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The best constant for the centered Hardy-Littlewood maximal inequality

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Abstract

We find the exact value of the best possible constant C for the weak-type $(1,1)$ inequality for the one-dimensional centered Hardy-Littlewood maximal operator. We prove that C is the largest root of the quadratic equation $12C^2 - 22C + 5 = 0$ thus obtaining $C = 1.5675208\ldots$. This is the first time the best constant for one of the fundamental inequalities satisfied by a *centered* maximal operator is precisely evaluated.

1. Introduction

Maximal operators play a central role in the theory of differentiation of functions and also in Complex and Harmonic Analysis. In general one considers a certain collection of sets $\mathcal{C}$ in $\mathbb{R}^n$ and then given any locally integrable function f, at each x one measures the maximal average value of f with respect to the collection $\mathcal{C}$, translated by x. Then it is of fundamental importance to obtain certain regularity properties of this operators such as weak-type inequalities or L^p-boundedness. These properties are well known if $\mathcal{C}$, for example, consists of all αD where $\alpha > 0$ is arbitrary and $D \subseteq \mathbb{R}^n$ is a fixed bounded convex set containing 0 in its interior. Such maximal operators are usually called *centered*.

However little is known about the deeper properties of centered maximal operators even in the simplest cases. And one way to acquire such a deeper understanding is to start asking for the best constants in the corresponding inequalities satisfied by them. In this direction let us mention the result of E. M. Stein and J.-O. Strömberg [13] where certain upper bounds are given for such constants in the case of centered maximal operators as described above, and the corresponding still open question raised there (see also [3, Problem 7.74b]), on whether the best constant in the weak-type $(1,1)$ inequality for certain centered maximal operators in $\mathbb{R}^n$ has an upper bound independent of n.

The simplest example of such a maximal operator is the centered Hardy-Littlewood maximal operator defined by

$$M\,f(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f| \tag{1.1}$$

for every $f \in L^1(\mathbb{R})$. The weak-type $(1,1)$ inequality for this operator says that there exists a constant $C > 0$ such that for every $f \in L^1(\mathbb{R})$ and every $\lambda > 0$,

$$|\{M\,f > \lambda\}| \leq \frac{C}{\lambda} \|f\|_1 . \tag{1.2}$$

However even in this case not much was known for the best constant C in the above inequality. This must be contrasted with the corresponding uncentered maximal operator defined similarly to (1.1) but by not requiring x to be the center but just any point of the interval of integration. Here the best constant in the analogous to (1.2) inequality is equal to 2 which corresponds to a single dirac delta. The proof follows from a covering lemma that depends on a simple topological property of the intervals of the real line and can be extended to the case of any measure of integration, not just the Lebesgue measure (see [2]). Moreover in this case the best constants in the corresponding L^p inequalities are also known (see [5]).

However in the case of the centered maximal operator the behavior is much more difficult and it seems to not only depend on the Lebesgue measure but to also involve a much deeper geometry of the real line. A. Carbery proposed that $C = 3/2$ ([3, Problem 7.74c]), a joint conjecture with F. Soria which also appears in [14] and corresponds to sums of equidistributed dirac deltas. This conjecture has been refuted by J. M. Aldaz in [1] who actually obtained the bounds $1.541\ldots = \frac{37}{24} \leq C \leq \frac{9+\sqrt{41}}{8} = 1.9253905\ldots < 2$ which also implies that C is strictly less than the constant in the uncentered case, thus answering a question that was asked in [14]. Then J. Manfredi and F. Soria improved the lower bound proving that ([9]; see also [1]): $C \geq \frac{5}{3} - \frac{2\sqrt{7}}{3} \sin\left(\frac{\arctan(3\sqrt{3})^{-1}}{3}\right) = 1.5549581\ldots$.

The proofs of these results use as a starting point the discretization technique introduced by M. de Guzmán [6] as sharpened by M. Trinidad Menárguez-F. Soria (see Theorem 1 in [14]). To describe it we define for any finite measure σ on $\mathbb{R}$ the corresponding maximal function

$$M\,\sigma(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |d\sigma| . \tag{1.3}$$

Then the best constant C in inequality (1.2) is equal to the corresponding best constant in the inequality

$$|\{M\,\mu > \lambda\}| \leq \frac{C}{\lambda}\int_{\mathbb{R}} d\mu \tag{1.4}$$

where $\lambda > 0$ and μ runs through all measures of the form $\sum_{i=1}^{n}\delta_{t_i}$ where $n \geq 1$ and $t_1, \ldots, t_n \in \mathbb{R}$. This technique allows us to apply arguments of combinatorial nature to get information or bounds for this constant.

The author (see [10]) using also this technique, obtained the following improved estimates for C:

$$1.5675208\ldots = \frac{11+\sqrt{61}}{12} \leq C \leq \frac{5}{3} = 1.66\ldots \tag{1.5}$$

and also made the conjecture that the lower bound in (1.5) is actually the exact value of C. Recently in [11] the author found the best constant in a related but more general covering problem on the real line. This implies the following improvement of the upper bound in (1.5): $C \leq 1 + \dfrac{1}{\sqrt{3}} = 1.57735\ldots$. None of these however tells us what the exact value of C is.

In this paper we will prove that the above conjecture is correct thus settling the problem of the computation of the best constant C completely. We will prove the following.

Theorem 1. *For the centered Hardy-Littlewood maximal operator M, for every measure μ of the form $k_1\delta_{y_1} + \cdots + k_{y_n}\delta_{y_n}$ where $k_i > 0$ for $i = 1, \ldots, n$ and $y_1 < \cdots < y_n$ and for every $\lambda > 0$ we have*

$$|\{M\,\mu > \lambda\}| \leq \frac{11+\sqrt{61}}{12\lambda}\,\|\mu\| \tag{1.6}$$

and this is sharp.

We will call the measures μ that appear in the statement of the above theorem, *positive linear combinations of dirac deltas.*

In view of the discretization technique described above Theorem 1 implies the following.

Corollary 1. *For every $f \in L^1(\mathbb{R})$ and for every $\lambda > 0$ we have*

$$|\{M\,f > \lambda\}| \leq \frac{11+\sqrt{61}}{12\lambda}\,\|f\|_1 \tag{1.7}$$

and this is sharp.

Hence

$$C = \frac{11+\sqrt{61}}{12} = 1.5675208\ldots \tag{1.8}$$

is the largest solution of the quadratic equation

$$12C^2 - 22C + 5 = 0. \tag{1.9}$$

By the lower bound in (1.5) proved in [10] we only have to prove inequality (1.6) to complete the proof of Theorem 1. The number appearing in equality (1.8) is probably not suggesting anything, nor is the equation (1.9). However this number is what one would get in the limit by computing the corresponding constants in the measures that are produced by applying an iteration based on the construction in [10] that leads to the lower bound. These measures, although rather complicated (much more complicated than single or equidistributed dirac deltas), have a very distinct inherent structure (see the appendix here). Thus it would be probably better to view Theorem 1 as a statement saying that this specific structure actually is one that produces configurations with optimal behavior.

Then, in a completely analogous manner as the result in [6], [14], we will also prove the following.

THEOREM 2. *For any finite Borel measure σ on $\mathbb{R}$ and for any $\lambda > 0$ we have*

$$|\{M\,\sigma > \lambda\}| \le \frac{11+\sqrt{61}}{12\lambda}\,\|\sigma\|\,. \tag{1.10}$$

We have included this here because it is then natural to ask whether there exists a function $f \in L^1(\mathbb{R})$, or more generally a measure σ, and a $\lambda > 0$ for which equality holds in the corresponding estimate (1.7) and (1.10). We will show here that such an extremal cannot be found in the class of all positive linear combinations of dirac deltas.

THEOREM 3. *For any measure μ that is a positive linear combination of dirac deltas and for any $\lambda > 0$ we have*

$$|\{M\,\mu > \lambda\}| < \frac{11+\sqrt{61}}{12\lambda}\,\|\mu\|\,. \tag{1.11}$$

For the proof of Theorem 1, that is of inequality (1.6), our starting point will be the related covering and overlapping problems that were introduced in [10] using the discretization technique. This proof is divided into several sections and will contain a mixture of combinatorial, geometric and analytic arguments. We start from the assumption that this upper bound is not correct and fix a certain combination of dirac deltas that violates it and contain the least possible number of positions. Then using the related covering problem from [10], studied in more detail here, we will prove that this assumed measure will contain, or can be used to produce, segments that share certain structural similarities with the examples leading to the lower bound. This needs some work and is better described if we further discretize the corresponding covering problem by assuming that all masses and positions of this measure are integers. Then elaborating on the structure of these segments combined with the assumed violation of (1.6) we will obtain a certain estimate for the central part of these segments. This estimate will then lead to a contradiction using the assumption that any measure of fewer positions will actually satisfy (1.6). This will complete the proof of Theorem 1. Then we will give the proofs of Theorems 2 and 3 and in the Appendix we will briefly describe the construction from [10] that leads to the lower bound and we will compare it with the proof of the upper bound.

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2. Preliminaries

We will start here by describing our basic reduction of the problem as was introduced in [10], where also further details and proofs can be found. We will consider measures μ of the form

$$\mu = \sum_{i=1}^{n} k_i \delta_{y_i} \tag{2.1}$$

where n is a positive integer, $k_1, \dots, k_n > 0$ are its *masses* and $y_1 < \cdots < y_n$ are its *positions*.

For any such measure as in (2.1) we define the intervals

$$I_{i,j} = I_{i,j}(\mu) = [\, y_j - k_i - \cdots - k_j \,,\, y_i + k_i + \cdots + k_j \,], \tag{2.2}$$

for $1 \le i \le j \le n$ (where $[a,b] = \emptyset$ if $b < a$) and the set

$$E\,(\mu) = \bigcup_{1 \le i \le j \le n} I_{ij}(\mu)\,. \tag{2.3}$$

This set can be seen to be equal to $\{x : \, M\,\mu(x) \ge 1/2\}$ (see [10]).

It will be convenient throughout this paper to use the following notation: We define

$$K_i^j = k_i + \cdots + k_j \tag{2.4}$$

if $1 \le i < j \le n$, $K_i^i = k_i$ if $1 \le i \le n$ and $K_i^j = 0$ if $j < i$. Thus we can write $I_{i,j}(\mu) = [\, y_j - K_i^j, \, y_i + K_i^j]$.

We will say that μ satisfies the *separability inequalities* if:

$$y_{i+1} - y_i > k_i + k_{i+1} \tag{2.5}$$

for all $i = 1, \ldots, n-1$. If this happens then it is easy to see that for any $1 \le i < j \le n$ we have

$$I_{i,j}(\mu) \subseteq (y_i, y_j) \tag{2.6}$$

(in fact this is equivalent to $K_i^j < y_j - y_i$ which follows by adding certain inequalities from (2.5)) and therefore $E(\mu) \subseteq [y_1 - k_1, y_n + k_n]$.

We also set

$$R(\mu) = \frac{|E(\mu)|}{2\,\|\mu\|} = \frac{|E(\mu)|}{2(k_1 + \cdots + k_n)} = \frac{|E(\mu)|}{2K_1^n}. \tag{2.7}$$

Then we have the following (see [10]).

PROPOSITION 1. (i) *The best constant C in the Hardy-Littlewood maximal inequality* (1.2) *is equal to the supremum of all numbers $R(\mu)$ when μ runs through all positive measures of the form* (2.1) *that satisfy* (2.5).

(ii) *C is also equal to the supremum of all numbers $R(\mu)$ when μ runs through all positive measures as in* (i) *that also satisfy the condition*:

$$E(\mu) = [y_1 - k_1, y_n + k_n]. \tag{2.8}$$

Any such measure that satisfies the conditions in Proposition 1(ii), that is the separability inequalities and the connectedness of $E(\mu)$, will be called *admissible*. It is clear that for any admissible μ the intervals $I_{i,j}(\mu)$, $1 \le i \le j \le n$ form a covering of the interval $[y_1 - k_1, y_n + k_n]$.

We will also use the following lemma whose proof is essentialy given in [10] (see also [1]).

LEMMA 1. *Suppose μ is a measure containing $n \ge 2$ positions that does not satisfy all separability inequalities* (2.5), *that is for at least one i we have $y_{i+1} - y_i \le k_{i+1} + k_i$. Then there exists an admissible measure μ^* containing at most $n-1$ positions and such that $R(\mu^*) \ge R(\mu)$.*

Hence, unless otherwise stated, we will only consider measures μ that satisfy all inequalities (2.5). It is easy then to see that for any such μ the intervals $I_{i,i}(\mu)$ for $1 \le i \le n$ are pairwise disjoint. We define the set of *covered gaps* of μ as follows:

$$G(\mu) = E(\mu) \setminus \bigcup_{i=1}^{n} I_{i,i}(\mu). \tag{2.9}$$

This is the set of points that must be covered by the intervals $I_{i,j}(\mu)$ for $i < j$ that come from interactions of distant masses and are nonempty if their positions are, in some sense, close together. We also have

$$R(\mu) = 1 + \frac{|G(\mu)|}{2K_1^n}. \tag{2.10}$$

To proceed further let us now fix an admissible measure μ as in (2.1). An important device that can describe efficiently the covering properties $I_{i,j}(\mu)$ for $i < j$ is the so called *gap interval* of μ that was introduced in [10]. We consider the positive numbers

$$x_i = y_{i+1} - y_i - k_{i+1} - k_i \tag{2.11}$$

for $1 \leq i \leq n$, the points

$$a_1 = 0,\ a_2 = x_1,\ a_3 = x_1 + x_2, \dots, a_n = x_1 + \cdots + x_{n-1} \tag{2.12}$$

and define the gap interval $J(\mu)$ of μ as follows

$$J(\mu) = [a_1, a_n]. \tag{2.13}$$

The gap interval can be obtained from $E(\mu) = [y_1 - k_1, y_n - k_n]$ by collapsing the central intervals $I_{i,i}(\mu) = [y_i - k_i, y_i + k_i]$, $1 \leq i \leq n$ into the points a_i. This can be described by defining a (measure-preserving and discontinuous) mapping

$$Q = Q_\mu : J(\mu) \to G(\mu) \tag{2.14}$$

that satisfies $Q(x) = y_i + k_i + (x - a_i)$ whenever $x \in (a_i, a_{i+1})$, $1 \leq i < n$. Thus Q maps each subinterval (a_i, a_{i+1}) of $J(\mu)$ onto the corresponding gap $(y_i + k_i, y_{i+1} - k_{i+1})$ of $G(\mu)$. It is also trivial to see that the mapping Q is distance nondecreasing and so Q^{-1} is distance nonincreasing.

We also consider the intervals

$$J_i = J_i(\mu) = [a_i - k_i, a_i + k_i] \tag{2.15}$$

around each of the points a_i, $1 \leq i \leq n$, of $J(\mu)$, let

$$\mathcal{F}(\mu) = \{J_1(\mu), \dots, J_n(\mu)\} \tag{2.16}$$

denote the corresponding family of all these intervals and let

$$J_i^+ = J_i^+(\mu) = [a_i, a_i + k_i] \text{ and } J_i^- = J_i^-(\mu) = [a_i - k_i, a_i] \tag{2.17}$$

denote the right and left half of J_i respectively. We also consider the families of intervals

$$\mathcal{F}^+(\mu) = \{J_1^+(\mu), \dots, J_n^+(\mu)\} \text{ and } \mathcal{F}^-(\mu) = \{J_1^-(\mu), \dots, J_n^-(\mu)\}. \tag{2.18}$$

The elements of $\mathcal{F}^+(\mu)$ will be called *right intervals* and the elements of $\mathcal{F}^-(\mu)$ will be called *left intervals*.

