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On the nonnegativity of $L(\frac{1}{2},\pi)$ for SO_{2n+1}

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Abstract

Let π be a cuspidal generic representation of $\mathrm{SO}(2n+1,\mathbb{A})$. We prove that $L(\frac{1}{2},\pi) \geq 0$.

1. Introduction

Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ where $\mathbb{A}$ is the ring of adèles of a number field F. Suppose that π is self-dual. Then the "standard" L-function ([GJ72]) $L(s,\pi)$ is real for $s \in \mathbb{R}$ and positive for $s > 1$. Assuming GRH we have $L(s,\pi) > 0$ for $\frac{1}{2} < s \leq 1$, except for the case where $n = 1$ and π is the trivial character. It would follow that $L(\frac{1}{2},\pi) \geq 0$. However, the latter is not known even in the case of quadratic Dirichlet characters. In general, if π is self-dual then π is either *symplectic* or *orthogonal*, i.e. exactly one of the (partial) L-functions $L^S(s,\pi,\wedge^2)$, $L^S(s,\pi,\mathrm{sym}^2)$ has a pole at $s = 1$. In the first case n is even and the central character of π is trivial ([JS90a]). In the language of the Tannakian formalism of Langlands ([Lan79]), any cuspidal representation π of $\mathrm{GL}_n(\mathbb{A})$ corresponds to an irreducible n-dimensional representation φ of a conjectural group $\mathcal{L}_F$ whose derived group is compact. Then π is self-dual if and only if φ is self-dual, and the classification into symplectic and orthogonal is compatible with (and suggested by) the one for finite dimensional representations of a compact group. Our goal in this paper is to show

Theorem 1. *Let π be a symplectic cuspidal representation of* $\mathrm{GL}_n(\mathbb{A})$. *Then $L(\frac{1}{2},\pi) \geq 0$.*

We note that the same will be true for the partial L-function. The value $L(\frac{1}{2},\pi)$ appears in many arithmetic, analytic and geometric contexts – among them, the Shimura correspondence ([Wal81]), or more generally – the theta

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correspondence ([Ral87]), the Birch-Swinnerton-Dyer conjecture, the Gross-Prasad conjecture ([GP94]), certain period integrals, and the relative trace formula ([JC01], [BM]). In all the above cases, the L-functions are of symplectic type. Moreover, all motivic L-functions which have the center of symmetry as a critical point in the sense of Deligne are necessarily of symplectic type. In the case $n = 2$, π is symplectic exactly when the central character of π is trivial. The above-mentioned interpretations of $L(\frac{1}{2}, \pi)$ were used to prove Theorem 1 in that case ([KZ81], [KS93], using the Shimura correspondence in special cases, and [Guo96], using a variant of Jacquet's relative trace formula, in general). The nonnegativity of $L(\frac{1}{2}, \pi)$ in the GL_2 case already has striking applications, for example to sub-convexity estimates for various L-functions ([CI00], [Ivi01]). We expect that the higher rank case will turn out to be useful as well. The nonnegativity of $L(\frac{1}{2}, \chi)$ for quadratic Dirichlet characters would have far-reaching implications to Gauss class number problem. Unfortunately, our method is not applicable to that case.

The Tannakian formalism suggests that the symplectic and orthogonal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ are functorial images from classical groups. In fact, it is known that every symplectic cuspidal automorphic representation π of $\mathrm{GL}_{2n}(\mathbb{A})$ is a functorial image of a cuspidal generic representation of $\mathrm{SO}(2n+1, \mathbb{A})$. Conversely, to every cuspidal generic representation of $\mathrm{SO}(2n+1, \mathbb{A})$ corresponds an automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ which is parabolically induced from cuspidal symplectic representations ([GRS01], [CKP-SS01]). As a consequence:

THEOREM 2. *Let σ be a cuspidal generic representation of* $\mathrm{SO}(2n+1, \mathbb{A})$. *Then* $L^S\left(\frac{1}{2}, \sigma\right) \geq 0$.

The L-function is the one pertaining to the imbedding of $\mathrm{Sp}(n, \mathbb{C})$, the L-group of $\mathrm{SO}(2n+1)$, in $\mathrm{GL}(2n, \mathbb{C})$. By the work of Jiang-Soudry ([JS]) Theorem 2 applies equally well to the completed L-function as defined by Shahidi in [Sha81].

We emphasize however that our proof of Theorem 1 is independent of the functorial lifting above. In fact, it turns out, somewhat surprisingly, that Theorem 1 is a simple consequence of the theory of Eisenstein series on classical groups. Consider the symplectic group Sp_n and the Eisenstein series $E(g, \varphi, s)$ induced from π viewed as a representation on the Siegel parabolic subgroup. If π is symplectic then for $E(g, \varphi, s)$ to have a pole at $s = \frac{1}{2}$ it is necessary and sufficient that $L(\frac{1}{2}, \pi) \neq 0$, in which case the pole is simple. In particular, in this case $\varepsilon(\frac{1}{2}, \pi) = 1$ by the functional equation. We refer the reader to the body of the paper for any unexplained notation. Let $E_{-1}(\cdot, \varphi)$ be the residue of $E(\cdot, \varphi, s)$ at $s = \frac{1}{2}$. It is a square-integrable automorphic form on Sp_n. A consequence of the spectral theory is that the inner product of two such residues is given by the residue $\mathfrak{M}_{-1}$ of the intertwining operator at $s=\frac{1}{2}$. Thus, $\mathfrak{M}_{-1}$ is a positive semi-definite operator. First assume that the local components of π are unramified at every place including the archimedean ones. Then by a well-known formula of Langlands ([Lan71]), the intertwining operator $\mathfrak{M}(s)$ satisfies

$$\mathfrak{M}(s)v_0 = L(s,\pi)/L(s+1,\pi)\cdot L(2s,\pi,\wedge^2)/L(2s+1,\pi,\wedge^2)\cdot v_0$$

for the unramified vector v_0. Therefore

$$\mathfrak{M}_{-1}v_0 = \frac{1}{2}\cdot L\left(\frac{1}{2},\pi\right)/L\left(\frac{3}{2},\pi\right)\cdot \mathrm{res}_{s=1}L(s,\pi,\wedge^2)/L(2,\pi,\wedge^2)\cdot v_0.$$

Since $L(s,\pi)$ is positive for $s>1$ and $L(s,\pi,\wedge^2)$ is real and nonzero for $s>1$ we obtain Theorem 1 in this case. In order to generalize this argument and avoid any local assumptions on π we have, as usual, to make some local analysis. For that, we use Shahidi's normalization of the intertwining operators ([Sha90b]) which is applicable since π is generic. Let $R(\pi,s)=R(s)=\otimes_v R_v(s): I(\pi,s)\to I(\pi,-s)$ be the normalized intertwining operator. Here we take into account a canonical identification of π with its contragredient and suppress the dependence of $R_v(s)$ on a choice of an additive character. Then $\mathfrak{M}(s)=m(s)\cdot R(s)$ where

$$m(s)=\frac{L(s,\pi)}{\varepsilon(s,\pi)L(s+1,\pi)}\cdot\frac{L(2s,\pi,\wedge^2)}{\varepsilon(2s,\pi,\wedge^2)L(2s+1,\pi,\wedge^2)}.$$

Hence, $\mathfrak{M}_{-1}=m_{-1}\cdot R\left(\frac{1}{2}\right)$, where m_{-1} is the residue of $m(s)$ at $s=\frac{1}{2}$, and the operator $R(\frac{1}{2})$ is semi-definite with the same sign as m_{-1}. On the other hand, the argument of Keys-Shahidi ([KS88]) shows that the Hermitian involution $R(\pi_v,0)$ has a nontrivial $+1$ eigenspace. The main step (Lemma 3, proved in §3) is to show that $R(\pi_v,\frac{1}{2})$ is *positive* semi-definite by "deforming" it to $R(\pi_v,0)$. This will imply that $m_{-1}>0$, i.e.

$$\frac{L\left(\frac{1}{2},\pi\right)}{L\left(\frac{3}{2},\pi\right)}\cdot\frac{\mathrm{res}_{s=1}L(s,\pi,\wedge^2)}{\varepsilon(1,\pi,\wedge^2)L(2,\pi,\wedge^2)}>0.$$

Similarly, working with the group $\mathrm{SO}(2n)$ we obtain

$$\frac{\mathrm{res}_{s=1}L(s,\pi,\wedge^2)}{\varepsilon(1,\pi,\wedge^2)L(2,\pi,\wedge^2)}>0$$

if π is symplectic. Altogether this implies Theorem 1 (see §2). We may work with the group $\mathrm{SO}(2n+1)$ as well. Using the relation $\varepsilon(\frac{1}{2},\pi\otimes\widetilde{\pi})=1$ ([BH99]) we will obtain the following:

THEOREM 3. *Let π be a self-dual cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$. Then $\varepsilon(\frac{1}{2},\pi,\wedge^2)=\varepsilon(\frac{1}{2},\pi,\mathrm{sym}^2)=1$.*

This is compatible with the Tannakian formalism. In general one expects that $\varepsilon(\frac{1}{2}, \pi, \rho) = 1$ if the representation $\rho \circ \varphi$ is orthogonal ([PR99]). This is inspired by results of Fröhlich-Queyrut, Deligne and Saito about epsilon factors of orthogonal Galois representations and motives ([FQ73], [Del76], [Sai95]).

The analysis of Section 3, the technical core of this article, relies on detailed information about the reducibility of induced representations of classical groups. This was studied extensively by Goldberg, Jantzen, Muic, Shahidi, Tadic, and others (see [Gol94], [Jan96], [Mui01], [Sha92], [Tad98]).

Note added in proof. Since the time of writing this paper Theorem 1 was generalized by the first-named author to tensor product L-functions of symplectic type ([Lap03]). Similarly, other root numbers of orthogonal type have shown to be 1 ([Lap02]).

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2. The setup

Let F be a number field, $\mathbb{A} = \mathbb{A}_F$ its adèles ring and let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. We say that π is symplectic (resp. orthogonal) if $L^S(s, \pi, \wedge^2)$ (resp. $L^S(s, \pi, \mathrm{sym}^2)$) has a pole at $s = 1$. If π is symplectic or orthogonal then π is self-dual. Conversely, if π is self-dual then π is either symplectic or orthogonal but not both. Moreover, if π is symplectic then n is even and the central character of π is trivial ([JS90a]). Our goal is to prove Theorems 1 and 3. In this section we will reduce them to a few local statements, namely Lemmas 1–4 below which will be proved in the next section. They all have some overlap with known results in the literature. We first fix some notation. By our convention, if X is an algebraic group over F we denote the F-points of X by X as well. Let J_n be $n \times n$ matrix with ones on the nonprincipal diagonal and zeros otherwise. Let G be either the split orthogonal group $\mathrm{SO}(2n+1)$ with respect to the symmetric form defined by

$$\begin{pmatrix} & & J_n \\ & 1 & \\ J_n & & \end{pmatrix}$$

or the symplectic group Sp_n with respect to the skew-symmetric form defined by the matrix

$$\begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}$$

or the split orthogonal group $\mathrm{SO}(2n)$ with respect to the symmetric form defined by $\begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix}$. Then G acts by right multiplication on the space $\mathbb{V}$ of row vectors of size $2n$ or $2n+1$. Let $P = M\cdot U$ be the Siegel parabolic subgroup of G with its standard Levi decomposition. It is the stabilizer of the maximal isotropic space $\mathbb{U}$ defined by the vanishing of all but the last n coordinates. We identify M with $\mathrm{GL}(\mathbb{V}/\mathbb{U}^{\perp}) \simeq \mathrm{GL}_n$ where $\mathbb{U}^{\perp}$ is the perpendicular of $\mathbb{U}$ in $\mathbb{V}$ with respect to the form defining G. We denote by $\nu : M(\mathbb{A}) \to \mathbb{R}_+$ the absolute value of the determinant in that identification. Let $\mathbf{K}$ be the standard maximal compact subgroup of $G(\mathbb{A})$. We extend ν to a left-$U(\mathbb{A})$ right-$\mathbf{K}$-invariant function on $G(\mathbb{A})$ using the Iwasawa decomposition. Let δ_P be the modulus function of $P(\mathbb{A})$. It is given by $\delta_P = \nu^n, \nu^{n+1}$ or ν^{n-1} according to whether $G = \mathrm{SO}(2n+1)$, Sp_n or $\mathrm{SO}(2n)$. Let π be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ and $\mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{\pi,s}$ be the space of automorphic forms φ on $U(\mathbb{A})M\backslash G(\mathbb{A})$ such that the function $m \to \nu^{-s}(m)\delta_P(m)^{-1/2}\varphi(mk)$ belongs to the space of π for any $k \in \mathbf{K}$. By multiplicity-one for GL_n, $\mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{\pi,s}$ depends only on the equivalence class of π and not on its automorphic realization. By choosing an automorphic realization for π (unique up to a scalar), we may identify $\mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{\pi,s}$ with (the $\mathbf{K}$-finite vectors in) the induced space $I(\pi, s)$. The Eisenstein series

$$E(g,\varphi,s) = \sum_{\gamma\in P\backslash G} \varphi(\gamma g)\nu^s(\gamma g)$$

converges when $\mathrm{Re}(s)$ is sufficiently large and admits a meromorphic continuation. Whenever it is regular it defines an intertwining map $\mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{\pi,s} \to \mathcal{A}(G\backslash G(\mathbb{A}))$. It is known that the only possible singularity of $E(g,\varphi,s)$ for $\mathrm{Re}(s) \geq 0$ is a simple pole at $s = \frac{1}{2}$ (except when π is the trivial character and $G = \mathrm{Sp}_1$, where there is a pole at $s = 1$).

