

Classification of simple C^* -algebras and higher dimensional noncommutative tori

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Abstract

We show that unital simple C^* -algebras with tracial topological rank zero which are locally approximated by subhomogeneous C^* -algebras can be classified by their ordered K -theory. We apply this classification result to show that certain simple crossed products are isomorphic if they have the same ordered K -theory. In particular, irrational higher dimensional noncommutative tori of the form $C(\mathbb{T}^k) \times_{\theta} \mathbb{Z}$ are in fact inductive limits of circle algebras.

Introduction

In recent years there has been rapid progress in classification of nuclear simple C^* -algebras. In the case that C^* -algebras are of real rank zero and finite, Elliott and Gong ([EG]) have proved that simple inductive limits of finite direct sums of homogeneous C^* -algebras (AH for brevity) of slow dimension growth with real rank zero can be completely classified up to isomorphism by their scaled ordered K -theory (with the reduction of dimension growth proved by [G2] and [D]). In their remarkable paper ([EG]), they also showed that the class of AH-algebras that they classified exhausts all possible invariants. So any general classification theorem for simple C^* -algebras of real rank zero and stable rank one with weakly unperforated K_0 will not expand their class. However, many interesting simple C^* -algebras, which are important in applications, do not arise as inductive limits of finite direct sums of homogeneous C^* -algebras. Therefore, it is extremely important to have a classification theorem which covers C^* -algebras that are *not* assumed to be AH-algebras. The main purpose of this paper is to establish such a theorem. Our general classification result covers at least some of the well-known interesting simple C^* -algebras that are not known to be AH-algebras.

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For example, certain simple C^* -algebras arising as dynamical systems with minimal diffeomorphisms can be classified by their ordered K -theory. More specifically, let X_i ($i = 1, 2$) be a smooth manifold (X_1 and X_2 may be the same) and $\sigma_i : X_i \rightarrow X_i$ be a minimal diffeomorphism. Suppose that (X_i, σ_i) ($i = 1, 2$) is uniquely ergodic. Then the resulting crossed products $C(X_i) \rtimes_{\sigma_i} \mathbb{Z}$ are simple C^* -algebras with unique traces and are isomorphic if they have the same scaled ordered K -theory (which is determined by σ_i only). A consequence of this is that the noncommutative tori of the form $C(\mathbb{T}^k) \rtimes_{\theta} \mathbb{Z}$, where θ is an irrational rotation, are isomorphic to unital simple inductive limits of circle algebras. This also generalizes an important result of Elliott and Evans that every irrational rotation algebra is an inductive limit of circle algebras ([EE]).

To classify a class of C^* -algebras, one often needs to establish a so-called uniqueness theorem and an existence theorem. Uniqueness is used to describe two maps from one C^* -algebra to another as approximately equivalent in an appropriate sense if they carry the same K -theory (or KK -theory) data. Existence is often involved in showing that given K -theory data (or KK -data) α , there is a map ϕ from one C^* -algebra to another which carries α .

In [Ln2], [Ln3] and [Ln4], we show that a uniqueness theorem holds for all nuclear C^* -algebras with a reasonable and mild restriction which, together with a “half” existence theorem (see also [DE]), gives a number of classification results which do not require that C^* -algebras considered to be AH-algebras. The “half” existence theorem we mentioned above does give us a map which carries most of the required KK -data but not all. The missing part is the order information from the required KK -data.

Suppose that A is a unital simple C^* -algebra with $\text{TR}(A) = 0$ (see 3.5). Then, in [Ln3], we show that A is always an inductive limit of A_n , where each A_n is a residually finite-dimensional C^* -algebra. If $\rho_{A_n} : K_0(A_n) \rightarrow \text{Aff}(T(A_n))$ is the homomorphism given by the traces of A_n , and every finitely generated subgroup of ρ_{A_n} can be order-embedded into \mathbb{Z}^k for some integer k , then the needed existence theorem holds. This certainly appears to be a very technical condition. However every AH-algebra obviously satisfies this condition. We show that if A is an inductive limit of subhomogeneous C^* -algebras, then this condition is also satisfied. In fact, a much broader class of C^* -algebras satisfies this condition. Combining this with our other recent results, we are able to prove that simple C^* -algebras with $\text{TR}(A) = 0$ can be classified up to isomorphism by their scaled ordered K -theory provided that they are locally approximated by subhomogeneous C^* -subalgebras. The recent results of Q. Lin and N. C. Phillips show that every simple C^* -algebra arising from a dynamical system of minimal diffeomorphisms is in fact an inductive limit of subhomogeneous C^* -algebras. Therefore, the general classification result mentioned above can be applied to those simple C^* -algebras when they have real rank zero.

The paper is organized as follows. In Section 1, we revisit AH-algebras. A few refinements of known results will be presented. These refinements are needed for construction of some maps that possess certain required KK -data. In Section 2, we present results which enable us to extend some positive homomorphisms from ordered subgroups of \mathbb{Z}^k to some other ordered groups which are not assumed to be divisible. In Section 3, we use results from Sections 1 and 2 together with the “half” existence theorem in [Ln4] (see also [DE]) to prove a full existence theorem. We then combine our recent results to establish the main classification theorem (3.9). A few examples of applications are presented at the end of the paper.

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1. AH-algebras revisited

The main purpose of this section is to prove Lemma 1.8.

LEMMA 1.1. *Let $G = \lim_{n \rightarrow \infty} (G_n, \alpha_n)$ be a countable unperforated simple ordered group with the Riesz interpolation property, where $G_n = \bigoplus_{i=1}^{l(n)} D_i^{(n)}$, $D_i^{(n)} = \mathbb{Z}$ and α_n is an order homomorphism. Suppose that $\alpha_{m,\infty}(D_i^{(m)}) \neq \{0\}$ for every i and m .*

Then for any $0 < m_1 < m_2$, there is an integer $N = N(m_1, m_2) > m_2$ satisfying the following: if $\pi_k \circ \alpha_{m_2, N}(G_{m_2}) \neq \{0\}$, then $\pi_k \circ \alpha_{m_1, N}(D_j^{(m_1)}) \neq \{0\}$ for $1 \leq j \leq l(m_1)$ and for all $k \leq l(N)$, where $\pi_k : G_N \rightarrow D_k^N$ is the standard projection.

Proof. Let $g_i \in D_i^{(m_2)}$ be the positive generator of $D_i^{(m_2)}$ ($\cong \mathbb{Z}$), $i = 1, 2, \dots, l(m_2)$. Let $x_i = \alpha_{m_2, \infty}(g_i)$. Let s_j be the positive generators for $D_j^{(m_1)}$ and let $f_j = \alpha_{m_1, \infty}(s_j)$, $j = 1, 2, \dots, l(m_1)$. From the assumption, $f_j \neq 0$. Since G is simple, there are positive integers k_{ij} such that $k_{ij}f_j \geq x_i$ for $i = 1, 2, \dots, l(m_2)$. Therefore, there is $N = N(m_1, m_2)$ such that

$$k_{ij}\alpha_{m_1, N}(s_j) \geq \alpha_{m_2, N}(g_i),$$

$i = 1, 2, \dots, l(m_2)$. Fix $k \leq l(N)$, define $\pi_k : G_N \rightarrow D_k^N$ to be the standard projection. If for some k , $\pi_k \circ \alpha_{m_2, N}(g_i) > 0$, then $k_{ij}\pi_k \circ \alpha_{m_1, N}(s_j) > 0$. Since $\{g_1, g_2, \dots, g_{l(m_2)}\}$ is a set of generators, we see that the conclusion holds. \square

Let G be a countable unperforated simple ordered group and T be the state space of G . Let $\rho_G : G \rightarrow \text{Aff}T$ be the map defined by evaluation, i.e., $\rho_G(g)(t) = t(g)$ ($g \in G$ and $t \in T$). It is known that

$$G_+ = \{g : \rho(g)(t) > 0 \text{ for all } t \in T\} \cup \{0\}.$$

In the following lemma, it is known that one can require that $\alpha_n^{i,j}$ have multiplicity at least 2 or $\alpha_n^{i,j} = 0$ ([Ell2]). So the only thing new is that we can always assume for every j , $\alpha_n^{i,j}$ has multiplicity at least 2 (not zero) for some i .

LEMMA 1.2. *Let G be a countable unperforated simple ordered group with the Riesz interpolation property. Suppose that $\ker \rho = 0$. Then there are $\{G_n\}$, where G_n is a finite sum of \mathbb{Z} with the usual order, and positive homomorphisms $\alpha_n : G_n \rightarrow G_{n+1}$ such that $G = \lim_{n \rightarrow \infty} (G_n, \alpha_n)$. Furthermore, nonzero $\alpha_n^{i,j}$ has multiplicity at least 2, and for each j , there is at least one $\alpha_n^{i,j} \neq 0$, and for each i , there is at least one $\alpha_n^{i,j} \neq 0$, where $\alpha_n^{i,j} : D_i^{(n)} \rightarrow D_j^{(n+1)}$ is the partial map of α and $D_i^{(m)}$ is the i^{th} summand of G_m ($D_i^{(m)} \cong \mathbb{Z}$).*

Proof. The first part of the lemma follows from [EHS]. It is the last part which needs a proof. We write $G_n = \bigoplus_{i=1}^{m(n)} D_i^{(n)}$, where $D_i^{(n)} \cong \mathbb{Z}$. Without loss of generality, we may assume that

$$\alpha_{m,\infty}(D_i^{(m)}) \neq \{0\}$$

for all i and m . By [Ell2], we may also assume that $\alpha_n^{i,j}$ is either zero or has multiplicity at least 2.

Let $G'_1 = G_1$ and let $G'_2 = \bigoplus \{D_j^{(2)} : \alpha_1^{i,j} \neq 0 \text{ for some } i\}$. If G'_n is defined, define

$$G'_{n+1} = \bigoplus \{D_j^{(n+1)} : \alpha_n^{i,j} \neq 0 \text{ for some } i \text{ such that } D_i^{(n)} \in G'_n\}.$$

Set $\beta_n = (\alpha_n)|_{G'_n}$. It is important to note that each nonzero partial map ($\alpha_n^{j,i}$) of β_n has multiplicity at least 2 and for each j , at least one $\alpha_n^{j,i} \neq 0$.

