Convex integration for Lipschitz mappings and counterexamples to regularity

By S. Müller and V. Šverák*

1. Introduction

In this paper we study Lipschitz solutions of partial differential relations of the form

$$(1) \qquad \nabla u(x) \in K \qquad \text{a.e. in } \Omega,$$

where u is a (Lipschitz) mapping of an open set $\Omega \subset \mathbf{R}^n$ into $\mathbf{R}^m$, $\nabla u(x)$ is its gradient (i.e. the matrix $\partial u_i(x)/\partial x_j$, $1 \leq i \leq m$, $1 \leq j \leq n$, defined for almost every $x \in \Omega$), and K is a subset of the set $M^{m \times n}$ of all real $m \times n$ matrices. In addition to relation (1), boundary conditions and other conditions on u will also be considered.

Relation (1) is a special case of partial differential relations which have been extensively studied in connection with certain geometrical problems, such as isometric immersions. For example, the celebrated results of Nash [Na 54] and Kuiper [Ku 55] and their far-reaching generalizations by Gromov [Gr 86] showed striking and completely unexpected features of the behavior of C^1-isometric immersions of $\mathbf{R}^n$ to $\mathbf{R}^{n+1}$, and Lipschitz isometric immersions of $\mathbf{R}^n$ to $\mathbf{R}^n$. A general result describing a large class of Lipschitz solutions of partial differential relations more general than (1) can be found in the book of Gromov [Gr 86, p. 218].

More recently, problems concerning solutions of relations of the form (1) have been studied in connection with the characterization of absolute minimizers of variational integrals describing the elastic energy of crystals exhibiting interesting microstructures ([BJ 87], [CK 88]). An important observation which came from this direction [Ba 90] is that relation (1) can have highly oscillatory solutions even when the difference of any two (nonidentical) matrices in K has rank ≥ 2. This situation, which does occur in some very interesting cases, is not covered by the theorem of Gromov mentioned above. In technical terms to be explained below, the reason is that Gromov's *P-convex hull* of the

set K is again K in that situation. The main result of this paper, Theorem 3.2, covers many of these cases and shows that in the Lipschitz case it seems to be more natural to work with a different hull, which is defined in terms of rank-one convex functions, and can be significantly larger than the P-convex hull.

As an application of the theorem we give a solution of a long-standing problem regarding regularity of weak solutions of elliptic systems. We construct an example of a variational integral $I(u) = \int_\Omega F(\nabla u)$, where Ω is an open disc in $\mathbf{R}^2$, u is a mapping of Ω into $\mathbf{R}^2$, and F is a smooth, strongly quasi-convex function with bounded second derivatives, such that the Euler-Lagrange equation of I has a large class of weak solutions which are Lipschitz but not C^1 in any open subset of Ω, and have some other "wild" features. This result should be compared with the well-known result of Evans [Ev 86] which says that *minimizers* of I are smooth outside a closed subset of Ω of measure zero. Our method also gives new conditions on F which are necessary for regularity. The conditions are expressed in terms of geometrical properties of the gradient mapping $X \to DF(X)$. We expect that the method is applicable to other interesting problems.

Our construction is quite different from well-known counterexamples to regularity of solutions of elliptic systems, such as [DG 68], [GM 68], or [HLN 96]. We should emphasize, however, that our method does not apply when F is convex. Very recently we became aware of the work of Scheffer [Sch 74], in which important partial results, including counterexamples, related to the regularity problem for the elliptic systems described above were obtained. It seems that the work was never published in a journal and has not received the attention it deserves. The point of view taken in that paper is implicitly quite similar to ours and in particular the T_4-configurations discussed in Section 4.2 play an important role in Scheffer's work. At the same time, the new techniques we develop enable us to answer questions which [Sch 74] left open.

2. Preliminaries

Let us first recall the various notions of convexity related to lower-semi-continuity of variational integrals of the form $I(u) = \int_\Omega f(\nabla u)$, where Ω is a bounded domain in $\mathbf{R}^n$, $u \colon \Omega \to \mathbf{R}^m$ is a (sufficiently regular) mapping, and $f \colon M^{m \times n} \to \mathbf{R}$ is a continuous function defined on the set $M^{m \times n}$ of all real $m \times n$ matrices.

A function $f \colon M^{m \times n} \to \mathbf{R}$ is *quasi-convex* if $\int_\Omega (f(A + \nabla\varphi) - f(A)) \geq 0$ for each $A \in M^{m \times n}$ and each smooth, compactly supported $\varphi \colon \Omega \to \mathbf{R}^m$. This definition was introduced by Morrey (see e.g. [Mo 66]) who also proved that the quasi-convexity of f is necessary and sufficient for the functional I to be

lower-semicontinuous with respect to the uniform convergence of uniformly Lipschitz functions. It is also necessary and sufficient for the weak sequential lower-semicontinuity of I on Sobolev spaces $W^{1,p}(\Omega, \mathbf{R}^m)$, if natural growth conditions are satisfied; see [Ma 85] and [AF 87]. The definition of quasi-convexity is independent of Ω, as can be seen by a simple scaling and covering argument ([Mo 66]). In fact, we have the following simple observation made by many authors:

LEMMA 2.1.   *Let $\mathbf{T}^n$ be a flat n-dimensional torus. A function $f\colon M^{m\times n} \to \mathbf{R}$ is quasi-convex if and only if $\int_{\mathbf{T}^n}(f(A + \nabla\varphi) - f(A)) \geq 0$ for each $A \in M^{m\times n}$ and each smooth $\varphi\colon \mathbf{T}^n \to \mathbf{R}^m$.*

The reader is referred to [Sv 92a] for a proof of this statement.

We also recall that, with the notation above, $f\colon M^{m\times n} \to \mathbf{R}$ is *strongly quasi-convex* if there exists $\gamma > 0$ such that $\int_\Omega(f(A+\nabla\varphi) - f(A)) \geq \gamma\int_\Omega |\nabla\varphi|^2$ for each $A \in M^{m\times n}$ and each smooth, compactly supported $\varphi\colon \Omega \to \mathbf{R}^m$. This notion appears naturally in the regularity theory; see for example [Ev 86].

A function $f\colon M^{m\times n} \to \mathbf{R}$ is *rank-one convex* if it is convex along any line whose direction is given by a matrix of rank one, i.e. $t \to f(A + tB)$ is convex for each $A \in M^{m\times n}$ and each $B \in M^{m\times n}$ with rank $B = 1$. This class of functions will play a particularly important rôle in our analysis. It can be proved that any quasi-convex function is rank-one convex, but the opposite implication fails when $n \geq 2$, $m \geq 3$ ([Sv 92a]). (The case $n \geq 2$, $m = 2$ is open.)

We will also deal with functions which are defined only on symmetric matrices. We will denote by $S^{n\times n}$ the set of all symmetric $n \times n$ matrices. The notions introduced above for functions on $M^{m\times n}$ can be modified in the obvious manner to apply to functions on symmetric matrices. For example, a function $f\colon S^{n\times n} \to \mathbf{R}$ is quasi-convex, if $\int_\Omega(f(A + \nabla^2\phi) - f(A)) \geq 0$ for each $A \in S^{n\times n}$ and each smooth, compactly supported $\phi\colon \Omega \to \mathbf{R}$. Again, the definition is independent of Ω and, in fact, Ω can be replaced by any flat n-dimensional torus.

In the rest of this section we examine in more detail facts related to rank-one convexity.

Let $\mathcal{O} \subset M^{m\times n}$ be an open set and let $f\colon \mathcal{O} \to \mathbf{R}$ be a function. We say that f is *rank-one convex in $\mathcal{O}$*, if f is convex on each rank-one segment contained in $\mathcal{O}$. It is easy to see that every rank-one convex function $f\colon \mathcal{O} \to \mathbf{R}$ is locally Lipschitz in $\mathcal{O}$.

We will use $\mathcal{P}$ to denote the set of all compactly supported probability measures in $M^{m\times n}$. For a compact set $K \subset M^{m\times n}$ we use $\mathcal{P}(K)$ to denote the set of all probability measures supported in K. For $\nu \in \mathcal{P}$ we denote by $\bar{\nu}$ the center of mass of ν, i.e. $\bar{\nu} = \int_{M^{m\times n}} X d\nu(X)$.

Following [Pe 93], we say that a measure $\nu \in \mathcal{P}$ is a *laminate* if $\langle \nu, f \rangle \geq f(\bar{\nu})$ for each rank-one convex function $f: M^{m \times n} \to \mathbf{R}$. At the center of our attention will be the sets $\mathcal{P}^{\mathrm{rc}}(K) = \{\nu \in \mathcal{P}(K), \ \nu \text{ is a laminate}\}$, which are defined for any compact set $K \subset M^{m \times n}$.

For $A \in M^{m \times n}$ we denote by δ_A the Dirac mass at A.

Let $\mathcal{O}$ be an open subset of $M^{m \times n}$. Assume $\nu \in \mathcal{P}$ is of the form $\nu = \sum_{j=1}^{j=r} \lambda_j \delta_{A_j}$, with $A_j \in \mathcal{O}$, $j = 1, \ldots, r$, and $A_j \neq A_k$ when $j \neq k$. We say that $\nu' \in \mathcal{P}$ can be obtained from ν by an *elementary splitting in $\mathcal{O}$* if, for some $j \in \{1, \ldots, r\}$, and some $\lambda \in [0,1]$, there exists a rank-one segment $[B_1, B_2] \subset \mathcal{O}$ containing A_j, with $A_j = (1-s)B_1 + sB_2$, such that $\nu' = \nu + \lambda \lambda_j((1-s)\delta_{B_1} + s\delta_{B_2} - \delta_{A_j})$.

We now define an important subset $\mathcal{L}(\mathcal{O})$ of laminates, called *laminates of a finite order in $\mathcal{O}$*. By definition, $\nu \in \mathcal{L}(\mathcal{O})$ if there exists a finite sequence of measures $\nu_1, \ldots, \nu_m$ such that $\nu_1 = \delta_A$ for some $A \in \mathcal{O}$, $\nu_m = \nu$, and ν_{j+1} can be obtained from ν_j by an elementary splitting in $\mathcal{O}$ for $j = 1, \ldots, m-1$. When $\mathcal{O} = M^{m \times n}$, the measures in $\mathcal{L}(\mathcal{O}) = \mathcal{L}(M^{m \times n})$ are called *laminates of a finite order* (i.e. we do not refer to the set $\mathcal{O}$ in that case).

Let K be a compact subset of $M^{m \times n}$. The *rank-one convex hull $K^{\mathrm{rc}} \subset M^{m \times n}$ of K* is defined as follows. A matrix X does not belong to K^{rc} if and only if there exists $f: M^{m \times n} \to \mathbf{R}$ which is rank-one convex such that $f \leq 0$ on K and $f(X) > 0$. We emphasize that this definition will be used only when K is compact. For open sets $\mathcal{O} \subset M^{m \times n}$, we define the rank-one convex hull $\mathcal{O}^{\mathrm{rc}}$ of $\mathcal{O}$ as $\mathcal{O}^{\mathrm{rc}} = \cup\{K^{\mathrm{rc}}, \ K \text{ is a compact subset of } \mathcal{O}\}$. With this definition we have the property that the rank-one convex hull of an open set is again an open set, which will be useful for our purposes.

We refer the reader to [MP 98] for interesting results about rank-one convex hulls of closed sets. The following theorem, which is a slight generalization of a result from [Pe 93], will play an important rôle.

THEOREM 2.1.   *Let K be a compact subset of $M^{m \times n}$ and let $\nu \in \mathcal{P}^{\mathrm{rc}}(K)$. Let $\mathcal{O} \subset M^{m \times n}$ be an open set such that $K^{\mathrm{rc}} \subset \mathcal{O}$. Then there exists a sequence $\nu_j \in \mathcal{L}(\mathcal{O})$ of laminates of a finite order in $\mathcal{O}$ such that $\bar{\nu}_j = \bar{\nu}$ for each j and the ν_j converge weakly* to ν in $\mathcal{P}$.*

As a preparation for the proof of the theorem, we prove the following lemma.