In the case $G = \mathrm{SO}(2n)$ let Σ be the outer automorphism obtained by conjugation by the element

$$\begin{pmatrix} 1_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1_{n-1} \end{pmatrix}$$

of $\mathrm{O}(2n) \setminus \mathrm{SO}(2n)$. For the other groups let $\Sigma = \mathbf{1}$. In all cases we set $\theta = \Sigma^n$. Then θ induces the principal involution on the root data of G. Note that $\{P, \theta(P)\}$ is the set of standard parabolic subgroups of G which are associate to P. Fix $w \in G \setminus M$ such that $wMw^{-1} = \theta(M)$; it is uniquely determined up to right multiplication by M. Let $^\sharp : M \to \theta(M)$ be defined by $m^\sharp = wmw^{-1}$. Denote by $w\pi$ the cuspidal automorphic representation of $\theta(M)(\mathbb{A})$ on $\{\varphi^\sharp : \varphi \in V_\pi\}$ where $\varphi^\sharp(m^\sharp) = \varphi(m)$. The "automorphic" intertwining operator

$$\mathfrak{M}(s) = \mathfrak{M}(\pi, s) : \mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{\pi,s} \to \mathcal{A}(\theta(U)(\mathbb{A})\theta(M)\backslash G(\mathbb{A}))_{w\pi,-s}$$

is defined by

$$[\mathfrak{M}(s)\varphi](g) = \int_{\theta(U)(\mathbb{A})} \varphi(w^{-1}ug)\nu^s(w^{-1}ug)\, du.$$

Let $E_{-1}(\bullet, \varphi)$ be the residue of $E(g, \varphi, s)$ at $s = \frac{1}{2}$. It is zero unless $w\pi = \pi$, and in particular, $\theta(M) = M$, i.e. $\theta = \mathbf{1}$. The latter means that P is conjugate to its opposite. We say that π is of G-type if $E_{-1} \not\equiv 0$, or what amounts to the same, that $\mathfrak{M}_{-1} \not\equiv 0$ where $\mathfrak{M}_{-1}$ is the residue of $\mathfrak{M}(s)$ at $\frac{1}{2}$. In this case E_{-1} defines an intertwining map $\mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{\pi,\frac{1}{2}} \to \mathcal{A}(G\backslash G(\mathbb{A}))$. The inner product formula for two residues of Eisenstein series is given by

$$(1) \qquad \int_{G\backslash G(\mathbb{A})} E_{-1}(g, \varphi_1)\overline{E_{-1}(g, \varphi_2)}\, dg = \int_{\mathbf{K}} \int_{M\backslash M(\mathbb{A})^1} \mathfrak{M}_{-1}\varphi_1(mk)\overline{\varphi_2(mk)}\, dm\, dk$$

up to a positive constant depending on normalization of Haar measures. This follows for example by taking residues in the Maass-Selberg relations for inner product of truncated Eisenstein series (cf. [Art80, §4]). Alternatively, this is a consequence of spectral theory ([MW95]).

We let $\pi^\sharp$ be the representation of $\theta(M)(\mathbb{A})$ on V_π defined by $\pi^\sharp(m^\sharp)v = \pi(m)v$. We may identify $\pi^\sharp$ with $w\pi$ by the map $\varphi \mapsto \varphi^\sharp$. Let $M(s) = M(\pi, s) : I(\pi, s) \to I(\pi^\sharp, -s)$ be the "abstract" intertwining operator given by

$$M(s)\varphi(g) = \int_{\theta(U)(\mathbb{A})} \varphi(w^{-1}ug)\nu^s(w^{-1}ug)\, du.$$

Under the isomorphisms

$$\begin{aligned} \mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{\pi,s} &\simeq I(\pi, s) \text{ and} \\ \mathcal{A}(\theta(U)(\mathbb{A})\theta(M)\backslash G(\mathbb{A}))_{w\pi,-s} &\simeq I(\pi^\sharp, -s), \end{aligned}$$

$\mathfrak{M}(s)$ becomes $M(s)$.

Let $^\flat : M \to M$ be the map defined by $m^\flat = \theta(m^\sharp)$. We will choose the representative w as in [Sha90b] so that when M is identified with GL_n, $^\flat$ becomes the involution $x \mapsto w_n^{-1}\, {}^t x^{-1} w_n$ where

$$(w_n)_{ij} = \begin{cases} (-1)^i & \text{if } i + j = n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular $^\flat$ does not depend on G. A direct computation shows that

(2)

$w^2 \in M$ corresponds to the central element $(-1)^n$(resp. $(-1)^{n+1}$) of GL_n

if G is symplectic (resp. orthogonal). We define $\varphi^\flat$ and $\pi^\flat$ as before. Since π is irreducible we have ([GK75])

(3) $\pi^\flat$ is equivalent to the contragredient $\widetilde{\pi}$ of π.

Thus, for π to be of G-type it is necessary that $\theta = \mathbf{1}$ and that π be self-dual. If π is self-dual we define the intertwining operator $\iota = \iota_\pi : \pi^\flat \to \pi$ by $\iota(\varphi) = \varphi^\flat$. It is well-defined by multiplicity-one and does not depend on the automorphic realization of π. We write $\iota(s) = \iota(\pi, s)$ for the induced map $I(\pi^\flat, s) \to I(\pi, s)$ given by $[\iota(s)(f)](g) = \iota(f(g))$. Note that when $\theta = \mathbf{1}$, $\iota(s)$ is the map $I(\pi^\flat, s) \to I(\pi, s)$ induced from the "physical" equality of the two spaces $\mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{w\pi,s}$ and $\mathcal{A}(U(\mathbb{A})M\backslash G(\mathbb{A}))_{\pi,s}$. Assume that π is self-dual and that $\theta = \mathbf{1}$. Then as a map from $I(\pi, s)$ to $I(\pi, -s)$ the intertwining operator $\mathfrak{M}(s)$ becomes $\iota(-s) \circ M(s)$. Let $(\cdot,\cdot)_\pi$ be the invariant positive-definite Hermitian form on π obtained through its automorphic realization. This gives rise to the invariant sesqui-linear form $(\cdot,\cdot) = (\cdot,\cdot)_s : I(\pi, -s) \times I(\pi, \overline{s}) \to \mathbb{C}$ given by

$$(\varphi_1, \varphi_2) = \int_{\mathbf{K}} (\varphi_1(k), \varphi_2(k))_\pi \, dk.$$

Thus, the right-hand side of (1), viewed as a positive-definite invariant Hermitian form on $I(\pi, \frac{1}{2})$, is $(\iota(-\frac{1}{2}) \circ M_{-1}\varphi_1, \varphi_2)_{\frac{1}{2}}$.

In the local case we can define ${\pi_v}^\sharp$, ${\pi_v}^\flat$ and the local intertwining operators

$$M_v(s) : I(\pi_v, s) \to I({\pi_v}^\sharp, -s)$$

in the same way. Fix a nontrivial character $\psi = \otimes_v \psi_v$ of $F\backslash\mathbb{A}_F$. For any v choose a Whittaker model for π_v with respect to the $^\flat$-stable character

$$\begin{pmatrix} 1 & x_1 & * & * \\ & 1 & \ddots & * \\ & & 1 & x_{n-1} \\ & & & 1 \end{pmatrix} \mapsto \psi_v(x_1 + \ldots + x_{n-1}).$$

If π_v is self-dual then we define the intertwining map $\iota_v = \iota_{\pi_v}^{\psi_v} : {\pi_v}^\flat \to \pi_v$ by

$$[\iota_v(W)](g) = W(g^\flat)$$

in the Whittaker model with respect to ψ_v. By uniqueness of the Whittaker model ι_v is well-defined and does not depend on choice of the Whittaker model. If we change ψ_v to $\psi_v(a\cdot)$ for $a \in F_v^*$ then ι_v is multiplied by the sign $\omega_{\pi_v}^{n-1}(a)$. If π_v and ψ_v are unramified then $\iota_v(u) = u$ for an unramified vector u since the unramified Whittaker vector is nonzero at the identity by the Casselman-Shalika formula.

Suppose that $\pi = \otimes_v \pi_v$ is an automorphic self-dual cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ where the restricted tensor product is taken with respect to a choice of unramified vectors e_v almost everywhere. We choose invariant positive definite Hermitian forms $(\cdot,\cdot)_{\pi_v}$ on π_v for all v so that $(e_v,e_v)_{\pi_v}=1$ almost everywhere. This gives rise to sesqui-linear forms $(\cdot,\cdot)_{v,s}: I(\pi_v,-s)\times I(\pi_v,\overline{s})\to\mathbb{C}$ as above. We have $(\cdot,\cdot)_\pi = c\otimes(\cdot,\cdot)_{\pi_v}$ and $(\cdot,\cdot)_s = c\otimes(\cdot,\cdot)_{v,s}$ in the obvious sense, for some *positive* scalar c, and $\iota_\pi=\otimes_v\iota_{\pi_v}$.

At this point it is useful to normalize $M_v(s)$ by the normalization factors $m_v^{\psi_v}(\pi_v,s)=m_v(s)$ defined by Shahidi in [Sha90b]. The latter are given by

$$m_v(s)=\begin{cases}\frac{L(2s,\pi_v,\mathrm{sym}^2)}{\varepsilon(2s,\pi_v,\mathrm{sym}^2,\psi_v^{-1})L(2s+1,\pi_v,\mathrm{sym}^2)} & G=\mathrm{SO}(2n+1),\\ \frac{L(s,\pi_v)}{\varepsilon(s,\pi_v,\psi_v^{-1})L(s+1,\pi_v)}\frac{L(2s,\pi_v,\wedge^2)}{\varepsilon(2s,\pi_v,\wedge^2,\psi_v^{-1})L(2s+1,\pi_v,\wedge^2)} & G=\mathrm{Sp}_n,\\ \frac{L(2s,\pi_v,\wedge^2)}{\varepsilon(2s,\pi_v,\wedge^2,\psi_v^{-1})L(2s+1,\pi_v,\wedge^2)} & G=\mathrm{SO}(2n),\end{cases}$$

where $L(s,\pi_v)$, $L(s,\pi_v,\wedge^2)$, $L(s,\pi_v,\mathrm{sym}^2)$ are the local L-functions pertaining to the standard, symmetric square and exterior square representations of $GL_n(\mathbb{C})$ respectively, and similarly for the epsilon factors. We write $M_v(\pi_v,s)=m_v^{\psi_v}(\pi_v,s)R_v^{\psi_v}(\pi_v,s)$ where $R_v(s)=R_v^{\psi_v}(\pi_v,s)$ are the normalized intertwining operators. Note that by changing ψ_v to $\psi_v(a\cdot)$ the scalar $m_v(s)$ is multiplied by $(\omega_{\pi_v}(a)\,|a|^{n(s-\frac12)})^k$ where $k=n+1,n$, or $n-1$ according to whether $G=\mathrm{SO}(2n+1),\mathrm{Sp}_n$ or $\mathrm{SO}(2n)$.