Let $G' = \lim_{n \rightarrow \infty} (G'_n, \beta_n)$. It suffices to show that G' is order isomorphic to G . We will use $\alpha_{n,m}$ ($m > n$) for $\alpha_m \circ \alpha_{m-1} \circ \dots \circ \alpha_n$ and $\beta_{n,m}$ for $\beta_m \circ \beta_{m-1} \circ \dots \circ \beta_n$. Clearly $\beta_{n,m} = (\alpha_{n,m})|_{G'_n}$.

Define $\phi_1 = \beta_1 : G'_1 \rightarrow G_2$. By Lemma 1.1, there exists an integer $N(1, 2) > 0$ such that the conclusion of 1.1 holds for $m_1 = 1$ and $m_2 = 2$. Let $k(2) = N(1, 2)$. Define $\psi_2 = \alpha_{2,k(2)}$. The conclusion of 1.1 shows that ψ_2 maps G_2 to $G'_{k(2)}$ and $\psi_2 \circ \phi_1 = \beta_{1,k(2)}$.

Define $\phi_2 = \beta_{k(2)} : G'_{k(2)} \rightarrow G_{k(2)}$. By definition of ψ_2 , $\alpha_{2,k(2)} = \phi_2 \circ \psi_2$. By applying 1.1 again, we obtain an integer $k(3) = N(1, k(2))$ such that the conclusion of 1.1 holds for 1 and $k(2)$. Define $\psi_3 = \alpha_{k(2),k(3)}$. The conclusion of 1.1 shows that ψ_3 maps $G_{k(2)}$ to $G'_{k(3)}$. Also, $\beta_{k(2),k(3)} = \psi_3 \circ \phi_2$. Thus we obtain the following commutative diagram:

$$\begin{array}{ccccc} G'_1 & \xrightarrow{\beta_{1,k(1)}} & G'_{k(2)} & \xrightarrow{\beta_{k(2),k(3)}} & G'_{k(3)} \\ \downarrow \phi_1 & \nearrow \psi_2 & \downarrow \phi_2 & \nearrow \psi_3 & \\ G_2 & \xrightarrow{\alpha_{2,k(2)}} & G_{k(2)} & \xrightarrow{\alpha_{k(2),k(3)}} & G_{k(3)}. \end{array}$$

Continuing this construction, we obtain the next commutative diagram:

$$\begin{array}{ccccccc}
 G'_1 & \xrightarrow{\beta_{1,k(1)}} & G'_{k(2)} & \xrightarrow{\beta_{k(2),k(3)}} & G'_{k(3)} & \xrightarrow{\beta_{k(3),k(4)}} & \cdots \longrightarrow G' \\
 \downarrow \phi_1 & \nearrow \psi_2 & \downarrow \phi_2 & \nearrow \psi_3 & \downarrow \phi_3 & \nearrow \psi_4 & \\
 G_2 & \xrightarrow{\alpha_{2,k(2)}} & G_{k(2)} & \xrightarrow{\alpha_{k(2),k(3)}} & G_{k(3)} & \xrightarrow{\alpha_{k(3),k(4)}} & \cdots \longrightarrow G.
 \end{array}$$

Therefore $G \cong G'$. Since each ϕ_n and ψ_n is positive, this isomorphism is in fact an order isomorphism. \square

Definition 1.3. Let $f : S^1 \rightarrow S^1$ be a degree k map ($k > 1$), i.e., a continuous map with winding number k . We let (following 4.2 in [EG]) $T_{II,k} = D^2 \cup_f S^1$, the finite connected CW complex obtained by attaching a 2-cell D^2 to S^1 via the map f . Note that $K_0(C(T_{II,k})) = \mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$ and $K_1(C(T_{II,k})) = \{0\}$. Let $g : S^2 \rightarrow S^2$ be a degree k map ($k > 1$). Let $T_{III,k} = D^3 \cup_g S^2$ be the connected finite CW complex obtained by attaching a 3-cell D^3 to S^2 via the map g . Note that $K_0(C(T_{III,k})) = \mathbb{Z}$ and $K_1(C(T_{III,k})) = \mathbb{Z}/k\mathbb{Z}$ (see 4.2 in [EG]).

Definition 1.4. Let $C = PM_n(C(X))P$, where $P \subset M_n(C(X))$ is a projection with rank $r(x)$ at point x . Note that if X is connected, $r(x)$ is a constant. Let B be another C^* -algebra. A map $\omega : C \rightarrow B$ is said to be a *point-evaluation*, if $\omega = h \circ \pi_x$, where $x \in X$ is a point, $\pi_x(f) = f(x)$ maps C to $M_{r(x)}$ and $h : M_{r(x)} \rightarrow B$ is a homomorphism. Suppose that $e \in M_{r(x)}$ is a minimal projection. We say that ω is a point-evaluation with a minimal projection $h(e)$.

The following is a refinement of a result in [EG]. Only part (3) is new.

THEOREM 1.5. *For any countable simple weakly unperforated scaled ordered group $(G, G_+, [u])$ with the Riesz interpolation property and any countable abelian group F , there exists a unital simple C^* -algebra A of real rank zero with the following properties:*

- (1) $A = \lim_{n \rightarrow \infty} (A_n, h_n)$, where each A_n is a finite direct sum of $P_i M_{m(i)}(C(X_i)) P_i$, where $X_i = Y_1 \vee Y_2 \vee \cdots \vee Y_m$ and each $Y_i = S^1, S^2, T_{II,k}, T_{III,l}$, or a point;
- (2) $(K_0(A), K_0(A)_+, [1_A]) = (G, G_+, u)$ and $K_1(A) = F$ and
- (3) $\ker \rho_{K_0(A)} = \lim_{n \rightarrow \infty} (\ker \rho_{K_0(A_n)}, (h_n)_*)$.

Proof. Let $G_0 = \rho_G(G)$. Then G_0 is an unperforated ordered group. By Lemma 1.2, we may write $G_0 = \lim_{n \rightarrow \infty} (G_n, \alpha_n)$, where $D_m = \bigoplus_{i=1}^{l(m)} \mathbb{Z}$ with the usual order and $\alpha_n^{(i,j)} : D_i^{(n)} \rightarrow D_j^{(n+1)}$ has order at least 2 and satisfies

the rest of the requirements of 1.2. Let $\ker \rho_G = \lim_{n \rightarrow \infty} (H_n, \beta_n)$ and let $F = \lim_{n \rightarrow \infty} (F_n, \gamma_n)$, where H_n and F_n are finitely generated abelian groups. Let $\tilde{G}_n = \alpha_{n, \infty}(G_n)$. We may write $\tilde{G}_n \subset \tilde{G}_{n+1}$, $n = 1, 2, \dots$.

Let \tilde{S}_n be the subgroup of G generated by $\ker \rho_G$ and G'_n such that $\rho_G(G'_n) = \tilde{G}_n$. Since \tilde{G}_n is free, we may write $\tilde{S}_n = \tilde{G}_n \oplus \ker \rho_G$. Let $\iota_n : \tilde{S}_n \rightarrow \tilde{S}_{n+1}$ be the embedding. Then $\iota_n(g \oplus h) = g \oplus (\iota_n^{(0)}(g) + h)$, where $\iota_n^{(0)} : \tilde{G}_n \rightarrow \ker \rho$ is a homomorphism (note that if $G = G_0 \oplus \ker \rho$, then one may choose $\iota_n^{(0)} = 0$). Set $S_n = G_n \oplus \ker \rho_G$. Define $j_n : S_n \rightarrow S_{n+1}$ by $j_n(g \oplus h) = \alpha_n(g) \oplus (\alpha_n^{(0)}(g) + h)$, where $\alpha_n^{(0)} = \iota_n^{(0)} \circ \alpha_{n, \infty}$. Then the following is commutative:

$$\begin{array}{ccc} \tilde{S}_n & \xrightarrow{\iota_n} & \tilde{S}_{n+1} \\ \uparrow \alpha_{n, \infty} & & \uparrow \alpha_{n+1, \infty} \\ S_n & \xrightarrow{j_n} & S_{n+1}. \end{array}$$

Set $\eta_n = (j_n)|_{G_n}$ and $\delta_n = \eta_n \oplus \beta_n$. Set

$$(G_n \oplus H_n)_+ = \{(g, x) : g > 0, x \in H_n\} \cup \{0\}.$$

Note that $(G, G_+) = \lim_{n \rightarrow \infty} (G_n \oplus H_n, (G_n \oplus H_n)_+, \delta_n)$.

Let $X_n = Y_1 \vee Y_2 \vee \dots \vee Y_{t(n)}$, where $Y_i = S^1, S^2, T_{II, k}$ or $T_{III, l}$, such that $K_0(C(X_n)) = \mathbb{Z} \oplus H_n$ with

$$\begin{aligned} \ker \rho_{C(X_n)} &= H_n, \\ K_0(C(X_n))_+ &\subset \{(z, x) : z > 0, x \in H_n\} \cup \{0\}, \\ \{(y, x) : y \geq 3, x \in H_n\} &\subset K_0(C(X_n))_+ \end{aligned}$$

(see 4.17 in [EG]) and $K_1(C(X_n)) = F_n$. Suppose that $u = \delta_n(u_n)$ and that $\pi_j(u_n) \geq 4$, where $\pi_j : G_n \rightarrow \mathbb{Z}$ is the projection to the j^{th} coordinate.

Let $A_1 = P_1 M_{s(1)}(C(X_1)) P_1 \oplus B_1$, where $B_1 = \oplus^{l(1)-1} M_2$, $P_1 \in M_{r(1)}(C(X_{1,1}))$ is a projection so that $[P_1] = \pi_1(u_1)$ ($s(1) > \pi_1(u_1)$). We have

$$\begin{aligned} K_0(A_1) &= G_1 \oplus H_1, \\ \ker \rho_{A_1} &= H_1, \\ K_0(A_1)_+ &\subset \{(g, x) : g \in (G_1)_+\} \oplus \{0\}, \\ \{(g, x) : g \in 3(D_1^{(1)})_+\} &\subset K_0(A_1)_+, \end{aligned}$$

and

$$K_1(A_1) = F_n.$$

Set $C_1 = P_1 M_{r(1)}(C(X_1)) P_1$. Denote by $r(1, 1)$ the rank of P_1 ($r(1, 1) \leq r(1)$), and let $r(1, i) = 2$ for $2 \leq i \leq l(1)$.

