

New upper bounds on sphere packings I

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Abstract

We develop an analogue for sphere packing of the linear programming bounds for error-correcting codes, and use it to prove upper bounds for the density of sphere packings, which are the best bounds known at least for dimensions 4 through 36. We conjecture that our approach can be used to solve the sphere packing problem in dimensions 8 and 24.

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1. Introduction

The sphere packing problem asks for the densest packing of spheres into Euclidean space. More precisely, what fraction of \mathbb{R}^n can be covered by congruent balls that do not intersect except along their boundaries? This problem fits into a broad framework of packing problems, including error-correcting codes

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and spherical codes. Linear programming bounds [D] are the most powerful known technique for producing upper bounds in such problems. In particular, [KL] uses this technique to prove the best bounds known for sphere packing density in high dimensions. However, [KL] does not study sphere packing directly, but rather passes through the intermediate problem of spherical codes. In this paper, we develop linear programming bounds that apply directly to sphere packing, and study these bounds numerically to prove the best bounds known¹ for sphere packing in dimensions 4 through 36. In dimensions 8 and 24, our bounds are very close to the densities of the known packings: they are too high by factors of 1.000001 and 1.0007071 in dimensions 8 and 24, respectively. (The best bounds previously known were off by factors of 1.01216 and 1.27241.) We conjecture that our techniques can be used to prove sharp bounds in 8 and 24 dimensions.

The sphere packing problem in \mathbb{R}^n is trivial for $n = 1$, and the answer has long been known for $n = 2$: the standard hexagonal packing is optimal. For $n = 3$, Hales [Ha] has proved that the obvious packing, known as the “face-centered cubic” packing (equivalently, the A_3 or D_3 root lattice), is optimal, but his proof is long and difficult, and requires extensive computer calculation; as of December, 2002, it has not yet been published, but it is widely regarded as being likely to be correct. For $n \geq 4$ the problem remains unsolved. Upper and lower bounds on the density are known, but they differ by an exponential factor as $n \rightarrow \infty$. Each dimension seems to have its own peculiarities, and it does not seem likely that a single, simple construction will give the best packing in every dimension.

We begin with some basic background on sphere packings; for more information, see [CS]. Recall that a *lattice* in \mathbb{R}^n is a subgroup consisting of the integer linear combinations of a basis of \mathbb{R}^n . One important way to create a sphere packing is to start with a lattice $\Lambda \subset \mathbb{R}^n$, and center the spheres at the points of Λ , with radius half the length of the shortest nonzero vectors in Λ . Such a packing is called a *lattice packing*. Not every sphere packing is a lattice packing, and in fact it is plausible that in all sufficiently large dimensions, there are packings denser than every lattice packing. However, many important examples in low dimensions are lattice packings.

A more general notion than a lattice packing is a *periodic packing*. In periodic packings, the spheres are centered on the points in the union of finitely many translates of a lattice Λ . In other words, the packing is still periodic under translations by Λ , but spheres can occur anywhere in a fundamental parallelotope of Λ , not just at its corners (as in a lattice packing).

¹W.-Y. Hsiang has recently announced a solution of the 8-dimensional sphere packing problem [Hs], but the details are not yet public. His methods are apparently quite different from ours.

The *density* Δ of a packing is defined to be the fraction of space covered by the balls in the packing. Density is not necessarily well-defined for pathological packings, but in those cases one can take a lim sup of the densities for increasingly large finite regions. One can prove that periodic packings come arbitrarily close to the greatest packing density, so when proving upper bounds it suffices to consider periodic packings. Clearly, density is well-defined for periodic packings, so we will not need to worry about subtleties. See Appendix A for more details.

For many purposes, it is more convenient to talk about the *center density* δ . It is the number of sphere-centers per unit volume, if unit spheres are used in the packing. Thus,

$$\Delta = \frac{\pi^{n/2}}{(n/2)!} \delta,$$

since a unit sphere has volume $\pi^{n/2}/(n/2)!$. Of course, for odd n we interpret $(n/2)!$ as $\Gamma(n/2 + 1)$.

In most dimensions, there are not even any plausible conjectures for the densest sphere packing. The only exceptions are low dimensions (up to perhaps 8 or 10), and a handful of higher dimensions (such as 12, 16, and 24). The most striking examples are 8 and 24 dimensions. In those dimensions, the densest packings are undoubtedly the E_8 root lattice and the Leech lattice, respectively. The E_8 lattice is easy to define. It consists of all points of \mathbb{R}^8 whose coordinates are either all integers or all halves of odd integers, and sum to an even integer. A more illuminating characterization is as follows: E_8 is the unique lattice in \mathbb{R}^8 of covolume 1 such that all vectors v in the lattice have even norm $\langle v, v \rangle$. Such a lattice is called an even unimodular lattice. Even unimodular lattices exist only in dimensions that are multiples of 8, and in \mathbb{R}^8 there is only one, up to isometries of \mathbb{R}^8 . The Leech lattice is harder to write down explicitly; see [CS] for a detailed treatment. It is the unique even unimodular lattice in \mathbb{R}^{24} with no vectors of length $\sqrt{2}$. These two lattices have many remarkable properties and connections with other branches of mathematics, but so far these properties have not led to a proof that they are optimal sphere packings. We conjecture that our linear programming bounds can be used to prove optimality.

If linear programming bounds can indeed be used to prove the optimality of these lattices, it would not come as a complete surprise, because other packing problems in these dimensions can be solved similarly. The most famous example is the kissing problem: how many nonoverlapping unit balls can be arranged tangent to a given one? If we regard the points of tangency as a spherical code, the question becomes how many points can be placed on a sphere with no angles less than $\pi/3$. Odlyzko and Sloane [OS] and Levenshtein [Lev] independently used linear programming bounds to solve the kissing problem in

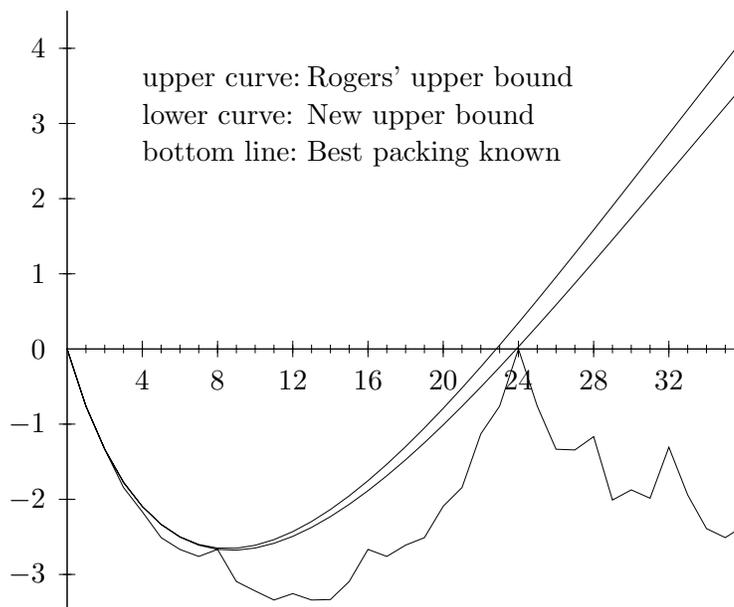


Figure 1. Plot of $\log_2 \delta + n(24 - n)/96$ vs. dimension n .