LEMMA 2.2.   *Let $\mathcal{O}$ be an open subset of $M^{m \times n}$. Let $f: \mathcal{O} \to \mathbf{R}$ be a continuous function and let $R_{\mathcal{O}} f: \mathcal{O} \to \mathbf{R} \cup \{-\infty\}$ be defined by*

$$R_{\mathcal{O}} f = \sup\{g, \ g: \mathcal{O} \to \mathbf{R} \text{ is rank-one convex in } \mathcal{O} \text{ and } g \leq f\}.$$

Then for each $X \in \mathcal{O}$, $R_{\mathcal{O}} f(X) = \inf\{\langle \nu, f \rangle, \ \nu \in \mathcal{L}(\mathcal{O}) \text{ and } \bar{\nu} = X\}$.

Proof. Let us denote by $\tilde{f}$ the function in $\mathcal{O}$ defined by $\tilde{f}(X) = \inf\{\langle \nu, f\rangle,$ $\nu \in \mathcal{L}(\mathcal{O})$ and $\bar{\nu} = X\}$. Clearly $R_{\mathcal{O}}f \leq \tilde{f}$ in $\mathcal{O}$. On the other hand, we see from the definition of the set $\mathcal{L}(\mathcal{O})$ that it has the following property: if $\nu_1, \nu_2 \in \mathcal{L}(\mathcal{O})$, and the segment $[\bar{\nu}_1, \bar{\nu}_2]$ is a rank-one segment contained in $\mathcal{O}$, then any convex combination of ν_1 and ν_2 is again in $\mathcal{L}(\mathcal{O})$. Using this, we see immediately from the definitions that $\tilde{f}$ is rank-one convex in $\mathcal{O}$ and hence $R_{\mathcal{O}}f = \tilde{f}$.

Proof of Theorem 2.1. Let $\nu \in \mathcal{P}^{\mathrm{rc}}(K)$ and let $\bar{\nu} = A$ be its center of mass. From the definitions we see that $A \in K^{\mathrm{rc}}$. We choose an open set $U \subset M^{m \times n}$ satisfying $K^{\mathrm{rc}} \subset U \subset \bar{U} \subset \mathcal{O}$ and define $\mathcal{F} = \{\mu \in \mathcal{L}(U),\ \bar{\mu} = A\}$. We claim that the weak* closure of $\mathcal{F}$ contains ν. To prove the claim, we argue by contradiction. Assume ν does not belong to the weak* closure of $\mathcal{F}$. Since $\mathcal{F}$ is clearly convex, we see from the Hahn-Banach theorem that there exists a continuous function $f : \bar{U} \to \mathbf{R}$ such that $\langle \nu, f\rangle < \inf\{\langle \mu, f\rangle,\ \mu \in \mathcal{L}(U)$ and $\bar{\mu} = A\}$. By Lemma 2.2, we have $\inf\{\langle \mu, f\rangle,\ \mu \in \mathcal{L}(U)$ and $\bar{\mu} = A\} = R_U f(A)$. We see that the function $\tilde{f} = R_U f : U \to \mathbf{R}$ is rank-one convex in U and satisfies $\langle \nu, \tilde{f}\rangle \leq \langle \nu, f\rangle < \tilde{f}(\bar{\nu})$. By Lemma 2.3 below, there exists a rank-one convex function $F : M^{m \times n} \to \mathbf{R}$ such that $F = \tilde{f}$ on K^{rc}. We conclude that ν cannot belong to $\mathcal{P}^{\mathrm{rc}}(K)$, a contradiction. The proof is finished.

LEMMA 2.3. *Let $K \subset M^{m \times n}$ be a compact set, let $\mathcal{O}$ be an open set containing K^{rc} (the rank-one convex hull of K) and let $f : \mathcal{O} \to \mathbf{R}$ be rank-one convex. Then there exists $F : M^{m \times n} \to \mathbf{R}$ which is rank-one convex and coincides with f in a neighborhood of K^{rc}.*

Proof. We claim there exists a nonnegative rank-one convex $g : M^{m \times n} \to \mathbf{R}$ such that $K^{\mathrm{rc}} = \{X, g(X) = 0\}$. To prove this, we choose $R > 0$ so that $K \subset B_{R/2} = \{X, |X| < R/2\}$ and define $g_1 : B_R \to \mathbf{R}$ by

$$g_1(X) \quad = \quad \sup\{f(X),\ f : B_R \to \mathbf{R},$$
$$f \text{ is rank-one convex in } B_R \text{ and } f \leq \mathrm{dist}\,(\cdot, K) \text{ in } B_R\}.$$

The function g_1 is obviously nonnegative and rank-one convex in B_R. Moreover, $\{X \in B_R, g_1(X) = 0\} \supset K$ and from the definition of K^{rc} we see that $g_1 > 0$ outside K^{rc}. We now define

$$g(X) = \begin{cases} \max\,(g_1(X), 12|X| - 9R) & \text{when } X \in B_R \\ 12|X| - 9R & \text{when } |X| \geq R. \end{cases}$$

Clearly g is rank-one convex in a neighborhood of any point X with $|X| \neq R$. Since $g_1(X) \leq 2|X|$ when $|X| = R$, we see that we have $g(X) = 12|X| - 9R$ in a neighborhood of $\{|X| = R\}$. We see that g is nonnegative, rank-one convex in $M^{m \times n}$, $\{X, g(X) = 0\} \supset K$, and $\{X, g(X) > 0\} \cap K^{\mathrm{rc}} = \emptyset$. Therefore $\{X, g(X) = 0\} = K^{\mathrm{rc}}$

We can now finish the proof of the lemma. Replacing f by $f + c$, if necessary, we can assume that $f > 0$ in a neighborhood of K^{rc}. For $k > 0$ we let $U_k = \{X \in \mathcal{O},\ f(X) > kg(X)\}$. We also let V_k be the union of the connected components of U_k which have a nonempty intersection with K^{rc}. It is easy to see that there exists $k_0 > 0$ such that $\bar{V}_{k_0} \subset \mathcal{O}$. We now let $F(X) = f(X)$ when $X \in V_{k_0}$ and $F(X) = k_0 g(X)$ when $X \in M^{m \times n} \setminus V_{k_0}$. It is easy to check that the function F defined in this way is rank-one convex on $M^{m \times n}$.

3. Constructions

Throughout this section, Ω denotes a fixed bounded open subset of $\mathbf{R}^n$. We will use the following terminology. A Lipschitz mapping $u \colon \Omega \to \mathbf{R}^m$ is *piecewise affine*, if there exists a countable system of mutually disjoint open sets $\Omega_j \subset \Omega$ which cover Ω up to a set of zero measure, and the restriction of u to each of the sets Ω_j is affine.

Following Gromov ([Gr 86, p. 18]) we also introduce the following concept. Let $\mathcal{F}(\Omega, \mathbf{R}^m)$ be a family of continuous mappings of Ω into $\mathbf{R}^m$. We say that a given continuous mapping $v_0 \colon \Omega \to \mathbf{R}^m$ admits *a fine C^0-approximation* by the family $\mathcal{F}(\Omega, \mathbf{R}^m)$ if there exists, for every continuous function $\varepsilon \colon \Omega \to (0, \infty)$, an element v of the family $\mathcal{F}(\Omega, \mathbf{R}^m)$ such that $|v(x) - v_0(x)| < \varepsilon(x)$ for each $x \in \Omega$.

3.1. *The basic construction.* The main building block of all the solutions of relation (1) which we construct in this paper is the following simple lemma.

LEMMA 3.1. *Let $A, B \in M^{m \times n}$ be two matrices with $\mathrm{rank}\,(B - A) = 1$, let $b \in \mathbf{R}^m$, $0 < \lambda < 1$ and $C = (1 - \lambda)A + \lambda B$. Then, for any $0 < \delta < |A - B|/2$, the affine mapping $x \to Cx + b$ admits a fine C^0-approximation by piecewise affine mappings $u \colon \Omega \to \mathbf{R}^m$ such that $\mathrm{dist}\,(\nabla u(x), \{A, B\}) < \delta$ almost everywhere in Ω, $\mathrm{meas}\,\{x \in \Omega,\ |\nabla u(x) - A| < \delta\} = (1 - \lambda)\,\mathrm{meas}\,\Omega$, and $\mathrm{meas}\,\{x \in \Omega,\ |\nabla u(x) - B| < \delta\} = \lambda\,\mathrm{meas}\,\Omega$.*

Proof. We first note that it is enough to prove the lemma only for a special case when the function $\varepsilon(x)$ appearing in the definition of a fine C^0-approximation is constant and the function approximating the function u satisfies the boundary condition $u(x) = Cx + b$ for $x \in \partial\Omega$. This can be seen by considering a sequence of open sets Ω_j which are mutually disjoint, satisfy $\bar{\Omega}_j \subset \Omega$, and cover Ω up to a set of measure zero.

To prove the special case, we note that we can assume without loss of generality that $A = -\lambda a \otimes e_n$, $B = (1 - \lambda)a \otimes e_n$, and $C = 0$, where $a \in \mathbf{R}^m$ and $e_n = (0, \ldots, 0, 1) \in \mathbf{R}^n$. We define $h \colon \mathbf{R} \to \mathbf{R}$ and $w \colon \mathbf{R}^n \to \mathbf{R}^m$ by $h(s) = (|s| + (2\lambda - 1)s)/2$ and $w(x) = a \max(0, 1 - |x_1| - \ldots - |x_{n-1}| - h(x_n))$. We

choose a small $\delta' > 0$, and set $v(x) = \delta' w(x_1, \ldots, x_{n-1}, x_n/\delta')$. We also let $\omega = \{x, v(x) > 0\}$. We check by a direct calculation that $\text{dist}\,(\nabla v(x), \{A, B\}) \leq (n-1)|a|\delta'$ for almost every $x \in \omega$. We clearly also have $v(x) = 0$ when $x \in \partial\omega$. By Vitali's theorem we can cover Ω up to a set of measure zero by a countable family $\{\omega_i\}$ of mutually disjoint sets of the form $\omega_i = y_i + r_i\omega$ (with $y_i \in \mathbf{R}^n$ and $r_i \in (0, \epsilon)$). We let $u(x) = r_i v(r_i^{-1}(x - y_i)$ when $x \in \omega_i$, and $u(x) = 0$ if $x \in \Omega \setminus \cup_i \omega_i$. It easy to check that u satisfies the required conditions, provided δ' is sufficiently small.

LEMMA 3.2.   *Let $\nu \in \mathcal{P}(M^{m \times n})$ be a laminate of a finite order, let $A = \bar{\nu}$ be its center of mass. Let us write $\nu = \sum_{j=1}^r \lambda_j \delta_{A_j}$ with $\lambda_j > 0$ and $A_i \neq A_j$ when $i \neq j$, and let*

$$\delta_1 = \min\{|A_i - A_j|/2; 1 \leq i < j \leq r\}.$$

Then, for each $b \in \mathbf{R}^m$, and each $0 < \delta < \delta_1$, the mapping $x \to Ax + b$ admits a fine C^0-approximation by piecewise affine mappings u satisfying $\text{dist}\,(\nabla u(x), \{A_1, \ldots, A_r\}) < \delta$ a.e. in Ω and

$$\text{meas}\,\{x \in \Omega, \text{dist}\,(\nabla u(x), A_j) < \delta\} = \lambda_j \,\text{meas}\,\Omega$$

for each $j \in \{1, \ldots, r\}$.

