Annals of Mathematics, **157** (2003), 125–257

Finite energy foliations of tight three-spheres and Hamiltonian dynamics

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Abstract

Surfaces of sections are a classical tool in the study of 3-dimensional dynamical systems. Their use goes back to the work of Poincaré and Birkhoff. In the present paper we give a natural generalization of this concept by constructing a system of transversal sections in the complement of finitely many distinguished periodic solutions. Such a system is established for nondegenerate Reeb flows on the tight 3-sphere by means of pseudoholomorphic curves. The applications cover the nondegenerate geodesic flows on $T_1S^2 \equiv \mathbb{R}P^3$ via its double covering S^3, and also nondegenerate Hamiltonian systems in $\mathbb{R}^4$ restricted to sphere-like energy surfaces of contact type.

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*The research of the first author was partially supported by an NSF grant, a Clay scholarship and the Wolfensohn Foundation. The research of the second author was partially supported by an Australian Research Council grant. The research of the third author was partially supported by TH-project.

1. Introduction

Pseudoholomorphic curves, in symplectic geometry introduced by Gromov [23], are smooth maps from Riemann surfaces into almost complex manifolds solving a system of partial differential equations of Cauchy-Riemann type. The use of such solutions in dynamical systems was demonstrated in the proofs of the V. I. Arnold conjectures in [15], [17] and [16] concerning forced oscillations of Hamiltonian systems on compact symplectic manifolds. The proofs are based on the structure of pseudoholomorphic cylinders having bounded energies and hence connecting periodic orbits. In his proof [24] of the A. Weinstein conjecture about existence of periodic orbits for Reeb flows, H. Hofer designed a theory of pseudoholomorphic curves for contact manifolds. This theory was extended in [35] in order to establish a global surface of section for special Reeb flows on tight three spheres. These flows include, in particular, Hamiltonian flows on strictly convex three-dimensional energy surfaces. In the following we consider a larger class of Reeb flows on the tight three sphere which do not necessarily admit a global surface of section. The aim is to construct an intrinsic global system of transversal sections bounded by finitely many very special periodic orbits of the Reeb flow. For this purpose we shall establish a smooth foliation $\mathcal{F}$ of $\mathbb{R} \times S^3$ in the nondegenerate case. The leaves are embedded pseudoholomorphic punctured spheres having finite energies. In order to formulate the main result and some consequences for dynamical systems we first recall the concepts from contact geometry and from the theory of pseudoholomorphic curves in symplectizations from [32], [30] and [36].

1.1. *Concepts from contact geometry and Reeb flows.* We consider a compact oriented three-manifold M equipped with the contact form λ. This is a one-form having the property that $\lambda \wedge d\lambda$ is a volume form on M. The contact form determines the plane field distribution $\xi =$ kernel $(\lambda) \subset TM$, called the associated contact structure. It also determines the so-called Reeb vector field $X = X_\lambda$ on M by

$$i_X \lambda = 1 \quad \text{and} \quad i_X d\lambda = 0. \tag{1.1}$$

The tangent bundle

$$TM = \mathbb{R} \cdot X \oplus \xi \tag{1.2}$$

splits into a line bundle having the section X and the contact bundle ξ carrying the symplectic structure fiberwise defined by $d\lambda$. By

$$\pi : TM \to \xi$$

we denote the projection along the Reeb vector field X. Since the contact form λ is invariant under the flow φ^t of the Reeb vector field, the restrictions of the tangent maps onto the contact planes,

$$T\varphi^t(m)_{|\xi_m} : \xi_m \to \xi_{\varphi^t(m)}$$

are symplectic maps.

In the following, periodic orbits (x, T) of the Reeb vector field X will play a crucial role. They are solutions of $\dot{x}(t) = X(x(t))$ satisfying $x(0) = x(T)$ for some $T > 0$. If T is the minimal period of $x(t)$, the periodic solution (x, T) will be called simply covered. A periodic orbit (x, T) is called nondegenerate, if the self map

$$T\varphi^T(x(0))\ _{|\xi_{x(0)}} : \xi_{x(0)} \to \xi_{x(0)}$$

does not contain 1 in its spectrum. If all the periodic solutions of X_λ are nondegenerate, the contact form is called nondegenerate. Such forms occur in abundance, as the following proposition from [35] indicates. Later on, the contact forms under consideration will all be nondegenerate.

Proposition 1.1. *Fix a contact form λ on the closed 3-manifold M and consider the subset $\Theta_1 \subset C^\infty(M, (0, \infty))$ consisting of those f for which $f\lambda$ is nondegenerate. Let Θ_2 consist of all those $f \in \Theta_1$ such that, in addition, the stable and unstable manifolds of hyperbolic periodic orbits of $X_{f\lambda}$ intersect transversally. Then Θ_1 and Θ_2 are Baire subsets of $C^\infty(M, (0, \infty))$.*

Nondegenerate periodic orbits (x, T) of X are distinguished by their μ-indices, sometimes called Conley-Zehnder indices, and their self-linking numbers $\mathrm{sl}(x, T)$. These integers are defined as follows. We take a smooth disc map $u : D \to M$ satisfying $u\left(e^{2\pi i t/T}\right) = x(t)$, where D is the closed unit disc in $\mathbb{C}$.

Then we choose a symplectic trivialization $\beta : u^*\xi \to D \times \mathbb{R}^2$ and consider the arc $\Phi : [0,T] \to$ Sp (1) of symplectic matrices $\Phi(t)$ in $\mathbb{R}^2$, defined by

$$\Phi(t) = \beta\Big(e^{2\pi it/T}\Big) \circ T\varphi^t_{|\xi_{x(0)}} \circ \beta^{-1}(1).$$

The arcs start at the identity $\Phi(0) = \mathrm{Id}$ and end at a symplectic matrix $\Phi(T)$ which does not contain 1 in its spectrum. To every such arc one associates the integer $\mu(\Phi) \in \mathbb{Z}$, recalled in Appendix 8.1. It describes how often nearby solutions wind around the periodic orbit with respect to a natural framing. The index of the periodic solution is then defined by

$$\mu\Big(x, T, [u]\Big) = \mu(\Phi) \in \mathbb{Z}.$$

This integer depends only on the homotopy class $[u]$ of the chosen disc map keeping the boundaries fixed. If, as in our study later on, $M = S^3$, the index is independent of all choices and will be denoted by

$$\mu(x, T) \in \mathbb{Z}.$$

To define the self-linking numbers $\mathrm{sl}(x,T)$ we take a disc map u as before and a nowhere-vanishing section Z of the bundle $u^*\xi \to D$. Then we push the loop $t \mapsto x(Tt)$ for $0 \le t \le 1$ in the direction of Z to obtain a new oriented loop $y(t)$. The oriented intersection number of u and y is, by definition, the self-linking number of x. This integer will be useful later on in the investigation of the minimality of the periods.

1.2. *Finite energy spheres in* S^3. We recall the concept of a finite energy sphere, choosing the special manifold $M = S^3$ dealt with later on. Here S^3 is the standard sphere $S^3 = \{z \in \mathbb{C}^2 \mid |z| = 1\}$, where $z = (z_1, z_2) = (q_1 + ip_1, q_2 + ip_2)$ with $z_j \in \mathbb{C}$ and q_j, $p_j \in \mathbb{R}$. Recalling the standard contact form on S^3,

$$\lambda_0 = \frac{1}{2}\sum_{j=1}^{2}\Big(q_j dp_j - p_j dq_j\Big)|_{S^3},$$

we choose a nondegenerate contact form $\lambda = f\lambda_0$ on S^3 and denote its Reeb vector field by X and the contact structure by ξ. Now we choose a smooth complex multiplication $J : \xi \to \xi$ on the contact planes satisfying

$$d\lambda(h, Jh) > 0 \quad \text{for all } h \in \xi \setminus \{0\}$$

and abbreviate by $\mathcal{J}$ the set of these admissible complex multiplications. With $J \in \mathcal{J}$ we associate a distinguished $\mathbb{R}$-invariant almost complex structure $\widetilde{J}$ on $\mathbb{R} \times S^3$ by extending J onto $\mathbb{R} \times \mathbb{R} \cdot X$ by $1 \mapsto X \mapsto -1$, in formulas,

$$\widetilde{J}(\alpha, k) = \Big(-\lambda(k), J\pi k + \alpha X\Big), \tag{1.3}$$

for $(\alpha, k) \in T(\mathbb{R} \times S^3)$, where $\pi : TS^3 \to \xi$ is the projection along the Reeb vector field X. The important property of $\widetilde{J}$ is the invariance along the fibers $\mathbb{R}$. Denote by Σ the set of all smooth functions $\varphi : \mathbb{R} \to [0,1]$ satisfying $\varphi' \geq 0$.

Definition 1.2 (Finite energy sphere). A (*nontrivial*) *finite energy sphere* for (S^3, λ, J) is a pair $(\Gamma, \widetilde{u})$ consisting of a finite subset Γ of the Riemann sphere S^2 and a smooth map

$$\widetilde{u} : S^2 \setminus \Gamma \to \mathbb{R} \times S^3$$

solving the partial differential equation

$$T\widetilde{u} \circ j = \widetilde{J} \circ T\widetilde{u} \quad \text{on } S^2 \setminus \Gamma \tag{1.4}$$

and satisfying the energy condition

$$0 < E(\widetilde{u}) < \infty,$$

where

$$E(\widetilde{u}) = \sup_{\varphi \in \Sigma} \int_{S^2 \setminus \Gamma} \widetilde{u}^* d(\varphi\lambda), \tag{1.5}$$

with the one-form $\varphi\lambda$ on $\mathbb{R} \times S^3$ defined by $(\varphi\lambda)(a,m)[\alpha,k] = \varphi(a) \cdot \lambda(m)[k]$. We call $\widetilde{u}$ a *finite energy plane* if $\Gamma = \{\infty\}$. A *finite energy sphere* will be called an embedding if $\widetilde{u}$ is an embedding.

We note that for a solution $\widetilde{u}$ of equation (1.4) the integrand of the energy (1.5) is nonnegative. The condition $E(\widetilde{u}) > 0$ implies that $\widetilde{u}$ is not a constant map.

A special example of a finite energy sphere is an orbit cylinder over a periodic solution (x, T) of X. It is parametrized by the map $\widetilde{u} : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$,

$$\widetilde{u}\Big(\mathrm{e}^{2\pi(s+\mathrm{i}t)}\Big) = \Big(Ts, x(Tt)\Big) \in \mathbb{R} \times S^3. \tag{1.6}$$

Its energy agrees with the period $T = E(\widetilde{u})$ while its $d\lambda$-energy vanishes,

$$\int_{\mathbb{C} \setminus \{0\}} u^* d\lambda = 0.$$

The punctures are $\Gamma = \{0, \infty\}$, where $S^2 = \mathbb{C} \cup \{\infty\}$. Orbit cylinders govern the asymptotic behavior of finite energy spheres near the punctures Γ as we recall next from [24], [32] and [30].

We begin with the distinction between positive and negative punctures.

PROPOSITION 1.3. *Let $(\Gamma, \widetilde{u})$ be a finite energy sphere and $z_0 \in \Gamma$. Then one of the following mutually exclusive cases holds, where $\widetilde{u} = (a, u) \in \mathbb{R} \times S^3$.*

- *positive puncture:* $\lim_{z \to z_0} a(z) = +\infty$;
- *negative puncture:* $\lim_{z \to z_0} a(z) = -\infty$;
- *removable puncture:* $\lim_{z \to z_0} a(z) = a(z_0)$ *exists in* $\mathbb{R}$.

In the third case one can show that $\widetilde{u}\,(U(z_0)\setminus\{z_0\})$ is bounded for a suitable neighborhood $U(z_0)$ and moreover, employing Gromov's removable singularity theorem from [23] one can extend $\widetilde{u}$ smoothly over z_0. For this reason we consider later on only positive and negative punctures, $\Gamma=\Gamma^+\cup\Gamma^-$. We note that $\Gamma\neq\emptyset$ since a finite energy surface defined on S^2 is necessarily constant. Indeed, from Stokes' theorem it follows that $E(\widetilde{u})=0$. There is always at least one positive puncture.

In order to describe the asymptotic behavior near the puncture $z_0\in\Gamma$ we introduce holomorphic polar coordinates. We take a holomorphic chart $h: D\subset\mathbb{C}\to U\subset S^2$ around z_0 satisfying $h(0)=z_0$ and define $\sigma:[0,\infty)\times S^1\to U\setminus\{z_0\}$ by

$$\sigma(s,t)=h\Big(\mathrm{e}^{-2\pi(s+\mathrm{i}t)}\Big) \tag{1.7}$$

so that $z_0=\lim_{s\to\infty}\sigma(s,t)$. In these coordinates the energy surface near z_0 becomes the positive half cylinder

$$\widetilde{v}=(b,v):=\widetilde{u}\circ\sigma:[0,\infty)\times S^1\to\mathbb{R}\times S^3.$$

The map $\widetilde{v}$ satisfies the Cauchy-Riemann equation

$$\widetilde{v}_s+\widetilde{J}(\widetilde{v})\widetilde{v}_t=0$$

and has bounded energy $E(\widetilde{v})\leq E(\widetilde{u})<\infty$. Because of this energy bound the following limit exists in $\mathbb{R}$,

$$m(\widetilde{u},z_0):=\lim_{s\to\infty}\int_{S^1}v(s,\cdot\,)^*\lambda. \tag{1.8}$$

The real number $m=m(\widetilde{u},z_0)$ is called the charge of the puncture $z_0\in\Gamma$. It is positive if z_0 is positive and negative for a negative puncture. Moreover, $m=0$ if the puncture is removable. The behavior of the sphere near z_0 is now determined by periodic solutions of the Reeb vector field X having periods $T=|m(\widetilde{u},z_0)|$. Every sequence $s_k\to\infty$ possesses a subsequence denoted by the same letters such that

$$\lim_{k\to\infty}v(s_k,t)=x(mt)\qquad\text{in } C^\infty(S^1)$$

for an orbit $x(t)$ of the Reeb vector field $\dot{x}(t)=X(x(t))$. Here m is the charge of z_0. If $m\neq0$, the solution x is a periodic orbit of X having period $T=|m|$. If this periodic orbit is nondegenerate then

$$\lim_{s\to\infty}v(s,t)=x(mt)\qquad\text{in } C^\infty(S^1) \tag{1.9}$$

and

$$\lim_{s\to\infty}\frac{b(s,t)}{s}=m\qquad\text{in } C^\infty(S^1). \tag{1.10}$$

Hence in the nondegenerate case there is a unique periodic orbit (x,T) associated with the puncture z_0. It has period $T=|m|$ and is called the asymptotic limit of z_0. In the nondegenerate case, the finite energy surface $\widetilde{v}$ approaches the special orbit cylinder $\widetilde{v}_\infty(s,t) = \big(sm, x(mt)\big)$ in $\mathbb{R} \times S^3$ as $s \to \infty$ in an exponential way. The asymptotic formula is recalled in the appendix. We visualize a finite energy sphere $\widetilde{u}$ in $\mathbb{R} \times S^3$ by Figure 1.

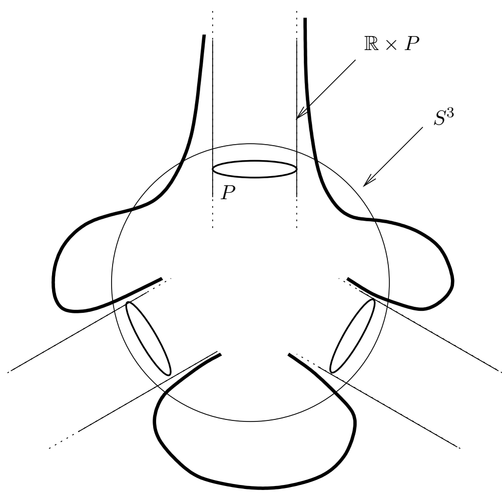

Figure 1. The figure shows a finite energy sphere with one positive and two negative punctures.

We next introduce the main concept.

1.3. *Finite energy foliations.* We consider the three-manifold M equipped with the contact form λ, choose an admissible $J \in \mathcal{J}$ and denote the associated $\mathbb{R}$-invariant almost complex structure on $\mathbb{R} \times M$ by $\widetilde{J}$.

Definition 1.4. A *spherical finite energy foliation* for (M, λ, J) is, by definition, a 2-dimensional smooth foliation $\mathcal{F}$ of $\mathbb{R} \times M$ having the following properties:

- There exists a universal constant $c > 0$ such that for every leaf $F \in \mathcal{F}$ there exists an embedded finite energy sphere $\widetilde{u} : S^2 \setminus \Gamma \to \mathbb{R} \times M$ for (M, λ, J) satisfying
$$F = \widetilde{u}\Big(S^2 \setminus \Gamma\Big) \quad \text{and} \quad E(\widetilde{u}) \leq c.$$
- The translation along the fiber $\mathbb{R}$ of $\mathbb{R} \times M$,
$$T_r(F) := r + F = \Big\{(r + a, m) \mid (a, m) \in F\Big\},$$
$F \in \mathcal{F}$ and $r \in \mathbb{R}$, defines an $\mathbb{R}$-action $T : \mathbb{R} \times \mathcal{F} \to \mathcal{F}$. Hence, in particular, $T_r(F) \in \mathcal{F}$ if $F \in \mathcal{F}$, and either $T_r(F_1) \cap F_2 = \emptyset$ or $T_r(F_1) = F_2$ for any two leaves in $\mathcal{F}$.

