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Fundamental groups of manifolds with positive isotropic curvature

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Abstract

A central theme in Riemannian geometry is understanding the relationships between the curvature and the topology of a Riemannian manifold. Positive isotropic curvature (PIC) is a natural and much studied curvature condition which includes manifolds with pointwise quarter-pinched sectional curvatures and manifolds with positive curvature operator. By the results of Micallef and Moore there is only one topological type of compact simply connected manifold with PIC; namely any such manifold must be homeomorphic to the sphere. On the other hand, there is a large class of nonsimply connected manifolds with PIC. An important open problem has been to understand the fundamental groups of manifolds with PIC. In this paper we prove a new result in this direction. We show that the fundamental group of a compact manifold M^n with PIC, $n \geq 5$, does not contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. The techniques used involve minimal surfaces.

1. Introduction

It is a fundamental problem in geometry to determine the relationships between the curvature and topology of manifolds. In this paper we study the fundamental groups of compact manifolds with positive isotropic curvature (PIC). PIC is a natural and much studied curvature condition, which first derived its importance from the following beautiful theorem of Micallef and Moore [MM]:

Theorem 1.1 (Micallef-Moore). *Let M be a compact n-dimensional Riemannian manifold with* PIC, $n \geq 4$. *Then $\pi_k(M) = 0$ for $k = 2, \ldots, [\frac{n}{2}]$ (where $[\cdot]$ denotes the integer part of the number). In particular, if M is simply connected, then M is homeomorphic to a sphere.*

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If M is an n-dimensional Riemannian manifold, one can consider the complexification $T_pM \otimes \mathbb{C}$ of the tangent space T_pM at the point p. The inner product on the tangent space T_pM can be extended to the complexified tangent space $T_pM \otimes \mathbb{C}$ as a complex bilinear form $(\cdot, \cdot)$ or as a Hermitian inner product $\langle \cdot, \cdot \rangle$. The relationship between these extensions is given by $\langle v, w \rangle = (v, \bar{w})$ for $v,\ w \in T_pM \otimes \mathbb{C}$. The curvature tensor extends to complex vectors by linearity, and the *complex sectional curvature* of a two-dimensional subspace π of $T_pM \otimes \mathbb{C}$ is defined by $K(\pi) = \langle R(v, w)\bar{w}, v \rangle$, where $\{v, w\}$ is any unitary basis of π. A subspace π is said to be *isotropic* if every vector $v \in \pi$ has square zero; that is, $(v, v) = 0$.

Definition 1.2. A Riemannian manifold M has *positive isotropic curvature* (PIC) if $K(\pi) > 0$ for every isotropic two-plane π.

This curvature condition is nonvacuous only for $n \geq 4$, since in dimensions less than four there are no two-dimensional isotropic subspaces. The classical conditions of pointwise quarter-pinched sectional curvatures and positive curvature operator are easily seen to imply PIC; and so in particular, the Micallef-Moore theorem gives a generalization of the classical sphere theorem. Just as positive sectional curvature is well suited to studying the stability of geodesics, positive isotropic curvature is in a similar way ideally suited to studying the stability of minimal surfaces. The proof of Micallef-Moore involves an amplified version of the celebrated existence theory for minimal two-spheres of Sacks-Uhlenbeck [SU] together with estimates giving lower bounds on the Morse index for the area function of any minimal two-sphere in a manifold with PIC.

By the result of Micallef and Moore, there is only one topological type of compact simply connected manifold with PIC. On the other hand there is a large class of nonsimply connected manifolds with PIC, the simplest being $S^1 \times S^{n-1}$. This example, with infinite fundamental group $\mathbb{Z}$, shows in particular that PIC does not imply positive Ricci curvature. PIC does however imply positive scalar curvature [MW]. It was shown by Micallef and M. Wang [MW] that the connected sum of manifolds of PIC also carries such a metric. Thus, the fundamental group of a manifold with PIC can be very large. It is conjectured that *the fundamental group of a manifold with* PIC *is a virtually free group* (*that is*, *contains a free subgroup of finite index*). Evidence for this is suggested by classification results of Hamilton [H], Micallef-Wang [MW] and Noronha [N1], [N2] for four-manifolds with PIC. For example, an important property of PIC is that it is preserved under the Ricci flow, and for $n = 4$ Hamilton used the Ricci flow to give a classification of compact four-manifolds with PIC with no essential incompressible space forms. In fact, he has shown that such manifolds are diffeomorphic to those which can be obtained from S^4, $S^1 \times S^3$ and RP^4 by the connected sum operation. Also note that for four-dimensional conformally flat manifolds, PIC is equivalent to positive scalar curvature, and it follows from results of Schoen and Yau [SY2] that the conjecture holds in this case. There are a few results in higher dimensions. Noronha and Mercuri [MN] for example, show that the fundamental group of a compact hypersurface M in $\mathbb{R}^k$ with nonnegative isotropic curvature is a free group on $b_1(M)$ elements, where $b_1(M)$ denotes the first Betti number of M. Another result in [MW] is that the second Betti number of any closed even-dimensional manifold of PIC vanishes, $b_2(M) = 0$ (in particular, a Kähler manifold can never have PIC). Little else has been known in dimensions greater than four. The main result of this paper is:

THEOREM 1.3. *Let M be a compact n-dimensional Riemannian manifold with positive isotropic curvature, $n \geq 5$. Then the fundamental group of M does not contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.*

This proves a weaker version of the above conjecture, that any *free* abelian subgroup of the fundamental group is cyclic. This may be thought of as a PIC version of the Chern conjecture for manifolds with positive sectional curvature (recently proven false by Shankar [Sh]); we note that the assumption that the abelian subgroup be *free* cannot be removed under the PIC condition (quotients of $S^1 \times S^{n-1}$).

The idea of the proof of Theorem 1.3 is as follows. If the fundamental group did contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, then by results of Schoen and Yau [SY1] it could be represented by an area minimizing torus T^2. There are manifolds with PIC that do have stable tori:

Example 1.4. $S^1 \times L$, where L is a lens space, admits a metric of PIC. We can define a map $h : T^2 \to S^1 \times L$ such that $h_* : \pi_1(T^2) \to \pi_1(S^1 \times L) = \mathbb{Z} \oplus \mathbb{Z}_p$ is onto by viewing T^2 as the square with sides identified, mapping the boundary to generators of $\pi_1(S^1 \times L)$ and extending to the interior via contractibility. Any simple closed curve on T^2 has the form $\pm ra \pm sb$ where a and b represent generators of $\pi_1(T^2)$ and r and s are relatively prime integers,$(r, s) = 1$. Assume a and b are mapped to generators α and β of $\pi_1(S^1 \times L)$. Then the image of the simple closed curve is $\pm r\alpha \pm s\beta$, which is nontrivial in $\pi_1(S^1 \times L)$ since $(r, s) = 1$. Thus, u maps any simple closed curve in T^2 to a noncontractible curve in $S^1 \times L$, and it then follows by the proof of [SY1] that there is a branched minimal immersion $u : T^2 \to S^1 \times L$ such that $u_* = h_*$ on $\pi_1(T^2)$ and the induced area of u is least among all maps with the same action on $\pi_1(T^2)$.

However, we argue that for sufficiently large k, any minimal torus representing a subgroup $k\mathbb{Z} \oplus k\mathbb{Z}$ of the fundamental group, must be unstable. To prove instability we use a complex formula for the second variation of area developed by Siu and Yau [SiY] in their proof of the Frankel conjecture, and by Micallef in [Mi], and Micallef and Moore [MM]. PIC is naturally suited to studying the stability of minimal surfaces because one of the terms in the complexified second variation formula involves isotropic curvature; and, in order to prove instability or estimate the index of a minimal surface one must find holomorphic isotropic deformations of the surface. The argument for the index estimate for minimal two-spheres in manifolds with PIC of Micallef and Moore uses Grothendieck's splitting theorem for holomorphic vector bundles over the Riemann sphere. In the case of a torus, there is not such a nice splitting theorem, and in fact holomorphic deformations will not exist in general. However, for large k we are able to find *almost holomorphic* isotropic deformations that violate the stability inequality. We expect to be able to extend our techniques from the torus to the case of higher genus surfaces. In fact the techniques seem promising for proving the full conjecture.

Although our techniques are different, the idea of finding almost holomorphic sections is reminiscent of Donaldson's [D] almost holomorphic sections used in his procedure for producing symplectic submanifolds of any even codimension within a symplectic manifold. Also, in finding our sections, we use the fact that for sufficiently large k, there is a distance decreasing, degree one mapping from the torus to the two-sphere. The existence of such a map is somewhat analogous to Gromov and Lawson's [GL] notion of 'enlargeability', which was used by them in studying manifolds with positive scalar curvature.

