi}) = \sum_{\gamma\in\Gamma_\infty\backslash\,\mathrm{SL}(2,\mathbb{Z})}\sum_{m\in\mathbb{Z}} f\Big((y_{1,\gamma}+m)v_\gamma^{1/2}\Big)\, v_\gamma\,\chi_R(v_\gamma)$$

which is invariant under $\mathrm{SL}(2,\mathbb{Z})\ltimes\mathbb{Z}^{2k}$ (Section 6) and thus also under Γ. Note that $F_R(\tau;\boldsymbol{\xi})$ is constant with respect to $x_2,\ldots,x_k$ and $y_2,\ldots,y_k$. Again, $f\in\mathrm{C}(\mathbb{R})$ is rapidly decreasing at $\pm\infty$, positive and even.

7.3. Theorem. *Let Γ be a subgroup of $\mathrm{SL}(2,\mathbb{Z})\ltimes\mathbb{Z}^{2k}$ of finite index. Let h be a continuous probability density $\mathbb{R}\to\mathbb{R}_+$ with compact support. Suppose the continuous function $F\ge 0$ is dominated by F_R. Fix some $\mathbf{y}\in\mathbb{T}^k$ such that* *the components of the vector* $({}^{\mathrm{t}}\mathbf{y},1)\in\mathbb{R}^{k+1}$ *are linearly independent over* $\mathbb{Q}$. *Then, for any* ε *with* $0<\varepsilon<1$,

$$\liminf_{v\to 0}\int_{|u|>v^{1-\varepsilon}} F\left(u+\mathrm{i}v,0;\begin{pmatrix}\mathbf{0}\\ \mathbf{y}\end{pmatrix}\right)\,h(u)\,du \geq \frac{1}{\mu(\Gamma\backslash G^k)}\int_{\Gamma\backslash G^k} F\,d\mu.$$

Assume furthermore that y_1 *is diophantine. Then, for any* ε *with* $0<\varepsilon<1$,

$$\limsup_{v\to 0}\int_{|u|>v^{1-\varepsilon}} F\left(u+\mathrm{i}v,0;\begin{pmatrix}\mathbf{0}\\ \mathbf{y}\end{pmatrix}\right)\,h(u)\,du \leq \frac{1}{\mu(\Gamma\backslash G^k)}\int_{\Gamma\backslash G^k} F\,d\mu.$$

Proof. We obtain the *lower bound* from the function

$$G_R(\tau,\phi;\boldsymbol{\xi}) := F(\tau,\phi;\boldsymbol{\xi})\,(1-X_R(\tau)) \leq F(\tau,\phi;\boldsymbol{\xi}).$$

Clearly, G_R is bounded. Therefore

$$\int_{|u|>v^{1-\varepsilon}} G_R(u+\mathrm{i}v,0;\boldsymbol{\xi})\,h(u)\,du = \int_{\mathbb{R}} G_R(u+\mathrm{i}v,0;\boldsymbol{\xi})\,h(u)\,du + O_R(v^{1-\varepsilon}),$$

and, by Theorem 5.7,[3]

$$\lim_{v\to 0}\int_{\mathbb{R}} G_R(u+\mathrm{i}v,0;\boldsymbol{\xi})\,h(u)\,du = \frac{1}{\mu(\Gamma\backslash G^k)}\int_{\Gamma\backslash G^k} G_R\,d\mu.$$

Now since $0\leq FX_R\leq LX_R+F_R$ for R large enough we have

$$\int_{\Gamma\backslash G^k} FX_R\,d\mu \leq \int_{\Gamma\backslash G^k}(LX_R+F_R)\,d\mu \ll LR^{-1}+R^{-1/2}$$

from 5.4 and 6.3, and hence

$$\int_{\Gamma\backslash G^k} G_R\,d\mu = \int_{\Gamma\backslash G^k} F\,d\mu + O(LR^{-1}+R^{-1/2}).$$

In summary

$$\liminf_{v\to 0}\int_{|u|>v^{1-\varepsilon}} F(u+\mathrm{i}v,0;\boldsymbol{\xi})\,h(u)\,du \geq \frac{1}{\mu(\Gamma\backslash G^k)}\int_{\Gamma\backslash G^k} F\,d\mu + O(R^{-1/2}),$$

for all R large enough. The assertion on the lower bound follows now from the fact that R can be chosen arbitrarily large.

For the *upper bound*, notice that for R large enough,

$$F(\tau,\phi;\boldsymbol{\xi}) \leq F(\tau,\phi;\boldsymbol{\xi})(1-X_R(\tau)) + LX_R(\tau) + F_R(\tau;\boldsymbol{\xi}).$$

[3] The fact that G_R is only piecewise continuous should not worry us: Theorem 5.7 can easily be extended to such functions by approximating these from above and from below by continuous functions. In any case, the argument presented here works as well if χ_R is smoothed slightly, which makes G_R continuous.

By virtue of the bound obtained in the previous paragraph, and by Proposition 6.5, we find that

$$\begin{aligned}\limsup_{v\to 0} \int_{|u|>v^{1-\varepsilon}} & F(u+\mathrm{i}v,0;\boldsymbol{\xi})\, h(u)\, du \\ & \leq \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\, d\mu + O(R^{-1/2}) + O(R^{-\eta})\end{aligned}$$

for some small constant $\eta > 0$. This holds again for arbitrarily large R, and the statement is proved. □

7.4. Corollary. *Let* Γ, h, $\mathbf{y}$ *be as in Theorem* 7.3, *and* $F : \Gamma\backslash G^k \to \mathbb{C}$ *be a continuous function which is dominated by* F_R. *If* y_1 *is diophantine, then, for any* ε *with* $0 < \varepsilon < 1$,

$$\lim_{v\to 0} \int_{|u|>v^{1-\varepsilon}} F\left(u+\mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}\right) h(u)\, du = \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\, d\mu.$$

Proof. Define

$$\mathrm{Re}_+ F(\tau,\phi;\boldsymbol{\xi}) = \begin{cases} \mathrm{Re}\, F(\tau,\phi;\boldsymbol{\xi}) & \text{if } \mathrm{Re}\, F(\tau,\phi;\boldsymbol{\xi}) > 0, \\ 0 & \text{if } \mathrm{Re}\, F(\tau,\phi;\boldsymbol{\xi}) \leq 0, \end{cases}$$

and $\mathrm{Re}_- F = \mathrm{Re}_+ F - \mathrm{Re}\, F$. We similarly define $\mathrm{Im}_\pm F$ as the positive/negative part of $\mathrm{Im}\, F$. Then

$$F = \mathrm{Re}_+ F - \mathrm{Re}_- F + \mathrm{i}\, \mathrm{Im}_+ F - \mathrm{i}\, \mathrm{Im}_- F$$

with

$$\begin{aligned} &0 \leq \mathrm{Re}_+ F X_R \leq L + F_R, \quad 0 \leq \mathrm{Re}_- F X_R \leq L + F_R, \\ &0 \leq \mathrm{Im}_+ F X_R \leq L + F_R, \quad 0 \leq \mathrm{Im}_- F X_R \leq L + F_R. \end{aligned}$$