8 and 24 dimensions. (The solutions in dimensions 8 and 24 are obtained from the minimal nonzero vectors in the E_8 and Leech lattices.) Because we know *a priori* that the answer must be an integer, any upper bound within less than 1 of the truth would suffice. Remarkably, the linear programming bound gives the exact answer, with no need to take into account its integrality. By contrast, in most dimensions it gives a noninteger. The remarkable exactness seems to occur only in dimensions 1, 2, 8, and 24. We observe the same numerically in our case, but can prove it only for dimension 1.

Figure 1 compares our results with the best packings known as of December, 2002 (see Tables I.1(a) and I.1(b) of [CS, pp. xix, xx]), and the best upper bounds previously known in these dimensions (due to Rogers [Ro]). The graph was normalized for comparison with Figure 15 from [CS, p. 14].

2. Lattices, Fourier transforms, and Poisson summation

Given a lattice $\Lambda \subset \mathbb{R}^n$, the *dual lattice* Λ^* is defined by

$$\Lambda^* = \{y \mid \langle x, y \rangle \in \mathbb{Z} \text{ for all } x \in \Lambda\};$$

it is easily seen to be the lattice with basis given by the dual basis to any basis of Λ . The *covolume* $|\Lambda| = \text{vol}(\mathbb{R}^n/\Lambda)$ of a lattice Λ is the volume of any fundamental parallelootope. It satisfies $|\Lambda||\Lambda^*| = 1$. Given any lattice Λ with

shortest nonzero vectors of length r , the density of the corresponding lattice packing is

$$\frac{\pi^{n/2}}{(n/2)!} \left(\frac{r}{2}\right)^n \frac{1}{|\Lambda|},$$

and the center density is therefore $(r/2)^n/|\Lambda|$.

The Fourier transform of an L^1 function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ will be defined by

$$\widehat{f}(t) = \int_{\mathbb{R}^n} f(x)e^{2\pi i\langle x,t \rangle} dx.$$

PROPOSITION 2.1. *Let $\alpha = n/2 - 1$. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a radial function, then*

$$\widehat{f}(t) = 2\pi|t|^{-\alpha} \int_0^\infty f(r)J_\alpha(2\pi r|t|)r^{n/2} dr,$$

where “ $f(r)$ ” denotes the common value of f on vectors of length r .

For a proof, see Theorem 9.10.3 of [AAR]. Here J_α denotes the Bessel function of order α .

We will deal with functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ to which the Poisson summation formula applies; i.e., for every lattice $\Lambda \subset \mathbb{R}^n$ and every vector $v \in \mathbb{R}^n$,

$$(2.1) \quad \sum_{x \in \Lambda} f(x + v) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} e^{-2\pi i\langle v,t \rangle} \widehat{f}(t),$$

with both sides converging absolutely. It is not hard to verify that the right-hand side of the Poisson summation formula is the Fourier series for the left-hand side (which is periodic under translations by elements of Λ), but of course even when the sum on the left-hand side converges, some conditions are needed to make it equal its Fourier series.

For our purposes, we need only the following sufficient condition:

Definition 2.2. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *admissible* if there is a constant $\delta > 0$ such that $|f(x)|$ and $|\widehat{f}(x)|$ are bounded above by a constant times $(1 + |x|)^{-n-\delta}$.

Admissibility implies that f and \widehat{f} are continuous, and that both sides of (2.1) converge absolutely. These two conditions alone do not suffice for Poisson summation to hold, but admissibility does. For a proof for the integer lattice \mathbb{Z}^n , see Corollary 2.6 of Chapter VII of [SW]. The general case can be proved similarly, or derived by a linear change of variables.

We could define admissibility more broadly, to include every function to which Poisson summation applies, but the restricted definition above appears to cover all the useful cases, and is more concrete.

3. Principal theorems

Our principal result is the following theorem. It is similar in spirit to work of Siegel [S], but is capable of giving much better bounds. Gorbachev [Go] has independently discovered essentially the same result, with a slightly different proof. (He concentrates on deriving Levenshtein's bound using functions f for which \widehat{f} has fairly small support, but mentions that one could let the size of the support go to infinity.)

THEOREM 3.1. *Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is an admissible function, is not identically zero, and satisfies the following two conditions:*

- (1) $f(x) \leq 0$ for $|x| \geq 1$, and
- (2) $\widehat{f}(t) \geq 0$ for all t .

Then the center density of n -dimensional sphere packings is bounded above by

$$\frac{f(0)}{2^n \widehat{f}(0)}.$$

Notice that because \widehat{f} is nonnegative and not identically zero, we have $f(0) > 0$. If $\widehat{f}(0) = 0$, then we treat $f(0)/\widehat{f}(0)$ as $+\infty$, so the theorem is still true, although only vacuously.

Proof. It is enough to prove this for periodic packings, since they come arbitrarily close to the greatest packing density (see Appendix A). In particular, suppose we have a packing given by the translates of a lattice Λ by vectors v_1, \dots, v_N , whose differences are not in Λ . If we choose the scale so that the radius of the spheres in our packing is $1/2$ (i.e., no two centers are closer than 1 unit), then the center density is given by

$$\delta = \frac{N}{2^n |\Lambda|}.$$

By the Poisson summation formula (2.1),

$$\sum_{x \in \Lambda} f(x + v) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} e^{-2\pi i \langle v, t \rangle} \widehat{f}(t)$$

for all $v \in \mathbb{R}^n$. It follows that

$$\sum_{1 \leq j, k \leq N} \sum_{x \in \Lambda} f(x + v_j - v_k) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \widehat{f}(t) \left| \sum_{1 \leq j \leq N} e^{2\pi i \langle v_j, t \rangle} \right|^2.$$

Every term on the right is nonnegative, so the sum is bounded from below by the summand with $t = 0$, which equals $N^2 \widehat{f}(0)/|\Lambda|$. On the left, the vector

$x+v_j-v_k$ is the difference between two centers in the packing, so $|x+v_j-v_k| < 1$ if and only if $x = 0$ and $j = k$. Whenever $|x+v_j-v_k| \geq 1$, the corresponding term in the sum is nonpositive, so we get an upper bound of $Nf(0)$ for the entire sum. Thus,

$$Nf(0) \geq \frac{N^2 \widehat{f}(0)}{|\Lambda|},$$

i.e.,

$$\delta \leq \frac{f(0)}{2^n \widehat{f}(0)},$$

as desired. \square

This theorem was first proved by a more complicated argument, which is given in the companion paper [C].