Let N be the required integer in Lemma 3.27 in [EG] corresponding to the space X_1 and $\varepsilon < 1/4$. Using 1.2, choose k_2 such that each nonzero partial map $\alpha_{1, k_2}^{i, j}$ has multiplicity $N + 3$. Since, for each j , $\alpha_{1, k_2}^{i, j} \neq 0$ for some i , $\pi_j(u_{k_2}) \geq$

(see the proof of 4.18 in [EG] for example) and $K_1(A) = F$. The simplicity of G also implies that A is simple. This also follows the fact that the multiplicity of each $\alpha_n^{(i,j)}$ is at least 4. So (1) and (2) follow. Part (3) also follows from the fact that $[\Phi_{1,2}]|_{K_0(A_1)} = \delta_{1,k_2}$. \square

Definition 1.6. Let C_n be a commutative C^* -algebra with $K_0(C_n) = \mathbb{Z}/n\mathbb{Z}$ and $K_1(C_n) = 0$. Suppose that A is a C^* -algebra. Then $K_i(A, \mathbb{Z}/k\mathbb{Z}) = K_i(A \otimes C_k)$. Let $\mathbf{P}(A)$ be the set of all projections in $M_\infty(A)$, $M_\infty(C(S^1) \otimes A)$, $M_\infty((A \otimes C_m))$ and $M_\infty((C(S^1) \otimes A \otimes C_m))$. We have the following commutative diagram ([Sc]):

$$\begin{array}{ccccc} K_0(A) & \rightarrow & K_0(A, \mathbb{Z}/k\mathbb{Z}) & \rightarrow & K_1(A) \\ & & \uparrow \mathbf{k} & & \downarrow \mathbf{k} \\ K_0(A) & \leftarrow & K_1(A, \mathbb{Z}/k\mathbb{Z}) & \leftarrow & K_1(A). \end{array}$$

As in [DL], we use the notation

$$\underline{K}(A) = \bigoplus_{i=0,1, n \in \mathbb{Z}_+} K_i(A; \mathbb{Z}/n\mathbb{Z}).$$

By $\text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respect the direct sum decomposition and the so-called Bockstein operations (see [DL]). It follows from [DL] that if A satisfies the Universal Coefficient Theorem, then $\text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) = KL(A, B)$.

Let A and B be two C^* -algebras and $L : A \rightarrow B$ a completely positive linear map. Then L induces maps from $A \otimes C_m$ to $B \otimes C_m$, from $C(S^1) \otimes A \otimes C_m$ to $C(S^1) \otimes B \otimes C_m$, namely, $L \otimes \text{id}$. For convenience, we will also denote the induced map by L . Let A and B be C^* -algebras, let $L : A \rightarrow B$ be a contractive completely positive linear map, let $\varepsilon > 0$ and let $\mathcal{F} \subset A$ be a subset. Now, L is said to be \mathcal{F} - ε -multiplicative, if

$$\|L(xy) - L(x)L(y)\| < \varepsilon$$

for all $x, y \in \mathcal{F}$. Given a projection $p \in \mathbf{P}(A)$, if L is \mathcal{G} - ε -multiplicative with sufficiently large \mathcal{G} and sufficiently small ε , $L(p)$ is close to a projection. Let $L(p)'$ be that projection. Fix a finite subset $\mathcal{P}_1 \subset \mathbf{P}(A)$. It is easy to see that $L(p)'$ and $L(q)'$ are in the same equivalence class of projections of $\mathbf{P}(A)$, if p and q are in \mathcal{P}_1 and are in the same equivalence class of projections of $\mathbf{P}(A)$, provided that \mathcal{F} is sufficiently large and ε is sufficiently small. We use $[L](p)$ for the class of projections containing $[L](p)'$. In what follows, whenever we write $[L](p)$, we assume that \mathcal{F} is sufficiently large and ε is sufficiently small so that $[L](p)$ is well-defined on \mathcal{P}_1 . Furthermore, abusing the language, we write $[L]([p])$ as well as $[L](p)$, where $[p]$ is the equivalence class containing p .

Suppose that q is in $\mathbf{P}(A)$ with $[q] = k[p]$ for some integer k ; by adding sufficiently many elements (partial isometries) in \mathcal{F} , we can assume that $[L](q) = k[L](p)$. Suppose that G is a finitely generated group generated by \mathcal{P} and

$G = \mathbb{Z}^n \oplus \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_m\mathbb{Z}$. Let g_1, g_2, \dots, g_n be free generators of \mathbb{Z}^n and $t_i \in \mathbb{Z}/k_i\mathbb{Z}$ be the generator with order k_i , $i = 1, 2, \dots, m$. Since every element in $K_0(C)$ (for any unital C^* -algebra C) may be written as $[p_1] - [p_2]$ for projections $p_1, p_2 \in A \otimes M_l$, for some $l > 0$, with sufficiently large \mathcal{F} and sufficiently small ε , one can define $[L](g_j)$ and $[L](t_i)$. Moreover (with sufficiently large \mathcal{F} and sufficiently small ε), the order of $[L](t_i)$ divides k_i . Then we can define a map $[L]|_G$ by defining $[L](\sum_i^n n_i g_i + \sum_j^m m_j t_j) = \sum_i^n n_i [L](g_i) + \sum_j^m m_j [L](t_j)$. Thus $[L]$ is a group homomorphism on G . Note, in general, that $[L]|_{\mathcal{P}}$ may not coincide with $[L]|_G$ on \mathcal{P} . However, if \mathcal{F} is large enough and ε is small enough, they coincide. In what follows, if \mathcal{P} is given, we say $[L]|_G$ is well-defined and write $[L]|_G$ if $[L]|_{\mathcal{P}}$ is well-defined, $[L]|_G$ is well-defined and is a homomorphism and $[L]|_{\mathcal{P}} = [L]|_G$ on \mathcal{P} .

Definition 1.7. We denote by \mathcal{C} the family of all unital simple C^* -algebras of real rank zero which are direct limits of finite direct sums of unital hereditary C^* -subalgebras of $M_n(C(X))$ (for various n), where X is a connected finite CW complex of dimension no more than 3 (and X may be different in the sums). This is precisely the class of simple C^* -algebras classified in [EG].

LEMMA 1.8. Let A be a unital simple C^* -algebra in \mathcal{C} . Let G_0 be a finitely generated subgroup of $K_0(A)$ with decomposition $G_0 = G_{00} \oplus G_{01}$, where $G_{00} \subset \ker \rho_A$ and G_{01} is a finitely generated free group such that $(\rho_A)|_{G_{01}}$ is injective. Suppose that $\mathcal{P} \subset \underline{K}(A)$ is a finite subset which generates a subgroup G such that $G \cap K_0(A) = G_0$.

Then, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, any $1 > r > 0$, and any integer K , there is an \mathcal{F} - ε -multiplicative map $L : A \rightarrow A$ satisfying the following:

- (1) $[L]|_{\mathcal{P}}$ and $[L]|_G$ are well-defined and $[L]|_G$ is positive on G ,
- (2) $[L]|_{G \cap \ker \rho_A} = \text{id}|_{G \cap \ker \rho_A}$, $[L]|_{G \cap K_0(A, \mathbb{Z}/k\mathbb{Z})} = \text{id}|_{G \cap K_0(A, \mathbb{Z}/k\mathbb{Z})}$, $[L]|_{G \cap K_1(A)} = \text{id}|_{G \cap K_1(A)}$ and $[L]|_{G \cap K_1(A, \mathbb{Z}/k\mathbb{Z})} = \text{id}|_{G \cap K_1(A, \mathbb{Z}/k\mathbb{Z})}$ for those k with $G \cap K_i(A, \mathbb{Z}/k\mathbb{Z}) \neq \emptyset$ ($i = 0, 1$),
- (3) $|\rho_A \circ [L](g)| \leq r |\rho_A(g)|$ for all $g \in G \cap K_0(A)$.
- (4) Let g_1, g_2, \dots, g_l be positive generators of G_{01} . Then, there are $f_1, \dots, f_l \in K_0(A)_+$ such that

$$g_i - [L](g_i) = K f_i, \quad i = 1, 2, \dots, l.$$

Proof. We may write $A = \lim_{n \rightarrow \infty} (A_n, \phi_{n, n+1})$, where $A_n = \bigoplus_{i=1}^{m(n)} B_{n,i}$, each

$$B_{n,i} = P_{n,i} M_{J(n,i)}(C(X_{n,i})) P_{n,i}$$

for some connected finite CW complex $X_{n,i}$ with $\dim X_{n,i} \leq 3$ and $P_{n,i} \in M_{J(n,i)}(C(X_{n,i}))$ is a projection, and $\phi_{n,n+1} : A_n \rightarrow A_{n+1}$ is a homomorphism, as constructed in 1.5. In particular, (3) in 1.5 holds.

Let $\phi_{n,i,j} : B_{n,i} \rightarrow B_{n+1,j}$ be the partial map determined by $\phi_{n,n+1}$. Fix $0 < r < 1$. By the proof of 1.5, we may assume that

$$\phi_{n,i,j}(f) = \text{diag}(\tilde{\phi}_{n,i,j}(f), \psi_{n,i,j}(f)),$$

where $\tilde{\phi}_{n,i,j}(1_{B_{n,i}})$ has rank no more than $12 \times \text{rank}(1_{B_{n,i}})$ and

$$\psi_{n,i,j}(f) = \sum_{s=1}^{l(n,i,j)} \omega_{n,i,j,s}(f),$$

where $\omega_{n,i,j,s}$ is a point-evaluation (see 1.4) such that $\omega_{n,i,j,s}(1_{B_{n,i}}) = q_{n,i,j,s}$, $\{q_{n,i,j,s}\}$ is a set of mutually orthogonal projections in $B_{n+1,j}$ and $\{q_{n,i,j,s}\}$ are equivalent (trivial) projections in $B_{n+1,j}$ with rank at least $12 \times \text{rank}(1_{B_{n,i}})$ for $s = 2, \dots, l(n, i, j)$, and $q_{n,i,j,1}$ has rank $\text{rank} 1_{B_{n+1,j}} - 12 \times l(n, i, j) 1_{B_{n,i}}$.