We can thus apply Theorem 7.3 to each term separately,

$$\lim_{v\to 0} \int_{|u|>v^{1-\varepsilon}} \mathrm{Re}_\pm F\left(u+\mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}\right) h(u)\, du = \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} \mathrm{Re}_\pm F\, d\mu$$

and likewise for $\mathrm{Im}_\pm F$. □

7.5. Since in our main application $\Gamma = \Gamma^k$, which is a subgroup of finite index in $\mathrm{SL}(2,\mathbb{Z}) \ltimes (\frac{1}{2}\mathbb{Z})^{2k}$ rather than in $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$ (Lemma 4.12), we restate Corollary 7.4 in the following equivalent way. Define the dominating function $\hat{F}_R$ on $\Gamma\backslash G^k$ by $\hat{F}_R(\tau;\boldsymbol{\xi}) = F_R(\tau;2\boldsymbol{\xi})$, with F_R as in 7.2.

7.6. Corollary. *Let* Γ *be a subgroup of* $\mathrm{SL}(2,\mathbb{Z}) \ltimes (\frac{1}{2}\mathbb{Z})^{2k}$ *of finite index*, h, $\mathbf{y}$ *be as in Theorem* 7.3, *and* $F : \Gamma\backslash G^k \to \mathbb{C}$ *a continuous function which is dominated by* $\hat{F}_R$. *If* y_1 *is diophantine, then, for any* ε *with* $0 < \varepsilon < 1$,

$$\lim_{v\to 0} \int_{|u|>v^{1-\varepsilon}} F\left(u+\mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}\right) h(u)\, du = \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\, d\mu.$$

Proof. Apply Corollary 7.4 with the test function $\tilde{F} : \tilde{\Gamma}\backslash G^k \to \mathbb{C}$ defined by

$$\tilde{F}(\tau, \phi; \boldsymbol{\xi}) = F(\tau, \phi; \tfrac{1}{2}\boldsymbol{\xi})$$

where

$$\tilde{\Gamma} = \left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} ; \mathbf{0} \right) \Gamma \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} ; \mathbf{0} \right)$$

is a subgroup of finite index in $\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^{2k}$ (compare Remark 4.13). □

8. The main theorem

8.1. Main Theorem. *Suppose $f(w_1, w_2) = \psi_1(w_1^2+w_2^2)$ and $g(w_1, w_2) = \psi_2(w_1^2 + w_2^2)$ with $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}_+)$. Let h be a continuous function $\mathbb{R} \to \mathbb{C}$ with compact support. Assume that $y_1, y_2, 1$ are linearly independent over $\mathbb{Q}$ and that y_1 is diophantine. Then, with $\boldsymbol{\xi} = {}^{\mathrm{t}}(0, 0, y_1, y_2)$,*

$$\lim_{v \to 0} \int_{\mathbb{R}} \Theta_f(u + \mathrm{i}v, 0; \boldsymbol{\xi}) \overline{\Theta_g(u + \mathrm{i}v, 0; \boldsymbol{\xi})}\, h(u)\, du$$
$$= \pi \Big\{ 2\pi h(0) + \int_{\mathbb{R}} h(u)\, du \Big\} \int_0^\infty \psi_1(r) \overline{\psi_2(r)}\, dr.$$

The proof of the main theorem requires the following two lemmas.

8.2. Lemma. *If $f, g \in \mathcal{S}(\mathbb{R}^2)$,*

$$\frac{1}{\mu(\Gamma^2\backslash G^2)} \int_{\Gamma^2\backslash G^2} \Theta_f(\tau, \phi; \boldsymbol{\xi}) \overline{\Theta_g(\tau, \phi; \boldsymbol{\xi})}\, d\mu = \iint f(w_1, w_2) \overline{g(w_1, w_2)}\, dw_1\, dw_2.$$

Note that if $f(w_1, w_2) = \psi_1(w_1^2 + w_2^2)$ and $g(w_1, w_2) = \psi_2(w_1^2 + w_2^2)$, then

$$\iint f(w_1, w_2) \overline{g(w_1, w_2)}\, dw_1\, dw_2 = \pi \int_0^\infty \psi_1(r) \overline{\psi_2(r)}\, dr.$$

Proof. A short calculation shows that

$$\int_{\mathbb{T}^4} \Theta_f(\tau, \phi; \boldsymbol{\xi}) \overline{\Theta_g(\tau, \phi; \boldsymbol{\xi})}\, d\boldsymbol{\xi} = \iint f_\phi(w_1, w_2) \overline{g_\phi(w_1, w_2)}\, dw_1\, dw_2.$$

Since $f_\phi = \tilde{R}(\mathrm{i}, \phi) f$ with $\tilde{R}(\mathrm{i}, \phi)$ unitary, we have

$$\iint f_\phi(w_1, w_2) \overline{g_\phi(w_1, w_2)}\, dw_1\, dw_2 = \iint f(w_1, w_2) \overline{g(w_1, w_2)}\, dw_1\, dw_2. \qquad \square$$

8.3. Lemma. *Suppose* $f(w_1, w_2) = \psi_1(w_1^2 + w_2^2)$ *and* $g(w_1, w_2) = \psi_2(w_1^2 + w_2^2)$, *with* $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}_+)$. *For any* $\frac{1}{2} < \gamma < 1$,

$$\lim_{v\to 0} \int_{|u|<v^\gamma} \Theta_f\left(u + \mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}\right) \overline{\Theta_g\left(u + \mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}\right)} h(u)\, du$$
$$= 2\pi^2 h(0) \int_0^\infty \psi_1(r)\overline{\psi_2(r)}\, dr.$$

Proof. Proposition 4.11 tells us that

$$\Theta_f\left(-\frac{1}{\tau}, \arg\tau; \begin{pmatrix} -\mathbf{y} \\ \mathbf{0} \end{pmatrix}\right) \overline{\Theta_g\left(-\frac{1}{\tau}, \arg\tau; \begin{pmatrix} -\mathbf{y} \\ \mathbf{0} \end{pmatrix}\right)}$$
$$= \frac{v}{|\tau|^2} f_{\arg\tau}(0,0)\overline{g_{\arg\tau}(0,0)} + O_R\left(\left(\frac{v}{|\tau|^2}\right)^{-R}\right)$$

holds uniformly for $\mathrm{Im}(-\tau^{-1}) = v|\tau|^{-2} > \frac{1}{2}$. This condition is met, e.g., when $|u| < v^{1/2} < 1$. For $|u| < v^\gamma < 1$, with $\frac{1}{2} < \gamma < 1$, the error term is bounded by

$$O_R\left(\left(\frac{v}{|\tau|^2}\right)^{-R}\right) = O_R(v^{R(2\gamma-1)}).$$