The hypotheses and conclusion of Theorem 3.1 are invariant under rotating the function f . Hence, we can assume without loss of generality that f has radial symmetry, since otherwise we can replace f with the average of its rotations. The Fourier transform maps radial functions to radial functions, and Proposition 2.1 gives us the corresponding one-dimensional integral transform.

As an example of how to apply Theorem 3.1 in one dimension, consider the function $(1 - |x|)\chi_{[-1,1]}(x)$. It satisfies the hypotheses of Theorem 3.1 in dimension $n = 1$, because it is the convolution of $\chi_{[-1/2,1/2]}(x)$ with itself, and therefore its Fourier transform is

$$\left(\frac{\sin \pi t}{\pi t}\right)^2.$$

Thus, this function satisfies the hypotheses of Theorem 3.1. We get a bound of $1/2$ for the center density in one dimension, which is a sharp bound. This example generalizes to higher dimensions by replacing $\chi_{[-1/2,1/2]}(x)$ with the characteristic function of a ball about the origin. However, the bound obtained is only the trivial bound (density can be no greater than 1), so we omit the details. In later sections we apply Theorem 3.1 to prove nontrivial bounds.

It will be useful later to have the following alternative form of Theorem 3.1:

THEOREM 3.2. *Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is an admissible function satisfying the following three conditions:*

- (1) $f(0) = \widehat{f}(0) > 0$,
- (2) $f(x) \leq 0$ for $|x| \geq r$, and
- (3) $\widehat{f}(t) \geq 0$ for all t .

Then the center density of sphere packings in \mathbb{R}^n is bounded above by $(r/2)^n$.

Theorem 3.2 can be obtained either from rescaling the variables in Theorem 3.1 or from the following direct proof. For simplicity we deal only with the case of lattice packings, but as in the proof of Theorem 3.1 the argument extends to all periodic packings (and hence to all packings).

Proof for lattice packings. For lattice packings, the density bound in the theorem statement simply amounts to the claim that every lattice of covolume 1 contains a nonzero vector of length at most r . We will prove this first for lattices Λ of covolume $1 - \varepsilon$, and then let $\varepsilon \rightarrow 0+$. For such lattices,

$$\sum_{x \in \Lambda} f(x) = \frac{1}{1 - \varepsilon} \sum_{t \in \Lambda^*} \widehat{f}(t),$$

by Poisson summation. If all nonzero vectors in Λ had length greater than r , then all terms except $f(0)$ on the left-hand side would be nonpositive. Because all terms on the right-hand side are nonnegative, we would have

$$f(0) \geq \sum_{x \in \Lambda} f(x) = \frac{1}{1 - \varepsilon} \sum_{t \in \Lambda^*} \widehat{f}(t) \geq \frac{\widehat{f}(0)}{1 - \varepsilon}.$$

However,

$$\frac{\widehat{f}(0)}{1 - \varepsilon} = \frac{f(0)}{1 - \varepsilon} > f(0),$$

which is a contradiction. Thus, every lattice of covolume strictly less than 1 must have a nonzero vector of length r or less, and it follows that the same holds for covolume 1. \square

It seems natural to try to prove Theorem 3.2 by applying Poisson summation directly to a lattice of covolume 1, but some sort of rescaling and limiting argument seems to be needed. We included the proof to illustrate how to do this.

Logan [Lo] has studied the optimization problem from Theorem 3.2 in the one-dimensional case (for reasons unconnected to sphere packing), but we do not know of any previous study of the higher-dimensional cases. Unfortunately, these cases seem much more difficult than the one-dimensional case.

4. Homogeneous spaces

The space \mathbb{R}^n is a 2-point homogeneous space; i.e., its isometry group acts transitively on ordered pairs of points a given distance apart. By studying packing problems in homogeneous spaces, one can put Theorem 3.1 into a broader context, in which it can be seen to be analogous to previously known theorems about compact homogeneous spaces.

We start by reviewing the theory of compact homogeneous spaces. See Chapter 9 of [CS] for a more detailed treatment of this material. Suppose X is a compact 2-point homogeneous space. We assume that X is a connected Riemannian manifold, of positive dimension. We can write X as G/H , where (G, H) is a Gelfand pair of Lie groups. Then $L^2(X)$ is a Hilbert space direct sum of distinct irreducible representations of G , say $\widehat{\bigoplus_{j=0}^{\infty} V_j}$. For each j , evaluation gives a map $f_j : X \rightarrow V_j^*$, because V_j turns out to consist of continuous functions. We define

$$K_j(x, y) = \langle f_j(x), f_j(y) \rangle.$$

This is a positive definite kernel: for every finite subset $C \subseteq X$, we have

$$\sum_{x, y \in C} K_j(x, y) = \left| \sum_{x \in C} f_j(x) \right|^2 \geq 0.$$

Because of G -invariance, $K_j(x, y)$ depends only on the distance between x and y . This function of the distance is a zonal spherical function; we can define a way of measuring distance $t(x, y)$ and an ordering of the V_j 's so that $K_j(x, y)$ is a polynomial P_j of degree j evaluated at $t(x, y)$. In general, t maps $X \times X$ to $[0, 1]$, and $t(x, y) = 1$ if and only if $x = y$ (note that it is not a metric). For the unit sphere in \mathbb{R}^n , we take $t(x, y) = (1 + \langle x, y \rangle)/2$, and the polynomial P_j is the Jacobi polynomial $P_j^{(\alpha, \beta)}(t)$, where $\alpha = \beta = (n - 3)/2$.

Now suppose C is a finite subset of X . We get inequalities on C from the fact that for each j , the sum $\sum_{x \in C} f_j(x)$ has nonnegative norm. We can apply these inequalities as follows to get an upper bound for the size of C , in terms of the minimal distance between points of C :

THEOREM 4.1 (Delsarte [D]). *Suppose*

$$f(t) = \sum_{j=0}^m a_j P_j(t)$$

with $a_j \geq 0$ for all j and $f(t) \leq 0$ for $0 \leq t \leq \tau$. If $t(x, y) \leq \tau$ whenever x and y are distinct points of C , then

$$|C| \leq f(1)/a_0.$$

Proof. Suppose C satisfies $t(x, y) \leq \tau$ for all distinct $x, y \in C$. Then consider

$$\sum_{x, y \in C} f(t(x, y)).$$

This sum is bounded above by $|C|f(1)$ since $t(x, y) \leq \tau$ unless $x = y$, and is bounded below by $|C|^2 a_0$ since $f - a_0$ is a positive definite kernel. Thus, $|C| \leq f(1)/a_0$. \square

Theorem 4.1 is the analogue of Theorem 3.1 for compact homogeneous spaces. To see the analogy clearly, we need to study \mathbb{R}^n as a homogeneous space.

We can write \mathbb{R}^n as G/H , where G is the group of isometries of \mathbb{R}^n and $H = O(n)$. Then we need to decompose $L^2(\mathbb{R}^n)$ in terms of irreducible representations of G . It is no longer a direct sum, but it can be written as a direct integral; specifically, $L^2(\mathbb{R}^n) = \int_0^\infty \pi_r dr$, where π_r is the irreducible representation of G consisting of functions whose Fourier transforms are distributions with support on the sphere of radius r .

