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Global existence and convergence for a higher order flow in conformal geometry

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1. Introduction

An important problem in conformal geometry is the construction of conformal metrics for which a certain curvature quantity equals a prescribed function, e.g. a constant. In two dimensions, the uniformization theorem assures the existence of a conformal metric with constant Gauss curvature. Moreover, J. Moser [20] proved that for every positive function f on S^2 satisfying $f(x) = f(-x)$ for all $x \in S^2$ there exists a conformal metric on S^2 whose Gauss curvature is equal to f.

A natural conformal invariant in dimension four is

$$Q = -\frac{1}{6}\,(\Delta R - R^2 + 3\,|\mathrm{Ric}|^2),$$

where R denotes the scalar curvature and Ric the Ricci tensor. This formula can also be written in the form

$$Q = -\frac{1}{6}\,(\Delta R - 6\,\sigma_2(A)),$$

where

$$A = \mathrm{Ric} - \frac{1}{6}\,Rg$$

is the Schouten tensor of M and

$$\sigma_2(A) = \frac{1}{2}\,(\mathrm{tr}\,A)^2 - \frac{1}{2}\,|A|^2$$

is the second elementary symmetric polynomial in its eigenvalues. Under a conformal change of the metric

$$g = e^{2w} g_0,$$

the quantity Q transforms according to

$$Q = e^{-4w}(Q_0 + P_0 w),$$

where P_0 denotes the Paneitz operator with respect to g_0. The Gauss-Bonnet-Chern theorem asserts that

$$\int_M Q\,dV + \int_M \frac{1}{4}\,|W|^2\,dV = 8\pi^2\chi(M).$$

Since the Weyl tensor W is conformally invariant, it follows that the expression

$$\int_M Q\, dV$$

is conformally invariant, too. The quantity Q plays an important role in four-dimensional conformal geometry; see [2], [3], [5], [16] (note that our notation differs slightly from that in [2], [3]). Moreover, the Paneitz operator plays a similar role as the Laplace operator in dimension two; compare [2], [3], [5], [11], [12]. We also note that the Paneitz operator is of considerable interest in mathematical physics, see [10, SSIV.4].

T. Branson, S.-Y. A. Chang and P. Yang [2] studied metrics for which the curvature quantity Q is constant. Since

$$\int_M Q\, dV$$

is conformally invariant, these metrics minimize the functional

$$\int_M Q^2\, dV$$

among all conformal metrics with fixed volume. In addition, these metrics are critical points of the functional

$$E_1[w] = \int_M 2\, w\, P_0 w\, dV_0 + \int_M 4\, Q_0\, w\, dV_0 - \int_M Q_0\, dV_0\, \log\Big(\int_M e^{4w}\, dV_0\Big),$$

where g_0 denotes a fixed metric on M and $g = e^{2w} g_0$.

According to the results in [2], one can construct conformal metrics of constant Q-curvature by minimizing the functional $E_1[w]$ provided that the Paneitz operator is weakly positive and the integral of the Q-curvature on M is less than that on the standard sphere S^n. In dimension four, M. Gursky [17] proved that both conditions are satisfied if

$$Y(g_0) \geq 0$$

and

$$\int_M Q_0\, dV_0 \geq 0,$$

and M is not conformally equivalent to the standard sphere S^4.

C. Fefferman and R. Graham [14], [15] established the existence of a conformally invariant self-adjoint operator with leading term $(-\Delta)^{\frac{n}{2}}$ in all even dimensions n. Moreover, there is a curvature quantity which transforms according to

$$Q = e^{-nw}(Q_0 + P_0 w)$$

for

$$g = e^{2w} g_0.$$

This implies that the expression

$$\int_M Q\,dV$$

is conformally invariant. Hence, a metric with $Q =$ constant minimizes the functional

$$\int_M Q^2\,dV$$

among all conformal metrics with fixed volume. Finally, the analogue of the functional $E_1[w]$ is given by

$$E_1[w] = \int_M \frac{n}{2}\,w\,P_0 w\,dV_0 + \int_M n\,Q_0\,w\,dV_0 - \int_M Q_0\,dV_0\,\log\Big(\int_M e^{nw}\,dV_0\Big).$$

Our aim is to construct conformal metrics for which the curvature quantity Q is a constant multiple of a prescribed positive function f on M. This equation is the Euler-Lagrange equation for the functional

$$E_f[w] = \int_M \frac{n}{2}\,w\,P_0 w\,dV_0 + \int_M n\,Q_0\,w\,dV_0 - \int_M Q_0\,dV_0\,\log\Big(\int_M e^{nw}\,f\,dV_0\Big).$$

We construct critical points of the functional $E_f[w]$ using the gradient flow for $E_f[w]$. A similar method was used by R. Ye [25] to prove Yamabe's theorem for locally conformally flat manifolds. K. Ecker and G. Huisken [13] used a variant of mean curvature flow to construct hypersurfaces with prescribed mean curvature in cosmological spacetimes.

The flow of steepest descent for the functional $E_f[w]$ is given by

$$\frac{\partial}{\partial t} g = -\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)\,g.$$

Here, $\overline{Q}$ and $\overline{f}$ denote the mean values of Q and f respectively, i.e.

$$\int_M (Q - \overline{Q})\,dV = 0 \quad \text{and} \quad \int_M (f - \overline{f})\,dV = 0.$$

This evolution equation preserves the conformal structure of M. Moreover, since

$$\int_M \Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)\,dV = \int_M \Big(\overline{Q} - \frac{\overline{Q}\,\overline{f}}{\overline{f}}\Big)\,dV = 0,$$

the volume of M remains constant. From this it follows that $\overline{Q}$ is constant in time. If we write $g = e^{2w} g_0$ for a fixed metric g_0, then the evolution equation takes the form

$$\frac{\partial}{\partial t} w = -\frac{1}{2}\,e^{-nw}\,P_0 w - \frac{1}{2}\,e^{-nw}\,Q_0 + \frac{1}{2}\,\frac{\overline{Q}\,f}{\overline{f}},$$

where P_0 denotes the Paneitz operator with respect to g_0. Therefore, the function w satisfies a quasilinear parabolic equation of order n involving the critical Sobolev exponent. Moreover, the reaction term is nonlocal, since $\overline{f}$ involves values of w on the whole of M.