Now replacing (w_1, w_2) by polar coordinates $(r\cos\zeta, r\sin\zeta)$ yields

$$\begin{aligned}
f_{\arg\tau}(0,0)\overline{g_{\arg\tau}(0,0)} &= \frac{|\tau|^2}{v^2}\left\{\iint e\left(\tfrac{1}{2}(w_1^2 + w_2^2)\frac{u}{v}\right) f(w_1, w_2)\, dw_1\, dw_2\right\} \\
&\quad \times \overline{\left\{\iint e\left(\tfrac{1}{2}(w_1^2 + w_2^2)\frac{u}{v}\right) g(w_1, w_2)\, dw_1\, dw_2\right\}} \\
&= \frac{|\tau|^2}{v^2}\pi^2 \iint_0^\infty e\left(\frac{(r_1 - r_2)u}{2v}\right) \psi_1(r_1)\overline{\psi_2(r_2)}\, dr_1 dr_2 \\
&= \frac{|\tau|^2}{v^2}\pi^2 \hat{\psi}_1\left(\frac{u}{2v}\right) \overline{\hat{\psi}_2\left(\frac{u}{2v}\right)},
\end{aligned}$$

where $\hat{\psi}$ denotes the Fourier transform

$$\hat{\psi}(u) = \int_0^\infty e(ur)\psi(r)\, dr.$$

Clearly $\hat{\psi} \in \mathrm{L}^2(\mathbb{R})$ for $\psi \in \mathcal{S}(\mathbb{R}_+) \subset \mathrm{L}^2(\mathbb{R}_+)$. Thus,

$$\begin{aligned}
&\int_{|u|<v^\gamma} \Theta_f\left(u + \mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}\right) \overline{\Theta_g\left(u + \mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}\right)} h(u)\, du \\
&\quad = \frac{\pi^2}{v} \int_{|u|<v^\gamma} \hat{\psi}_1\left(\frac{u}{2v}\right) \overline{\hat{\psi}_2\left(\frac{u}{2v}\right)} h(u)\, du + O_R(v^{\gamma + R(2\gamma-1)}) \\
&\quad = 2\pi^2 \int_{2|u|<v^{\gamma-1}} \hat{\psi}_1(u)\overline{\hat{\psi}_2(u)} h(2vu)\, du + O_R(v^{\gamma+R(2\gamma-1)}).
\end{aligned}$$

Since h is continuous, for any given $\varepsilon > 0$ we find a $v_0 > 0$, such that

$$|h(2vu) - h(0)| < \varepsilon, \text{ uniformly for all } 2|u| < v^{\gamma-1},\ 0 < v < v_0.$$

Thus for any $\varepsilon > 0$

$$\begin{aligned}
\lim_{v\to 0} \int_{|u|<v^\gamma} \Theta_f &\left(u + \mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}\right) \overline{\Theta_g \left(u + \mathrm{i}v, 0; \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}\right)} h(u)\, du \\
&= \lim_{v\to 0} 2\pi^2 \{h(0) + O(\varepsilon)\} \int_{2|u|<v^{\gamma-1}} \hat{\psi}_1(u) \overline{\hat{\psi}_2(u)}\, du \\
&= 2\pi^2 \{h(0) + O(\varepsilon)\} \int_{\mathbb{R}} \hat{\psi}_1(u) \overline{\hat{\psi}_2(u)}\, du \\
&= 2\pi^2 \{h(0) + O(\varepsilon)\} \int_0^\infty \psi_1(r) \overline{\psi_2(r)}\, dr
\end{aligned}$$

by Parseval's equality. Because $\varepsilon > 0$ can be arbitrarily small, the claim is proved. □

8.4. *Proof of the main theorem.*

8.4.1. Due to the linearity in h of the integrals in 8.1, we may assume without loss of generality that (i) h is positive (compare the argument used in the proof of Corollary 7.4) and (ii) that h is normalized as a probability density.

8.4.2. Let us split the integration on the left-hand side of 8.1 into

$$\int_{\mathbb{R}} = \int_{|u|<v^{1-\varepsilon}} + \int_{|u|>v^{1-\varepsilon}},$$

for some small $\varepsilon > 0$. The first integral gives, by virtue of Lemma 8.3, the contribution

$$2\pi^2 h(0) \int_0^\infty \psi_1(r) \overline{\psi_2(r)}\, dr.$$

8.4.3. In order to apply Corollary 7.6, we need to construct a function F_R of the form studied in 7.2, which dominates $|F|$. Let us define

$$f^*(w_1) = \sup_{w_2\in\mathbb{R}} \sup_{\phi\in\mathbb{R}} |f_\phi(w_1, w_2)\, g_\phi(w_1, w_2)|,$$

which is clearly rapidly decreasing at $\pm\infty$ since, for every $T > 1$, there is a constant $c_T > 0$ such that

$$f^*(w_1) \le \sup_{w_2\in\mathbb{R}} c_T \left(1 + \sqrt{w_1^2 + w_2^2}\right)^{-2T} \le c_T (1 + |w_1|)^{-2T},$$

holds (cf. Lemma 4.3).

Choosing (compare 7.2)

$$F_R(\tau; \boldsymbol{\xi}) = \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}(2,\mathbb{Z})} \sum_{m\in\mathbb{Z}} f^*\Big(-\tfrac{1}{2}(y_{1,\gamma} + m) v_\gamma^{1/2}\Big)\, v_\gamma\, \chi_R(v_\gamma),$$

we have for all $v > R$

$$\hat{F}_R(\tau;\boldsymbol{\xi}) = F_R(\tau;2\boldsymbol{\xi}) = v\sum_{m\in\mathbb{Z}}\left\{f^*\left(\left(\tfrac{m}{2}-y_1\right)v^{1/2}\right) + f^*\left(\left(\tfrac{m}{2}+y_1\right)v^{1/2}\right)\right\};$$

that is,

$$\hat{F}_R(\tau;\boldsymbol{\xi}) = v\left\{f^*\left(\left(\tfrac{n}{2}-y_1\right)v^{1/2}\right) + f^*\left(\left(-\tfrac{n}{2}+y_1\right)v^{1/2}\right)\right\} + O(v^{-T}),$$

for all $y_1 \in \frac{n}{2}+[-\frac14,\frac14]$, $n\in\mathbb{Z}$. By construction, for $\mathbf{n} = {}^{\mathrm{t}}(n_1,n_2)$,