We can find the zonal spherical functions as follows. The representation π_r is generated by the functions $x \mapsto e^{2\pi i \langle x, y \rangle}$ with $|y| = r$, so π_r^* consists of functions on the sphere of radius r . The evaluation map from \mathbb{R}^n to π_r^* takes a point $x \in \mathbb{R}^n$ to the function $y \mapsto e^{2\pi i \langle x, y \rangle}$ on the sphere of radius r . Thus, the zonal spherical functions are given by

$$K_r(x_1, x_2) = \int_{|y|=r} e^{2\pi i \langle y, x_1 - x_2 \rangle} dy.$$

(This of course depends only on $|x_1 - x_2|$, and can be evaluated explicitly in terms of Bessel functions using Proposition 2.1.) In other words, they are given by functions whose Fourier transforms are delta functions on spheres centered at the origin. See Section 4.15 of [DM] for a more detailed discussion of this point of view.

Now the analogue of positive combinations of the zonal spherical functions $P_j(t)$ from the compact case is radial functions with nonnegative Fourier transform, and we can see that Theorem 3.1 corresponds to 4.1.

5. Conditions for a sharp bound

In one dimension, we have already seen how to use Theorem 3.1 to solve the (admittedly trivial) sphere packing problem. Based on numerical evidence and analogy with the kissing problem, we conjecture that it can also be used to get sharp bounds in dimensions 2, 8, and 24. For reasons to be explained shortly, it is more convenient to work with Theorem 3.2 instead of Theorem 3.1, so we shall do so; we can convert everything to the framework of Theorem 3.1 by rescaling the variables.

In each of dimensions 1, 2, 8, and 24, the densest known packing is a lattice packing, given by a lattice that is homothetic to its dual. This lattice is \mathbb{Z} in dimension 1, the A_2 root lattice (i.e., the hexagonal lattice) in dimension 2, the E_8 root lattice in dimension 8, and the Leech lattice in dimension 24. See [CS] for information about these lattices. Each of these lattices except A_2 actually equals its dual, but that is not true for A_2 . However, we can rescale A_2 so that the rescaled lattice is *isodual*, i.e., isometric with its own dual (in this case, via a rotation).

Suppose Λ is any lattice of covolume 1, such as an isodual lattice, and f is a radial function giving a sharp bound on Λ via Theorem 3.2 (i.e., r is the length of the shortest nonzero vector of Λ). By Poisson summation, we have

$$\sum_{x \in \Lambda} f(x) = \sum_{x \in \Lambda^*} \widehat{f}(x).$$

Given the inequalities on f and \widehat{f} , the only way this equation can hold is if f vanishes on $\Lambda \setminus \{0\}$ and \widehat{f} vanishes on $\Lambda^* \setminus \{0\}$. This puts strong constraints on f and \widehat{f} . When Λ is isodual, the vector lengths in Λ and Λ^* are the same, so f and \widehat{f} must both vanish on $\Lambda \setminus \{0\}$.

Of course, there are similar constraints on f for a sharp bound in Theorem 3.1 (as opposed to Theorem 3.2), but we prefer to work with this context, since the isodual normalizations are more pleasant, and are the standard normalizations for E_8 and the Leech lattice.

It is natural to try to guess f from our knowledge of its roots. For example, in one dimension we could try

$$f(x) = (1 - x^2) \prod_{k \geq 2} \left(1 - \frac{x^2}{k^2}\right)^2 = \frac{1}{1 - x^2} \left(\frac{\sin \pi x}{\pi x}\right)^2,$$

which clearly satisfies $f(x) \leq 0$ for $|x| \geq 1$ and has the right zeros. In fact, one can compute its Fourier transform and check that \widehat{f} is nonnegative everywhere (it has support $[-1, 1]$ and is positive in $(-1, 1)$), so it solves the sphere packing problem in dimension 1, in a different way from the function in the previous section.

Unfortunately, it seems difficult to generalize this approach to higher dimensions. One can generalize this function by replacing the sine function with a Bessel function (see Proposition 6.1), but that does not yield a sharp bound in dimensions greater than 1. Attempts to write down a product with zeros at the right places for a sharp bound lead to products that seem intractable.

One important thing to note is that for a sharp bound above dimension 1, it is not possible for \widehat{f} to have compact support, as it does in the examples involving sine and Bessel functions. If it did, then f could not have sufficiently densely-spaced zeros. To be precise, if \widehat{f} is a radial function with support in the ball $B(0, R)$ of radius R about the origin, then the common value $f(r)$ on vectors of radius r satisfies

$$f(r) = \int_{B(0,R)} \widehat{f}(t) e^{2\pi i \langle rx, t \rangle} dt,$$

where x is any vector with $|x| = 1$. This defines an entire function of r , and for all complex r ,

$$|f(r)| \leq e^{2\pi R|r|} \int_{B(0,R)} |\widehat{f}(t)| dt,$$

so f is a function of exponential type, and Jensen's formula implies that f can have at most linearly spaced zeros (see Section 15.20 of [Ru]). However, the nonzero vectors in the Leech lattice have lengths $\sqrt{2k}$ for integers $k > 1$, and those in E_8 have lengths $\sqrt{2k}$ for integers $k > 0$. The function f must vanish at those vector lengths, and these roots are too densely spaced for \hat{f} to have compact support. Of course, f also cannot have compact support (because \hat{f} vanishes on $\Lambda^* \setminus \{0\}$).

One might wonder whether the restriction to radial functions is misleading: perhaps a nonradial function could be constructed more naturally. We cannot rule out that possibility, but consider it unlikely. Even if f is not radial, a sharp bound implies that f and \hat{f} must vanish on concentric spheres centered at the origin and passing through the nonzero points of Λ and Λ^* , respectively. The simplest reason is that if f proves that Λ is optimal, then it proves the same for every rotation of Λ . Alternatively, after rotational symmetrization f and \hat{f} must vanish on these spheres, and the inequalities on their values then imply that they must have vanished before symmetrization (the average of nonnegative values vanishes if and only if the values all do). It would seem strange for f to vanish on these spheres without being radial, but of course we cannot rule it out.

6. Stationary points

We do not know how to use Theorem 3.1 to match the best density bound known in high dimensions, that of Kabatiansky and Levenshtein [KL]. However, it provides a new proof of the second-best bound known, due to Levenshtein [Lev]:

$$\Delta \leq \frac{j_{n/2}^n}{(n/2)!^2 4^n},$$

where j_t is the smallest positive zero of the Bessel function J_t . (For more information about the asymptotics of this bound and how it compares with other bounds, see page 19 of [CS], but note that equation (42) is missing the exponent in $j_{n/2}^n$.) We will show how to use a calculus of variations argument to find functions that prove that bound. This approach is analogous to that used by Levenshtein. Yudin [Y] has also given a proof of Levenshtein's bound that seems reminiscent of our general approach, but not identical.