The following lemma will be proved in the next section, together with the other lemmas below.

LEMMA 1. *For all v, $R_v(s)$, $M_v(s)$, $L_v(2s,\pi_v,\mathrm{sym}^2)$, $L_v(2s,\pi_v,\wedge^2)$, $L_v(s,\pi_v)$ and $m_v(s)$ are holomorphic and nonzero for* $\mathrm{Re}(s)\geq\frac12$.

In fact, the holomorphy and nonvanishing of $R_v(s)$ for $\mathrm{Re}(s)\geq\frac12$ is proved more generally in a recent paper of Kim ([Kim02]).

Let $m(s)=m(\pi,s)=\prod_v m_v^{\psi_v}(\pi_v,s)$ and $R(s)=\otimes_v R_v(s)$ so that $M(s)=m(s)R(s)$. If $G=\mathrm{SO}(2n+1)$ then

$$m(s)=\frac{L(2s,\pi,\mathrm{sym}^2)}{\varepsilon(2s,\pi,\mathrm{sym}^2)L(2s+1,\pi,\mathrm{sym}^2)}=\frac{L(1-2s,\pi,\mathrm{sym}^2)}{L(1+2s,\pi,\mathrm{sym}^2)}.$$

If $G=\mathrm{Sp}_n$ then

$$\begin{aligned}m(s)&=\frac{L(s,\pi)}{\varepsilon(s,\pi)L(s+1,\pi)}\frac{L(2s,\pi,\wedge^2)}{\varepsilon(2s,\pi,\wedge^2)L(2s+1,\pi,\wedge^2)}\\&=\frac{L(1-s,\pi)}{L(1+s,\pi)}\frac{L(1-2s,\pi,\wedge^2)}{L(1+2s,\pi,\wedge^2)}.\end{aligned}$$

If $G=\mathrm{SO}(2n)$,

$$m(s)=\frac{L(2s,\pi,\wedge^2)}{\varepsilon(2s,\pi,\wedge^2)L(2s+1,\pi,\wedge^2)}=\frac{L(1-2s,\pi,\wedge^2)}{L(1+2s,\pi,\wedge^2)}.$$

In particular, the residue m_{-1} at $s=\frac{1}{2}$ is equal to $\frac{1}{2}$ times

$$\begin{cases} \frac{\mathrm{res}_{s=1} L(s,\pi,\mathrm{sym}^2)}{\varepsilon(1,\pi,\mathrm{sym}^2)L(2,\pi,\mathrm{sym}^2)} & G=\mathrm{SO}(2n+1) \\ \frac{L(\frac{1}{2},\pi)}{\varepsilon(\frac{1}{2},\pi)L(\frac{3}{2},\pi)} \frac{\mathrm{res}_{s=1} L(s,\pi,\wedge^2)}{\varepsilon(1,\pi,\wedge^2)L(2,\pi,\wedge^2)} & G=\mathrm{Sp}_n \\ \frac{\mathrm{res}_{s=1} L(s,\pi,\wedge^2)}{\varepsilon(1,\pi,\wedge^2)L(2,\pi,\wedge^2)} & G=\mathrm{SO}(2n). \end{cases}$$

By Lemma 1, π is of G-type if and only if $m(s)$ has a pole (necessarily simple) at $s=\frac{1}{2}$. Thus, π is of Sp_n type if and only if π is symplectic and $L(\frac{1}{2},\pi)\neq 0$; π is of $\mathrm{SO}(2n+1)$ type if and only if π is orthogonal; π is of $\mathrm{SO}(2n)$ type if and only if π is symplectic. Suppose that π is of G-type. Let $\mathfrak{B}(s)=\mathfrak{B}(\pi,s)$ be the operator $\iota(-s)\circ R(s): I(\pi,s)\to I(\pi,-s)$ for $s\in\mathbb{R}$ and let $\mathfrak{I}(\pi,s)$ be the form on $I(\pi,s)$ defined by $(\mathfrak{B}(s)\varphi,\varphi)$. Since $\mathfrak{M}_{-1}=m_{-1}\cdot\mathfrak{B}\left(\frac{1}{2}\right)$, it follows from (1) that $\mathfrak{I}(\pi,\frac{1}{2})$ is semi-definite with the same sign as m_{-1}. We will show that

$$\mathfrak{I}(\pi,\frac{1}{2}) \text{ is positive semi-definite} \tag{4}$$

and thus

$$m_{-1}>0. \tag{5}$$

2.1. *Proof of Theorem* 1. We will use (5) for the groups Sp_n and $\mathrm{SO}(2n)$. Together, this implies that if π is symplectic and $L(\frac{1}{2},\pi)\neq 0$ then $\frac{L(\frac{1}{2},\pi)}{\varepsilon(\frac{1}{2},\pi)L(\frac{3}{2},\pi)}>0$. By the functional equation and the fact that $L(\frac{1}{2},\pi)\neq 0$ we must have $\varepsilon(\frac{1}{2},\pi)=1$. On the other hand $L(s,\pi)$ is a convergent Euler product for $s>1$, all factors of which are real and positive. Indeed, $L(s,\pi_v)=\overline{L(\bar{s},\pi_v)}$ since π_v is equivalent to its Hermitian dual. In the nonarchimedean case, $L(s,\pi_v)\to 1$ as $s\to+\infty$ (s real). In the archimedean case $L(s,\pi_v)=\prod_{i=1}^n \Gamma_{\mathbb{R}}(s-s_i)$ for some $s_i\in\mathbb{C}$ where $\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma(s/2)$. We have $\sum \mathrm{Im} s_i=0$ since $\pi_v=\overline{\pi}_v$. It is easily deduced from Stirling's formula that $L(s,\pi_v)\to+\infty$ as $s\to+\infty$. In both cases $L(s,\pi_v)$ is holomorphic and nonzero for $s\geq\frac{1}{2}$. The claim follows. Hence $L(\frac{3}{2},\pi)>0$, and therefore, $L(\frac{1}{2},\pi)>0$.

It remains to prove (4). The operator $\mathfrak{B}(\pi,s)$ and the form $\mathfrak{I}(\pi,s)$ admit a local analogue and we have $\mathfrak{B}(\pi,s)=\otimes_v\mathfrak{B}^{\psi_v}(\pi_v,s)$ and $\mathfrak{I}(\pi,s)=c\otimes_v \mathfrak{I}^{\psi_v}(\pi_v,s)$.

We will prove the following purely local Lemmas. Recall the assumption that $\theta=\mathbf{1}$.

LEMMA 2. *Let π_v be a generic irreducible unitary self-dual representation of* GL_n *over a local field of characteristic* 0. *Then $\mathfrak{B}^{\psi_v}(\pi_v,s)$ is Hermitian for $s\in\mathbb{R}$ and holomorphic near $s=0$. Moreover, $\mathfrak{B}^{\psi_v}(\pi_v,0)$ is an involution with a nontrivial* $+1$-*eigenspace.*

Lemma 3. *Under the same assumptions, suppose further that* $\mathfrak{I}^{\psi_v}(\pi_v, \frac{1}{2})$ *is semi-definite. Then* $\mathfrak{I}^{\psi_v}(\pi_v, 0)$ *is definite with the same sign as* $\mathfrak{I}^{\psi_v}(\pi_v, \frac{1}{2})$. *Hence, by Lemma* 2, $\mathfrak{B}^{\psi_v}(\pi_v, 0) = \mathbf{1}$ *and* $\mathfrak{I}^{\psi_v}(\pi_v, \frac{1}{2})$ *is positive semi-definite.*

These two lemmas, together with the fact that $\mathfrak{I}(\pi, \frac{1}{2})$ is semi-definite, imply (4), even locally.

We remark that in the case where G is an orthogonal group then up to a positive scalar $\mathfrak{B}^{\psi_v}(\pi_v, s)$ is independent of ψ_v. This is no longer true in the Sp_n case if the central character of π_v is nontrivial. In that case, Lemma 2 actually implies the well-known fact that $I(\pi_v, 0)$ is reducible.

Note also that the very last (and most important) conclusion of Lemma 3 is trivial in the unramified case. Finally, let us mention that a property related (and ultimately, equivalent) to the conclusion of Lemma 3 for the local components of a symplectic cuspidal representation was proved by Jiang-Soudry using the descent construction ([JS]). We will not use their result.

2.2. *Proof of Theorem* 3. We first observe that $L(s, \pi, \mathrm{sym}^2)$ and $L(s, \pi, \wedge^2)$ are holomorphic and nonzero for $\mathrm{Re}(s) > 1$. Indeed, the partial L-functions $L^S(s, \pi, \mathrm{sym}^2)$, $L^S(s, \pi, \wedge^2)$ are holomorphic for $\mathrm{Re}(s) > 1$ ([JS90a], [BG92]) and their product is $L^S(s, \pi \otimes \pi)$, which is nonzero for $\mathrm{Re}(s) > 1$, since the Euler product converges absolutely ([JS81]). The statement now follows from Lemma 1.

Suppose that π is orthogonal. Applying (5) to the group $\mathrm{SO}(2n + 1)$ we obtain $\frac{\mathrm{res}_{s=1} L(s,\pi,\mathrm{sym}^2)}{\varepsilon(1,\pi,\mathrm{sym}^2) L(2,\pi,\mathrm{sym}^2)} > 0$. Since $L(s, \pi, \mathrm{sym}^2)$ is real and nonzero for $s > 1$ we obtain $\frac{\mathrm{res}_{s=1} L(s,\pi,\mathrm{sym}^2)}{L(2,\pi,\mathrm{sym}^2)} > 0$. Hence $\varepsilon(1, \pi, \mathrm{sym}^2) > 0$. Since $\varepsilon(s, \pi, \mathrm{sym}^2)$ is nonzero and real for $s \in \mathbb{R}$ we get $\varepsilon(\frac{1}{2}, \pi, \mathrm{sym}^2) > 0$. On the other hand, $\varepsilon(\frac{1}{2}, \pi, \mathrm{sym}^2) = \pm 1$ by the functional equation and hence, $\varepsilon(\frac{1}{2}, \pi, \mathrm{sym}^2) = 1$. Similarly, if π is symplectic then using the group $G = \mathrm{SO}(2n)$ and the same argument we obtain $\varepsilon(\frac{1}{2}, \pi, \wedge^2) = 1$. Since any self-dual cuspidal representation π is either symplectic or orthogonal, the above argument shows that either $\varepsilon(\frac{1}{2}, \pi, \wedge^2) = 1$ or $\varepsilon(\frac{1}{2}, \pi, \mathrm{sym}^2) = 1$. On the other hand for any π (self-dual or not)

$$\varepsilon(s, \pi \otimes \pi) = \varepsilon(s, \pi, \wedge^2) \varepsilon(s, \pi, \mathrm{sym}^2). \tag{6}$$

Indeed, this follows from the corresponding equality of L-functions, which is in fact true locally. In the archimedean case this follows from the compatibility of L-factors with Langlands classification ([Sha90b]). For p-adic fields this is Corollary 8.2 of [Sha92] in the square-integrable case and follows from multiplicativity ([Sha90a]) in the general case. Note that on the left-hand side we may take the epsilon factor as defined by Jacquet, Piatetski-Shapiro and Shalika ([JP-SS83], [JS90b]); it coincides with the one defined by Shahidi; see [Sha84]. To finish the proof of Theorem 3 it remains to note that $\varepsilon(\frac{1}{2}, \pi \otimes \widetilde{\pi}) = 1$

for any cuspidal representation π of $\mathrm{GL}_n(\mathbb{A})$. This follows at once from the next lemma which, at least in the nonarchimedean case, was proved (even without the genericity assumption) by Bushnell and Henniart ([BH99]).

LEMMA 4. *For any generic representation π_v of GL_n over a local field of characteristic* 0,

$$\varepsilon\left(\frac{1}{2}, \pi_v \otimes \widetilde{\pi_v}, \psi_v\right) = \omega_{\pi_v}(-1)^{n-1}, \tag{7}$$

where ω_{π_v} is the central character of π_v.