Remark. Most of our results and definitions will be given for right intervals only. The corresponding facts for left intervals can be easily obtained in a symmetrical way or by applying the given ones to the reflected measure $\tilde{\mu} = \sum_{i=1}^n k_i \delta_{-y_i}$.

The role of the gap interval in the covering properties of the $I_{i,j}$'s can be seen by the following (see [10]):

PROPOSITION 2. (i) *Let* $1 \leq i < j \leq n$. *Then* $I_{i,j} \neq \emptyset$ *if and only if* $J_i^+ \cap J_j^- \neq \emptyset$.

(ii) *If* $a_j \notin J_i^+$ *and* $a_i \notin J_j^-$ *then* $|I_{i,j}| = \left| J_i^+ \cap J_j^- \right|$.

(iii) *If* μ *is admissible then* $|J(\mu)| = |G(\mu)|$ *and* $J(\mu) \subseteq J_1 \cup \cdots \cup J_n$.

Any interval $I_{i,j}$ as in Proposition 2(ii) will be called *special*. We also have the following.

LEMMA 2. *The interval* $I_{i,j} \neq \emptyset$ *is special if and only if* $|I_{i,j}| < \min(k_i, k_j)$.

Proof. It is easy to see that $|I_{i,j}| = \max(k_i + k_j - (a_j - a_i), 0)$. Hence if nonempty it would be special if and only if $a_j > a_i + k_i$ and $a_i < a_j - k_j$ and this easily completes the proof. □

To proceed further for each fixed i we set $l_i = \min\{l \leq i : a_l \in J_i^-\}$, $r_i = \max\{r \geq i : a_r \in J_i^+\}$ and define the intervals

$$F_i = F_i(\mu) = [y_i - K_{l_i}^i,\, y_i + K_i^{r_i}]. \tag{2.19}$$

Then the following holds (see [10]).

PROPOSITION 3. (i) *We have* $F_i = I_{l_i,i} \cup I_{i,l_i+1} \cup \cdots \cup I_{i,i} \cup I_{i,i+1} \cup \cdots \cup I_{i,r_i}$.

(ii) *For any* i *the nonempty of the closed intervals* $I_{1,i}, \ldots, I_{l_i-1,i}$ *and* $I_{i,r_i+1}, \ldots, I_{i,n}$ *(if any) are pairwise disjoint and each of them is disjoint from* F_i.

(iii) *The set* $E(\mu)$ *is covered by the* n *main intervals* F_i, $1 \leq i \leq n$ *together with the nonempty (if any) special intervals* $I_{p,q}$ *where* $a_q \notin J_p^+$ *and* $a_p \notin J_q^-$.

By exploiting the above structure of the gap interval we will prove the following basic for our developments (see also [11]).

PROPOSITION 4. (i) *The set* $G(\mu)$ *can be covered by appropriately placing certain parts of the nonempty of the intervals* $J_i^+ \cap J_j^-$ *over* $[y_i + k_i, y_j - k_j]$ *for* $1 \leq i < j \leq n$, *each such part used at most once.*

(ii) *In particular if* μ *is admissible* $J(\mu)$ *can be also covered as in* (i), *where each used part of* $J_i^+ \cap J_j^-$ *is placed appropriately over* $[a_i, a_j]$.

Proof. (i) Consider an i with $1 \leq i \leq n$. If $a_i \notin J_s$ for every $l_i \leq s \leq r_i$ with $s \neq i$, then clearly $|J_i^+ \cap J_s^-| = k_s$ for any $i < s \leq r_i$ (respectively $|J_s^+ \cap J_i^-| = k_s$ for any $l_i \leq s < i$) and so writing $\tilde{I}_{i,s} = [y_i + K_i^{s-1},\, y_i + K_i^s]$ $\subseteq I_{i,s}$ (respectively $\tilde{I}_{s,i} = [y_i - K_s^i,\, y_i - K_{s+1}^i] \subseteq I_{s,i}$) we easily conclude that these intervals cover $F_i \backslash I_{i,i}$ and have lengths equal to $|J_i^+ \cap J_s^-|$ (respectively $|J_s^+ \cap J_i^-|$) and using (2.6) each such $\tilde{I}_{i,s}$ (respectively $\tilde{I}_{s,i}$) is contained in $[y_i, y_s]$ (respectively $[y_s, y_i]$).

Now assume that there is a largest possible s such that $i < s \leq r_i$ and $a_i \in J_s^-$. Then since also $a_s \in J_i^+$ we conclude that $[a_i, a_s] = J_i^+ \cap J_s^-$ and so the part of $G(\mu)$ that lies in $[y_i + k_i, y_s - k_s]$ can be obviously covered by using certain parts of just $J_i^+ \cap J_s^-$. The remaining part of the $F_i \cap (y_i, +\infty)$ that is $F_i \backslash (-\infty, y_s + k_s)$ (if any) has length

$$(y_i + K_i^{r_i}) - (y_s + k_s) = K_i^{r_i} - (a_s - a_i + 2K_i^s - k_i) < K_{s+1}^{r_i}$$

and is thus covered by the intervals

$$\tilde{I}_{i,j} = [y_i + K_i^{j-1},\, y_i + K_i^j] \subseteq I_{i,j}$$

where $s < j \leq r_i$ each contained in the corresponding $[y_i, y_j]$ and having length $\left|J_i^+ \cap J_j^-\right|$ since $a_i \notin J_j$ for every $s < j \leq r_i$. Similar considerations can be applied if $a_i \in J_s^+$ for some $l_i \leq s < i$.

Finally for any special interval $I_{p,q}$ where $a_q \notin J_p^+$ and $a_p \notin J_q^-$ we know that $|I_{p,q}| = \left|J_p^+ \cap J_q^-\right|$.

These, combined with Proposition 3(iii), complete the proof of (i), observing that any part of any used piece that is contained in

$$\bigcup_{i=1}^n I_{i,i} = \bigcup_{i=1}^n [y_i - k_i, y_i + k_i]$$

can be ignored.

(ii) If μ is admissible then all gaps in $[y_1 - k_1, y_n + k_n] \backslash (I_{1,1} \cup \cdots \cup I_{n,n})$ are covered and so $|G(\mu)| = |J(\mu)|$. Therefore we can via the mapping Q^{-1} transport the way $G(\mu)$ is covered to cover $J(\mu)$ and this completes the proof observing that any piece placed over $[y_i + k_i, y_j - k_j]$ when transported via Q^{-1} will lie over $[a_i, a_j]$. □

Remarks. (i) When the covering of $G(\mu)$ that is described in the above proof is transported via Q^{-1} to cover $J(\mu)$ some intervals might shrink due to existence of intermediate masses. Here the fact that Q^{-1} is distance nonincreasing is used.

(ii) It is evident from the proof of Proposition 4 that in the case $a_j \in J_i^+$ and $a_i \in J_j^-$ the whole part $[a_i, a_j]$ of the gap interval is equal and hence completely covered by $J_i^+ \cap J_j^-$. However due to the possible existence of masses between y_i and y_j, it might be necessary to break $J_i^+ \cap J_j^-$ into several pieces before placing it over $[y_i + k_i, y_j - k_j]$. Actually this is the only case where such a breaking occurs.

It would be important to keep track of exactly how the parts of the $J_i^+ \cap J_j^-$'s are placed to cover $G(\mu)$ and $J(\mu)$. This has been more or less analysed in the above proof except for the case of special intervals. Related to this we have the following (where by $\mathfrak{l}(I)$, $\mathfrak{r}(I)$ we will denote the left and right endpoints of the interval I).

LEMMA 3. *Suppose that $1 \le i \le n$, that $r_i \le r < s$ and that both $I_{i,r}$ and $I_{i,s}$ are nonempty. Then*

$$\mathfrak{l}(I_{i,s}) - \mathfrak{r}(I_{i,r}) = \mathrm{dist}(a_s, J_i) + K_{r+1}^{s-1} \tag{2.20}$$

and a similar relation holds when $s < r \le l_i$.

Proof. We have $\mathfrak{l}(I_{i,s}) - \mathfrak{r}(I_{i,r}) = (y_s - K_i^s) - (y_i + K_i^r)$ and using the relation $y_s - y_i = a_s - a_i + k_i + 2k_{i+1} + \cdots + 2k_{s-1} + k_s$ we easily get $\mathfrak{l}(I_{i,s}) - \mathfrak{r}(I_{i,r}) = a_s - a_i - k_i + k_{r+1} + \cdots + k_{s-1} = a_s - \mathfrak{r}(J_i) + K_{r+1}^{s-1}$ which completes the proof since $a_s > a_i$ and $a_s \notin J_i$. □

Remarks. (i) Clearly $\mathfrak{l}(I_{i,r}) = \mathfrak{l}(F_i)$ if $r = r_i$. Thus Lemma 3 shows where the special intervals are located after the related F_i's. For example it shows that there is a gap between F_i and the first special interval of the form $I_{i,s}$ (if any) that is at least $\mathrm{dist}(a_s, J_i)$ and in case μ is admissible has to be covered by intervals of the form $I_{p,q}$ where $p \ne i$ and $q \ne i$. This exact location will be important in our proof of Theorem 1.

(ii) Actually the above results show how one can read off the covering properties of the family of intervals $I_{i,j}(\mu)$ for $i < j$ from the corresponding overlappings of the families $\mathcal{F}^+(\mu)$ and $\mathcal{F}^-(\mu)$ over the gap interval. In particular they show that the length and exact location in $E(\mu)$ of the special intervals $I_{i,r}$ (if any) depend only on the behavior of the gap interval and the corresponding J_m^-'s that are located to the right of the *right endpoint* of J_i^+.

Notation. (i) In this paper we will use the notation $|\cdots|$ in two different contexts: If S is a subset of $\mathbb{R}$ (which will ususaly be the union of finitely many closed intervals) then $|S|$ will denote its Lebesgue measure. If on the other hand T is a finite set (that will usually consist of a finite number of intervals) then $|T|$ will denote the cardinality of T.

(ii) For every family $\mathcal{U}$ of intervals by $\bigcup \mathcal{U}$ we will denote the union of all elements of $\mathcal{U}$.

(iii) As above for any interval $I \subseteq \mathbb{R}$ by $\mathfrak{l}(I)$, $\mathfrak{r}(I)$ we will denote its left and right endpoints respectively.

3. The measure μ

Let

$$\gamma = \frac{-1+\sqrt{61}}{12} = 0.5675208\ldots \tag{3.1}$$

be the positive solution of the quadratic equation

$$12\gamma^2 + 2\gamma - 5 = 0. \tag{3.2}$$

Assuming that $C > 1+\gamma$ there must exist measures μ as in (2.1) such that $R(\mu) > 1+\gamma$. We then consider the smallest possible integer n such that there exists a measure $\mu = \sum_{i=1}^{n} k_i \delta_{y_i}$ such that $R(\mu) > 1+\gamma$. Then $R(\nu) \leq 1+\gamma$ for any measure as in (2.1) that contains less than n positions. Hence using Lemma 1 and Proposition 1(ii) we may assume that μ is admissible; that is, it satisfies (2.5) and (2.6).

Moreover we may assume that all the y_i's and all the k_i's are positive *integers*. Indeed we can find rational numbers $k_i' > k_i$ and y_i' for $1 \leq i \leq n$ such that $0 < y_{i+1}' - y_i' < y_{i+1} - y_i$, the y_i' and k_i' satisfy (2.5) and the (as it is easy to see) admissible measure $\mu' = \sum_{i=1}^{n} k_i' \delta_{y_i}'$ still satisfies $R(\mu') > 1+\gamma$. Then by multiplying all y_i' and k_i' by an appropriate integer we get a measure with all entries integers.

From now on we will fix such a measure μ and let its gap interval $J(\mu)$ and its corresponding cover $\mathcal{F}(\mu) = \{J_1, \ldots, J_n\}$ be as in Section 2.

Then we write

$$J(\mu) = [0, N] = \omega_1 \cup \cdots \cup \omega_N, \tag{3.3}$$

where N is a positive integer and $\omega_p = [p-1, p]$ for $p = 1, 2, \ldots, N$. Each ω_p will be called a *place* in the gap interval $J(\mu)$. Also since the corresponding x_i and k_i's are integers to each such ω_p there correspond three nonnegative integers h_p^+, h_p^- and h_p such that

$$h_p^+ = \sum_{i=1}^{n} \chi_{J_i^+}(x),\ h_p^- = \sum_{i=1}^{n} \chi_{J_i^-}(x) \text{ and } h_p = h_p^+ + h_p^- \tag{3.4}$$

for any $x \in \operatorname{int}(\omega_p)$. Clearly

$$2K_1^n = \sum_{i=1}^{n} |J_i| \geq h_1 + \cdots + h_N. \tag{3.5}$$

(We write $\geq$ since $J_1 \cup \cdots \cup J_n$ might contain points outside $J(\mu)$.)

We will be considering that over each place ω_p there are h_p distinct intervals of length 1 which we call *bricks* h_p^+ corresponding to the right intervals that contain ω_p and h_p^- to the left. It is clear that $h_1 + \cdots + h_N$ is the total number of bricks.

We also let

$$P = \{a_1, \ldots, a_n\} \tag{3.6}$$

denote the set of all *positions* (centers of the J_i's) in the gap interval.

Now we consider the set of places

$$E_1 = \{\omega_p \subseteq J(\mu) : h_p = 1\} \tag{3.7}$$

over which exactly one interval from the family $\mathcal{F}^+(\mu) \cup \mathcal{F}^-(\mu)$ passes. It is then easy to see, using (3.5) and Proposition 2(iii) that the places in E_1 are the only ones that have the property of pushing $R(\mu)$ to something bigger than $\frac{1}{2}$. Thus it would be important to analyze the behavior of the intervals of $\mathcal{F}^+(\mu) \cup \mathcal{F}^-(\mu)$ that contain such places. We will consider only right intervals the corresponding statements for left intervals being symmetrical. It is clear, by Proposition 4(ii), that if a J_i^+ contains an $\omega_p \in E_1$ then ω_p can be covered only through the involvement of this J_i^+.

There are essentially two cases to consider. The first is treated in the following.

Proposition 5. *Suppose that for some $i \geq 1$ there exist $\omega_p \in E_1$ and $x \in \operatorname{int}(\omega_p) \subseteq J_i^+$ such that $Q(x) \leq \mathfrak{r}(F_i)$. Then we have*

$$(a_i, x] \cap P = \emptyset \tag{3.8}$$

and

$$a_{i+1} - a_i \leq K_{i+1}^{r_i} = \left| F_i \backslash (-\infty, y_i + k_i) \right|. \tag{3.9}$$

Proof. Suppose that $(a_i, x] \cap P = \{a_{i+1}, \ldots, a_s\} \neq \emptyset$ and so $a_s \leq x < a_{s+1}$. Since $h_p = 1$ it is clear that no interval other than J_i^+ contains x and so by Proposition 2(i) we have $I_{s,r} = \emptyset$ whenever $r > s$. Hence moving $k_s \delta_{y_s}$ to the left by $a_s - a_{s-1}$ will not change the connectivity of $E(\mu)$ since this mass does not interact with any mass to its left, since the inequality $Q(x) \leq \mathfrak{r}(F_i)$ implies that y_s belongs to F_i that will hence not change, as long as $a_s \in J_i^+$, and since this movement can only enlarge the intervals $I_{l,s}$ for $l < i$. But then the resulting measure μ' will have the same $E(\mu')$ but will not satisfy the separability condition (2.5) for the $s-1$ position. However in view of Lemma 1 this implies that there is a measure μ'' containing at most $n-1$ positions with $R(\mu'') \geq R(\mu') = R(\mu)$ and this contradicts our choice of μ. Hence $(a_i, x] \cap P = \emptyset$.