Proof. This can be easily proved by applying iteratively Lemma 3.1 in a way which is naturally suggested by the definition of the laminate of a finite order. We outline some details for the convenience of the reader. Let $\delta_A = \nu_1, \nu_2, \ldots, \nu_m = \nu$ be a sequence of measures such that ν_{j+1} can be obtained from ν_j by an elementary splitting in $M^{m \times n}$. If $m = 1$, there is nothing to prove, if $m = 2$, our statement is exactly Lemma 3.1. Proceeding by induction on m, let us assume that the lemma has been proved for ν replaced by ν_{m-1}. Let us write $\nu_{m-1} = \sum_{j=1}^{j=r'} \lambda_j' \delta_{A_j'}$, with $A_k' \neq A_l'$ when $k \neq l$. Since $\nu = \nu_m$ can be obtained from ν_{m-1} by an elementary splitting,

$$\nu = \nu_{m-1} + \lambda\lambda_{j_0}'((1 - s)\delta_{B_1} + s\delta_{B_2} - \delta_{A_{j_0}'})$$

for some $\lambda \in [0, 1]$, $s \in [0, 1]$, $j_0 \in \{1, \ldots, r'\}$, and a rank-one segment $[B_1, B_2]$ containing A_{j_0}'. By our assumptions, for any sufficiently small $0 < \delta' < \delta/2$, the map $x \to Ax + b$ admits a fine C^0-approximation by piecewise affine maps u' satisfying $\text{dist}\,(\nabla u'(x), \{A_1', \ldots, A_{r'}'\}) < \delta'$ a.e. in Ω and

$$\text{meas}\,\{x \in \Omega; \text{dist}\,(\nabla u'(x), A_j') < \delta'\} = \lambda_j' \,\text{meas}\,\Omega.$$

For any such u' we can find an open set $\Omega' \subset \Omega$ such that $\text{dist}\,(\nabla u'(x), A_{j_0}') < \delta'$ in Ω', $\text{meas}\,\Omega' = \lambda\,\text{meas}\,\{x \in \Omega; \text{dist}\,(\nabla u'(x), A_{j_0}') < \delta'\} = \lambda\lambda_{j_0}'\,\text{meas}\,\Omega$, and u' is piecewise affine in Ω'. Let $\Omega_k' \subset \Omega'$, $k = 1, 2, \ldots$ be mutually disjoint open sets which cover Ω' up to a set of measure zero such that $\nabla u' = \tilde{A}_k = \text{const}$ in

Ω'_k, with $|\tilde{A}_k - A'_{j_0}| < \delta'$. We now adjust u' by applying Lemma 3.1 on each Ω'_k with $A = B_1 + \tilde{A}_k - A'_{j_0}$, $B = B_2 + \tilde{A}_k - A'_{j_0}$, $C = \tilde{A}_k$, $\delta = \delta'$, and the proof is easily finished.

3.2. *Open relations.* We recall that the rank-one convex hull $\mathcal{O}^{\mathrm{rc}}$ of an open set $\mathcal{O} \subset M^{m \times n}$ is, by definition, the union of the rank-one convex hulls of all compact subsets of $\mathcal{O}$. The main result of this subsection is the following.

THEOREM 3.1.   *Let $\mathcal{O} \subset M^{m \times n}$ be open, and let $P \subset \mathcal{O}^{\mathrm{rc}}$ be compact. Let $u_0 : \Omega \to \mathbf{R}^m$ be a piecewise affine Lipschitz mapping such that $\nabla u_0(x) \in P$ for a.e. $x \in \Omega$. Then u_0 admits a fine C^0-approximation by piecewise affine Lipschitz mappings $u : \Omega \to \mathbf{R}^m$ satisfying $\nabla u(x) \in \mathcal{O}$ a.e. in Ω.*

Proof. As a first step, we prove the following lemma.

LEMMA 3.3.   *Let $K \subset M^{m \times n}$ be a compact set and let $U \subset M^{m \times n}$ be an open set containing K. Let $\nu \in \mathcal{P}^{\mathrm{rc}}(K)$ and denote $A = \bar{\nu}$. Let $b \in \mathbf{R}^m$. Then, for any given $\delta > 0$, the mapping $x \to Ax + b$ admits a fine C^0-approximation by piecewise affine mappings u satisfying $\nabla u(x) \in U^{\mathrm{rc}}$ a.e. in Ω and $\mathrm{meas}\,\{x \in \Omega,\, \nabla u(x) \in U\} > (1 - \delta)\,\mathrm{meas}\,\Omega$.*

Proof. By Theorem 2.1 there exists a laminate μ of a finite order which is supported in a finite subset of U^{rc} and satisfies $\bar{\mu} = \bar{\nu}$ and $\mu(U) > (1 - \delta)$. Let us write $\mu = \sum_{j=1}^{j=r} \lambda_j \delta_{A_j}$, so that $\delta_1 = \min\{|A_k - A_l|/2;\, 1 \le k < l \le r\} > 0$. We choose $0 < \delta' < \delta_1$ so that each $A_k \in U$ is at distance at least δ' from the boundary ∂U. From Lemma 3.2 we see that the map $x \to Ax + b$ admits a fine C^0-approximation by piecewise maps u such that $\mathrm{dist}\,(\nabla u(x), \{A_1, \ldots, A_r\}) < \delta'$ a.e. in Ω and $\mathrm{meas}\,\{x \in \Omega;\, \mathrm{dist}\,(\nabla u(x), A_j) < \delta'\} = \lambda_j \,\mathrm{meas}\,\Omega$ for $j = 1, \ldots, r$, and our lemma immediately follows.

Theorem 3.1 can now be proved by repeatedly applying Lemma 3.3 in the following way. We first choose a sequence of compact sets $K_1, K_2, \ldots \subset M^{m \times n}$, a sequence of open sets $U_1, U_2, \ldots \subset M^{m \times n}$, and a compact set $Q \subset M^{m \times n}$ such that $P = K_1 \subset U_1 \subset K_2 \subset U_2 \subset \ldots \subset Q \subset \mathcal{O}^{\mathrm{rc}}$. We also choose $0 < \delta < 1$. Let $\varepsilon = \varepsilon(x) > 0$ be a continuous function on Ω. In the first step we apply Lemma 3.3 to approximate u_0 up to $\varepsilon/2$ by a mapping u_1 satisfying $\nabla u_1(x) \in U_1^{\mathrm{rc}}$ a.e. in Ω, together with $\mathrm{meas}\,\{x \in \Omega,\, \nabla u_1(x) \in U_1\} > (1 - \delta)\mathrm{meas}\,\Omega$. We now modify u_1 on those subregions of Ω where $\nabla u_1(x)$ does not belong to U_1 by applying Lemma 3.3 again. We obtain a new mapping, u_2, which approximates u_1 up to $\varepsilon/4$, coincides with u_1 a.e. in the set $\{x \in \Omega,\, \nabla u_1(x) \in U_1\}$, and satisfies $\nabla u_2(x) \in U_2^{\mathrm{rc}}$ a.e. in Ω together with $\mathrm{meas}\,\{x \in \Omega,\, \nabla u_2(x) \in U_2\} > ((1 - \delta) + \delta(1 - \delta))\,\mathrm{meas}\,\Omega$. By continuing this procedure we get a sequence u_k of mappings which is easily seen to converge to a mapping u which gives the required approximation of u_0.

Remark. From the proofs of Lemma 3.2, Lemma 3.3, and Theorem 3.1 it is easy to see that Lemma 3.2 remains true if ν is a laminate (not necessarily of finite order) which can be written as a finite convex combination of Dirac masses.

3.3. *Closed relations and in-approximations.* When considering relation (1) for closed sets K, it is natural to try to construct solutions by combining Theorem 3.1 and a suitable limit procedure. For simplicity we will assume in this section that K is compact. Following Gromov ([Gr 86, p. 218]) we say that a sequence of open sets $\{U_i\}_{i=1}^{\infty}$ is an *in-approximation* of K if $U_i \subset U_{i+1}^{\mathrm{rc}}$ for each i, and $\sup_{X \in U_i} \mathrm{dist}\,(X, K) \to 0$ as $i \to \infty$. (The definition does not require that each point of K can be reached by a sequence $X_j \in U_j$.)

THEOREM 3.2. *Assume that a compact set $K \subset M^{m \times n}$ admits an in-approximation by open sets U_i in the sense of the definition above. Then any C^1-mapping $v \colon \Omega \to \mathbf{R}^m$ satisfying $\nabla v(x) \in U_1$ in Ω admits a fine C^0-approximation by Lipschitz mappings $u \colon \Omega \to \mathbf{R}^m$ satisfying $\nabla u(x) \in K$ a.e. in Ω.*

Proof. By the same argument as in the proof of Lemma 3.1 it is enough to prove the statement only in the case when the function $\varepsilon = \varepsilon(x)$ in the definition of a fine C^0-approximation is constant.

Let $\rho \colon \mathbf{R}^n \to \mathbf{R}$ be the usual mollifying kernel, i.e. we assume that ρ is smooth, nonnegative, supported in $\{x,\ |x| < 1\}$, and $\int \rho = 1$. For $\varepsilon > 0$ we let $\rho_\varepsilon = \varepsilon^{-n} \rho(x/\varepsilon)$. For a function $w \in L^1(\Omega)$ we define $\rho_\varepsilon * w$ in the usual way, by considering w as a function on $\mathbf{R}^n$ with $w = 0$ outside Ω. In other words, $\rho_\varepsilon * w(x) = \int_\Omega w(y) \rho_\varepsilon(x - y)\, dy$.

We start the proof by choosing $\delta_1 > 0$ (the exact value of which will be specified later) and by approximating v by a piecewise affine $u_1 \colon \Omega \to \mathbf{R}^m$ with $|u_1 - v| < \delta_1$ in Ω, $u_1 = v$ on $\partial\Omega$, and $\nabla u_1 \in U_1$ a.e. in Ω. (We recall that in this paper "piecewise affine" allows for countably many affine pieces.) We also choose $\varepsilon_1 > 0$ so that $\|\nabla u_1 * \rho_{\varepsilon_1} - \nabla u_1\|_{L^1(\Omega)} \leq 2^{-1}$.

Using Theorem 3.1 together with an obvious inductive argument, we construct a sequence of mappings $u_i \colon \Omega \to \mathbf{R}^m$ and numbers $0 < \varepsilon_i < 2^{-i}$, $\delta_i > 0$ satisfying

$$
\begin{aligned}
\nabla u_i &\in U_i & \text{a.e. in } \Omega\,, \\
u_i &= v & \text{on } \partial\Omega\,, \\
\|\nabla u_i * \rho_{\varepsilon_i} - \nabla u_i\|_{L^1(\Omega)} &\leq 2^{-i}\,, \\
\delta_{i+1} &= \varepsilon_i \delta_i\,, \\
|u_{i+1} - u_i| &\leq \delta_{i+1} & \text{in } \Omega\,.
\end{aligned}
$$

The mappings u_i converge uniformly to a Lipschitz function $u \colon \Omega \to \mathbf{R}^m$. We also have $|u - v| \leq \sum_i |u_{i+1} - u_i| + |u_1 - v| \leq 2\delta_1$. It remains to prove that

$\nabla u \in K$ a.e. in Ω. This will be clear if we establish that $\nabla u_i \to \nabla u$ in $L^1(\Omega)$. We can write

$$
\begin{aligned}
||\nabla u_i - \nabla u||_{L^1(\Omega)} \quad &\leq \quad ||\nabla u_i - \nabla u_i * \rho_{\varepsilon_i}||_{L^1(\Omega)} \\
&+ ||\nabla u * \rho_{\varepsilon_i} - \nabla u||_{L^1(\Omega)} \\
&+ ||\nabla u_i * \rho_{\varepsilon_i} - \nabla u * \rho_{\varepsilon_i}||_{L^1(\Omega)}.
\end{aligned}
$$

The first two terms on the right-hand side of this inequality clearly converge to zero as $i \to \infty$. Defining $\Omega_i = \{x \in \Omega, \operatorname{dist}(x, \partial\Omega) > 2\varepsilon_i\}$ we can estimate the third term as

$$
||(u_i - u) * \nabla \rho_{\varepsilon_i}||_{L^1(\Omega)} + ||\nabla u_i - \nabla u||_{L^1(\Omega \setminus \Omega_i)} \leq \frac{c}{\varepsilon_i}||u_i - u||_\infty + C \operatorname{meas}(\Omega \setminus \Omega_i),
$$

where c and C are constants depending only on ρ and the Lipschitz constant of $u_i - u$, respectively.

We have

$$
||u_i - u||_\infty \quad \leq \quad \sum_{j=i}^{\infty} ||u_j - u_{j+1}||_\infty \leq \sum_{j=i+1}^{\infty} \delta_j \leq 2\delta_{i+1}.
$$

Hence the third term can be estimated by

$$
2c\delta_{i+1}/\varepsilon_i + C \operatorname{meas}(\Omega \setminus \Omega_i) \leq 2c\delta_i + C \operatorname{meas}(\Omega \setminus \Omega_i)
$$

which converges to zero as $i \to \infty$. The proof is finished.