3. Local analysis

In this section we prove Lemmas 1–4 which were left out in the discussion of the previous section.

For the rest of the paper let F be a local field of characteristic 0. We will suppress the subscript v from all notation and fix a nontrivial character ψ of F throughout. As before, the F-points of an algebraic group X over F will often be denoted by X. We denote by ν the absolute value of the determinant, viewed as a character on any one of the groups $\mathrm{GL}_n(F)$. If π is a representation of GL_n and $s \in \mathbb{C}$ we let $\pi\nu^s$ be the representation obtained by twisting π by the character ν^s. Let Irr_n be the set of equivalence classes of irreducible (admissible) representations of GL_n. Given representations π_i, $i = 1, \ldots, k$ of GL_{n_i} we denote by $\pi_1 \times \ldots \times \pi_k$ the representation on GL_n with $n = n_1 + \ldots + n_k$ induced from the representation $\pi_1 \otimes \ldots \otimes \pi_k$ on the parabolic subgroup of GL_n of type $(n_1, \ldots, n_k)$.

3.1. *Proof of Lemma* 4. For completeness we include a proof which was communicated to us by Hervé Jacquet. We are very grateful to him.

By the functional equation the left-hand side of (7) is ± 1. We prove the lemma by induction on n. If π is not essentially square-integrable then we can write $\pi = \pi_1 \times \pi_2$ where $\pi_i \in \mathrm{Irr}_{n_i}$ are generic. We have

$$\begin{aligned}
&\varepsilon\left(\frac{1}{2}, \pi_1 \otimes \widetilde{\pi_2}, \psi\right) \varepsilon\left(\frac{1}{2}, \pi_2 \otimes \widetilde{\pi_1}, \psi\right) \\
&\qquad = \varepsilon\left(\frac{1}{2}, \pi_1 \otimes \widetilde{\pi_2}, \psi\right) \varepsilon\left(\frac{1}{2}, \pi_2 \otimes \widetilde{\pi_1}, \overline{\psi}\right) \omega_{\pi_1}^{n_2}(-1)\omega_{\pi_2}^{n_1}(-1) \\
&\qquad = \omega_{\pi_1}^{n_2}(-1)\omega_{\pi_2}^{n_1}(-1)
\end{aligned}$$

by the functional equation ([JP-SS83, p. 396]) and the dependence of epsilon on ψ. By "multiplicativity" of epsilon factors (loc. cit., p. 452) we get

$$\varepsilon\left(\frac{1}{2}, \pi \otimes \widetilde{\pi}, \psi\right) = \varepsilon\left(\frac{1}{2}, \pi_1 \otimes \widetilde{\pi_1}, \psi\right) \varepsilon\left(\frac{1}{2}, \pi_2 \otimes \widetilde{\pi_2}, \psi\right) \omega_{\pi_1}^{n_2}(-1)\omega_{\pi_2}^{n_1}(-1)$$

and we may use the induction hypothesis. Thus, it remains to consider the case where π is essentially square-integrable, which immediately reduces to the case where π is square-integrable. In this case the zeta integral

$$\Psi(s, W, W', \Phi) = \int_{N_n \backslash \mathrm{GL}_n} W(g)W'(g)\Phi((0, \dots, 0, 1)g) \, |\det g|^s \, dg$$

converges for $\mathrm{Re}(s) > 0$ (loc. cit., (8.3)). Here W, W' are elements in the Whittaker spaces of π and $\widetilde{\pi}$ respectively, and Φ is a Schwartz function on F^n. In particular, $L(s, \pi \otimes \widetilde{\pi})$ has no pole (or zero) for $\mathrm{Re}(s) > 0$ and by the local functional equation (loc. cit., p. 391) we get

$$\Psi\left(\frac{1}{2}, \widetilde{W}, \overline{\widetilde{W}}, \hat{\Phi}\right) = \varepsilon\left(\frac{1}{2}, \pi \otimes \widetilde{\pi}, \psi\right) \omega_\pi(-1)^{n-1} \Psi\left(\frac{1}{2}, W, \overline{W}, \Phi\right) \tag{8}$$

for any W and Φ. Choose $W \not\equiv 0$ and let g be such that $W(g) \neq 0$. We may choose $\Phi \geq 0$ such that $\Phi((0, \dots, 0, 1)g) \neq 0$ and $\hat{\Phi} \geq 0$. For example, we may take Φ of the form $\Phi_1 \star \Phi_1^\vee$ where $\Phi_1 \geq 0$. Then clearly, both zeta integrals in (8) are nonnegative and the one on the right-hand side is nonzero. Hence $\varepsilon(\frac{1}{2}, \pi \otimes \widetilde{\pi}, \psi)$ has the same sign as $\omega_\pi(-1)^{n-1}$ and consequently, it is equal to it. This finishes the proof of Lemma 4

If $\pi \in \mathrm{Irr}_n$ we denote by $\mathrm{e}(\pi)$ the (central) *exponent* of π. It is the unique real number so that $\pi\nu^{-\mathrm{e}(\pi)}$ has a unitary central character. If π_1, π_2 are generic and irreducible we let $\mathcal{M}(\pi_1, \pi_2)$ be the normalized intertwining operator $\pi_1 \times \pi_2 \to \pi_2 \times \pi_1$ (depending on ψ) as defined by Shahidi ([Sha90b]) provided that it is holomorphic there.

We recall that if π and π' are essentially square-integrable and $|\mathrm{e}(\pi) - \mathrm{e}(\pi')| < 1$ then $\pi \times \pi'$ is irreducible and $\pi \times \pi' \simeq \pi' \times \pi$.

Recall the classification of the irreducible generic unitarizable representations of GL_n. (This is a very special case of [Tad86] in the p-adic case and [Vog86] in the archimedean case; cf. [JS81] for the unramified case.) These are the representations of the form

$$\sigma_1 \times \ldots \times \sigma_s \times \tau_1\nu^{\gamma_1} \times \tau_1\nu^{-\gamma_1} \times \ldots \times \tau_t\nu^{\gamma_t} \times \tau_t\nu^{-\gamma_t} \tag{9}$$

where the σ_i's and the τ_j's are square integrable (unitary), the σ_i's are mutually inequivalent and $0 \leq \gamma_j < \frac{1}{2}$. Moreover, the data $(\sigma_i)_{i=1}^s, (\tau_j, \gamma_j)_{j=1}^t$ are uniquely determined up to permutation. Clearly, π is self-dual if and only if $\{\sigma_i, \tau_j\nu^{\gamma_j}\} = \{\widetilde{\sigma_i}, \widetilde{\tau_j}\nu^{\gamma_j}\}$ as multi-sets. Let $\Pi^{\mathrm{s.d.u.}}$ be the set of self-dual generic irreducible unitarizable representations of GL_n.

Let $\mathcal{S} = \{S_n\}_{n \geq 0}$ be any one of the families $\mathcal{B} = \mathrm{SO}(2n+1)$, $\mathcal{C} = \mathrm{Sp}_n$ or $\mathcal{D} = \mathrm{SO}(2n)$ (with $S_0 = 1$). The family will be fixed throughout. In each case, except for $\mathrm{SO}(2)$, the group $G = S_n$ is semisimple of rank n and we enumerate its simple roots $\{\alpha_1, \dots, \alpha_n\}$ in the standard way. Recall the automorphisms θ and Σ of G defined in the previous section. If π is a representation of G we let

$\theta(\pi)$ be the representation obtained by twisting by θ. Similarly for $\Sigma(\pi)$. We let $\mathrm{Irr}(S_n)$ be the set of equivalence classes of irreducible representations of S_n. Let π_i, $i=1,\dots,k$, be representations of GL_{n_i} and σ a representation of S_m. Let $n=n_1+\ldots+n_k+m$ and Q be the parabolic subgroup of S_n obtained by "deleting" the simple roots $\alpha_{n_1},\alpha_{n_1+n_2},\dots,\alpha_{n_1+\ldots+n_k}$, as well as α_n in the case where $\mathcal{S}=\mathcal{D}$ and $m=1$. The Levi subgroup L of Q is isomorphic to $\mathrm{GL}_{n_1}\times\ldots\times\mathrm{GL}_{n_k}\times S_m$. As in [Tad98] we denote by $\pi_1\times\ldots\times\pi_k\rtimes\sigma$ the representation of S_n induced from the representation $\pi_1\otimes\ldots\otimes\pi_k\otimes\sigma$ of Q. We have, $\pi\times\tau\rtimes\sigma=\pi\rtimes(\tau\rtimes\sigma)$. In the case $\mathcal{S}=\mathcal{D}$ we have $\Sigma(\pi\rtimes\sigma)=\pi\rtimes\Sigma(\sigma)$ for $\pi\in\mathrm{Irr}_n$ and $\sigma\in\mathrm{Irr}(S_m)$ with $m\geq 1$.

Let L be a Levi subgroup of G and let w_0 (resp. w_0^L) be the longest element in the Weyl group of G (resp. L). We denote by w_L the Weyl group element $w_0w_0^L$. In particular w_M is defined, where we recall that $M\simeq\mathrm{GL}_n$ is the Siegel Levi.

Suppose that $\pi_i\in\mathrm{Irr}_{n_i}$ are essentially square integrable with $\mathrm{e}(\pi_1)>\mathrm{e}(\pi_2)>\ldots>\mathrm{e}(\pi_k)>0$ and that $\sigma\in\mathrm{Irr}(S_m)$ is square integrable. Let Q and L be as before. Then

1. $\pi_1\times\ldots\times\pi_k\rtimes\sigma$ admits a unique irreducible quotient.

2. The multiplicity of this quotient in the semi-simplification of $\pi_1\times\ldots\times\pi_k\rtimes\sigma$ is one.

3. The quotient is isomorphic to the image of the (unnormalized) intertwining operator

   $$M_w:\pi_1\times\ldots\times\pi_k\rtimes\sigma\to\Sigma^{n_1+\ldots+n_k}({\pi_1}^\flat\times\ldots\times{\pi_k}^\flat\rtimes\sigma)$$

   with respect to Q and w where $w=w_L$.

4. M_w is given by a convergent integral.

This is the Langlands quotient in this setup. For all this see [BW00]. Let Q' be the parabolic subgroup with Levi subgroup L' isomorphic to $\mathrm{GL}_{n_1+\ldots+n_k}\times S_m$ and let $\pi=\pi_1\times\ldots\times\pi_k$. The operator M_w is obtained as the composition of the intertwining operator

$$\pi\rtimes\sigma\to\Sigma^{n_1+\ldots+n_k}(\pi^\flat\rtimes\sigma) \tag{10}$$

with respect to Q' and $w_{L'}$, and an intertwining operator M_2 "inside" $\mathrm{GL}_{n_1+\ldots+n_k}$. Under the weaker hypothesis that $\mathrm{e}(\pi_1)\geq\ldots\geq\mathrm{e}(\pi_k)>0$ the statements 1–3 will continue to hold provided that M_2 is normalized. This is because the R-groups for general linear groups are trivial. In particular, if π is irreducible then the Langlands quotient is isomorphic to the image of the intertwining operator (10).

If π is a representation of GL_n we let $I(\pi,s)=I^G(\pi,s)$ be the induced representation $\pi\nu^s\rtimes\mathbf{1}$. Similar notation will be used for induction from the parabolic subgroup $\theta(P)$. We denote by $M(\pi,s)=M(s):I(\pi,s)\to I(\pi^\sharp,-s)=\theta(I(\pi^\flat,-s))$ the unnormalized intertwining operator with respect to P and w_M. If π is generic we denote by $R(\pi,s)=R(s)$ the normalized intertwining operator (with respect to P and w_M). In the case $G=\mathrm{SO}(2)$ we set $M(s)=R(s)=\mathbf{1}$.