Next we will show that $\mathfrak{r}(F_i) < y_{i+1} - k_{i+1}$ is impossible. Indeed if this happened then since $x < a_{i+1}$ it is easy to see that $I_{l,s} = \emptyset$ whenever $l < i < s$ and so the interval $[\mathfrak{r}(F_i), \mathfrak{r}(F_i) + 1]$ must be covered by some $I_{i,s}$ where necessarily $s > r_i$ and so $I_{i,s}$ is a special interval. Thus $\mathfrak{l}(I_{i,s}) \le \mathfrak{r}(F_i)$ which contradicts Lemma 3. Hence $\mathfrak{r}(F_i) = y_i + K_i^{r_i} \ge y_{i+1} - k_{i+1}$ and since $y_{i+1} - y_i = a_{i+1} - a_i + k_{i+1} + k_i$ we get (3.9). □

If for the right interval J_i^+ there exist $\omega_p \in E_1$ and $x \in \mathrm{int}(\omega_p) \subseteq J_i^+$ such that $Q(x) \le \mathfrak{r}(F_i)$ (and so $(a_i, x] \cap P = \emptyset$) then the right interval J_i^+ will be called *clean.* A symmetrical definition applies for the left intervals J_j^-.

Suppose now that for some $m \ge 1$ the right interval J_m^+ contains at least one place from E_1 but is not clean. Then defining

$$w = \min\{q : \omega_{q+1} \subseteq J_m^+ \text{ and } h_{q+1} = 1\} \ge a_m \tag{3.10}$$

we must have $(a_m, w] \cap P \ne \emptyset$. Indeed if $(a_m, w] \cap P = \emptyset$ then clearly $w+1 \le a_{m+1}$ and moreover since $[w, w+1] \in E_1$ the *interval* $Q((a_m, w+1]) \subseteq [y_m, y_{m+1}]$ must be covered only by intervals of the form $I_{m,r}$ for $r > m$ (because by Proposition 2(i), $I_{l,r} = \emptyset$ whenever $l < m < m+1 \le r$). However Proposition 3(ii) now implies that we must have $Q((a_m, w+1]) \subseteq F_m$ and so $Q(w + \frac{1}{2}) < \mathfrak{r}(F_m)$, which contradicts the assumption that J_m^+ is not clean. Hence we may write

$$(a_m, w] \cap P = \{a_{m+1}, \dots, a_s\} \ne \emptyset. \tag{3.11}$$

Clearly $h_p \ge 2$ for all $a_m \le p \le w$. Now let

$$g(J_m^+) = a_s - a_m,\ K(J_m^+) = K_{m+1}^s. \tag{3.12}$$

Then we have the following.

Lemma 4. *The interval $(y_s + k_s, y_s + k_s + 1]$ must be covered by a special interval $I_{m,t}$ for some $t > r_m$. Moreover we must have*

$$g(J_m^+) + K(J_m^+) \ge \mathrm{dist}(a_t, J_m) + K_{s+1}^{t-1}. \tag{3.13}$$

Proof. By a similar reasoning as in the proof of Proposition 5, we conclude that F_m cannot cover the point $y_s + k_s + \frac{1}{2}$. Since for any $l \le s < r$ we have $I_{l,r} = \emptyset$ unless $l = m$ we conclude that it must be covered by some special interval $I_{m,t}$ for some $t > r_m$ and so $a_t > a_m + k_m = \mathfrak{r}(J_m)$. Since the y_l's and the k_l's are integers we have

$$y_s + k_s \ge \mathfrak{l}(I_{m,t}) = y_t - k_m - \cdots - k_t. \tag{3.14}$$

Writing now

$$y_s + k_s = y_m + a_s - a_m + k_m + 2k_{m+1} + \cdots + 2k_s$$

and

$$y_t - k_m - \cdots - k_t = y_m + a_t - a_m + k_{m+1} + \cdots + k_{t-1}$$

we get (3.13). □

Remark. In the above lemma we may actually assume that equality holds in (3.14) and hence also in (3.13). Indeed clearly the mass $k_s\delta_{y_s}$ interacts with no mass to the right of it (meaning that $I_{s,j} = \emptyset$ for every $j > s$). Hence as in the proof of Proposition 5 it can be moved to the left until either equality in (3.14) occurs or the separability inequality (2.5) for $i = s - 1$ is violated. But as in the proof of that proposition the second alternative cannot happen.

4. Further covering properties of μ

By Proposition 4 and since μ is admissible to each ω_p we can associate an $\omega_{c(p)}$ and certain $i(p) < j(p)$ such that $\omega_p \in [a_{i(p)}, a_{j(p)}]$, $\omega_{c(p)} \subseteq J^+_{i(p)} \cap J^-_{j(p)}$ and such that the part $\omega_{c(p)}$ of $J^+_{i(p)} \cap J^-_{j(p)}$ is used (corresponds to the part of $I_{i(p),j(p)}$ used) to cover $\omega_p \subseteq J(\mu)$ (equivalently $Q(\omega_p) \subseteq [y_1, y_n]$) according to above mentioned proposition. Moreover it is clear that the mapping

$$p \to (c(p), i(p), j(p)) \tag{4.1}$$

is *one-to-one*. We will write $\omega_{c(p)} \to \omega_p$ and we will say that that $\omega_{c(p)}$ *covers* ω_p. Also to indicate the exact way this covering takes place we will say that ω_p is covered by $(\omega_{c(p)}, J^+_{i(p)}, J^-_{j(p)})$ and we will say that ω_p is covered by $\omega_{c(p)}$ through the *interaction* of the right interval $J^+_{i(p)}$ with the left interval $J^-_{j(p)}$.

Remark. It may happen that ω_p is covered by more than one way according to Proposition 4. In such a case we choose exactly one of these ways arbitrarily to make the mapping c well defined.

For any ω_p that covers at least one place we let

$$l(p) = \min\{i : \omega_p \subseteq J^+_i\} < r(p) = \max\{j : \omega_p \subseteq J^-_j\} \tag{4.2}$$

(both well defined) and we define the intervals

$$L_p = J^+_{l(p)} \text{ and } R_p = J^-_{r(p)}. \tag{4.3}$$

Now except for E_1 we will more generally consider for any nonnegative integers s, t the sets

$$E_{s,t} = \{\omega_p \subseteq J(\mu) : h^+_p = s \text{ and } h^-_p = t\} \tag{4.4}$$

and

$$E_t = \{\omega_p \subseteq J(\mu) : h_p = t\} = \bigcup_{a+b=t} E_{a,b}. \tag{4.5}$$

We have the following.

Lemma 5. (i) *$\omega_p \in E_{a,b}$ can cover at most $a.b$ places in $J(\mu)$.*

(ii) *Any ω_p can cover at most $h_p - 1$ places in $E_1 \cup E_{1,1}$.*

Proof. For (i) obviously $a.b$ is equal to the number of all possible pairs (A, B) of a right interval A and a left interval B such that $\omega_p \subseteq A \cap B$. We will now prove (ii). If ω_p covers at least one place then $l(p), r(p)$ are well defined. Suppose that for some i, j with $l(p) < i < j < r(p)$ a place $\omega_q \in E_1 \cup E_{1,1}$ is covered through (ω_p, J_i^+, J_j^-). Then we have $\omega_q \subseteq [a_i, a_j]$. However $\omega_p \subseteq J_i^+ \cap J_j^-$ so it is clear that $\chi_{J_i^+} + \chi_{J_{l(p)}^+} \geq 2$ on $[a_i, p]$ and $\chi_{J_j^-} + \chi_{J_{r(p)}^-} \geq 2$ on $[p-1, a_j]$. Therefore $h_q^+ \geq 2$ if $q \leq p$ and $h_q^- \geq 2$ if $q \geq p-1$ and both lead to a contradiction. Hence the possible $\omega_q \in E_1 \cup E_{1,1}$ covered by ω_p can come only from interactions in which at least one of the intervals L_p and R_p is involved and it easy to see that there are $(h_p^+ - 1) + (h_p^- - 1) + 1 = h_p - 1$ such interactions. □

Remark. This lemma in particular implies that an ω_p in E_1 does not cover any place, an ω_p in E_2 covers at most one place (and this can happen only if $h_p^+ = h_p^- = 1$) and an ω_p in E_3 covers at most two places. Also an $\omega_p \in E_{3,1} \cup E_{1,3}$ can cover at most three places whereas an $\omega_p \in E_{2,2}$ can cover at most four places at most three of which can belong to $E_1 \cup E_{1,1}$.

We will introduce now the following notation: Suppose, for example, that an $\omega_p \in E_3$ covers an $\omega_q \in E_1$ and also an $\omega_a \in E_{1,1} \subseteq E_2$ that in turn covers an $\omega_b \in E_1$. Then we will say that ω_p is the *head* of an $E_3 \to (E_1, (E_2 \to E_1))$ *pattern*. We will consider the following nine types of such patterns:

Type 1	:	E_1
Type 2	:	$E_2 \to E_1$
Type 3	:	$E_2 \to E_2 \to E_1$
Type 4	:	$E_2 \to E_2 \to E_2 \to E_1$
Type 5	:	$E_2 \to (E_3 \to (E_1, E_1))$
Type 6	:	$E_3 \to (E_1, E_1)$
Type 7	:	$E_3 \to ((E_2 \to E_1), E_1)$
Type 8	:	$E_{1,3} \cup E_{3,1} \to (E_1, E_1, E_1)$
Type 9	:	$E_4 \to ((E_3 \to (E_1, E_1)), (E_2 \to E_1), E_1, E_1)$.

It is required that the E_1's appearing in the Types $5, 6, 8$ and 9 patterns are referring to distinct places. It is also clear that if ω_p is the head of a Type j pattern then for $1 \leq j \leq 5$ we must have $\omega_p \in E_{1,1}$ and for $j = 6, 7$ we must have $\omega_p \in E_{1,2} \cup E_{2,1}$. The possibility $\omega_p \in E_{2,2}$ has been excluded from the Type 8 pattern.

Moreover we have the following.

Lemma 6. *Consider any Type j pattern where $1 \leq j \leq 9$ and let T be the set of all places involved in it. Then*:

(i) *All places indicated in this pattern are distinct*; *hence T has as many elements as the E_t's appearing in the pattern.*

(ii) *No $\omega_q \in T$ can cover any place outside T.*

(iii) *If an ω_q covers the head of this pattern, then $\omega_q \notin T$.*

(iv) *Given $\omega_q \in T$ and a pair (A, B) of a right interval A and a left interval B such that $\omega_q \subseteq A \cap B$ then there exists $\omega_s \in T$ such that (ω_q, A, B) covers ω_s.*

Proof. For (i) it obviously suffices to consider only places in the same E_t that are covered by places in the same E_s. Hence by the requirements set for the Types $5, 6, 8$ and 9 it only remains to treat the Types 3 and 4. Suppose for example that a Type 4 pattern involves $\omega_a \to \omega_b \to \omega_p \to \omega_q$ but $\omega_a = \omega_p$. Then $\omega_a \in E_2$ would have to cover the two different places $\omega_b \in E_2$ and $\omega_q \in E_1$ contradicting Lemma 5 The proof for the other cases is similar. The assertion (ii) follows again by Lemma 5, (iii) can be proved in a similar way as (i) and (iv) can be proved by examining each considered pattern. □

Let u_j denote the number of places in a Type j pattern and v_j the corresponding number of bricks. Then clearly $u_1 = v_1 = 1$, $u_2 = 2$, $v_2 = 3$, $u_3 = u_6 = 3$, $v_3 = v_6 = 5$, $u_4 = u_5 = u_7 = u_8 = 4$, $v_4 = v_5 = v_7 = v_8 = 7$, $u_9 = 8$ and $v_9 = 14$. Also for $1 \leq j \leq 9$ let

$$\lambda_j = u_j - \gamma v_j. \tag{4.6}$$

It is easy to see that

$$0 < \lambda_4 = \lambda_5 = \lambda_7 = \lambda_8 < \lambda_9 < \lambda_3 = \lambda_6 < \lambda_2 < \lambda_1. \tag{4.7}$$

Now for any ω_p that is not the head of any Type j pattern for any $1 \leq j \leq 9$ we let T_p be the set that consists of ω_p and all places from all (maximal) patterns whose head is covered by ω_p and let

$$H_p = \sum_{\omega_s \in T_p} h_s \tag{4.8}$$

be the corresponding number of bricks that lie over all such places.

If now ω_p is the head of a Type j pattern for some $1 \leq j \leq 9$ we let T_p be the set of all places involved in this pattern, so $|T_p| = u_j$, but let

$$H_p = v_j + 1 \tag{4.9}$$

in this case (instead of v_j). This modification, whose use will be made clear later, results in the following estimate

$$|T_p| \leq \frac{8}{15} H_p < \gamma H_p \tag{4.10}$$

whenever ω_p is the head of such a pattern.

We also define $T_p = \emptyset$ and $H_p = 0$ if ω_p does not fall into one of the above two categories (for example an $\omega_p \in E_2$ that say covers an $\omega_q \in E_4$).

We now have the following.

LEMMA 7. *For any $p \neq q$ the sets T_p and T_q (if defined) are either disjoint or one of them is contained in the other.*

Proof. We will associate to each $\omega_s \in T_p$ an integer $r = r(s)$, called its rank, to be the length of the chain $\omega_p \to \cdots \to \omega_s$ that leads to ω_s. This is well defined since Lemma 6 implies that exactly one such chain can exist. Then if $T_p \cap T_q$ were nonempty we choose an $\omega_s \in T_p \cap T_q$ whose rank in T_p is as small as possible. It is then clear that $\omega_{c(s)}$ cannot be contained in both T_p and T_q. Suppose that $\omega_{c(s)} \notin T_p$ (the argument will show that the other case is impossible by the choice of ω_s). Then ω_s cannot be contained in any Type j pattern whose head is covered by ω_q since this would easily imply that $\omega_{c(s)}$ is either contained in the same pattern or is equal to ω_q and in both cases $\omega_{c(s)} \in T_p$. The only alternative is that $\omega_s = \omega_q$ and so that $\omega_q \in T_p$ must be the head of a Type j pattern. This easily implies that $T_q \subseteq T_p$ and completes the proof. $\square$

In the next two propositions we will show that any set T_p will not contribute significally to $R(\mu) > 2(1+\gamma)$ unless L_p and R_p satisfy certain strong restrictions in relation with the set E_1.