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2. Proof of the theorem

In this section we prove the main result, Theorem 1.3. Let (M, g) be a compact n-dimensional Riemannian manifold with PIC, $n \geq 5$, and assume that $\pi_1(M)$ contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Let $\kappa > 0$ denote a lower positive bound on the isotropic curvature of M. Choose ε such that $0 < 2(C\varepsilon)^2 < \kappa$ where C is to be chosen later. Given an integer $k \geq 1$, consider the subgroup $G = k\mathbb{Z} \oplus k\mathbb{Z}$ of $\pi_1(M)$. By the results of Schoen-Yau [SY1] (and Sacks-Uhlenbeck [SU2]) there is a conformal branched minimal immersion of the torus $u : T^2 \to M$ whose induced map u_* on the fundamental group has image G, with least area among branched immersions with the same action as u_*. However, we claim that for k sufficiently large, such a minimal torus must be *unstable*. A key point in proving this is the following: For k sufficiently large, there is a distance decreasing, degree one map $f : (T^2, u^*g) \to (S^2, h)$, where h is the standard metric on S^2, with $|df| < \varepsilon$. The existence of this map follows from Lemma 3.1 from the Appendix. From now on, we fix k so that the lemma holds.

Let $\Sigma = u(T^2)$ and let $N\Sigma$ be the normal bundle to the surface Σ in M. Consider the pull-back of the normal bundle $u^*(N\Sigma)$ with the pull back of the metric and normal connection $\nabla^\perp$. Let $E = u^*(N\Sigma) \otimes \mathbb{C}$ be the complexified bundle. The metric on $u^*(N\Sigma)$ extends as a complex bilinear form $(\cdot,\cdot)$ or as a Hermitian metric $\langle \cdot,\cdot \rangle$ on E, and the connection $\nabla^\perp$ and curvature tensor extend complex linearly to sections of E. There is a unique holomorphic structure on E such that the $\bar\partial$ operator

$$\bar\partial : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E),$$

where $\mathcal{A}^{p,q}(E)$ denotes the space of (p,q)-forms on T^2 with values in E, is given by

$$\bar\partial \omega = (\nabla^\perp_{\frac{\partial}{\partial \bar z}} \omega) d\bar z$$

where $\frac{\partial}{\partial \bar z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$, for local coordinates x, y on T^2. Suppose Σ is *stable*. Then the complexified stability inequality (see [SiY], [Mi], [MM], [F]) holds:

$$\int_{T^2} \langle R(s, \frac{\partial u}{\partial z})\frac{\partial u}{\partial \bar z}, s\rangle \; dxdy \le \int_{T^2} [|\nabla^\perp_{\frac{\partial}{\partial \bar z}} s|^2 - |\nabla^\top_{\frac{\partial}{\partial z}} s|^2] \; dxdy$$

for all $s \in \Gamma(E)$. Assume now that s is isotropic. Since u is conformal, $\frac{\partial u}{\partial z}$ is isotropic and $\{s, \frac{\partial u}{\partial z}\}$ span an isotropic two-plane. Using the lower bound on the isotropic curvature and throwing away the second term on the right, we get

$$\kappa \int_{T^2} |s|^2 \; da \le \int_{T^2} |\bar\partial s|^2 \; da \tag{2.1}$$

where da denotes the area element for the induced metric u^*g on T^2. We now argue that we can find an "almost holomorphic" isotropic section of E that violates this stability inequality. That is, we will find $s \in \Gamma(E)$ such that

$$\int_{T^2} |\bar\partial s|^2 \; da << \int_{T^2} |s|^2 \; da.$$

Recall from [MM, p. 209], that the bilinear pairing $(\cdot,\cdot)$ establishes a holomorphic isomorphism between E and its dual E^*, and hence $c_1(E) = 0$. Let L be a positive holomorphic line bundle over S^2 with metric and connection, with $c_1(L) \ge 2$. Pull back L to a line bundle $\xi = f^*L$ over T^2 with pull-back metric and connection. The pull-back metric and connection on ξ determine a holomorphic structure on ξ, and

$$c_1(\xi) = (\deg f)c_1(L) = c_1(L) \ge 2.$$

Now consider the tensor product holomorphic vector bundle $\xi \otimes E$ of rank $n-2$ over T^2. Let $\mathcal{H}$ denote the complex vector space of holomorphic sections of