By choosing larger n , we may assume that $\mathcal{F} \subset \phi_{n,\infty}(A_n)$. Furthermore, we may also assume that

$$G \subset [\phi_{n,\infty}](\underline{K}(A_n)).$$

Let G' be a finitely generated subgroup of $\underline{K}(A_n)$ such that $[\phi_{n,\infty}](G') = G$. To save notation without loss of generality, we may assume that $G_0 = [\phi_{n,\infty}](K_0(A_n))$. Write $K_0(A_n) = F_0 \oplus F_1$, where $F_0 = \ker \rho_{A_n}$. By (3) of 1.5, we may assume, without loss of generality, that $[\phi_{n,\infty}](F_0) = G_{00}$. Let K_0 be the integer such that

$$G' \cup K_i(A, \mathbb{Z}/k\mathbb{Z}) = \{0\}$$

whenever $k \geq K_0$ and $i = 0, 1$. Put $K_1 = K(K_0)!$. By replacing $n + 1$ by a larger integer, if necessary, we may assume that

$$l(n, i, j) > (K_1 + 1)K_1(1/r) \quad \text{for all } n, i, j.$$

Let

$$l(n, i, j) - 1 = K_1 v(n, i, j)' + r(n, i, j),$$

where $v(n, i, j)$ and $r(n, i, j)$ are nonnegative integers with $r(n, i, j) < K_1$.

Define

$$\Phi_{n,i,j}(f) = \text{diag}(\tilde{\phi}_{n,i,j}(f), \psi'_{n,i,j}(f)), \quad \text{where } \psi'_{n,i,j}(f) = \sum_{s=1}^{1+r(n,i,j)} \omega_{n,i,j,s}(f)$$

and define

$$\Phi'_{n,i,j}(f) = \sum_{s=2+r(n,i,j)}^{l(n,i,j)} \omega_{n,i,j,s}(f) \quad \text{for all } f \in B_{n,i}.$$

Define $\Phi(f) = \bigoplus_{i,j} \Phi_{n,i,j}(f)$ and $\Phi'(f) = \bigoplus_{i,j} \Phi'_{n,i,j}(f)$ for $f \in A_n$. Since Φ' has finite-dimensional range, we know that

$$[\Phi']|_{K_0(A_n) \cap \ker \rho_{A_n}} = 0 \quad \text{and} \quad [\Phi']|_{K_1(A_n)} = 0.$$

For each i and j , $\Phi'_{n,i,j}(f)$ is a direct sum of $K_1 v(n, i, j)$ many point-evaluations. Since $X_{n,i}$ is connected and $(K_0)!K_1$, by considering each $[\Phi'_{n,i,j}]$, we conclude that (for $k = 2, \dots, K_0$)

$$[\Phi']|_{K_0(A_n, \mathbb{Z}/k\mathbb{Z})} = 0 \quad \text{and} \quad [\Phi']|_{K_1(A_n, \mathbb{Z}/k\mathbb{Z})} = 0.$$

Therefore

$$\begin{aligned} [\Phi]|_{F_0} &= [\phi_{n,n+1}]|_{F_0}, & [\Phi]|_{K_0(A_n, \mathbb{Z}/k\mathbb{Z})} &= [\phi_{n,n+1}]|_{K_0(A_n, \mathbb{Z}/k\mathbb{Z})}, \\ [\Phi]|_{K_1(A_n)} &= [\phi_{n,n+1}]|_{K_1(A_n)} \quad \text{and} \quad [\Phi]|_{K_1(A_n, \mathbb{Z}/k\mathbb{Z})} &= [\phi_{n,n+1}]|_{K_1(A_n, \mathbb{Z}/k\mathbb{Z})} \end{aligned}$$

for every $k = 1, 2, \dots, K_0$. Thus

$$\begin{aligned} [\phi_{n+1, \infty} \circ \Phi]|_{F_0} &= [\phi_{n, \infty}]|_{F_0}, \\ [\phi_{n+1, \infty} \circ \Phi]|_{K_0(A_n, \mathbb{Z}/k\mathbb{Z})} &= [\phi_{n, \infty}]|_{K_0(A_n, \mathbb{Z}/k\mathbb{Z})}, \\ [\phi_{n+1, \infty} \circ \Phi]|_{K_1(A_n)} &= [\phi_{n, \infty}]|_{K_1(A_n)} \end{aligned}$$

and

$$[\phi_{n+1, \infty} \circ \Phi]|_{K_1(A_n, \mathbb{Z}/k\mathbb{Z})} = [\phi_{n, \infty}]|_{K_1(A_n, \mathbb{Z}/k\mathbb{Z})}$$

for every $k = 1, 2, \dots, K_0$. It is clear that

$$\ker \phi_{n+1, \infty} \circ \Phi = \ker \phi_{n, n+1}.$$

(In fact, we could assume that $\ker \phi_{n+1, \infty} \circ \Phi = \ker \phi_{n, n+1} = \{0\}$.) Thus, $\phi_{n+1, \infty} \circ \Phi$ induces a map $\Psi : \phi_{n, \infty}(A_n) \rightarrow A$. Set $A'_n = \phi_{n, \infty}(A_n)$. Denote by $\iota : A'_n \rightarrow A$ the embedding. From above, we have

$$\begin{aligned} [\Psi]|_{\ker \rho_{A'_n}} &= [\iota]_{\ker \rho_{A'_n}}, & [\Psi]|_{K_1(A'_n)} &= [\iota]_{K_1(A'_n)}, \\ [\Psi]|_{K_0(A'_n, \mathbb{Z}/k\mathbb{Z})} &= [\iota]_{K_0(A'_n, \mathbb{Z}/k\mathbb{Z})}, \quad \text{and} \quad [\Psi]|_{K_1(A'_n, \mathbb{Z}/k\mathbb{Z})} &= [\iota]_{K_1(A'_n, \mathbb{Z}/k\mathbb{Z})} \end{aligned}$$

for $k = 1, 2, \dots$.

Since A is unclear, for any finite subset $\mathcal{F}_1 \subset A$ and $\delta > 0$, there exists (see 4.1 in [Ln4]) a completely positive linear map $L : A \rightarrow A$ such that

$$\|L(a) - \Psi(a)\| < \delta$$

for all $a \in \mathcal{F}_1$. We may assume that $\mathcal{F} \subset \mathcal{F}_1$ and $\delta < \varepsilon$. Furthermore, by choosing even larger \mathcal{F}_1 and small δ , we may assume that

$$\begin{aligned} [L]|_{G_{00}} &= \text{id}|_{G_{00}}, & [L]|_{G \cap K_1(A)} &= \text{id}|_{G \cap K_1(A)} \\ [L]|_{G \cap K_0(A, \mathbb{Z}/k\mathbb{Z})} &= \text{id}|_{G \cap K_0(A, \mathbb{Z}/k\mathbb{Z})} \quad \text{and} \quad [L]|_{G \cap K_1(A, \mathbb{Z}/k\mathbb{Z})} &= \text{id}|_{G \cap K_1(A, \mathbb{Z}/k\mathbb{Z})} \end{aligned}$$

for all k so that $G \cap K_i(A, \mathbb{Z}/k\mathbb{Z}) \neq \emptyset$ ($i = 0, 1$). From $l(n, i, j) \geq (K_1 + 1)K_1(1/r)$ we conclude that

$$\tau(L(a)) < r\tau(a)$$

for all $a \in A$ and $\tau \in T(A)$. This implies that L satisfies (1), (2) and (3). For $s \geq 2$, $[\omega_{n,i,j,s}] = [\omega_{n,i,j,2}]$. Since, for any $z \in K_0(B_{n,i})$,

$$[\phi_{n,i,j}](z) - [\Phi_{n,i,j}](z) = K(K_0)!v(n, i, j)[\omega_{n,i,j,2}](z),$$

(4) also follows. □

2. Extensions of positive homomorphisms on ordered groups

Let G be a group, G_0 be a subgroup of G and F be another group. In general, to extend a homomorphism $\phi : G_0 \rightarrow F$ to a homomorphism $\tilde{\phi} : G \rightarrow F$ requires F to be divisible. If G is an ordered group and ϕ is positive, much more is required of F to obtain a positive extension. In this section, we will present a positive extension theorem which neither requires divisibility nor completeness of F . This result is rather special but is essential for us to construct maps in 3.4.

LEMMA 2.1. *Let $G \subset \mathbb{Z}^k$ be an ordered subgroup and let*

$$e_1 = (1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1) \in \mathbb{Z}^k.$$

Suppose that $S \subset \{1, 2, \dots, k\}$ such that if $i \in S$, then there exists a positive integer m_i such that $m_i e_i \in G$ and if $i \notin S$, then $m e_i \notin G$ for any $m \in \mathbb{Z} \setminus \{0\}$. Then for any positive homomorphism $\phi : G \rightarrow \mathbb{R}$ with $\phi(G_+ \setminus \{0\}) \subset \mathbb{R}_+ \setminus \{0\}$,

$$\inf\{\phi(g)/n : g \in G, g \geq n e_i\} > \sup\{\phi(g)/m : g \in G, g \leq m e_i\} > 0$$

for all $i \notin S$.