We illustrate the concept with an explicit example for (S^3, λ_0, i). The Reeb vector field X on $S^3 \subset \mathbb{C}^2$ is for the standard contact form λ_0 given by

$$X(z) = 2iz, \qquad z \in S^3.$$

The contact plane ξ_z, $z \in S^3$, agrees with the complex line in $T_z S^3$. As complex multiplication we choose $J = i_{|\xi}$ and denote by $\widetilde{J}$ the associated $\mathbb{R}$-invariant almost complex structure on $\mathbb{R} \times S^3$. Then the inverse of the diffeomorphism $(t, z) \mapsto \mathrm{e}^{2t} z$ from $\mathbb{R} \times S^3$ onto $\mathbb{C}^2 \setminus \{0\}$ is given by

$$\Phi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{R} \times S^3, \qquad z \mapsto \Big(\frac{1}{2} \ln |z|, \frac{z}{|z|}\Big).$$

It satisfies

$$T\Phi \circ i = \widetilde{J} \circ T\Phi$$

and hence is biholomorphic. Consider the planes

$$\Phi\Big(\mathbb{C} \times \{c\}\Big) \quad \text{for all } c \in \mathbb{C} \setminus \{0\}$$

and the special cylinder

$$F_0 = \Phi\Big((\mathbb{C} \setminus \{0\}) \times \{0\}\Big)$$

in $\mathbb{R} \times S^3$. The union $\mathcal{F}$ of these sets constitutes a smooth foliation of $\mathbb{R} \times S^3$ consisting of finite energy planes and the finite energy cylinder F_0. The action of $\mathbb{R}$ is represented by $T_r\Big(\Phi\Big(\mathbb{C} \times \{c\}\Big)\Big) = \Phi\Big(\mathbb{C} \times \{e^{2r} c\}\Big)$ if $c \neq 0$ while $T_s F_0 = F_0$ for every $s \in \mathbb{R}$. Clearly, $T_r F \cap F = \emptyset$ for every $r \neq 0$ and $F \neq F_0$. Consequently, the only fixed point of the $\mathbb{R}$-action is the cylinder F_0. It is the orbit cylinder of the special solution $x_0(t) = (\mathrm{e}^{2\mathrm{i}t}, 0)$ of X on S^3 having period π. The map $\widetilde{u} : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ parametrizing F_0 is given by $\Phi\Big((\mathrm{e}^{2\pi(s+\mathrm{i}t)}, 0)\Big) = \Big(\pi s, (\mathrm{e}^{2\pi \mathrm{i}t}, 0)\Big)$. The periodic orbit (x_0, π) is the asymptotic limit of all the finite energy planes. Indeed, $\Phi\Big((\mathrm{e}^{2\pi(s+\mathrm{i}t)}, c)\Big) \to \Big(\pi s, (\mathrm{e}^{2\pi \mathrm{i}t}, 0)\Big)$ as $s \to \infty$, for every $c \neq 0$. Let now

$$p : \mathbb{R} \times S^3 \to S^3$$

be the projection map. Then $p(F_0) = x_0(\mathbb{R})$ and for every $F \neq F_0$, the subset $p(F)$ is an embedded plane transversal to the Reeb vector field X. Moreover, if F_1 and $F_2 \in \mathcal{F}$ do not belong to the same orbit of the $\mathbb{R}$-action, then $p(F_1) \cap p(F_2) = \emptyset$. Therefore, the projection $p(\mathcal{F}) = \widehat{\mathcal{F}}$ is a singular foliation of S^3. It is a smooth foliation of $S^3 \setminus x_0(\mathbb{R}) = p(\mathcal{F} \setminus \{F_0\})$ into planes transversal to X and asymptotic to x_0. Hence the periodic orbit x_0 is the binding orbit of an open book decomposition of S^3 illustrated by Figure 2.

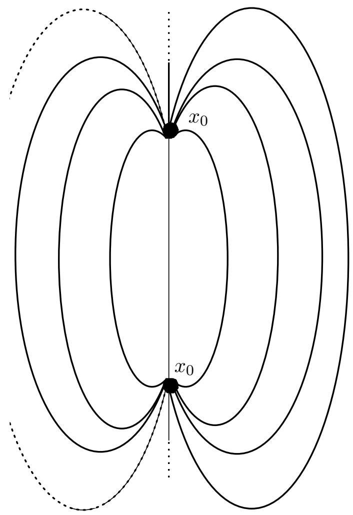

Figure 2. The figure illustrates a section through an open book decomposition of S^3 viewed as $\mathbb{R}^3 \cup \{\infty\}$. The two black dots represent the periodic orbit perpendicular to the plane. The curves represent pages of an open book decomposition.

Although this example is not nondegenerate, the fact that a finite energy foliation on $\mathbb{R} \times M$ leads to a geometric decomposition of the manifold M is of quite general nature as we shall see below where we strengthen the concept of finite energy foliation. We should remark that there are other finite energy foliations for (S^3, λ_0, i). For example, the collection of all cylinders $\mathbb{R} \times P$, where P runs over all Hopf circles on S^3. Here a small perturbation, taking the contact form $f\lambda_0$ for f close to the constant function equal to one will destroy most periodic orbits so that this second foliation is rather unstable.

1.4. *Stable finite energy foliations, the main result.* Let $M = S^3$ be the standard sphere equipped with the nondegenerate contact form $\lambda = f\lambda_0$ and consider an embedded finite energy sphere $\widetilde{u} = (a, u) : S^2 \setminus \Gamma \to \mathbb{R} \times S^3$ for (S^3, λ, J). The punctures Γ split into the positive and the negative punctures, $\Gamma = \Gamma^+ \cup \Gamma^-$. With every $z_0 \in \Gamma$ we associate the index $\mu(z_0)$ of its asymptotic limit, which is a nondegenerate periodic orbit of the Reeb vector field X. Following [30] we can associate with the sphere $\widetilde{u}$ the integer

$$\mu(\widetilde{u}) = \sum_{z \in \Gamma^+} \mu(z) - \sum_{z \in \Gamma^-} \mu(z).$$

If $\widetilde{u}\Big(S^2 \setminus \Gamma\Big) =: F$, we set

$$\mu(F) = \mu(\widetilde{u}) \in \mathbb{Z}.$$

The definition does not depend on the choices involved. Finally, we define the index of the embedded finite energy sphere F by

$$\text{Ind}(F) := \mu(F) - \chi(S^2) + \sharp F, \tag{1.11}$$

where $\sharp F = \sharp\Gamma$ is the number of the punctures and $\chi(S^2) = 2$ is the Euler characteristic of the two-sphere. The integer $\text{Ind}(F)$ will be important in the analysis later on. It has an interpretation as the Fredholm index describing the dimension of the moduli space of nearby embedded finite energy spheres having the same topological type and the same number of punctures which are allowed to move on S^2; see [36]. The following definition is crucial.

Definition 1.5 (Stable finite energy foliation). Assume the contact form $\lambda = f\lambda_0$ to be nondegenerate. Let $\mathcal{F}$ be a *spherical finite energy foliation* for (S^3, λ, J). We call $\mathcal{F}$ *stable* if it has the following properties:

- Every leaf of $\mathcal{F}$ is the image of an embedded finite energy sphere.
- For every leaf the asymptotic limits are simply covered, their Conley-Zehnder indices are contained in $\{1, 2, 3\}$ and their self-linking numbers are equal to -1.
- Every leaf has precisely one positive puncture but an arbitrary number of negative punctures. For every leaf $F \in \mathcal{F}$ which is not a fixed point of the $\mathbb{R}$-action, $\text{Ind}(F) \in \{1, 2\}$.

We deduce some immediate consequences from this definition. Consider a leaf $F \in \mathcal{F}$ which is not a fixed point of the $\mathbb{R}$-action. Its punctures are $\Gamma = \Gamma^+ \cup \Gamma^- = \Gamma^+ \cup \Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3^-$, where Γ_j^- are the punctures having μ-index equal to j, and where $\sharp\Gamma^+ = 1$. Denoting by μ^+ the index of the unique positive puncture we have, recalling (1.11),

$$\begin{aligned} \text{Ind}(F) &= \mu^+ - 3\sharp\Gamma_3^- - 2\sharp\Gamma_2^- - \sharp\Gamma_1^- - 2 + 1 + \sharp\Gamma_1^- + \sharp\Gamma_2^- + \sharp\Gamma_3^- \\ &= \mu^+ - 1 - 2\sharp\Gamma_3^- - \sharp\Gamma_2^-. \end{aligned}$$

Since, by definition of $\mathcal{F}$, $\text{Ind}(F) \geq 1$ we find $2\sharp\Gamma_3^- + \sharp\Gamma_2^- \leq \mu^+ - 2$ from which we conclude,

$$\mu^+ \in \{2, 3\}, \quad \sharp\Gamma_3^- = 0, \qquad \sharp\Gamma_2^- \leq 1.$$

There is no restriction on $\sharp\Gamma_1^-$. In order to represent different types of leaves $F \in \mathcal{F}$ which are not fixed points of the $\mathbb{R}$-action on $\mathcal{F}$ we introduce the vectors

$$\alpha = (\mu^+, \mu_1^-, \ldots, \mu_N^-).$$

Here N is the number of negative punctures of F, μ^+ the Conley-Zehnder index of its unique positive puncture and μ_j^- the indices of the negative punctures ordered so that $\mu_j^- \geq \mu_{j+1}^-$. The leaves $F \in \mathcal{F}$ with $\text{Ind}(F) \in \{1,2\}$ and N negative punctures are of the following types

$$\begin{aligned}
\alpha &= (3, 1_1, \ldots, 1_N), && \text{Ind}(F) = 2 \\
\alpha &= (3, 2, 1_1, \ldots, 1_{N-1}), && \text{Ind}(F) = 1 \\
\alpha &= (2, 1_1, \ldots, 1_N), && \text{Ind}(F) = 1.
\end{aligned}$$

The number N of negative punctures can, of course, be zero. If this happens, the first and the third case represent finite energy planes. The second type represents for $N = 1$ a finite energy cylinder connecting a periodic orbit of index 3 (f.e. elliptic) with a periodic orbit of index 2 (hyperbolic).

Postponing the nontrivial consequences of the definition of a stable finite energy foliation we first formulate our main existence result.

THEOREM 1.6 (Existence of a stable finite energy foliation). *For every choice of $f \in \Theta_1$, there exists a Baire set of admissible complex multiplications J admitting a stable finite energy foliation $\mathcal{F}$ of $(S^3, f\lambda_0, J)$ containing a finite energy plane.*

Since, by hypothesis, the energies $E(\widetilde{u})$ are uniformly bounded and since the periods of the asymptotic limits are bounded by the energy we conclude from the nondegeneracy of λ, that the number of all asymptotic limits appearing in $\mathcal{F}$ is finite. It follows from Fredholm theory that a leaf $F \in \mathcal{F}$ satisfying $\text{Ind}(F) = 2$ belongs to a 2-parameter family of leaves all having the same asymptotic limits. One parameter is defined by $\mathbb{R}$-action on $\mathcal{F}$. The image of the 2-parameter family under the projection map

$$p : \mathbb{R} \times S^3 \to S^3,$$

where the $\mathbb{R}$-action is divided out, is a 1-parameter family of embedded punctured Riemann spheres. In contrast, a leaf $F \in \mathcal{F}$ satisfying $\text{Ind}(F) = 1$ belongs to a 1-parameter family, namely the orbit of F under the $\mathbb{R}$-action. The projection of this orbit into S^3 is an isolated embedded punctured sphere, in the following called a rigid surface. Clearly, if F is an orbit cylinder, its projection in S^3 agrees with its asymptotic limit.

The stable finite energy foliation $\mathcal{F}$ of $\mathbb{R} \times S^3$ gives rise to the following geometric decomposition of S^3.

PROPOSITION 1.7. *The stable finite energy foliation $\mathcal{F}$ established in Theorem* 1.6 *has the following properties*:

- *If $T_r(F) = F$ for some $r \neq 0$ and $F \in \mathcal{F}$, then $T_s(F) = F$ for all $s \in \mathbb{R}$ and $F = \mathbb{R} \times P$ is an orbit cylinder. Hence the fixed points of the $\mathbb{R}$-action on $\mathcal{F}$ are orbit cylinders.*

- *If two leaves F and $G \in \mathcal{F}$ do not belong to the same orbit of the action, then $p(F) \cap p(G) = \emptyset$.*
- *If $F \in \mathcal{F}$ is not a fixed point of the $\mathbb{R}$-action, the projection $p(F)$ is a smooth embedded punctured two-sphere in S^3 which is transversal to the Reeb vector field X and which converges at the punctures to the asymptotic limits of F.*
- *Denote by $\mathcal{P}$ the finite set of asymptotic limits of $\mathcal{F}$. Then the projection $p(\mathcal{F})$ is a singular foliation of S^3 having the singularities $\mathcal{P}$. Moreover, $p(\mathcal{F} \setminus \{$ fixed points of the $\mathbb{R}$-action $\})$ is a smooth foliation of $S^3 \setminus \{\mathcal{P}\}$. The leaves of the foliation are embedded punctured spheres transversal to X and at the punctures asymptotic to elements in $\mathcal{P}$.*

Important for our applications to the Reeb flows is the global system of transversal sections of the Reeb vector field which is an immediate consequence of Theorem 1.6 and Proposition 1.7.

COROLLARY 1.8 (Global system of transversal sections). *If $f\lambda_0$ is a nondegenerate contact form on the standard sphere S^3 with associated Reeb vector field X, then there exists a nonempty set $\mathcal{P}$ consisting of finitely many distinguished periodic orbits of X which are simply covered, have self-linking number -1 and μ-indices in the set $\{1, 2, 3\}$ so that the complement*

$$S^3 \setminus \mathcal{P}$$

is smoothly foliated into leaves which are embedded punctured Riemann spheres, transversal to the Reeb vector field X and converging at the punctures to periodic orbits in $\mathcal{P}$.

We illustrate the situation in Figure 3 which sketches the projection of a stable finite energy foliation into S^3.

The 3-sphere is viewed as $\mathbb{R}^3 \cup \{\infty\}$. The figure shows the trace of the projection $p(\mathcal{F})$ in a 2-dimensional plane. There are three distinguished asymptotic limits $\mathcal{P} = \{P_1, P_2, P_3\}$, two of them P_1 and P_3 have index 3 (elliptic) and P_2 with index 2 (hyperbolic). There are four rigid leaves in S^3, namely two cylinders connecting the elliptic periodic orbit with the hyperbolic orbit, and two planes asymptotic to the hyperbolic orbit. The nonrigid leaves are two 1-parameter families of planes.

1.5. *Outline of the proof.* The origin of the foliation lies in the structure of Gromov's pseudoholomorphic spheres homologous to $\mathbb{C}P^1$ in the compact symplectic manifold $(\mathbb{C}P^2, \omega)$ represented as $\mathbb{C}P^2 = \mathbb{C}^2 \cup \mathbb{C}P^1$ and equipped with a compatible almost complex structure. It will be recalled in Section 2.4 below. Our contact manifold $(S^3, f\lambda_0)$ can be identified with (M, λ_0) where

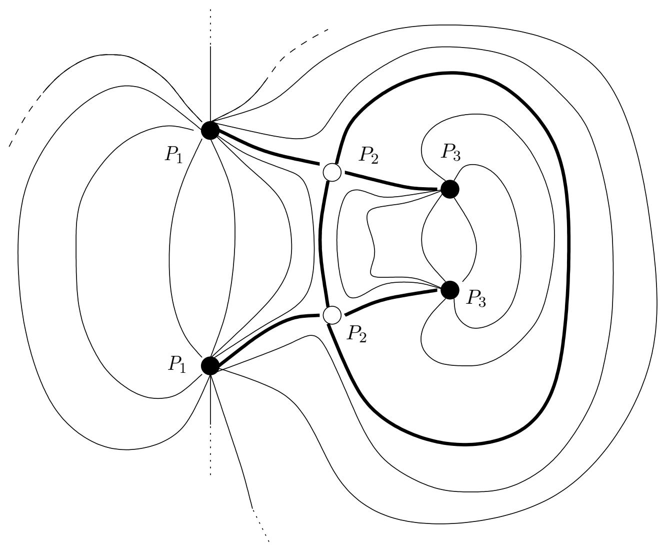

Figure 3. Stable finite energy foliation of S^3.

$M \subset \mathbb{C}^2$ is a star-like hypersurface defined by the function f. Denoting by $\omega_0 = d\lambda_0$ the standard symplectic structure in $\mathbb{C}^2$, the hypersurface M is of contact type, and an open neighborhood of the region bounded by M in $\mathbb{C}^2$ can be symplectically embedded into $\mathbb{C}P^2 \setminus \mathbb{C}P^1$ to obtain the decomposition

$$\mathbb{C}P^2 = \widetilde{W} \cup M \cup \widetilde{V}$$

into the inside $\widetilde{W}$ of M whose closure has M as convex contact boundary and the outside $\widetilde{V}$ of M, containing $\mathbb{C}P^1$, whose closure has M as concave contact boundary. Adding, for $N \geq 1$, the necks $[-N, N] \times M$ in the complement of the sphere at infinity, we obtain a sequence of symplectic manifolds (A_N, ω_N) which are symplectomorphic to $(\mathbb{C}P^2, \omega)$ and which have compatible almost complex structures $\widehat{J}_N$ agreeing on the necks with the distinguished $\mathbb{R}$-invariant structure $\widetilde{J}$. Given a point $(0, m) \in [-N, N] \times M \subset A_N$ there exists a unique $\widehat{J}_N$-holomorphic sphere C^N homologous to $\mathbb{C}P^1$ and containing the two points $(0, m)$ and $o_\infty \in \mathbb{C}P^1$ in A_N. The spheres C^N are embedded and automatically generic. Two such spheres are either identical or intersect transversally at the point o_∞ having intersection numbers equal to 1. In the limit as $N \to \infty$ singularities show up and the spheres C^N disintegrate into a tree of different types of punctured finite energy spheres in the target spaces $\widetilde{W}$, $\mathbb{R} \times M$ and $\widetilde{V}$. In particular, one obtains leaves C^{m_k} of the desired foliation in $\mathbb{R} \times M$ through a dense set of points $(0, m_k) \in M$ which are in the complement of the periodic orbits of the Reeb vector field on M. The leaves are embedded and either identical or disjoint. The limit procedure as $N \to \infty$

is based on a technical bubbling off analysis and uses our Fredholm theory for symplectified contact manifolds and Gromov-McDuff's intersection theory of pseudoholomorphic curves in 4-dimensional symplectic manifolds. By means of a second round of bubbling off analysis we find, as $m_k \to m$, leaves through every point $(0, m) \in \mathbb{R} \times M$ and translating these leaves by the $\mathbb{R}$-action establishes the desired foliation of $\mathbb{R} \times M$ into pseudoholomorphic punctured spheres of uniformly bounded energies.