$\xi \otimes E$. By the Riemann-Roch theorem for holomorphic vector bundles over a Riemann surface ([Gu, p. 64]) we have

$$\begin{aligned} \dim \mathcal{H}(\xi \otimes E) &\geq c_1(\det(\xi \otimes E)) \\ &= \operatorname{rank}(E) c_1(\xi) + c_1(E) \\ &= (n-2) c_1(\xi). \end{aligned} \tag{2.2}$$

Therefore, there is at least a $2n-4$ dimensional space of holomorphic sections of the tensor product bundle $\xi \otimes E$. We now prove existence of a holomorphic section of $\xi \otimes E$ which is *isotropic*. Define a complex bilinear pairing

$$\mathcal{H}(\xi \otimes E) \times \mathcal{H}(\xi \otimes E) \to \mathcal{H}(\xi \otimes \xi)$$

given by

$$(t \otimes s_1, t \otimes s_2) = t \otimes t \ (s_1, s_2)$$

on any local basis of $\mathcal{H}(\xi \otimes E)$ of the form $\{t \otimes s_i\}_{i=1}^{n-2}$, where t, s_i $(i = 1, \dots, n-2)$ are local holomorphic sections giving a local basis for holomorphic sections of ξ and E respectively. Given any $x \in T^2$ we obtain a homogeneous polynomial on $\mathbb{C}^m \cong \mathcal{H}(\xi \otimes E)$ where $m = \dim \mathcal{H}(\xi \otimes E)$, given by $P_x(\sigma) = (\sigma, \sigma)(x)$. The zero set

$$V(P_x) = \{\sigma \in \mathbb{P}^{m-1} : P_x(\sigma) = 0\}$$

is a hypersurface in $\mathbb{P}^{m-1}$. Now given $m-1$ distinct points, we obtain $m-1$ hypersurfaces in $\mathbb{P}^{m-1}$, and observe that $m-1$ hypersurfaces in $\mathbb{P}^{m-1}$ intersect in a nonempty set of points. The intersection is a set of holomorphic sections of $\xi \otimes E$ which are isotropic at $m-1$ distinct points. Let $\sigma \in \mathcal{H}(\xi \otimes E)$ be such a section. Then (σ, σ) is a holomorphic section of $\xi \otimes \xi$ with at least $m-1$ zeros. But the number of zeros of a holomorphic section of $\xi \otimes \xi$ is $2c_1(\xi)$. From (2.2)

$$m - 1 \geq (n-2) c_1(\xi) - 1 > 2c_1(\xi)$$

since $n \geq 5$ and we chose $c_1(\xi) \geq 2$. It follows that $(\sigma, \sigma) \equiv 0$ and so σ is isotropic.

Finally, we show that any holomorphic isotropic section of $\xi \otimes E$ produces an *almost* holomorphic isotropic section of E. Let $\tilde{s} \in \Gamma(\xi \otimes E)$ be holomorphic and isotropic. First, we choose t_1^*, $t_2^* \in \Gamma(L^*)$ that satisfy

$$|t_1^*| + |t_2^*| \geq 1$$

and

$$|t_1^*| + |t_2^*| \leq 2.$$

This is possible since one can for example let t_1^* be a trivialization of the dual bundle L^* over $S^2 - U_-$, where U_- is some neighborhood of the south pole in the southern hemisphere S^2_- of S^2, with pointwise norm $|t_1^*| = 1$. Now extend

t_1^* to a global section of L^* and multiply by a cut-off function φ, $0 \leq \varphi \leq 1$ with $\varphi \equiv 1$ on S^2_+ (the northern hemisphere of S^2) and $\varphi \equiv 0$ on U_-. Similarly, we obtain $t_2^* \in \Gamma(L^*)$ with $|t_2^*| \leq 1$ on S^2 and $|t_2^*| = 1$ on S^2_-.