We will often use the following fact. Suppose that $\pi=\pi_1\times\ldots\times\pi_k$ is a generic representation of GL_n with $\pi_i\in\mathrm{Irr}_{n_i}$. We may identify $I(\pi,s)$ with $\mathrm{Ind}_Q^G(\pi_1\nu^s\otimes\ldots\otimes\pi_k\nu^s)$ where $\pi_1\otimes\ldots\otimes\pi_k$ is viewed as a representation of the parabolic subgroup Q of G whose Levi subgroup is the Levi subgroup of GL_n of type $(n_1,\ldots,n_k)$. We may also identify $\pi^\flat$ with ${\pi_k}^\flat\times\ldots\times{\pi_1}^\flat$ and $I(\pi^\flat,s)$ with $\mathrm{Ind}_{Q^\flat}^G{\pi_k}^\flat\nu^s\otimes\ldots\otimes{\pi_1}^\flat\nu^s$. Under these identifications $R(s)$ becomes the normalized intertwining operator

$$\mathrm{Ind}_Q^G(\pi_1\nu^s\otimes\ldots\otimes\pi_k\nu^s)\to\theta(\mathrm{Ind}_{Q^\flat}^G({\pi_k}^\flat\nu^{-s}\otimes\ldots\otimes{\pi_1}^\flat\nu^{-s}))$$

with respect to Q and w_M. This is merely a reformulation of the multiplicativity of L and ε-factors ([Sha90a]). As a result, we may decompose the operator $R(\pi,s)$ as a product of "basic" intertwining operators according to the reduced decomposition of w_M. Each basic intertwining operator is obtained by inducing an operator of the form $R(\pi_i,s)$ or $\mathcal{M}(\pi_i,\pi_j)$ or $\mathcal{M}(\pi_i,{\pi_j}^\flat)$ with $i>j$.

3.2. *Proof of Lemma* 1. Let $\pi\in\Pi^{\text{s.d.u.}}$ and $\mathrm{Re}(s)\geq\frac{1}{2}$. We may write $\pi\nu^s$ as $\pi_1\times\ldots\times\pi_k$ with π_i essentially square-integrable and $\mathrm{e}(\pi_1)\geq\ldots\geq\mathrm{e}(\pi_k)>0$. Hence $I(\pi,s)$ admits a Langlands quotient, which by the discussion above, is given by the image of $M(\pi,s)$. By multiplicativity, the statements about the L-functions follow from the holomorphy of $L(s,\pi_i)$, $L(s,\pi_i,\mathrm{sym}^2)$, $L(s,\pi_i,\wedge^2)$ and $L(s,\pi_i\otimes\pi_j)$ at $s=0$, which in turn follows from [Sha90b, Prop. 7.2]. The statements about the normalizing factors and $R(\pi,s)$ follow immediately. □

3.3. *Proof of Lemma* 2. Recall the definition of the operators $\mathfrak{B}^\psi(\pi,s)$. (We assume that $\theta=\mathbf{1}$ and that $\pi\in\mathrm{Irr}_n$ is self-dual.) We first note that ι_π is Hermitian since, being an intertwining operator of order two, it must preserve the inner product. We conclude that $\iota_{\pi,s}^*=\iota_{\pi^\sharp,-\overline{s}}$ where * denotes the Hermitian dual. Also, a direct calculation shows the relation

$$M(\pi^\sharp,s)\iota_{\pi^\sharp,s}=\iota_{\pi,-s}M(\pi,s).$$

Moreover, by (2) the Hermitian dual of $M(\pi,s)$ is given by $\omega_\pi(-1)^kM(\pi^\sharp,\overline{s})$ where $k=n$ if G is symplectic and $k=n+1$ if G is orthogonal. On the other hand, by the dependence of root numbers on the additive character it is easily deduced that

$$\overline{m^\psi(\pi,s)}=\omega_\pi(-1)^km^\psi(\pi,\overline{s}).$$

The Hermitian property of $\mathfrak{B}^\psi(\pi,s)$ for s real follows.

To prove the second part we use the argument of [KS88, Prop. 6.3]. Let $W_\pi^\psi(\cdot,s)$ be the Whittaker functional on $I(\pi,s)$ and let $W_{\pi^\sharp}^\psi(\cdot,s)$ be the Whittaker functional on $I(\pi^\sharp,s)$ obtained through ι_π. They are holomorphic, nonzero ([Sha81]), and satisfy the functional equation

$$W_\pi^\psi(\varphi,s)=c(\pi,s,\psi)W_{\pi^\sharp}^\psi(M(\pi,s)\varphi,-s) \tag{11}$$

where $c^\psi(\pi,s)$ is the "local coefficient" which was studied by Shahidi. By [Sha91, Th. 3.5] it is given by

$$c^\psi(\pi,s)=\begin{cases}\frac{\varepsilon(2s,\pi,r,\psi^{-1})L(1-2s,\pi,r)}{L(2s,\pi,r)} & \mathcal{S}=\mathcal{B},\mathcal{D}\\ \frac{\varepsilon(s,\pi,\psi^{-1})L(1-s,\pi)}{L(s,\pi)}\cdot\frac{\varepsilon(2s,\pi,r,\psi^{-1})L(1-2s,\pi,r)}{L(2s,\pi,r)} & \mathcal{S}=\mathcal{C}\end{cases} \tag{12}$$

where $r=\mathrm{sym}^2$ for $\mathcal{S}=\mathcal{B}$ and $r=\wedge^2$ for $\mathcal{S}=\mathcal{C},\mathcal{D}$. (Here we use that π is self-dual.) By the identification $\iota_\pi:\pi^\sharp\to\pi$, (11) becomes

$$W_\pi^\psi(\varphi,s)=c^\psi(\pi,s)m^\psi(\pi,s)W_\pi^\psi(\mathfrak{B}^\psi(\pi,s)\varphi,-s).$$

The term $c^\psi(\pi,s)m^\psi(\pi,s)$ is either $L(1-2s,\pi,r)/L(1+2s,\pi,r)$ if $\mathcal{S}=\mathcal{B},\mathcal{D}$ or $L(1-2s,\pi,r)/L(1+2s,\pi,r)\cdot L(1-s,\pi)/L(1+s,\pi)$ if $\mathcal{S}=\mathcal{C}$. It follows that

$$\mathfrak{B}^\psi(\pi,-s)\mathfrak{B}^\psi(\pi,s)=I.$$

We infer that $\mathfrak{B}^\psi(\pi,s)$ is unitary, and in particular, holomorphic at $s=0$. Moreover, $\mathfrak{B}^\psi(\pi,0)$ fixes the ψ-generic irreducible constituent of $I(\pi,0)$, since $L(s,\pi)$ and $L(s,\pi,r)$ are holomorphic at $s=1$ by Lemma 1.

The rest of the paper is devoted to the proof of Lemma 3. Since the lemma is evidently independent of the choice of the character ψ, we will suppress it from the notation.

3.4. *Representations of G-type.* Let σ be a self-dual square-integrable representation of GL_n and suppose that $\theta=\mathbf{1}$. By the theory of R-groups (e.g. [Gol94]) the following conditions are equivalent.

1. $I(\sigma,0)$ is irreducible.
2. $\mathfrak{B}(\sigma,0)$ is a scalar.
3. The Plancherel measure $\mu(\sigma,s)$ is zero at $s=0$.

Definition 1. An essentially square-integrable representation σ of GL_n will be called of *G-type* (or of $\mathcal{S}$*-type* if we do not want to specify n) if it is self-dual (in particular, $\mathrm{e}(\sigma)=0$), $\theta=\mathbf{1}$, and the conditions above are satisfied.

PROPOSITION 1. *Let σ be a square-integrable representation of GL_n. Then $I^G(\sigma,s)$ is irreducible for $0<s<1$ except possibly for $s=\frac{1}{2}$. Moreover, if $I(\sigma,\frac{1}{2})$ is reducible then σ is of G-type.*

Proof. By the results of Muic ([Mui01]) we may use Proposition 5.3 of [CS98]. Thus the reducibility points of $I(\sigma, s)$ for $s > 0$ are the poles of $L(1-2s, \sigma, r)$ (if $\mathcal{S} = \mathcal{B}$ or $\mathcal{D}$) or $L(1-s, \sigma)L(1-2s, \sigma, r)$ (if $\mathcal{S} = \mathcal{C}$). These L-functions are computed in [Sha92, Prop. 8.1]. In particular, $L(s, \sigma, r)$ is holomorphic for $s > -1$ except possibly for $s = 0$ and $L(s, \sigma)$ is holomorphic for $s > 0$. Therefore $I(\sigma, s)$ is irreducible for $0 < s < 1$ except possibly for $s = \frac{1}{2}$ and moreover, if $I(\sigma, \frac{1}{2})$ is reducible then $L(s, \sigma, r)$ has a pole at $s = 0$. In the latter case $\theta = \mathbf{1}$, σ is self-dual and the local coefficient vanishes at 0 (loc. cit.). By [Sha90b, (1.4)] the same will be true for the Plancherel measure. $\square$

Remark 1. Shahidi also proved the following in ([Sha92]). Suppose that σ is a self-dual square-integrable representation of GL_n which is not the trivial character of GL_1. Then the following are equivalent:

1. σ is of Sp_n type.
2. σ is not of $\mathrm{SO}(2n+1)$ type.
3. σ is of $\mathrm{SO}(2n)$ type.

In particular, in this case n must be even. We will not use this fact.

For convenience, we consider the set $\Pi^{\text{s.d.}}$ of all representations of the form

$$\sigma_1 \times \ldots \times \sigma_s \times \tau_1 \times {\tau_1}^\flat \times \ldots \times \tau_t \times {\tau_t}^\flat \tag{13}$$

where the σ_i's are square-integrable, self-dual and (as we may assume) mutually inequivalent, and the τ_j's are essentially square-integrable with $0 \le \mathrm{e}(\tau_j) < \frac{1}{2}$. Any element of $\Pi^{\text{s.d.}}$ is irreducible, generic and self-dual. Clearly, $\Pi^{\text{s.d.}} \supset \Pi^{\text{s.d.u.}}$. The condition on $\pi \in \Pi^{\text{s.d.}}$ to belong to $\Pi^{\text{s.d.u.}}$ (i.e. to be unitarizable) is that each τ_j which is not square-integrable appears in (13) the same number of times as ${\tau_j}^\flat \nu^{2\mathrm{e}(\tau_j)}$. If $\pi \in \Pi^{\text{s.d.}}$ then by the discussion of subsection 3.2, $I(\pi, \frac{1}{2})$ admits a Langlands quotient, which will be denoted by $\mathcal{LQ}(\pi)$. It is obtained as the image of $R(\pi, \frac{1}{2})$ (or $M(\pi, \frac{1}{2})$).

Also, if χ is an essentially square-integrable representation of GL_n we denote by $\mathcal{SP}(\chi)$ the unique irreducible quotient of $\chi\nu^{\frac{1}{2}} \times \chi\nu^{-\frac{1}{2}}$. It is isomorphic to the image of the intertwining operator $\mathcal{M}(\chi\nu^{\frac{1}{2}}, \chi\nu^{-\frac{1}{2}})$.