Remark. The explanation of the strong convergence of ∇u_i is more or less the following. We can achieve a very fast convergence of u_i in the sup-norm. It may seem that this is not enough to say much about the convergence of ∇u_i. However, in the proof we choose the parameters in such a way that $||u_i - u||_\infty$ is very small in comparison with a typical length over which ∇u_i changes significantly (in an integral sense). Therefore, as regards the convergence of ∇u_i, we get a situation which is in a certain sense similar to the simple case when the functions u_i are affine in Ω. This is the main reason we get the strong convergence. The above argument is taken from [MS 96]. A different approach can be found in [DM 97].

4. Applications to elliptic systems

Let $\Omega \subset \mathbf{R}^2$ be a disc. For (sufficiently regular) mappings $u: \Omega \to \mathbf{R}^2$ we consider the functional $I(u) = \int_\Omega F(\nabla u(x))\, dx$, where F is a (smooth) function on the set $M^{2 \times 2}$ of all real 2×2 matrices, which satisfies certain "ellipticity conditions". More precisely, we will require that F be strongly quasiconvex and that its second derivatives be uniformly bounded in $M^{2 \times 2}$.

The purpose of this section is to show how we can apply the results above to construct weak solutions of the Euler-Lagrange equation

$$\text{(2)} \qquad\qquad \text{div}\, DF(\nabla u) = 0$$

of the functional I which are Lipschitz, but not continuously differentiable on any open subset of Ω. This is in sharp contrast with regularity properties of minimizers of I, see, for example [Ev 86]. In fact, we prove the following slightly stronger statement.

THEOREM 4.1. *There exists a smooth strongly quasiconvex function $F_0 \colon M^{2\times 2} \to \mathbf{R}$ with $|D^2 F_0| \leq c$ in $M^{2\times 2}$, four matrices $A_1, \ldots, A_4 \in M^{2\times 2}$, $\varepsilon > 0$ and $\delta > 0$ such that the following is true. Let $F \colon M^{2\times 2} \to \mathbf{R}$ be a C^2-function satisfying $|DF(A_j) - DF_0(A_j)| \leq \delta$ and $|D^2 F(A_j) - D^2 F_0(A_j)| \leq \delta$ for $j = 1, 2, 3, 4$. Then each piecewise C^1-function $v \colon \Omega \to \mathbf{R}^2$ satisfying $|\nabla v| < \varepsilon$ a.e. in Ω admits a fine C^0-approximation by Lipschitz mappings $u \colon \Omega \to \mathbf{R}^2$ which are not C^1 on any open subset of Ω and are weak solutions of the equation $\text{div}\, DF(\nabla u) = 0$ in Ω.*

The theorem will be proved in Section 4.4, after we establish some useful facts about quasiconvex functions and rank-one convex hulls. The idea of the construction is the following. We rewrite equation (2) as a first-order system

$$\text{(3)} \qquad\qquad \nabla w \in K$$

and then show that the strong quasiconvexity does not prevent the rank-one convex hull of K from being large. (We note that the strong quasi-convexity does exclude any nontrivial rank-one connections in K; see [Ba 80].) We can then use the methods developed in the previous sections to construct the desired solutions. Moreover, it turns out that the situation is stable under the perturbations of F_0 which are allowed in the theorem.

Remark. In [Sch 74] Scheffer constructs counterexamples to partial regularity of solutions of equation (2) with F rank-one convex and with u in the Sobolev space $W^{1,1}$.

One way to write equation (2) in the form (3) is the following. We denote by J the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The condition that the 2×2 tensor $DF(\nabla u)$ be divergence-free is equivalent to the condition that $DF(\nabla u)J$ be the gradient of a function $\tilde{u} \colon \Omega \to \mathbf{R}^2$. We now introduce $w \colon \Omega \to \mathbf{R}^4$ by $w = \begin{pmatrix} u \\ \tilde{u} \end{pmatrix}$. We

also let K be the set of all 4×2 matrices of the form $\begin{pmatrix} X \\ DF(X)J \end{pmatrix}$, where X runs through all 2×2 matrices. It is clear that, in this notation, system (2) is equivalent to system (3).

4.1. *Quasiconvex functions.* We begin by describing a quasi-convex function which will play an important role in our construction using notation introduced in Section 2. We define $f_0 \colon S^{2 \times 2} \to \mathbf{R}$ by $f_0(X) = \det X$ when X is positive definite and by $f_0(X) = 0$ otherwise.

LEMMA 4.1. *The function f_0 is quasiconvex on $S^{2 \times 2}$.*

Proof. This result is proved in [Sv 92b]. In that paper the proof is actually carried out for a more general class of functions. We give a simple version of the proof here, for the convenience of the reader. Let $\Omega = \{x \in \mathbf{R}^2, |x| < 1\}$ and let $\phi \colon \Omega \to \mathbf{R}$ be smooth and compactly supported in Ω. We must prove that for each $A \in S^{2 \times 2}$ we have $\int_\Omega (f_0(A + \nabla^2\phi) - f_0(A)) \geq 0$. This is obvious if A is not positive definite, since then we integrate a nonnegative function. If A is positive definite, we can assume $A = I$ by a simple change of variables. Let $u_0(x) = |x|^2/2$ and $u(x) = u_0(x) + \phi(x)$. We also set $\varphi = \nabla u$, which will be viewed as a map $\varphi \colon \Omega \to \mathbf{R}^2$. Finally, we let $E = \{x \in \Omega, \det \nabla\varphi(x) \geq 0\}$. We must prove that $\int_E \det \nabla\varphi \geq \mathrm{meas}\,(\Omega)$. Since $\det \varphi \geq 0$ on E, we can use the area formula ([Fe 69]) to infer that it is enough to prove $\Omega \subset \varphi(E)$. Consider an arbitrary $b \in \Omega$ and let $a \in \bar\Omega$ be a point where the function $x \to u(x) - b \cdot x$ attains its minimum in $\bar\Omega$. It is easy to verify that $a \in \Omega$ and hence $\varphi(a) = b$ and $a \in E$. We see that $\Omega \subset \varphi(E)$ and the proof is finished.

In what follows we will use the following notation: for $X \in M^{2 \times 2}$ we let $X_{\mathrm{sym}} = (X + X^t)/2$ and $X_{\mathrm{asym}} = (X - X^t)/2$.

LEMMA 4.2. *Let $f \colon S^{2 \times 2} \to \mathbf{R}$ be a smooth function such that $|D^2 f| \leq c$ in $S^{2 \times 2}$. Assume that f is strongly quasi-convex in the sense that for some $\gamma > 0$ we have $\int_{\mathbf{R}^2}(f(A + \nabla^2\phi) - f(A)) \geq \gamma \int_{\mathbf{R}^2}|\nabla^2\phi|^2$ for all smooth, compactly supported $\phi \colon \mathbf{R}^2 \to \mathbf{R}$. Then for sufficiently large $\kappa > 0$ the function $\tilde{f} \colon M^{2 \times 2} \to \mathbf{R}$ defined by $\tilde{f}(X) = f(X_{\mathrm{sym}}) + \kappa|X_{\mathrm{asym}}|^2$ is strongly quasi-convex.*

Proof. Let $\mathbf{T}^2$ be the two-dimensional torus $\mathbf{R}^2/\mathbf{Z}^2$. Let $\varphi \colon \mathbf{T}^2 \to \mathbf{R}^2$ be a smooth function and let $A \in M^{2 \times 2}$. We want to prove that

$$\int_{\mathbf{T}^2}(\tilde{f}(A + \nabla\varphi) - \tilde{f}(A)) \geq \gamma/2 \int_{\mathbf{T}^2}|\nabla\varphi|^2.$$

Let us consider the Helmholtz decomposition $\varphi = \nabla\phi + \nabla^\perp\eta + a$ of φ, where ϕ and η are scalar functions, $\nabla^\perp\eta = J\nabla\eta$ (with J as above), and a is a constant vector. We have $\nabla\varphi = \nabla^2\phi + \nabla\nabla^\perp\eta$. Set $Y = (\nabla\nabla^\perp\eta)_{\mathrm{sym}}$. A standard calcu-

lation (involving integration by parts and the use of the identity $\int_{\mathbf{T}^2} \det \nabla^2 \eta = 0$) gives $\int_{\mathbf{T}^2} |Y|^2 = \int_{\mathbf{T}^2} |\nabla^2 \eta|^2/2 = \int_{\mathbf{T}^2} (\Delta \eta)^2/2 = \int_{\mathbf{T}^2} |(\nabla \nabla^\perp \eta)_{\text{asym}}|^2$. We can write

$$\int_{\mathbf{T}^2} (\tilde{f}(A + \nabla \varphi) - \tilde{f}(A))$$

$$= \int_{\mathbf{T}^2} (f(A_{\text{sym}} + \nabla^2 \phi) - f(A_{\text{sym}}))$$

$$+ \int_{\mathbf{T}^2} (\kappa |A_{\text{asym}} + (\nabla \nabla^\perp \eta)_{\text{asym}}|^2 - \kappa |A_{\text{asym}}|^2)$$

$$+ \int_{\mathbf{T}^2} (f(A_{\text{sym}} + \nabla^2 \phi + Y) - f(A_{\text{sym}} + \nabla^2 \phi))$$

$$= I + II + III.$$

We have $I \geq \gamma \int_{\mathbf{T}^2} |\nabla^2 \varphi|^2$ by our assumptions and Lemma 2.1. The second term can be evaluated as $II = \int_{\mathbf{T}^2} \kappa |Y|^2$ by using the calculation above and the fact that $\int_{\mathbf{T}^2} \nabla^2 \eta = 0$. Finally, the third term can be written as

$$III = \int_{\mathbf{T}^2} (f(A_{\text{sym}} + \nabla^2 \phi + Y) - f(A_{\text{sym}} + \nabla^2 \phi) - Df(A_{\text{sym}} + \nabla^2 \phi)Y)$$

$$+ \int_{\mathbf{T}^2} (Df(A_{\text{sym}} + \nabla^2 \phi) - Df(A_{\text{sym}}))Y$$

$$\geq - \int_{\mathbf{T}^2} (c/2|Y|^2 + c|\nabla^2 \phi||Y|)$$

$$\geq - \int_{\mathbf{T}^2} (\gamma/2|\nabla^2 \phi|^2 + c/2|Y|^2 + c^2/(2\gamma)|Y|^2).$$

We obtain the right inequality when $\kappa \geq \gamma/2 + c/2 + c^2/(2\gamma)$. The proof is finished.

Lemma 4.2 cannot be directly applied to the function f_0 from Lemma 4.1. However, we can modify f_0 in the following way. We consider a smooth mollifier ω on $S^{2\times 2}$ which is supported in the ball of radius $1/8$ centered at 0 and satisfying $\int_{S^{2\times 2}} \omega = 1$, $\int_{S^{2\times 2}} X \omega(X)\, dX = 0$, and $\int_{S^{2\times 2}} \det(X)\omega(X)\, dX = 0$. We let $f_1(X) = \max(f_0(X), |X|^2 - 25)$ and $f_2 = f_1 * \omega$. We note that $f_2(X) = f_0(X)$ when $|X| \leq 5$ and the open ball $B_{X, \frac{1}{8}}$ is contained in the set of the positive definite matrices. Choosing a small $\gamma > 0$ (to be specified later) and setting $f_3(X) = f_2(X) + \gamma |X|^2$, we denote by $\tilde{f}_3$ the strongly quasi-convex extension of f_3 to $M^{2\times 2}$ obtained in Lemma 4.2 (for a suitable κ).

Let $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We define $\theta \colon M^{2\times 2} \to M^{2\times 2}$ by $\theta \cdot X = TXJ^t$, where J is the rotation by $\pi/2$ introduced above. Note that the diagonal matrices are invariant under θ and that θ restricted to the diagonal matrices can be thought of as a rotation by $\pi/2$. The same is true for anti-diagonal matrices, by which we mean the matrices of the form TX, where X is diagonal. Therefore $\theta^2 = -\text{Id}$.