To construct a function f for use in Theorem 3.1, we begin by supposing that there is a function g such that $f(x) = (1 - |x|^2)\hat{g}(x)^2$, so that f automatically satisfies the inequality $f(x) \leq 0$ for $|x| \geq 1$. (We write \hat{g} instead of g for convenience later.) Assume that g is radial, and has support in the ball of radius R about the origin; we discuss these assumptions later. Notice that nothing in our setup requires \hat{f} to be nonnegative, so we must check for this property later.

We have $\widehat{f} = g * (g + Lg)$, where $*$ denotes convolution and

$$L = \frac{1}{4\pi^2} \sum_{j=1}^n \frac{\partial^2}{\partial t_j^2},$$

so that under the Fourier transform, L corresponds to multiplication by $-|x|^2$. We require $f(0) = 1$, i.e., $\int_{\mathbb{R}^n} g = 1$. We want to maximize $\widehat{f}(0)$, subject to this constraint. Notice that

$$\widehat{f}(0) = \int_{\mathbb{R}^n} g(0 - t)(g + Lg)(t) dt = \int_{\mathbb{R}^n} g(t)(g + Lg)(t) dt = \int_{\mathbb{R}^n} g(g + Lg),$$

because g is radial (and hence even).

Perturb g to $g + h$, where h has integral zero (so that $f(0)$ does not change) and has support in the ball of radius R . Then the first order change in $\widehat{f}(0)$ is

$$\int_{\mathbb{R}^n} (gh + hg + gLh + hLg) = 2 \int_{\mathbb{R}^n} h(g + Lg)$$

(where the equality comes from integration by parts).

In order to have this vanish whenever $\int_{\mathbb{R}^n} h = 0$, the function $g + Lg$ must be constant (within the support of g), so for some constant c we have

$$g + Lg = c\chi_R,$$

where χ_R is the characteristic function of the ball of radius R about 0. Then it follows from Proposition 2.1 that

$$(1 - |x|^2)\widehat{g}(x) = c(R/|x|)^{n/2} J_{n/2}(2\pi R|x|).$$

Now $2\pi R$ must be a zero of $J_{n/2}$ for the right-hand side to vanish at $|x| = 1$, and c is determined (given R) by

$$1 = \int_{\mathbb{R}^n} g = \int_{\mathbb{R}^n} g + Lg = \int_{\mathbb{R}^n} c\chi_R = cR^n \pi^{n/2} / (n/2)!.$$

(The function $g + Lg$ has the same integral as g , because integration by parts shows that Lg has integral 0.)

In fact, we take $2\pi R$ to be the first positive root $j_{n/2}$ of $J_{n/2}$, in order to make \widehat{f} nonnegative everywhere. To check that it is indeed nonnegative everywhere then, we use the equation

$$\widehat{f} = g * (g + Lg) = g * (c\chi_R).$$

If g is nonnegative everywhere, then so is \widehat{f} . We can now determine g explicitly:

We know that $g + Lg = c\chi_R$. The differential operator $1 + L$ takes a radial function $u(|t|)$ to

$$\frac{u''(|t|)}{4\pi^2} + \frac{n-1}{4\pi^2|t|} u'(|t|) + u(|t|).$$

It follows from this and the differential equation satisfied by the Bessel functions that

$$J_\alpha(2\pi|t|)/|t|^\alpha$$

is in the kernel of $1 + L$, if $\alpha = n/2 - 1$. From that, one can check that

$$g(t) = \left(-\frac{R^\alpha}{J_\alpha(2\pi R)} \frac{J_\alpha(2\pi|t|)}{|t|^\alpha} + 1 \right) \chi_R(t).$$

One can check that the minimum of $J_\alpha(2\pi|t|)/|t|^\alpha$ occurs at

$$|t| = j_{n/2}/(2\pi) = R,$$

so g is nonnegative everywhere, as desired, since $J_\alpha(2\pi R) = J_\alpha(j_{n/2}) < 0$ (the roots of the functions J_α and $J_{\alpha+1} = J_{n/2}$ are interlaced).

Thus, we have constructed a function f satisfying the hypotheses of Theorem 3.1. It has $f(0) = 1$ and

$$\widehat{f}(0) = \int_{\mathbb{R}^n} g(g + Lg) = c \int_{\mathbb{R}^n} g = c = \frac{(n/2)!}{\pi^{n/2} R^n}.$$

Because $R = j_{n/2}/(2\pi)$, this function proves Levenshtein's sphere packing bound:

PROPOSITION 6.1. *The function*

$$f(x) = \frac{J_{n/2}(j_{n/2}|x|)^2}{(1 - |x|^2)|x|^n}$$

satisfies the hypotheses of Theorem 3.1, and leads to the upper bound

$$\frac{j_{n/2}^n}{(n/2)!^2 4^n}$$

for the densities of n -dimensional sphere packings.

This function does not optimize the bound in Theorem 3.1, but it does optimize it within the class of functions whose Fourier transforms have support in the ball of radius $j_{n/2}/\pi$ about the origin. This was first proved by Gorbachev [Go]. For another proof, see [C].

It is not a coincidence that this proves exactly the same bound as in Levenshtein's paper [Lev]. Levenshtein studies spherical codes with minimal angular separation θ , and derives his sphere packing bound from letting $\theta \rightarrow 0$. Under that limit, the functions he uses in the linear programming bounds for spherical codes become our Bessel function example. This can be proved using the limit

$$(6.1) \quad \lim_{j \rightarrow \infty} j^{-\alpha} P_j^{(\alpha, \beta)} \left(\cos \frac{z}{j} \right) = (z/2)^{-\alpha} J_\alpha(z),$$

which is 10.8 (41) in [EMOT].

The functions we have obtained are not optimal in any dimension above 1. There are two reasons for this. First, we restricted our attention to functions such that \hat{f} has compact support, and as we have seen in Section 5, that cannot be true if we are to get sharp bounds. Second, and more importantly, we implicitly considered only functions such that \hat{f} is positive within its support. The problem is that if \hat{f} vanishes somewhere, then the perturbations h must be chosen so as not to push \hat{f} below zero. We made no attempt to do so, and therefore could not find any stationary points for which \hat{f} has zeros within its support. We have seen in Section 5 that zeros are essential for sharp bounds.

Unfortunately, it seems difficult to adapt the stationary point argument to deal with these difficulties. One approach is to assume that \hat{f} has zeros at certain locations, and look at only the perturbations h that, up to first order, do not push \hat{f} below zero. Although we can set up such problems, we have not been able to solve them.

7. Numerical results

It is possible to get numerical results by using linear programming to find functions for use in Theorem 3.1, as was done for the kissing problem by Odlyzko and Sloane in [OS]. The idea is to fix one of $f(0)$ and $\hat{f}(0)$, and view extremizing the other as an infinite-dimensional linear programming problem. One can try to approximate it with a finite-dimensional problem, and solve it on a computer. Although we obtained some numerical results this way, it was cumbersome and generally ineffective. Instead, we use the following approach.