PROPOSITION 6. *If ω_p is not the head of a Type j pattern for any $1 \leq j \leq 9$ and is such that at least one of the intervals L_p and R_p does not contain any place from E_1, then we have*

$$|T_p| < \gamma H_p. \tag{4.11}$$

Proof. We may assume that R_p does not contain any place from E_1, the proof for L_p being symmetrical. Let $h_p^+ = a+1$ and $h_p^- = b+1$ and number the the right intervals containing ω_p as $A_0 = L_p, A_1, \ldots, A_a$ and the left intervals containing ω_p as $B_0 = R_p, B_1, \ldots, B_b$ so that

$$\mathfrak{l}(A_0) < \mathfrak{l}(A_1) < \cdots < \mathfrak{l}(A_a) \text{ and } \mathfrak{r}(B_0) > \mathfrak{r}(B_1) > \cdots > \mathfrak{r}(B_b). \tag{4.12}$$

Suppose first that $a, b > 0$. By Lemma 5(ii), ω_p can cover the head of a Type j pattern with $1 \leq j \leq 5$ only if A_0 or B_0 is involved (of course other patterns could also be so covered). However since $\chi_{A_0} + \chi_{A_1} + \chi_{B_0} \geq 2$ on $[\mathfrak{l}(A_1), \min(\mathfrak{r}(A_1), \mathfrak{r}(B_0))]$ the triples (ω_p, A_i, B_0) for $i \geq 1$ cannot cover an E_1 (since it should be contained in B_0). Also since for any ω_q that is the head of a Type $6, 7$ or 9 pattern there are exactly *two* intervals of the same direction that contain it we conclude, using a similar argument as in the proof of Lemma 5, that ω_p can cover the head of such a pattern only if at least one of the intervals

A_0, A_1, B_0, B_1 is involved. However if $i \geq 2$ (so $b > 1$) and (ω_p, A_1, B_i) covers the head ω_q of a Type 6 pattern then we must have $\omega_q \in A_1 \backslash B_2$ (and so $q < p$) since $h_q^+ \geq 2$ if $\mathfrak{l}(A_1) \leq q \leq p$ and $h_q^- \geq 3$ if $p-1 \leq q \leq \mathfrak{r}(B_2)$. Therefore ω_q would be contained in A_0 and A_1 and in exactly one other interval J of the opposite direction and moreover (ω_p, A_1, J) must cover a place in E_1. But since B_0 doesn't contain places from E_1 we clearly must have $\mathfrak{r}(J) > \mathfrak{r}(B_0)$ and since $q < p$ this implies that also $\omega_p \in J$. This contradicts the choice of $B_0 = R_p$. Hence (ω_p, A_1, B_i) can cover only in Types $7, 8$ or 9.

Now similarly ω_p covers the head of a Type 8 pattern only if at least one of the intervals $A_0, A_1, A_2, B_0, B_1, B_2$ is involved. However if $i \geq 2$ then (ω_p, A_i, B_2) cannot cover the head of a Type 9 pattern since $h_q^+ \geq 3$ if $\mathfrak{l}(A_2) \leq q \leq p$ and $h_q^- \geq 3$ if $p-1 \leq q \leq \mathfrak{r}(B_2)$. Also if $i \geq 3$ then (ω_p, A_2, B_i) cannot cover the head of a Type 8 (or 9) pattern for as before this would imply that this place must be in $A_2 \backslash B_i$ and this leads in a similar manner to a contradiction.

Hence the patterns covered by ω_p fall into exactly one of the following categories:

(1) With A_0 involved ω_p covers at most $b+1$ patterns of Type 1–9.

(2) With B_0, but not A_0, involved ω_p covers at most a patterns of Type 2–9.

(3) With B_1, but not A_0, involved ω_p covers at most a patterns of Type 6–9.

(4) With A_1, but not B_0, B_1, involved ω_p covers at most $b-1$ patterns of Type 7–9.

(5) With B_2, but not A_0, A_1, involved ω_p covers at most $a-1$ patterns of Type 8.

Let now $d_{i,j}$ the number of heads of Type j patterns covered by ω_p in the way described in category (i) where $1 \leq i \leq 5, 1 \leq j \leq 9$. Some of those are of course 0 as explained above, for example $d_{4,6} = d_{5,9} = 0$. Also we have given bounds for all five sums $\sum_j d_{i,j}$, for example $\sum_j d_{4,j} \leq b-1$. Now it is clear that

$$|T_p| = 1 + \sum_{i,j} u_j d_{i,j} \text{ and } H_p = a + b + 2 + \sum_{i,j} v_j d_{i,j}. \tag{4.13}$$

Hence using (4.7) the bounds for the sums $\sum_j d_{i,j}$ and the zero $d_{i,j}$'s we have

$$\begin{aligned} |T_p| - \gamma H_p &= 1 + \sum_{i,j} \lambda_j d_{i,j} - \gamma(a+b+2) \\ &\leq\ 1 + \lambda_1 \sum_j d_{1,j} + \lambda_2 \sum_j d_{2,j} + \lambda_6 \sum_j d_{3,j} + \lambda_9 \sum_j d_{4,j} \\ &\quad + \lambda_7 \sum_j d_{5,j} - \gamma(a+b+2) \leq\ (9-16\gamma)(a+b) - (10-18\gamma) \end{aligned}$$

and so if $a+b \geq 3$ we have

$$|T_p| - \gamma H_p \leq 17 - 30\gamma < 0. \tag{4.14}$$

If on the other hand $a = b = 1$ and so $\omega_p \in E_{2,2}$ examining the five categories it is easy to see that $|T_p| - \gamma H_p < 0$ unless $d_{1,1} = 2, d_{2,2} = d_{3,6} = 1$ which implies that ω_p is the head of a Type 9 pattern, thus contradicting our assumption.

Suppose now that $a = 0$ (the case $b = 0$ is similar). Then ω_p covers at most $b+1$ places and if d_j of them are heads of Type j patterns then $\sum_j d_j \leq b+1$ and in a similar way we have

$$\begin{aligned} |T_p| - \gamma H_p &= 1 + \sum_j \lambda_j d_j - \gamma(b+2) \\ &= 1 - \gamma - (2\gamma - 1)(d_1 + 2d_2 + 3(d_3 + d_6) + 4(d_4 + d_5 + d_7 + d_8)) \\ &\quad - (15\gamma - 8)d_9 - \gamma \left(b + 1 - \sum_j d_j \right) \end{aligned}$$

and this would be negative unless $\sum_j d_j = b + 1$ and

$$d_1 + 2d_2 + 3(d_3 + d_6) + 4(d_4 + d_5 + d_7 + d_8) + 3.5d_9 \leq 3$$

(and so $b \leq 2$) since $\dfrac{15\gamma - 8}{2\gamma - 1} > 3.5$, $\dfrac{1-\gamma}{2\gamma - 1} < 3.3$ and the d_j's are integers. These however easily imply that ω_p must be the head of one of the Types 1–8 pattern which is a contradiction. This completes the proof. □

Proposition 7. *If ω_p is not the head of a Type j pattern for any $1 \leq j \leq 9$ and is such that there is no $\omega_s \in L_p \cap R_p$ such that (ω_s, L_p, R_p) covers a place in E_1, then we have*

$$|T_p| < \gamma H_p. \tag{4.15}$$

Proof. By Propostion 6 both L_p and R_p contain places from E_1. Also by the proof of that proposition we may assume that $h_p^+ = a + 1 \geq 2$ and $h_p^- = b + 1 \geq 2$. We number the the right and left intervals containing ω_p as $A_0 = L_p, A_1, \ldots, A_a$ and $B_0 = R_p, B_1, \ldots, B_b$ as in the proof of that proposition. By our assumption (ω_p, A_0, B_0) cannot cover the head of a Type 1 pattern.

Suppose now that for some $i \geq 1$, (ω_p, A_1, B_i) covers the head ω_q of a Type j pattern for some $1 \leq j \leq 9$. If $\omega_q \subseteq A_1 \backslash B_0$ then clearly $h_q^+ \geq 2$ and so $h_q \geq 3$ and also there is no left interval F such that (ω_q, A_1, F) covers a place in E_1 (since the only possible such F would be B_0 which does not contain ω_q). A similar statement holds if $\omega_q \subseteq B_1 \backslash A_0$. If $\omega_q \subseteq A_0 \cap B_0$ then also $h_q \geq 3$ (since $\omega_q \subseteq A_1 \cup B_1$) and by our assumption (ω_q, A_0, B_0) cannot cover any place in E_1. Therefore the only possible values for j are $7, 8$ or 9 and a similar statement holds if (ω_p, A_i, B_1) covers the head ω_q of a Type j pattern.

Suppose now that for some $i \geq 2$, (ω_p, A_2, B_i) or (ω_p, A_i, B_2) covers the head ω_q of a Type j pattern for some $1 \leq j \leq 9$. Then $h_q^+ \geq 3$ or $h_q^- \geq 3$ and so $j = 8$. If $\omega_q \subseteq (A_2 \backslash B_0) \cup (B_2 \backslash A_0)$ then as before it cannot happen that all places covered by ω_q are in E_1, contradiction. Also if $\omega_q \subseteq A_0 \cap B_0$ then (ω_q, A_0, B_0) cannot cover any place in E_1. Hence no such covering can occur.

Therefore the patterns covered by ω_p fall into exactly one of the following categories:

(1) With A_0 or B_0, but not both, involved ω_p covers at most $a+b$ patterns of Type 1–9.

(2) With both A_0 and B_0 involved ω_p covers at most 1 pattern of Type $2-9$.

(3) With A_1 or B_1 (or both), but not A_0 or B_0, involved ω_p covers at most $a+b-1$ patterns of Type 7–9.

Letting now $d_{i,j}$ denote the number of heads of Type j patterns covered by ω_p in the way described in category (i) where $1 \leq i \leq 3, 1 \leq j \leq 9$ and using (4.7) the bounds for the sums $\sum_j d_{i,j}$ and the zero $d_{i,j}$'s we have, as in the proof of Proposition 6,

$$\begin{aligned} |T_p| - \gamma H_p &\leq 1 + \lambda_1 \sum_j d_{1,j} + \lambda_2 \sum_j d_{2,j} + \lambda_9 \sum_j d_{3,j} - \gamma(a+b+2) \\ &\leq (9-16\gamma)(a+b) - (5-9\gamma) \leq 13 - 23\gamma < 0 \end{aligned}$$

since $a+b \geq 2$. This completes the proof. □

Remark. The above proofs explain why we have only considered only those nine types of patterns. For example it is now easy to show that if ω_p covers the head of a pattern looking like $E_{2,2} \to (E_1, E_1, E_1, *)$ (which has not been included) then L_p and R_p will have the properties mentioned in the above propositions.

5. Good pairs

We will say that a pair (A, B) of a right interval $A \in \mathcal{F}^+(\mu)$ and a left interval $B \in \mathcal{F}^-(\mu)$ is *good* if there exists $\omega_p \subseteq A \cap B$ such that $A = L_p$, $B = R_p$ and

$$|T_p| - \gamma H_p > 0. \tag{5.1}$$

Using Propositions 6 and 7 we now conclude that any good pair (A, B) must satisfy the following:

(i) Both A and B contain places from E_1.

(ii) There exists $\omega_s \subseteq A \cap B$ such that (ω_s, A, B) covers an $\omega_t \in E_1$.

Suppose now that (A, B) is a good pair. Then clearly A uniquely determines B and vice versa. We define

$$w(A) = \min(A \cap \bigcup E_1) < w(B) = \max(B \cap \bigcup E_1). \tag{5.2}$$

Clearly by (i) above we must have $\omega_p \subseteq A \cap B \subseteq (w(A), w(B))$. Moreover we have the following.

LEMMA 8. *Suppose (A, B) is a good pair. Then*:

(i) *No $\omega_q \subseteq [\mathfrak{l}(A), w(A)] \cup [w(B), \mathfrak{r}(B)]$ can be the head of a Type j pattern for any $1 \leq j \leq 9$.*

(ii) *For every $\omega_q \subseteq [w(A), w(B)]$ we have $\bigcup T_q \subseteq [w(A), w(B)]$.*

(iii) *Suppose that $\omega_q \subseteq [\mathfrak{l}(A), w(A)]$ covers the head of a Type j pattern for some $1 \leq j \leq 9$. Then this can happen only through the involvement of L_q, which is then uniquely determined. A symmetrical statement holds if $\omega_q \subseteq [w(B), \mathfrak{r}(B)]$. (Here R_q must be involved.)*

Proof. (i) Suppose $\omega_q \subseteq [\mathfrak{l}(A), w(A)]$. Clearly $h_p \geq 2$ and $\omega_q \subseteq A$. Using Lemma 6 it easily follows that there must exist a left interval I_1 such that (ω_q, A, I_1) covers an ω_{q_1} that is the head of a Type $1, 2, 3$ or 6 pattern. Since $q \leq w(A)$ and $[w(A), w(A)+1] \in E_1$ we must have $\mathfrak{r}(I_1) \leq w(A)$ and therefore $\omega_{q_1} \subseteq [\mathfrak{l}(A), w(A)]$. Arguing similarly there must exist an $\omega_{q_2} \subseteq [\mathfrak{l}(A), w(A)]$ (covered by ω_{q_1}) that is the head of a Type 1 or 2 pattern and hence an $\omega_{q_3} \subseteq [\mathfrak{l}(A), w(A)] \cap \bigcup E_1$, which is a contradiction. The proof for $[w(B), \mathfrak{r}(B)]$ is similar.

(ii) Let G, H be a pair such that (ω_q, G, H) covers ω_s which is the head of some pattern. It is clear that $G \cup H \subseteq A \cup B$ and so $\omega_s \in [\mathfrak{l}(A), \mathfrak{r}(B)]$. But also by (i) ω_s cannot be contained in $[\mathfrak{l}(A), w(A)] \cup [w(B), \mathfrak{r}(B)]$. Hence $\omega_s \subseteq [w(A), w(B)]$ and this completes the proof.

(iii) Suppose that there is a right interval I different from L_q and so with $\mathfrak{l}(L_q) < \mathfrak{l}(I)$ and a left interval H such that (ω_q, I, H) covers an ω_s which is the head of some pattern. As in (i) $\mathfrak{r}(H) \leq w(A)$ and so $\omega_s \subseteq [\mathfrak{l}(I), w(A)]$. However (i) now implies that $\omega_s \subseteq [\mathfrak{l}(I), \mathfrak{l}(A)] \subseteq I$. As in (i) there must exist a right interval H_1 such that (ω_s, I, H_1) covers an ω_{s_1} which is the head of a Type $1, 2, 3$ or 6 pattern. Again we get $\omega_{s_1} \subseteq [\mathfrak{l}(I), w(A)]$ and so by (i) $\omega_{s_1} \subseteq [\mathfrak{l}(I), \mathfrak{l}(A)] \subseteq I$. Now as in (i) there must exist an $\omega_t \subseteq [\mathfrak{l}(I), \mathfrak{l}(A)] \cap \bigcup E_1$ and this is a contradiction since $\chi_I + \chi_{L_q} = 2$ on $[\mathfrak{l}(I), \mathfrak{l}(A)]$. Thus in any such covering L_q must be involved.

To show that L_q is uniquely defined suppose that for some other $\omega_{q'} \subseteq [\mathfrak{l}(A), w(A)]$ that covers the head of some pattern we had $L_q \neq L_{q'}$. We may assume that $\mathfrak{l}(L_q) < \mathfrak{l}(L_{q'})$. Then as before $\omega_{q'}$ must cover the head ω_s of some pattern, where $\omega_s \subseteq [\mathfrak{l}(L_{q'}), \mathfrak{l}(A)]$ and this leads to a similar contradiction. Hence L_q, if it exists, is uniquely defined. $\square$

Remark. If an ω_q as in Lemma 8(iii) exists then it is easy to see that there is no left interval G such that (L_q, G) is a good pair. Indeed if such a G existed then $L_q \cap G \subseteq [w(L_q), w(G)]$ and so since $\omega_q \in L_q \cap A$ we must have $G = J_r^-$ for some r with $a_r < w(A)$ which implies that $G \subseteq L_q \cap A$ and this is a contradiction.