Lemma 5. *Let χ be an essentially square-integrable representation of GL_n with $0 \le \mathrm{e}(\chi) < \frac{1}{2}$. Assume that χ is not of $\mathcal{S}$-type. Then $\mathcal{LQ}(\chi \times \chi^\flat) \simeq \Sigma^n(\mathcal{SP}(\chi) \rtimes \mathbf{1})$.*

Proof. The Langlands quotient is obtained as the image of the longest intertwining operator, which is the composition of the following intertwining operators:

$$I\left(\chi\times\chi^\flat,\frac{1}{2}\right)\xrightarrow{\mathbf{1}\rtimes R(\chi^\flat,\frac{1}{2})}\chi\nu^{\frac{1}{2}}\rtimes\Sigma^n\left(\chi\nu^{-\frac{1}{2}}\rtimes\mathbf{1}\right)\simeq\Sigma^n\left(\chi\nu^{\frac{1}{2}}\times\chi\nu^{-\frac{1}{2}}\rtimes\mathbf{1}\right)$$
$$\xrightarrow{\Sigma^n(I(R_1,0))}\Sigma^n\left(\chi\nu^{-\frac{1}{2}}\times\chi\nu^{\frac{1}{2}}\rtimes\mathbf{1}\right)\simeq\chi\nu^{-\frac{1}{2}}\rtimes\Sigma^n\left(\chi\nu^{\frac{1}{2}}\rtimes\mathbf{1}\right)$$
$$\xrightarrow{\mathbf{1}\rtimes\Sigma^n(R(\chi,\frac{1}{2}))}\chi\nu^{-\frac{1}{2}}\times\chi^\flat\nu^{-\frac{1}{2}}\rtimes\mathbf{1}$$

where $R_1=\mathcal{M}(\chi\nu^{\frac{1}{2}},\chi\nu^{-\frac{1}{2}})$. By Proposition 1 the only map which is not an isomorphism is $\Sigma^n(I(R_1,0))$, whose image is $\Sigma^n(\mathcal{SP}(\chi)\rtimes\mathbf{1})$ as required. □

Any $\pi\in\Pi^{\text{s.d.}}$ can be written uniquely as $\pi^{\text{non-}\mathcal{S}\text{-type}}\times\pi^{\text{non-}\mathcal{S}\text{-pairs}}\times\pi^{\text{pure-}\mathcal{S}\text{-type}}$ with

- $\pi^{\text{pure-}\mathcal{S}\text{-type}}$ of the form $\sigma_1\times\ldots\times\sigma_s$ where the σ_i's are square-integrable, self-dual and of $\mathcal{S}$-type;
- $\pi^{\text{non-}\mathcal{S}\text{-type}}$ of the form $\rho_1\times\ldots\times\rho_r$ where the ρ_i's are square-integrable, mutually inequivalent, self-dual and not of $\mathcal{S}$-type;
- $\pi^{\text{non-}\mathcal{S}\text{-pairs}}$ of the form $\tau_1\times{\tau_1}^\flat\times\ldots\times\tau_t\times{\tau_t}^\flat$ where the τ_j's are essentially square-integrable, not of $\mathcal{S}$-type (self-dual or not), and $0\le\mathrm{e}(\tau_j)<\frac{1}{2}$.

Note that $\pi^{\text{non-}\mathcal{S}\text{-type}}$ and $\pi^{\text{pure-}\mathcal{S}\text{-type}}$ are tempered.

Definition 2. We say that $\pi\in\Pi^{\text{s.d.}}$ is of G-type if $\pi^{\text{non-}\mathcal{S}\text{-type}}=0$.

The definition is suggested by the local Langlands reciprocity. Note that if π is of $\mathrm{SO}(2n)$ type then n is even.

The crucial property of representations of G-type is the following.

LEMMA 6. *If $\pi\in\Pi^{\text{s.d.}}$ is of G-type then $\mathfrak{B}(0)$ is a nonzero scalar.*

Proof. We use induction on n, the case $n=0$ being trivial. For the induction step, we can assume that $\pi=\pi'\times\omega$ where $\pi'\in\Pi^{\text{s.d.}}$ is of $\mathcal{S}$-type and $\omega\in\mathrm{Irr}_l$ is either square-integrable and of $\mathcal{S}$-type or of the form $\tau\times\tau^\flat$ where $\tau\in\mathrm{Irr}_m$ is essentially square-integrable. Note that l is even if $\mathcal{S}=\mathcal{D}$. The operator $R(0)$ can be written as the composition of the following intertwining operators:

$$I(\pi'\times\omega,0)\xrightarrow{\mathbf{1}\rtimes R(\omega,0)}I(\pi'\times\omega^\flat,0)\xrightarrow{I(R_1,0)}$$
$$I(\omega^\flat\times\pi',0)\xrightarrow{\mathbf{1}\rtimes R(\pi',0)}I(\omega^\flat\times{\pi'}^\flat,0)\simeq I((\pi'\times\omega)^\flat,0)\tag{14}$$

where $R_1=\mathcal{M}(\pi',\omega^\flat)$. The last identification is induced by the isomorphism $\omega^\flat\times{\pi'}^\flat\simeq(\pi'\times\omega)^\flat$. By the induction hypothesis the third map is a nonzero scalar multiple of $\mathbf{1}\rtimes\iota(\pi',0)^{-1}$. Also, by uniqueness, $\iota_\pi : \omega^\flat \times \pi'^\flat \to \pi$ is a scalar multiple of $(\mathbf{1}\times\iota_\omega)R_1^{-1}(\mathbf{1}\times\iota'_\pi)$. All in all, the map $\mathfrak{B}(\pi,0)$ is a scalar multiple of

$$I((\mathbf{1}\times\iota_\omega)R_1^{-1}(\mathbf{1}\times\iota'_\pi),0)\circ\mathbf{1}\rtimes\iota(\pi',0)^{-1}\circ I(R_1,0)\circ\mathbf{1}\rtimes R(\omega,0)=\mathbf{1}\rtimes\mathfrak{B}(\omega,0).$$

It remains to show that $\mathfrak{B}(\omega,0)$ is a scalar in the two cases above. In the first case, this follows from the definition of $\mathcal{S}$-type. In the second case, we decompose $R(\omega,0)$ as before as

$$\begin{aligned} I(\tau\times\tau^\flat,0) &\xrightarrow{\mathbf{1}\rtimes R(\tau^\flat,0)} \Sigma^m(I(\tau\times\tau,0)) \xrightarrow{\Sigma^m(I(R_2,0))} \\ &\Sigma^m(I(\tau\times\tau,0)) \xrightarrow{\mathbf{1}\rtimes\Sigma^m(R(\tau,0))} I(\tau\times\tau^\flat,0)\simeq I((\tau\times\tau^\flat)^\flat,0). \end{aligned}$$

Note that the map $R_2=\mathcal{M}(\tau,\tau)$ is a scalar, and similarly for the map $\iota_\omega:\tau\times\tau^\flat\simeq(\tau\times\tau^\flat)^\flat\to\tau\times\tau^\flat$. Thus, $\mathfrak{B}(\omega,0)$ is a scalar multiple of $\mathbf{1}\rtimes\Sigma^m(R(\tau,0))\circ R(\tau^\flat,0)$ which is $\mathbf{1}$ by the properties of the normalized intertwining operator. □

Remark 2. The converse to Lemma 6 is also true.

3.5. *Langlands quotient.*

We extract a few results from [MW89] (cf. I.6.3 for the p-adic case and I.7 for the archimedean case).

LEMMA 7. *Let π and π' be irreducible representations of* GL_n *and* $\mathrm{GL}_{n'}$ *respectively.*

1. *If $\pi\times\pi'$ is irreducible then $\pi\times\pi'\simeq\pi'\times\pi$.*
2. *Let π and π' be essentially square-integrable. Suppose that* $|\mathrm{e}(\pi)-\mathrm{e}(\pi')|<1$. *Then $\pi\times\pi'$, $\pi\nu^{\frac12}\times\mathcal{SP}(\pi')$ and $\mathcal{SP}(\pi)\times\mathcal{SP}(\pi')$ are irreducible.*
3. *Suppose that π and π' are inequivalent square-integrable representations. Then $\pi\nu^\gamma\times\pi'\nu^{-\frac12}$ is irreducible for $-1<\gamma<1$.*

We will also need the following lemma which is based on [Jan96].

LEMMA 8. *Let $\pi_i\in\mathrm{Irr}_{n_i}$ for $i=1,\dots,k$ and $\sigma\in\mathrm{Irr}(S_m)$. Suppose that $\pi_i\times\pi_j$, $\pi_i\times{\pi_j}^\flat$ are irreducible for all $i\neq j$ and $\pi_i\rtimes\sigma$ is irreducible for all i. Then*

$$\pi_1\times\ldots\times\pi_k\rtimes\sigma\simeq\Sigma^{n_1+\ldots+n_k}({\pi_1}^\flat\times\ldots\times{\pi_k}^\flat\rtimes\sigma). \tag{15}$$

Suppose in addition that the π_i's are essentially square-integrable with $\mathrm{e}(\pi_i)>0$ *and σ is square-integrable. Then $\pi_1\times\ldots\times\pi_k\rtimes\sigma$ is irreducible.*

Proof. In the case where $k = 1$ we note that if $\pi \in \mathrm{Irr}_n$ and $\sigma \in \mathrm{Irr}(S_m)$ then $\pi \rtimes \sigma = \Sigma^n(\pi^\flat \rtimes \sigma)$ in the Grothendieck group since $\pi \otimes \sigma$ and $\Sigma^n(\pi^\flat \otimes \sigma)$ are associate. The case $k > 1$ and the last statement are proved in ([Jan96]) for the cases $\mathcal{S} = \mathcal{B}, \mathcal{C}$. The proof carries over almost literally (except for putting in some Σ's) to the case $\mathcal{S} = \mathcal{D}$ (cf. Proposition 2 below). □

Let $\pi \in \Pi^{\text{s.d.}}$. Recall that $I(\pi, \frac{1}{2})$ admits a Langlands quotient, denoted by $\mathcal{LQ}(\pi)$, which is isomorphic to the image under $R(\pi, \frac{1}{2})$.

PROPOSITION 2. *Let $\pi = \pi^{\text{non-}\mathcal{S}\text{-type}} \times \pi^{\text{non-}\mathcal{S}\text{-pairs}} \times \pi^{\text{pure-}\mathcal{S}\text{-type}} \in \Pi^{\text{s.d.}}$ be as above. Then*

$$\mathcal{LQ}(\pi) \simeq \Sigma^\varepsilon(\mathcal{SP}(\tau_1)\times\ldots\times\mathcal{SP}(\tau_t)\times\pi^{\text{non-}\mathcal{S}\text{-type}}\nu^{\frac{1}{2}}\rtimes\mathcal{LQ}(\pi^{\text{pure-}\mathcal{S}\text{-type}})) \tag{16}$$

for ε either 0 *or* 1 (*depending only on $\pi^{\text{non-}\mathcal{S}\text{-pairs}}$*). *Hence*,

$$\mathcal{LQ}(\pi) \simeq \pi^{\text{non-}\mathcal{S}\text{-type}}\nu^{\frac{1}{2}}\rtimes\mathcal{LQ}(\pi^{\text{non-}\mathcal{S}\text{-pairs}} \times \pi^{\text{pure-}\mathcal{S}\text{-type}}).$$

Proof. Clearly, the second statement follows from the first. Let Λ be the right-hand side of (16). Following the argument of [Jan96, Th. 3.3] we will argue that

$$\Lambda \text{ is a quotient of } I(\pi, \tfrac{1}{2}) \text{ for } \varepsilon \text{ either 0 or 1.} \tag{17}$$

$$\Lambda \text{ is irreducible.} \tag{18}$$

The first statement is proved by induction on n, as in subsection 3.4. Since the case where $\pi^{\text{non-}\mathcal{S}\text{-pairs}} = 0$ is immediate, we may assume for the induction step that $\pi = \pi' \times \tau \times \tau^\flat$ where $\pi' \in \Pi^{\text{s.d.}}$, $\tau \in \mathrm{Irr}_m$ is essentially square-integrable, $0 \le \mathrm{e}(\tau) < \frac{1}{2}$ and τ is not of $\mathcal{S}$-type.

It follows from Lemma 5 that up to Σ, $I(\pi, \frac{1}{2})$ has a quotient isomorphic to $I(\pi'\nu^{\frac{1}{2}} \times \mathcal{SP}(\tau), 0)$. It follows from part 2 of Lemma 7 that $\pi'\nu^{\frac{1}{2}} \times \mathcal{SP}(\tau)$ is irreducible, and hence, that $\pi'\nu^{\frac{1}{2}} \times \mathcal{SP}(\tau) \simeq \mathcal{SP}(\tau) \times \pi'\nu^{\frac{1}{2}}$. We deduce that up to Σ, $I(\pi, \frac{1}{2})$ admits $I(\mathcal{SP}(\tau) \times \pi'\nu^{\frac{1}{2}}, 0)$, and thus also $\mathcal{SP}(\tau)\rtimes\mathcal{LQ}(\pi')$, as a quotient. This implies (17) by the induction hypothesis.

To prove (18), it suffices to show that $\theta(\Lambda) \simeq \widetilde{\Lambda}$. (This condition does not depend on ε.) Indeed, we have $\theta(\mathcal{LQ}(\pi)) \simeq \widetilde{\mathcal{LQ}(\pi)}$ (cf. [Jan96]) since both sides are the unique irreducible subrepresentation of $I(\widetilde{\pi}, -\frac{1}{2})$ by (3). We would conclude that $\mathcal{LQ}(\pi)$ is both a quotient and a subrepresentation of Λ. However, $\mathcal{LQ}(\pi)$ is the unique irreducible quotient of Λ, and it has multiplicity-one in the semi-simplification of Λ. Thus, $\Lambda \simeq \mathcal{LQ}(\pi)$.