1.6. *Application to dynamical systems.* The system of transversal sections established is a natural generalization of the concept of a global surface of section. Recall that a global surface of section for a vector field X on a 3-dimensional manifold M is an embedded compact surface $\Sigma \subset M$ whose boundary components are periodic orbits of X, whose interior $\text{int}\Sigma$ is transversal to X and has the property that every orbit of X other than the boundary components intersects $\text{int}\Sigma$ in forward and backward time. The flow φ^t of X induces a diffeomorphism $\psi : \text{int}\Sigma \to \text{int}\Sigma$, the so called Poincaré section map. It is defined by following a point $p \in \text{int}\Sigma$ along its solution $\varphi^t(p)$ until the first time it hits $\text{int}\Sigma$ again. This way the study of the solutions of X is reduced to the study of the section map ψ and its iterates.

THEOREM 1.9. *Let $\mathcal{F}$ be a stable finite energy foliation for (S^3, λ, J) established in Theorem* 1.6. *Assume $\lambda = f\lambda_0$ with $f \in \Theta_2$ as specified in Proposition* 1.1. *If $\mathcal{F}$ has precisely one fixed point of the $\mathbb{R}$-action, then the Reeb vector field X_λ possesses a global surface of section of disc type. If $\mathcal{F}$ has at least two fixed points, then X_λ possesses a hyperbolic periodic orbit and an orbit homoclinic to this periodic orbit.*

The dynamical consequences of the first alternative are the following. The singular foliation on S^3 obtained by $\mathcal{F}$ in this case is an open book decomposition into pages of disc type all having the distinguished periodic orbit P (of the fixed point of the $\mathbb{R}$-action) as asymptotic limit. The index of this periodic solution is $\mu(P) = 3$. Therefore, by the arguments in [35], every page is a global surface of section. Moreover, the section map ψ is conjugated to an area-preserving diffeomorphism $\hat{\psi}$ of the open unit disc. Since the area of the disc is finite we conclude by means of Brouwer's translation theorem that $\hat{\psi}$ possesses a fixed point p. It is the initial condition to a periodic solution of the Reeb vector field X_λ which is different from P. If $\hat{\psi}$ has another periodic point different from the fixed point p already established, then by the remarkable theorem of John Franks in [20], the map $\hat{\psi}$ has infinitely many periodic points, so that X_λ has infinitely many periodic solutions. Summarizing, if there is a global surface of section, the Reeb vector field possesses either 2 or ∞ many periodic orbits.

Assume now that $\mathcal{F}$ has more than one fixed point of the $\mathbb{R}$-action. In this case the Reeb vector field X_λ possesses necessarily a hyperbolic periodic orbit of index $\mu(P) = 2$ and an orbit homoclinic to this periodic orbit. The stable and unstable invariant manifolds of the hyperbolic orbit intersect transversally giving rise to a Bernoulli-system and hence, in particular, to infinitely many periodic solutions. Therefore, we conclude from Theorem 1.9 the following:

Corollary 1.10. *For every contact form $\lambda = f\lambda_0$ on S^3 satisfying $f \in \Theta_2$ the associated Reeb vector field X_λ possesses either two or infinitely many periodic solutions.*

An interesting class of contact forms is the so-called class of dynamically convex contact forms.

Definition 1.11. The contact form $\lambda = f\lambda_0$ with $f \in \Theta_1$ is called *dynamically convex* if $\mu(P) \geq 3$ for all periodic solutions P of the associated Reeb vector field X_λ.

It turns out that the finite energy foliation for $(S^3, f\lambda_0, J)$ in case of a dynamically convex contact form has precisely one fixed point of the $\mathbb{R}$-action, and we conclude from Theorem 1.9 the following corollary.

Corollary 1.12. *The Reeb vector field X_λ associated with a nondegenerate and dynamically convex contact form $\lambda = f\lambda_0$ possesses a global surface of section.*

It is shown in [35] that the statement holds true without the nondegeneracy condition on the periodic orbits replacing in the definition of dynamically convex the requirement $\mu(P) \geq 3$ by $\widetilde{\mu}(P) \geq 3$ for the generalized index $\widetilde{\mu}$ introduced in [35].

The constructions and results are applicable to Hamiltonian systems on $(\mathbb{R}^4, \omega_0)$ restricted to sphere-like energy surfaces. Here ω_0 denotes the standard symplectic form $\omega_0 = d\lambda_0$ with the Liouville form $\lambda_0 = \frac{1}{2}\sum_{j=1}^{2}(q_j dp_j - p_j dq_j)$. We shall use the complex notation $z = (z_1, z_2) = (q_1 + ip_1, q_2 + ip_2) \in \mathbb{C}^2 \equiv \mathbb{R}^4$. Consider a regular energy surface $E = \{z \in \mathbb{C}^2 \mid H(z) = \text{ constant}\}$ for the Hamiltonian vector field X_H defined by $i_{X_H}\omega_0 = -dH$. If E is star-like, i.e., if

$$E = \left\{ z\sqrt{f(z)} \mid z \in S^3 \right\}$$

for some $f \in C^\infty(S^3, \mathbb{R}^+)$, then the restriction of the Hamiltonian flow on E is equivalent to the Reeb flow on S^3 associated with the contact form $\lambda = f\lambda_0$. If E bounds a strictly convex domain in $\mathbb{C}^2$, then $\lambda = f\lambda_0$ is a dynamically convex contact form, provided that periodic orbits on E are nondegenerate; see [35]. We conclude from Corollary 1.12 that a Hamiltonian flow on a strictly convex energy surface in $\mathbb{R}^4$ possesses a global surface of section in the nondegenerate case. Again, the nondegeneracy requirement on the periodic solutions can be dropped, see [35].

Theorem 1.9 is also applicable to the geodesic problem of a Riemannian metric g on S^2. The geodesic flow restricted to the unit sphere bundle $T_1S^2 \equiv \mathbb{R}P^3$ is a Reeb flow. The unit sphere bundle has the double covering S^3 and it can be shown that the double covered geodesic flow is equivalent to the Reeb flow on S^3 associated with a tight contact form $\lambda = f\lambda_0$. The flow is invariant under the symmetry $z \mapsto -z$ on S^3.

By the classical result due to Lyusternik and Schnirelmann there are at least three geometrically distinct closed geodesics on S^2 so that the associated Reeb flow on S^3 possesses at least three distinct periodic orbits. We therefore conclude from Corollary 1.10 that there are ∞ many closed geodesics for a generic metric g on S^2. The result is, of course, not new and even holds true for every Riemannian metric g as proved by V. Bangert and J. Franks [2], [20]. The new aspect in the generic case lies in the proof which shows that either there is a disc-like surface of section (for the doubly covered geodesic flow) or there exists a hyperbolic periodic orbit having orientable stable and unstable manifolds intersecting transversally in a homoclinic orbit.

Conjecture 1.13. A tight Reeb flow on S^3 has either precisely two or infinitely many geometrically distinct periodic orbits.

As already mentioned, the conjecture is true for dynamically convex contact forms, $f\lambda_0$ for f constituting an open subset of $C^\infty(S^3,(0,\infty))$, and also for every generic $f \in \Theta_2$, in view of Corollary 1.10.

2. The main construction

We shall identify the contact manifold $(S^3, f\lambda_0)$ with $(M, \lambda_{0|M})$ where $M \subset \mathbb{C}^2$ is a star-like hypersurface defined by means of the function f. The hypersurface M is of contact-type in $\mathbb{C}^2$ equipped with the standard symplectic structure ω_0 and an open neighborhood of the domain bounded by M in $\mathbb{C}^2$ will be symplectically embedded in $\mathbb{C}P^2$ away from its sphere at infinity S_∞. Adding the neck $[-N,N]\times M$ to the embedded hypersurface we shall obtain a compact manifold (A_N, ω_N), symplectomorphic to $(\mathbb{C}P^2, \omega)$, which has a special compatible almost complex structure $\widehat{J}_N$ which in particular agrees on the neck with the $\mathbb{R}$-invariant structure $\widetilde{J}$. We then show that there is a unique $\widehat{J}_N$-holomorphic sphere in A_N containing the two given points $(0,m) \in [-N,N]\times M$ and $o_\infty \in S_\infty$. The sphere is embedded and generic. The desired finite energy foliation on $\mathbb{R}\times M$ will be the result of a limit procedure as $N\to\infty$ carried out in Chapters 3–6.

For later reference we shall first collect in Section 2.1 some results on finite energy spheres in $\mathbb{R} \times M$ in the generic situation. The manifolds (A_N, ω_N), $N \geq 1$, will be constructed in Sections 2.2 and 2.3. For the convenience of the reader, Gromov's theory of pseudoholomorphic curves in $\mathbb{C}P^2$, homologous to $\mathbb{C}P^1$, will be outlined in Section 2.4.

2.1. *The problem* **(M)**. In the following M is a closed 3-manifold equipped with the contact form λ which is assumed to be nondegenerate in the sense of Proposition 1.1. The aim of this section is to collect some information about finite energy spheres in $\mathbb{R} \times M$. In order to formulate the generic properties of such maps we first recall Floer's $\mathcal{C}_\varepsilon$-space.

We choose a compatible complex multiplication $J^0 : \xi \to \xi$ of the contact planes and denote by $\widetilde{J}^0$ the associated $\mathbb{R}$-invariant almost complex structure on $\mathbb{R} \times M$. The Fréchet space $\mathcal{C}^\infty$ consists of all smooth maps $m \mapsto Y(m)$, where $m \in M$, and

$$Y(m) \in \operatorname{Hom}_{\mathbb{R}}(\xi_m)$$

satisfying

$$Y(m) \circ J^0(m) + J^0(m) \circ Y(m) = 0. \tag{2.1}$$

The map $Y(m)$ has the following property:

$$d\lambda\Big(Y(m)h, k\Big) + d\lambda\Big(h, Y(m)k\Big) = 0 \tag{2.2}$$

for h, $k \in \xi_m$. Indeed, if $h \neq 0$ we set $k = \gamma h + \delta J^0(m)h$ and obtain from (2.1) (since $d\lambda(\cdot, J^0(m)\cdot)$ is an inner product on ξ_m)

$$\begin{aligned} d\lambda\Big(Y(m)h, k\Big) &= \gamma d\lambda\Big(Y(m)h, h\Big) + \delta d\lambda\Big(Y(m)h, J^0(m)h\Big) \\ &= -\gamma d\lambda\Big(h, Y(m)h\Big) + \delta d\lambda\Big(h, J^0(m)Y(m)h\Big) \\ &= -\gamma d\lambda\Big(h, Y(m)h\Big) - \delta d\lambda\Big(h, Y(m)J^0(m)h\Big) \\ &= -d\lambda\Big(h, Y(m)k\Big). \end{aligned}$$

If $\varepsilon = (\varepsilon_k)$ is a sequence of positive numbers converging to 0 we denote by $\mathcal{C}_\varepsilon$ the subspace of $\mathcal{C}^\infty$ consisting of Y satisfying (2.1) and such that

$$\|Y\|_\varepsilon = \sum_{k=0}^{\infty} \varepsilon_k \|Y\|_{C^k} < \infty. \tag{2.3}$$

If (ε_k) converges sufficiently fast to 0, the subset $\mathcal{C}_\varepsilon$ is dense in $\mathcal{C}^\infty$; see A. Floer [16]. For $\delta > 0$ we denote by $\mathcal{U}_\delta$ the set of $\mathbb{R}$-invariant almost complex structures $\widetilde{J} : T(\mathbb{R} \times M) \to T(\mathbb{R} \times M)$ of the form

$$\widetilde{J}(a, m)(\gamma, k) = \Big(-\lambda(m)(k), J(m)\pi k + \gamma X(m)\Big), \tag{2.4}$$

where

$$J(m) = J^0(m)\exp\Big[-J^0(m)Y(m)\Big],$$

with $Y \in \mathcal{C}_\varepsilon$ satisfying $\|Y\|_\varepsilon < \delta$. The map $Y \to \widetilde{J} \in \mathcal{U}_\delta$ constitutes the global chart for $\mathcal{U}_\delta$ defining a separable Banach manifold structure.

We consider finite energy spheres in $\mathbb{R} \times M$ for generic $\widetilde{J}$, i.e., $\widetilde{J} \in \Xi$, where the set $\Xi \subset \mathcal{U}_\delta$ is as defined in Theorem 2.1 below,

$$\begin{aligned} \widetilde{u} : S^2 \setminus \Gamma \to \mathbb{R} \times M, \\ T\widetilde{u} \circ i = \widetilde{J} \circ T\widetilde{u}, \\ 0 < E(\widetilde{u}) < \infty. \end{aligned} \tag{2.5}$$

Later on we shall refer to this nonlinear problem as problem (M). From Fredholm theory in [36] we recall that the finite energy spheres in the neighborhood of an embedded finite energy sphere $\widetilde{u}$ are described by a nonlinear Fredholm equation having the Fredholm index $\mathrm{Ind}(\widetilde{u}) = \mu(\widetilde{u}) - 2 + \sharp\Gamma$. The index is computed for unparametrized spheres. This means that the positions of the punctures Γ are allowed to vary and the group of Möbius transformations is divided out. Due to the $\mathbb{R}$-action, the kernel of the linearized Fredholm operator is at least one-dimensional unless the image of $\widetilde{u}$ is a cylinder over a periodic orbit, in which case $\pi \circ Tu = 0$. If $\widetilde{J}$ is generic we have the following result, proved for embedded finite energy surfaces in [36], and for somewhere injective surfaces in [7].

Theorem 2.1. *There exists a Baire subset $\Xi \subset \mathcal{U}_\delta$ such that for every $\widetilde{J} \in \Xi$ the following holds. If $\widetilde{u} : S^2 \setminus \Gamma \to \mathbb{R} \times M$ is a somewhere injective finite energy sphere for $\widetilde{J}$, then*

$$\mathrm{Ind}(\widetilde{u}) = \mu(\widetilde{u}) - 2 + \sharp\Gamma \geq 1$$

provided $\pi \circ Tu$ does not vanish identically.

The number 1 on the right-hand side of the estimate stems from the $\mathbb{R}$-invariance of $\widetilde{J}$. Theorem 2.1 has the following consequence already mentioned in the introduction.

Corollary 2.2. *Assume $\widetilde{J} \in \Xi$. Let $\widetilde{u} = (a, u) : S^2 \setminus \Gamma \to \mathbb{R} \times M$ be a somewhere injective finite energy sphere with precisely one positive puncture and an arbitrary number of negative punctures. If all the occurring Conley-Zehnder indices for the asymptotic limits of the punctures (computed with respect to a suitable symplectic trivialization of $u^*\xi$) are contained in $\{1, 2, 3\}$ and if $\pi \circ Tu \neq 0$, then*

$$\sharp\Gamma_2 \leq 1.$$

Here Γ_2 is the set of those punctures whose asymptotic limits have Conley-Zehnder indices equal to 2.

Proof. Denote by Γ_j^- the set of negative punctures whose asymptotic limits have index $j \in \{1,2,3\}$. By assumption, $\sharp\Gamma = 1 + \sharp\Gamma_1^- + \sharp\Gamma_2^- + \sharp\Gamma_3^-$. From Theorem 2.1 we deduce, using the definition of $\mu(\widetilde{u})$,

$$\begin{aligned} 1 &\leq \mu^+ - \mu^- - 2 + \sharp\Gamma \\ &= \mu^+ - \sharp\Gamma_1^- - 2\sharp\Gamma_2^- - 3\sharp\Gamma_3^- - 2 + \sharp\Gamma \\ &= [\mu^+ - 1] - \sharp\Gamma_2^- - 2\sharp\Gamma_3^-. \end{aligned}$$

Consequently,

$$\sharp\Gamma_2^- + 2\sharp\Gamma_3^- \leq \mu^+ - 2,$$

which leads to the following conclusions:

1. $\mu^+ \in \{2,3\}$.
2. If $\mu^+ = 2$, then all the negative punctures have index 1.
3. If $\mu^+ = 3$, then there is at most one negative puncture with index 2 and all other negative punctures have index 1. □

COROLLARY 2.3. *Assume $\widetilde{J}$ and $\widetilde{u} : S^2 \setminus \Gamma \to \mathbb{R} \times M$ meet the hypotheses of Corollary* 2.2. *Then the Fredholm index of $\widetilde{u}$ satisfies*

$$\mathrm{Ind}(\widetilde{u}) \in \{1,2\}.$$

More precisely, the following situations are possible, where μ^+ is the Conley-Zehnder index of the positive puncture:

- $\mu^+ = 2$ *and every negative puncture has Conley-Zehnder index equal to* 1. *In this case* $\mathrm{Ind}(\widetilde{u}) = 1$.
- $\mu^+ = 3$ *and there is one negative puncture with index equal to* 2 *while all other negative punctures have index equal to* 1. *In this case* $\mathrm{Ind}(\widetilde{u}) = 1$.
- $\mu^+ = 3$ *and all negative punctures have indices equal to* 1. *In this case* $\mathrm{Ind}(\widetilde{u}) = 2$.

Proof. The statement is an immediate consequence of the conclusions 1, 2, 3 above and the formula $\mathrm{Ind}(\widetilde{u}) = \mu^+ - \mu^- - 2 + \sharp\Gamma$ for the Fredholm index. □

The nature of the punctures strongly influences the geometry of the finite energy sphere. In this context it is useful to recall Proposition 4.1 in [30].

PROPOSITION 2.4. *If $\widetilde{u} = (a,u) : S^2 \setminus \Gamma \to \mathbb{R} \times M$ is a finite energy sphere*, *the section $\pi \circ Tu$ of the bundle*

$$\mathrm{Hom}_{\mathbb{C}}\Big(T(S^2 \setminus \Gamma), \widetilde{u}^*\xi\Big) \to S^2 \setminus \Gamma$$

either vanishes identically or has only a finite number of zeros. Every zero has a positive index.