Suppose now that (A, B) is a good pair and define

$$T(A, B) = \{\omega_s : \omega_s \subseteq A \cup B\} \tag{5.3}$$

and so $|T(A, B)| = \mathfrak{r}(B) - \mathfrak{l}(A) = |A \cup B|$.

Next we consider A. If A is clean then let $g(A) = K(A) = K^*(A) = 0$. If A is not clean then we write (see §3) $P \cap (\mathfrak{l}(A), w(A)] = \{a_s, \ldots, a_t\} \neq \emptyset$ (and so $A = J_{s-1}^+$) and with

$$g(A) = a_t - \mathfrak{l}(A) \text{ and } K(A) = k_s + \cdots + k_t \tag{5.4}$$

we now define $K^*(A)$ as follows:

(i) if there exists at least one $\omega_q \subseteq [\mathfrak{l}(A), w(A)]$ as in the statement of Lemma 8(iii), $K^*(A)$ is equal to the total number of bricks that correspond to the left intervals $J_t^-, \ldots, J_s^-$ or to right intervals J_l^+ with $l < i$ and lie over $[\mathfrak{l}(A), w(A)] \backslash L_q$ plus the length of the interval $L_q \cap A$ (note that L_q is uniquely determined and that we must have $\mathfrak{r}(L_q) \leq w(A)$), and

(ii) if no such ω_q exists, $K^*(A)$ is equal to the total number of bricks that lie over $[\mathfrak{l}(A), w(A)]$ and correspond to either the left intervals $J_t^-, \ldots, J_s^-$ or to right intervals J_l^+ with $l < i$.

Note that in both cases bricks that correspond to A are not counted in $K^*(A)$.

We also consider B and define $g(B), K(B), K^*(B)$ in a completely symmetrical way.

Regarding the masses that lie in $(w(A), w(B))$ we set

$$K(A, B) = \sum_{w(A) < a_r < w(B)} k_r \tag{5.5}$$

and now we define

$$H(A, B) = |A| + K^*(A) + K(A) + 2K(A, B) + K(B) + K^*(B) + |B| \,. \tag{5.6}$$

It is easy to see that by our construction

$$H(A,B) \leq \sum_{\omega_p \subseteq A \cup B} h_p. \tag{5.7}$$

(For example if $a_r \in (w(A), w(B))$ then we must have $J_r \subseteq (w(A), w(B))$ and so all the $2k_r$ bricks corresponding to J_r lie over $A \cup B$.) Also if A is *not clean* then $K^*(A) > 0$ and each place in $[\mathfrak{l}(A), w(A)]$ contributes at least *two* bricks in $H(A,B)$ (one from A and at least one counted in $K^*(A) + K(A)$), in particular $K^*(A) + K(A) \geq g(A)$.

The main thing now is to prove the following basic.

Proposition 8. *There exists at least one good pair (A,B) such that*

$$|T(A,B)| > \gamma H(A,B). \tag{5.8}$$

Proof. First of all we have the following.

Lemma . *Given any two good pairs (A,B) and (A',B') with $\mathfrak{l}(A) < \mathfrak{l}(A')$ we must have*

$$\mathfrak{r}(B) \leq \mathfrak{l}(A'). \tag{5.9}$$

Proof. Assume that $A = J_i^+$, $B = J_j^-$, $A' = J_s^+$ and $B' = J_r^-$ and moreover that $a_i < a_s$ but $a_j > a_s$. We must have $a_j < a_r$; otherwise, since both $A \cap B$ and $A' \cap B'$ are nonempty we would have $\chi_A + \chi_B + \chi_{A'} + \chi_{B'} \geq 2$ on $[a_s, a_r]$ contradiction. Considering now the symmetric of A' and B intervals $H = J_s^-$ and $G = J_j^+$ we have $\chi_H + \chi_{A'} + \chi_{B'} \geq 1$ on $[a_s - k_s, a_r]$ and $\chi_G + \chi_A + \chi_B \geq 1$ on $[a_i, a_j + k_j]$. Consider now an $\omega_q \in E_1$ contained in B. Then we must have $q \leq a_s - k_s$ and so $a_j - k_j = \mathfrak{l}(B) < q \leq a_s - k_s$. In a similar way we obtain $a_s + k_s > a_j + k_j$. These give $a_j - a_s < k_j - k_s < a_s - a_j$ contradiction since $a_j > a_s$. This completes the proof. □

In view of the above lemma we can number all the good pairs of μ (if any) as $(A_1, B_1), \ldots, (A_d, B_d)$ so that $\mathfrak{r}(B_i) \leq \mathfrak{l}(A_{i+1})$ for $i = 1, \ldots, d-1$. This implies that the sets $T(A_1, B_1), \ldots, T(A_d, B_d)$ are pairwise disjoint. Let

$$W = \{\omega_1, \ldots, \omega_N\} \setminus \bigcup_{i=1}^{d} T(A_i, B_i) \tag{5.10}$$

and consider the collection $\mathcal{S}$ of all T_p's where either: (i) $\omega_p \in W$ and is not the head of any Type j pattern for any $1 \leq j \leq 9$ or (ii) ω_p is the head of some such pattern but there is $1 \leq i \leq d$ such that ω_p is covered by an $\omega_q \subseteq [\mathfrak{l}(A_i), w(A_i)] \cup [w(B_i), \mathfrak{r}(B_i)]$ (through the involvement of L_q). We then have the following.

LEMMA 10. (i) *Any* $T_p \in \mathcal{S}$ *is disjoint from* $\bigcup_{i=1}^d T(A_i, B_i)$.

(ii) *We have*

$$E_1 \subseteq \bigcup_{T_p \in \mathcal{S}} T_p \cup \bigcup_{i=1}^{d} T(A_i, B_i). \tag{5.11}$$

(iii) *For every* $T_p \in \mathcal{S}$ *we have* $|T_p| < \gamma H_p$.

Proof. (i) Suppose that $T_p \in \mathcal{S}$ and $\omega_q \in T_p \cap T(A_i, B_i)$ for some i. If $q = p$ then Lemma 8 and the definition of $\mathcal{S}$ easily imply that $T_p \notin \mathcal{S}$. If $q \neq p$ then ω_q is the head of some Type j pattern and so by Lemma 8 we must have $\omega_q \subseteq [w(A_i), w(B_i)]$. But then it is easy to see that ω_q can be covered only if A_i,B_i or some of the masses corresponding to positions in $[w(A_i), w(B_i)]$ are involved and this would give $\omega_{c(q)} \in T(A_i, B_i)$. Continuing this (for at most three steps) we conclude that $\omega_p \in T(A_i, B_i)$ which as we have seen is a contradiction.

(ii) Suppose that $\omega_{q_0} \in E_1 \backslash \bigcup_{i=1}^d T(A_i, B_i)$ and let $q_1 = c(q_0), q_2 = c(q_1), \ldots$ (that is ω_q is covered by ω_{q_1} which is covered by ω_{q_2} and so on). Clearly $h_{q_r} \geq 2$ for all $r \geq 1$. Let $m \geq 1$ be the smallest possible integer such that ω_{q_m} is not the head of a Type j pattern for any $1 \leq j \leq 9$ (note that ω_{q_0} is the head of a Type 1 pattern). Such an m exists since each such pattern contains at most eight places and by Lemma 6 no cycles (that is chains of the form $\omega_{p_1} \to \omega_{p_2} \to \cdots \to \omega_{p_s} = \omega_{p_1}$). By Lemma 8 we conclude that $\omega_{q_r} \in W$ for all $0 \leq r \leq m-1$. If $\omega_{q_m} \in T(A_i, B_i)$ for some i then we must have $\omega_{q_m} \subseteq [\mathfrak{l}(A_i), w(A_i)] \cup [w(B_i), \mathfrak{r}(B_i)]$ (otherwise Lemma 8(ii) would imply that $\omega_{q_0} \in T_{q_m} \subseteq T(A_i, B_i)$) and so $\omega_{q_0} \in T_{q_{m-1}} \in \mathcal{S}$. If $\omega_{q_m} \in W$ then $\omega_{q_0} \in T_{q_m} \in \mathcal{S}$.

(iii) Consider $T_p \in \mathcal{S}$. Suppose that $\omega_p \in W$ is not the head of any Type j pattern. Then by (i), (L_p, R_p) is not a good pair hence we have $|T_p| < \gamma H_p$. If $\omega_p \in W$ is the head of such a pattern then the definition of H_p (see (4.9)) shows that $|T_p| < \gamma H_p$. □

We next let

$$D = \{\omega_1, \ldots, \omega_N\} \backslash \left(\bigcup_{T_p \in \mathcal{S}} T_p \cup \bigcup_{i=1}^{d} T(A_i, B_i) \right) \tag{5.12}$$

and note that by Lemma 10(ii) we have $h_q \geq 2$ for every $\omega_q \in D$. Then by letting $T_{p_1}, \ldots, T_{p_m}$ be all the maximal T_p's from $\mathcal{S}$, which by Lemma 7 are pairwise disjoint and cover $\bigcup_{T_p \in \mathcal{S}} T_p$ we have

$$|J(\mu)| = N = \sum_{r=1}^{m} |T_{p_r}| + \sum_{i=1}^{d} |T(A_i, B_i)| + |D|\,. \tag{5.13}$$

Now the following holds.

LEMMA 11. *We have*

$$\sum_{p=1}^{N} h_p \geq \sum_{r=1}^{m} H_{p_r} + \sum_{i=1}^{d} H(A_i, B_i) + 2\,|D|\,. \tag{5.14}$$

Proof. It is enough to show that the right-hand side of (5.14) is at most as large as the total number of bricks that lie over all ω_s's. Using that $h_p \geq 2$ for all $\omega_p \in D$, Lemma 9, Lemma 10(i), (5.7), the remark following Lemma 8 and the definitions of the H_p's and the $H(A_i, B_i)$'s we easily see that the only case that should be considered is when ω_p is the head of a Type j pattern and is covered by an $\omega_q \subseteq [\mathfrak{l}(A_i), w(A_i)] \cup [w(B_i), \mathfrak{r}(B_i)]$ for some i in which case H_p counts one more brick than the ones involved. Assume $\omega_q \subseteq [\mathfrak{l}(A_i), w(A_i)]$. Then L_q is uniquely determined and ω_q can cover at most as many such heads ω_p as there are bricks lying over $\omega_q \subseteq L_q$ that correspond to *left* intervals whose right endpoints are contained in $[\mathfrak{l}(A_i), w(A_i)]$. However by the definition of $K^*(A_i)$ it is clear that all these bricks are not counted in $H(A_i, B_i)$. A similar reasoning for the case $\omega_q \subseteq [w(B_i), \mathfrak{r}(B_i)]$ completes the proof of (5.14). □

Now since $R(\mu) > 1 + \gamma$ we have $2\gamma \sum_{p=1}^{N} h_p \leq 2\gamma K_1^n < |J(\mu)|$ and so using Lemma 10(iii), (5.13) and (5.14) we conclude that there must exist at least one i (hence at least one good pair) such that $|T(A_i, B_i)| > \gamma H(A_i, B_i)$. This completes the proof of the proposition. □

6. The core of a good pair

Now, using the theorem, we can find and fix a good pair (A, B) that satisfies (5.8).

LEMMA 12. *The interval $A \cap B$ (corresponding to the pair (A, B)) cannot cover places in both $A \cap \bigcup E_1$ and $B \cap \bigcup E_1$. Moreover if it covers at least one place in $A \cap \bigcup E_1$ then it cannot cover any place in $B \backslash A$.*

Proof. Suppose $A = J_i^+$ and $B = J_j^-$. Then clearly $I_{i,j}(\mu)$ is a special interval; therefore $|I_{i,j}(\mu)| = |A \cap B|$ and so $A \cap B$ is placed, without breaking it, over $E(\mu)$. Going to the gap interval $J(\mu)$ if $x, y \in J(\mu)$ are covered by $A \cap B$ then since Q is distance nondecreasing we must have $|x - y| \leq |A \cap B|$. However if $\omega_p \subseteq A \cap \bigcup E_1$ and $\omega_q \subseteq B \cap \bigcup E_1$, or $\omega_q \subseteq B \backslash A$, then it is easy to see that $|q - p| > |A \cap B|$ and this completes the proof. □

In view of the above lemma and the properties shared by any good pair we may assume that $A \cap B$ covers at least one place in $A \cap \bigcup E_1$ and so no place in $B \cap \bigcup E_1$ or $B \backslash A$.

We then let

$$z = z(A, B) = \max\{s : \omega_s \subseteq A \cap \bigcup E_1\} > w(A). \tag{6.1}$$

Since $A \cap B$ does not contain any place from E_1 we have $A \cap B \subseteq [z(A, B), w(B)]$. Next we write

$$[z(A, B), w(B)] \cap P = \{a_p, a_{p+1}, \ldots, a_q\} \tag{6.2}$$

and define the *core* of (A, B) to be the measure

$$\sigma = \sigma(A, B) = \sum_{r=p}^{q} k_r \delta_{y_r} \tag{6.3}$$

that corresponds to these positions.

Remark. (i) The set $[z(A, B), w(B)] \cap P$ must be nonempty. If it were empty then B would not interact with any right interval other than A to the left of $w(B)$ and also A would not interact with any left interval other than B after z. This would imply that $A \cap B \subseteq [z(A, B), w(B)]$ must be covered only by the intersection $A \cap B$. But this is impossible since $A \cap B$ must cover at least one place in E_1 and this place must be outside $A \cap B$.

(ii) Note the nonsymmetrical way with respect to A and B the core interval is defined (a max for right intervals would correspond to a min for left intervals). This is forced because of the location of the special interval corresponding to (A, B) (see also the construction in the Appendix).

We will now show that without affecting the core of (A, B) we may assume that both intervals A and B are *clean*. This would be important in the next section and is furnished by the following.

PROPOSITION 9. *For the good pair (A, B) considered above there exists an admissible measure $\bar{\mu}$ (which in general might contain more positions than μ) and a good pair $(\bar{A}, \bar{B})$ associated to the families $\mathcal{F}^{\pm}(\bar{\mu})$ corresponding to the gap interval of $\bar{\mu}$ such that*:

(i) $|T(\bar{A}, \bar{B})| > \gamma H(\bar{A}, \bar{B})$.