We now define a function $f_4 \colon M^{2\times 2} \to \mathbf{R}$, which will play an important rôle in our construction. Let $H = \begin{pmatrix} \frac{5}{4} & 0 \\ 0 & -\frac{5}{4} \end{pmatrix}$, and set

$$f_4(X) = \sum_{k=0}^{3} \tilde{f}_3(\theta^{-k} \cdot X - H).$$

It is easy to see that f_4 satisfies $f_4(\theta \cdot X) = f_4(X)$ for each $X \in M^{2\times 2}$ and therefore $Df_4(\theta \cdot X) = \theta \cdot Df_4(X)$ for each $X \in M^{2\times 2}$. (We note that the restriction of f_4 to the diagonal matrices vanishes in the square given by the matrices $\theta^k \cdot H$, $k = 0, 1, 2, 3$, and on the half-lines originating at $\theta^k \cdot H$ and passing through $\theta^{k+1} \cdot H$, where $k = 0, 1, 2, 3$.)

We now let

$$A_1 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \, A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \, A_3 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}, \, A_4 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix},$$

noting that $A_{k+1} = \theta^k \cdot A_1$, $k = 1, 2, 3$. By a direct calculation, $Df_4(A_1) = \begin{pmatrix} \frac{1}{4} + 14\gamma & 0 \\ 0 & \frac{7}{4} + 2\gamma \end{pmatrix}$. By considering functions of the form $\frac{1}{2}\alpha|X|^2 + \beta f_4(X)$ we can easily obtain the following lemma, by choosing suitable positive α, β, and γ.

LEMMA 4.3. *There exist a smooth, strongly quasi-convex function* $F_1 \colon M^{2\times 2} \to \mathbf{R}$ *with uniformly bounded* $D^2 F_1$ *which satisfies* (*in the notation introduced above*) $F_1(\theta \cdot X) = F_1(X)$ *for each* X *and* $DF_1(A_1) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

Proof. See above.

The set K corresponding to the function $F = F_1$ (see the beginning of the section) contains the matrices $\begin{pmatrix} A_k \\ DF_1(A_k)J \end{pmatrix}$, $k = 1, \ldots, 4$. These are the matrices

$$M_1^0 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \\ 0 & -1 \\ 3 & 0 \end{pmatrix}, \, M_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 3 \\ 1 & 0 \end{pmatrix}, \, M_3^0 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \\ 0 & 1 \\ -3 & 0 \end{pmatrix}, \, M_4^0 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \\ 0 & -3 \\ -1 & 0 \end{pmatrix}.$$

4.2. *Deformations of* T_4-*configurations.* Let us consider four $m \times n$ matrices $M_1, \ldots, M_4$. We say that $M_1, \ldots, M_4$ are in T_4-configuration (see Figure 1) if $\operatorname{rank}(M_i - M_j) \neq 1$ for all i, j, and if there exist rank-one matrices $C_1, \ldots, C_4$ with $\sum_k C_k = 0$, real numbers $\kappa_1, \cdots \kappa_4 > 1$, and a matrix $P \in M^{m\times n}$ such

that

$$
\begin{aligned}
M_1 &= P + \kappa_1 C_1\,, \\
M_2 &= P + C_1 + \kappa_2 C_2\,, \\
M_3 &= P + C_1 + C_2 + \kappa_3 C_3\,, \\
M_4 &= P + C_1 + C_2 + C_3 + \kappa_4 C_4.
\end{aligned}
$$

This configuration was discovered independently by several authors. We are aware of [Sch 74], where it is used in a similar context as below, [AH 86], and [Ta 93], where it is used in a different context. Slightly different examples exhibiting similar features were also independently discovered in [NM 91] and [CT 93]. The paper [BFJK 94] contains an interesting example using a T_4-configuration. The following observation appears in [AH 86], [Ta 93] and implicitly also in the other papers.
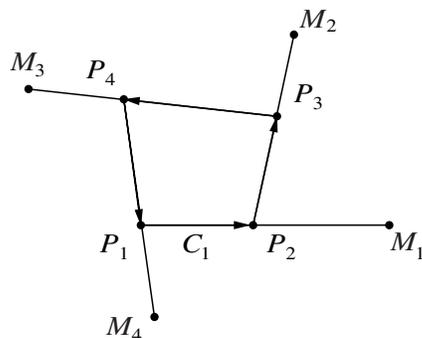

Figure 1. A T_4 configuration with $P_1 = P$, $P_2 = P + C_1$, $P_3 = P + C_1 + C_2$, $P_4 = P + C_1 + C_2 + C_3$. The lines indicate rank-1 connections. Note that the figure need not be planar.

LEMMA 4.4. *If $M_1, \ldots, M_4$ are in T_4-configuration, the rank-one convex hull of the set $\{M_1, \ldots, M_4\}$ contains the points $P_1 = P, P_2 = P + C_1, P_3 = P + C_1 + C_2, P_4 = P + C_1 + C_2 + C_3$. For each point X in the rank-one convex hull there exists a unique laminate $\mu = \sum \mu_l \delta_{M_l}$ with center of mass X.*

Proof. To see this, let us consider a rank-one convex function $f \colon M^{m \times n} \to \mathbf{R}$ which vanishes at the points $M_1, \ldots, M_4$. We have

$$f(P_{i+1}) \le 1/\kappa_i f(M_i) + (1 - 1/\kappa_i) f(P_i) = (1 - 1/\kappa_i) f(P_i)$$

for each i, where the indices are considered modulo 4. Applying this recursively, we get that $f(P_i) \le 0$ for each i. Uniqueness is obvious if the M_l span a three dimensional affine space. If all four matrices lie in a plane one can introduce coordinates x, y along the rank-one directions in this plane and exploit the fact that the function $g(x, y) = xy$ satisfies $\langle \mu, g \rangle = g(\bar{\mu})$.

Example. For future reference, let us calculate the coefficients μ_l above for $X = P_1$. We let $\beta_i = 1 - 1/\kappa_i$, $i = 1, \ldots, 4$. Using recursively the identity $P_{i+1} = (1 - \beta_i)M_i + \beta_i P_i$ (where the indices are considered modulo 4), we get easily the following expression for the laminate μ supported in $\{M_1, \ldots, M_4\}$ with $\bar{\mu} = P_1$:

$$(4) \qquad \mu = \sum_{i=1}^{4} \frac{(1 - \beta_i)\beta_1\beta_2\beta_3\beta_4}{\beta_1 \ldots \beta_i (1 - \beta_1\beta_2\beta_3\beta_4)} \delta_{M_i}.$$

The matrices M_k^0 at the end of subsection 4.1 are in T_4-configuration, as one can see by taking

$$P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 2 \\ 0 & 0 \end{pmatrix},$$

and $C_3 = -C_1$, $C_4 = -C_2$, $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = 2$. The matrices also lie in the set

$$K_1 = \left\{ \begin{pmatrix} X \\ DF_1(X)J \end{pmatrix} ; X \in M^{2\times 2} \right\} \subset M^{4\times 2}$$

given by the quasi-convex function F_1 constructed in Lemma 4.3. This shows that the rank-one convex hull K_1^{rc} of K_1 is nontrivial. We now wish to establish that K_1^{rc} is sufficiently large, so that we can apply Theorem 3.2. We will see later that rather than trying to work with the specific function F_1, it is more convenient to work with a small perturbation $F = F_1 + \varepsilon V$ of F_1, where V is a compactly supported smooth function, the properties of which will be specified later. For the moment we will only assume that F satisfies $DF(A_k) = DF_1(A_k)$ for $k = 1, 2, 3, 4$, where the matrices A_k are the same as in Subsection 4.1 . We also denote by $K \subset M^{4\times 2}$ the set corresponding to F. By our assumptions we know that K contains a T_4-configuration given by the matrices M_k^0, $k = 1, 2, 3, 4$ defined above. It is natural to investigate deformations of this T_4-configuration. In other words, we will investigate four-tuples $M_1, \ldots M_4$ such that, for $k = 1, \ldots, 4$, M_k is close to M_k^0, $M_k \in K$, and $M_1, \ldots M_4$ are in T_4-configuration.

We introduce the following notation.

$$\begin{aligned}
e_1 &= (1, 0), & e_2 &= (0, 1), \\
f_1 &= (2, 0, 0, 2), & f_2 &= (0, 2, 2, 0), \\
C_1^0 &= f_1 \otimes e_1, & C_2^0 &= f_2 \otimes e_2, \\
C_3^0 &= -C_1^0, & C_4^0 &= -C_2^0, \\
P^0 &= -(C_1^0 + C_2^0)/2, & \\
\kappa_1^0 &= \kappa_2^0 = \kappa_3^0 = \kappa_4^0 = 2.
\end{aligned}$$

We parametrize the rank-one matrices C_k in a small neighborhood of C_k^0 as follows.

$$
\begin{aligned}
C_1 &= (f_1 + a_1) \otimes (e_1 + \beta_1 e_2), \\
C_2 &= (f_2 + a_2) \otimes (e_2 - \beta_2 e_1), \\
C_3 &= (-f_1 + a_3) \otimes (e_1 + \beta_3 e_2), \\
C_4 &= (-f_2 + a_4) \otimes (e_2 - \beta_4 e_1),
\end{aligned}
$$

where $a_1, \ldots, a_4$ are (small) vectors in $\mathbf{R}^4$, and $\beta_1, \ldots, \beta_4$ are (small) real numbers. We linearize the equation $\sum_k C_k = 0$ around the solution C_k^0. The linearized equation is equivalent to

$$
\begin{aligned}
a_1 + a_3 + (\beta_4 - \beta_2) f_2 &= 0, \\
a_2 + a_4 + (\beta_1 - \beta_3) f_1 &= 0.
\end{aligned}
$$

Using these formulae and the above expressions for M_k, we easily check (with the help of the implicit-function theorem) that the four-tuples $(M_1, \ldots, M_4)$ of the 4×2 matrices which are close to $(M_1^0, \ldots, M_4^0)$ and form T_4-configuration such that the parameters P, C_j, κ_j are close to P^0, C_j^0, κ_j^0 form a 24-dimensional manifold $\mathcal{M}$. The tangent space $L_{\mathcal{M}}$ of $\mathcal{M}$ at the point $(M_1^0, \ldots, M_4^0)$ can be identified with four-tuples $(Z_1, \ldots, Z_4)$ of 4×2 matrices of the form

$$
Z_1 = \begin{pmatrix}
p_{11} + 2a_{11} + \kappa_1' & p_{12} + 2\beta_1' \\
p_{21} + 2a_{21} & p_{22} \\
p_{31} + 2a_{31} & p_{32} \\
p_{41} + 2a_{41} + \kappa_1' & p_{42} + 2\beta_1'
\end{pmatrix},
$$

$$
Z_2 = \begin{pmatrix}
p_{11} + a_{11} & p_{12} + 2a_{12} + \beta_1' \\
p_{21} + a_{21} - 2\beta_2' & p_{22} + 2a_{22} + \kappa_2' \\
p_{31} + a_{31} - 2\beta_2' & p_{32} + 2a_{32} + \kappa_2' \\
p_{41} + a_{41} & p_{42} + 2a_{42} + \beta_1'
\end{pmatrix},
$$

$$
Z_3 = \begin{pmatrix}
p_{11} - a_{11} - \kappa_3' & p_{12} + a_{12} - 2\beta_3' + \beta_1' \\
p_{21} - a_{21} + \beta_2' - 2\beta_4' & p_{22} + a_{22} \\
p_{31} - a_{31} + \beta_2' - 2\beta_4' & p_{32} + a_{32} \\
p_{41} - a_{41} - \kappa_3' & p_{42} + a_{42} - 2\beta_3' + \beta_1'
\end{pmatrix},
$$

$$
Z_4 = \begin{pmatrix}
p_{11} & p_{12} - a_{12} + \beta_3' - \beta_1' \\
p_{21} + \beta_4' & p_{22} - a_{22} - \kappa_4' \\
p_{31} + \beta_4' & p_{32} - a_{32} - \kappa_4' \\
p_{41} & p_{42} - a_{42} + \beta_3' - \beta_1'
\end{pmatrix},
$$

where the values of all the 24 parameters run through all real numbers. Moreover, there is a well-defined mapping $(M_1, \ldots, M_4) \to (P_1, \ldots, P_4)$ from $\mathcal{M}$ to

the four-tuples of 4×2 matrices, where (in the notation introduced in the definition of T_4-configuration) $P_1 = P$, $P_2 = P_1 + C_1$, $P_3 = P_2 + C_2$, $P_4 = P_3 + C_3$ as above.