THEOREM 1.1. *Assume that the Paneitz operator P_0 is weakly positive with kernel consisting of the constant functions. Moreover, assume that*

$$\int_M Q_0 \, dV_0 < (n-1)! \, \omega_n.$$

Then the evolution equation

$$\frac{\partial}{\partial t} g = -\Big(Q - \frac{\overline{Q} \, f}{\overline{f}}\Big) \, g$$

has a solution which is defined for all times and converges to a metric with

$$\frac{Q}{f} = \frac{\overline{Q}}{\overline{f}}.$$

On the standard sphere S^n, we have

$$\int_M Q \, dV = (n-1)! \, \omega_n \, ;$$

hence Theorem 1.1 cannot be applied. In fact, the conclusion of Theorem 1.1 fails for $M = S^n$. To see this, one can consider the Kazdan-Warner identity

$$\int_{S^n} \langle \nabla_0 Q, \nabla_0 x_j \rangle \, e^{nw} \, dV_0 = 0 \, ;$$

see [3]. If f is an increasing function of x_j, then

$$\int_{S^n} \langle \nabla_0 Q, \nabla_0 x_j \rangle \, e^{nw} \, dV_0 > 0.$$

Consequently, there is no conformal metric on S^n satisfying

$$\frac{Q}{f} = \frac{\overline{Q}}{\overline{f}}.$$

Nevertheless, the conclusion of Theorem 1.1 holds if $f(x) = f(-x)$ and $w(x) = w(-x)$ for all $x \in S^n$. This is a generalization of Moser's theorem [20].

THEOREM 1.2. *Suppose that $M = \mathbf{RP}^n$. Then the evolution equation*

$$\frac{\partial}{\partial t} g = -\Big(Q - \frac{\overline{Q} \, f}{\overline{f}}\Big) \, g$$

has a solution which is defined for all times and converges to a metric with

$$\frac{Q}{f} = \frac{\overline{Q}}{\overline{f}}.$$

Combining Theorem 1.2 with M. Gursky's result [17] gives

THEOREM 1.3. *Suppose that M is a compact manifold of dimension four satisfying*

$$Y(g_0) \geq 0 \quad \textit{and} \quad \int_M Q_0 \, dV_0 \geq 0.$$

Moreover, assume that M is not conformally equivalent to the standard sphere S^4. Then the evolution equation

$$\frac{\partial}{\partial t} g = -\Big(Q - \frac{\overline{Q}\, f}{\overline{f}} \Big)\, g$$

has a solution which is defined for all times and converges to a metric with

$$\frac{Q}{f} = \frac{\overline{Q}}{\overline{f}}.$$

Finally, we prove a compactness theorem for conformal metrics on S^n. In two dimensions, the corresponding result was first established by X. Chen [6] (see also [24]).

PROPOSITION 1.4. *Let $g_k = e^{2w_k}\, g_0$ be a sequence of conformal metrics on S^n with fixed volume such that*

$$\int_{S^n} Q_k^2 \, dV_k \leq C.$$

Assume that for every point $x \in S^n$ there exists $r > 0$ such that

$$\lim_{r \to 0} \lim_{k \to \infty} \int_{B_r(x)} |Q_k| \, dV_k < \frac{1}{2}\,(n-1)!\,\omega_n.$$

Then the sequence w_k is uniformly bounded in H^n.

The evolution equation can be viewed as a generalization of the Ricci flow on compact surfaces. In dimension four, the quantity Q plays a similar role as the Gauss curvature in dimension two. Moreover, the energy functional $E_1[w]$ corresponds to the Liouville energy studied by B. Osgood, R. Phillips and P. Sarnak in [21].

It was shown by R. Hamilton [18] and B. Chow [8] that every solution of the Ricci flow on a compact surface exists for all time and converges exponentially to a metric with constant Gauss curvature. A different approach was introduced by X. Chen [6] in his work on the Calabi flow. Similar methods were used by M. Struwe [24] to establish global existence and exponential convergence for the Ricci flow on compact surfaces, and by X. Chen and G. Tian [7] to prove convergence of the Kähler-Ricci flow on Kähler-Einstein surfaces. For the Ricci flow, the situation is more complicated since the Calabi energy is not decreasing along the flow. H. Schwetlick [23] used similar arguments to deduce global existence and convergence for a natural sixth order flow on surfaces. The approach used in [6] and [24] is based on integral estimates and does not rely on the maximum principle. These ideas are also useful in our situation. This is due to the fact that the equation studied in this paper has higher order, hence the maximum principle is not available.

In Section 2 we derive the evolution equation for the conformal factor and the curvature quantity Q. In Section 3 we show that the solution is bounded in $H^{\frac{n}{2}}$. In Sections 4 and 5 we show that the solution exists for all time, and in Section 6 we prove that the evolution equation converges to a stationary solution. Finally, the proof of Proposition 1.4 is carried out in Section 8.

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2. The evolution equations for w and $Q - \frac{\overline{Q}\,f}{\overline{f}}$

Since the evolution equation preserves the conformal structure, we may write $g = e^{2w}\, g_0$ for a fixed metric g_0 and some real-valued function w. Then we have the formula

$$Q = e^{-nw}\,(Q_0 + P_0 w),$$

where P_0 denotes the Paneitz operator with respect to the metric g_0. Hence, the function w obeys the evolution equation

$$\frac{\partial}{\partial t} w = -\frac{1}{2}\, e^{-nw}\, P_0 w - \frac{1}{2}\, e^{-nw}\, Q_0 + \frac{1}{2}\, \frac{\overline{Q}\, f}{\overline{f}}.$$

Differentiating both sides with respect to t, we obtain

$$\frac{\partial}{\partial t}\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big) = -\frac{1}{2}\, P\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big) + \frac{n}{2}\, Q\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big) + \frac{\overline{Q}\,f}{\overline{f}^{\,2}}\, \frac{d}{dt}\overline{f},$$

where $P = e^{-nw}\, P_0$ is the Paneitz operator with respect to the metric g. It follows from the evolution equation for w that

$$\frac{d}{dt}\overline{f} = -\int_M \frac{n}{2}\, f\,\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)\, dV.$$

This implies

$$\begin{aligned} \frac{\partial}{\partial t}\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big) \quad &= \quad -\frac{1}{2}\, P\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big) + \frac{n}{2}\, Q\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big) \\ &\quad\; -\frac{n}{2}\, \frac{\overline{Q}\,f}{\overline{f}} \int_M \frac{f}{\overline{f}}\,\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)\, dV, \end{aligned}$$

where P denotes the Paneitz operator with respect to the metric g.