(ii) *Both the right interval $\bar{A}$ and the left interval $\bar{B}$ are clean.*

(iii) *The core $\sigma(\bar{A}, \bar{B})$ of the good pair $(\bar{A}, \bar{B})$ is identical to the core $\sigma(A, B)$ of (A, B).*

(iv) *For any measure ν formed from masses of $\bar{\mu}$ whose associated positions in $J(\bar{\mu})$ are contained in the interior of $\bar{A} \cup \bar{B}$ we have $|E(\nu)| \leq 2(1+\gamma)\,\|\nu\|$.*

Proof. If both A and B are clean there is nothing to prove. Suppose that A is not clean. Define then $w(A), g = g(A), K = K(A)$ and $K^* = K^*(A)$ as in Section 5, write $A = J_p^+$ and suppose that for some $i > p$

$$(w(A), +\infty) \cap P = \{a_{i+1}, \dots, a_n\} \tag{6.4}$$

(it is obviously nonempty) and so $K = K(A) = K_{p+1}^i$. We will not change anything in the part of the gap interval of μ that lies to the right of $w(A)$. Let $s \geq i$ be such that

$$a_s \leq \mathfrak{l}(A) = a_p + k_p < a_{s+1}. \tag{6.5}$$

Then the considerations in Section 3 and Lemma 4 imply that $[y_i + k_i, y_i + k_i + 1]$ is covered by a special interval $I_{p,t}(\mu)$ for some $t > s$ and moreover using the remark following Lemma 4 we may and will assume that

$$y_i + k_i = \mathfrak{l}(I_{p,t}(\mu)) = y_t - k_p - K - K_{i+1}^t \tag{6.6}$$

and so

$$g + K = \operatorname{dist}(A, a_t) + K_{i+1}^{t-1}. \tag{6.7}$$

Now we fix an admissible measure τ all whose entries are rational numbers such that

$$E(\tau) = [y_i + k_i - \operatorname{dist}(A, a_t) - K_{s+1}^{t-1}, y_i + k_i], \tag{6.8}$$
$$|E(\tau)| = 2(1 + \gamma - \varepsilon)\,\|\tau\|, \tag{6.9}$$

where $\varepsilon > 0$ is small to be fixed later and such that the maximum (individual) mass appearing in the positions of τ is so small that no mass of τ interacts with any $k_r\delta_r$ for any $r > i$. Such a measure can be constructed for example by the proceedure that leads to the lower bound for C (see [10] or the Appendix here) and an appropriate scaling-translation.

Let

$$\bar{K} = \|\tau\| \text{ and } \bar{g} = |G(\tau)| = |E(\tau)| - 2\bar{K} = 2(\gamma - \varepsilon)\bar{K} = |J(\tau)|\,. \tag{6.10}$$

Next we define

$$\bar{k}_p = k_p + K - \bar{K}, \tag{6.11}$$

noticing that $\bar{k}_p > 0$ since $2\bar{K} < g + K < k_p + K$.

Consider now the measure

$$\bar{\mu} = \bar{k}_p\delta_{y_p} + \tau + \sum_{r=i}^{n} k_r\delta_{y_r}. \tag{6.12}$$

Here the index p is used for convenience only, since we have no control on the number of positions in τ. Consequently we will not associate indices to the positions of τ.

Also by multiplying all entries in μ and $\bar{\mu}$ by the same appropriately chosen large integer we may assume that all such entries are *integers*.

Also consider in the gap interval $J(\bar{\mu})$ the pair $(\bar{A}, B)$ (B as before) where

$$\bar{A} = J_p^+(\bar{\mu}) \tag{6.13}$$

is the right interval corresponding to $\bar{k}_p \delta_{y_p}$. We will show that $\bar{\mu}$ is admissible, that the pair $(\bar{A}, B)$ is good with $\bar{A}$ clean and also that (i), (iii) and (iv) are satisfied. This will actually complete the proof since in case B is also not clean we can apply a similar symmetrical construction with B and the measure $\bar{\mu}$ to satisfy all conditions.

Since y_p and y_t have not been altered and since $\bar{k}_p + \bar{K} = k_p + K(A)$ we have

$$I_{p,t}(\mu) = I_{p,t}(\bar{\mu}). \tag{6.14}$$

Consequently in view of Lemma 2 and since $I_{p,t}(\mu)$ is a special interval we conclude that $I_{p,t}(\bar{\mu})$ must also be a special interval (with respect to $\bar{\mu}$) and therefore in the gap intervals $J(\mu)$ and $J(\bar{\mu})$ the right endpoints $\mathfrak{r}(A)$ and $\mathfrak{r}(\bar{A})$ must respectively be located at the same point of $J_t^-(\mu)$ and $J_t^-(\bar{\mu})$. This in view of Proposition 2(ii) and Lemma 3 and, since we have not altered μ to the right of y_i, implies that we must have

$$I_{p,r}(\mu) = I_{p,r}(\bar{\mu}) \tag{6.15}$$

for every $r \geq t$ and since (the nonempty of) these intervals together with $E(\sum_{r=i+1}^n k_r \delta_r)$ cover the space $[y_i + k_i, y_n]$ of $E(\mu)$ (note that $I_{l,r}(\mu) = \emptyset$ if $l \leq i < r$ with $l \neq p$ and that the nonempty, if any, of the intervals $I_{p,r}(\mu)$ for $r < t$ are located to the right of $y_i + k_i$) we conclude that

$$[y_i + k_i, y_n] \subseteq E(\bar{\mu}). \tag{6.16}$$

Also it is clear that $E(\tau) \subseteq E(\bar{\mu})$. Now as remarked above in the gap interval of $\bar{\mu}$ the interval $\bar{A} = J_p^+(\bar{\mu})$ must contain all positions that correspond to the masses $k_{i+1}\delta_{i+1}, \ldots, k_s\delta_s$ (and obviously all the positions corresponding to τ) we have

$$\mathfrak{r}(F_p(\bar{\mu})) = y_p + \bar{k}_p + \bar{K} + K_{i+1}^s = y_p + k_p + K + K_{i+1}^s. \tag{6.17}$$

Hence in view of (6.7) and (6.8)

$$\mathfrak{l}(E(\tau)) - \mathfrak{r}(F_p(\bar{\mu})) = y_i + k_i - g - K - y_p - k_p - K = 0 \tag{6.18}$$

and this now implies that $E(\bar{\mu})$ is connected, therefore that $\bar{\mu}$ is admissible (the separability inequalities being here obvious).

Now by the way τ is chosen (iv) is satisfied and also, since nothing has changed after $w(A)$, it is clear, using also Lemma 8(ii), that the pair $(\bar{A}, B)$ is good and that its core satisfies $\sigma(\bar{A}, B) = \sigma(A, B)$.

To prove (i) we form the gap intervals of μ and $\bar{\mu}$ simultaneously shrinking the corresponding central intervals $I_{r,r}$ of μ and $\bar{\mu}$ in such a way that in both cases the point $b = y_i + k_i$ is kept fixed. In this way in both gap intervals the segments that lie in $[b, +\infty)$ are identical and also $\mathfrak{r}(A) = \mathfrak{r}(\bar{A})$. Now in μ, as we already know, a gap of exactly g will be formed between $\mathfrak{l}(A)$ and b. In $\bar{\mu}$ however $E(\tau)$ will shrink to the interval $[b - |J(\tau)|, b]$ and between $\mathfrak{l}(\bar{A})$ and $b - |J(\tau)|$ a gap of exactly $\bar{K} + K^s_{i+1}$ will be formed, proving thus that in particular $\bar{A}$ is clean (since the individual masses of τ have been chosen very small). Hence it is easy to see that

$$X = |T(\bar{A}, B)| - |T(A, B)| = \bar{K} + K^s_{i+1} + \bar{g} - g \tag{6.19}$$

and

$$Y = |H(\bar{A}, B)| - |H(A, B)| = \bar{K} + K^s_{i+1} + \bar{g} + 2\bar{K} - g - (K + K^*). \tag{6.20}$$

In view of (5.8) to prove (i) it is enough to show that $X > \gamma Y$. We have

$$X - \gamma Y = (1-\gamma)K^s_{i+1} + \gamma(K + K^*) - (1-\gamma)g + (1-\gamma)\bar{g} - (3\gamma - 1)\bar{K}. \tag{6.21}$$

Using (6.8) it is now easy to compute that

$$(1-\gamma)\bar{g} - (3\gamma - 1)\bar{K} = \left(\frac{1}{2} - \gamma - \varepsilon'\right)(K^{t-1}_{s+1} + \operatorname{dist}(A, a_t)) \tag{6.22}$$

where $\varepsilon' = \dfrac{\varepsilon}{2(\gamma - \varepsilon + 1)}$. Moreover we have $K + K^* \geq g$, since obviously each place in g contributes at least one brick counted in $K + K^*$. Hence (since $\gamma > \frac{1}{2}$),

$$\gamma(K + K^*) - (1-\gamma)g \geq \left(\gamma - \frac{1}{2}\right)(g + K + K^*). \tag{6.23}$$

Now using (6.22), (6.23) and (6.7) in (6.21) and observing that we must have $K^* > 0$ we get

$$\begin{aligned} X - \gamma Y \quad &\geq \quad (1-\gamma)K^s_{i+1} + \left(\frac{1}{2} - \gamma - \varepsilon'\right)(K^{t-1}_{s+1} + \operatorname{dist}(A, a_t)) \\ &\quad + \left(\gamma - \frac{1}{2}\right)(K^{t-1}_{i+1} + \operatorname{dist}(A, a_t) + K^*) \\ &= \quad \frac{1}{2}K^s_{i+1} + \left(\gamma - \frac{1}{2}\right)K^* - \varepsilon'(K^{t-1}_{s+1} + \operatorname{dist}(A, a_t)) > 0 \end{aligned} \tag{6.24}$$

if $\varepsilon > 0$ has been choosen small enough. This completes the proof. $\square$

7. The basic estimate for the core

We will now consider a good pair (A, B) in which both A and B are clean and is such that (5.8) is satisfied. This pair can be a part of μ or be produced as in Proposition 9. In both cases its core $\sigma(A, B)$ is a part of μ and contains less n positions. For convenience we will change the numbering of the y_i, a_i and k_i's, introducing negative indices and also introduce if necessary (at most) two positions in μ (or $\bar{\mu}$) with masses 0 in such a way that

$$\sigma = \sigma(A, B) = \sum_{i=1}^{m} k_i \delta_{y_i}, \tag{7.1}$$

where $m \leq n$ and moreover so that there are $1 \leq r < s \leq n$ ($r < s$ since $A \cap B$ covers at least one place) with

$$\mathfrak{r}(A) = a_s \text{ and } \mathfrak{l}(B) = a_r. \tag{7.2}$$

It is easy to see that these new zero mass positions will not affect any of the covering properties of $\mathcal{F}(\mu)$ or related estimates, but will make our computations easier.

We will also use the following notation: For any $i < j$ we will let

$$\alpha_i^j = a_j - a_i \tag{7.3}$$

and we will let $\alpha_i^j = 0$ if $j \leq i$.

Now the gap interval of σ is $J(\sigma) = [a_1, a_m]$. Doing that we would have

$$A = J_{-p}^{+} \tag{7.4}$$

for some integer $p > 0$ and we will also consider the intermediate measure

$$\nu = \sum_{i=-p+1}^{0} k_i \delta_{y_i}. \tag{7.5}$$

As for B since it is also clean it is easy to see that Proposition 5 implies that

$$B = J_{m+1}^{-} \text{ and } S = a_{m+1} - a_m \leq K_r^m. \tag{7.6}$$

We will now analyse A. Let

$$\rho = \left| E(\nu) \backslash \bigcup_{i=-p+1}^{0} I_{i,i}(\nu) \right| \text{ and } K = \sum_{i=-p+1}^{0} k_i = \|\nu\|. \tag{7.7}$$

Since $[a_{-p+1}, a_0] \subseteq A$ is surrounded by places in E_1 we conclude that no interval of $\mathcal{F}(\nu)$ interacts with any interval other than A and the interactions with A produce an interval of length K in $F_{-p} \backslash (-\infty, y_{-p} + k_{-p}]$ (where $F_{-p} = F_{-p}(\mu)$). Actually we have

$$|F_{-p} \backslash (-\infty, y_{-p} + k_{-p}]| = K + K_1^s. \tag{7.8}$$

The interval $I_{-p,m+1}(\mu)$ (or $\bar{\mu}$) that corresponds to $A \cap B$ can cover, by Lemma 12, only points $x \in G(\mu)$ such that $Q^{-1}(x) \in A$ and moreover it covers at least one place of $G(\mu)$ that corresponds to some place in $A \cap \bigcup E_1$. In particular,

$$I_{-p,m+1}(\mu) \subseteq (-\infty, y_s]. \tag{7.9}$$

Therefore denoting by D the part of $I_{-p,m+1}(\mu)$ that lies in $(-\infty, y_1]$ and also corresponds to the places in $(-\infty, a_1)$ in the gap interval covered by $A \cap B$ and by h the part that lies in $[y_1, y_s]$, that is the, possibly empty, space in $[y_1, y_s] \backslash E(\sigma)$ covered by $I_{-p,m+1}(\mu)$, we have (since Q is distance nondecreasing)

$$D > 0 \text{ and } D + h \leq |A \cap B| = \alpha_r^s. \tag{7.10}$$

Now we thus have $\mathfrak{l}(I_{-p,m+1}(\mu)) < y_1 - k_1$ and by Lemma 3 we see that

$$g = \mathfrak{l}(I_{-p,m+1}(\mu)) - \mathfrak{r}(F_{-p}) = \operatorname{dist}(a_{m+1}, J_{-p}) + K_{s+1}^m = \alpha_s^m + S + K_{s+1}^m, \tag{7.11}$$

where (7.11) defines g. Hence by the above considerations and Proposition 3(iii) the interval $(\mathfrak{r}(F_{-p}), \mathfrak{l}(I_{-p,m+1}(\mu)))$ in $E(\mu)$ must be covered by $E(\nu)$ and some of the nonempty special intervals $I_{-p,j}(\mu)$ for $s+1 \leq j \leq m$. Hence there is $\lambda \geq 0$ such that

$$\lambda \leq \sum_{j=s+1}^{m} \left| A \cap J_j^- \right| \text{ and } g = \lambda + |E(\nu)| = \lambda + \rho + 2K. \tag{7.12}$$

This in turn implies that the total space in the gap interval $E(\mu)$, between $a_{-p} + K + K_1^s$ and $a_1 - D$, is at most $\lambda + \rho$. Hence

$$\alpha_{-p}^1 = a_1 - a_{-p} \leq K + K_1^s + D + \lambda + \rho. \tag{7.13}$$

Moreover since ν has less than n positions (or see Proposition 9(iv)) we have $|E(\nu)| \leq 2(1+\gamma)N(\nu)$ and so

$$\frac{\rho}{2K} \leq \gamma. \tag{7.14}$$

Turning now to the core σ we have that since no mass of σ interacts with any mass outside σ other than those corresponding to A and B and since all nonempty $I_{-p,j}(\mu)$ for $s+1 \leq j \leq m$ are situated to the left of $I_{-p,m+1}(\mu)$ whose left endpoint is smaller than $y_1 - k_1$, the interval $[y_1 - k_1, y_m + k_m]$ can be covered only by $E(\sigma)$, the part h of $I_{-p,m+1}(\mu)$ and possibly some of the nonempty special intervals $I_{i,m+1}(\mu)$ for $1 \leq i \leq r-1$. Hence denoting by $u \geq 0$ the measure of $[y_1, y_m] \backslash (E(\sigma) \cup I_{-p,m+1}(\mu))$ we have

$$u \leq \sum_{j=1}^{r-1} \left| B \cap J_j^+ \right| \text{ and } (y_m + k_m) - (y_1 - k_1) \leq |E(\sigma)| + u + h. \tag{7.15}$$

Therefore since σ contains less than n (nonzero mass) positions we have $|E(\sigma)| \leq 2(1+\gamma)\,\|\sigma\|$ and so

$$\alpha_1^m \leq 2\gamma K_1^m + u + h. \tag{7.16}$$

Now to use the above information efficiently we introduce the estimate (5.8) satisfied by the pair (A, B). This gives

$$\begin{aligned} a_{m+1} - a_{-p} &= T(A,B) > \gamma H(A,B) \\ &\geq \gamma(a_{m+1} - a_r + a_s - a_{-p} + 2K + 2K_1^m) \end{aligned} \tag{7.17}$$

and so

$$(1-\gamma)(S + \alpha_1^m + \alpha_{-p}^1) > \gamma(2K + 2K_1^m + \alpha_r^s). \tag{7.18}$$