We now consider the additional constraint $M_k \in K$, where K is the set determined by F. The four-tuples $(M_1, \ldots, M_4)$ satisfying $M_k \in K$ clearly form a 16-dimensional manifold $\mathcal{K} = K \times K \times K \times K$. The tangent space $L_\mathcal{K}$ of $\mathcal{K}$ at $(M_1^0, \ldots, M_4^0)$ can be identified with the four-tuples

$$\begin{pmatrix} X_1 \\ D^2F(A_1)X_1J \end{pmatrix}, \begin{pmatrix} X_2 \\ D^2F(A_2)X_2J \end{pmatrix}, \begin{pmatrix} X_3 \\ D^2F(A_3)X_3J \end{pmatrix}, \begin{pmatrix} X_4 \\ D^2F(A_4)X_4J \end{pmatrix}$$

where $X_1, \ldots, X_4$ run through all 2×2 matrices.

We now consider the maps $(M_1, \ldots, M_4) \rightarrow (M_k, P'_k)$, where P_k is defined as above and where we denote (with a slight abuse of notation) by P'_k the orthogonal projection of the point P_k into the space $(T_{A_k}K)^\perp$, the normal space of K at A_k. We would like to establish the following nondegeneracy conditions, which will be important later when we construct in-approximations.

Condition (C). $\mathcal{M}$ and $\mathcal{K}$ intersect transversely at $(M_1^0, \ldots, M_4^0)$ and, (after $\mathcal{M}$ is perhaps replaced by a sufficiently small neighborhood of $(M_1^0, \ldots, M_4^0)$ in $\mathcal{M}$) the map $(M_1, \ldots, M_4) \rightarrow (M_k, P'_k)$ is, for each k, a nondegenerate diffeomorphism of $\mathcal{M} \cap \mathcal{K}$ and a neighborhood of $(M_k^0, (P_k^0)')$ in $K \times (T_{A_k}K)^\perp$.

Rather than trying to decide whether these nondegeneracy conditions are satisfied for an explicitly given function F, it seems to be more natural to verify that the conditions are satisfied in the generic case. More specifically, we note that for each smooth compactly supported function $V \colon M^{4\times 2} \rightarrow \mathbf{R}$ the function $F = F_1 + \varepsilon V$ is strongly quasi-convex for sufficiently small ε. By choosing V in a suitable way, we can perturb $D^2F(A_1), \ldots D^2F(A_4)$ to any prescribed values which are close enough to the original values, without changing the values of $DF(A_1), \ldots, DF(A_4)$, and without affecting the strong quasi-convexity. For the purpose of the construction of the counterexample announced at the beginning of this section, we can therefore restrict our considerations to the generic case.

LEMMA 4.5. *Assume that $DF(A_k) = DF_1(A_k)$ for $k = 1, 2, 3, 4$. Then condition* (C) *above is satisfied for the generic values of $D^2F(A_k)$, $k = 1, \ldots, 4$.*

Proof. The condition that $\mathcal{M}$ and $\mathcal{K}$ intersect transversely at $(M_1^0, \ldots, M_4^0)$ and that the map $(M_1, \ldots, M_4) \rightarrow (M_1, P'_1)$ is a nondegenerate diffeomorphism of a small neighborhood of $(M_1^0, \ldots, M_4^0)$ in $\mathcal{M} \cap \mathcal{K}$ and a neighborhood of $(M_1^0, (P_1^0)')$ in $K \times (T_{A_1}K)^\perp$ is easily seen to be equivalent to the condition that the following linear homogeneous system of 40 equations for 40 unknowns has no nontrivial solutions.

$$Z_j = \begin{pmatrix} X_j \\ D^2 F(A_j) X_j J \end{pmatrix}, \quad j = 1, 2, 3, 4,$$

$$\begin{pmatrix} p_{31} & p_{32} \\ p_{41} & p_{42} \end{pmatrix} = D^2 F(A_1) \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} J,$$

$$X_1 = 0,$$

where $Z_j = Z_j(p_{kl}, a_{kl}, \beta'_k, \kappa'_k)$ (with $k = 1, 2, 3, 4$, $l = 1, 2$) are the 4×2 matrices introduced above and X_1, X_2, X_3, X_4 are 2×2 matrices. The determinant of the corresponding 40×40 matrix is a polynomial expression in the entries of the matrices $D^2 F(A_j)$ (which are now considered as parameters), and will be denoted by Q_1. The polynomial Q_1 is not identically zero, since for

$$D^2 F(A_1) = I, \quad D^2 F(A_2) = I, \quad D^2 F(A_3) = 0, \quad D^2 F(A_4) = I$$

we can check by a straightforward calculation that the system has no nontrivial solutions.

By using symmetry we see that, for each $k = 1, 2, 3, 4$, the condition that $\mathcal{M}$ and $\mathcal{K}$ intersect transversely at $(M_1^0, \dots, M_4^0)$ and that the map $(M_1, \dots, M_4) \to (M_k, P'_k)$ is a nondegenerate diffeomorphism of a small neighborhood of $(M_1^0, \dots, M_4^0)$ in $\mathcal{M} \cap \mathcal{K}$ and a neighborhood of $(M_k^0, (P_k^0)')$ in $K \times (T_{A_k} K)^\perp$ can be expressed as $Q_k \neq 0$, where Q_k is a suitable nonzero polynomial in the entries of the matrices $D^2 F(A_j)$. Hence all of our nondegeneracy conditions will be satisfied at all values of $D^2 F(A_j)$ where the polynomial $Q = Q_1 Q_2 Q_3 Q_4$ does not vanish. Since Q is not identically zero, the result follows.

4.3. *In-approximation.* To be able to use Theorem 3.2, we need to have a suitable in-approximation.

LEMMA 4.6. *Using the notation above, assume that condition* (C) *is satisfied. Let $r > 0$. Then there exists an in-approximation $\{\mathcal{U}_i\}_{i=1}^\infty$ of*

$$K_r = \cup_{j=1}^4 \{X \in M^{4 \times 2}, |X - M_j^0| \le r\} \cap K$$

such that $\mathcal{U}_1$ contains a (small) neighborhood of the rank-one convex hull of the points $P_1^0, \dots, P_4^0$.

Proof. Let $\mathcal{O}$ be a sufficiently small neighborhood of $(M_1^0, M_2^0, M_3^0, M_4^0)$ in $\mathcal{M} \cap \mathcal{K} \subset (M^{4 \times 2})^4$. The main point is that, for each $k = 1, 2, 3, 4$, the image of $\mathcal{O}$ under the map $(M_1, M_2, M_3, M_4) \to P_k(M_1, M_2, M_3, M_4)$ is open in $M^{4 \times 2}$, whereas the image of $\mathcal{O}$ under the projections $(M_1, M_2, M_3, M_4) \to M_k$ is not (since $M_k \in K$). We will therefore consider convex combinations $(1 - \lambda) P_k + \lambda M_k$ with $\lambda \to 1$ to construct an in-approximation of K (see Fig. 2). We now describe the details.

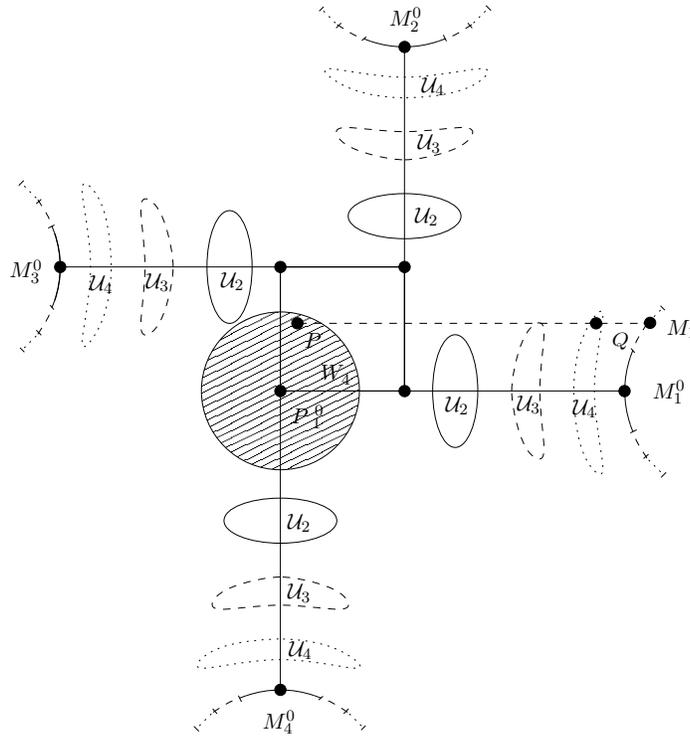Figure 2. Schematic illustration of the sets $\mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4 \subset M^{4\times2}$. The solid (resp. dashed, or dotted) lines through the point M_1^0 are the projections of the set $\mathcal{O}_2$(resp. $\mathcal{O}_3$, or $\mathcal{O}_4$) $\subset \mathcal{M} \cap \mathcal{K} \subset (M^{4\times2})^4$ to the first component. They are not open in $M^{4\times2}$ since they are contained in K. The shaded set W_4 is the image of $\mathcal{O}_4$ under the map $(M_1, M_2, M_3, M_4) \to P_1(M_1, M_2, M_3, M_4)$ and it is open in $M^{4\times2}$. By $P = P_1(M_1, M_2, M_3, M_4)$ we denote a typical point in W_4. A typical point Q in $\mathcal{U}_{1,4} \subset \mathcal{U}_4$ is given by $(1-\lambda_4)P_1(M_1, M_2, M_3, M_4)+\lambda_4 M_1$, where $(M_1, M_2, M_3, M_4) \in \mathcal{O}_4$.

We consider a sequence $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2 \ldots \subset \mathcal{O}$ of open neighborhoods of $(M_1^0, \ldots, M_4^0)$ in $\mathcal{M} \cap \mathcal{K}$, such that each $\mathcal{O}_j$ is diffeomorphic to the eight-dimensional unit ball and that, for each $j = 0, 1, 2, \ldots$ we have $\bar{\mathcal{O}}_j \subset \mathcal{O}_{j+1}$. We also consider a sequence of numbers $0 = \lambda_0$, $1/2 < \lambda_1 < \ldots < \lambda_j < \ldots < 1$ converging to 1 as $j \to \infty$. For $j = 0, 1, 2, \ldots$ we let

$$\mathcal{U}_{k,j} = \{(1 - \lambda_j)P_k + \lambda_j M_k, \ (M_1, \ldots, M_4) \in \mathcal{O}_j\},$$

where $P_k = P_k(M_1, \ldots, M_4)$ is the map considered in subsection 4.2. We also let $\mathcal{U}_j = \cup_{k=1}^{k=4}\mathcal{U}_{k,j}$. Condition (C) implies that there exists j_0 such that the sets $\mathcal{U}_j$ are open when $j \geq j_0$ and $\mathcal{O}$ is sufficiently small. To see this, consider for example $k = 1$ and let us write points $M_1 \in K$ which are close to M_1^0 as

$M_1 = M_1^0 + X + \xi(X)$, with $X \in T_{A_1}K$ and $\xi(X) \in (T_{A_1}K)^\perp$. We can also write $P_1 = P_1^0 + Y + \eta$ with $Y \in (T_{A_1}K)^\perp$ and $\eta \in T_{A_1}K$. If condition (C) is satisfied, we know that, in a small neighborhood of $(M_1^0, \ldots, M_4^0)$, we can take X and Y as local coordinates in $\mathcal{M} \cap \mathcal{K}$. For $(M_1, \ldots, M_4) \in \mathcal{M} \cap \mathcal{K}$ which is close to $(M_1^0, \ldots, M_4^0)$ and $P_1 = P_1(M_1, \ldots, M_4)$, we can therefore write the η–component of P_1 in the above decomposition as $\eta = \eta(X, Y)$, where η is a smooth function of X and Y with $\eta(0,0) = 0$. In the coordinates (X, Y), the derivative of the map $(X, Y) \to (1 - \lambda)P_1 + \lambda M_1$ is given by the block matrix

$$\begin{pmatrix} \lambda I + (1 - \lambda)\partial_X \eta & (1 - \lambda)\partial_Y \eta \\ \lambda \partial_X \xi & (1 - \lambda)I \end{pmatrix}.$$

Since $\partial_X \xi(0) = 0$, we see that the matrix is regular when X is small and λ is close to 1. The openess of $\mathcal{U}_{1,j}$ for large j, λ close (but not equal) to 1, and small $\mathcal{O}$ follows.