We shall write π_3 for $\pi^{\text{pure-}\mathcal{S}\text{-type}}$. To show that $\theta(\Lambda) \simeq \widetilde{\Lambda}$ we note once more that $\widetilde{\mathcal{LQ}(\pi_3)} = \mathcal{LQ}(\pi_3)$. By Lemmas 7 and 8 it suffices to show that both $\rho\nu^{\frac{1}{2}}\rtimes\mathcal{LQ}(\pi_3)$ and $\mathcal{SP}(\tau)\rtimes\mathcal{LQ}(\pi_3)$ are irreducible where $\rho \in \mathrm{Irr}_l$ is square-integrable self-dual not of $\mathcal{S}$-type and τ is as before.

To prove this, consider the representation $\pi' = \tau \times \tau^\flat \times \pi_3$. The Langlands quotient of $I(\pi', \frac{1}{2})$ is the image of the operator M_{w_0} which is the composition of the intertwining operators

$$\begin{aligned}
& I\left(\tau\nu^{\frac{1}{2}} \times \tau^\flat\nu^{\frac{1}{2}} \times \pi_3\nu^{\frac{1}{2}}, 0\right) \\
\longrightarrow\ & I\left(\tau\nu^{\frac{1}{2}} \times \pi_3\nu^{\frac{1}{2}} \times \tau^\flat\nu^{\frac{1}{2}}, 0\right) \\
\longrightarrow\ & \Sigma^m\left(I\left(\tau\nu^{\frac{1}{2}} \times \pi_3\nu^{\frac{1}{2}} \times \tau\nu^{-\frac{1}{2}}, 0\right)\right) \\
\longrightarrow\ & \Sigma^m\left(I\left(\tau\nu^{\frac{1}{2}} \times \tau\nu^{-\frac{1}{2}} \times \pi_3\nu^{\frac{1}{2}}, 0\right)\right) \\
\longrightarrow\ & \Sigma^m\left(I\left(\tau\nu^{-\frac{1}{2}} \times \tau\nu^{\frac{1}{2}} \times {\pi_3}^\flat\nu^{-\frac{1}{2}}, 0\right)\right) \\
\longrightarrow\ & \Sigma^m\left(I\left(\tau\nu^{-\frac{1}{2}} \times {\pi_3}^\flat\nu^{-\frac{1}{2}} \times \tau\nu^{\frac{1}{2}}, 0\right)\right) \\
\longrightarrow\ & I\left(\tau\nu^{-\frac{1}{2}} \times {\pi_3}^\flat\nu^{-\frac{1}{2}} \times \tau^\flat\nu^{-\frac{1}{2}}, 0\right) \\
\longrightarrow\ & I\left(\tau\nu^{-\frac{1}{2}} \times \tau^\flat\nu^{-\frac{1}{2}} \times {\pi_3}^\flat\nu^{-\frac{1}{2}}, 0\right) \\
\longrightarrow\ & I\left(\tau^\flat\nu^{-\frac{1}{2}} \times \tau\nu^{-\frac{1}{2}} \times {\pi_3}^\flat\nu^{-\frac{1}{2}}, 0\right).
\end{aligned}$$

Again by Lemma 7 and Proposition 1, all arrows except the fourth one are isomorphisms. Thus, the Langlands quotient is isomorphic to the image of the fourth map, which is $\Sigma^m(\mathcal{SP}(\tau) \rtimes \mathcal{LQ}(\pi_3))$. Hence, the latter is irreducible. Similarly, if $\pi' = \rho \times \pi_3$ then $\mathcal{LQ}(\pi')$ is the image of the composition of the intertwining operators

$$\begin{aligned}
& I\left(\rho\nu^{\frac{1}{2}} \times \pi_3\nu^{\frac{1}{2}}, 0\right) \\
\longrightarrow\ & I\left(\rho\nu^{\frac{1}{2}} \times {\pi_3}^\flat\nu^{-\frac{1}{2}}, 0\right) \\
\longrightarrow\ & I\left({\pi_3}^\flat\nu^{-\frac{1}{2}} \times \rho\nu^{\frac{1}{2}}, 0\right) \\
\longrightarrow\ & \Sigma^l\left(I\left({\pi_3}^\flat\nu^{-\frac{1}{2}} \times \rho^\flat\nu^{-\frac{1}{2}}, 0\right)\right).
\end{aligned}$$

Again, all maps except the first are isomorphisms. Thus, as before, $\rho\nu^{\frac{1}{2}} \rtimes \mathcal{LQ}(\pi_3)$ is irreducible. □

For future reference, let us reformulate the conclusion of Proposition 2. Using a decomposition of w_0 we may decompose $R(\pi, \frac{1}{2})$ as

(19)

$$\begin{aligned}
& I\left(\pi, \frac{1}{2}\right) = I\left(\mathsf{X}_{i=1}^r\left(\tau_i\nu^{\frac{1}{2}} \times {\tau_i}^\flat\nu^{\frac{1}{2}}\right) \times \pi^{\text{non-}\mathcal{S}\text{-type}}\nu^{\frac{1}{2}} \times \pi^{\text{pure-}\mathcal{S}\text{-type}}\nu^{\frac{1}{2}}, 0\right) \\
& \xrightarrow{R_{w_1}} \Sigma^\varepsilon\left(I\left(\mathsf{X}_{i=1}^r\left(\tau_i\nu^{\frac{1}{2}} \times \tau_i\nu^{-\frac{1}{2}}\right) \times \pi^{\text{non-}\mathcal{S}\text{-type}}\nu^{\frac{1}{2}} \times \pi^{\text{pure-}\mathcal{S}\text{-type}}\nu^{\frac{1}{2}}, 0\right)\right) \\
& \xrightarrow{R_{w_2}} \Sigma^\varepsilon\left(I\left(\mathsf{X}_{i=1}^r\left(\tau_i\nu^{-\frac{1}{2}} \times \tau_i\nu^{\frac{1}{2}}\right) \times \pi^{\text{non-}\mathcal{S}\text{-type}}\nu^{\frac{1}{2}} \times \pi^{\text{pure-}\mathcal{S}\text{-type}^\flat}\nu^{-\frac{1}{2}}, 0\right)\right)
\end{aligned}$$

$$\xrightarrow{R_{w_3}} \Sigma^{\varepsilon'}\left(I\left(\mathsf{X}_{i=1}^{r}\left(\tau_i^{\flat}\nu^{-\frac{1}{2}}\times\tau_i\nu^{-\frac{1}{2}}\right)\times\pi^{\text{non-}\mathcal{S}\text{-type}^{\flat}}\nu^{-\frac{1}{2}}\times\pi^{\text{pure-}\mathcal{S}\text{-type}^{\flat}}\nu^{-\frac{1}{2}},0\right)\right)$$
$$=\theta\left(I\left(\pi,-\frac{1}{2}\right)\right)$$

where R_{w_i} are normalized intertwining operators. We observe that the image of R_{w_2} (of the whole induced space) is isomorphic to the right-hand side of (16), and hence it is the Langlands quotient. By irreducibility and multiplicity-one of Langlands quotient $\mathrm{im}(R_{w_2}\circ R_{w_1})=\mathrm{im}(R_{w_2})$ and $\ker(R_{w_3}\circ R_{w_2})=\ker(R_{w_2})$.

3.6. *Reduction to the tempered case.* Let $\pi\in\Pi^{\text{s.d.u.}}$. We may write $\pi=\pi^{\text{temp}}\times\pi^{\text{n.t.}}$ where $\pi^{\text{temp}}\in\Pi^{\text{s.d.u.}}$ is tempered and $\pi^{\text{n.t.}}$ is of the form $\mathsf{X}_i(\omega_i\nu^{\beta_i}\times{\omega_i}^{\flat}\nu^{-\beta_i})$ with ω_i square-integrable and $0<\beta_i<\frac{1}{2}$. Clearly, $\pi^{\text{n.t.}}$ appears as a factor of $\pi^{\text{non-}\mathcal{S}\text{-pairs}}$. We will deform the nontempered parameters of π. For $0\le t\le 1$ let

$$\pi_t=\pi^{\text{temp}}\times\mathsf{X}_i\left(\omega_i\nu^{t\beta_i}\times{\omega_i}^{\flat}\nu^{-t\beta_i}\right)=\pi^{\text{temp}}\times\pi_t^{\text{n.t.}}.$$

Then π_t is a "deformation" in $\Pi^{\text{s.d.u.}}$ from $\pi\simeq\pi_1$ to the tempered representation π_0. Clearly $\pi_t^{\text{non-}\mathcal{S}\text{-type}}=\pi^{\text{non-}\mathcal{S}\text{-type}}$ for all t and $\pi_t^{\text{pure-}\mathcal{S}\text{-type}}=\pi^{\text{pure-}\mathcal{S}\text{-type}}$ for $t\neq 0$ although not necessarily for $t=0$. The form $\mathfrak{I}(\pi,s)$ depends on the unitary structure on π, or what amounts to the same, on a GL_n-invariant positive-definite Hermitian form on π. We identify the ambient vector spaces of π_t with that of π in the usual way. The K-action does not depend on t, where K denotes the standard maximal compact. We may choose a family of GL_n-invariant positive-definite Hermitian forms on π_t which depends continuously on t (using intertwining operators for example).

The following lemma will reduce Lemma 3 to the tempered case.

LEMMA 9. 1. *The definiteness of* $\mathfrak{I}(\pi_t,0)$ *does not depend on* t.

2. *If* $\mathfrak{I}(\pi,\frac{1}{2})$ *is semi-definite then* $\mathfrak{I}(\pi_0,\frac{1}{2})$ *is semi-definite with the same sign.*

We will use the following elementary lemma.

LEMMA 10. *Let* $\{l_\beta\}_{a\le\beta\le b}$ *be a continuous family of Hermitian forms on* $\mathbb{C}^m$. *Suppose that* $\mathrm{rank}(l_\beta)$ *is constant for* $a<\beta\le b$ *and that* l_b *is positive semi-definite. Then* l_a *is positive semi-definite.*

Indeed, both parameters of the signature $(s_+(\beta),s_-(\beta))$ of l_β are lower semi-continuous functions. By the conditions of the lemma, $s_+(\beta)+s_-(\beta)$ is constant on $(a,b]$, and hence the same is true for $s_\pm(\beta)$.

Proof of Lemma 9. Since $R(\pi_t,0)$ is invertible, $\mathfrak{I}(\pi_t,0)$ is a nondegenerate Hermitian form on $I(\pi_t,0)$ for any t. Thus, the first statement follows from Lemma 10, after passing to any K-type.

To prove the second part, we will apply the discussion following Proposition 2 to the representations π_t. We may identify all the induced spaces in (19) with the ones for $t = 0$ in the usual manner. The K-action will be independent of t. We obtain a decomposition of the operator $\iota(\pi_t, -\frac{1}{2})R(\pi_t, \frac{1}{2})$ defining the form $\mathfrak{I}(\pi_t, \frac{1}{2})$ as $C_t \circ B \circ A_t$ such that for $t \neq 0$ we have $\operatorname{im}(B \circ A_t) = \operatorname{im}(B)$ and $\ker(C_t \circ B) = \ker(B)$. The crucial point is that the operator B (denoted by R_{w_2} in (19) does not depend on t. Thus on each K-type of $I(\pi_t, \frac{1}{2})$ the rank of $\mathfrak{I}(\pi_t, \frac{1}{2})$ is equal to the rank of B, as long as $t \neq 0$. Thus, we may apply Lemma 10 to conclude the second statement of the lemma. □

3.7. *The tempered Case.* We continue the proof of Lemma 3. By virtue of the last section, we may assume that π is tempered. In this case, the representations $I(\pi, s)$ are irreducible for $0 < s < \frac{1}{2}$ by Lemma 8 and Proposition 1. Thus $\mathfrak{I}(\pi, s)$ is nondegenerate for $0 < s < \frac{1}{2}$. We will show below that if $\mathfrak{I}(\pi, \frac{1}{2})$ is semi-definite then π is of G-type. Then by Lemma 6, $\mathfrak{I}(\pi, 0)$ is definite. We may use Lemma 10 on each K-type to conclude Lemma 3.