$$\left|f_\phi\left((\mathbf{n}-\mathbf{y})v^{1/2}\right)g_\phi\left((\mathbf{n}-\mathbf{y})v^{1/2}\right)\right| \le f^*\left((n_1-y_1)v^{1/2}\right)$$

which implies that, for all $v > R$, R large enough,

$$\begin{aligned} v\Bigg|\sum_{\mathbf{m}\in\mathbb{Z}^2} & f_\phi\left((\mathbf{m}-\mathbf{y})v^{1/2}\right)g_\phi\left((\mathbf{m}-\mathbf{y})v^{1/2}\right)\Bigg| \\ &= v\left|f_\phi\left((\mathbf{n}-\mathbf{y})v^{1/2}\right)g_\phi\left((\mathbf{n}-\mathbf{y})v^{1/2}\right)\right| + O(v^{-T}) \\ &\le vf^*\Big((n_1-y_1)v^{1/2}\Big) + O(v^{-T}) = v\sum_{m_1\in\mathbb{Z}} f^*\Big((m_1-y_1)v^{1/2}\Big) + O(v^{-T}), \end{aligned}$$

uniformly for $\mathbf{y} = {}^{\mathrm{t}}(y_1,y_2) \in \mathbf{n}+[\frac12,\frac12]^2$, $\mathbf{n}\in\mathbb{Z}^2$.

Therefore, by virtue of Proposition 4.10, we have, for all sufficiently large R,

$$|\Theta_f(\tau,\phi;\boldsymbol{\xi})\Theta_g(\tau,\phi;\boldsymbol{\xi})| \le 1 + v\sum_{\substack{m\in\mathbb{Z}\\ m\ \text{even}}} f^*\left(\left(\tfrac{m}{2}-y_1\right)v^{1/2}\right) \le 1+\hat{F}_R(\tau;\boldsymbol{\xi})$$

for $v \ge R$, and so $|\Theta_f\Theta_g|X_R \le 1+\hat{F}_R$. We can now apply Corollary 7.6, and thus obtain the second term on the right-hand side of 8.1 (recall Lemma 8.2). □

8.5. *Proof of Theorem* 2.2. Recall that

$$\begin{aligned} \int_{\mathbb{R}} \Theta_f\left(u+\mathrm{i}\frac{1}{\lambda},0;\,{}^{\mathrm{t}}(0,0,\alpha,\beta)\right) & \overline{\Theta_g\left(u+\mathrm{i}\frac{1}{\lambda},0;\,{}^{\mathrm{t}}(0,0,\alpha,\beta)\right)}\,h(u)\,du \\ &= \pi R_2(\psi_1,\psi_2,h,\lambda). \end{aligned}$$

We have furthermore

$$\hat{h}(s) = \int_{\mathbb{R}} h(u)e\left(\tfrac12 us\right)du, \quad h(u) = \tfrac12\int_{\mathbb{R}}\hat{h}(s)e\left(-\tfrac12 us\right)ds;$$

hence $2h(0) = \int\hat{h}(s)ds$ and $\int h(u)du = \hat{h}(0)$. □

8.6. *Proof of Theorem* 1.8.

8.6.1. Let $\chi[a,b]$ be the characteristic function of the interval $[a,b]$. Given any $\varepsilon > 0$, we approximate $\chi[a,b]$ from above and below by functions $\chi_\pm \in \mathrm{C}^\infty(\mathbb{R})$ with compact support so that

$$\chi_-(s) \le \chi[a,b](s) \le \chi_+(s), \qquad \int_{\mathbb{R}}(\chi_+(s)-\chi_-(s))\,ds < \varepsilon.$$

Put
$$\hat{h}_\pm(s) = \chi_\pm(s) \pm \frac{\delta}{1+s^2},$$
where $\delta > 0$ is chosen such that
$$\int_{\mathbb{R}} \frac{4\delta}{1+s^2}\, ds < \varepsilon.$$
Then
$$\hat{h}_-(s) + \frac{\delta}{1+s^2} \le \chi[a,b](s) \le \hat{h}_+(s) - \frac{\delta}{1+s^2},$$
$$\int_{\mathbb{R}} \left(\hat{h}_+(s) - \hat{h}_-(s) + \frac{2\delta}{1+s^2} \right) ds < 2\varepsilon.$$
The inverse Fourier transform
$$h_\pm(u) = \tfrac{1}{2} \int_{\mathbb{R}} \hat{h}_\pm(s) e\left(-\tfrac{1}{2} us\right)\, ds$$
is continuous on $\mathbb{R}$, infinitely differentiable on $\mathbb{R} - \{0\}$ and decreases, together with its derivatives, rapidly at $\pm\infty$.

8.6.2. We fix a smoothed characteristic function $\chi \in \mathrm{C}^\infty(\mathbb{R})$ of compact support in $[-2,2]$, with $0 \le \chi \le 1$ and $\chi(u) = 1$ if $u \in [-1,1]$. Define
$$h_{T,\pm}(u) = h_\pm(u)\, \chi\left(\frac{u}{T}\right),$$
which is continuous and has compact support in $[-2T, 2T]$. For the Fourier transform
$$\hat{h}_{T,\pm}(s) = \int_{\mathbb{R}} h_{T,\pm}(u) e\left(\tfrac{1}{2} us\right)\, du$$
we have, for some constant C,
$$|\hat{h}_\pm(s) - \hat{h}_{T,\pm}(s)| \le \int_{\mathbb{R}} |h_\pm(u)| \left| 1 - \chi\left(\frac{u}{T}\right) \right| du \le \int_{|u|>T} |h_\pm(u)|\, du \le \frac{C}{T},$$
and (integrate by parts twice)
$$\begin{aligned} |\hat{h}_\pm(s) - \hat{h}_{T,\pm}(s)| \quad &\le \quad \frac{1}{(\pi s)^2} \Bigg\{ \int_{|u|>T} |h''_\pm(u)|\, du + \frac{2}{T} \int_{\mathbb{R}} \left| h'_\pm(u) \chi'\left(\frac{u}{T}\right) \right| du \\ &\qquad + \frac{1}{T^2} \int_{\mathbb{R}} \left| h_\pm(u) \chi''\left(\frac{u}{T}\right) \right| du \Bigg\} \\ &\le \quad \frac{C}{Ts^2}. \end{aligned}$$
Therefore we find some $T > 1$ such that
$$|\hat{h}_\pm(s) - \hat{h}_{T,\pm}(s)| < \frac{\delta}{1+s^2}.$$

Hence

$$\hat{h}_{T,-}(s) \leq \chi[a,b](s) \leq \hat{h}_{T,+}(s), \qquad \int_{\mathbb{R}} (\hat{h}_{T,+}(s) - \hat{h}_{T,-}(s))ds < 2\varepsilon.$$