By Lemma 4.7 below, the closure of $\mathcal{U}_j$ (and hence the closure of its rank-one convex hull) is contained in the rank-one convex hull of $\mathcal{U}_{j+1}$. Moreover, the rank-one convex hull of $\mathcal{U}_0$ contains a neighborhood of the square given by the convex hull of the points $P_1^0, \ldots, P_4^0$ (which coincides with the rank-one convex hull of these points, since the points lie in a two-dimensional plane). The required in-approximation has therefore been established.

LEMMA 4.7.   *Using the notation introduced in the proof of Lemma 4.6 the following is true. For each $j = 1, 2, \ldots$, the set $\overline{\mathcal{U}_j}$ is contained in $\mathcal{U}_{j+1}^{rc}$, and each $A \in \overline{\mathcal{U}_{j,k}}$ is the center of mass of a laminate $\mu = \sum_{l=1}^4 \mu_l \delta_{Y_l}$, with $Y_l \in \mathcal{U}_{l,j+1}$. Moreover, when λ_j is sufficiently close to 1 and $\mathcal{O}$ is sufficiently small, we can achieve in addition that*

$$\begin{aligned} \mu_k &\geq 1 - (\lambda_{j+1} - \lambda_j), \\ |Y_k - A| &\leq 2|M_1^0 - P_1^0|(\lambda_{j+1} - \lambda_j), \\ \mu_l &\geq (\lambda_{j+1} - \lambda_j)/8, \quad \text{for } l \neq k. \end{aligned}$$

Proof. To simplify the notation suppose $A \in \overline{\mathcal{U}_{1,j}}$. Then there exist $(M_1, M_2, M_3, M_4) \in \overline{\mathcal{O}_j} \subset \mathcal{O}_{j+1}$ such that $A = (1 - \lambda_j)P_1 + \lambda_j M_1$, where $P_1 = P_1(M_1, M_2, M_3, M_4)$. Let $Y_l = (1 - \lambda_{j+1})P_l + \lambda_{j+1}M_l$ (see Fig. 3). Then A is the center of mass of the laminate

$$\tilde{\mu} = \frac{\lambda_j}{\lambda_{j+1}}\delta_{Y_1} + (1 - \frac{\lambda_j}{\lambda_{j+1}})\delta_{P_1}.$$

and $|Y_1 - A| = |M_1 - P_1|(\lambda_{j+1} - \lambda_j) \leq 2|M_1^0 - P_1^0|(\lambda_{j+1} - \lambda_j)$.

By Lemma 4.4 the point P_1 is the center of mass of a unique laminate $\eta = \sum_{l=1}^4 \alpha_l \delta_{Y_l}$ supported on the T_4 configuration (Y_1, Y_2, Y_3, Y_4), where
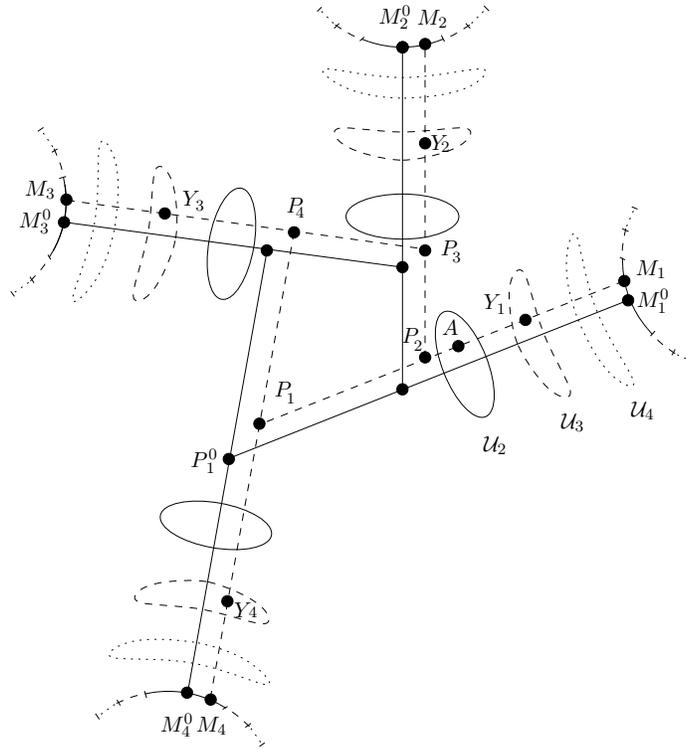
Figure 3. The point $A \in \mathcal{U}_{1,2}$ lies in the rank-1 convex hull of the points $Y_1, Y_2, Y_3, Y_4 \in \mathcal{U}_3$.

the coefficients α_l are given by equation (4). Since P_1 and Y_1 differ by a rank-one matrix the measure

$$\mu = \frac{\lambda_j}{\lambda_{j+1}}\delta_{Y_1} + (1 - \frac{\lambda_j}{\lambda_{j+1}})\eta$$

is a laminate with center of mass A. If $\mathcal{O}$ is small and λ_j is close to 1, the numbers β_i in (4) are close to $1/2$, and an elementary calculation gives our estimates.

4.4. *Solutions with nowhere continuous gradients.*

Proof of Theorem 4.1. The main idea of the proof is described in heuristic terms in the remarks immediately following the theorem. In the proof below we will be freely using the notation introduced earlier in Section 4.

We recall that $A_1, \ldots, A_4$ are defined as follows:

$$A_1 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \ A_3 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}, \ A_4 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix};$$

see Section 4.1 . We let F_0 be a suitable small perturbation of the quasiconvex function F_1 from Lemma 4.3 such that $DF_0(A_k) = DF_1(A_k)$ for $k = 1, \ldots, 4$

and condition (C) is satisfied. Since the transversality and the other non-degeneracy conditions are stable under small perturbations, a version of (C) with $M_1^0, \ldots, M_4^0$ replaced by close-by matrices $\tilde{M}_1^0 \ldots, \tilde{M}_4^0$ will also be satisfied for any F as in the statement of the theorem, provided δ is sufficiently small. Moreover, we see easily that by choosing δ sufficiently small we can also achieve that Lemma 4.6 can be applied (with $M_1^0, \ldots, M_4^0$ replaced by close-by matrices $\tilde{M}_1^0 \ldots, \tilde{M}_4^0$) with a fixed small $r > 0$ to any set K arising from a function F satisfying the assumptions of the theorem. In addition, we see easily that the in-approximations can be constructed so that $\mathcal{U}_1$ contains a fixed small neighborhood of the zero matrix for any F satisfying the assumptions. Let us choose $\varepsilon > 0$ so that the ball of radius ε centered at the zero matrix is contained in this fixed small neighbourhood. We see that the assumptions of Theorem 3.2 are satisfied in our situation. However, it does not seem to be immediately clear that the solutions obtained from Theorem 3.2 are not continuously differentiable in any open subset of Ω. To obtain such solutions in a simple way, we make the construction more explicit and impose some additional conditions on the approximations so that the nowhere differentiability of the limit is easy to see.

Let $\{\lambda_j\}$ and $r > 0$ be as in Lemma 4.6, and assume (as we can without loss of generality) that r is sufficiently small. Let $\mathcal{U}_j$ denote the in-approximation constructed in Lemma 4.6. Let $\phi: M^{4 \times 2} \to \mathbf{R}$ be be a continuous function which is $\equiv 1$ in $\{X; |X| \leq 2r\}$ and vanishes outside $\{X, |X| \leq 3r\}$. For $l = 1, 2, 3, 4$ set $\phi_l(X) = \phi(X - M_l^0)$. Assume now that ε is as above, $v: \Omega \to \mathbf{R}^2$ is as in Theorem 4.1 and let $\varepsilon_1: \Omega \to \mathbf{R}$ be a continuous function in Ω which is > 0. Let $\tilde{w} = \begin{pmatrix} v \\ 0 \end{pmatrix}$. We will now go through constructions involved in the proof of Theorem 3.2 in more detail and construct a sequence of functions $w_j: \Omega \to \mathbf{R}^4$ together with a sequence $\mathcal{F}_j$ of families of open subsets of Ω, so that the following conditions are satisfied.

(i) The sets in $\mathcal{F}_j$ are open, mutually disjoint, contained in Ω together with their closures, and cover Ω up to a set of measure zero;

(ii) Each set of $\mathcal{F}_{j+l}$ is contained in a set of $\mathcal{F}_j$ (where $j, l \geq 1$);

(iii) $\sup \{\operatorname{diam} V; V \in \mathcal{F}_j\} \to 0$ as $j \to \infty$;

(iv) ∇w_j is constant on V for each $V \in \mathcal{F}_j$;

(v) $\nabla w_j \in \mathcal{U}_j$ a.e. in Ω;

(vi) $|w_1 - \tilde{w}| < \varepsilon_1/2$ in Ω and $|w_{j+1} - w_j| \leq 2^{-j-2}\varepsilon_1$ in Ω, $(j = 1, 2, \ldots)$.

In addition, the following conditions, which are crucial for the desired behavior, are satisfied when j is sufficiently large.

(vii) (L^1-*convergence of* ∇w_j) We have $\int_\Omega |\nabla w_{j+1} - \nabla w_j| \leq L(\lambda_{j+1} - \lambda_j) \operatorname{meas} \Omega$ for a suitable constant L;

(viii) (*Persistence of oscillations*) For each $V \in \mathcal{F}_j$ and each $l \in \{1, 2, 3, 4\}$,

$$(5) \qquad \int_V \phi_l(\nabla w_{j+1}) \geq \frac{1}{8}(\lambda_{j+1} - \lambda_j) \operatorname{meas} V \quad \text{and}$$

$$(6) \qquad \int_V \phi_l(\nabla w_{j+1}) \geq (1 - (\lambda_{j+1} - \lambda_j)) \int_V \phi_l(\nabla w_j).$$

Once the existence of $\{w_j\}$ and $\{\mathcal{F}_j\}$ satisfying (i)–(viii) is established, we can consider $w_\infty = \lim_{j \to \infty} w_j$. From (v)–(vii) we infer that w_∞ is Lipschitz, with $\nabla w_\infty \in K$ a.e. in Ω. Moreover, using (ii), (vii), and (viii) we see that, for each sufficiently large j and $V \in \mathcal{F}_j$,

$$\int_V \phi_l(\nabla w_\infty) = \lim_{m \to \infty} \int_V \phi_l(\nabla w_m)$$

$$\geq \lim_{m \to \infty} (1 - (\lambda_m - \lambda_{m-1})) \ldots (1 - (\lambda_{j+2} - \lambda_{j+1})) \int_V \phi_l(\nabla w_{j+1})$$

$$\geq \frac{1}{16} \lambda_{j+1}(\lambda_{j+1} - \lambda_j) \operatorname{meas} V.$$

This, together with (iii) implies that the essential oscillation of ∇w_∞ over any open set is at least $\max_{1 \leq k < l \leq 4} |M_k^0 - M_l^0|/2$, and therefore w_∞ cannot be continuously differentiable in any open subset of Ω.