3. Boundedness of w in $H^{\frac{n}{2}}$

We consider the functional

$$E_f[w] = \int_M \frac{n}{2}\, w\, P_0 w\, dV_0 + \int_M n\, Q_0\, w\, dV_0 - \int_M Q_0\, dV_0\, \log\Big(\int_M f\, e^{nw}\, dV_0\Big).$$

Since P_0 is self-adjoint,

$$\begin{aligned}
\frac{d}{dt}E_f[w] &= \int_M \frac{n}{2}\, \frac{\partial}{\partial t}w\, P_0 w\, dV_0 + \int_M \frac{n}{2}\, w\, P_0 \frac{\partial}{\partial t}w\, dV_0 + \int_M n\, Q_0\, \frac{\partial}{\partial t}w\, dV_0 \\
&\quad - \int_M n\, \frac{\overline{Q}\, f}{\overline{f}}\, \frac{\partial}{\partial t}w\, dV \\
&= \int_M n\, P_0 w\, \frac{\partial}{\partial t}w\, dV_0 + \int_M n\, Q_0\, \frac{\partial}{\partial t}w\, dV_0 - \int_M n\, \frac{\overline{Q}\, f}{\overline{f}}\, \frac{\partial}{\partial t}w\, dV \\
&= \int_M n\, Q\, \frac{\partial}{\partial t}w\, dV - \int_M n\, \frac{\overline{Q}\, f}{\overline{f}}\, \frac{\partial}{\partial t}w\, dV \\
&= \int_M n\, \Big(Q - \frac{\overline{Q}\, f}{\overline{f}}\Big)\, \frac{\partial}{\partial t}w\, dV.
\end{aligned}$$

Since the time derivative of w is given by

$$\frac{\partial}{\partial t}w = -\frac{1}{2}\Big(Q - \frac{\overline{Q}\, f}{\overline{f}}\Big),$$

we obtain

$$\frac{d}{dt}E_f[w] = -\int_M \frac{n}{2}\Big(Q - \frac{\overline{Q}\, f}{\overline{f}}\Big)^2\, dV.$$

In particular, the functional $E_f[w]$ is decreasing under the evolution equation. This implies

$$E_f[w] \leq C.$$

In the first step, we consider the case

$$\int_M Q_0\, dV_0 < 0.$$

Using Jensen's inequality we obtain

$$\log\Big(\int_M e^{n(w-\overline{w})}\, dV_0\Big) \geq -C.$$

This implies

$$\begin{aligned}
E_f[w] &\geq \int_M \frac{n}{2}\, w\, P_0 w\, dV_0 + \int_M n\, Q_0\, w\, dV_0 \\
&\quad - \int_M Q_0\, dV_0\, \log\Big(\int_M e^{nw}\, dV_0\Big) - C
\end{aligned}$$

$$\begin{aligned}
&= \int_M \frac{n}{2}\, w\, P_0 w\, dV_0 + \int_M n\, Q_0\, (w - \overline{w})\, dV_0 \\
&\quad - \int_M Q_0\, dV_0\, \log\Big(\int_M e^{n(w-\overline{w})}\, dV_0\Big) - C \\
&\geq \int_M \frac{n}{2}\, w\, P_0 w\, dV_0 + \int_M n\, Q_0\, (w - \overline{w})\, dV_0 - C \\
&\geq 2\delta \int_M \Big((-\Delta_0)^{\frac{n}{4}} w\Big)^2\, dV_0 + \int_M n\, Q_0\, (w - \overline{w})\, dV_0 - C \\
&\geq \delta \int_M \Big((-\Delta_0)^{\frac{n}{4}} w\Big)^2\, dV_0 - C.
\end{aligned}$$

In the second step, we consider the case

$$0 \leq \int_M Q_0\, dV_0 < (n-1)!\, \omega_n.$$

Since the Paneitz operator P_0 is self-adjoint and weakly positive, it has a square root $P_0^{\frac{1}{2}}$. Moreover, the kernel of $P_0^{\frac{1}{2}}$ coincides with the kernel of P_0, which consists of the constant functions. Thus, we conclude that

$$w(y) - \overline{w} = \int_M P_0^{\frac{1}{2}} w(z)\, H(y,z)\, dV_0(z)$$

for a suitable function $H(y,z)$. The leading term in the asymptotic expansion of the kernel $H(y,z)$ coincides with that of the Green's function for the operator $(-\Delta)^{\frac{n}{4}}$ in $\mathbf{R}^n$. Hence, we can apply an inequality of D. Adams (see [1, Theorems 1 and 2]). This implies

$$\int_M e^{\frac{2^n \pi^n n}{\omega_{n-1}} \frac{(w-\overline{w})^2}{\int_M (P_0^{\frac{1}{2}} w)^2\, dV_0}}\, dV_0 \leq C,$$

hence

$$\int_M e^{\frac{2^n \pi^n n}{\omega_{n-1}} \frac{(w-\overline{w})^2}{\int_M w\, P_0 w\, dV_0}}\, dV_0 \leq C.$$

Since

$$\omega_{n-1}\omega_n = \frac{2^{n+1}\pi^n}{(n-1)!},$$

we obtain

$$\int_M e^{n(w-\overline{w})}\, dV_0 \leq C\, e^{\int_M \frac{n}{2(n-1)!\,\omega_n}\, w\, P_0 w\, dV_0}.$$

From this it follows that

$$\begin{aligned}
E_f[w] &\geq \int_M \frac{n}{2}\, w\, P_0 w\, dV_0 + \int_M n\, Q_0\, w\, dV_0 \\
&\quad - \int_M Q_0\, dV_0\, \log\Big(\int_M e^{nw}\, dV_0\Big) - C
\end{aligned}$$

$$\begin{aligned}
&= \int_M \frac{n}{2}\, w\, P_0 w\, dV_0 + \int_M n\, Q_0\, (w - \overline{w})\, dV_0 \\
&\quad - \int_M Q_0\, dV_0\, \log\Big(\int_M e^{n(w-\overline{w})}\, dV_0\Big) - C \\
&\geq \Big(1 - \frac{\int_M Q_0\, dV_0}{(n-1)!\,\omega_n}\Big) \int_M \frac{n}{2}\, w\, P_0 w\, dV_0 + \int_M n\, Q_0\, (w - \overline{w})\, dV_0 - C \\
&\geq 2\delta \int_M \Big((-\Delta_0)^{\frac{n}{4}} w\Big)^2 dV_0 + \int_M 4\, Q_0\, (w - \overline{w})\, dV_0 - C \\
&\geq \delta \int_M \Big((-\Delta_0)^{\frac{n}{4}} w\Big)^2 dV_0 - C.
\end{aligned}$$