Now pull back t_1^*, t_2^* to $\alpha_1 = f^* t_1^*$, $\alpha_2 = f^* t_2^* \in \Gamma(\xi^*)$. We can think of α_i $(i = 1, 2)$ as a linear mapping

$$\alpha_i : \Gamma(\xi \otimes E) \to \Gamma(E)$$

where α_i is defined on any local basis section $t \otimes s_j$ by

$$\alpha_i(t \otimes s_j) = \alpha_i(t) s_j.$$

Note that the image of an isotropic section of $\xi \otimes E$ under α_i is an isotropic section of E. Define $s_1 = \alpha_1(\tilde{s})$, $s_2 = \alpha_2(\tilde{s}) \in \Gamma(E)$ and observe that

$$\begin{aligned} |s_1|^2 + |s_2|^2 &= |\alpha_1(\tilde{s})|^2 + |\alpha_2(\tilde{s})|^2 \\ &= |\alpha_1|^2 |\tilde{s}|^2 + |\alpha_2|^2 |\tilde{s}|^2 \\ &= (|\alpha_1|^2 + |\alpha_2|^2) |\tilde{s}|^2 \\ &\geq |\tilde{s}|^2. \end{aligned}$$

Integrating this pointwise inequality we obtain

$$\int\limits_{T^2} |s_1|^2 \, da + \int\limits_{T^2} |s_2|^2 \, da \geq \int\limits_{T^2} |\tilde{s}|^2 \, da.$$

Therefore, for either $i = 1$ or $i = 2$, say $i = 1$, we must have

$$\int\limits_{T^2} |s_1|^2 \, da \geq \frac{1}{2} \int\limits_{T^2} |\tilde{s}|^2 \, da. \tag{2.3}$$

Now,

$$\begin{aligned} |\bar{\partial} s_1| &= |\bar{\partial}(\alpha_1(\tilde{s}))| \\ &= |(\bar{\partial}\alpha_1)\tilde{s} + \alpha_1(\bar{\partial}\tilde{s})| \\ &= |(\bar{\partial}(f^* t_1^*)\tilde{s}| \\ &\leq C |\bar{\partial} f| |\tilde{s}| \\ &\leq C\varepsilon |\tilde{s}| \end{aligned}$$

where C depends only on L. Combining this with (2.3) we have

$$\int\limits_{T^2} |\bar{\partial} s_1|^2 \, da \leq 2(C\varepsilon)^2 \int\limits_{T^2} |s_1|^2 \, da < \kappa \int\limits_{T^2} |s_1|^2 \, da.$$

Therefore s_1 is an *almost holomorphic* isotropic section of E, that violates the stability inequality (2.1). This completes the proof of Theorem 1.3.

By a similar argument we also obtain the following.

Theorem 2.1. *There exists a sufficiently high finite cover of any stable incompressible minimal torus in a manifold with positive isotropic curvature which is unstable.*

3. Appendix: Distance decreasing map

Lemma 3.1. *Let M be a compact Riemannian manifold whose fundamental group contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Given $\varepsilon > 0$, there exists k sufficiently large such that for any torus Σ in M representing the subgroup $G = k\mathbb{Z} \oplus k\mathbb{Z}$ of $\pi_1(M)$, there is a distance decreasing, degree one mapping $f : \Sigma \to S^2$ with $|df| < \varepsilon$.*

Proof. Let $\tilde{M}$ denote the universal cover of M. First observe that for k large, any torus Σ in M representing the group $G = k\mathbb{Z} \oplus k\mathbb{Z}$ has large systole. Recall that the systole (see [G]) of Σ is defined to be the number

$$\mathcal{L} = \inf\{l(\gamma) : \gamma \text{ a noncontractible closed curve in } \Sigma\}.$$

To see this, define the systole associated to the group element $\gamma \in \pi_1(M)$,

$$\delta(\gamma) = \inf_{x \in \tilde{M}} d(x, \gamma x);$$

that is, $\delta(\gamma)$ is the length of the shortest closed curve in M freely homotopic to γ. Note that since M is compact, the set $S = \{\gamma \in \pi_1(M) : \delta(\gamma) < C\}$ is finite for any fixed constant C. Thus, for k large, $G \cap S = \phi$. That is, given L, there exists k such that the systole of Σ is greater than L.

Let $\tilde{\Sigma}$ be the universal cover of Σ, and $p : \tilde{\Sigma} \to \Sigma$ the covering map. Since $\tilde{\Sigma}$ is complete and noncompact with compact quotient Σ, there is a geodesic line $r : \mathbb{R} \to \tilde{\Sigma}$. Let $\mathcal{C}_1$ be the component of $\tilde{\Sigma} - r(\mathbb{R})$ which is on the side of $\tilde{\Sigma}$ in the direction of the unit normal ν to r such that $\{r'(0), \nu(0)\}$ is positively oriented. Choose T very large, $T >> L$. Define $D_1 : \tilde{\Sigma} \to \mathbb{R}$ by,

$$D_1(x) = d(x, r(T)) - T$$

and let $D_2 : \tilde{\Sigma} \to \mathbb{R}$ be the signed distance function to r,

$$D_2(x) = \begin{cases} d(x, r) & \text{for } x \in \mathcal{C}_1 \\ -d(x, r) & \text{for } x \in \tilde{\Sigma} - \mathcal{C}_1. \end{cases}$$