First, consider trying to use our techniques to bound the density of an isodual lattice. There is no reason for optimal sphere packings to be isodual lattices, and for example in three dimensions they are known not to be, but it is convenient to use this case as a stepping stone.

PROPOSITION 7.1. *Suppose $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is a radial, admissible function, is not identically zero, and satisfies the following three properties:*

- (1) $g(0) = 0$,
- (2) $g(x) \geq 0$ for $|x| \geq r$, and
- (3) $\hat{g} = -g$.

Then every isodual lattice in dimension n must contain a nonzero vector of length at most r .

Proof of special case. For simplicity, we deal only with the case in which $g(x) > 0$ for $|x| \gg 0$. Let $\Lambda \cong \Lambda^*$ be an isodual lattice. By Poisson summation,

$$\sum_{x \in \Lambda} g(x) = \sum_{x \in \Lambda^*} \hat{g}(x) = - \sum_{x \in \Lambda} g(x),$$

so

$$\sum_{x \in \Lambda} g(x) = 0.$$

In order for the sum not to be positive, the lattice Λ must contain some nonzero vector of length at most r . \square

We can actually remove the hypothesis that $g(x) > 0$ for $|x| \gg 0$ from the proof of Proposition 7.1, by using a scaling trick, as in the proof of Theorem 3.2. However, we omit the details, because the hypothesis holds in all our numerical examples.

Notice that given any function f that proves a bound in Theorem 3.2, we can produce a g that proves the same bound in Proposition 7.1, by taking $g = \hat{f} - f$. Thus, the bound we get for isodual lattices is at least as good as that for general sphere packings. In principle, it could be better, but in practice we find that it is not (as we explain below).

We can find excellent functions for use in Proposition 7.1 as follows. Let $L_k^\alpha(x)$ be the Laguerre polynomials orthogonal with respect to the measure $e^{-x}x^\alpha dx$ on $[0, \infty)$. Set $\alpha = n/2 - 1$, and define

$$g_k(x) = L_k^\alpha(2\pi|x|^2)e^{-\pi|x|^2}$$

for $k \geq 0$; we suppress the dependence on n in our notation. These functions form a basis for the radial eigenfunctions of the Fourier transform, with eigenvalues $(-1)^k$ (see Section 4.23 and equation (4.20.3) of [Leb]).

To find a function g for use in Proposition 7.1, we consider a linear combination of $g_1, g_3, \dots, g_{4m+3}$, and require it to have a root at 0 and m double roots at z_1, \dots, z_m . (Counting degrees of freedom suggests that there should be a unique such function, up to scaling.) We then choose the locations of z_1, \dots, z_m to minimize the value r of the last sign change of g . To make this choice, we do a computer search. Specifically, we make an initial guess for the locations of z_1, \dots, z_m , and then see whether we can perturb them to decrease r . We repeat the perturbations until reaching a local optimum. Strictly speaking, we cannot prove that it ever converges, or comes anywhere near the global optimum. However, it works well in practice. At any rate this cannot affect the validity of our bounds, only their optimality given m . As we increase m , this method should give better and better bounds, which should converge to the best bounds obtainable using Proposition 7.1.

This method gives good functions for use in Proposition 7.1, but naturally bounds on all sphere packings would be better than bounds only on isodual lattices. We can turn these functions into functions satisfying the hypotheses of Theorem 3.2, without changing r , as follows.

Let h be a linear combination of $g_0, g_2, \dots, g_{4m+2}$ with double zeros at z_1, \dots, z_m , such that $g + h$ has a double zero at r . Then in the examples we have computed, $g + h$ has constant sign, which we can take to be positive. We

end up with a function $f = -g + h$ that is nonpositive outside radius r , and whose Fourier transform $\widehat{f} = g + h$ is nonnegative everywhere; furthermore, $f(0) = \widehat{f}(0)$. Thus, we can apply Theorem 3.2 to get the same bound for general sphere packings that g proves for isodual lattices. Notice that it is *not* clear *a priori* that f must satisfy all the hypotheses of Theorem 3.2, but it happens in all of our numerical examples.

Figure 1 (from §1) and Table 3 (from Appendix C) illustrate the bounds this method produces, using $m = 6$ as the number of forced double zeros (except in dimension 7 and lower, where $m = 5$ suffices for the accuracy we desire). We also list for comparison Rogers’ bound [Ro], which was previously the best bound known in dimensions 4 through 36. The choices of forced double roots are described in Table 4 (from Appendix C).

These bounds were calculated using a computer. However, the mathematics behind the calculations is rigorous. In particular, we use exact rational arithmetic, and apply Sturm’s theorem to count real roots and make sure we do not miss any sign changes. More precisely, we take the quantities $2\pi z_i^2$ to be rational numbers. That is convenient, because the functions $g_k(x)$ are polynomials in $2\pi|x|^2$ with rational coefficients (times Gaussians, which have no sign changes). The one subtlety is that in constructing h , we want $g + h$ to have a double root at r , and $2\pi r^2$ is generally not rational. Instead of doing this computation using floating point arithmetic, we replace $2\pi r^2$ by a nearby rational number and then determine h exactly, using that value of r . To do so, we must increase r slightly, but of course we can make the increase as small as we wish.

In dimensions 8 and 24, we carried out the calculations for $m = 11$. The resulting upper bounds are within factors of 1.000001 and 1.0007071 of equality, respectively. More precisely, in dimension 8 we take

$$2\pi r^2 = 12.56637375,$$

and in dimension 24 we take

$$2\pi r^2 = 25.1342216.$$

The forced double roots we used to achieve these bounds are given in Tables 1 and 2.

i	1	2	3	4	5	6
$2\pi z_i^2$	25.1328	37.6995	50.2678	62.8463	75.4579	88.2463
i	7	8	9	10	11	
$2\pi z_i^2$	101.3618	115.6443	131.0298	150.0861	174.2876	

Table 1. Forced double roots for $m = 11, n = 8$.

i	1	2	3	4	5	6
$2\pi z_i^2$	37.705	50.285	62.893	75.578	88.454	101.737
i	7	8	9	10	11	
$2\pi z_i^2$	115.776	131.035	148.162	168.215	193.766	

Table 2. Forced double roots for $m = 11$, $n = 24$.

Our numerical results lead us to the following conjectures.

CONJECTURE 7.2. *The smallest possible value of r in Proposition 7.1 equals that in Theorem 3.2, and for each optimal g from Proposition 7.1, there exists an optimal f from Theorem 3.2 such that $g = \hat{f} - f$.*

CONJECTURE 7.3. *There exist functions that satisfy the hypotheses of Theorem 3.2 and solve the sphere packing problem in dimensions 2, 8, and 24.*

CONJECTURE 7.4. *The numerical method described above gives bounds that converge (as $m \rightarrow \infty$) to the optimal bounds obtainable using Theorem 3.2.*

8. Uniqueness

It is natural to ask whether the densest sphere packing in \mathbb{R}^n is unique. Of course, it is trivially not unique, since for example removing a single sphere does not change the global density. However, it is conjectured that E_8 and the Leech lattice are unique among periodic packings. By contrast, in \mathbb{R}^3 there are infinitely many distinct periodic packings of maximal density. Our techniques can be used to prove this uniqueness in 8 and 24 dimensions, given the following slight strengthening of Conjecture 7.3. At the same time, we will deal with the hexagonal lattice, although uniqueness in that case has long been known.