Since $E_f[w]$ is bounded from above, we conclude that

$$\int_M \Big((-\Delta_0)^{\frac{n}{4}} w\Big)^2 dV_0 \leq C;$$

hence

$$\|w - \overline{w}\|_{H^{\frac{n}{2}}} \leq C.$$

Using an inequality of N. Trudinger, we obtain

$$\int_M e^{\alpha(w-\overline{w})}\, dV_0 \leq C$$

for all real numbers α. In particular, we have

$$\int_M e^{n(w-\overline{w})}\, dV_0 \leq C.$$

Since $\int_M e^{nw}\, dV_0 = 1$, we conclude that $e^{-n\overline{w}} \leq C$; hence $-C \leq \overline{w} \leq C$. This implies $\|w\|_{H^{\frac{n}{2}}} \leq C$ and $\int_M e^{\alpha w}\, dV_0 \leq C$ for all real numbers α. Since the functional $E_f[w]$ is bounded from below, we finally obtain

$$\int_0^T \int_M \Big(Q - \frac{\overline{Q}\, f}{\overline{f}}\Big)^2 dV\, dt \leq C.$$

4. Boundedness of w in H^n for $0 \leq t \leq T$

Let T be a fixed, positive real number. We claim that

$$\|w\|_{H^n} \leq C$$

for all $0 \leq t \leq T$. For the sake of brevity, we put

$$\begin{aligned}
v &= -\frac{1}{2}\, e^{\frac{nw}{2}} \Big(Q - \frac{\overline{Q}\, f}{\overline{f}}\Big) = e^{\frac{nw}{2}}\, \frac{\partial}{\partial t} w \\
&= -\frac{1}{2}\, e^{-\frac{nw}{2}}\, P_0 w - \frac{1}{2}\, e^{-\frac{nw}{2}}\, Q_0 + \frac{1}{2}\, e^{\frac{nw}{2}}\, \frac{\overline{Q}\, f}{\overline{f}}.
\end{aligned}$$

This implies

$$\frac{\partial}{\partial t} w = e^{-\frac{nw}{2}} v \quad \text{and} \quad P_0 w = -2\, e^{\frac{nw}{2}} v - Q_0 + e^{nw} \frac{\overline{Q}\, f}{\overline{f}}.$$

From this it follows that

$$\begin{aligned} \frac{d}{dt}\Big(\int_M (P_0 w)^2 \, dV_0 \Big) &= -\int_M 4\, (e^{\frac{nw}{2}} v)\, P_0(e^{-\frac{nw}{2}} v)\, dV_0 \\ &\quad - \int_M 2\, Q_0\, P_0(e^{-\frac{nw}{2}} v) \\ &\quad + \int_M \frac{2\,\overline{Q}}{\overline{f}}\, (e^{nw} f)\, P_0(e^{-\frac{nw}{2}} v)\, dV_0. \end{aligned}$$

This implies

$$\begin{aligned} \frac{d}{dt}\Big(\int_M (P_0 w)^2 \, dV_0 \Big) &= -\int_M 4\, (-\Delta_0)^{\frac{n}{4}} (e^{\frac{nw}{2}} v)\, (-\Delta_0)^{\frac{n}{4}} (e^{-\frac{nw}{2}} v)\, dV_0 \\ &\quad - \int_M 2\, P_0 Q_0\, (e^{-\frac{nw}{2}} v) \\ &\quad + \int_M \frac{2\,\overline{Q}}{\overline{f}}\, (-\Delta_0)^{\frac{n}{4}} (e^{nw} f)\, (-\Delta_0)^{\frac{n}{4}} (e^{-\frac{nw}{2}} v)\, dV_0 \\ &\quad + \text{lower order terms.} \end{aligned}$$

Here, we adopt the convention that

$$(-\Delta_0)^{m+\frac{1}{2}} = \nabla_0\, (-\Delta_0)^m$$

for all integers m (see [1]). The right-hand side involves derivatives of v and w of order at most $\frac{n}{2}$. Moreover, the total number of derivatives is at most n. Therefore, we obtain

$$\begin{aligned} \frac{d}{dt}\Big(\int_M (P_0 w)^2 \, dV_0 \Big) &= -\int_M 4\, \Big((-\Delta_0)^{\frac{n}{4}} v \Big)^2 \, dV_0 \\ &\quad + C \sum_{k_1,\ldots,k_m} \int_M |\nabla_0^{k_1} v| \cdot |\nabla_0^{k_2} v| \cdot |\nabla_0^{k_3} w| \cdots |\nabla_0^{k_m} w| \, dV_0 \\ &\quad + C \sum_{l_1,\ldots,l_m} \int_M |\nabla_0^{l_1} v| \cdot |\nabla_0^{l_2} w| \cdots |\nabla_0^{l_m} w|\, e^{\alpha w} \, dV_0. \end{aligned}$$

The first sum is taken over all m-tuples $k_1, \ldots, k_m$ with $m \geq 3$ satisfying the conditions

$$\begin{aligned} 0 \leq k_i \leq \frac{n}{2} \qquad &\text{for } 1 \leq i \leq 2, \\ 1 \leq k_i \leq \frac{n}{2} \qquad &\text{for } 3 \leq i \leq m, \end{aligned}$$

and

$$k_1 + \cdots + k_m \leq n.$$

To estimate this term, we choose real numbers $p_1, \dots, p_m \in [2, \infty[$ such that

$$k_i \le \frac{n}{p_i} \qquad \text{for} \quad 1 \le i \le 2,$$
$$\frac{n}{p_i} < k_i \qquad \text{for} \quad 3 \le i \le m$$

and

$$\frac{1}{p_1} + \dots + \frac{1}{p_m} = 1.$$

Moreover, we define real numbers $\theta_1, \dots, \theta_m$ by

$$\theta_i = \frac{k_i - \frac{n}{p_i} + \frac{n}{2}}{\frac{n}{2}} \in [0,1] \quad \text{for } 1 \le i \le 2$$

and

$$\theta_i = \frac{k_i - \frac{n}{p_i}}{\frac{n}{2}} \in]0,1[\quad \text{for} \quad 3 \le i \le m.$$