Denote by $\text{wind}_\pi(u)$ the number of zeros (counting multiplicities) of $\pi \circ Tu$. This integer is related to the asymptotic data of the punctures and we recall Theorem 5.8 in [30].

THEOREM 2.5. *If $\widetilde{u} = (a, u) : S^2 \setminus \Gamma \to \mathbb{R} \times M$ is a finite energy sphere satisfying $\pi \circ Tu \neq 0$, then*

$$2\text{wind}_\pi(u) \leq \mu(\widetilde{u}) - 2\chi(S^2) + 2\sharp\Gamma_{\text{even}} + \sharp\Gamma_{\text{odd}}.$$

Here Γ_{even} is the subset of Γ consisting of punctures with even Conley-Zehnder index and Γ_{odd} the subset of punctures with odd Conley-Zehnder index, computed with respect to $u^*\xi$. Moreover, $\mu(\widetilde{u})$ is the difference between the sum of the indices associated with positive punctures and the sum of the indices belonging to the negative punctures.

Theorem 2.5 is very useful whenever more information about the nature of punctures is available as the following corollary shows.

COROLLARY 2.6. *Assume $\widetilde{u} = (a, u) : S^2 \setminus \Gamma \to \mathbb{R} \times M$ is a finite energy sphere satisfying $\pi \circ Tu \neq 0$. If $\text{Ind}(\widetilde{u}) \leq 2$ and $\sharp\Gamma_{\text{even}} \leq 1$, then*

$$\pi \circ Tu(z) \neq 0$$

for every point $z \in S^2 \setminus \Gamma$.

Proof. We compute, using Theorem 2.5,

$$\begin{aligned} 2\ \text{wind}_\pi(\widetilde{u}) &\leq \mu(\widetilde{u}) - 4 + 2\sharp\Gamma_{\text{even}} + \sharp\Gamma_{\text{odd}} \\ &= (\mu(\widetilde{u}) - 2 + \sharp\Gamma) - 2 - \sharp\Gamma + 2\sharp\Gamma_{\text{even}} + \sharp\Gamma_{\text{odd}} \\ &= \text{Ind}(\widetilde{u}) - 2 + \sharp\Gamma_{\text{even}} \leq 2 - 2 + 1 = 1. \end{aligned}$$

Hence $\text{wind}_\pi(\widetilde{u}) = 0$ implying the desired result. □

We make use of the corollary in the proof of the following result.

THEOREM 2.7. *Let $\widetilde{J} \in \Xi$ be generic and assume $\widetilde{u} := (a, u) : S^2 \setminus \Gamma \to \mathbb{R} \times M$ is an embedded finite energy sphere with simply covered asymptotic limits and $\pi \circ Tu$ not vanishing identically. If $\text{Ind}(\widetilde{u}) \in \{1, 2\}$ and $\sharp\Gamma_{\text{even}} \leq 1$, then the map $u : S^2 \setminus \Gamma \to M$ is an embedding transversal to the Reeb vector field X. Moreover, the image of u does not intersect the periodic orbits associated with the punctures Γ.*

Proof. By the results in [36], the given sphere $\widetilde{u}$ lies in an $\text{Ind}(\widetilde{u})$-dimensional family of embedded finite energy spheres. A member of this family can be described by means of a graph of a section of the normal bundle of $\widetilde{u}$ in $\mathbb{R} \times M$

satisfying a Monge-Ampère-type equation. Clearly, a zero of the section is an intersection point with $\widetilde{u}$. The linearization at the zero-section is a Cauchy-Riemann type operator L. Our first aim is to show that the family consists of mutually disjoint spheres. Since the asymptotic limits are, by assumption, simply covered, it is sufficient to prove that the nontrivial elements in the kernel of L do not admit any zero. Indeed, due to the special asymptotic behavior near a puncture, a neighboring sphere can be homotoped to an element in the kernel without introducing zeros near the punctures. Since $\Gamma \neq \emptyset$, the normal bundle of $\widetilde{u}$ is trivial and hence can be identified with $\mathbb{R}^2$. So, let $h : S^2 \setminus \Gamma \to \mathbb{R}^2$ be a nontrivial element in the kernel of L. The crucial observation now is, that as a solution of the perturbed Cauchy-Riemann operator L, the map h admits only isolated zeros having, in addition, positive indices. Denoting the sum of the local indices by ℓ it remains to show that $\ell = 0$. In order to do so, we make use of the asymptotic behavior of h near every puncture in Γ which is similar to the z-part studied in [32], [36]. This follows since the asymptotic operators near the punctures are, in suitable coordinates, the same as those describing the sphere $\widetilde{u}$ near the punctures. Let $\mathcal{D}$ be a holomorphic disc centered at a positive puncture corresponding to $z = 0$ and introduce holomorphic polar coordinates $\sigma : \mathbb{R}^+ \times S^1 \to \mathcal{D} \setminus \{0\}$ by $z = \mathrm{e}^{-2\pi(s+\mathrm{i}t)}$. Then $v = h \circ \sigma : \mathbb{R}^+ \times S^1 \to \mathbb{R}^2$ has the following asymptotic representation:

$$v(s,t) = \mathrm{e}^{\int_{s_0}^{s} \lambda^+(\tau)d\tau} \Big[e^+(t) + r(s,t) \Big],$$

where $\partial^\alpha r(s,t) \to 0$ uniformly in $t \in S^1$ for all derivatives as $s \to \infty$ and where $\lambda^+(s)$ converges to a negative eigenvalue λ^+ of the asymptotic self-adjoint operator

$$-J_0 \frac{d}{dt} - S_\infty(t) \quad \text{on } L^2(S^1, \mathbb{R}^2) \tag{2.6}$$

associated with the periodic solution of the puncture. The periodic function $e^+(t+1) = e^+(t)$ is an eigenvector belonging to λ^+. Since it does not vanish anywhere it possesses a winding number $\operatorname{wind}(e^+) \in \mathbb{Z}$. For the behavior of h near a negative puncture there is an analogous formula, where $s \to -\infty$ and $\lambda^-(s) \to \lambda^-$ for a positive eigenvalue λ^- of (2.6) with associated eigenvector e^- and winding number $\operatorname{wind}(e^-)$. Clearly,

$$\ell = \sum_{\Gamma^+} \operatorname{wind}(e^+) - \sum_{\Gamma^-} \operatorname{wind}(e^-). \tag{2.7}$$

The winding numbers $\operatorname{wind}(e)$ are related to the normal Conley-Zehnder indices μ_N computed with respect to the above trivialization of the normal bundle. Recall from Theorem 3.10 in [30] the formula

$$\mu_N = 2\alpha + p. \tag{2.8}$$

Here $p \in \{0,1\}$ and α is the maximal winding number of eigenvectors belonging to the negative eigenvalues of the asymptotic operator (2.6). Since the winding numbers are monotone increasing with the eigenvalues we conclude for a positive puncture 2 wind$(e^+) \leq \mu_N$ if μ_N is even and $\leq \mu_N - 1$ if μ_N is odd, while for a negative puncture 2 wind$(e^-) \geq \mu_N$ for μ_N even and $\geq \mu_N + 1$ for μ_N odd. Therefore, in view of (2.7),

$$
\begin{aligned}
(2.9) \qquad 2\ell &\leq \sum_{\Gamma^+_{\text{odd}}} \Big(\mu_N - 1\Big) + \sum_{\Gamma^+_{\text{even}}} \mu_N - \sum_{\Gamma^-_{\text{odd}}} \Big(\mu_N + 1\Big) - \sum_{\Gamma^-_{\text{even}}} \mu_N \\
&= \Big(\mu_N^+ - \mu_N^-\Big) - \Big(\sharp\Gamma^+_{\text{odd}} + \sharp\Gamma^-_{\text{odd}}\Big) \\
&= \mu_N(\widetilde{u}) - \sharp\Gamma_{\text{odd}}.
\end{aligned}
$$

The relationship between the normal Conley-Zehnder index $\mu_N(\widetilde{u})$ and the usual index $\mu(\widetilde{u})$ computed with respect to a trivialization of $u^*\xi$ is, by Theorem 1.8 in [36], given by the formula

$$\mu(\widetilde{u}) = \mu_N(\widetilde{u}) + 4 - 2\sharp\Gamma.$$

In view of Theorem 2.1,

$$\operatorname{Ind}(\widetilde{u}) = \mu(\widetilde{u}) - 2 + \sharp\Gamma.$$

We can estimate, using (2.9),

$$
\begin{aligned}
\operatorname{Ind}(\widetilde{u}) &= \mu_N + 2 - \sharp\Gamma \\
&\geq 2\ell + 2 - \sharp\Gamma + \sharp\Gamma_{\text{odd}} \\
&= 2\ell + 2 - \sharp\Gamma_{\text{even}}.
\end{aligned}
$$

By our assumptions, $\sharp\Gamma_{\text{even}} \leq 1$ and $\operatorname{Ind}(\widetilde{u}) \leq 2$, so that $2 \geq \operatorname{Ind}(\widetilde{u}) \geq 2\ell + 1$. Consequently, $\ell \leq 0$ and hence $\ell = 0$, as we set out to prove.

Summing up, we conclude that the spheres near $\widetilde{u}$ in the $\operatorname{Ind}(\widetilde{u})$-dimensional family are mutually disjoint. As shown in the Fredholm theory [36], the $\mathbb{R}$-action accounts for one dimension in this family. The $\mathbb{R}$-action is defined by $\widetilde{u}_c(z) := (a(z) + c, u(z))$. Hence, for $0 < |c|$ small the intersection number $\operatorname{int}(\widetilde{u}, \widetilde{u}_c)$ is well defined and 0. If $c \neq 0$ the intersection set of $\widetilde{u}$ and $\widetilde{u}_c$ is compact in view of the asymptotic behavior near the punctures, and by homotopy invariance we conclude

$$\operatorname{int}(\widetilde{u}, \widetilde{u}_c) = 0, \quad c \neq 0.$$

In view of the positivity of intersections of pseudoholomorphic curves we deduce that the images of $\widetilde{u}$ and $\widetilde{u}_c$ for $c \neq 0$ are disjoint. This implies that u is injective. Since, by Corollary 2.6, the section $\pi \circ Tu$ does not vanish anywhere, $u : S^2 \setminus \Gamma \to M$ is an injective immersion transversal to X and so, by the asymptotic behavior near the punctures, the map u must be an embedding.

Moreover, an intersection point of u with an asymptotic limit would have to be transversal, and hence would imply a self intersection of u contradicting the injectivity of u. Therefore, the image of u does not intersect the asymptotic limits of the punctures Γ and the proof of Theorem 2.7 is complete. □

2.2. *Gluing almost complex half cylinders over contact boundaries.* Let (A, ω) be a compact 4-dimensional symplectic manifold with boundary $\partial A \neq \emptyset$, in the following denoted by

$$B := \partial A.$$

We assume the boundary to be of contact type. This requires the existence of a one-form λ on B satisfying

$$d\lambda = \omega_{|B}, \quad \lambda \wedge d\lambda = \text{ volume form on } B. \tag{2.10}$$

In particular, λ is a contact form on the 3-manifold B and determines on B the contact structure ξ by $\xi = \ker \lambda$ and the Reeb vector field X by $\lambda(X) = 1$ and $d\lambda(X, \cdot) = 0$, so that the tangent space

$$T_pB = \mathbb{R}X(p) \oplus \xi_p, \quad p \in B,$$

splits into a line bundle with section X and the plane bundle $\xi \to B$ having the symplectic form $d\lambda_{|\xi}$. We denote by $\pi : TB \to \xi$ the projection along the Reeb vector field. The one-form λ on B can be extended to a one-form on an open neighborhood U of B in such a way that still $d\lambda = \omega_{|U}$, where we denote the extension by λ again; see, for example, [40]. On U we define the vector field η by

$$i_\eta \omega = \lambda.$$

Then $i_\eta \lambda = 0$ and, in view of Cartan's formula $L_\eta = d \circ i_\eta + i_\eta \circ d$ for the Lie derivative of the vector field, we have $L_\eta \omega = \omega$ and $L_\eta \lambda = \lambda$. Consequently, the flow φ_t of η satisfies on its domain of definition in U,

$$\varphi_t^* \omega = \mathrm{e}^t \omega, \qquad \varphi_t^* \lambda = \mathrm{e}^t \lambda.$$

The vector field η is transversal to B,

$$T_pA = \mathbb{R}\eta \oplus T_pB, \quad p \in B \subset A,$$

since otherwise $\eta \in T_pB$, leading to the contradiction $0 = d\lambda(\eta, X) = \lambda(X) = 1$. Conversely, of course, a vector field η on U transversal to B and satisfying $L_\eta \omega = \omega$, defines the one-form $\lambda = i_\eta \omega$ meeting the properties (2.10). The boundary B splits into two parts

$$B = B^+ \cup B^-,$$

where the vector field η points outward on B^+, and inward on B^-. (One of the parts might, of course, be empty.) We shall use the flow φ_t in order to define useful collars of $B^{\pm}$. If $\varepsilon > 0$ is sufficiently small we define the embeddings $\Phi^{\pm}$ by

$$\Phi^{+} : [-\varepsilon, 0] \times B^{+} \to A, \quad (t, b^{+}) \mapsto \varphi_t(b^{+}) \tag{2.11}$$

if $-\varepsilon \leq t \leq 0$ and $b^{+} \in B^{+}$;

$$\Phi^{-} : [0, \varepsilon,] \times B^{-} \to A, \quad (t, b^{-}) \mapsto \varphi_t(b^{-}) \tag{2.12}$$

if $0 \leq t \leq \varepsilon$ and $b^{-} \in B^{-}$. This way a neighborhood of $B \subset A$ is foliated by conformally symplectomorphic leaves

$$B^{+}_{\tau} = \varphi_{\tau}(B^{+}), \quad -\varepsilon \leq \tau \leq 0,$$

with $B^{+}_{0} = B^{+}$, and analogously for B^{-}_{τ}. If $\varphi \in C^{\infty}(\mathbb{R}, \mathbb{R})$, the two-form $d(\varphi\lambda)$ on $\mathbb{R} \times B$ evaluated at the tangent vectors (α, a) and $(\beta, b) \in T_{(s,p)}(\mathbb{R} \times B)$ results in

$$\begin{aligned} d(\varphi\lambda)\Big[(\alpha, a), (\beta, b)\Big] &= \varphi'(s)\Big[\alpha\lambda(b) - \beta\lambda(a)\Big] + \varphi(s)d\lambda(a, b) \\ &= \varphi'(s)\Big[\alpha b_1 - \beta a_1\Big] + \varphi(s)d\lambda(a_2, b_2). \end{aligned} \tag{2.13}$$

We have used the representations $a = a_1 X(p) + a_2$ and $b = b_1 X(p) + b_2$ according to the splitting $T_p B = \mathbb{R}X(p) \oplus \xi_p$. We see that $d(\varphi\lambda)$ is a symplectic form if $\varphi > 0$ and $\varphi' > 0$ on $\mathbb{R}$. In particular, $d(\mathrm{e}^s\lambda)$ is symplectic on $\mathbb{R} \times B$, and a computation shows that

$$(\Phi^{\pm})^{*}\omega = d(\mathrm{e}^s\lambda) \tag{2.14}$$

on $[-\varepsilon, 0] \times B^{+}$, resp. on $[0, \varepsilon] \times B^{-}$.

Recall that an almost complex structure $\widehat{J}$ on A is called compatible with ω if

$$g_{\widehat{J}}(h, k) := \omega(h, \widehat{J}k)$$

is a Riemannian metric on A. The set of compatible almost complex structures is nonempty and contractible. This is, of course, well known and we refer to [40].

Definition 2.8. The almost complex structure $\widehat{J}$ on A is called *admissible* if it is compatible with ω and if, in addition,

$$\begin{aligned} T\Phi^{+} \circ \widetilde{J} &= \widehat{J} \circ T\Phi^{+} \quad \text{on } [-\varepsilon, 0] \times B^{+} \\ T\Phi^{-} \circ \widetilde{J} &= \widehat{J} \circ T\Phi^{-} \quad \text{on } [0, \varepsilon] \times B^{-}, \end{aligned} \tag{2.15}$$

where $\widetilde{J}$ is the standard $\mathbb{R}$-invariant almost complex structure on $\mathbb{R} \times B$,

$$\widetilde{J}(s, p)[h, k] = \Big[-\lambda(p)k, J(p)\pi k + hX(p)\Big], \tag{2.16}$$

$[h, k] \in T_{(s,p)}(\mathbb{R} \times B)$. Here $J : \xi \to \xi$ is an *almost complex structure* on the contact planes.

If $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$ satisfies $\varphi > 0$ and $\varphi' > 0$, the almost complex structure $\widetilde{J}$ is compatible with the symplectic form $d(\varphi\lambda)$ on $\mathbb{R}\times B$. Indeed, by (2.13),

$$d(\varphi\lambda)\Big[(\alpha,a),\widetilde{J}(\beta,b)\Big] = \varphi'(s)\Big[\alpha\beta + a_1 b_1\Big] + \varphi(s)d\lambda(a_2, Jb_2)$$

and $d\lambda(a_2, Jb_2)$ is a symmetric and positive definite bilinear form on ξ.