To construct $\{w_j\}$ and $\{\mathcal{F}_j\}$, we proceed by induction. The existence of w_1 and $\mathcal{F}_1$ satisfying (i)–(v) and the first inequality of (vi) follows from Theorem 3.1. Assume that, for some $j \geq 1$ there exist w_j and $\mathcal{F}_j$ satisfying (i), (iv), and (v). Let $V \in \mathcal{F}_j$ and assume that $\nabla w_j = A$ in V, with $A \in \mathcal{U}_j$. Assume that $A \in \mathcal{U}_{1,j}$, for example. By Lemma 4.7, the matrix A is the center of mass of a laminate $\mu = \sum_{l=1}^4 \mu_l \delta_{Y_l}$, with $Y_l \in \mathcal{U}_{l,j+1}$. In addition, by the same lemma, if j is sufficiently large,

$$(7) \qquad |Y_1 - A| \leq 2|M_1^0 - P_1^0|(\lambda_{j+1} - \lambda_j),$$

$$(8) \qquad \mu_1 \geq 1 - (\lambda_{j+1} - \lambda_j) \quad \text{and}$$

$$(9) \qquad \mu_l \geq (\lambda_{j+1} - \lambda_j)/8 \quad \text{for each } l = 1, 2, 3, 4.$$

From the remark following the proof of Theorem 3.1 we see that there exists a piecewise affine $w_{j+1}^V : V \to \mathbf{R}^4$ such that $\nabla w_{j+1}^V \in \mathcal{U}_{j+1}$ a.e. in V, $|w_{j+1}^V - w_j| \leq \varepsilon_1 2^{-j-2}$ in V, $w_{j+1}^V = w_j$ at the boundary of V, and $\operatorname{meas}\{x \in V; \nabla w_{j+1}^V \in \mathcal{U}_{l,j+1}\} = \mu_l \operatorname{meas} V$ for each $l = 1, 2, 3, 4$. We choose a family $\mathcal{F}_{j+1}^V$ of mutually disjoint open sets of radius $< 1/(j+1)$ which cover V up to a set of measure zero and ∇w_{j+1}^V is constant on each of them. We can now define $\mathcal{F}_{j+1} = \cup_{V \in \mathcal{F}_j} \mathcal{F}_{j+1}^V$ and $w_{j+1} : \Omega \to \mathbf{R}^4$ by $w_{j+1} = w_{j+1}^V$ in the closure of V for each $V \in \mathcal{F}_j$.

From (7) and (8) we see that (vii) is satisfied with

$$L = 2|M_1^0 - P_1^0| + 2 \max_{1 \leq k < l \leq 4} |M_k^0 - M_l^0|.$$

In addition, (8) and (9) imply that (viii) is satisfied. The rest of the properties (i)-(viii) are immediate consequences of our construction.

Remark. The above construction is quite similar to the following simpler example. Let us consider a sequence $0 < \lambda_0 < \lambda_1 < \ldots \lambda_j < \ldots < 1$, with $\lim_{j \to \infty} \lambda_j = 1$. Let $X \subset L^\infty(0,1)$ be the space of all piecewise constant functions. For a function $f \in X$ with $|f| \leq \lambda_j$ we define $T_j f \in X$ in the following way. Let (a, b) be a maximal open interval on which f is constant. Let $c = (a + b)/2$. We find $d \in (a, c)$ and $e \in (c, b)$ such that the function $g: (a, b) \to \mathbf{R}$ defined by $g(x) = -\lambda_j$ when $x \in (a, d)$, $g(x) = \lambda_j$ when $x \in (d, c)$, $g(x) = -\lambda_j$ when $x \in (c, e)$, and $g(x) = \lambda_j$ when $x \in (e, b)$ has the same average as f over the intervals (a, c) and (c, b). We then set $T_j f(x) = g(x)$ for $x \in (a, b)$, and repeat the same construction on the other maximal intervals on which f is constant. Let $0 < A < \lambda_0$ and let $f_0 \equiv A$ in $(0, 1)$. Set $f_{j+1} = T_{j+1} f_j$. It is not difficult to see that the sequence f_j converges in $L^1(0, 1)$ to a function f_∞. Moreover, the essential oscillation of f_∞ over any open set is 2.

4.5. *Linear systems.* The examples above can be used to answer open questions (raised in [GS 85]) concerning solutions of linear 2×2 systems of the form

$$\partial_\alpha a_{ij}^{\alpha\beta}(x) \partial_\beta v_j = 0, \qquad i = 1, 2 \tag{10}$$

where the coefficients are in L^∞ and satisfy the strong Legendre-Hadamard condition

$$a_{ij}^{\alpha\beta}(x)\xi_\alpha\xi_\beta \bar{u}^i \bar{u}^j \geq \nu|\xi|^2|\bar{u}|^2$$

for each $\xi, \bar{u} \in \mathbf{R}^2$ and almost every x. (As usual, $\nu > 0$.) In what follows we will write the system (10) as div $A(x)\nabla v = 0$.

There is a well known procedure for passing from solutions of nonlinear equations to solutions of linear equations with measurable coefficients (see e.g. [Mo 66]). We will use it to construct our examples. These examples will be based on the following proposition.

PROPOSITION 4.1.   *There exist a smooth strictly quasiconvex function* $F: M^{2\times2} \to \mathbf{R}$ *with uniformly bounded* $D^2 F$ *and a nontrivial Lipschitz function* $u: \mathbf{R}^2 \to \mathbf{R}^2$ *which vanishes for* $|x| > 1$ *and satisfies (weakly) the equation* div $DF(\nabla u) = 0$ *is* $\mathbf{R}^2$.

Proof. We will use the notation introduced earlier in Section 4. We note that the function F_1 from Lemma 4.3 satisfies $DF_1(0) = 0$ and therefore the zero matrix belongs to the set $K_1 \subset M^{4\times2}$ corresponding to F_1. Thus, we

see that the function F_0 in Theorem 4.1 can be taken so that $DF_0(0) = 0$. Hence the set K corresponding to $F = F_0$ in Theorem 4.1 can be taken so that it contains the zero matrix. We know that there are nontrivial solutions of $\nabla w \in K$ a.e. in Ω which vanish at $\partial\Omega$. Extending w by zero outside Ω, we get solutions with the required properties.

PROPOSITION 4.2. *There exist L^∞-coefficients $A(x)$ defined in $\mathbf{R}^2$ which satisfy the strong Legendre-Hadamard condition such that weak solutions of the linear system* div $A(x)\nabla v = 0$ *exhibit the following behavior.*

(i) *There exists a compactly supported solution v belonging to the Sobolev space $W^{1,2}$ but not to $W^{1,2+\delta}$ for any $\delta > 0$.*

(ii) *There exists a sequence v_j, $j = 1, 2, \ldots$ of Lipschitz solutions which are supported in $\{x, |x| < 1\}$, and converge to zero weakly but not strongly in $W^{1,2}$.*

Proof. Let F and u be as in Proposition 4.1 and let

$$\tilde{A}(x) = \int_0^1 D^2 F(t\nabla u(x))\, dt.$$

Since F is smooth and $|D^2 F| \leq c$, $\tilde{A}(x)$ is a well-defined L^∞-function. Since F is strongly quasiconvex, it is also strongly rank-one convex, and therefore $\tilde{A}(X)$ satisfies the Legendre-Hadamard condition. Moreover, we have

$$\text{div } \tilde{A}(x)\nabla u = \text{div}\left(DF(\nabla u(x)) - DF(0)\right) = 0 \quad \text{in } \mathbf{R}^2$$

in the weak sense.

Let us consider a sequence $B_{a_j, r_j} \subset \{x \in \mathbf{R}^2, |x| < 1\}$ of mutually disjoint balls centered at a_j with radius $r_j > 0$ so that $a_j \to 0$ in $\mathbf{R}^2$ and $r_j \to 0$. We let

$$A(x) = D^2 F(0) + \sum_{j=1}^\infty \left(\tilde{A}(r_j^{-1}(x - a_j)) - D^2 F(0)\right) \qquad \text{and}$$

$$v_j(x) = u(r_j^{-1}(x - a_j)), \qquad j = 1, 2, \ldots .$$

The coefficients $A(x)$ are again bounded and satisfy the strong Legendre-Hadamard condition. We also have div $A(x)\nabla v_j = 0$, $\quad j = 1, 2, \ldots$. The sequence $v_1, v_2, \ldots$ gives (ii). To obtain (i), we consider a sequence $c_1, c_2, \ldots$ satisfying $\sum_{j=1}^\infty c_j^2 < \infty$ and $\sum_{j=1}^\infty c_j^{2+\delta} = \infty$ for each $\delta > 0$. Then $v = \sum_{j=1}^\infty c_j v_j$ has the required properties.

MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, LEIPZIG, GERMANY
E-mail address: sm@mis.mpg.de

UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN
E-mail address: sverak@math.umn.edu

References

[AF 87]   E. Acerbi and N. Fusco, A regularity theorem for minimizers of quasiconvex integrals, *Arch. Rational Mech. Anal.* **99** (1987), 261–281.

[AH 86]   R. Aumann and S. Hart, Bi-convexity and bi-martingales, *Israel J. Math.* **54** (1986), 159–180.

[Ba 80]   J. M. Ball, Strict convexity, strong ellipticity and regularity in the calculus of variations, *Math. Proc. Cambridge Philos. Soc.* **87** (1980), 501–513.

[Ba 90]   ———, Sets of gradients with no rank-one connections, *J. Math. Pures Appl.* (9) **69** (1990), 241–259.

[BJ 87]   J. M. Ball and R. D. James, Fine phase mixtures as minimizers of energy, *Arch. Rational Mech. Anal.* **100** (1987), 13–52.

[BFJK 94] K. Bhattacharya, N. Firoozye, R. D. James, and R. V. Kohn, Restrictions on microstructure, *Proc. Roy. Soc. Edinburgh A* **124** (1994), 843–878.

[CT 93]   E. Casadio-Tarabusi, An algebraic characterization of quasi-convex functions, *Ricerche Mat.* **42** (1993), 11–24.

[CK 88]   M. Chipot and D. Kinderlehrer, Equilibrium configurations of crystals, *Arch. Rational Mech. Anal.* **103** (1988), 237–277.

[DM 97]   B. Dacorogna and P. Marcellini, General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases, *Acta Math.* **178** (1997), 1–37.

[DG 68]   E. De Giorgi, Un esempio di estremali discontinue per un problema variazionale di tipo ellittico, *Boll. Un. Mat. Ital.* **4** (1968), 135–137.

[Ev 86]   L. C. Evans, Quasiconvexity and partial regularity in the calculus of variations, *Arch. Rational Mech. Anal.* **95** (1986), 227–252.

[Fe 69]   H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[GS 85]   M. Giaquinta and J. Souček, Caccioppoli's inequality and Legendre-Hadamard condition, *Math. Ann.* **270** (1985), 105–107.

[GM 68]   E. Giusti and M. Miranda, Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni, *Boll. Un. Mat. Ital.* **2** (1968), 219–226.

[Gr 86]   M. Gromov, *Partial Differential Relations*, Springer-Verlag, New York, 1986.

[HLN 96]   W. Hao, S. Leonardi, and J. Nečas, An example of irregular solution to a nonlinear Euler-Lagrange elliptic system with real analytic coefficients. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **23** (1996), 57–67.

[Ku 55]   N. H. Kuiper, On C^1-isometric embeddings. I, *Nederl. Akad. Wetensch. Proc.* **A 58** (1955), 545–556.

[Ma 85]   P. Marcellini, Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals, *Manuscripta Math.* **51** (1985), 1–28.

[MP 98]   J. Matoušek and P. Plecháč, On functional separately convex hulls, *Discrete Comput. Geom.* **19** (1998), 105–130.

[Mo 66]   C. B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, New York, 1966.

[MS 96]   S. Müller and V. Šverák, Attainment results for the two-well problem by convex integration, in *Geometric Analysis and the Calculus of Variations* (J. Jost, ed.), Internat. Press, Cambridge, MA, 1996, 239–251.

[Na 54]   J. Nash, C^1 isometric imbeddings, *Ann. of Math.* **60** (1954), 383–396.

[NM 91]   V. Nesi and G. W. Milton, Polycrystalline configurations that maximize electrical resistivity, *J. Mech. Phys. Solids* **39** (1991), 525–542.

[Pe 93]   P. Pedregal, Laminates and microstructure, *European J. Appl. Math.* **4** (1993), 121–149.

[Sch 74]   V. Scheffer, Regularity and irregularity of solutions to nonlinear second order elliptic systems of partial differential equations and inequalities, Dissertation, Princeton University, 1974 (unpublished).

[Sv 90]   V. Šverák, Examples of rank-one convex functions, *Proc. Roy. Soc. Edinburgh* **A 114** (1990), 237–242.

[Sv 92a]   ———, Rank-one convexity does not imply quasiconvexity, *Proc. Roy. Soc. Edinburgh* **A 120** (1992), 185–189.

[Sv 92b]   ———, New examples of quasiconvex functions, *Arch. Rational Mech. Anal*. **119** (1992), 293–300.

[Ta 93]   L. Tartar, Some remarks on separately convex functions, in: *Microstructure and Phase Transitions*, *IMA Vol. Math. Appl*. **54** (D. Kinderlehrer, R. D. James, M. Luskin and J. L. Ericksen, eds.), Springer-Verlag, New York (1993), 191–204.