Let Λ_2 , Λ_8 , and Λ_{24} denote the isodual scaling of the hexagonal lattice, the E_8 root lattice, and the Leech lattice, respectively.

CONJECTURE 8.1. *For $n \in \{2, 8, 24\}$, there exists a function that satisfies the hypotheses of Theorem 3.2 to prove that Λ_n is the densest packing in \mathbb{R}^n . Furthermore, this function and its Fourier transform have roots only at the vector lengths in Λ_n .*

First, let n be 8 or 24, so that Λ_n is an even unimodular lattice (we will discuss the $n = 2$ case below). Let f be a function satisfying the conclusions of Conjecture 8.1. Suppose we have a maximally dense packing given by the translates of a lattice Λ by vectors v_1, \dots, v_N , whose differences are not in Λ .

Without loss of generality, we can assume that $|\Lambda| = N$ and $v_1 = 0$. Note that $|\Lambda| = N$ implies that the packing uses balls of the same radius as those in Λ_n . By Poisson summation,

$$\sum_{1 \leq j, k \leq N} \sum_{x \in \Lambda} f(x + v_j - v_k) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \widehat{f}(t) \left| \sum_{1 \leq j \leq N} e^{2\pi i \langle v_j, t \rangle} \right|^2,$$

and the $f(0)$ terms cancel the $\widehat{f}(0)$ term, so we can draw the usual conclusions from the inequalities on each side. In particular, each vector $x + v_j - v_k$ must occur at a root of f . Now we can apply the following lemma:

LEMMA 8.2. *Suppose S is a subset of \mathbb{R}^n such that $0 \in S$, there are n linearly independent vectors in S , and for all $x, y \in S$, the distance $|x - y|$ is the square root of an even integer. Then the subgroup of \mathbb{R}^n generated by S is an even integral lattice.*

Recall that an *integral lattice* is a lattice in which the inner product of each pair of vectors is an integer; it is *even* if every vector has even norm.

Proof. For all $x, y \in S$, their inner product $\langle x, y \rangle$ is an integer, because

$$\langle x, y \rangle = (|x - 0|^2 + |y - 0|^2 - |x - y|^2)/2 \in \mathbb{Z},$$

and the norm $|x|^2$ of any element $x \in S$ is an even integer. It follows that the same facts hold for integer linear combinations of elements of S . The restriction on norms implies that S generates a discrete subgroup of \mathbb{R}^n , and hence a lattice (because S spans \mathbb{R}^n by assumption). Now the conditions above on inner products and norms amount to the definition of an even integral lattice. \square

Let L be the subgroup of \mathbb{R}^n generated by our periodic packing. By Lemma 8.2, it is an even integral lattice. In any integral lattice, the covolume is always the square root of an integer, since its square is the determinant of a Gram matrix, which is an integral matrix. Thus, L has at most one point per unit volume in \mathbb{R}^n , with equality if and only if L is unimodular. However, the periodic packing has one sphere per unit volume in \mathbb{R}^n , because $|\Lambda| = N$. It follows that the periodic packing is in fact the lattice packing determined by L : if any spheres from the lattice packing were missing from the periodic packing, then by periodicity the numbers of spheres per unit volume would be strictly smaller. Thus, our packing comes from an even unimodular lattice, and that lattice must have minimal norm 2 in \mathbb{R}^8 and 4 in \mathbb{R}^{24} for the density to be right. Such lattices are unique (see Chapters 16 and 18 of [CS]). Thus, we have shown that Conjecture 8.1 implies that E_8 and the Leech lattice are the only periodic packings of maximal density in \mathbb{R}^8 and \mathbb{R}^{24} , respectively.

For \mathbb{R}^2 , this argument requires a slight modification, because the isodual scaling of the hexagonal lattice is not an even unimodular lattice. However, the modification is not hard. As above, in any maximally dense periodic packing, the distances between points must occur among the distances in the hexagonal lattice. Of course, we can choose any scaling we prefer; the most convenient is that of the A_2 root lattice, given by

$$A_2 = \{(x_0, x_1, x_2) \in \mathbb{Z}^3 : x_0 + x_1 + x_2 = 0\},$$

because A_2 is an even integral lattice. As above, our periodic packing must then be contained in an even integral lattice L . However, the unimodularity argument no longer applies. Fortunately, we do not need it: because L is even, its minimal norm is at least 2, so L determines a sphere packing with spheres of the same radius as in our periodic packing. This sphere packing contains the original periodic packing. If the periodic packing did not use all these spheres, then its density would be lower than that of L . Thus, it is a lattice packing, and it is well known and easy to prove that A_2 is the unique densest lattice in two dimensions.

It is worth noting that the arguments above do not require the full strength of Conjecture 8.1. In particular, we never make use of restrictions on the roots of the Fourier transform. Furthermore, in the $n = 2$ case, we do not even require as strong a condition on the function's roots: A_2 does not have vectors of every even norm, but our argument allows the function to have roots corresponding to the missing norms.

Appendix A. Technicalities about density

In this appendix, we provide precise statements and references for what it means for a packing to have a density and whether there is a maximally dense packing. Much of our discussion closely follows Section I of [K].

Let \mathcal{P} be any sphere packing in \mathbb{R}^n . We say that \mathcal{P} has density Δ if for all $p \in \mathbb{R}^n$, we have

$$\Delta = \lim_{r \rightarrow \infty} \frac{\text{vol}(B(p, r) \cap \mathcal{P})}{\text{vol } B(p, r)},$$

where $B(p, r)$ is the ball of radius r centered at p , and $B(p, r) \cap \mathcal{P}$ consists of those parts of the balls in \mathcal{P} that lie within $B(p, r)$. It is proved in [Gr] that if this limit exists for one p , then it exists for all p and is equal for all p . We say that the packing has uniform density if the limit exists uniformly for all p . In that case, [Gr] shows that for every compact set R that is the closure of its interior and every point p ,

$$\Delta = \lim_{r \rightarrow \infty} \frac{\text{vol}((rR + p) \cap \mathcal{P})}{\text{vol } rR},$$

where of course $rR + p$ denotes R scaled by a factor of r and translated by p .

Although not every packing has a density, every packing has an upper density, defined by

$$\Delta = \limsup_{r \rightarrow \infty} \sup_{p \in \mathbb{R}^n} \frac{\text{vol}(B(p, r) \cap \mathcal{P})}{\text{vol} B(p, r)}.$$

It is proved in [Gr] that the supremum of all upper densities is achieved by a uniformly dense packing.