It will be important for our considerations later on that an admissible $\widehat{J}$ be compatible not only with ω but with a variety of other symplectic forms. To see this, introduce the collections $\Sigma^\pm$ of smooth functions $\varphi^\pm$ satisfying

$$\begin{aligned} &\varphi^+ : [-\varepsilon,\infty) \to (0,1), \\ &\varphi^- : (-\infty,\varepsilon] \to (1,\infty), \\ &\tfrac{d}{ds}\varphi^\pm(s) > 0, \\ &\varphi^+(s) = \mathrm{e}^s \qquad \text{if } s\in[-\varepsilon,-\varepsilon/2], \\ &\varphi^-(s) = \mathrm{e}^s \qquad \text{if } s\in[\varepsilon/2,\varepsilon]. \end{aligned}$$

Denote by Σ the collection of pairs $\varphi = (\varphi^+,\varphi^-)$ with $\varphi^\pm \in \Sigma^\pm$. With $\varphi\in\Sigma$ we associate the following 2-form ω_φ on A:

$$\begin{aligned} \omega_\varphi = \omega \quad &\text{on } A\setminus\Big(\Phi^+([-\varepsilon/2,0]\times B^+)\cup\Phi^-([0,\varepsilon/2]\times B^-)\Big), \\ (\Phi^+)^*\omega_\varphi = d(\varphi^+\lambda) \qquad &\text{on } \Phi^+\left([-\varepsilon/2,0]\times B^+\right), \\ (\Phi^-)^*\omega_\varphi = d(\varphi^-\lambda) \qquad &\text{on } \Phi^-\left([0,\quad \varepsilon/2]\times B^-\right). \end{aligned}$$

In view of (2.14) the 2-form ω_φ is a smooth symplectic form on A. Assume now that $\widehat{J}$ is admissible; then

$$\omega_\varphi\Big(T\Phi^+u,\widehat{J}T\Phi^+v\Big) = \omega_\varphi\Big(T\Phi^+u, T\Phi^+\widetilde{J}v\Big) = d\Big(\varphi^+\lambda\Big)\Big(u,\widetilde{J}v\Big),$$

and similarly for Φ^-. Since $\widetilde{J}$ is compatible with $d(\varphi^\pm\lambda)$ we have proved:

LEMMA 2.9. *An admissible almost complex structure $\widehat{J}$ on A is compatible with every symplectic structure ω_φ, $\varphi\in\Sigma$.*

We next define the noncompact almost complex manifold

$$(\widetilde{A},\widehat{J})$$

having no boundary. It will be constructed from A, equipped with an admissible structure $\widehat{J}$, by gluing cylinders to the collars of the boundaries $B^\pm$. Recalling the symplectic embeddings from (2.11) and (2.12) we introduce the compact manifold

$$\widehat{A} = A\setminus\Big(\Phi^+([-\varepsilon/2,0]\times B^+)\cup\Phi^-([0,\varepsilon/2]\times B^-)\Big)$$

and equip the manifolds $[-\varepsilon,\infty)\times B^+$ and $(-\infty,\varepsilon]\times B^-$ with the $\mathbb{R}$-invariant structure $\widetilde{J}$ in (2.16). Identifying $[-\varepsilon,-\varepsilon/2]\times B^+$ with its image in A under

Φ^+, and $[\varepsilon/2, \varepsilon] \times B^-$ with its image under Φ^- we obtain the manifold $\widetilde{A}$ as the disjoint union

$$\widehat{A} \cup \big([-\varepsilon, \infty) \times B^+ \cup (-\infty, \varepsilon] \times B^-\big)$$

with the pointwise identifications:

$$\begin{aligned} \Phi^+\Big([-\varepsilon, -\varepsilon/2] \times B^+\Big) &\equiv [-\varepsilon, -\varepsilon/2] \times B^+, \\ \Phi^-\Big([\varepsilon/2, \varepsilon] \times B^-\Big) &\equiv [\varepsilon/2, \varepsilon] \times B^-. \end{aligned}$$

In view of the compatibility conditions (2.15) we can equip $\widetilde{A}$ with an almost complex structure $\widehat{J}$ by choosing on A an admissible structure and on the two cylindrical ends the corresponding standard structure $\widetilde{J}$. We denote the almost complex manifold obtained this way by $(\widetilde{A}, \widehat{J})$.

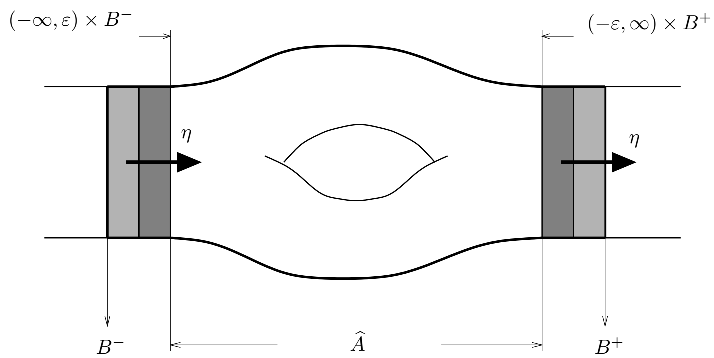

Figure 4. Construction of the almost complex manifold $(\widetilde{A}, \widehat{J})$.

We define for $\varphi \in \Sigma$ the closed 2-form ω_φ on $\widetilde{A}$ by the requirements

$$\omega_{\varphi|\widehat{A}} = \omega, \qquad \omega_{\varphi|\text{end}^\pm} = d(\varphi^\pm \lambda) \tag{2.17}$$

where we abbreviated $\text{end}^+ = [-\varepsilon, \infty) \times B^+$ and $\text{end}^- = (-\infty, \varepsilon] \times B^-$. In view of the compatibility condition (2.14), this defines a smooth symplectic form on $\widetilde{A}$ which, moreover, is compatible with $\widehat{J}$. This proves the first part of the following lemma.

Lemma 2.10. *If $\varphi \in \Sigma$, then the map $\omega_\varphi \circ (\text{Id} \times \widehat{J})$ defines a Riemannian metric on $\widetilde{A}$. If, in addition,*

$$\lim_{s \to \infty} \varphi^+(s) = 1 \quad \textit{and} \quad \lim_{s \to -\infty} \varphi^-(s) = 1,$$

then $(\widetilde{A}, \omega_\varphi)$ is symplectomorphic to $(\text{int}(A), \omega)$ by means of a symplectic diffeomorphism inducing the identity on $\widehat{A}$.

Proof. In order to prove the second part of the lemma we observe that, by assumption, $\varphi^+(s) = \mathrm{e}^{\beta^+(s)}$, where the smooth function $\beta^+ : [-\varepsilon, \infty) \to [-\varepsilon, 0)$ satisfies $\frac{d}{ds}\beta^+ > 0$ and $\beta^+(s) = s$ if $-\varepsilon \le s \le -\varepsilon/2$ and $\beta^+(s) \to 0$ as $s \to \infty$. Similarly $\varphi^- : (-\infty, \varepsilon] \to \mathbb{R}$ has the form $\varphi^-(s) = \mathrm{e}^{\beta^-(s)}$ with $\frac{d}{ds}\beta^- > 0$ and $\beta^-(s) = s$ if $\varepsilon/2 \le s \le \varepsilon$ and $\beta^-(s) \to 0$ as $s \to -\infty$. Let $\varphi = (\varphi^+, \varphi^-)$ and define the diffeomorphism $\Psi : (\widetilde{A}, \omega_\varphi) \to (\text{ int } A, \omega)$ by setting $\Psi_{|\widehat{A}} = \mathrm{Id}$. Moreover, using the flow φ_t of η we define $\Psi : [-\varepsilon, \infty) \times B^+ \to \text{ int}A$ at the positive end by $\Psi(s, m) = \varphi_{\beta^+(s)}(m)$. Then

$$\Psi^*\omega = d(\varphi^+\lambda).$$

Similarly, taking $\Psi(s, m) = \varphi_{\beta^-(s)}(m)$ at the negative end, the statement follows. □

A general finite energy sphere in $(\widetilde{A}, \widehat{J})$ is a smooth map $u : S^2 \setminus \Gamma \to \widetilde{A}$ solving the partial differential equation

$$Tu \circ i = \widehat{J}(u) \circ Tu$$

and satisfying the energy requirement

$$0 < E(u) < \infty,$$

where

$$E(u) := \sup_{\varphi \in \Sigma} \int_{S^2 \setminus \Gamma} u^* \omega_\varphi.$$

In order to describe the behavior of u near a puncture $z_0 \in \Gamma$ we observe that a suitable neighborhood $U \subset S^2$ of z_0 looks biholomorphically like the closed unit disc $D \subset \mathbb{C}$, i.e., $U = \varphi(D)$ and $z_0 = \varphi(0)$. Introduce the holomorphic parametrization $\psi : \mathbb{R}^+ \times S^1 \to D \setminus \{0\}$ by $\psi(s, t) = \mathrm{e}^{-2\pi(s+\mathrm{i}t)}$ and parametrize $U \setminus \{z_0\}$ by the holomorphic parametrization $\sigma = \varphi \circ \psi : \mathbb{R}^+ \times S^1 \to U \setminus \{z_0\}$. The composition $v = u \circ \sigma : \mathbb{R}^+ \times S^1 \to \widetilde{A}$ solves the Cauchy-Riemann equation on $\mathbb{R}^+ \times S^1$ and has finite energy,

$$\partial_s v + \widehat{J}(v)\partial_t v = 0, \qquad E(v) < \infty.$$

PROPOSITION 2.11. *Consider a general finite energy sphere $u : S^2 \setminus \Gamma \to \widetilde{A}$ in the neighborhood $U \setminus \{z_0\}$ of the punctures $z_0 \in \Gamma$. Then for $v = u \circ \sigma : \mathbb{R}^+ \times S^1 \to \widetilde{A}$ one of the following three alternatives holds*:

1. *There exists a compact $K \subset \widetilde{A}$ such that $v(\mathbb{R}^+ \times S^1) \subset K$.*

2. *The image of v is unbounded and there exists $R > 0$ such that*

$$v\Big([R, \infty) \times S^1\Big) \subset \mathbb{R}^+ \times B^+.$$

3. *The image of v is unbounded and there exists $R>0$ such that*

$$v\Big([R,\infty)\times S^1\Big)\subset \mathbb{R}^-\times B^-.$$

Moreover, in the first case the solution u has a smooth extension over z_0. In the second case there exist a sequence $R_k\to\infty$ and a positive number $T>0$ such that $v(s,t)=(a(s,t),v_{B^+}(s,t))\in\mathbb{R}^+\times B^+$ satisfies

$$a(R_k,t)\to\infty,\quad v_{B^+}(R_k,t)\to x(Tt),$$

in $C^\infty(S^1)$, where x is a T-periodic solution of the Reeb vector field $\dot{x}=X(x)$ on B^+. In the third case there exist a sequence $R_k\to\infty$ and a negative number $T<0$ such that

$$a(R_k,t)\to-\infty,\quad v_{B^-}(R_k,t)\to x(Tt),$$

for a $|T|$-periodic solution of X on B^-.

Definition 2.12. The puncture $z_0\in\Gamma$ is called *removable* in the first case, *positive* in the second and *negative* in the third case.

Proof. In order to estimate the derivatives of the finite energy cylinder $v:[0,\infty)\times S^1\to\widetilde{A}$ it is convenient to embed $\widetilde{A}$ in some $\mathbb{R}\times\mathbb{R}^m$ so that the cylindrical ends have the form $(-\infty,0]\times B^-$ and $[0,\infty)\times B^+$. Using the metric induced from the Euclidean inner product we first note that the gradient and all higher order derivatives are uniformly bounded. This is proved by means of a standard bubbling off argument precisely as in [24]. If the image of v is bounded hence contained in the compact domain Ω, we can take a symplectic form ω_φ restricted to Ω and apply Gromov's removable singularity theorem from [23] in order to conclude that v can be extended smoothly over the puncture z_0. Assume next the image of v to be unbounded and assume that there exists a sequence $s_k\to\infty$ such that $v(s_k,0)=(a_k,m_k)\in\mathbb{R}^+\times B^+$ and $a_k\to\infty$. We claim that $v([R,\infty)\times S^1)\subset\mathbb{R}^+\times B^+$ for some $R>0$. Indeed, arguing indirectly we find a sequence s_k' satisfying $v(s_k',0)\in A$ and, going over to subsequences, we can assume

$$s_k<s_k'<s_{k+1}<s_{k+1}'.$$

Take a smooth function $\tau:\mathbb{R}\to\mathbb{R}$ satisfying $\tau(s)=0$ if $s\le 0$ and $\tau''(s)>0$ for $s>0$. Define the smooth function $H:[0,\infty)\times S^1\to\mathbb{R}$ by

$$\begin{aligned} H(s,t)&=\tau(a(s,t)) \quad \text{if } v(s,t)=(a(s,t),w(s,t))\in\mathbb{R}^+\times B^+\\ H(s,t)&=0 \quad \text{if } v(s,t)\notin\mathbb{R}^+\times B^+. \end{aligned}$$

A straightforward calculation shows that

$$\Delta H(s,t)\ge 0$$

on $\mathbb{R}^+ \times S^1$. Using the uniform gradient bounds for v we have, however,

$$\max\left\{\sup_{S^1} H(s_k', t), \sup_{S^1} H(s_{k+1}', t)\right\} < \inf_{S^1} \ H(s_k, t).$$

This contradicts the maximum principle and proves the claim. Turning to the remaining alternative there is a sequence $s_k \to \infty$ such that $v(s_k, 0) = (a_k, m_k) \in \mathbb{R}^- \times B^-$ and $a_k \to -\infty$. Again the maximum principle shows that $v([R,\infty) \times S^1) \subset \mathbb{R}^- \times B^-$ for some $R > 0$. Since in the unbounded cases, the images $v([R,\infty) \times S^1)$ are completely contained in almost complex cylinders equipped with the $\mathbb{R}$-invariant standard structure $\widetilde{J}$, the remaining statements of the proposition are proved in [24]. □

Remark 2.13 (Uniqueness of the foliation). If λ and λ_1 are two contact forms on B satisfying $d\lambda = d\lambda_1 = \omega_{|B}$, then $d(\lambda - \lambda_1) = 0$, so that $\lambda - \lambda_1$ defines a cohomology class in $H^1(B, \mathbb{R})$. We call λ and λ_1 equivalent if this class vanishes and denote by $[\lambda]$ the equivalence class $[\lambda] = \{\lambda_1 = \text{ contact form on } B \mid \ d\lambda_1 = \omega_{|B} \text{ and } \lambda_1 = \lambda + dh \text{ for } h : B \to \mathbb{R}\}$. We shall show that the equivalence class $[\lambda]$ determines the foliation near B up to a Hamiltonian diffeomorphism of A preserving the leaves.

PROPOSITION 2.14. *Assume that $\Phi^{\pm}$ and $\Phi_1^{\pm}$ are two conformally symplectic foliations near B associated with the equivalent contact forms λ and λ_1. Then there exist an open neighborhood $U \subset A$ of B and a Hamiltonian diffeomorphism $\Psi : A \to A$ which preserves the leaves of the foliation in U:*

$$\Psi : \Phi^{\pm}(\tau \times B^{\pm}) \to \Phi_1^{\pm}(\tau \times B^{\pm}),$$

for $\tau \in [-\varepsilon, 0]$ resp. $\tau \in [0, \varepsilon]$.

Proof. We first prove the statement locally near one of the connected components of the boundary which we assume to be a negative one (for λ) and which we call B. The equivalent contact form $\lambda_1 = \lambda + dh$ satisfies $d\lambda_1 = d\lambda = \omega_{|B}$. If X is the Reeb vector field of λ then $\mathbb{R}X$ is the kernel of $\omega_{|B}$ and hence $\lambda_1(X) > 0$ on B. Therefore, $\lambda_s := \lambda + s dh$, $0 \le s \le 1$, is an arc of contact forms connecting $\lambda_0 = \lambda$ with λ_1. The associated Reeb vector fields are denoted by X_s. Introduce the s-dependent vector field $Y_s := -hX_s$ and denote by ψ_s its flow, satisfying $\frac{d}{ds}\psi_s = Y_s(\psi_s)$ and $\psi_0 = \text{id}$. Then $\frac{d}{ds}(\psi_s^*\lambda_s) = 0$ and hence $\psi_s^*\lambda_s = \lambda_0$ for $0 \le s \le 1$. Defining the family of one-forms $\widehat{\lambda}_s$ on $[0,\varepsilon] \times B$ by

$$\widehat{\lambda}_s(\tau, m)[\alpha, a] = \mathrm{e}^{\tau}\lambda_s(m)[a],$$

$0 \le s \le 1$, we conclude for the induced diffeomorphisms $\widehat{\psi}_s : [0,\varepsilon] \times B \to [0,\varepsilon] \times B$, defined by $\widehat{\psi}_s(\tau, m) = (\tau, \psi_s(m))$, that

$$\widehat{\psi}_s^*\widehat{\lambda}_s = \widehat{\lambda}_0, \quad 0 \le s \le 1.$$

Next we extend the family λ_s of one-forms on B to the family $\lambda_s = \lambda + sd\widehat{h}$ of one-forms on a neighborhood U of B, still satisfying $d\lambda_s = \omega$ on U. Define the family of vector fields η_s on U transversal to B by

$$\omega(\eta_s, \cdot) = \lambda_s, \quad 0 \leq s \leq 1,$$

and denote their flows by φ_t^s so that $\frac{d}{dt}\varphi_t^s = \eta_s(\varphi_t^s)$ and $\varphi_0^s = \mathrm{id}$. As before, the foliations near B are introduced by means of the maps

$$\Phi_s : [0, \varepsilon] \times B \to U, \quad (\tau, m) \mapsto \varphi_\tau^s(m),$$

$0 \leq s \leq 1$. Using $(\varphi_t^s)^* \lambda_s = \mathrm{e}^t \lambda_s$ one verifies that $\Phi_s^* \lambda_s = \widehat{\lambda}_s$. Consequently, the composition $\Psi_s := \Phi_s \circ \widehat{\psi}_s \circ \Phi_0^{-1} : U \to U$ satisfies

$$\Psi_s^* \lambda_s = \lambda_0, \quad 0 \leq s \leq 1. \tag{2.18}$$

Since $d\lambda_s = d\lambda = \omega$ on U, the maps are symplectic, $\Psi_s^* \omega = \omega$. By construction,

$$\Psi_s : \Phi_0(\tau \times B) \to \Phi_s(\tau \times B)$$

for $\tau \in [0, \varepsilon]$ and $0 \leq s \leq 1$. Note that $\Psi_0 = \mathrm{id}$ so that Ψ_s is the flow of the s-dependent vector field Z_s on U,

$$Z_s(x) = \left(\frac{d}{ds}\Psi_s\right) \circ \Psi_s^{-1}(x), \quad x \in U.$$

Differentiating (2.18) we obtain

$$\begin{aligned} 0 = \frac{d}{ds}\Big(\Psi_s^* \lambda_s\Big) &= \Psi_s^*\Big(L_{Z_s}\lambda_s + d\widehat{h}\Big) \\ &= \Psi_s^*\Big(i_{Z_s}\omega + d(i_{Z_s}\lambda_s) + d\widehat{h}\Big). \end{aligned}$$

Hence $i_{Z_s}\omega = -dH_s$, so that Z_s is a time-dependent Hamiltonian vector field. The time-one map Ψ_1 is the desired local Hamiltonian diffeomorphism on U. Carrying out the above construction near every boundary component and extending these local Hamiltonians to a function on A we obtain a time-dependent Hamiltonian vector field whose time-one map Ψ is the desired diffeomorphism. $\square$

2.3. *Embeddings into* $\mathbb{C}P^2$, *the problems* **(V)** *and* **(W)**. We consider a hypersurface $M \subset \mathbb{R}^4$ which is star-like with respect to the origin. In complex notation $\mathbb{C}^2 = \mathbb{R}^4$ it is represented by

$$M = \Big\{z\sqrt{f(z)} \mid z \in S^3\Big\},$$

for a positive smooth function $f : S^3 \to \mathbb{R}$, with the sphere $S^3 = \{z \in \mathbb{C}^2 \mid |z| = 1\}$. Recall the standard symplectic form ω_0 on $\mathbb{C}^2$ (viewed as a real vector space):

$$\omega_0 = \sum_{j=1}^{2} dq_j \wedge dp_j.$$

Then $\omega_0 = d\lambda_0$, with the Liouville form

$$\lambda_0 = \frac{1}{2}\sum_{j=1}^{2}\Big(q_j dp_j - p_j dq_j\Big).$$

The star-like hypersurface M bounds the compact domain $W \subset \mathbb{C}^2$. The boundary $\partial W = M$ of the symplectic manifold (W, ω_0) is of contact type, the contact form on ∂W is $\lambda_{0|\partial W}$. Viewed from the inside, the boundary is convex. Indeed, the Liouville vector field η_0 on $\mathbb{C}^2$, defined by $i_{\eta_0}\omega_0 = \lambda_0$, is equal to $\eta_0(z) = z/2$ and satisfies $L_{\eta_0}\omega_0 = \omega_0$. Moreover, η_0 is transversal to ∂W where it points to the outside of W.