8.6.3. We will assume in the following that $\psi_1, \psi_2 \geq 0$. Then

$$\begin{aligned}
&\frac{1}{\pi\lambda} \sum_{j \neq k} \psi_1\left(\frac{\lambda_j}{\lambda}\right) \psi_2\left(\frac{\lambda_k}{\lambda}\right) \hat{h}_{T,-}(\lambda_j - \lambda_k) \\
&\qquad\qquad \leq \frac{1}{\pi\lambda} \sum_{j \neq k} \psi_1\left(\frac{\lambda_j}{\lambda}\right) \psi_2\left(\frac{\lambda_k}{\lambda}\right) \chi[a,b](\lambda_j - \lambda_k) \\
&\qquad\qquad \leq \frac{1}{\pi\lambda} \sum_{j \neq k} \psi_1\left(\frac{\lambda_j}{\lambda}\right) \psi_2\left(\frac{\lambda_k}{\lambda}\right) \hat{h}_{T,+}(\lambda_j - \lambda_k).
\end{aligned}$$

The functions $h_{T,\pm}$ satisfy the assumptions in Theorem 2.2, so the limits of left- and right-hand sides exist, and differ by less than

$$2\pi\varepsilon \left| \int_0^\infty \psi_1(r)\psi_2(r)dr \right|$$

for arbitrarily small $\varepsilon > 0$. Hence

$$\lim_{\lambda\to\infty} \frac{1}{\pi\lambda} \sum_{j \neq k} \psi_1\left(\frac{\lambda_j}{\lambda}\right) \psi_2\left(\frac{\lambda_k}{\lambda}\right) \chi[a,b](\lambda_j - \lambda_k) = \pi(b-a) \int \psi_1(r)\psi_2(r)dr.$$

8.6.4. Analogous arguments allow us to replace first ψ_1 and then ψ_2 by characteristic functions. □

For detailed discussions of approximation functions of the type used above, see [29] and references therein.

9. Counterexamples

9.1. Put

$$Q_{\alpha,\beta}(m,n) = (m-\alpha)^2 + (n-\beta)^2.$$

For $(\alpha, \beta) \in \mathbb{Q}^2$ we find (see Appendix A.10 for details) for $\lambda \to \infty$,

$$\begin{aligned}
R^{(\alpha,\beta)}[0,0] = \frac{1}{\pi\lambda} \#\{&(m_1, m_2, n_1, n_2) \in \mathbb{Z}^4 : (m_1, n_1) \neq (m_2, n_2), \\
&Q_{\alpha,\beta}(m_1, n_1) \leq \lambda,\ Q_{\alpha,\beta}(m_1, n_1) = Q_{\alpha,\beta}(m_2, n_2)\} \sim c_{\alpha,\beta} \log \lambda,
\end{aligned}$$

for some constant $c_{\alpha,\beta} > 0$. This fact will be the key in proving the first half of Theorem 1.13.

9.2. *Proof of Theorem* 1.13 (i). Enumerate the rational forms Q_{α_j,β_j} with $(\alpha_j, \beta_j) \in \mathbb{Q}^2$ as $P_1, P_2, P_3, \ldots$. Because of the asymptotics 9.1, given any

$\lambda > 1$, there exists an $M_j > \lambda$ such that

$$\frac{1}{\pi M_j} \#\{(m_1, m_2, n_1, n_2) \in \mathbb{Z}^4 : (m_1, n_1) \neq (m_2, n_2),$$
$$P_j(m_1, n_1) \leq M_j,\ P_j(m_1, n_1) = P_j(m_2, n_2)\} \geq \frac{\log M_j}{\log \log \log M_j}.$$

Now since

$$Q_{\alpha,\beta}(m_1, n_1) - Q_{\alpha,\beta}(m_2, n_2) = Q_{\alpha_j,\beta_j}(m_1, n_1) - Q_{\alpha_j,\beta_j}(m_2, n_2) + 2(\alpha_j - \alpha)(m_1 - m_2) + 2(\beta_j - \beta)(n_1 - n_2),$$

we have that

$$R_2^{(\alpha,\beta)}[-a, a](M_j) \geq R_2^{(\alpha_j,\beta_j)}[0, 0](M_j)$$

when $|\alpha - \alpha_j| < \frac{a}{8(\sqrt{M_j}+1)}$ and $|\beta - \beta_j| < \frac{a}{8(\sqrt{M_j}+1)}$. Denote by $B_j \subset \mathbb{T}^2$ the open set of such (α, β).

To summarize, given any $\lambda > 1$, there exists an $M_j > \lambda$ such that

$$R_2^{(\alpha,\beta)}[-a, a](M_j) \geq \frac{\log M_j}{\log \log \log M_j}$$

for all $(\alpha, \beta) \in B_j$. Individually, the sets B_j shrink to a point as $\lambda \to \infty$. Note, however, that for every fixed λ the union

$$\bigcup_{j:M_j \geq \lambda} B_j$$

is open and dense in $\mathbb{T}^2$, and therefore

$$B = \bigcap_{\lambda=1}^{\infty} \bigcup_{j:M_j \geq \lambda} B_j$$

is of second Baire category.

So if $(\alpha, \beta) \in B$, then, given any $\lambda > 1$, there exists some $M > \lambda$, such that

$$R_2^{(\alpha,\beta)}[-a, a](M) \geq \frac{\log M}{\log \log \log M}. \qquad \square$$

Note that the proof remains valid if $\log \log \log$ is replaced by any slowly increasing positive function $\nu \leq \log \log \log$ with $\nu(M) \to \infty$ $(M \to \infty)$.

9.3. *Proof of Theorem* 1.13 (ii). By virtue of Theorem 1.8, there exists a countable dense set $\{(\xi_j, \zeta_j) \in \mathbb{T}^2 : j \in \mathbb{N}\}$ for which the pair correlation density of the forms $O_j := Q_{\xi_j,\zeta_j}$ is uniform. That is, for any $\lambda > 1$, we find some $L_j > \lambda$ such that

$$2\pi a - \frac{1}{\lambda} < R_2^{(\xi_j,\zeta_j)}[-a, a](L_j) < 2\pi a + \frac{1}{\lambda}.$$

Let $A_j \subset \mathbb{T}^2$ be the open set of (α, β), which satisfy $|\alpha - \xi_j| < \varepsilon_j$ and $|\beta - \zeta_j| < \varepsilon_j$, where $\varepsilon_j > 0$ can be chosen in such a way that

$$2\pi a - \frac{2}{\lambda} < R_2^{(\alpha,\beta)}[-a, a](L_j) < 2\pi a + \frac{2}{\lambda}$$

for all $(\alpha, \beta) \in A_j$. Now,

$$A = \bigcap_{\lambda=1}^{\infty} \bigcup_{j:L_j \geq \lambda} A_j$$

is again of second category. We conclude that if $(\alpha, \beta) \in A$, then, given any $\lambda > 1$, there exists some $L > \lambda$, such that

$$2\pi a - \frac{2}{\lambda} < R_2^{(\alpha,\beta)}[-a, a](L) < 2\pi a + \frac{2}{\lambda}. \qquad \square$$