Proof. This follows from Lemma 2.10 in [Ln6]. □

LEMMA 2.2. *Let $G \subset \mathbb{Z}^k$, let*

$$e_1 = (1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1) \in \mathbb{Z}^k,$$

and $S \subset \{1, 2, \dots, k\}$ be as in Lemma 2.1. Let T be a Choquet simplex and $F \subset \text{Aff}(T)$ be a dense subgroup such that $f \in F_+ \setminus \{0\}$, $f(t) > 0$ for all $t \in T$. Suppose that $\phi : G \rightarrow F$ is a positive homomorphism. Let

$$U_j(t) = \inf\{\phi(g)(t)/n : n \in \mathbb{N}, g \in G, n e_j \leq g\},$$

$$L_j(t) = \sup\{\phi(g)(t)/n : n \in \mathbb{N}, g \in G, g \leq n e_j\}$$

and $H_j(t) = U_j(t) - L_j(t)$. Then

$$\liminf_{t \rightarrow t_0} H_j(t) > 0$$

for every $t_0 \in T$, if $j \notin S$.

Proof. Clearly $H_j(t) \geq 0$ for all $t \in T$. Suppose that $j \notin S$. By Lemma 2.1, if $H_j(t) = 0$ for some $t \in T$, then $j \in S$. Therefore $H_j(t) > 0$ for every $t \in T$.

We extend ϕ on $\mathbb{Q}G$ by defining $\phi(rg) = r\phi(g)$ for all $r \in \mathbb{Q}$ and $g \in G$.

Note that, since $\mathbb{Q}G$ is finite-dimensional, if $\{x_n\} \subset \mathbb{Q}G$ is a bounded sequence, so is $\{\phi(x_n)\}$. Note also that $H_j(t) > 0$ for all $t \in T$. Without loss of generality, to simplify notation, we may assume that $j = 1$. There are $g_n \in \mathbb{Q}G$ with $g_n \geq e_1$, $g_n = (1, r_1^{(n)}, r_2^{(n)}, \dots, r_k^{(n)})$, where $r_i^{(n)} \in \mathbb{Q}_+$ for $i = 2, 3, \dots, k$ and $t_n \in T$ such that $t_n \rightarrow t_0$ and $\phi_n(g_n)(t_n)(t) \rightarrow \liminf_{t \rightarrow t_0} U_j(t)$; and, there are $y_n \in \mathbb{Q}G$ with $y_n \leq e_1$, $y_n = (1, q_1^{(n)}, \dots, q_k^{(n)})$, where $-q_i^{(n)} \in \mathbb{Q}_+$ for $i = 2, 3, \dots, k$ and $s_n \in T$ such that $s_n \rightarrow t_0$ and $\phi(y_n)(s_n) \rightarrow \limsup_{t \rightarrow t_0} L_j(t)$.

By 2.9 in [Ln6], for each $t \in T$, there exist $\alpha_i > 0$, $i = 1, 2, \dots, k$, such that $\phi(z)(t) = \langle z, \omega \rangle$ for $z \in G$, where $\omega = (\alpha_1, \dots, \alpha_k)$. Therefore $\{r_i^{(n)}\}$ is a bounded sequence for every $i = 2, \dots, k$. Similarly, $\{q_i^{(n)}\}$ is a bounded sequence for every $i = 2, \dots, k$. Thus $\{\phi(g_n)\}$ and $\{\phi(y_n)\}$ are (uniformly) bounded (on T). Since $\mathbb{Q}G$ is finite-dimensional, $\{\phi(g_n)\}$ and $\{\phi(y_n)\}$ are precompact subsets of $\text{Aff}(T)$. Thus, without loss of generality, we may assume that $\phi(g_n) \rightarrow g$ and $\phi(y_n) \rightarrow x$ uniformly on T , where $g, x \in \text{Aff}(T)$. Since each $g_n \geq e_1$ and $y_n \leq e_1$, we conclude that $g(t) \geq U_j(t) > 0$ and $x(t) \leq L_j(t)$ for all $t \in T$. Furthermore,

$$g(t_0) = \liminf_{t \rightarrow t_0} U_j(t) \quad \text{and} \quad x(t_0) = \limsup_{t \rightarrow t_0} L_j(t).$$

We assume that $H_j(t) > 0$ for all $t \in T$. Therefore $g(t) > x(t)$ for all $t \in T$. Since x and g are continuous and T is compact,

$$\inf\{g(t) - x(t) : t \in T\} > 0.$$

This implies that

$$\liminf_{t \rightarrow t_0} H_j(t) > 0 \quad \text{for all } t_0 \in T. \quad \square$$

LEMMA 2.3. *In Lemma 2.2, if $K > 0$ is a previously given integer, then there exists $f \in F \subset \text{Aff}(T)$ such that*

$$L_j(t) < Kf < U_j(t)$$

for all $t \in T$.

Proof. Since T is compact and H_j is upper semi-continuous, $\liminf_{t \rightarrow t_0} H_j(t) > 0$ for all $t_0 \in T$ and the fact that $H_j(t) > 0$ for all $t \in T$ implies that $\inf\{H_j(t) : t \in T\} > 0$. Let $a = (1/32) \inf\{H_j(t) : t \in T\} > 0$. Then $0 < 31a < \liminf_{t' \rightarrow t} H_j(t')$ for all $t \in T$. Fix $t_0 \in T$, let g and x be as in the proof of 2.2. Then

$$\begin{aligned} \limsup_{t' \rightarrow t_0} L_j(t') &< x(t_0) + a/8 < x(t_0) + a/4 \\ &< g(t_0) - a/2 < g(t_0) - a/4 < \liminf_{t' \rightarrow t_0} U_j(t'). \end{aligned}$$

We have

$$x(t') \leq L_j(t') \quad \text{and} \quad U_j(t') \leq g(t')$$

for any $t' \in T$. So, in particular,

$$x(t') \leq g(t')$$

for all $t' \in T$. Therefore there is a neighborhood $O(t_0)$ such that the following holds:

$$\begin{aligned} L_j(t') &< \limsup_{t'' \rightarrow t} L_j(t'') + a/16 < x(t') + a/8 < x(t') + a/4 < g(t') - a/2 \\ &< g(t') - a/4 < \liminf_{t'' \rightarrow t} U(e_1)(t'') - a/8 < U(e_1)(t') \end{aligned}$$

for all $t' \in O(t_0)$.

Since T is compact, there are $O(t_1), O(t_2), \dots, O(t_l)$, such that $\cup_{i=1}^l O(t_i) \supset T$. Note that (with x_i and g_i corresponding to $O(t_i)$) $x_i(t) + a/4, g_i - a/2 \in \text{Aff}(T)$, $i = 1, 2, \dots, l$. Set

$$\tilde{x} = (x_1 + a/4) \vee (x_2 + a/4) \vee \dots \vee (x_l + a/4)$$

and

$$\hat{g} = (g_1 - a/2) \wedge (g_2 - a/2) \wedge \dots \wedge (g_l - a/2).$$

Since $x_i \leq L_j$ and $U_j \leq g_i$ for $i, j = 1, 2, \dots, l$, $\tilde{x} \leq \hat{g}$. Since T is a Choquet simplex, $\text{Aff}(T)$ has the Riesz interpolation property (see II.3.11 in [Alf]). Thus there is $h \in \text{Aff}(T)$ such that

$$\tilde{x} \leq h \leq \hat{g}.$$

Therefore (by considering each $O(t_i)$), we have

$$L_j(t) + a/16 < \tilde{x}(t) \leq h(t) \leq \hat{g}(t) < U_j(t) - a/16$$

for all $t \in T$. Hence,

$$L_j(t) < h(t) - a/32 < h(t) < U_j(t) - a/16.$$

Since F is dense in $\text{Aff}(T)$, there exists $f \in F$ such that

$$\|f - (1/K)h\| < a/128K.$$

Therefore,

$$L_j(t) < h(t) - a/32 < Kf(t) < h(t) + a/32 < U_j(t)$$

for all $t \in T$. □

LEMMA 2.4. *Let G and D be ordered groups. Suppose that there is a surjective homomorphism $\rho : G \rightarrow D$ such that $g \geq 0$ if and only if $\rho(g) \geq 0$. Then, for any other ordered group G' and a homomorphism $\psi : G' \rightarrow G$, ψ is positive if and only if $\rho \circ \psi$ is positive.*

Proof. This is evident. □

LEMMA 2.5. *Let $G \subset \mathbb{Z}^k$ be an ordered subgroup, T be a Choquet simplex and $F \subset \text{Aff}(T)$ be an ordered dense subgroup with the strict ordering (i.e., $f \in F_+ \setminus \{0\}$ implies $f(t) > 0$ for all $t \in T$). Suppose that G' is an ordered group with a surjective map $\rho : G' \rightarrow F$ such that $g \in G'_+$ if and only if $\rho(g) \in F_+$. There exists an integer K depending only on G satisfying the following: Suppose that $\phi : G \rightarrow G'$ is a positive homomorphism such that $\phi(g) > 0$ for all $g \in G_+ \setminus \{0\}$ and such that $\phi(g_i) = Kf_i$ for the generating set $\{g_1, \dots, g_m\} \subset G$, where $f_i \in G'$; then there is a positive homomorphism $\tilde{\phi} : \mathbb{Z}^k \rightarrow G'$ such that $\tilde{\phi}|_G = \phi$ and $\rho \circ \tilde{\phi}(e_j) \in G'_+ \setminus \{0\}$, where $e_1 = (1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1)$ are the standard generators of \mathbb{Z}^k .*

Proof. Let $P_j : \mathbb{Z}^k \rightarrow \mathbb{Z}^j$ be the projection on the first j coordinates. Let G_j be the subgroup generated by $P_j(\mathbb{Z}^k)$ and G , $j = 1, 2, \dots, k$. Set $G_0 = G$. Let S_0 be the subset of $\{1, 2, \dots, k\}$ such that there is a positive integer m_t with $m_t e_t \in G_0$ whenever $t \in S_0$. We may assume that $S_0 = \{1, 2, \dots, l\}$ ($l \geq 0$). Let

$$I_{ji} = \{m \in \mathbb{Z} : m_i e_i \in G_j\}.$$

Then I_{ji} is a subgroup of \mathbb{Z} . Let $m_{ji} = \min\{|m| \in I_{ji} \setminus \{0\}\}$ and

$$K_j = \prod_{i \in S_j} m_{ji}.$$

Set $J = \prod_{j=0}^k K_j$ and $K = J^k$.