Then we have $\theta_1 + \dots + \theta_m \le 2$; hence

$$\theta_3 + \dots + \theta_m \le (1 - \theta_1) + (1 - \theta_2).$$

Since w is bounded in $H^{\frac{n}{2}}$, this implies

$$\begin{aligned}
& -\int_M 2\left((-\Delta_0)^{\frac{n}{4}} v\right)^2 dV_0 \\
& +C \sum_{k_1,\dots,k_m} \int_M |\nabla_0^{k_1} v| \cdot |\nabla_0^{k_2} v| \cdot |\nabla_0^{k_3} w| \cdots |\nabla_0^{k_m} w| \, dV_0 \\
\le \quad & -\|v\|_{H^{\frac{n}{2}}}^2 + C \sum_{k_1,\dots,k_m} \|\nabla_0^{k_1} v\|_{L^{p_1}} \cdot \|\nabla_0^{k_2} v\|_{L^{p_2}} \cdot \|\nabla_0^{k_3} w\|_{L^{p_3}} \cdots \|\nabla_0^{k_m} w\|_{L^{p_m}} \\
\le \quad & -\|v\|_{H^{\frac{n}{2}}}^2 + C \sum_{k_1,\dots,k_m} \|v\|_{H^{k_1 - \frac{n}{p_1} + \frac{n}{2}}} \\
& \cdot \|v\|_{H^{k_2 - \frac{n}{p_2} + \frac{n}{2}}} \cdot \|w\|_{H^{k_3 - \frac{n}{p_3} + \frac{n}{2}}} \cdots \|w\|_{H^{k_m - \frac{n}{p_m} + \frac{n}{2}}} \\
\le \quad & -\|v\|_{H^{\frac{n}{2}}}^2 + C \sum_{k_1,\dots,k_m} \|v\|_{L^2}^{(1-\theta_1)+(1-\theta_2)} \\
& \cdot \|v\|_{H^{\frac{n}{2}}}^{\theta_1+\theta_2} \|w\|_{H^{\frac{n}{2}}}^{(1-\theta_3)+\dots+(1-\theta_m)} \|w\|_{H^n}^{\theta_3+\dots+\theta_m} \\
\le \quad & -\|v\|_{H^{\frac{n}{2}}}^2 + C \sum_{k_1,\dots,k_m} \|v\|_{L^2}^{(1-\theta_1)+(1-\theta_2)} \|v\|_{H^{\frac{n}{2}}}^{\theta_1+\theta_2} \|w\|_{H^n}^{\theta_3+\dots+\theta_m} \\
\le \quad & C \sum_{k_1,\dots,k_m} \|v\|_{L^2}^2 \|w\|_{H^n}^{\frac{2(\theta_3+\dots+\theta_m)}{(1-\theta_1)+(1-\theta_2)}} \\
\le \quad & C \, \|v\|_{L^2}^2 \, (\|w\|_{H^n}^2 + 1).
\end{aligned}$$

The second sum is taken over all m-tuples $l_1, \dots, l_m$ with $m \geq 1$ satisfying the conditions

$$\begin{aligned} &0 \leq l_1 \leq \frac{n}{2}, \\ &1 \leq l_i \leq \frac{n}{2} \qquad \text{for } 2 \leq i \leq m \end{aligned}$$

and

$$l_1 + \cdots + l_m \leq n.$$

To estimate this term, we choose real numbers $q_1, \dots, q_m \in [2, \infty[$ such that

$$l_1 \leq \frac{n}{q_1}, \qquad \frac{n}{q_i} < l_i \quad \text{for} \quad 2 \leq i \leq m$$

and

$$\frac{1}{2} \leq \frac{1}{q_1} + \cdots + \frac{1}{q_m} < 1.$$

Moreover, we define real numbers $\rho_2, \dots, \rho_m$ by

$$\rho_1 = \frac{l_1 - \frac{n}{q_1} + \frac{n}{2}}{\frac{n}{2}} \in [0,1]$$

and

$$\rho_i = \frac{l_i - \frac{n}{q_i}}{\frac{n}{2}} \in]0,1[\quad \text{ for } 2 \leq i \leq m.$$

Then we have $\rho_1 + \cdots + \rho_m \leq 2$; hence $\rho_2 + \cdots + \rho_m \leq 2 - \rho_1$. Since w is bounded in $H^{\frac{n}{2}}$, this implies

$$\begin{aligned}
& -\int_M 2\left((-\Delta_0)^{\frac{n}{4}} v\right)^2 dV_0 + C \sum_{l_1,\dots,l_m} \int_M |\nabla_0^{l_1} v| \cdot |\nabla_0^{l_2} w| \cdots |\nabla_0^{l_m} w| \, e^{\alpha w} \, dV_0 \\
\leq \; & -\|v\|_{H^{\frac{n}{2}}}^2 + C \sum_{l_1,\dots,l_m} \|\nabla_0^{l_1} v\|_{L^{q_1}} \cdot \|\nabla_0^{l_2} w\|_{L^{q_2}} \cdots \|\nabla_0^{l_m} w\|_{L^{q_m}} \\
\leq \; & -\|v\|_{H^{\frac{n}{2}}}^2 + C \sum_{l_1,\dots,l_m} \|v\|_{H^{l_1 - \frac{n}{q_1} + \frac{n}{2}}} \cdot \|w\|_{H^{l_2 - \frac{n}{q_2} + \frac{n}{2}}} \cdots \|w\|_{H^{l_m - \frac{n}{q_m} + \frac{n}{2}}} \\
\leq \; & -\|v\|_{H^{\frac{n}{2}}}^2 + C \sum_{l_1,\dots,l_m} \|v\|_{L^2}^{1-\rho_1} \, \|v\|_{H^{\frac{n}{2}}}^{\rho_1} \, \|w\|_{H^{\frac{n}{2}}}^{(1-\rho_2)+\cdots+(1-\rho_m)} \, \|w\|_{H^n}^{\rho_2+\cdots+\rho_m} \\
\leq \; & -\|v\|_{H^{\frac{n}{2}}}^2 + C \sum_{l_1,\dots,l_m} \|v\|_{L^2}^{1-\rho_1} \, \|v\|_{H^{\frac{n}{2}}}^{\rho_1} \, \|w\|_{H^n}^{\rho_2+\cdots+\rho_m} \\
\leq \; & C \sum_{l_1,\dots,l_m} \|v\|_{L^2}^{\frac{2-2\rho_1}{2-\rho_1}} \, \|w\|_{H^n}^{\frac{2(\rho_2+\cdots+\rho_m)}{2-\rho_1}} \\
\leq \; & C \, (\|v\|_{L^2}^2 + 1) \, (\|w\|_{H^n}^2 + 1).
\end{aligned}$$