9.4. *Conclusion of the proof of Theorem* 1.13. Since A and B are of second Baire category, so is the intersection $C = A \cap B$. $\square$

Appendix A. Symmetries

A.1. We have seen in Section 5 that in the case when $\alpha, \beta, 1$ are linearly independent over $\mathbb{Q}$,

$$\int_{\mathbb{R}} F(u + \mathrm{i}v, 0; {}^{\mathrm{t}}(0, 0, \alpha, \beta))h(u)du$$

converges for all suitably nice test functions F on $\Gamma^2 \backslash G^2$ to the average of F over $\Gamma^2 \backslash G^2$, as $v \to 0$. This is no longer true when $\alpha, \beta, 1$ are linearly dependent over $\mathbb{Q}$, i.e., if we find integers $(k, l, m) \in \mathbb{Z}^3 - \{(0, 0, 0)\}$ such that $k\alpha + l\beta + m = 0$. One of k, l must be nonzero, and we will assume in the following (without loss of generality) that $l \neq 0$, i.e., $\beta = -\frac{1}{l}(k\alpha + m)$.

A.2. Suppose $\alpha \notin \mathbb{Q}$. For any given function $F \in \mathrm{C}(G^2)$ which is invariant under the left action of Γ^2, we define a function $\tilde{F} \in \mathrm{C}(G^1)$ by

$$\tilde{F}\left(\tau, \phi; \begin{pmatrix} x \\ y \end{pmatrix}\right) = F\left(\tau, \phi; \begin{pmatrix} x \\ -\frac{1}{l}(kx + m) \\ y \\ -\frac{1}{l}(ky + m) \end{pmatrix}\right).$$

Since $\tilde{F}$ is invariant under the left action of the subgroup

$$\Gamma_{2l}^1 = \left\{(\gamma, \mathbf{n}) \in \mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2 : \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2l,\ \mathbf{n} = \mathbf{0} \bmod 2l\right\} \subset \Gamma^1,$$

we can identify $\tilde{F}$ as a function on $\Gamma^1_{2l}\backslash G^1$. The congruence group Γ^1_{2l} is of finite index in $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$, and hence $\Gamma^1_{2l}\backslash G^1$ has finite volume with respect to Haar measure.

A.3. If $\alpha = \frac{p}{q}$ and $\beta = \frac{r}{s}$ are rational, we define instead

$$\tilde{F}(\tau,\phi) = F(\tau,\phi;\, {}^{\mathrm{t}}(0,0,\tfrac{p}{q},\tfrac{r}{s}))$$

which is a function on G^0 invariant under the left action of the subgroup

$$\Gamma^0_{2qs} = \left\{ \gamma \in \Gamma_\theta : \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2qs \right\}.$$

Again, $\Gamma^0_{2qs}\backslash G^0$ has finite measure.

A.4. *Example* 1. Consider the case $\alpha = \beta \notin \mathbb{Q}$. In order to remove the two-fold degeneracy we consider the symmetry-reduced set

$$\{(m-\alpha)^2 + (n-\alpha)^2 : (m,n) \in \mathbb{Z}^2,\ m \ge n\}$$

whose elements we label by $\lambda_1 < \lambda_2 < \cdots$. The pair correlation function of this sequence is now again Poissonian:

A.5. Theorem. *Assume $\alpha = \beta$ is diophantine. Then*

$$\lim_{\lambda\to\infty} R_2[a,b](\lambda) = \frac{\pi}{2}(b-a).$$

Notice that the mean density is now $\frac{\pi}{2}$ since we count only distinct elements.

A.6. *Sketch of the proof.* The smoothed correlation function is

$$\begin{aligned} R_2(\psi_1,\psi_2,h,\lambda) \quad &= \quad \frac{2}{\pi\lambda} \sum_{\substack{(m_1,n_1)\in\mathbb{Z}^2 \\ m_1 \ge n_1}} \sum_{\substack{(m_2,n_2)\in\mathbb{Z}^2 \\ m_2 \ge n_2}} \\ &\times\, \psi_1\left(\frac{(m_1-\alpha)^2 + (n_1-\alpha)^2}{\lambda} \right) \\ &\times\, \psi_2\left(\frac{(m_2-\alpha)^2 + (n_2-\alpha)^2}{\lambda} \right) \\ &\times\, \hat{h}((m_1-\alpha)^2 + (n_1-\alpha)^2 - (m_2-\alpha)^2 - (n_2-\alpha)^2). \end{aligned}$$

This is asymptotic for large λ:

$$
\begin{aligned}
R_2(\psi_1,\psi_2,h,\lambda) \quad \sim \quad & \frac{1}{2\pi\lambda} \sum_{(m_1,n_1)\in\mathbb{Z}^2} \sum_{(m_2,n_2)\in\mathbb{Z}^2} \psi_1\left(\frac{(m_1-\alpha)^2+(n_1-\alpha)^2}{\lambda}\right) \\
& \times\ \psi_2\left(\frac{(m_2-\alpha)^2+(n_2-\alpha)^2}{\lambda}\right) \\
& \times \hat{h}((m_1-\alpha)^2+(n_1-\alpha)^2-(m_2-\alpha)^2-(n_2-\alpha)^2),
\end{aligned}
$$

since the diagonal terms $m_1 = n_1$ or $m_2 = n_2$ give lower order contributions. The right-hand side of the above expression is equal to

$$
\frac{1}{2\pi}\int_{\mathbb{R}} \Theta_f\left(u+\mathrm{i}\frac{1}{\lambda},0;\,{}^{\mathrm{t}}(0,0,\alpha,\alpha)\right)\overline{\Theta_g\left(u+\mathrm{i}\frac{1}{\lambda},0;\,{}^{\mathrm{t}}(0,0,\alpha,\alpha)\right)}\,h(u)\,du.
$$