This integer does not depend on ϕ but only on G . Let $\{g_1, g_2, \dots, g_n\}$ be a generating set for G . Suppose that there are $f_1, \dots, f_n \in G$ such that $\phi(g_i) = Kf_i$, $i = 1, 2, \dots$.

The condition that $\phi(g_i) = Kf_i$ for some $f_i \in G'$ implies that $(1/K)\phi(m_i e_i) \in G'$ for some positive m_i and for all $i \leq l$. So we define $\tilde{\phi}(e_i) = (1/m_i)\phi(m_i e_i)$ ($i = 1, 2, \dots, l$). Note that $(1/J^{k-1})\tilde{\phi}(g) \in G'_+$ for $g \in G_+$. Since (for $j \leq l$)

$$\begin{aligned} \sup\{\rho \circ \phi(g)/m : g \in G, m > 0 \quad \text{and} \quad g \leq m e_j\} &> 0, \\ \sup\{\rho \circ \tilde{\phi}(g)/m : g \in G_l, m > 0 \quad \text{and} \quad g \leq m e_j\} &> 0. \end{aligned}$$

So $\rho \circ \tilde{\phi}$ maps $(G_l)_+ \setminus \{0\} \subset F_+ \setminus \{0\}$.

It follows from 2.3 that there is $f \in F_+$ such that

$$L_{l+1}(t) < Kf(t) < U_{l+1}(t)$$

for all $t \in T$, where

$$L_{l+1}(t) = \sup\{\rho \circ \tilde{\phi}(g)/m : m \in \mathbb{N}, g \in G_l \text{ and } g \leq me_{l+1}\}$$

and

$$U_{l+1}(t) = \inf\{\rho \circ \tilde{\phi}(g)/m : m \in \mathbb{N}, g \in G_l \text{ and } g \geq me_{l+1}\}.$$

Let $g \in G'$ such that $\rho(g) = f$. Define $\tilde{\phi}(e_{l+1}) = Kg$. It follows from [GH] that $\tilde{\phi}$ extends ϕ and $\rho \circ \tilde{\phi}$ is positive. By 2.4, $\tilde{\phi}$ is positive. Furthermore, $\rho \circ \tilde{\phi}$ maps $(G_{l+1})_+ \setminus \{0\}$ into $F_+ \setminus \{0\}$. Let S_1 be a subset of $\{1, 2, \dots, k\}$ such that $m_t e_t \in G'_{l+1}$ for some positive integer m_t whenever $t \in S_1$. Note that for any $g \in (G_{l+1})_+$, $(1/J^{k-1})\tilde{\phi}(g) \in G'_+$. If $l+2 \in S_2$, define $\tilde{\phi}(e_{l+2}) = (1/m_{l+1})\tilde{\phi}(m_{l+1}e_{l+2})$. Otherwise, by 2.3, there is $f_{l+2} \in F_+$ such that

$$L_{l+2}(t) < Kf_{l+2} < U_{l+2}(t)$$

for all $t \in T$. Choose $g_{l+2} \in G'$ such that $\rho(g_{l+2}) = f_{l+2}$. Define $\tilde{\phi}(e_{l+2}) = Kg_{l+2}$. In either case we obtain a positive extension $\tilde{\phi}$ on G'_{l+2} and $(1/J^{k-2})\tilde{\phi}(g) \in F_+$ for all $g \in (G_{l+1})_+$. So by an induction argument we obtain $\tilde{\phi} : \mathbb{Z}^k \rightarrow G'$ as desired. □

3. A classification theorem for simple nuclear C^* -algebras with tracial topological rank zero

Definition 3.1. A C^* -algebra A is said be in \mathcal{BD} if there is an integer $k > 0$ such that every irreducible representation of A is finite-dimensional and its dimension is no more than k . The integer k is called the bound. A C^* -algebra A is said be in \mathcal{LBD} (locally \mathcal{BD}) if for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists $B \in \mathcal{BD}$ such that

$$\text{dist}(x, B) < \varepsilon \quad \text{for all } x \in \mathcal{F}.$$

LEMMA 3.2. *Let A be a unital separable C^* -algebra in \mathcal{BD} with the bound k , let $1 = f_1, f_2, \dots, f_m \in \rho_A(K_0(A)_+)$ and let G be the subgroup generated by f_1, \dots, f_m . Then there exists finite-dimensional irreducible representations $\pi_1, \pi_2, \dots, \pi_N$ of A such that*

$$g \mapsto (\text{tr} \circ (\pi_1)_*(g), \dots, \text{tr} \circ (\pi_N)_*(g))$$

is an order embedding from G to \mathbb{Q}^N , where tr is the normalized trace on matrix algebras.

Remark 3.3. In Lemma 3.2, set $x = \max\{\text{tr} \circ \pi_i([1_A]) : i = 1, \dots, N\}$. Then the composition

$$g \mapsto (\text{tr} \circ (\pi_1)_*(g), \dots, \text{tr} \circ (\pi_N)_*(g)) \mapsto x(\text{tr} \circ (\pi_1)_*(g), \dots, \text{tr} \circ (\pi_N)_*(g))$$

gives an order embedding from G into \mathbb{Z}^N .

Proof of Lemma 3.2. This is proved in [Ln7]. Denote by $s(A)$ the set of normalized traces on A defined by $t(a) = \text{tr} \circ \pi(a)$ ($a \in A$), where π is a finite-dimensional irreducible representation of A and tr is the standard normalized trace on matrix algebras, equipped with the weak*-topology. It follows from Corollary 2.7 in [Ln7] that the closure of the convex hull of $s(A)$ is $T(A)$, the tracial space of A . Therefore the map $f \mapsto f|_{s(A)}$ is an order embedding from $\text{Aff}(T(A))$ to $C(s(A))$. Let $D_A : K_0(A) \rightarrow C(s(A))$ be defined by $p \mapsto p(t)$ ($t \in s(A)$), where p is a projection in $A \otimes \mathcal{K}$. Then D_A is a positive homomorphism. Therefore the map from $\rho_A(K_0(A))$ ($\subset \text{Aff}(T(A))$) to $D_A(K_0(A)_+)$ defined by restriction is an order isomorphism. The lemma then follows from Lemma 2.4 in [Ln7]. \square

LEMMA 3.4. *Let A be a unital C^* -algebra in \mathcal{LBD} which is a simple C^* -algebra with stable rank one and weakly unperforated $K_0(A)$ and let $B \in \mathcal{C}$ which is a unital simple C^* -algebra with real rank zero. Suppose that $\alpha \in \text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ which gives an order isomorphism from $(K_0(A), K_0(A)_+, [1_A], K_1(A))$ to $(K_0(B), K_0(B)_+, [1_B], K_1(B))$. Then there is a sequence of contractive completely positive linear maps $L_n : A \rightarrow B$ such that, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$,*

$$[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$$

for all sufficiently large n and

$$\|L_n(ab) - L_n(a)L_n(b)\| \rightarrow 0$$

for all $a, b \in A$.

Proof. Fix a finite subset $\mathcal{F} \subset A$ and a finite subset $\mathcal{P} \subset \mathbf{P}(A)$. Since A is in \mathcal{LBD} , we may assume that there exists $A_m \subset A$ such that $A_m \in \mathcal{BD}$, $\mathcal{F} \subset A_m$ and $\mathcal{P} \subset [j](\mathcal{G})$, where $\mathcal{G} \subset \mathbf{P}(A_m)$ is a finite subset and $j : A_m \rightarrow A$ is the embedding. Set $\alpha = \beta \circ [j] \in \text{Hom}_\Lambda(\underline{K}(A_m), (\underline{K}(B))_+)$. Since both A and B are simple, and $\alpha|_{K_0(A)}$ is an order isomorphism, for any $g \in K_0(A) \setminus \{0\}$, $\rho_B \circ \beta(g) \neq 0$. Note that A_m satisfies the Universal Coefficient Theorem. It follows from 5.9 in [Ln4] that there exist a sequence of contractive completely positive linear maps $\psi_n : A_m \rightarrow B \otimes \mathcal{K}$ and a homomorphism $H_n : A_m \rightarrow B \otimes \mathcal{K}$ with finite-dimensional range such that

$$(e3.1) \quad [\psi_n]|_{\mathcal{G}} = \beta|_{\mathcal{G}} + [H_n]|_{\mathcal{G}}.$$

Let $G' = K_0(A_m) \cap G$, where $G = G(\mathcal{G})$ is the finitely generated subgroup generated by \mathcal{G} . We may assume that there are projections $p_1, \dots, p_l \in M_k(A_m)$ for some k such that G' is generated by $[p_1], \dots, [p_l]$. We may assume that (e3.1) above holds for G' . Let $G_0 = \rho_{A_m}(G')$. It follows from 3.2 (and 3.3) that $G_0 \subset K_0(\pi(A_m)) \subset \mathbb{Z}^k$ for some integer $k > 0$, where $\pi : A_m \rightarrow C$ is a (surjective) homomorphism from A_m to a finite-dimensional C^* -algebra C . Let K be the integer associated with G_0 defined by 2.5.

Let K_1 be the integer such that $G \cap K_0(A, \mathbb{Z}/k\mathbb{Z}) = \emptyset$ for all $k > K_1$.

Let $\Psi_n = \psi_n \oplus H_n \oplus \dots \oplus H_n$, the direct sum of $K(K_1)! - 1$ copies of H_n .