The Hamiltonian flow on M is equivalent to the Reeb flow on S^3 defined by the tight contact form $\lambda = f\lambda_{0|S^3}$. Indeed, the diffeomorphism

$$\varphi : S^3 \to M$$

defined by $\varphi(z) = z\sqrt{f(z)}$ satisfies

$$\varphi^*(\lambda_{0|M}) = f\lambda_{0|S^3}$$

and hence induces an isomorphism between $\ker \omega_{0|M}$ and $\ker d[f\lambda_{0|S^3}]$. Therefore, the Hamiltonian flow on M is equivalent (up to reparametrization of the orbits) to the flow of the Reeb vector field X determined by the contact form $f\lambda_{0|S^3}$. Multiplying f by a positive constant only changes the parametrization of the orbits and hence does not change the structure of the flow and we may assume that $0 < f \leq 1/4$. Hence $W \subset D_{1/2}(0)$, the closed Euclidean ball of radius $1/2$ in $\mathbb{C}^2$.

In the following the complex projective space $\mathbb{C}P^2$ will be equipped with the standard symplectic structure ω related to the Fubini-Study metric. There is a symplectic diffeomorphism

$$\Psi : (B_1(0), \omega_0) \to \Big(\mathbb{C}P^2 \setminus \mathbb{C}P^1, \omega\Big),$$

where $B_1(0) \subset \mathbb{C}^2$ is the open unit ball and where $\mathbb{C}P^1$ is the sphere at ∞ so that $\mathbb{C}P^2 = \mathbb{C}^2 \cup \mathbb{C}P^1$. In homogeneous coordinates of $\mathbb{C}P^2$ the map Ψ is given by

$$\Psi(z_1, z_2) = \Big[z_1, z_2, \sqrt{1 - |z_1|^2 - |z_2|^2}\Big].$$

We shall denote the sphere at infinity by S_∞; in the homogeneous coordinates above it corresponds to $[z_1, z_2, 0]$. Identifying for convenience the symplectomorphic manifolds under consideration (by means of Ψ) we have the following inclusions

$$M = \partial W \subset W \subset D_{\frac{1}{2}}(0) \subset B_1(0) \subset \mathbb{C}P^2.$$

Introduce now $V \subset \mathbb{C}P^2$,

$$V = \mathbb{C}P^2 \setminus \ \mathrm{int}(W). \tag{2.19}$$

The compact symplectic manifold (V, ω) has a concave contact type boundary $\partial V = M$. Moreover,

$$V \cap W = M, \qquad V \cup W = \mathbb{C}P^2, \qquad S_\infty \subset \operatorname{int}(V). \tag{2.20}$$

In view of the symplectic embedding Ψ we have on an open neighborhood U of M in $\mathbb{C}P^2$, a distinguished one-form λ and a vector field η transversal to M satisfying $d\lambda = \omega$ and $i_\eta \omega = \lambda$. We use the flow φ_t of the transversal vector field η to define for ε sufficiently small the embedding

$$\Phi : [-\varepsilon, \varepsilon] \times M \to U \subset \mathbb{C}P^2 \tag{2.21}$$

by setting $\Phi(t, m) = \varphi_t(m)$, and denote its restrictions by

$$\Phi^+ : [-\varepsilon, 0] \times M \to W \quad \text{ and } \quad \Phi^- : [0, \varepsilon] \times M \to V.$$

In the notation of Section 2.2,

$$[\partial V]^+ = [\partial W]^- = \emptyset \quad \text{ and } \quad [\partial W]^+ = [\partial V]^- = M.$$

We observe that $\lambda_{|M}$ is a contact form on M satisfying $d(\lambda_{|M}) = \omega_{|M}$. Since M is diffeomorphic to S^3 we conclude $[\tau] = [\lambda_{|M}]$, for every contact form τ on M satisfying $d\tau = \omega_{|M}$.

Recalling the construction in Section 2.2 we shall glue half-cylinders over the boundaries of V and W and define the associated compatible almost complex structures. Identifying $(M, \lambda_{0|M})$ with $(S^3, f\lambda_{0|S^3})$ we first fix an almost complex structure $J : \xi \to \xi$ on the contact bundle defined by λ on M. We choose J so that the associated $\mathbb{R}$-invariant almost complex structure $\widetilde{J}$ on $\mathbb{R} \times M$ is generic, i.e., $\widetilde{J} \in \Xi$ (Section 2.1).

We first glue a cylinder over the boundary M of W in order to obtain a symplectic noncompact manifold $(\widetilde{W}, \omega_\varphi)$ without boundary. Set $\widehat{W} := W \setminus \Phi^+\Big((-\varepsilon/2, 0] \times M\Big)$, take the disjoint union

$$\widehat{W} \cup \Big([-\varepsilon, \infty) \times M\Big)$$

and identify the points as follows:

$$\Phi^+\Big([-\varepsilon, -\varepsilon/2] \times M\Big) \equiv [-\varepsilon, -\varepsilon/2] \times M.$$

On $\widetilde{W}$ we choose an admissible, almost complex structure $\widehat{J}$ which agrees on the cylindrical end $[-\varepsilon, \infty) \times M$ with the $\mathbb{R}$-invariant structure $\widetilde{J}$. Recall the set Σ^+ from Section 2.2. Then the closed two-form ω_φ on $\widetilde{W}$, defined by (2.17) in Section 2.2 (with $\widehat{W}$ replacing $\widehat{A}$), is a symplectic form. If, in addition $\lim_{s \to \infty} \varphi(s) = 1$, then

$$(\widetilde{W}, \omega_\varphi) \quad \text{ and } \quad (W \setminus \partial W, \omega) \tag{2.22}$$

are symplectomorphic by Lemma 2.10 in Section 2.2. Similarly, a half cylinder is glued over the boundary M of V in order to obtain the symplectic manifold $(\widetilde{V}, \omega_\varphi)$. Here $\widehat{V} := V \setminus \Phi^-([0, \varepsilon/2) \times M)$ and in the disjoint union

$$\widehat{V} \cup (-\infty, \varepsilon] \times M$$

the points are identified by means of $\Phi^-([\varepsilon/2, \varepsilon] \times M) \equiv [\varepsilon/2, \varepsilon] \times M$. The symplectic forms ω_φ on $\widetilde{V}$ for $\varphi \in \Sigma^-$ are defined as above. If $\lim_{s\to-\infty} \varphi(s) = 1$, then

$$(\widetilde{V}, \omega_\varphi) \quad \text{and} \quad (V \setminus \partial V, \omega) \tag{2.23}$$

are symplectomorphic.

Later on we shall study in detail generalized finite energy spheres in $\widetilde{W}$ and $\widetilde{V}$ referred to as problem (W) and problem (V). By definition, a generalized finite energy sphere in $\widetilde{W}$, (respectively $\widetilde{V}$), is a smooth map

$$u : S^2 \setminus \Gamma \to \widetilde{W} \quad (\text{ resp. } \widetilde{V})$$

solving the equation

$$Tu \circ i = \widehat{J} \circ Tu$$

and satisfying the estimate

$$E(u) := \sup_{\varphi\in\Sigma^+} \int_{S^2\setminus\Gamma} u^*\omega_\varphi < \infty$$

(resp. $\varphi \in \Sigma^-$).

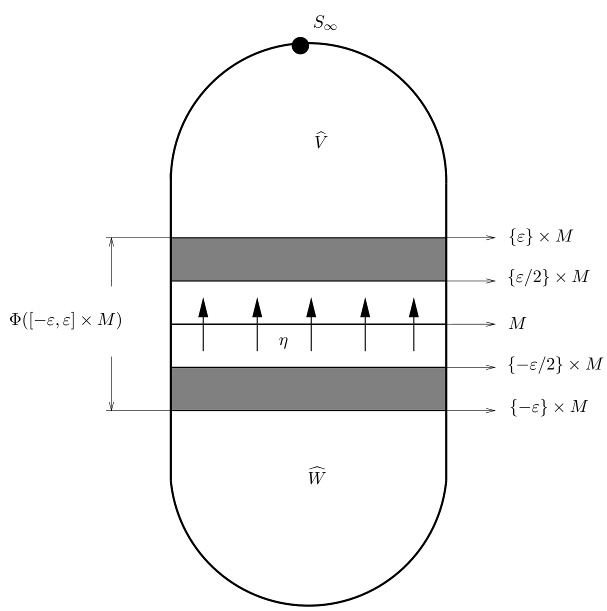

Figure 5. Construction of the compact symplectic manifold (A_R, ω_φ).

In order to change the almost complex structure in an open neighborhood of $M \subset \mathbb{C}P^2$ we add a neck to M in $\mathbb{C}P^2$ and define the compact symplectic manifold (A_R, ω_φ) as follows. We take

$$\widehat{V} \cup \widehat{W} = \mathbb{C}P^2 \setminus \Phi\Big((-\varepsilon/2, \varepsilon/2) \times M\Big)$$

and identify in the disjoint union $(\widehat{V} \cup \widehat{W}) \cup [-R-\varepsilon, R+\varepsilon] \times M$ the points as follows:

$$\begin{aligned} u = \Phi(s,m) &\equiv (s-R, m) \quad \text{if } (s,m) \in [-\varepsilon, -\varepsilon/2] \times M, \\ u = \Phi(s,m) &\equiv (s+R, m) \quad \text{if } (s,m) \in [\varepsilon/2, \varepsilon] \times M. \end{aligned} \tag{2.24}$$

On A_R we take the almost complex structure $\widehat{J}_R$ defined by $\widehat{J}_R = \widetilde{J}$ on $[-R-\varepsilon, R+\varepsilon] \times M$ and $\widehat{J}_R = \widehat{J}$ on $\widehat{V} \cup \widehat{W}$. These structures are compatible in view of the identification (2.24).

If $R \geq 0$, the set Σ_R consists of all smooth functions $\varphi : [-R-\varepsilon, R+\varepsilon] \to (0, \infty)$ satisfying

$$\varphi'(s) \geq 0$$

and

$$\varphi(s) = \begin{cases} e^{s+R} & \text{if } s \in [-R-\varepsilon, -R-\varepsilon/2] \\ e^{s-R} & \text{if } s \in [R+\varepsilon/2, R+\varepsilon]. \end{cases}$$

By C_R we shall abbreviate the set of all closed two-forms ω_φ on A_R, with $\varphi \in \Sigma_R$, defined by

$$\begin{aligned} \omega_\varphi &= \omega && \text{on } \mathbb{C}P^2 \setminus \Phi\Big((-\varepsilon/2, \varepsilon/2) \times M\Big), \\ \omega_\varphi &= d(\varphi\lambda) && \text{on } [-R-\varepsilon, R+\varepsilon] \times M. \end{aligned}$$

If $\varphi \in \Sigma_R$ has a positive derivative, the two-form ω_φ is a symplectic form on A_R compatible with $\widehat{J}_R$ and the symplectic manifolds

$$(A_R, \omega_\varphi) \quad \text{and} \quad (\mathbb{C}P^2, \omega) \tag{2.25}$$

are symplectomorphic. Indeed, observe that $\varphi(s) = e^{\beta(s)}$ with a function $\beta : [-R-\varepsilon, R+\varepsilon] \to [-\varepsilon, \varepsilon]$ satisfying $\beta' > 0$ and $\beta(s) = s + R$ if $s \in [-R-\varepsilon, -R-\varepsilon/2]$ and $\beta(s) = s - R$ if $s \in [R+\varepsilon/2, R+\varepsilon]$. Define the map $\Phi : [-R-\varepsilon, R+\varepsilon] \times M \to \mathbb{C}P^2$ by $\Phi(s,m) = \varphi_{\beta(s)}(m)$, where φ_t is the flow of the transversal vector field η. Then $\Phi^*\omega = d(\varphi\lambda)$. Hence setting $\Phi_{|\widehat{V} \cup \widehat{W}} = \text{id}$, the map $\Phi : A_R \to \mathbb{C}P^2$ is the desired diffeomorphism.

The energy of a $\widehat{J}_R$-holomorphic sphere $\widetilde{u} : S^2 \to A_R$ is defined by

$$E(\widetilde{u}) := \sup_{\varphi \in \Sigma_R} \int_{S^2} \widetilde{u}^* \omega_\varphi$$

and satisfies

$$E(\widetilde{u}) = \int_{S^2} \widetilde{u}^* \omega_\varphi$$

for every $\varphi \in \Sigma_R$ as is readily verified. Later on it will be convenient that the energy can be computed for the special functions φ_R defined as follows. Pick a smooth function

$$\varphi_0 : [-\varepsilon, \varepsilon] \to \mathbb{R}$$

satisfying

$$\varphi_0(s) = \begin{cases} e^s & \text{if } s \in [-\varepsilon, -\varepsilon/2] \\ 1 & \text{if } s \in [-\varepsilon/4, \varepsilon/4] \\ e^s & \text{if } s \in [\varepsilon/2, \varepsilon] \end{cases}$$

and

$$\varphi_0'(s) \geq 0.$$

If $R > 0$, the function φ_R is defined by

$$\varphi_R(s) = \begin{cases} \varphi_0(s+R) & \text{if } s \in [-R-\varepsilon, -R] \\ 1 & \text{if } s \in [-R, R] \\ \varphi_0(s-R) & \text{if } s \in [R, R+\varepsilon]. \end{cases}$$

We associate with every $R \geq 0$ the closed two-form ω_R on A_R which is defined as above by:

$$\begin{aligned} \omega_R &= \omega && \text{on } \widehat{V} \cup \widehat{W}, \\ \omega_R &= d(\varphi_R \lambda) && \text{on } [-R-\varepsilon, R+\varepsilon] \times M. \end{aligned} \tag{2.26}$$

The two-form ω_R is compatible with $\widehat{J}_R$. The energy of a $\widehat{J}_R$-holomorphic sphere $\widetilde{u} : S^2 \to A_R$ becomes

$$E(\widetilde{u}) = \int_{S^2} \widetilde{u}^* \omega_R.$$

2.4. *Pseudoholomorphic spheres in* $\mathbb{C}P^2$. In this section we recall some well known results about pseudoholomorphic spheres in $\mathbb{C}P^2$, referring to [23], [41], [43], [44], [52], [53] for more details. We consider $\mathbb{C}P^2$ equipped with its standard symplectic form ω related to the Fubini-Study metric. The almost complex structures J on $\mathbb{C}P^2$ considered are compatible with ω and, moreover, equal to i in a neighborhood of $\mathbb{C}P^1 \subset \mathbb{C}P^2$. We consider J-holomorphic spheres $u : S^2 \to \mathbb{C}P^2$ which are homologous to $\mathbb{C}P^1 \subset \mathbb{C}P^2$. Their virtual genus is defined by

$$\widetilde{g}(u) = 1 + \frac{1}{2}\Big[C \cdot C - c(u)\Big],$$

where $C = u(S^2)$, and $C \cdot C$ is the self-intersection number of the sphere. The latter is equal to 1. Moreover, $c(u) = c_1\Big(u^*(T\mathbb{C}P^2, J)\Big)[S^2]$ is the first Chern number which is equal to 3. As a special case of the adjunction formula for pseudoholmomorphic curves in symplectic 4-manifolds we have

$$\widetilde{g}(u) \geq \text{ genus}(S^2) = 0$$

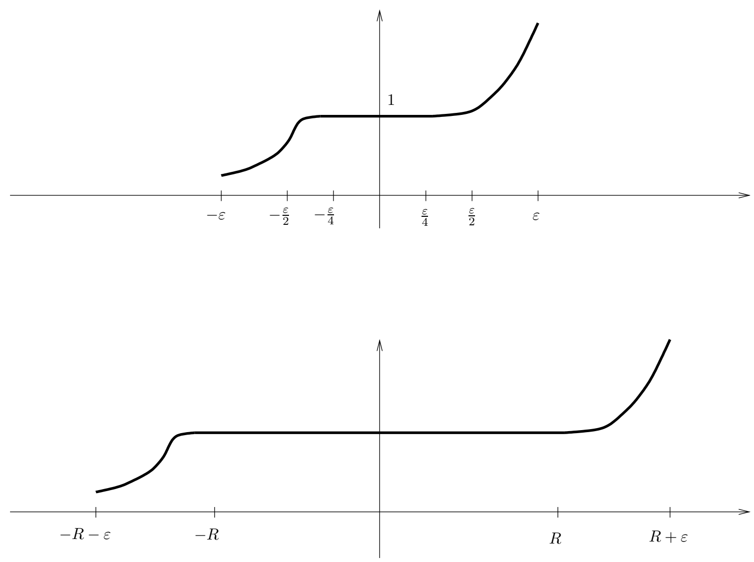

Figure 6. The functions φ_0 and φ_R.