The corresponding test function

$$
\tilde{F}(\tau,\phi;\,{}^{\mathrm{t}}(x,y)) = \Theta_f(\tau,\phi;\,{}^{\mathrm{t}}(x,x,y,y))\overline{\Theta_g(\tau,\phi;\,{}^{\mathrm{t}}(x,x,y,y))}
$$

is a function on $\Gamma^1\backslash G^1$; compare A.2. Starting from Theorem 7.3 we can apply the same string of arguments as before. The only main difference is that Lemma 8.2 has to be replaced by the one given below. This yields (compare the main Theorem 8.1; we assume here that ψ_1,ψ_2 are real-valued)

$$
\begin{aligned}
&\lim_{v\to 0}\int_{\mathbb{R}} \Theta_f(u+\mathrm{i}v,0;\,{}^{\mathrm{t}}(0,0,y,y))\overline{\Theta_g(u+\mathrm{i}v,0;\,{}^{\mathrm{t}}(0,0,y,y))}\,h(u)\,du \\
&\qquad = 2\pi\left\{\pi h(0)+\int_{\mathbb{R}} h(u)\,du\right\}\int_0^\infty \psi_1(r)\psi_2(r)\,dr,
\end{aligned}
$$

and hence

$$
\lim_{\lambda\to\infty} R_2(\psi_1,\psi_2,h,\lambda) = \left\{\hat{h}(0)+\frac{\pi}{2}\int_{\mathbb{R}}\hat{h}(s)\,ds\right\}\int \psi_1(r)\psi_2(r)\,dr,
$$

as needed. □

A.7. Lemma. *If $f,g\in\mathcal{S}(\mathbb{R}^2)$,*

$$
\begin{aligned}
&\frac{1}{\mu(\Gamma^1\backslash G^1)}\int_{\Gamma^1\backslash G^1} \Theta_f(\tau,\phi;\,{}^{\mathrm{t}}(x,x,y,y))\overline{\Theta_g(\tau,\phi;\,{}^{\mathrm{t}}(x,x,y,y))}\,d\mu \\
&\qquad = \iint \left\{f(w_1,w_2)\overline{g(w_1,w_2)}+f(w_1,w_2)\overline{g(w_2,w_1)}\right\}\,dw_1\,dw_2.
\end{aligned}
$$

When $f(w_1,w_2)=\psi_1(w_1^2+w_2^2)$ and $g(w_1,w_2)=\psi_2(w_1^2+w_2^2)$, this yields

$$
\iint\left\{f(w_1,w_2)\overline{g(w_1,w_2)}+f(w_1,w_2)\overline{g(w_2,w_1)}\right\}dw_1\,dw_2 = 2\pi\int_0^\infty \psi_1(r)\psi_2(r)\,dr;
$$

compare 8.2.

Proof. Consider the function

$$F(\tau,\phi)=\iint_{\mathbb{T}^2}\Theta_f(\tau,\phi;\,{}^{\mathrm{t}}(x,x,y,y))\overline{\Theta_g(\tau,\phi;\,{}^{\mathrm{t}}(x,x,y,y))}\,dx\,dy.$$

This function may be viewed as a function on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$. By virtue of Proposition 4.10 one finds that asymptotically in the cusp ($v\to\infty$) we have

$$F(\tau,\phi)=v^{1/2}\int f_\phi(w,w)\overline{g_\phi(w,w)}dw+O_R(v^{-R}).$$

It follows from the classical equidistribution of closed horocycles [24], [6] in the case of unbounded test functions (cf. Proposition 4.3 in [13]) that as $v\to 0$

$$\begin{aligned}&\int_0^1 F(u+\mathrm{i}v,0)du\\ &\qquad=\int_0^1\iint_{\mathbb{T}^2}\Theta_f(u+\mathrm{i}v,\phi;\,{}^{\mathrm{t}}(x,x,y,y))\overline{\Theta_g(\tau,\phi;\,{}^{\mathrm{t}}(x,x,y,y))}\,dx\,dy\,du\end{aligned}$$

converges to the left-hand side in Lemma A.7. The right-hand side of the above equation can, however, be worked out straightforwardly: The series representation of Θ_f gives a natural Fourier expansion with respect to x and u. The zeroth Fourier coefficient, which we want to calculate, is given by those summands for which

$$\begin{cases} m_1+n_1=m_2+n_2\\ m_1^2+n_1^2=m_2^2+n_2^2.\end{cases}$$

This set of equations is equivalent to

$$\begin{cases} m_1-m_2=n_2-n_1\\ m_1^2-m_2^2=n_2^2-n_1^2,\end{cases}$$

whose only solutions are obviously ($m_1=m_2, n_1=n_2$) or ($m_1=n_2, m_2=n_1$). In the limit $v\to 0$, the zeroth Fourier coefficient is now easily seen to converge to the right-hand side in Lemma A.7. □

A further special case of interest is the following.

A.8. *Example* 2. When $\beta=0$ or $\beta=\frac12$ we consider the symmetry-reduced sequences $\lambda_1<\lambda_2<\cdots$ given by the sets

$$\{(m-\alpha)^2+n^2:(m,n)\in\mathbb{Z}^2,\ n\ge 0\}$$

or

$$\{(m-\alpha)^2+(n-\tfrac12)^2:(m,n)\in\mathbb{Z}^2,\ n>\tfrac12\},$$

respectively.

A.9. Theorem. *Assume α is diophantine. Then*

$$\lim_{\lambda\to\infty}R_2[a,b](\lambda)=\frac{\pi}{2}(b-a).$$

The proof of this theorem is analogous to that of Theorem A.5.

A.10. *Example* 3. If $\alpha = \frac{p}{q}$, $\beta = \frac{r}{s}$ are both rational, the integral

$$\int_{\mathbb{R}} \tilde{F}(u + \mathrm{i}v, 0) h(u) du$$

of the corresponding test function

$$\tilde{F}(\tau,\phi) = \Theta_f(\tau,\phi;\, {}^{\mathrm{t}}(0,0,\tfrac{p}{q},\tfrac{r}{s}))\, \overline{\Theta_g(\tau,\phi;\, {}^{\mathrm{t}}(0,0,\tfrac{p}{q},\tfrac{r}{s}))}$$

is diverging as $v \to 0$; one finds in particular that in this limit

$$\frac{1}{2qs} \int_0^{2qs} \tilde{F}(u + \mathrm{i}v, 0) du \sim b_{\alpha,\beta} \log v^{-1}$$

for some constant $b_{\alpha,\beta} > 0$. This follows from arguments analogous to those given in [13, Th. 6.1].

Therefore, for $\lambda \to \infty$,

$$\begin{aligned} R_2[0,0](\lambda) = \frac{1}{\pi\lambda} \#\{&(m_1,m_2,n_1,n_2) \in \mathbb{Z}^4 : (m_1-\alpha)^2 + (n_1-\beta)^2 \le \lambda, \\ &(m_1-\alpha)^2 + (n_1-\beta)^2 = (m_2-\alpha)^2 + (n_2-\beta)^2\} \sim c_{\alpha,\beta} \log\lambda, \end{aligned}$$

for some constant $c_{\alpha,\beta} > 0$. In the case $\alpha = \beta = 0$ this yields, of course, Landau's well known result on the asymptotic number of ways of writing an integer as a sum of two squares.