Thus

$$[\Psi_n]_{\mathcal{G}} = \alpha|_{\mathcal{G}} + K(K_1)! [H_n]_{\mathcal{G}}.$$

If F is a finite-dimensional C^* -algebra then one has the following commutative diagram:

$$\begin{array}{ccccc} K_0(F) & \longrightarrow & K_0(F, \mathbb{Z}/k\mathbb{Z}) & \longrightarrow & K_1(F) \\ \uparrow & & & & \downarrow \\ K_0(F) & \longleftarrow & K_1(F, \mathbb{Z}/k\mathbb{Z}) & \longleftarrow & K_1(F) \end{array}$$

where $K_0(F, \mathbb{Z}/k\mathbb{Z}) = K_0(F)/kK_0(F)$, $K_1(F) = 0$, $K_1(F, \mathbb{Z}/k\mathbb{Z}) = 0$. Since H_n factors through a finite-dimensional C^* -subalgebra, it is easy to check that

$$[H_n]_{K_1(A) \cap G} = 0, \quad [H_n]_{K_1(A, \mathbb{Z}/k\mathbb{Z}) \cap G} = 0 \quad \text{and} \quad [H_n]_{\ker \rho_A(K_0(A)) \cap G} = 0.$$

Moreover,

$$(K_1)! [H_n]_{K_0(A, \mathbb{Z}/k\mathbb{Z}) \cap G} = 0 \quad (k \leq K_1).$$

Therefore

$$\begin{aligned} [\Psi_n]_{K_1(A) \cap G} &= \alpha|_{K_1(A) \cap G}, \\ [\Psi_n]_{K_1(A, \mathbb{Z}/k\mathbb{Z}) \cap G} &= \alpha|_{K_1(A, \mathbb{Z}/k\mathbb{Z}) \cap G} \\ [\Psi_n]_{\ker \rho_A(K_0(A)) \cap G} &= \alpha|_{\ker \rho_A(K_0(A)) \cap G} \end{aligned}$$

and

$$[\Psi_n]_{K_0(A, \mathbb{Z}/k\mathbb{Z}) \cap G} = \alpha|_{K_0(A, \mathbb{Z}/k\mathbb{Z}) \cap G}.$$

Choose $r > 0$ such that $r < \frac{1}{JK(K_1)!+1}$, where J is an integer so that $[H_n(1_A)] \leq [1_{M_J(B)}]$. Set $G'_1 = [\Psi_n](\mathcal{G})$. Let e_i be projections such that $[\bar{e}_i] = [\Psi_n]([p_i]) - [\bar{q}_i]$ ($= \alpha([p_i])$), where $[\bar{q}_i] = K(K_1)! [H_n]([p_i])$, $i = 1, 2, \dots, l$. Set $G_1 = [\Psi_n]\mathcal{G} \cup \{[\bar{e}_i], \alpha([p_i]), [\bar{q}_i], i = 1, \dots, l\}$.

Let $L : B \rightarrow B$ be as described in 1.8 associated with G_1 (with $\varepsilon > 0$ and \mathcal{F} to be determined later). Set $\Phi_n = L \circ \Psi_n$. Then

$$\begin{aligned} [\Phi_n]_{K_1(A_m) \cap G} &= \alpha|_{K_1(A_m) \cap G}, \\ [\Phi_n]_{K_1(A_m, \mathbb{Z}/k\mathbb{Z}) \cap G} &= \alpha|_{K_1(A_m, \mathbb{Z}/k\mathbb{Z}) \cap G} \\ [\Phi_n]_{\ker \rho_{A_m}(K_0(A_m)) \cap G} &= \alpha|_{\ker \rho_{A_m}(K_0(A_m)) \cap G} \end{aligned}$$

and

$$[\Phi_n]|_{K_0(A_m, \mathbb{Z}/k\mathbb{Z}) \cap G} = \alpha|_{K_0(A_m, \mathbb{Z}/k\mathbb{Z}) \cap G}.$$

We also have

$$\rho_B \circ [\Psi_n](g) \leq r \rho_B \circ \alpha(g) \text{ for } g \in K_0(A) \cap G$$

and

$$\alpha([p_i]) - [\Phi_n]([p_i]) = K(K_1)! [p_i], \quad i = 1, \dots, l,$$

where $f_i \in K_0(B)$. Note that $(\alpha - [\Psi_n])(g) > 0$ for all $G_0 \setminus \{0\}$, since $r < 1/(KJ(K_1)! + 1)$. Note that also with the order embedding $G_0 \subset K_0(\pi(A)) \subset \mathbb{Z}^k$ and the choice of K , by 2.5, there is a positive homomorphism $\Psi : K_0(\pi(A_m)) \rightarrow K_0(B)$ such that

$$\Psi|_{G_0} = (\alpha - [\Phi_n])|_{G_0}.$$

Since B has real rank zero and stable rank one, we obtain a homomorphism $h_n : \pi(A_m) \rightarrow (1 - p)B(1 - p)$ such that $[h_n] = \Psi$, where $p = 1_B - \Phi_n(1_A)$ (we may assume that $\Psi_n(1_A) \leq 1_B$, since $r < 1/2(KJ(K_1)! + 1)$). We set $L_n = \Phi_n \oplus h_n \circ \pi$ with sufficiently small ε (depends on n) and the finite subset \mathcal{F}_1 (which is larger than $\Phi_n(\mathcal{F})$). Now

$$[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}. \quad \square$$

Definition 3.5. Recall ([Ln6]) that a unital simple C^* -algebra A has tracial topological rank zero (written $\text{TR}(A) = 0$) if, for any $\varepsilon > 0$, any finite subset \mathcal{F} and any nonzero positive element a , there exists a nonzero projection $p \in A$ and a finite-dimensional C^* -subalgebra $B \subset A$ with $1_B = p$ such that

- (1) $\|px - xp\| < \varepsilon$ for all $x \in \mathcal{F}$,
- (2) $pxp \in_\varepsilon B$ for all $x \in \mathcal{F}$ and
- (3) $1 - p$ is equivalent to $q \in \overline{aAa}$.

In [Ln3], a simple unital C^* -algebra A with $\text{TR}(A) = 0$ is called TAF. It is shown in [Ln3] that a simple C^* -algebra A with $\text{TR}(A) = 0$ is quasidiagonal and has real rank zero, stable rank one and weakly unperforated $K_0(A)$. Every simple AH-algebra A with slow dimension growth and real rank zero has $\text{TR}(A) = 0$ (this was proved in [EG]. See 2.6 in [Ln3]).

One could prove the following result directly. But this follows from the main result in [Ln5].

THEOREM 3.6 (cf. [Ln5]). *Let $A \in \mathcal{LBD}$ be a unital separable simple C^* -algebra with unique normalized trace. Suppose that A has real rank zero, stable rank one and weakly unperforated $K_0(A)$. Then $\text{TR}(A) = 0$.*

Definition 3.7. Let $L_1, L_2 : A \rightarrow B$ be two linear maps, $\varepsilon > 0$ and $\mathcal{F} \subset A$ be a subset. We write

$$L_1 \approx_\varepsilon L_2 \quad \text{on } \mathcal{F}$$

if $\|L_1(x) - L_2(x)\| < \varepsilon$ for all $x \in \mathcal{F}$.

THEOREM 3.8 (Theorem 2.3 in [Ln4]). *Let A be a separable unital nuclear simple C^* -algebra with $\text{TR}(A) = 0$ satisfying the UCT. Then, for any $\varepsilon > 0$, and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, a finite subset $\mathcal{P} \subset P(A)$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any unital C^* -algebra B of real rank zero and stable rank one with weakly unperforated $K_0(B)$, and any two \mathcal{G} - δ -multiplicative morphisms $L_1, L_2 : A \rightarrow B$ with*

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}},$$

there exists a unitary $U \in B$ such that

$$\text{ad}(U) \circ L_1 \approx_\varepsilon L_2 \quad \text{on } \mathcal{F}.$$

THEOREM 3.9. *Let A and B be two unital C^* -algebras in \mathcal{LBD} with $\text{TR}(A) = \text{TR}(B) = 0$ satisfying the UCT. Suppose that there is an order isomorphism $\alpha : (K_0(A), K_0(A)_+, [1_A], K_1(A)) \rightarrow (K_0(B), K_0(B)_+, [1_B], K_1(B))$, then there is an isomorphism $h : A \rightarrow B$ such that $h_* = \alpha$.*

Proof. Since A satisfies the UCT, there is an (invertible) element $z \in KK(A, B)$ such that $z|_{K_i(A)} = \alpha$. We will use α for the corresponding element in $KL(A, B)$.

Fix a dense sequence $\{x_n\}$ of the unit ball of A and a dense sequence $\{y_n\}$ of the unit ball of B . Set $\varepsilon_n = 1/2^n$. Let $\mathcal{P}_1 = \mathcal{P}(\varepsilon_1/2, \{x_1\}) \subset \mathbf{P}(A)$, $\delta_1 = \delta(\varepsilon_1/2, \{x_1\}) > 0$ and $\mathcal{F}_1 = \mathcal{G}(\varepsilon_1/2, \{x_1\})$ be as in 3.8 associated with $\varepsilon_1/2 > 0$ and finite subset $\{x_1\}$. We assume that $x_1 \in \mathcal{F}_1$. By 3.4, there is a contractive completely positive linear map $L_1 : A \rightarrow B$ such that

$$\|L_1(ab) - L_1(a)L_1(b)\| < \delta_1/2$$

for all $a, b \in \mathcal{F}_1$ and $[L_1]|_{\mathcal{P}_1} = \alpha|_{\mathcal{P}_1}$. Let $\mathcal{F}'_1 = L_1(\mathcal{F}_1) \cup \{y_1\}$. Let $\mathcal{Q}_1 = \mathcal{P}(\varepsilon_2/2, \mathcal{F}'_1) \subset \mathbf{P}(A)$, $\mathcal{G}_1 = \mathcal{G}(\varepsilon_2/2, \mathcal{F}'_1) \subset B$ and $d_1 = \delta(\varepsilon_2/2, \mathcal{F}'_1) > 0$ be as in 3.8 (for B) associated with $\varepsilon_2/2$ and \mathcal{F}'_1 . We may assume that $\mathcal{Q}_1 \supset [L_1](\mathcal{P}_1)$, $\mathcal{G}_1 \supset \mathcal{F}'_1$ and $d_1 < \min\{\varepsilon_2/2, \delta_1/2\}$.