Thus, we conclude that

$$\frac{d}{dt}\left(\int_M (P_0 w)^2 \, dV_0 \right) \leq C \, (\|v\|_{L^2}^2 + 1) \, (\|w\|_{H^n}^2 + 1).$$

From the positivity of P_0 it follows that

$$\|w\|_{H^n}^2 \leq C \int_M (P_0 w)^2 \, dV_0.$$

Moreover, the function v satisfies

$$\|v\|_{L^2}^2 = \int_M \frac{1}{4} \, e^{nw} \Big(Q - \frac{\overline{Q} \, f}{\overline{f}} \Big)^2 \, dV_0 = \int_M \frac{1}{4} \Big(Q - \frac{\overline{Q} \, f}{\overline{f}} \Big)^2 \, dV.$$

Therefore, we obtain

$$\frac{d}{dt} \bigg(\int_M (P_0 w)^2 \, dV_0 + 1 \bigg) \leq C \bigg(\int_M (P_0 w)^2 \, dV_0 + 1 \bigg) \bigg(\int_M \Big(Q - \frac{\overline{Q} \, f}{\overline{f}} \Big)^2 \, dV + 1 \bigg).$$

Since

$$\int_0^T \int_M \Big(Q - \frac{\overline{Q} \, f}{\overline{f}} \Big)^2 \, dV \, dt \leq C,$$

we deduce that

$$\int_M (P_0 w)^2 \, dV_0 \leq C \quad \text{for all} \quad 0 \leq t \leq T.$$

This implies

$$\|w\|_{H^n} \leq C$$

for all $0 \leq t \leq T$. Using the Sobolev inequality, we obtain

$$|w| \leq C \quad \text{for all} \quad 0 \leq t \leq T.$$

5. Boundedness of w in H^{2k} for $0 \leq t \leq T$

We now establish bounds for the higher order derivatives:

$$\begin{aligned} \frac{d}{dt} \bigg(\int_M |(-\Delta_0)^k w|^2 \, dV_0 \bigg) \quad &\leq \quad - \int_M e^{-nw} \, |(-\Delta_0)^{k+\frac{n}{4}} w|^2 \, dV_0 \\ &\quad + C \sum_{k_1, \ldots, k_m} \int_M |\nabla_0^{k_1} w| \cdots |\nabla_0^{k_m} w| \, dV_0 \, ; \end{aligned}$$

hence

$$\begin{aligned} \frac{d}{dt} \bigg(\int_M |(-\Delta_0)^k w|^2 \, dV_0 \bigg) \quad &\leq \quad -\frac{1}{C} \int_M |(-\Delta_0)^{k+\frac{n}{4}} w|^2 \, dV_0 \\ &\quad + C \sum_{k_1, \ldots, k_m} \int_M |\nabla_0^{k_1} w| \cdots |\nabla_0^{k_m} w| \, dV_0. \end{aligned}$$

Here, the sum is taken over all m-tuples $k_1, \ldots, k_m$, with $m \geq 3$, which satisfy the conditions

$$1 \leq k_i \leq 2k + \frac{n}{2} \quad \text{and} \quad k_1 + \cdots + k_m \leq 4k + n.$$

We now choose real numbers $p_1, \dots, p_m \in [2, \infty[$ such that

$$k_i \leq 2k + \frac{n}{p_i} \quad \text{and} \quad \frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1.$$

Moreover, we define real numbers $\theta_1, \dots, \theta_m$ by

$$\theta_i = \max\left\{\frac{k_i - \frac{n}{p_i} - \frac{n}{2}}{2k - \frac{n}{2}}, 0\right\}.$$

Since $m \geq 3$, we can choose $p_1, \dots, p_m \in [2, \infty[$ such that

$$\theta_1 + \cdots + \theta_m < 2.$$

From this it follows that

$$\begin{aligned}
\frac{d}{dt}\|w\|_{H^{2k}}^2 &\leq -\frac{1}{C}\,\|w\|_{H^{2k+\frac{n}{2}}}^2 + C \sum_{k_1,\dots,k_m} \|\nabla_0^{k_1} w\|_{L^{p_1}} \cdots \|\nabla_0^{k_m} w\|_{L^{p_m}} \\
&\leq -\frac{1}{C}\,\|w\|_{H^{2k+\frac{n}{2}}}^2 + C \sum_{k_1,\dots,k_m} \|w\|_{H^{k_1 - \frac{n}{p_1} + \frac{n}{2}}} \cdots \|w\|_{H^{k_m - \frac{n}{p_m} + \frac{n}{2}}} \\
&\leq -\frac{1}{C}\,\|w\|_{H^{2k+\frac{n}{2}}}^2 + C \sum_{k_1,\dots,k_m} \|w\|_{H^n}^{(1-\theta_1)+\cdots+(1-\theta_m)} \|w\|_{H^{2k+\frac{n}{2}}}^{\theta_1+\cdots+\theta_m} \\
&\leq -\frac{1}{C}\,\|w\|_{H^{2k+\frac{n}{2}}}^2 + C \sum_{k_1,\dots,k_m} \|w\|_{H^{2k+\frac{n}{2}}}^{\theta_1+\cdots+\theta_m} \\
&\leq -\frac{1}{C}\,\|w\|_{H^{2k+\frac{n}{2}}}^2 + C \\
&\leq -\frac{1}{C}\,\|w\|_{H^{2k}}^2 + C
\end{aligned}$$

for all $0 \leq t \leq T$. Thus, we conclude that

$$\|w\|_{H^{2k}} \leq C \quad \text{for all} \quad 0 \leq t \leq T.$$

Therefore, the evolution equation has a solution which is defined for all time.