Using now the estimates (7.10) and (7.13) and since $\gamma < 1$ we get

$$\begin{aligned} &(1-\gamma)(K_1^s + S + \lambda + \alpha_1^m - h) + (1-\gamma)\rho - (3\gamma - 1)K \\ &\quad > 2\gamma K_1^m + (2\gamma - 1)\alpha_r^s. \end{aligned} \tag{7.19}$$

Moreover using (7.12) and (7.14) we may write

$$\rho = \eta(g - \lambda) \text{ and } K = \frac{1-\eta}{2}(g-\lambda), \tag{7.20}$$

where

$$\eta \leq \frac{\gamma}{\gamma+1}, \tag{7.21}$$

and so by (7.11)

$$(1-\gamma)\rho - (3\gamma-1)K \leq \left(\frac{1}{2} - \gamma\right)(g - \lambda) = \left(\frac{1}{2} - \gamma\right)(\alpha_s^m + S + K_{s+1}^m - \lambda). \tag{7.22}$$

Putting this into (7.19) and using (7.6) we obtain the following estimate

$$\begin{aligned} (1-\gamma)\alpha_1^r + (2 - 3\gamma)a_r^s + \left(\frac{3}{2} - 2\gamma\right)\alpha_s^m &> \frac{1}{2}(K_{s+1}^m - \lambda) \\ &+ (3\gamma-1)K_1^{r-1} + \frac{5}{2}(2\gamma - 1)K_r^m + (1-\gamma)h. \end{aligned} \tag{7.23}$$

Multiplying (7.16) by $(2 - 3\gamma) > 0$ and subtracting from (7.23), and noticing that $\frac{5}{2}(2\gamma - 1) = 2\gamma(2 - 3\gamma)$ we obtain

$$(2\gamma-1)(\alpha_1^r - h) + \left(\gamma - \frac{1}{2}\right)\alpha_s^m + (2-3\gamma)u > \frac{1}{2}(K_{s+1}^m - \lambda) + \left(\frac{3}{2} - 2\gamma\right)K_1^{r-1}, \tag{7.24}$$

and dividing by $2\gamma - 1 > 0$ and using the, equivalent to (3.2), equations $(3\gamma+1)(2\gamma-1) = \dfrac{3}{2} - 2\gamma$, $(3\gamma + \dfrac{1}{2})(2\gamma - 1) = 2 - 3\gamma$ and $(6\gamma+4)(2\gamma-1) = 1$ we obtain the following basic estimate for the (two tails of the) core measure σ:

$$[a_1^r - h + (3\gamma+1)(u - K_1^{r-1})] + \frac{1}{2}[\alpha_s^m - u + (6\gamma+4)(\lambda - K_{s+1}^m)] > 0, \tag{7.25}$$

where we have added and subtracted the term $\frac{1}{2}u$ for reasons that will become clear in the next section.

This estimate will lead to a contradiction and thus will prove Theorem 1. We will do this in the following section.

8. End of the proof of Theorem 1

Here we will show that both terms in brackets in (7.25) must be nonpositive. This contradicts (7.25) and will thus prove Theorem 1.

Consider any measure τ of the form

$$\tau = \sum_{i=1}^{m} \bar{k}_i \delta_{z_i}, \tag{8.1}$$

where $\bar{k}_1, \ldots, \bar{k}_m > 0$ and the $z_1 < z_2 < \cdots < z_m$ satisfy the separability inequalities $z_{i+1} - z_i > \bar{k}_{i+1} + \bar{k}_i$ for all $1 \leq i \leq m-1$ and suppose that the number of positions m in τ is at most n (the n we have defined in §3). The set $E(\tau)$ is not assumed connected. Consider the set

$$G(\tau) = E(\tau) \backslash \bigcup_{i=1}^{m} I_{i,i}(\tau) \subseteq [z_1, z_m] \tag{8.2}$$

that is covered by the nonempty of the intervals $I_{i,j}(\tau)$ where $1 \leq i < j \leq m$. Define the $\bar{K}_i^j$ similarly to (2.4). Then we have the following.

Lemma 13. *For every h such that $1 < h \leq m$ we have*

$$|G(\tau) \cap [z_1, z_h]| \leq (2\gamma + 1)\bar{K}_1^{h-1}. \tag{8.3}$$

Proof. The set $G(\tau) \cap [z_1, z_h]$ is covered by certain intervals $I_{i,j}(\tau)$ where $1 \leq i < j \leq m$. However we know that $I_{i,j}(\tau) \subseteq (z_i, z_j)$ if $i < j$ and so it would be disjoint from $[z_1, z_h]$ unless $i < h$. Therefore

$$G(\tau) \cap [z_1, z_h] \subseteq \bigcup_{1 \leq i < j \leq h-1} I_{i,j}(\tau) \cup \bigcup_{1 \leq i \leq h-1 < j} I_{i,j}(\tau). \tag{8.4}$$

Consider the measure $\tau' = \sum_{i=1}^{h-1} \bar{k}_i \delta_{z_i}$. Then $\bigcup_{1 \leq i < j \leq h-1} I_{i,j}(\tau) = G(\tau')$ and since τ' contains less than n positions we have

$$\left| \bigcup_{1 \leq i < j \leq h-1} I_{i,j}(\tau) \right| = |G(\tau')| \leq 2\gamma \bar{K}_1^{h-1}. \tag{8.5}$$

Now for the other part consider any interval of the form $I_{i,j}(\tau)$ where $1 \leq i \leq h-1 < j$. We have, since τ satisfies the separability inequalities,

$$\mathfrak{l}(I_{i,j}(\tau)) = z_j - \bar{K}_i^j > z_h + \bar{k}_h + 2\bar{K}_{h+1}^{j-1} + \bar{k}_j - \bar{K}_i^j \geq z_h - \bar{K}_1^{h-1} \tag{8.6}$$

if $j > h$ and

$$\mathfrak{l}(I_{i,h}(\tau)) = z_h - \bar{k}_h - \bar{K}_i^{h-1} \geq z_h - \bar{k}_h - \bar{K}_1^{h-1}. \tag{8.7}$$

Therefore since $G(\tau) \cap [z_1, z_h] \subseteq [z_1, z_h - \bar{k}_h]$ we have

$$G(\tau) \cap [z_1, z_h] \cap \bigcup_{1 \leq i \leq h-1 < j} I_{i,j}(\tau) \subseteq [z_h - \bar{k}_h - \bar{K}_1^{h-1}, z_h - \bar{k}_h] \tag{8.8}$$

and so its measure is at most $\bar{K}_1^{h-1}$. Combining (8.8) with (8.4) and (8.5) we get (8.3). □

Remarks. (i) A analogous symmetrical statement holds for $G(\tau) \cap [z_h, z_m]$ if $1 \leq h < m$.

(ii) After Theorem 1 is proved, the above lemma holds for any measure, without the restriction on the number of positions, and as it can be easily seen is best possible.

Now we can show that both terms in (7.25) are nonpositive.

LEMMA 14. *For the core measure σ we have*

$$\alpha_1^r - h + (3\gamma + 1)(u - K_1^{r-1}) \leq 0. \tag{8.9}$$

Proof. We may assume that $r > 1$ otherwise there is nothing to prove.
We have by (7.15)

$$u \leq \sum_{i=1}^{r-1} \max(a_i + k_i - a_r, 0). \tag{8.10}$$

Let

$$q = \min\{i : 1 \leq i \leq r \text{ and } a_i + k_i \geq a_r\}. \tag{8.11}$$

(Note that if $q = r$ then $u = 0$.) Then using (8.10) it is easy to see that

$$a_r - a_q + u \leq K_q^{r-1}. \tag{8.12}$$

Therefore we have

$$\alpha_1^r - h + (3\gamma + 1)(u - K_1^{r-1}) \leq \alpha_1^q - h - (3\gamma + 1)K_1^{q-1}. \tag{8.13}$$

But then, from the considerations in Section 7 and since the definition of q implies that $I_{i,m+1}(\mu) = \emptyset$ for all $1 \leq i < q$, it follows that the space in $[y_1, y_q]$ not covered by $E(\sigma)$ has measure at most h. Therefore using Lemma 13 we have $\alpha_1^q - h \leq (2\gamma + 1)K_1^{q-1}$, which in view of (8.13) easily implies (8.9). □

In a similar symmetrical manner we prove that

$$\alpha_s^m - u + (6\gamma + 4)(\lambda - K_{s+1}^m) \leq 0, \tag{8.14}$$

noticing that the part of $[y_s, y_m]$ not covered by $E(\sigma)$ has measure at most u (in view of (7.9)) and using (7.12).

But now the inequalities (8.9) and (8.14) contradict the basic core estimate (7.25). Therefore this completes the proof of Theorem 1.

9. Proof of Theorem 2

It is clearly sufficient to fix a finite positive Borel measure σ and prove (1.10) for $\lambda = 1$. The functions $F^+(x) = \sigma((-\infty, x])$ and $F^-(x) = \sigma((-\infty, x))$ are measurable as nondecreasing. Hence for each $h > 0$ the set

$$A(h) = \{x : \sigma([x-h, x+h]) > 2h\} \tag{9.1}$$

is measurable. Letting $E = \{x : M\sigma(x) > 1\}$ it is easy to see that

$$E = \bigcup_{h>0} A(h) = \bigcup \{A(h) : h \in \mathbb{Q} \text{ and } h > 0\}. \tag{9.2}$$

Hence setting

$$E_n = \bigcup \left\{A(h) : h \in \mathbb{Q} \text{ and } h > \frac{1}{n}\right\} \tag{9.3}$$

we conclude that E is the union of the increasing sequence (E_n) of measurable sets. Thus it is enough to show that for any fixed large $n > 1$ and every compact set $K \subseteq E_n$ we have

$$|K| \leq C(1 + \frac{1}{n}) \|\sigma\| \tag{9.4}$$

where C is the constant given in (1.8).

Fixing n and K as above we can find an interval $[a, b]$ containing K and such that $b - \sup K, \inf K - a > \|\sigma\|$ and $\sigma(\{a, b\}) = 0$ and a partition

$$a = c_0 < c_1 < \cdots < c_N = b \tag{9.5}$$

of this interval such that

$$\max_{1 \leq j \leq N} (c_j - c_{j-1}) < \frac{1}{n^2} \text{ and } \sigma(\{c_0, c_1, \ldots, c_N\}) = 0. \tag{9.6}$$

This is possible since there are at most countably many $x \in \mathbb{R}$ such that $\sigma(\{x\}) > 0$.

Consider now the following positive linear combination of dirac deltas

$$\mu = \sum_{j=1}^{N} \sigma([c_{j-1}, c_j]) \delta_{\frac{c_{j-1}+c_j}{2}} \, . \tag{9.7}$$

Then for every $x \in K$ there exists an $h > \frac{1}{n}$ such that

$$\sigma([x-h, x+h]) > 2h. \tag{9.8}$$

Clearly $h < \|\sigma\|$ and so $[x-h, x+h] \subseteq (a,b)$. Choose j and s such that $c_j < x-h \le c_{j+1}$ and $c_{s-1} \le x+h < c_s$ and let $h' = \max(c_s - x, x - c_j) > h$. Clearly $h' - h < \frac{1}{n^2}$. We have

$$\mu([x-h', x+h']) \ge \sigma([c_j, c_s] \ge \sigma([x-h, x+h]) > 2h > 2\frac{n}{n+1}h' \tag{9.9}$$

and so

$$K \subseteq \{x : M\mu(x) > \frac{n}{n+1}\} \tag{9.10}$$

and so since $\|\mu\| \le \|\sigma\|$ by applying Theorem 1 we get (9.4). This completes the proof of Theorem 2.

10. Proof of Theorem 3

To prove Theorem 3 we assume (in view of Theorem 1) that there exists an admissible positive linear combination of dirac deltas μ such that

$$|E(\mu)| = 2C\,\|\mu\| \tag{10.1}$$

and such that $|E(\nu)| < 2C\,\|\nu\|$ for every positive linear combination of dirac deltas ν that contains less positions than μ, where C is the constant given in (1.8).

Now we fix an integer $n > 1$ and consider the set

(10.2)

$$\Omega = \{\mathbf{b} = (y_1, \ldots, y_n; k_1, \ldots, k_n) \in \mathbb{R}^{2n} : y_1 \le \cdots \le y_n \text{ and } k_1, \ldots, k_n \ge 0\}.$$

Then to every $\mathbf{b} = (y_1, \ldots, y_n; k_1, \ldots, k_n) \in \Omega$ we associate the measure

$$\sigma(\mathbf{b}) = \sum_{i=1}^{n} k_i \delta_{y_i} \tag{10.3}$$

and the intervals

$$I_{i,j}(\mathbf{b}) = [y_j - k_i - \cdots - k_j, y_i + k_i + .. + k_j] \tag{10.4}$$

for all $1 \le i \le j \le n$ (where as usual $[a,b] = \emptyset$ if $b < a$).

Of course the mapping $\mathbf{b} \to \sigma(\mathbf{b})$ is not one-to-one. But it is easy to see (for example using a limiting argument) that for any measure $\tau = \sum_{i=1}^{m} h_i \delta_{z_i}$ where $z_1 < \cdots < z_m$ and $h_1, \ldots, h_m > 0$ and for any $\mathbf{b} \in \Omega$ such that $\tau = \sigma(\mathbf{b})$ we have $E(\tau) = \bigcup_{1 \le i \le j \le n} I_{i,j}(\mathbf{b})$.

We will use the following well-known lemma.

Lemma 15. *Let $\mathcal{C}$ be a finite collection of closed intervals in $\mathbb{R}$ such that their union $\bigcup\mathcal{C}$ is an interval $[x,y]$ where $x<y$. Then there is a subcollection $\mathcal{C}_0=\{[a_1,b_1],\dots,[a_N,b_N]\}$ of $\mathcal{C}$ such that $\bigcup\mathcal{C}_0=\bigcup\mathcal{C}$, satisfying the following*

$$a=a_1<a_2<\dots<a_N=y \tag{10.5}$$

and

$$a_2\le b_1<b_2,\dots,a_N\le b_{N-1}<b_N. \tag{10.6}$$

As it is well known to prove the above lemma it suffices to pick $\mathcal{C}_0$ of minimal cardinality among all subcollections $\mathcal{C}'$ of $\mathcal{C}$ satisfying $\bigcup\mathcal{C}'=\bigcup\mathcal{C}$, and so no element of $\mathcal{C}_0$ is contained in any union of other elements of $\mathcal{C}_0$. The intervals of $\mathcal{C}_0$ can be arranged so that (10.5) is satisfied; then (10.6) follows easily from the fact that $\bigcup\mathcal{C}_0$ is the interval $[x,y]$.

Then we will apply the following proposition.