Appendix B. Closed connected subgroups of $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^{2k}$

B.1. Suppose H is a subgroup of $G^k = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^{2k}$. Then

$$H = \{(M;\boldsymbol{\xi}) \in G^k :\ M \in L,\ \boldsymbol{\xi} \in \mathcal{C}(M)\}$$

where L is a subgroup of $\mathrm{SL}(2,\mathbb{R})$ and $\mathcal{C}(M)$ is a family of sets, which are suitably chosen such that H is a group, but are otherwise arbitrary.

B.2. Clearly $\Omega = \mathcal{C}(1)$ is a subgroup of $\mathbb{R}^{2k}$, because $(1;\boldsymbol{\xi})(1;\boldsymbol{\xi}')^{\pm1} = (1;\boldsymbol{\xi} \pm \boldsymbol{\xi}')$ for all $\boldsymbol{\xi},\boldsymbol{\xi}' \in \Omega$ implies $\boldsymbol{\xi} \pm \boldsymbol{\xi}' \in \Omega$.

Moreover,

$$(M;\boldsymbol{\xi})(1;\boldsymbol{\xi}')(M;\boldsymbol{\xi})^{-1} = (1;M\boldsymbol{\xi}'),\ \text{for any } (M;\boldsymbol{\xi}) \in H,$$

says that if $\boldsymbol{\xi}' \in \Omega$, then $M\boldsymbol{\xi}' \in \Omega$; hence Ω is invariant under the action of L. This means also that $\{1\} \ltimes \Omega$ is a normal subgroup of H. Thus if $(M;\boldsymbol{\zeta}(M))$ is a set of representatives from the coset $(\{1\} \ltimes \Omega)\backslash H$,

$$(M;\boldsymbol{\zeta}(M))(M';\boldsymbol{\zeta}(M')) = (1,\boldsymbol{\sigma}(M,M'))(MM';\boldsymbol{\zeta}(MM'))$$

with cocycle

$$\boldsymbol{\sigma}(M,M') = \boldsymbol{\zeta}(M) + M\boldsymbol{\zeta}(M') - \boldsymbol{\zeta}(MM') \in \Omega.$$

We choose $\boldsymbol{\zeta}(M)$ in such a way that $\boldsymbol{\zeta}(1) = \mathbf{0}$.

B.3. If H is a closed connected subgroup of G^k, then L is a connected Lie subgroup of $\mathrm{SL}(2,\mathbb{R})$. Since all such subgroups are closed in $\mathrm{SL}(2,\mathbb{R})$, L is a closed connected subgroup of $\mathrm{SL}(2,\mathbb{R})$.

B.4. Let us assume in the following that the subgroup

$$\Psi_0^{\mathbb{R}} = \left(\begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix} ; \mathbf{0} \right).$$

is contained in H, and that $L = \mathrm{SL}(2,\mathbb{R})$. Then

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in L$$

and thus $(R;\boldsymbol{\xi}_0) \in H$ for some vector

$$\boldsymbol{\xi}_0 = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{y}_0 \end{pmatrix}.$$

Since conjugation by

$$g = \left(1; \tfrac{1}{2} \begin{pmatrix} \mathbf{x}_0 - \mathbf{y}_0 \\ \mathbf{0} \end{pmatrix} \right)$$

yields

$$g^{-1} \left(R; \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{y}_0 \end{pmatrix} \right) g = \left(R; \tfrac{1}{2} \begin{pmatrix} \mathbf{x}_0 + \mathbf{y}_0 \\ \mathbf{x}_0 + \mathbf{y}_0 \end{pmatrix} \right)$$

and

$$g^{-1}\, \Psi_0^t\, g = \Psi_0^t$$

for any $t \in \mathbb{R}$, we may assume without loss of generality that $\mathbf{x}_0 = \mathbf{y}_0$ (replace H with $g^{-1}Hg$).

Note that

$$\left(R; \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_0 \end{pmatrix} \right)^2 = \left(-1; \begin{pmatrix} \mathbf{0} \\ 2\mathbf{x}_0 \end{pmatrix} \right)$$

is in H, and so is the conjugate

$$\left(-1; \begin{pmatrix} \mathbf{0} \\ 2\mathbf{x}_0 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} ; \mathbf{0} \right) \left(-1; \begin{pmatrix} \mathbf{0} \\ 2\mathbf{x}_0 \end{pmatrix} \right) = \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} ; \begin{pmatrix} -2t\mathbf{x}_0 \\ \mathbf{0} \end{pmatrix} \right).$$

This implies, however, that

$$\left(1, \begin{pmatrix} -2t\mathbf{x}_0 \\ \mathbf{0} \end{pmatrix} \right) \in H$$

for all $t \in \mathbb{R}$, and so

$$\left(1; \begin{pmatrix} -\mathbf{x}_0 \\ \mathbf{0} \end{pmatrix}\right) \left(R; \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_0 \end{pmatrix}\right) \left(1; \begin{pmatrix} -\mathbf{x}_0 \\ \mathbf{0} \end{pmatrix}\right) = (R; \mathbf{0}) \in H.$$

Because the elements

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \ (t \in \mathbb{R}), \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

generate $\mathrm{SL}(2,\mathbb{R})$, we find trivially that Ψ_0^t and $(R;\mathbf{0})$ generate $\mathrm{SL}(2,\mathbb{R}) \ltimes \{\mathbf{0}\}$, and thus $H = \mathrm{SL}(2,\mathbb{R}) \ltimes \Omega$.

B.5. Since Ω is invariant under the action of $\mathrm{SL}(2,\mathbb{R})$ and H is closed and connected, it is a closed connected subgroup of $\mathbb{R}^{2k}$, i.e., Ω is a closed linear subspace.

B.6. We conclude that any closed connected subgroup H of G^k, for which $L = \mathrm{SL}(2,\mathbb{R})$ and which contains a conjugate of $\Psi_0^{\mathbb{R}}$, is conjugate to $\mathrm{SL}(2,\mathbb{R}) \ltimes \Omega$, where Ω is a closed connected subgroup of $\mathbb{R}^{2k}$. That is,

$$H = g_0 \, (\mathrm{SL}(2,\mathbb{R}) \ltimes \Omega) g_0^{-1},$$

for some $g_0 = (M_0; \boldsymbol{\xi}_0) \in G^k$. Because

$$(M_0, \mathbf{0})(\mathrm{SL}(2,\mathbb{R}) \ltimes \Omega)(M_0, \mathbf{0})^{-1} = \mathrm{SL}(2,\mathbb{R}) \ltimes \Omega$$

we may take $M_0 = 1$ without loss of generality, and hence

$$H = (1; \boldsymbol{\xi}_0)(\mathrm{SL}(2,\mathbb{R}) \ltimes \Omega)(1; -\boldsymbol{\xi}_0).$$

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