with equality if and only if u is an embedding; see [43]. In our case, $\widetilde{g}(u) = 0$, so that the pseudoholomorphic spheres homologous to $\mathbb{C}P^1$ are all embedded. Hence neighboring (unparametrized) pseudoholomorphic spheres $S^2 \to \mathbb{C}P^2$ can be viewed as sections of the complex normal bundle of $u(S^2)$ satisfying a nonlinear first order elliptic system, whose linearization L_u at the zero section is a Cauchy-Riemann-type operator. Its Fredholm index is, by the Atiyah-Singer index theorem, equal to

$$\mathrm{Ind}(L_u) = 2 + 2c_N(u),$$

with the Chern number $c_N(u)$ of the normal bundle. The total Chern number $c(u)$ is the sum of the normal Chern number and the Chern number of the tangent bundle of $u(S^2)$. Since the latter is equal to 2 we have $c(u) = 3 = 2 + c_N(u)$ and hence $c_N(u) = 1$. Consequently,

$$\mathrm{Ind}(L_u) = 4. \tag{2.27}$$

The results in [27] imply that the Fredholm operator L_u is surjective so that for every J we have genericity automatically. In the parametrized description the family of pseudoholomorphic curves is 10-dimensional since the reparametrization group of S^2 is the (real) 6-dimensional Möbius group. We also note that u defines a generator of $H_2(\mathbb{C}P^2) = \mathbb{Z}$. The space of unparametrized pseudoholomorphic spheres homologous to $\mathbb{C}P^1$ is compact. This follows from the fact that bubbling off is not possible since there is no smaller class with positive ω-area other than $\mathbb{C}P^1$. Moreover, any two pseudoholomorphic spheres homologous to $\mathbb{C}P^1$ are either identical or have precisely one transversal intersection point. Indeed, they are homotopic and therefore their intersection number is equal to the self-intersection number which is equal to 1. Recall that the intersection number is always positive.

In view of (2.27) and the surjectivity the Cauchy-Riemann type operator, L_u has a 4-dimensional kernel K. Consider a nonvanishing section $h \in K$. By the similarity principle the zeros of h are isolated and of positive index. The sum of the indices has to be the Chern number $c_N(u) = 1$ of the normal bundle of $u(S^2)$. Consequently, a nontrivial section of the normal bundle in K is transversal to the zero section and has one positive intersection with the zero section of the normal bundle. Let h_j, $1 \leq j \leq 4$ be a basis of the kernel K. Take two different points z_0 and z_1 of S^2, and define the map $\Phi : \mathbb{R}^4 \to N_{u(z_1)} \oplus N_{u(z_2)}$ by

$$\Phi(\lambda_1, \ldots, \lambda_4) = \Big(\sum \lambda_j h_j(z_0), \sum \lambda_j h_j(z_1)\Big).$$

The map Φ is injective since otherwise we would have a nontrivial section in K with two zeros, which in view of the positivity of the index contradicts the fact that the intersection index is equal to 1.

Given two different points p and $q \in \mathbb{C}P^2$ there exists at most one curve passing through these points. Indeed, if there were two such curves, their intersection number would be greater than or equal to 2. Assume now a given curve passes through the points p and q, $p \neq q$. Then we obtain curves passing through nearby points p' and q' by means of the implicit function theorem using the property that the map Φ is an isomorphism. Fix now $p \neq q$ in $\mathbb{C}P^2$ and take a homotopy $J(s)$ from i to J through admissible almost complex structures. For the structure i there is a unique curve $\mathbb{C}P^1$ connecting p with q. Now changing the parameter one obtains a 1-parameter family of $J(s)$-pseudoholomorphic spheres homologous to $\mathbb{C}P^1$ and passing through p and q, by means of the implicit function theorem. For s fixed the curve is unique; moreover, it depends smoothly on p and q.

If two J-holomorphic curves homologous to $\mathbb{C}P^1$ have a tangential intersection they must agree, since otherwise they would have an intersection number greater than or equal to 2. From this it follows that a point q and a complex line in $T_q(\mathbb{C}P^2)$ determine a unique pseudoholomorphic curve homologous to $\mathbb{C}P^1$ through the point q whose tangent space agrees with the complex line.

We apply these results to the symplectic manifolds (A_N, ω_N) which are symplectomorphic to $(\mathbb{C}P^2, \omega)$ and which are equipped with the compatible almost complex structure $\widehat{J}_N$ introduced in the previous section. We denote by $\mathcal{M}_N = \mathcal{M}_{\widehat{J}_N}$ the space of unparametrized $\widehat{J}_N$-holomorphic spheres in $A_N \equiv \mathbb{C}P^2$ homologous to $\mathbb{C}P^1$. We may identify an element of $\mathcal{M}_N$ with the image of a representative which is an embedded $\widehat{J}_N$-pseudoholomorphic sphere. Summing up part of the above discussion we have:

THEOREM 2.15. *Assume $N \in \mathbb{N}$ is fixed and consider A_N. Given C_1 and C_2 in $\mathcal{M}_N$, either $C_1 = C_2$ or C_1 and C_2 have one point of intersection, and the intersection index is equal to* 1. *In particular, the intersection is transversal. In addition, given two different points q_1 and q_2 in A_N, there exists precisely one curve in $\mathcal{M}_N$ containing q_1 and q_2.*

3. Stretching the neck

We denote by $\mathcal{M}_N = \mathcal{M}_{\widehat{J}_N}$ the moduli space of unparametrized $\widehat{J}_N$-holomorphic spheres in $A_N \equiv \mathbb{C}P^2$ homologous to $\mathbb{C}P^1 \subset \mathbb{C}P^2$. We may identify an element of $\mathcal{M}_N$ with the image of a representative which is an embedded $\widehat{J}_N$-holomorphic sphere. We shall abbreviate by S_∞ the sphere at infinity. It is also $\widehat{J}_N$-holomorphic; recall that near the sphere at infinity all the almost complex structures under consideration coincide and are equal to i. We now fix a point $o_\infty \in S_\infty$ on the sphere at infinity, and denote by $\mathcal{M}_N^0$ the subset of all elements in $\mathcal{M}_N$ consisting of spheres different from S_∞ which intersect S_∞ at the point o_∞. Hence the elements of $\mathcal{M}_N^0$ intersect S_∞ in precisely one point, namely o_∞. The intersection index is equal to 1.

The main part of the desired finite energy foliation for M will be the result of a bubbling off analysis applied to $\mathcal{M}_N^0$ in a limiting process as $N \to \infty$.

We choose $(r, m) \in \mathbb{R} \times M$ and denote, for $N \geq |r|$, by C_N the unique sphere in $\mathcal{M}_N^0$ containing the point (r, m). In the following we shall describe the decomposition of a suitable subsequence of (C_N) into solutions of the Problems (M), (V) and (W).

If $\mathcal{C} = (C_N)$ is the above distinguished sequence in $\mathcal{M}_N^0$ and if $\varphi : \mathbb{N} \to \mathbb{N}$ is injective and monotone we shall abbreviate by $\mathcal{C}^\varphi$ the subsequence $(C_{\varphi(j)})_{j \in \mathbb{N}}$.

Let $w_N : S^2 \to A_N$ be a $\widehat{J}_N$-holomorphic parametrization of $C_N = w_N(S^2)$. Setting $S^2 = \mathbb{C} \cup \{\infty\}$ we normalize the parametrization as follows. We require that

$$w_N(\infty) = o_\infty, \tag{3.1}$$

and either $w_N(0) \in \widehat{W}$, or the point $w_N(0)$ belongs to $[-N, N] \times M$ and has the lowest possible $\mathbb{R}$-value. In addition, we require, denoting by D the closed unit disc in $\mathbb{C}$, that

$$\int_D w_N^* \omega_N = \pi - \gamma, \tag{3.2}$$

with the closed two-form ω_N introduced in (2.26) and with a number $0 < \gamma < \pi$. Recall that, the sphere being homologous to $\mathbb{C}P^1 \subset \mathbb{C}P^2$,

$$\int_{S^2} w_N^* \omega_N = \int_{\mathbb{C}P^1} \omega = \pi.$$

The number γ is chosen as follows. It is smaller than every period T of periodic solutions of the Reeb vector field X_λ on M. Moreover, γ is smaller than the minimum of all numbers $|T - T'|$, where T and T' are mutually different periods of periodic orbits of X_λ, the periods not exceeding π. Finally, γ is smaller than the area of a compact $\widehat{J}_N$-holomorphic curve through a point in S_∞ having its boundary outside of a fixed open neighborhood U of S_∞. This neighborhood U is assumed to be so small that it does not intersect the image of the map Φ defined in (2.21). We observe that on U the almost complex structure $\widehat{J}_N$ is independent of N if U is small enough. The existence of such an area bound is guaranteed by Gromov's isoperimetric inequality; see [23].

We choose a Riemannian metric g_N on A_N which is independent of N on the piece $\widehat{V} \cup \widehat{W}$ and which is translation invariant on $[-N, N] \times M$ inducing on every $\{c\} \times M \equiv M$ the same metric g. On S^2 we choose the Fubini-Study metric. Let D_δ denote the closed disc centered at the origin in $\mathbb{C}$ of radius δ.

LEMMA 3.1. *For any $\delta > 1$ there exists a constant C_δ such that*

$$|Tw_N(z)| \le C_\delta$$

for every $z \in S^2 \setminus D_\delta$ and $N \ge |r|$. Moreover, for any injective monotonic map $\varphi : \mathbb{N} \to \mathbb{N}$ and a sequence $(z_j) \subset S^2$ such that $|Tw_{\varphi(j)}(z_j)| \to \infty$ and for every $\varepsilon > 0$

$$\liminf_{j \to \infty} \int_{D_\varepsilon(z_j)} w_{\varphi(j)}^* \omega_{\varphi(j)} > \gamma.$$

Proof. In order to prove the first part of Lemma 3.1 we argue indirectly and assume that there exists a converging sequence $z_j \to z_0$ for $z_0 \notin D$ satisfying $R_j := |Tw_{\varphi(j)}(z_j)| \to \infty$ for some injective monotonic function φ. We choose a sequence $\rho_j \to 0$ such that still

$$|Tw_{\varphi(j)}(z_j)|\rho_j \to \infty.$$

By the arguments in [24, Lemma 26] we can assume that, in addition,

$$\Big|Tw_{\varphi(j)}(z)\Big| \le 2\Big|Tw_{\varphi(j)}(z_j)\Big| \quad \text{if} \quad \Big|z - z_j\Big| \le \rho_j,$$

after replacing z_j by a sequence still converging to z_0 and replacing ρ_j by another sequence converging to 0. We have to distinguish the following three cases:

1. There exists a subsequence such that

$$w_{\varphi \circ \psi(j)}(z_{\psi(j)}) = (r_j, m_j) \in [-\varphi \circ \psi(j), \varphi \circ \psi(j)] \times M \quad \text{and}$$
$$\min\{|r_j - \varphi \circ \psi(j)|, |r_j + \varphi \circ \psi(j)|\} \to \infty.$$

2. There exist a subsequence and $R > 0$ such that

$$w_{\varphi\circ\psi(j)}(z_{\psi(j)}) \in \widehat{W} \cup ([-\varepsilon - \varphi\circ\psi(j), R - \varphi\circ\psi(j)] \times M).$$

3. There exist a subsequence and $R > 0$ such that

$$w_{\varphi\circ\psi(j)}(z_{\psi(j)}) \in \widehat{V} \cup ([\varphi\circ\psi(j) - R, \varepsilon + \varphi\circ\psi(j)] \times M).$$

Simplifying the notation we set $\widetilde{u}_j(z) = w_{\varphi\circ\psi(j)}(z)$ and $z_j = z_{\psi(j)}$, and $\omega_j = \omega_{\varphi\circ\psi(j)}$. Considering first the case 1 we recall the standard bubbling off analysis from [24]. In a small neighborhood of z_j we have $\widetilde{u}_j = (a_j, u_j) \in \mathbb{R}\times M$. Introduce

$$\delta_j = \inf\Big\{\delta > 0 \mid \text{ there exists } \zeta_j \text{ with } |\zeta_j - z_j| = \delta \text{ and } a_j(\zeta_j) = \pm\varphi\circ\psi(j)\Big\}.$$

Note that if $\delta_j \le \rho_j$, then

$$\begin{aligned} |a_j(\zeta_j) - a_j(z_j)| &= |a_j(\zeta_j) - r_j| \le \left[\int_0^1 |\nabla a_j(\tau\zeta_j + (1-\tau)z_j)| d\tau\right] |\zeta_j - z_j| \\ &\le 2R_j\delta_j. \end{aligned}$$

Since the left side is equal to either $|r_j - \varphi\circ\psi(j)|$ or $|r_j + \varphi\circ\psi(j)|$ we conclude that $R_j\delta_j \to \infty$. Abbreviate $\varepsilon_j = \min\{\rho_j, \delta_j\}$. The mappings $\widetilde{u}_j = (a_j, u_j) : D_{\varepsilon_j}(z_j) \to \mathbb{R}\times M$ solve the equations

$$\partial_s\widetilde{u}_j + \widetilde{J}(\widetilde{u}_j)\partial_t\widetilde{u}_j = 0 \quad \text{ on } D_{\varepsilon_j}(z_j),$$

where $\widetilde{J}$ is the $\mathbb{R}$-invariant almost complex structure on $\mathbb{R}\times M$ and $z = s + \mathrm{i}t$. Moreover,

$$\begin{gathered} z_j \to z_0 \\ R_j = |T\widetilde{u}_j(z_j)| \to \infty \quad \text{ and } \quad R_j\varepsilon_j \to \infty, \\ |T\widetilde{u}_j(z)| \le 2|T\widetilde{u}_j(z_j)| \quad \text{ if } |z - z_j| \le \varepsilon_j. \end{gathered}$$

Rescaling the sequence in increasingly smaller neighborhoods of z_j we define for $z \in D_{R_j\varepsilon_j}$ the mappings

$$\widetilde{v}_j(z) = (b_j(z), v_j(z)) = \left(a_j\left(z_j + \frac{z}{R_j}\right) - a_j(z_j), u_j\left(z_j + \frac{z}{R_j}\right)\right).$$

Since $\widetilde{J}$ is $\mathbb{R}$-invariant, the sequence $\widetilde{v}_j$ has the properties

$$\begin{gathered} \partial_s\widetilde{v}_j + \widetilde{J}(\widetilde{v}_j)\partial_t\widetilde{v}_j = 0, \\ |T\widetilde{v}_j(z)| \le 2 \quad \text{ on } D_{R_j\varepsilon_j}, \qquad |T\widetilde{v}_j(0)| = 1. \end{gathered}$$

In view of these gradient bounds we conclude, by the Arzela-Ascoli theorem, that

$$\widetilde{v}_j \to \widetilde{v} \quad \text{ in } C^\infty_{\mathrm{loc}}(\mathbb{C}, \mathbb{R}\times M)$$

for a nonvanishing map $\widetilde{v}=(a,v):\mathbb{C}\to\mathbb{R}\times M$ solving

$$\partial_s\widetilde{v}+\widetilde{J}(\widetilde{v})\partial_t\widetilde{v}=0 \quad \text{on } \mathbb{C}.$$

From the definition of the energy $E(\widetilde{u}_j)$ it follows immediately that

$$E(\widetilde{v})\le\pi.$$

Hence the map $\widetilde{v}$ is a nontrivial finite energy plane in $\mathbb{R}\times M$. In order to estimate its $d\lambda$-energy we note that it was obtained by rescaling maps defined on small discs with area not exceeding γ (this follows from the normalization condition (3.2) and the fact that $z_0\notin D$). Hence

$$\int_{\mathbb{C}} v^*d\lambda\le\gamma.$$

Since $\widetilde{v}$ is nontrivial, the M-part v is asymptotic to a periodic orbit with period $0<T\le\gamma$. This, however, contradicts the definition of γ being smaller than any period T of X_λ and shows that the first case cannot occur.