Both D_1 and D_2 are Lipschitz continuous with derivative bounded by 1. Consider the region

$$\tilde{\mathcal{R}} = \left\{ x \in \tilde{\Sigma} : |D_1(x)| \leq \frac{L}{4}, |D_2(x)| \leq \frac{L}{4} \right\}.$$

Define the map $f : \tilde{\mathcal{R}} \to \mathbb{R}^2$ by

$$f(x) = (D_1(x), D_2(x)).$$

Then the boundary of $\tilde{\mathcal{R}}$ is mapped to the boundary of the rectangle $[-\frac{L}{4}, \frac{L}{4}] \times [-\frac{L}{4}, \frac{L}{4}]$ in $\mathbb{R}^2$. Also, $r(0)$ is the only point in $\tilde{\mathcal{R}}$ which is mapped under f to the origin in $\mathbb{R}^2$. In fact, f is a local diffeomorphism in a neighborhood of $r(0)$, and hence the degree of f is equal to one on the component of $\mathbb{R}^2 - f(\partial\tilde{\mathcal{R}})$ containing the origin. In the definition of f, now replace D_1, D_2 by smooth approximations of these functions, which we will denote by $\bar{D}_1$ and $\bar{D}_2$, with $\|D_i - \bar{D}_i\|_{C^1} < \delta$ for $i = 1, 2$, where δ may be chosen arbitrarily small. For this smoothly redefined f also, by the above argument, $f(\tilde{\mathcal{R}})$ covers a disk of radius at least $\frac{L}{5}$ about the origin in $\mathbb{R}^2$.

Let $\lambda : D(0, \frac{L}{5}) \to D(0, \pi)$ be the contraction $\lambda(x) = \frac{5\pi}{L}x$ (where $D(0, s)$ denotes the disk of radius s centered at the origin in $\mathbb{R}^2$). Let $e : D(0, \pi) \to S^2$ be the exponential map at the north pole n of the sphere S^2 with the standard metric, $e(x) = \exp_n(x)$. Extend $g = e \circ \lambda$ to $f(\tilde{\mathcal{R}})$ by defining $g \equiv s$ (the south pole) on $f(\tilde{\mathcal{R}}) - D(0, \frac{L}{5})$. Then $\tilde{F} = g \circ f : \tilde{\mathcal{R}} \to S^2$ is smooth with derivative

$$|d\tilde{F}| \le |df||d\lambda||de| \le \frac{C}{L}$$

where C is a constant (independent of k).

Observe that diam $\tilde{\mathcal{R}} \le L$. Given $x, y \in \tilde{\mathcal{R}}$, we have $d(x, r) = d(x, r(t_1)) \le \frac{L}{4}$ and $d(y, r) = d(y, r(t_2)) \le \frac{L}{4}$ for some t_1, t_2 with $r(t_1), r(t_2) \in \tilde{\mathcal{R}}$, and

$$d(r(t_1), r(t_2)) \le D_1(r(t_1)) + D_1(r(t_2)) \le \frac{L}{2}.$$

Now,

$$d(x, y) \le d(x, r(t_1)) + d(r(t_1), r(t_2)) + d(r(t_2), y) \le L.$$

Hence no two points of $\tilde{\mathcal{R}}$ are identified in the quotient Σ. If two points were identified, then any minimal curve in $\tilde{\mathcal{R}}$ joining the two points would project to a nontrivial closed curve of length less than or equal to L, which is less than the systole, a contradiction.

Therefore, $\tilde{\mathcal{R}}$ projects one to one into Σ, and we may define $F : \mathcal{R} \to S^2$, where $\mathcal{R} = p(\tilde{\mathcal{R}})$, by $F(x) = \tilde{F}(p^{-1}(x))$. Since F maps $\partial\mathcal{R}$ to s, we can extend F from $\mathcal{R}$ to a map $F : \Sigma \to S^2$ with the desired properties, by defining $F \equiv s$ on $\Sigma - \mathcal{R}$. $\square$

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