Periodic packings are the most convenient ones for our purposes. Using the above results, it is easy to see that they come arbitrarily close to the greatest possible density, as follows. Suppose Δ is the maximum packing density in \mathbb{R}^n , and let \mathcal{P} be a uniformly dense packing of density Δ . Let R be the fundamental parallelotope of any lattice $\Lambda \subset \mathbb{R}^n$. We know that

$$\Delta = \lim_{r \rightarrow \infty} \frac{\text{vol}(rR \cap \mathcal{P})}{\text{vol} rR}.$$

Let $\varepsilon > 0$. If we choose r large enough, then the total volume of the spheres in \mathcal{P} that lie *entirely* within rR is within $\varepsilon \text{vol} rR$ of $\Delta \text{vol} rR$, because only a negligible fraction of the spheres can intersect the sides of rR . Define a periodic packing \mathcal{P}' by taking all the spheres of \mathcal{P} that lie entirely within rR , and also including all translations of them by $r\Lambda$. Then this periodic packing has density at least $\Delta - \varepsilon$.

Appendix B. Other convex bodies

Our methods are not limited to studying sphere packings, but instead apply to packings with translates of any convex, symmetrical body. In fact, it is straightforward to prove the following generalization of our main theorem:

THEOREM B.1. *Let C be a convex body in \mathbb{R}^n , symmetric with respect to the origin. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is an admissible function, is not identically zero, and satisfies the following two conditions:*

- (1) $f(x) \leq 0$ for $x \notin C$, and
- (2) $\widehat{f}(t) \geq 0$ for all t .

Then all packings with translates of C have density bounded above by

$$\frac{\text{vol}(C)f(0)}{2^n \widehat{f}(0)}.$$

Unfortunately, when C is not a sphere, there does not seem to be a good analogue of the reduction to radial functions in Theorem 3.1. That makes these cases somewhat less convenient to deal with. There always exist functions that

prove the trivial density bound of 1: let $\chi_{C/2}$ be the characteristic function of the scaled body $C/2$, and let $f = \chi_{C/2} * \chi_{C/2}$.

Appendix C. Numerical data

Table 3 compares the best packings known, the previous best upper bound known, and our bound. The packings and previous bounds are from [CS] (see in particular Tables I.1(a) and I.1(b) on pages xix and xx). Table 4 lists the choices of forced double roots that lead to our new bounds.

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Dimension	Best Packing Known	Rogers' Bound	New Upper Bound
1	0.5	0.5	0.5
2	0.28868	0.28868	0.28868
3	0.17678	0.1847	0.18616
4	0.125	0.13127	0.13126
5	0.08839	0.09987	0.09975
6	0.07217	0.08112	0.08084
7	0.0625	0.06981	0.06933
8	0.0625	0.06326	0.06251
9	0.04419	0.06007	0.05900
10	0.03906	0.05953	0.05804
11	0.03516	0.06136	0.05932
12	0.03704	0.06559	0.06279
13	0.03516	0.07253	0.06870
14	0.03608	0.08278	0.07750
15	0.04419	0.09735	0.08999
16	0.0625	0.11774	0.10738
17	0.0625	0.14624	0.13150
18	0.07508	0.18629	0.16503
19	0.08839	0.24308	0.21202
20	0.13154	0.32454	0.27855
21	0.17678	0.44289	0.37389
22	0.33254	0.61722	0.51231
23	0.5	0.87767	0.71601
24	1.0	1.27241	1.01998
25	0.70711	1.8798	1.48001
26	0.57735	2.8268	2.18614
27	0.70711	4.3252	3.28537
28	1.0	6.7295	5.02059
29	0.70711	10.642	7.79782
30	1.0	17.094	12.30390
31	1.2095	27.880	19.71397
32	2.5658	46.147	32.06222
33	2.2220	77.487	52.90924
34	2.2220	131.94	88.55925
35	2.8284	227.71	150.29783
36	4.4394	398.25	258.54994

Table 3. Lower and upper bounds on center density, using $m = 6$.

Dimension	$2\pi r^2$	$2\pi z_1^2$	$2\pi z_2^2$	$2\pi z_3^2$	$2\pi z_4^2$	$2\pi z_5^2$	$2\pi z_6^2$
2	7.25520	21.77	29.02	50.79	65.34	90.19	
3	8.19385	22.00	31.63	49.10	61.80	80.01	
4	9.10543	22.46	33.49	48.69	62.03	79.73	
5	9.99512	23.04	34.87	48.90	63.04	80.44	
6	10.86682	23.70	35.98	49.43	64.07	81.61	
7	11.72351	24.41	36.94	50.13	65.04	82.94	
8	12.56674	25.14	37.74	50.50	64.08	79.03	99.37
9	13.39945	25.89	38.57	51.30	64.99	80.13	100.44
10	14.22261	26.66	39.38	52.13	65.89	81.24	101.56
11	15.03741	27.44	40.18	52.97	66.79	82.34	102.71
12	15.84481	28.23	40.98	53.82	67.69	83.44	103.89
13	16.64565	29.02	41.77	54.67	68.61	84.51	105.08
14	17.44063	29.82	42.57	55.53	69.53	85.58	106.28
15	18.23036	30.61	43.37	56.39	70.46	86.65	107.48
16	19.01538	31.42	44.18	57.25	71.39	87.71	108.68
17	19.79617	32.22	44.99	58.12	72.34	88.77	109.88
18	20.57315	33.02	45.80	58.99	73.29	89.84	111.08
19	21.34670	33.82	46.62	59.86	74.24	90.90	112.28
20	22.11717	34.63	47.45	60.74	75.20	91.97	113.48
21	22.88488	35.44	48.27	61.63	76.17	93.04	114.68
22	23.65012	36.24	49.10	62.51	77.14	94.12	115.88
23	24.41314	37.05	49.94	63.41	78.12	95.20	117.08
24	25.17419	37.86	50.78	64.30	79.09	96.28	118.29
25	25.93349	38.66	51.62	65.20	80.08	97.37	119.50
26	26.69125	39.47	52.46	66.11	81.07	98.46	120.71
27	27.44766	40.28	53.31	67.02	82.06	99.55	121.92
28	28.20290	41.09	54.17	67.93	83.06	100.64	123.13
29	28.95711	41.90	55.02	68.85	84.06	101.74	124.34
30	29.71046	42.72	55.88	69.77	85.06	102.84	125.56
31	30.46307	43.53	56.74	70.69	86.07	103.94	126.77
32	31.21508	44.34	57.61	71.62	87.08	105.05	127.99
33	31.96659	45.16	58.47	72.56	88.09	106.16	129.21
34	32.71772	45.97	59.34	73.49	89.10	107.27	130.43
35	33.46857	46.79	60.21	74.43	90.12	108.38	131.65
36	34.21922	47.60	61.09	75.37	91.14	109.49	132.88

Table 4. Values of $2\pi r^2$ and $2\pi z_i^2$ for $1 \leq i \leq 6$.

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