By 3.4, there exists $\Psi'_1 : B \rightarrow A$ such that

$$\|\Psi'_1(cd) - \Psi'_1(c)\Psi'_1(d)\| < d_1/2$$

for all $c, d \in \mathcal{F}'_1$ and $[\Psi'_1]|_{\mathcal{Q}_1} = \alpha^{-1}|_{\mathcal{Q}_1}$. Then $\Psi'_1 \circ L_1$ is δ_1 -multiplicative on \mathcal{F}_1 and $[\Psi'_1 \circ L_1]|_{\mathcal{P}_1} = [\text{id}]_{\mathcal{P}_1}$. It follows from 3.8 that there is a unitary $u_1 \in A$ such that

$$\text{adu}_1 \circ (\Psi'_1 \circ L_1) \approx_{\varepsilon_1/2} \text{id}_A \quad \text{on } \{x_1\}.$$

Define $\Psi_1 = \text{adu}_1 \circ \Psi'_1$. Set $\mathcal{S}_2 = \Psi_1(\mathcal{G}_1) \cup \{x_1, x_2\}$. Let $\mathcal{F}_2 = \mathcal{G}(\varepsilon_2/2, \mathcal{S}_2)$, $\mathcal{P}_2 = \mathcal{P}(\varepsilon_2/2, \mathcal{S}_2)$ and $\delta_2 = \delta(\varepsilon_2/2, \mathcal{S}_2) > 0$ (for A) associated with $\varepsilon_2/2$ and \mathcal{S}_2 . We may assume that $\mathcal{F}_2 \supset \mathcal{S}_2$, $\mathcal{P}_2 \supset [\Psi_1](\mathcal{Q}_1)$ and $\delta_2 < \min\{\varepsilon_2/2, d_1/2\}$.

It follows from 3.4 that there exists $L'_2 : A \rightarrow B$ such that

$$\|L'_2(ab) - L'_2(a)L'_2(b)\| < \delta_2/2$$

for all $a, b \in \mathcal{F}_2$ and

$$[L'_2]|_{\mathcal{P}_2} = \alpha|_{\mathcal{P}_2}.$$

Note that $\alpha(\alpha^{-1}) = [\text{id}]$. By 3.8 there is a unitary $v_2 \in B$ such that

$$\text{adv}_1 \circ L'_2 \circ \Psi_1 \approx_{\varepsilon_2/2} \text{id}_B \quad \text{on } \mathcal{G}_1.$$

Set $L_2 = \text{adv}_1 \circ L'_2$.

Let $\mathcal{F}'_2 = L_2(\mathcal{F}_2) \cup \{y_1, y_2\}$. Let $\mathcal{G}_2 = \mathcal{G}(\varepsilon_3/2, \mathcal{F}'_2)$, $\mathcal{Q}_2 = \mathcal{P}(\varepsilon_3/2, \mathcal{F}'_2)$ and $d_2 = \delta(\varepsilon_3/2, \mathcal{F}'_2)$ be as in 3.8 (for B) associated with $\varepsilon_3/2$ and \mathcal{F}'_2 . We may assume that $\mathcal{F}'_2 \supset \mathcal{F}_2$, $\mathcal{G}_2 \supset [L_2](\mathcal{P}_2)$ and $d_2 < \min\{\varepsilon_3/2, \delta_2/2\}$.

It follows from 3.4 that there is a contractive completely positive linear map $\Psi'_2 : B \rightarrow A$ such that

$$\|\Psi'_2(cd) - \Psi'_2(c)\Psi'_2(d)\| < d_2/2$$

for all $c, d \in \mathcal{G}_2$ and

$$[\Psi'_2]|_{\mathcal{Q}_2} = \alpha^{-1}|_{\mathcal{Q}_2}.$$

By 3.8 there is a unitary $u_2 \in A$ such that

$$\text{adu}_2 \circ \Psi'_2 \circ L_2 \approx_{\varepsilon_3/2} \text{id}_A \quad \text{on } \mathcal{F}_2.$$

Set $\Psi_2 = \text{adu}_2 \circ \Psi'_2$.

Continuing in this fashion, we construct a sequence of contractive completely positive linear maps $L_n : A \rightarrow B$ and $\Psi_n : B \rightarrow A$ such that the following diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\text{id}_A} & A & \xrightarrow{\text{id}_A} & A & \xrightarrow{\text{id}_A} & \dots A \\ \downarrow L_1 & \nearrow \Psi_1 & \downarrow L_2 & \nearrow \Psi_2 & \downarrow L_3 & \nearrow \Psi_3 & \\ B & \xrightarrow{\text{id}_A} & B & \xrightarrow{\text{id}_A} & B & \xrightarrow{\text{id}_A} & \dots B \end{array}$$

is approximately intertwining. It follows from an argument of Elliott (see for example 2.1, 2.2 and 2.3 in [Ell1] and also 3.1 in [Ln1] for the case that the maps are not homomorphisms) that there are isomorphisms $h : A \rightarrow B$ and $h^{-1} : B \rightarrow A$ (each determined by $\{L_n\}$ and $\{\Psi_n\}$). □

THEOREM 3.10. *Let A and B be two unital C^* -algebras which are inductive limits of C^* -algebras in \mathcal{BD} with $\text{TR}(A) = \text{TR}(B) = 0$. Suppose that there is an order isomorphism*

$$\alpha : (K_0(A), K_0(A)_+, [1_A], K_1(A)) \rightarrow (K_0(B), K_0(B)_+, [1_B], K_1(B));$$

then there is an isomorphism $h : A \rightarrow B$ such that $h_ = \alpha$.*

Proof. Since each C^* -algebra is in \mathcal{BD} , both A and B satisfy the UCT. So the theorem follows from 3.9. \square

Recently, it was shown by Q. Lin and N.C. Phillips that the following structure theorem about smooth minimal dynamical systems holds:

THEOREM 3.11 (Q. Lin and N.C. Phillips [LP]). *Let M be a compact manifold and $\delta : M \rightarrow M$ be a minimal diffeomorphism. Then $A = C^*(\mathbb{Z}, M, \delta)$, the simple crossed product arising from the smooth minimal dynamical system, is in \mathcal{LBD} . In fact, A is a direct limit of subhomogeneous C^* -algebras. Furthermore A has stable rank one and $K_0(A)$ is weakly unperforated.*

Thus we have the following:

THEOREM 3.12. *Let M_1 and M_2 be compact manifolds, $\delta_i : M_i \rightarrow M_i$ be a minimal diffeomorphism ($i = 1, 2$) and let $A_i = C^*(\mathbb{Z}, M_i, h_i)$. Suppose that $\text{TR}(A) = 0$. Then $A_1 \cong A_2$ if and only if*

$$(K_0(A_1), K_0(A_1), [1_{A_1}], K_1(A_1)) \cong (K_0(A_2), K_0(A_2)_+, [1_{A_2}], K_1(A_2)).$$

An important case for dynamical systems is the unique ergodic case, where systems admit unique invariant measures. The resulting simple crossed products admit unique normalized traces.

COROLLARY 3.13. *Let M_1 and M_2 be compact manifolds, $h_i : M_i \rightarrow M_i$ be a minimal diffeomorphism ($i = 1, 2$). Let $A_i = C^*(\mathbb{Z}, M_i, h_i)$. Suppose that (M_i, h_i) has a unique invariant measure ($i = 1, 2$) and the range of $K_0(A_i)$ under the trace is dense in \mathbb{R} . If*

$$(K_0(A_1), K_0(A_1), [1_{A_1}], K_1(A_1)) \cong (K_0(A_2), K_0(A_2)_+, [1_{A_2}], K_1(A_2)),$$

then

$$A_1 \cong A_2.$$

Proof. By [LP], the A_i are simple C^* -algebras that have stable rank one and have weakly unperforated $K_0(A_i)$. It follows from [Ph] that both A_i have real rank zero. Thus the corollary follows from 3.6 and 3.12. \square

There is a class of exciting unital simple C^* -algebras called (higher) irrational noncommutative tori. The irrational rotation C^* -algebra A_θ is generated by two unitaries u and v with relation $uv = e^{i\pi\theta}vu$, where θ is irrational, and is a very interesting unital simple C^* -algebra. It was shown that it is a direct limit of circle algebras with real rank zero (see [EE]). Therefore $\text{TR}(A_\theta) = 0$. Also A_θ can be realized as the crossed product $C(\mathbb{T}) \times_\theta \mathbb{Z}$

resulting from the minimal and unique ergodic dynamical system by a diffeomorphism $\alpha : \mathbb{T} \rightarrow \mathbb{T}$ which maps z to $ze^{i\pi\theta}$, where θ is irrational. One class of $k + 1$ -dimensional noncommutative tori called unital simple crossed products $C(\mathbb{T}^k) \times_{\theta} \mathbb{Z}$ result from a minimal and unique ergodic dynamical system by a diffeomorphism $\alpha : \mathbb{T}^k \rightarrow \mathbb{T}^k$ which maps (z_1, \dots, z_k) to $(z_1e^{i\pi\theta_1}, \dots, z_ke^{i\pi\theta_k})$, where θ_i are irrational.

We have the following theorem:

THEOREM 3.14. *Let $A = C(\mathbb{T}^k) \times_{\theta} \mathbb{Z}$ and $B = C(\mathbb{T}^k) \times_{\alpha} \mathbb{Z}$ be two irrational noncommutative $k + 1$ -tori. Then $A \cong B$ if and only if $[\alpha] = [\theta]$ on $K_i(C(\mathbb{T}^k))$ ($i = 0, 1$). Furthermore, A is an inductive limit of circle algebras.*

Proof. By [PV], $K_i(A) = K_i(B)$ ($i = 0, 1$). It follows from 3.13 that $A \cong B$. They are also isomorphic to a unital simple C^* -algebra in \mathcal{C} . Since $K_i(A)$ are torsion free ($i = 0, 1$), they are isomorphic to an inductive limit of circle algebras.

Remark 3.15. One should note that the condition that A_m are subhomogeneous in 3.4 is not necessary. It suffices to have the following: any finitely generated subgroup $G \subset \rho_{A_n}(K_0(A))$ can be embedded into \mathbb{Z}^k for some integer $k > 0$ as an ordered subgroup. It remains open whether this condition is automatically satisfied by residually finite-dimensional algebras, or any simple nuclear C^* -algebras which can be written as inductive limits of RFD C^* -algebras satisfying this condition. \square

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