6. Convergence

For the sake of brevity, we put

$$y(t) = \int_M \left(Q - \frac{\overline{Q}\,f}{\overline{f}}\right)^2 dV$$

and we show that

$$y(t) \to 0 \quad \text{for} \quad t \to \infty.$$

Let ε be an arbitrary positive number. We choose $t_0 \geq 0$ such that $y(t_0) \leq \varepsilon$. We claim that $y(t) \leq 3\varepsilon$ for all $t \geq t_0$. Otherwise, we define

$$t_1 = \inf\{t \geq t_0 : y(t) \geq 3\varepsilon\}.$$

This implies $y(t) \leq 3\varepsilon$ for all $t_0 \leq t \leq t_1$. From this it follows that

$$\int_M e^{-nw}\,(Q_0 + P_0 w)^2\,dV_0 \leq C$$

for all $t_0 \leq t \leq t_1$. Moreover, it follows from results in Section 3 that

$$\int_M e^{3nw}\,dV_0 \leq C \quad \text{for all} \quad t_0 \leq t \leq t_1.$$

Using Hölder's inequality, we obtain

$$\int_M |Q_0 + P_0 w|^{\frac{3}{2}}\,dV_0 \leq \Big(\int_M e^{-nw}\,(Q_0 + P_0 w)^2\,dV_0\Big)^{\frac{3}{4}}\,\Big(\int_M e^{3nw}\,dV_0\Big)^{\frac{1}{4}}.$$

From this it follows that

$$\int_M |P_0 w|^{\frac{3}{2}}\,dV_0 \leq C \quad \text{for all} \quad t_0 \leq t \leq t_1.$$

Using the Sobolev inequality, we obtain

$$|w| \leq C \quad \text{for all} \quad t_0 \leq t \leq t_1.$$

We have shown in Section 2 that the function $Q - \frac{\overline{Q}\,f}{\overline{f}}$ satisfies the evolution equation

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big) &= -\frac{1}{2}\,P\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big) + \frac{n}{2}\,Q\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big) \\
&\quad -\frac{n}{2}\,\frac{\overline{Q}\,f}{\overline{f}}\int_M \frac{f}{\overline{f}}\,\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)\,dV,
\end{aligned}$$

where P denotes the Paneitz operator with respect to the metric g. From this it follows that

$$\begin{aligned}
\frac{d}{dt}\Big(\int_M \Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)^2\,dV\Big) &= -\int_M \Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)\,P\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)\,dV \\
&\quad + \int_M \frac{n}{2}\,\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)^3\,dV. \\
&\quad + \int_M n\,\frac{\overline{Q}\,f}{\overline{f}}\,\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)^2\,dV \\
&\quad - n\,\overline{Q}\,\Big(\int_M \frac{f}{\overline{f}}\,\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)\,dV\Big)^2.
\end{aligned}$$

Using the Gagliardo-Nirenberg inequality, we can bound

$$\Big\|Q - \frac{\overline{Q}\,f}{\overline{f}}\Big\|_{L^3} \leq C\,\Big\|Q - \frac{\overline{Q}\,f}{\overline{f}}\Big\|_{L^2}^{\frac{2}{3}}\,\Big\|Q - \frac{\overline{Q}\,f}{\overline{f}}\Big\|_{H^{\frac{n}{2}}}^{\frac{1}{3}},$$

where the norms are taken with respect to the background metric g_0. This implies

$$\int_M \Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)^3\,dV_0 \leq C\,\Big(\int_M \Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)^2\,dV_0\Big)\,\Big(\int_M \Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)\,P_0\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)\,dV_0\Big)^{\frac{1}{2}}.$$

Since w is uniformly bounded for $t_0 \leq t \leq t_1$, we obtain

$$\int_M \Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)^3\,dV \leq C\,\Big(\int_M \Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)^2\,dV\Big)\,\Big(\int_M \Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)\,P\Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)\,dV\Big)^{\frac{1}{2}}.$$

Thus, we conclude that

$$\begin{aligned}\frac{d}{dt}\Big(\int_M \Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)^2\,dV\Big) \quad &\leq \quad C\,\Big(\int_M \Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)^2\,dV\Big)^2 \\ &\quad +C\,\Big(\int_M \Big(Q - \frac{\overline{Q}\,f}{\overline{f}}\Big)^2\,dV\Big);\end{aligned}$$

hence

$$\frac{d}{dt}y(t) \leq C\,y(t)^2 + C\,y(t).$$

Therefore, we obtain

$$2\varepsilon \leq y(t_1) - y(t_0) \leq C\int_{t_0}^{t_1} y(t)\,dt.$$

If we choose t_0 large enough, then we have

$$C\int_{t_0}^{\infty} y(t)\,dt \leq \varepsilon.$$

Hence, we obtain $2\varepsilon \leq \varepsilon$ which is a contradiction. Thus, we conclude that

$$y(t) \to 0 \quad \text{ for } \quad t \to \infty.$$

From this it follows that

$$|w| \leq C \quad \text{for all} \quad t \geq 0.$$

Moreover, we have

$$\int_M e^{-nw}\,(Q_0 + P_0 w)^2\,dV_0 \leq C$$

for all $t \geq 0$. From this it follows that

$$\int_M (Q_0 + P_0 w)^2\,dV_0 \leq C;$$

hence

$$\|w\|_{H^n} \leq C$$

for all $t \geq 0$. Arguing as above, we obtain

$$\|w\|_{H^k} \leq C \quad \text{for all} \quad t \geq 0.$$

It remains to show that the flow converges to a metric satisfying

$$\frac{Q}{f} = \frac{\overline{Q}}{\overline{f}}.$$

The evolution equation

$$\frac{\partial}{\partial t} g = -\Big(Q - \frac{\overline{Q}\, f}{\overline{f}}\Big)\, g$$

is the gradient flow for the functional

$$E_f[w] = \int_M \frac{n}{2}\, w\, P_0 w\, dV_0 + \int_M n\, Q_0\, w\, dV_0 - \int_M Q_0\, dV_0\, \log\Big(\int_M e^{nw}\, f\, dV_0\Big).$$

Since the functional $E_f[w]$ is real analytic, the assertion follows from a general result of L. Simon [22].

7. The case $M = \mathbf{RP}^n$

In this section, we consider the special case $M = \mathbf{RP}^n$. We normalize the metric such that the volume of M is equal to $\frac{1}{2}\,\omega_n$ and the mean value of the function Q is equal to $(n-1)!$. By Theorem 1.1, the flow converges to a limit metric g satisfying

$$\frac{Q}{f} = \frac{(n-1)!}{\overline{f}}.$$

In particular, for every positive function f on $\mathbf{RP}^n$, there exists a metric g on $\mathbf{RP}^n$ such that

$$\frac{Q}{f} = \frac{(n-1)!}{\overline{f}}.$$

We now consider the case $f = 1$. In this case, the limit metric g satisfies $Q = (n-1)!$. It follows from a result of S.-Y. A. Chang and P. Yang [4] (see also C. S. Lin's paper [19]) that the limit metric is the standard metric on $\mathbf{RP}^n$.