Proposition 10. *Let τ be an admissible positive linear combination of dirac deltas containing exactly $n>1$ positions such that $R(\nu)<R(\tau)$ for every positive linear combination of dirac deltas ν that contains less than n positions. Then there exists an admissible measure*

$$\tau^*=\sum_{i=1}^{n}k_i^*\delta_{y_i^*}, \tag{10.7}$$

where all $k_1^*,\dots,k_n^*>0$ and all $y_1^*<\dots<y_n^*$ are rational numbers and such that

$$R(\tau^*)\ge R(\tau). \tag{10.8}$$

Proof. Suppose that $\tau=\sigma(\mathbf{b}_0)$ where

$$\mathbf{b}_0=(y_1^{(0)},\dots,y_n^{(0)};k_1^{(0)},\dots,k_n^{(0)})\in\Omega$$

is uniquely determined. By scaling we may assume that

$$E(\tau)=[y_1^{(0)}-k_1^{(0)},y_n^{(0)}+k_n^{(0)}]=[0,1]. \tag{10.9}$$

Note that then $k_1^{(0)}+\dots+k_n^{(0)}=\|\tau\|\le 1$ for otherwise $R(\tau)<R(\delta_0)$.

Now applying Lemma 15 to the collection $\mathcal{C}=\{I_{i,j}(\tau):1\le i\le j\le n$ and $I_{i,j}(\tau)\ne\emptyset\}$ we can find a subcollection $\{I_{i_1,j_1}(\tau),\dots,I_{i_N,j_N}(\tau)\}$ of $\mathcal{C}$ that still covers $[0,1]$ and satisfies

$$0=\mathfrak{l}(I_{i_1,j_1}(\tau))<\dots<\mathfrak{l}(I_{i_N,j_N}(\tau))=1 \tag{10.10}$$

and

$$\mathfrak{l}(I_{i_p,j_p}(\tau))\le\mathfrak{r}(I_{i_{p-1},j_{p-1}}(\tau))<\mathfrak{r}(I_{i_p,j_p}(\tau)) \tag{10.11}$$

for all $p = 2, \dots, N$. It is easy to see that we must have $(i_1, j_1) = (1, 1)$ and $(i_N, j_N) = (n, n)$. Fixing the set of pairs $\{(i_1, j_1), \dots, (i_N, j_N)\}$ we now consider the following set

$$\begin{aligned}(10.12)\quad \Omega^* = \{\mathbf{b} &= (y_1, \dots, y_n; k_1, \dots, k_n) \in \Omega : y_1 - k_1 = 0, y_n + k_n = 1,\\ &y_{j_p} - K_{i_p}^{j_p} \le y_{j_{p+1}} - K_{i_{p+1}}^{j_{p+1}} \text{ for all } 1 \le p \le N-1,\\ &y_{j_p} - K_{i_p}^{j_p} \le y_{i_{p-1}} + K_{i_{p-1}}^{j_{p-1}} \le y_{i_p} + K_{i_p}^{j_p} \text{ for all } 2 \le p \le N \text{ and}\\ &k_1 + \cdots + k_n \le 1\}.\end{aligned}$$

It is easy to see that Ω^* is a nonempty (since $\mathbf{b}_0 \in \Omega^*$) compact convex polyhedron contained in a codimension 2 affine subspace of $\mathbb{R}^{2n}$. Moreover, it is easy to find nonzero vectors $\mathbf{v}_1, \dots, \mathbf{v}_M$ such that all the conditions that define Ω^* (including the conditions defining Ω) can be written as

$$(10.13)\qquad \mathbf{v}_1.\mathbf{b} = 0, \quad \mathbf{v}_2.\mathbf{b} = 1, \quad \mathbf{v}_3.\mathbf{b} \le 0, \dots, \mathbf{v}_M.\mathbf{b} \le 0,$$

and, moreover, $\mathbf{v}_1 = \mathbf{e}_1 - \mathbf{e}_{n+1}$, $\mathbf{v}_2 = \mathbf{e}_n + \mathbf{e}_{2n}$, and all the entries in all $\mathbf{v}_1, \dots, \mathbf{v}_M$ are from the set $\{-1, 0, 1\}$. Considering the linear functional F with

$$(10.14)\qquad F(\mathbf{b}) = k_1 + \cdots + k_n = (\mathbf{e}_{n+1} + \cdots + \mathbf{e}_{2n}).\mathbf{b}$$

and applying the standard result from the theory of linear programming we conclude that there exists an extreme point (vertex) $\mathbf{b}^* = \{y_1^*, \dots, y_n^*; k_1^*, \dots, k_n^*\}$ of Ω^* such that

$$(10.15)\qquad F(\mathbf{b}^*) = \min\{F(\mathbf{b}) : \mathbf{b} \in \Omega^*\} \le F(\mathbf{b}_0).$$

Let now $m_1 = 1 < m_2 = 2 < \cdots < m_s \le M$ be all the indices such that equality holds in the corresponding relation from (10.13) when $\mathbf{b}$ is replaced by $\mathbf{b}^*$. Then it is clear that since $\mathbf{b}^*$ is a vertex of Ω^* the linear system

$$(10.16)\qquad \mathbf{v}_{m_1}.\mathbf{b} = 0, \mathbf{v}_{m_2}.\mathbf{b} = 1, \mathbf{v}_{m_3}.\mathbf{b} = 0, \dots, \mathbf{v}_{m_s}.\mathbf{b} = 0$$

must have $\mathbf{b}^*$ as its unique solution and since all coefficients are integers we conclude that all the $2n$ coordinates of $\mathbf{b}^*$ must be *rational numbers*.

Consider now the measure $\tau^* = \sigma(\mathbf{b}^*)$. Since $\mathbf{b}^* \in \Omega^*$ it is easy to see that

$$(10.17)\qquad [0, 1] \subseteq \bigcup_{1 \le i \le j \le n} I_{i,j}(\mathbf{b}^*) = E(\tau^*);$$

moreover,

$$(10.18)\qquad \|\tau^*\| = F(\mathbf{b}^*) \le F(\mathbf{b}_0) = \|\tau\|.$$

Hence $R(\tau^*) \ge R(\tau)$ and the assumptions on τ combined with Lemma 1 (and its proof given in [10]) now imply that τ^* must contain exactly n positions and may assumed admissible (without changing its basic property that all its positions and masses are rational). This completes the proof of the proposition. □

Using now the above proposition we can find an admissible measure μ^* whose masses and positions are rational numbers and such that $R(\mu^*) \geq 2C$. But then $R(\mu^*) > 2C$ violates Theorem 1 and also $R(\mu^*) = 2C$ leads to a contradiction since $R(\mu^*)$ must be a rational number whereas C is irrational. This completes the proof of Theorem 3.

11. Appendix

Here we will briefly sketch the construction from [10] that leads to the lower bound in (1.5) thus showing that the inequality in Theorem 1 is actually best possible.

For any admissible measute μ as in (2.1) we consider the following modified norm

$$\|\mu\|^* = k_0 + 2k_1 + \cdots + 2k_n + k_{n+1} \tag{11.1}$$

and the corresponding modified ratio

$$R^*(\mu) = \frac{|E(\mu)| - k_0 - k_{n+1}}{\|\mu\|^*} = \frac{y_{n+1} - y_0}{k_0 + 2k_1 + \cdots + 2k_n + k_{n+1}}. \tag{11.2}$$

It is easy to see that $R^*(\mu) > R(\mu) > 1$ for any admissible μ. Moreover by applying a reflection-translation procedure one can show (see [10]) that for any admissible measure μ and every $\varepsilon > 0$ there exists a measure $\widetilde{\mu}$ such that $R(\widetilde{\mu}) \geq R^*(\mu) - \varepsilon$. This measures $\tilde{\mu}$ will consist of a large number of translated copies of μ (and its symmetric one). Hence any admissible measure μ also satisfies $R^*(\mu) \leq C$.

Then we consider any measure ν that satisfies the separability condition (2.5). We do not assume that $E(\nu)$ is connected. Writing ν as $\sum_{i=1}^{n} k_i \delta_{y_i}$ where $k_i > 0$ and $y_1 < \cdots < y_n$, we fix integers $1 \leq s, r \leq n$ and define the measure

$$T_{s,r}\nu = k_0 \delta_{y_0} + \nu + k_{n+1} y_{n+1}, \tag{11.3}$$

where

$$y_0 = 2y_1 - y_s - 2k_1 - k_s\,,\ k_0 = y_s - y_1 - K_2^{s-1} \tag{11.4}$$

and

$$y_{n+1} = 2y_n - y_r + 2k_n + k_r,\ k_{n+1} = y_n - y_r - K_{r+1}^{n-1}. \tag{11.5}$$

It is easy to show (see [10]) that $E(T_{s,r}\nu)$ does not have more gaps than $E(\nu)$. That is, the added intervals $(y_0, y_1 - k_1), (y_n + k_n, y_{n+1})$ are contained in $E(T_{s,r}\nu)$. Hence the operation $T_{s,r}$ does not create any new gaps. However we have the advantage of using the special interval $I_{0,n+1}(T_{s,r}\nu)$, which will be nonempty if $s > t$, to possibly cover gaps of our initial set $E(\nu)$. For this purpose we argue as follows.

Let μ be any, admissible now, measure written as $\mu = \sum_{i=1}^{m} k_i' \delta_{z_i}$ where $k_i' > 0$ and $z_1 < \cdots < z_n$ where for simplicity we assume that $z_1 = 0$. Fixing now two positive real numbers $A, \alpha > 0$ we consider the scaled measure $\alpha.\mu$ defined by

$$\alpha.\mu = \sum_{i=1}^{m} \alpha k_i' \delta_{\alpha z_i}. \tag{11.6}$$

Clearly the measure $\alpha.\mu$ is also admissible and so the measure

$$\nu = \mu + \mathrm{trasl}_A(\alpha.\mu) = \sum_{i=1}^{m} k_i' \delta_{z_i} + \sum_{i=1}^{m} \alpha k_i' \delta_{\alpha z_i + A} = \sum_{i=1}^{n} k_i \delta_{y_i}, \tag{11.7}$$

where $n = 2m$, satisfies the separability inequalities as long as $A > k_m' + \alpha k_1'$. We will next take as s the last position of μ, so $s = n = 2m$, and as r the first position of the translated $\alpha.\mu$, so $t = m + 1$ and consider the measure

$$T\mu = T_{2m,m+1}\nu. \tag{11.8}$$

Then in [10] it is shown that by choosing

$$\alpha = 2R(\mu) \text{ and } A = (\alpha^2 - \alpha) \|\mu\| + (\alpha - 1)k_1' \tag{11.9}$$

the measure $T\mu$ will be admissible (hence $E(T\mu)$ is connected) and moreover

$$R^*(T\mu) = \frac{20R(\mu)^2 - 4R(\mu)}{12R(\mu)^2 - 2R(\mu) + 1}. \tag{11.10}$$

Let now $f(x) = \dfrac{20x^2 - 4x}{12x^2 - 2x + 1}$. Starting from the admissible measure $\mu_0 = \delta_0 + \delta_3$ we define the sequence of positive linear combinations of dirac deltas $(\mu_p)_{p \geq 0}$ (all whose masses and positions are rational numbers) as follows. Having defined μ_p consider $T\mu_p$ and apply the reflection-translation procedure to obtain a measure μ_{p+1} such that $R(\mu_{p+1}) \geq R^*(T\mu_p) - \varepsilon_p = f(R(\mu_p)) - \varepsilon_p$ where the $\varepsilon_p > 0$ tend to 0 sufficiently fast. Then we will have $R(\mu_p) \to \dfrac{11 + \sqrt{61}}{12} = 1.5675208\ldots$ as $p \to \infty$. This implies the lower bound in (1.5). After the first few steps these measures will be rather complicated.

However each such measure μ_p will contain a large number of translated copies of $T\mu_{p-1}$ (and its symmetric one) so it will have a specific structure. To study this structure let us consider the gap interval $[a_0, a_{n+1}]$ of the measure $T\mu$ defined in (11.8). It is easy to see that it starts with $a_1 - a_0 = (1 + \alpha) \|\mu\|$ followed by a copy of $J(\mu)$, then by a gap of length $\alpha |J(\mu)|$ (that is completely covered by $I_{0,n+1}(T\mu)$), then by a copy of $J(\alpha.\mu) = \alpha J(\mu)$ and then by $a_{n+1} - a_n = \alpha \|\mu\|$. These easily imply that the pair (J_0^+, J_{n+1}^-) has the same structure as the good pairs described in Section 5 and that both of its intervals are *clean*. Moreover its core is equal to a copy of the measure $a.\mu$ and the μ corresponds to the intermediate measure ν considered in Section 7. Also (assuming all positions and masses integers), it is easy to see that $\left|T(J_0^+, J_{n+1}^-)\right| = |J(T\mu)|$ and $H(J_0^+, J_{n+1}^-) = \|\mu\|^*$; thus their ratio is equal to $R^*(T\mu) - 1$. So compared with the considerations in Section 7 we conclude that $T\mu$ shows, in a sense, the tightest possible structure.

In our proof of Theorem 1 we have actually shown that certain measures τ with $R(\tau) > C$ must have (or can be used to produce) segments that behave in a structurally similar fashion as the $T\mu$'s. However to prove the sharp upper bound we had to consider the effect of the more general operator $T_{s,r}$ with $r < s$ which makes it necessary to also study certain aspects of the internal structure of the core, which leads to the basic core estimate (7.25). The fact that in a sense r must be as small as possible and s as large as possible is reflected by the inability to satisfy (7.25). This is what actually leads to the proof of the upper bound.

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References

[1] J. M. Aldaz, Remarks on the Hardy-Littlewood maximal function, *Proc. Roy. Soc. Edinburgh* **128A** (1998), 1–9.

[2] A. Bernal, A note on the one-dimensional maximal function, *Proc. Roy. Soc. Edinburgh* **111A** (1989), 325–328.

[3] D. A. Brannan and W. K. Hayman, Research problems in complex analysis, *Bull. London Math. Soc*. **21** (1989), 1–35.

[4] H. Carlsson, A new proof of the Hardy-Littlewood maximal theorem, *Bull. London Math. Soc*. **16** (1984), 595–596.

[5] L. Grafakos and S. Montgomery-Smith, Best constants for uncentred maximal functions, *Bull. London Math. Soc*. **29** (1997), 60–64.

[6] M. de Guzmán, *Real Variable Methods in Fourier Analysis*, *North-Holland Math. Studies* **46**, *Notas de Matemática* **75**, North-Holland Publ. Co., New York, 1981.

[7] ———, *Differentiation of Integrals in* $\mathbb{R}^n$, *Lecture Notes in Math*. **481**, Springer-Verlag, New York, 1975.

[8] C. Fefferman and E. M. Stein, Some maximal inequalities, *Amer. J. Math*. **93** (1971), 107–115.

[9] J. Manfredi and F. Soria, On a dynamical system related to estimating the best constant in an inequality of Hardy and Littlewood, unpublished manuscript.

[10] A. Melas, On the centered Hardy-Littlewood maximal operator, *Trans. Amer. Math. Soc*. **354** (2002), 3263–3273.

[11] ———, On a covering problem related to the centered Hardy-Littlewood maximal inequality, *Ark. Mat.*, to appear.

[12] E. M. Stein, *Harmonic Analysis*: *Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, *Princeton Math. Series* **43**, Princeton Univ. Press, Princeton, NJ, 1993.

[13] E. M. Stein and J.-O. Strömberg, Behavior of maximal functions in $\mathbb{R}^n$ for large n, *Ark. Mat*. **21** (1983), 259–269.

[14] M. Trinidad Menarguez and F. Soria, Weak type $(1,1)$ inequalities of maximal convolution operators, *Rend. Circ. Mat. Palermo* **41** (1992), 342–352.

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