It remains to show that π is of G-type if $\pi \in \Pi^{\text{s.d.u.}}$ is tempered and $\mathfrak{I}(\pi, \frac{1}{2})$ is semi-definite. To shorten notation, let $\pi_1 = \pi^{\text{non-}\mathcal{S}\text{-pairs}} \times \pi^{\text{pure-}\mathcal{S}\text{-type}} \in \Pi^{\text{s.d.u.}}$ and $\pi_2 = \pi^{\text{non-}\mathcal{S}\text{-type}}$ so that $\pi = \pi_1 \times \pi_2$. Note that π_1 is of G-type and hence $\mathfrak{B}(\pi_1, 0)$ is a scalar by Lemma 6. Since $I(\pi_1, s)$ is irreducible for $0 < s < \frac{1}{2}$ it follows from Lemma 10 that

$$\mathfrak{B}\left(\pi_1, \frac{1}{2}\right) \text{ is semi-definite.} \tag{20}$$

We need to show that $\pi_2 = 0$. Consider the family

$$I\left(\pi_1 \otimes \pi_2, \left(\frac{1}{2}, \gamma\right)\right) := I\left(\pi_1 \nu^{\frac{1}{2}} \times \pi_2 \nu^{\gamma}, 0\right).$$

Let

$$\mathfrak{B}'(\gamma) : I\left(\pi_1 \otimes \pi_2, \left(\frac{1}{2}, \gamma\right)\right) \to I\left(\pi_1 \otimes \pi_2, \left(-\frac{1}{2}, -\gamma\right)\right)$$

be the operator $\kappa(\gamma) \circ \mathcal{R}(\gamma)$ where

$$\mathcal{R}(\gamma) : I\left(\pi_1 \otimes \pi_2, \left(\frac{1}{2}, \gamma\right)\right) \longrightarrow I\left({\pi_1}^\flat \otimes {\pi_2}^\flat, \left(-\frac{1}{2}, -\gamma\right)\right)$$

is the normalized intertwining operator and

$$\kappa(\gamma) = I\left(\iota_{\pi_1} \otimes \iota_{\pi_2}, \left(-\frac{1}{2}, -\gamma\right)\right) : I\left({\pi_1}^\flat \otimes {\pi_2}^\flat, \left(-\frac{1}{2}, -\gamma\right)\right) \longrightarrow I\left(\pi_1 \otimes \pi_2, \left(-\frac{1}{2}, -\gamma\right)\right).$$

As usual we identify the underlying K-module of each family of induced representations, so that it does not depend on γ. The same argument as in Proposition 2 with the exponent $\gamma > 0$ instead of $\frac{1}{2}$ gives:

Proposition 3. *If $\gamma > 0$ then $I\left(\pi_1 \otimes \pi_2, \left(\frac{1}{2}, \gamma\right)\right)$ admits a Langlands quotient which is given by the image of $\mathcal{R}(\gamma)$. It is isomorphic to $\pi_2\nu^\gamma \times \mathcal{LQ}(\pi_1)$.*

The operator $\mathcal{R}(\gamma)$ can be written as the composition of the intertwining maps

$$\begin{array}{rl} & I\left(\pi_1 \otimes \pi_2, \left(\frac{1}{2}, \gamma\right)\right) \\ \longrightarrow & I\left({\pi_1}^\flat \otimes \pi_2, \left(-\frac{1}{2}, \gamma\right)\right) \\ \xrightarrow{\mathbf{1} \rtimes R(\pi_2,\gamma)} & I\left({\pi_1}^\flat \otimes {\pi_2}^\flat, \left(-\frac{1}{2}, -\gamma\right)\right) \end{array}$$

where the first map is

$$I\left(\mathcal{M}\left(\pi_2\nu^\gamma, {\pi_1}^\flat\nu^{-\frac{1}{2}}\right), 0\right) \circ \left(\mathbf{1} \rtimes R\left(\pi_1, \frac{1}{2}\right)\right) \circ I\left(\mathcal{M}\left(\pi_1\nu^{\frac{1}{2}}, \pi_2\nu^\gamma\right), 0\right).$$

As before, the intermediate map $\mathbf{1} \rtimes R(\pi_1, \frac{1}{2})$, which does not depend on γ, already gives the Langlands quotient as its image (on the full induced representation) for $\gamma > 0$. Hence the rank of $\mathfrak{B}'(\gamma)$ on each K-type is independent of γ. Since $\mathfrak{B}'(\frac{1}{2}) = \mathfrak{B}(\frac{1}{2})$ we conclude by Lemma 10 that $\mathfrak{B}'(0)$ is a semi-definite operator.

Now,

$$\begin{aligned} \kappa(0) \circ \mathbf{1} \rtimes R(\pi_2, 0) &= \iota_{\pi_1}\nu^{-\frac{1}{2}} \times \iota_{\pi_2} \rtimes \mathbf{1} \circ \mathbf{1} \rtimes R(\pi_2, 0) \\ &= \iota_{\pi_1}\nu^{-\frac{1}{2}} \rtimes \mathfrak{B}(\pi_2, 0) = \mathbf{1} \rtimes \mathfrak{B}(\pi_2, 0) \circ \iota_{\pi_1}\nu^{-\frac{1}{2}} \rtimes \mathbf{1}. \end{aligned}$$

Also, since $\pi_2 \times {\pi_1}^\flat\nu^{-\frac{1}{2}}$ is irreducible,

$$\left(\iota_{\pi_1}\nu^{-\frac{1}{2}} \times \mathbf{1}\right) \circ \mathcal{M}\left(\pi_2, {\pi_1}^\flat\nu^{-\frac{1}{2}}\right) = \mathcal{M}\left(\pi_2, \pi_1\nu^{-\frac{1}{2}}\right) \circ \left(\mathbf{1} \times \iota_{\pi_1}\nu^{-\frac{1}{2}}\right)$$

up to a scalar. All in all, $\mathfrak{B}'(0)$ is equal up to a scalar to

$$\begin{aligned} &\mathbf{1} \rtimes \mathfrak{B}\left(\pi_2, 0\right) \circ \iota_{\pi_1}\nu^{-\frac{1}{2}} \rtimes \mathbf{1} \circ I\left(\mathcal{M}\left(\pi_2, {\pi_1}^\flat\nu^{-\frac{1}{2}}\right), 0\right) \circ \left(\mathbf{1} \rtimes R\left(\pi_1, \frac{1}{2}\right)\right) \circ M_1 \\ &\qquad = \mathbf{1} \rtimes \mathfrak{B}\left(\pi_2, 0\right) \circ I\left(\mathcal{M}\left(\pi_2, \pi_1\nu^{-\frac{1}{2}}\right), 0\right) \circ \left(\mathbf{1} \rtimes \mathfrak{B}\left(\pi_1, \frac{1}{2}\right)\right) \circ M_1 \end{aligned}$$

where $M_1 = I(\mathcal{M}(\pi_1\nu^{\frac{1}{2}}, \pi_2), 0)$. Note that by the properties of the normalized intertwining operators $I\left(\mathcal{M}\left(\pi_2, \pi_1\nu^{-\frac{1}{2}}\right), 0\right)$ is the Hermitian dual of M_1 up to a scalar, and hence

$$I\left(\mathcal{M}\left(\pi_2, \pi_1\nu^{-\frac{1}{2}}\right), 0\right) \circ \left(\mathbf{1} \rtimes \mathfrak{B}\left(\pi_1, \frac{1}{2}\right)\right) \circ M_1$$

is semi-definite. On the other hand, $\mathfrak{B}(\pi_2, 0)$ is a Hermitian involution. Thus, for $\mathfrak{B}'(0)$ to be semi-definite it is necessary and sufficient that

$$M_1^{-1}\left(\ker\left(\mathbf{1}\rtimes R\left(\pi_1,\frac{1}{2}\right)\right)\right)\supset \pi_1\nu^{\frac{1}{2}}\rtimes\Omega^{\pm} \tag{21}$$

where $\Omega^{\pm}$ are the ± 1 eigenspaces of $\mathfrak{B}(\pi_2, 0)$ on $I(\pi_2, 0)$. Indeed, $\mathfrak{B}'(0)$ is semi-definite, of opposite signs, on the subspaces $\pi_1\nu^{\frac{1}{2}}\otimes\Omega^{\pm}$. We will show that (21) is impossible if $\pi_2 \neq 0$. Let ω be any irreducible constituent of $I(\pi_2, 0)$. The Langlands quotient of $\pi_1\nu^{\frac{1}{2}}\rtimes\omega$ is obtained as the image of the corresponding intertwining operator (with respect to a maximal parabolic of G)

$$M_2 : \pi_1\nu^{\frac{1}{2}}\rtimes\omega \longrightarrow {\pi_1}^{\flat}\nu^{-\frac{1}{2}}\rtimes\omega$$

which is given by convergent integral. On the other hand, M_2 is also the restriction to $\pi_1\nu^{\frac{1}{2}}\rtimes\omega$ of the intertwining operator (with respect to a co-rank two parabolic subgroup of G, but the same Weyl element)

$$M_3 : I\left(\pi_1\otimes\pi_2,\left(\frac{1}{2},0\right)\right)\to I\left({\pi_1}^{\flat}\otimes\pi_2,\left(-\frac{1}{2},0\right)\right).$$

Thus, we conclude that the image of $\pi_1\nu^{\frac{1}{2}}\rtimes\omega$ under M_3 is nonzero. On the other hand M_3 is obtained as the composition of

$$\begin{aligned}
I\left(\pi_1\otimes\pi_2,\left(\frac{1}{2},0\right)\right) &\xrightarrow{M_1} I\left(\pi_2\otimes\pi_1,\left(0,\frac{1}{2}\right)\right)\\
&\xrightarrow{\mathbf{1}\rtimes R\left(\pi_1,\frac{1}{2}\right)} I\left(\pi_2\otimes{\pi_1}^{\flat},\left(0,-\frac{1}{2}\right)\right)\\
&\xrightarrow{I\left(\mathcal{M}\left(\pi_2,{\pi_1}^{\flat}\nu^{-\frac{1}{2}}\right),0\right)} I\left({\pi_1}^{\flat}\otimes\pi_2,\left(-\frac{1}{2},0\right)\right).
\end{aligned}$$

Thus the left-hand side of (21) does not contain $\pi_1\nu^{\frac{1}{2}}\rtimes\omega$ for any irreducible constituent of $I(\pi_2, 0)$. It remains to show that:

LEMMA 11. *$\Omega^{\pm}\neq 0$ if $\pi_2 \neq 0$.*

Proof. This follow from the theory of R-groups (cf. [Gol94]). Indeed, let $\sigma = \rho_1\otimes\ldots\otimes\rho_r$ considered as a square-integrable representation of a Levi subgroup L of $M = \mathrm{GL}_m$ and let $Q = LV$ be the corresponding standard parabolic subgroup of S_m. Thus $\pi_2 = \mathrm{Ind}_{Q\cap M}^{M}\sigma$. By our conditions on σ, the R-group of σ in S_m is isomorphic to

$$W(\sigma) = \{w \in W/W_L : wLw^{-1} = L, w\sigma \simeq \sigma\}.$$

Thus any nontrivial element in $W(\sigma)$ gives rise to a nonscalar intertwining operator R_w. Since the operator $\mathfrak{B}(\pi_2, 0)$ is up to a scalar R_w for $w = w_0w_0^L$ we get the result. □

Remark. Suppose that $\theta = \mathbf{1}$ and consider the following conditions on a self-dual generic representation of GL_n.

1. π is of G-type.
2. $\mathfrak{B}(\pi, 0)$ is a scalar.
3. $I(\pi, \frac{1}{2})$ has a unitarizable quotient.
4. $\mathfrak{B}(\pi, \frac{1}{2})$ is semi-definite.

We remarked above that conditions 1 and 2 are equivalent. Similarly, conditions 3 and 4 are equivalent. In the tempered case, all the conditions are equivalent, although in general 3 is stronger than 1. Any local component of a cuspidal representation of G-type satisfies 3. It seems that 3 reflects the fact that π is a functorial image of a unitarizable representation of a classical group.

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