We claim that the flow converges exponentially. To show this, we denote by g_0 the standard metric on $\mathbf{RP}^n$. Then the conformal factor satisfies the evolution equation

$$\frac{\partial}{\partial t} w = -\frac{1}{2}\, e^{-nw}\, P_0 w + \frac{1}{2}\,(n-1)!\,(1 - e^{-nw}).$$

Linearizing this equation, we obtain

$$\frac{\partial}{\partial t} w = -\frac{1}{2}\, P_0 w + \frac{1}{2}\, n!\, w.$$

The Paneitz operator on $\mathbf{RP}^n$ is given by

$$P_0 = \prod_{k=1}^{\frac{n}{2}} (-\Delta_0 + (k-1)(n-k)).$$

The first eigenvalue of the Laplace operator $-\Delta_0$ on $\mathbf{RP}^n$ is strictly greater than n. Hence, the first eigenvalue of the Paneitz operator P_0 is strictly greater than $n!$. Therefore, the first eigenvalue of the linearized operator is strictly less than 0. Thus, we conclude that the flow converges exponentially to the standard metric on $\mathbf{RP}^n$.

8. A compactness result for conformal metrics on S^n

In this section, we give a proof for Proposition 1.4. Let $g_k = e^{2w_k}\, g_0$ be a sequence of conformal metrics on S^n with fixed volume such that

$$\int_{S^n} Q_k^2 \, dV_k \leq C.$$

Since

$$Q_k = e^{-nw_k}\, (Q_0 + P_0 w_k),$$

we obtain

$$\int_{S^n} e^{-nw_k}\, (Q_0 + P_0 w_k)^2 \, dV_0 \leq C.$$

Moreover, we have

$$\int_{S^n} |Q_k| \, dV_k \leq C.$$

Hence

$$\int_{S^n} |P_0 w_k| \, dV_0 \leq C.$$

Finally, we have

$$\lim_{r \to 0} \lim_{k \to \infty} \int_{B_r(x)} |Q_k| \, dV_k < \frac{1}{2}\, (n-1)!\, \omega_n.$$

This implies

$$\lim_{r \to 0} \lim_{k \to \infty} \int_{B_r(x)} |P_0 w_k| \, dV_0 < \frac{1}{2}\, (n-1)!\, \omega_n.$$

Choosing r sufficiently small, we obtain

$$\lim_{k \to \infty} \int_{B_r(x)} |P_0 w_k| \, dV_0 < \frac{1}{2}\, (n-1)!\, \omega_n.$$

Let

$$I_k = \int_{B_r(x)} |P_0 w_k| \, dV_0.$$

We now use the formula

$$w_k(y) - \overline{w}_k = \int_{S^n} P_0 w_k(z) \, K(y,z) \, dV_0(z).$$

This implies

$$np(w_k(y) - \overline{w}_k) \leq \int_{B_r(x)} np \, |P_0 w_k(z)| \, |K(y,z)| \, dV_0(z) + C$$

for all $y \in B_{\frac{r}{2}}(x)$. Using Jensen's inequality, we obtain

$$e^{np(w_k(y) - \overline{w}_k)} \leq \frac{C}{I_k} \int_{B_r(x)} |P_0 w_k(z)| \, e^{npI_k \, |K(y,z)|} \, dV_0(z)$$

for all $y \in B_{\frac{r}{2}}(x)$. Since

$$\lim_{k \to \infty} I_k < \frac{1}{2} \, (n-1)! \, \omega_n,$$

we can find a real number $p > 1$ such that

$$\lim_{k \to \infty} pI_k < \frac{1}{2} \, (n-1)! \, \omega_n.$$

We now use an asymptotic formula of the function $K(y,z)$ for $|y - z| \to 0$. To derive this formula, we use the identity

$$(-\Delta)^{\frac{n}{2}} \log |y - z| = -2^{n-2} \left(\left(\frac{n-2}{2} \right)! \right)^2 \omega_{n-1} \, \delta(y - z).$$

This implies

$$(-\Delta)^{\frac{n}{2}} \log |y - z| = -\frac{1}{2} \, (n-1)! \, \omega_n \, \delta(y - z).$$

Therefore, we obtain

$$\frac{1}{2} \, (n-1)! \, \omega_n \, K(y,z) \sim -\log |y - z|;$$

hence

$$e^{\frac{1}{2} \, (n-1)! \, \omega_n \, |K(y,z)|} \sim \frac{1}{|y - z|}.$$

From this it follows that

$$\int_{S^n} e^{npI_k \, |K(y,z)|} \, dV_0(y) \leq C.$$

Since

$$\frac{1}{I_k} \int_{B_r(x)} |P_0 w_k| \, dV_0 = 1,$$

we conclude that
$$\int_{B_{\frac{r}{2}}(x)} e^{np(w_k(y)-\overline{w}_k)}\, dV_0(y) \leq C.$$
Covering S^n with finitely many balls $B_{\frac{r}{2}}(x)$, we obtain
$$\int_{S^n} e^{np(w_k-\overline{w}_k)}\, dV_0 \leq C$$
for some $p > 1$. In particular, we have
$$\int_{S^n} e^{n(w_k-\overline{w}_k)}\, dV_0 \leq C.$$
Since $\int_{S^n} e^{nw_k}\, dV_0 = 1$, we conclude that $e^{-n\overline{w}_k} \leq C$; hence $-C \leq \overline{w}_k \leq C$. This implies
$$\int_{S^n} e^{npw_k}\, dV_0 \leq C.$$
By Hölder's inequality,
$$\int_{S^n} |Q_0+P_0w_k|^{\frac{2p}{p+1}}\, dV_0 \leq \left(\int_{S^n} e^{-nw_k}\,(Q_0+P_0w_k)^2\, dV_0\right)^{\frac{p}{p+1}} \left(\int_{S^n} e^{npw_k}\, dV_0\right)^{\frac{1}{p+1}}.$$
From this it follows that
$$\int_{S^n} |P_0w_k|^{\frac{2p}{p+1}}\, dV_0 \leq C.$$
Using the Sobolev inequality, we obtain $|w_k| \leq C$. Thus, we conclude that
$$\int_{S^n} |P_0w_k|^2\, dV_0 \leq C.$$
Therefore, the sequence w_k is uniformly bounded in H^n.

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