Next we consider the second case. Identifying $[-\varepsilon-N,R-N]$ with $[-\varepsilon,R]$ we represent the map $\widetilde{u}_j$ by $\widetilde{u}_j=(a_j,u_j)$ whenever $\widetilde{u}_j(z)\in\mathbb{R}\times M$. Define $\delta_j=\inf\{\delta>0\mid \text{ there exists } \zeta_j \text{ with } |\zeta_j-z_j|=\delta \text{ and } a_j(\zeta_j)=\varphi\circ\psi(j)\}$ and set $\varepsilon_j=\min\{\rho_j,\delta_j\}$. Then the sequence of mappings $\widetilde{u}_j:D_{\varepsilon_j}(z_j)\to\widehat{W}\cup([-\varepsilon,\varphi\circ\psi(j)]\times M)$ solves the equations

$$\widehat{J}(\widetilde{u}_j)\circ T\widetilde{u}_j=T\widetilde{u}_j\circ i \quad \text{on } D_{\varepsilon_j}(z_j).$$

Moreover,

$$\begin{gathered} z_j\to z_0,\\ R_j=|T\widetilde{u}_j(z_j)|\to\infty \quad \text{and} \quad R_j\varepsilon_j\to\infty,\\ |T\widetilde{u}_j(z)|\le 2|T\widetilde{u}_j(z_j)| \quad \text{if } |z-z_j|\le\varepsilon_j. \end{gathered}$$

Rescaling the sequence $\widetilde{u}_j$ we define the mappings

$$\widetilde{v}_j(z)=\widetilde{u}_j\left(z_j+\frac{z}{R_j}\right)$$

for $z\in D_{R_j\varepsilon_j}$. At any point $z\in D_{R_j\varepsilon_j}$ satisfying $\widetilde{v}_j(z)\in[-\varepsilon,\infty)\times M$,

$$\widetilde{v}_j(z)=(b_j(z),v_j(z))=\left(a_j\left(z_j+\frac{z}{R_j}\right),u_j\left(z_j+\frac{z}{R_j}\right)\right).$$

The sequence $\widetilde{v}_j$ has the following properties on $D_{R_j\varepsilon_j}$:

$$\widehat{J}(\widetilde{v}_j)\circ T\widetilde{v}_j=T\widetilde{v}_j\circ i,\quad |T\widetilde{v}_j(z)|\le 2,\quad |T\widetilde{v}_j(0)|=1.$$

Since $\widetilde{v}_j(0)\in\widehat{W}\cup([-\varepsilon,R]\times M)$, the sequence $\widetilde{v}_j$ is, in view of the gradient bounds, uniformly bounded on compact subsets of $\mathbb{C}$. By applying the Arzela-

Ascoli theorem we obtain a nonconstant generalized finite energy plane $\widetilde{v}$: $\mathbb{C} \to \widetilde{W} := \widehat{W} \cup \Big([-\varepsilon, \infty) \times M\Big)$,

$$\widetilde{v}_j \to \widetilde{v} \quad \text{in } C^\infty_{\text{loc}}\Big(\mathbb{C}, \widehat{W} \cup ([-\varepsilon, \infty) \times M)\Big).$$

If $\widetilde{v}(\mathbb{C}) \subset \widehat{W} \cup \Big([-\varepsilon, L) \times M\Big)$ for some $L > 0$, then the puncture at ∞ is removable. In this case $\widetilde{v}$ can be extended smoothly over ∞ to the pseudoholomorphic map $\widetilde{v} : S^2 \to \widehat{W} \cup ([-\varepsilon, L] \times M)$. We will show that $\widetilde{v}$ is constant in this case, contradicting $|T\widetilde{v}(0)| = 1$. Indeed, ω_φ is exact on $\widehat{W} \cup \Big([-\varepsilon, L] \times M\Big)$ and therefore

$$\int_{S^2} \widetilde{v}^* \omega_\varphi = 0$$

implying that $\widetilde{v} =$ constant, contradicting $|T\widetilde{v}(0)| = 1$. The contradiction shows that the puncture at ∞ is not removable. Arguing as in Proposition 2.11 (Section 2.2) we have

$$\widetilde{v}(z) \in [0, \infty) \times M$$

for $|z|$ large. Since its energy is finite, $\widetilde{v}$ is asymptotic to a T- periodic solution of X_λ implying

$$\int_{\mathbb{C}} \widetilde{v}^* \omega_\varphi = \lim_{s \to \infty} \int_{S^1} v(\mathrm{e}^{2\pi(s+\mathrm{i}t)})^* \lambda = T \tag{3.3}$$

for every $\varphi \in \Sigma^+$ satisfying $\lim_{s \to \infty} \varphi(s) = 1$. However, the bubbling off analysis and the normalization show that the first term in (3.3) is bounded above by γ. Hence $T \le \gamma$. This contradicts the definition of γ and shows that the second case cannot occur either.

Next we exclude case 3. Identifying $[-R + N, \varepsilon + N]$ with $[-R, \varepsilon]$ we obtain in this case a nonconstant generalized finite energy plane $\widetilde{v} : \mathbb{C} \to \widetilde{V} = \widehat{V} \cup \Big((-\infty, \varepsilon] \times M\Big)$, where the points in $[\varepsilon/2, \varepsilon] \times M$ are identified with points in $\widehat{V}$ via the map Φ^-. As a consequence of the bubbling off analysis together with the normalization condition we again find for the energy of $\widetilde{v}$ the estimate

$$0 < \int_{\mathbb{C}} \widetilde{v}^* \omega_\varphi \le \gamma \tag{3.4}$$

for every $\varphi \in \Sigma^-$ satisfying $\lim_{s \to -\infty} \varphi(s) = 1$. If the puncture ∞ was removable, the map $\widetilde{v}$ would compactify to a nonconstant holomorphic sphere. This would imply that the above integral would have a value of at least π which is not possible. Consequently $\widetilde{v}$ has a nonremovable puncture at ∞, which has to be negative, since $\widehat{V}$ has a concave boundary. By Proposition 2.11,

$$\widetilde{v}(z) \in (-\infty, 0] \times M \tag{3.5}$$

for $|z|$ large. If $\widetilde{v}(\mathbb{C})$ does not intersect the sphere S_∞ it belongs to the complement of S_∞ in $\widetilde{V}$. Note that the symplectic forms ω_φ are exact on $\widetilde{V} \setminus S_\infty$. By

Stokes' theorem we obtain, in view of (3.5) and using the asymptotic behavior described in Proposition 2.11,

$$\int_{\mathbb{C}} \widetilde{v}^* \omega_\varphi = -|T|,$$

contradicting (3.4). So, $\widetilde{v}(\mathbb{C})$ must intersect the sphere at infinity. The intersection number cannot exceed 1, since all the intersection numbers of C_N with this sphere are equal to 1. In view of (3.5) we therefore conclude from the last property of γ that

$$\int_{\mathbb{C}} \widetilde{v}^* \omega_\varphi > \gamma,$$

in contradiction to (3.4). This finishes the proof of the first part of Lemma 3.1.

Next we proof the second part of the lemma. We again carry out a bubbling off analysis for the three different cases. This time $z_j \to z_0 \in D$ because of the first part. In case 1 we obtain a finite energy plane in $\mathbb{R} \times M$. In the second case we obtain a generalized finite energy plane in $\widetilde{W}$ and $\widetilde{V}$ respectively. The energies of these planes are, in view of the asymptotic behavior and the property $\gamma < T$, estimated as follows:

In the first case: $$\int_{\mathbb{C}} \widetilde{v}^* d\lambda > \gamma.$$

In the second case: $$\int_{\mathbb{C}} \widetilde{v}^* \omega_{\widetilde{W}} > \gamma.$$

In the third case: $$\int_{\mathbb{C}} \widetilde{v}^* \omega_{\widetilde{V}} > \gamma.$$

Here, $\omega_{\widetilde{W}} = \omega_\varphi$ for every $\varphi \in \Sigma^+$ satisfying $\lim_{s\to\infty} \varphi(s) = 1$ and $\omega_{\widetilde{V}} = \omega_\varphi$ for $\varphi \in \Sigma^-$ satisfying $\lim_{s\to-\infty} \varphi(s) = 1$. The desired estimates in the lemma now follow, unwinding the scaling in the bubbling off analysis from the estimates (in the short notation from above, for j sufficiently large)

$$\int_{D_R} \widetilde{v}_j^* \omega_j \leq \int_{D_{R_j \varepsilon_j}} \widetilde{v}_j^* \omega_j = \int_{B_{\varepsilon_j}(z_j)} \widetilde{u}_j^* \omega_j \leq \int_{B_\varepsilon(z_j)} \widetilde{u}_j^* \omega_j.$$

The left hand side converges to the energy of the corresponding finite energy plane as $j \to \infty$ and then as $R \to \infty$. The proof of Lemma 3.1 is complete. □

A point $z_0 \in S^2$ is called a bubbling off point of the sequence $w_{\varphi(j)}$ of mappings if there exists a converging sequence $z_j \to z_0$ satisfying $|Tw_{\varphi(j)}(z_j)| \to \infty$ as $j \to \infty$. The proof of the previous lemma shows that a sequence admits at most finitely many bubbling off points.

LEMMA 3.2. *Assume that $\varphi : \mathbb{N} \to \mathbb{N}$ is an injective monotonic map. Assume that $w_{\varphi(j)} : S^2 \to A_{\varphi(j)}$ is $\widehat{J}_{\varphi(j)}$-holomorphic and homologous to $\mathbb{C}P^1$, so that it parametrizes an element in $\mathcal{M}_{\varphi(j)}$. Suppose there exist sequences (z_j^i) for $i = 1, \ldots, m$ with $z_j^i \to z_0^i$ for $j \to \infty$, where the z_0^i are mutually different, so that $|Tw_{\varphi(j)}(z_j^i)| \to \infty$. Then the following estimate holds*:

$$m \cdot \gamma \leq \pi.$$

Proof. A bubbling off analysis shows that every bubbling off point takes a certain amount of energy away. This is $d\lambda$-energy in case 1, $\omega_{\widetilde{V}}$-energy in the second and $\omega_{\widetilde{W}}$-energy in the third case. All these energies exceed γ. Choose $\varepsilon > 0$ such that ε-discs around the points z_0^i are disjoint. Then for j sufficiently large

$$\pi = \int_{S^2} w_{\varphi(j)}^* \omega_{\varphi(j)} \geq \sum_{i=1}^m \int_{D_\varepsilon(z_0^i)} w_{\varphi(j)}^* \omega_{\varphi(j)}.$$

Taking the lim inf as $j \to \infty$, the right-hand side is, by Lemma 3.1, greater or equal to $m\gamma$ hence proving the lemma. □

We recall from [24] that on a compact almost complex manifold, gradient bounds of a solution of our elliptic system imply C^∞-bounds. In the case at hand the target manifolds A_N depend on a parameter and their diameters converge to ∞ as $N \to \infty$. Nevertheless gradient bounds imply C^∞-bounds in our case also since the metrics g_N are, by construction, quite special. In order to estimate the derivatives we embed $\widehat{V}$, $\widehat{W}$ into some $\mathbb{R}^m$ and M into $\mathbb{R}^{m-1}$ for m sufficiently large in such a way that $[-N, N] \times M$ lies in $\mathbb{R} \times \mathbb{R}^{m-1} = \mathbb{R}^m$ for every N. If $w : S^2 \to A_N$ is a smooth map and $p \in S^2$, then $w(p) \in \widehat{W} \setminus \partial \widehat{W}$ or $w(p) \in \widehat{V} \setminus \partial \widehat{V}$ or $w(p) \in (-\varepsilon - N, \varepsilon + N) \times M$. Using charts we may assume that a neighborhood of p is mapped into $\mathbb{R}^m$ and we can apply the differential operators $D^\alpha = \partial_s^{\alpha_1} \partial_t^{\alpha_2}$, where $s + it$ are suitable holomorphic coordinates near $p \in S^2$.

Using these particular embeddings, the higher order (≥ 1) derivatives of a sequence w_j near a point are understood in the usual sense. Since our system is elliptic we conclude from Lemma 3.1, in particular:

LEMMA 3.3. *Given any $\delta > 1$, the derivatives of order at least 1 of the sequence $w_N : S^2 \to A_N$ are uniformly bounded on $S^2 \setminus D_\delta$.*

Assume now that the sequence $w_j : S^2 \to A_j$ has the finite set $\Gamma = \{z^1, \ldots, z^m\}$ of bubbling off points. Then, in view of Lemma 3.3, $\Gamma \subset D$. In the complement of every open neighborhood of $\Gamma \subset S^2$ we have C^∞ bounds. If $z \notin \Gamma$ then $w_j(z) \notin \widehat{W}$ for all j sufficiently large. Indeed, arguing by contradiction we assume that $w_{\varphi(j)}(z) \in \widehat{W}$. We find a path in S^2 connecting z with ∞ and avoiding the points in Γ. Since the gradients are uniformly bounded, the images of the path under the mappings $w_{\varphi(j)}$ must have a uniformly bounded length. By assumption $w_{\varphi(j)}(\infty) = o_\infty$ for all j. The distance between $w_{\varphi(j)}(z)$ and o_∞ must exceed the distance between (r, m) and o_∞, and by the choice of the metric $\operatorname{dist}_N((r, m), o_\infty) \to \infty$ as $N \to \infty$. This contradiction proves our claim that $w_j(z) \notin \widehat{W}$ if $z \notin \Gamma$ for large j. Since $w_j(\infty) = o_\infty \in \widehat{V}$ we can identify $[-j, j] \times M$ with $[-2j, 0] \times M$ and conclude for a subsequence

$$w_{\varphi(j)} \to w_\infty \quad \text{in } C^\infty_{\text{loc}}\Big(S^2 \setminus \Gamma, \widetilde{V}\Big).$$

In particular, $w_\infty(\infty) = o_\infty$. The map $w_\infty : S^2 \setminus \Gamma \to \widetilde{V}$ is a generalized finite energy sphere and our next aim is to show that it is not constant. We first observe that the set Γ of bubbling off points of the sequence $w_{\varphi(j)}$ is not empty.

Lemma 3.4. *The set $\Gamma \subset S^2$ contains* 0.

Proof. Assume $0 \notin \Gamma$, then we find a path in S^2 connecting 0 with ∞ and avoiding the points in Γ. The image of the path under the mapping $w_{\varphi(j)}$ must have a uniformly bounded length, since the gradients are uniformly bounded. By assumption, $w_{\varphi(j)}(0)$ is either in $\widehat{W}$ or has an $\mathbb{R}$-component lower than r. We see that $\operatorname{dist}_{\varphi(j)}(w_{\varphi(j)}(0), o_\infty) \geq \operatorname{dist}_{\varphi(j)}((r, m), o_\infty)$. However, $\operatorname{dist}_{\varphi(j)}((r, m), o_\infty) \to \infty$ as $j \to \infty$. This contradiction proves the assertion. ∎

Up to this point we have found a subsequence $\mathcal{C}^\varphi = (C_{\varphi(j)})$ of $\mathcal{C}$, suitable parametrizations $w_{\varphi(j)} : S^2 \to A_{\varphi(j)}$ of $C_{\varphi(j)}$ with a nonempty set Γ of bubbling off points containing 0 such that $w_{\varphi(j)} \to w_\infty$ in $C^\infty_{\text{loc}}(S^2 \setminus \Gamma, \widetilde{V})$ and $w_\infty(\infty) = o_\infty$. Moreover, for $\varepsilon > 0$ sufficiently small and $z \in \Gamma$

$$\liminf_{j \to \infty} \int_{D_\varepsilon(z)} w^*_{\varphi(j)} \omega_{\varphi(j)} > \gamma, \tag{3.6}$$

in view of Lemma 3.1.

Lemma 3.5. *The generalized finite energy sphere $w_\infty : S^2 \setminus \Gamma \to \widetilde{V}$ is not constant.*

Proof. Arguing by contradiction we assume $w_\infty(z) = p$ for all $z \in S^2 \setminus \Gamma$. We choose $\varepsilon > 0$ so small that the closed ε-discs around the points in Γ are mutually disjoint. Let $z \in \Gamma$ and recall the estimate (3.6) on $D_\varepsilon(z)$. On the boundary $S_\varepsilon(z) = \partial D_\varepsilon(z)$, the map $w_{\varphi(j)}$ converges to the constant map p in C^∞. Hence we can extend $w_{\varphi(j)|D_\varepsilon(z)}$ to a smooth map $e_{z,j} : S^2 \to A_{\varphi(j)}$ satisfying

$$\int_{S^2 \setminus D_\varepsilon(z)} e^*_{z,j} \omega_{\varphi(j)} \leq \sigma_j \quad \text{and} \quad \sigma_j \to 0.$$

For j large the spheres $e_{z,j}$ have the positive area

$$\int_{S^2} e^*_{z,j} \omega_{\varphi(j)} > 0,$$

in view of (3.6). This implies that this integral has a value which is a positive integer multiple of π. Consequently,

$$\begin{aligned}\pi &= \int_{S^2} w^*_{\varphi(j)}\omega_{\varphi(j)} \geq \sum_{z\in\Gamma}\int_{D_\varepsilon(z)} w^*_{\varphi(j)}\omega_{\varphi(j)} \\ &= \sum_{z\in\Gamma}\int_{S^2} e^*_{z,j}\omega_{\varphi(j)} - \sum_{z\in\Gamma}\int_{S^2\setminus D_\varepsilon(z)} e^*_{z,j}\omega_{\varphi(j)} \\ &\geq \sharp\Gamma\ \pi - \sharp\Gamma\ \sigma_j.\end{aligned}$$

Therefore, $\sharp\Gamma \leq 1$ if w_∞ is constant, and so $\Gamma = \{0\}$ by Lemma 3.4. Consequently, $w_{\varphi(j)}$ converges in C^∞ on $S^2 \setminus D_{1/2}$ to the constant map $p \in \widetilde{V}$. By normalization,

$$\gamma = \int_{S^2\setminus D} w^*_{\varphi(j)}\omega_{\varphi(j)}.$$

Taking the limit as $j \to \infty$ we find $\gamma = 0$. This contradicts the definition of γ and Lemma 3.5 is proved. □

LEMMA 3.6. *The punctures* Γ *of* $w_\infty : S^2 \setminus \Gamma \to \widetilde{V}$ *are nonremovable. There exists* $\varepsilon > 0$ *such that*

$$w_\infty\Big(D_\varepsilon(\Gamma)\setminus\Gamma\Big) \subset \mathbb{R}^- \times M.$$

Proof. In view of Proposition 2.11 it is enough to show that the punctures are not removable. Since $w_\infty \not\equiv$ constant we know that

$$\int_{S^2\setminus\Gamma} w^*_\infty\omega_{\widetilde{V}} > 0. \tag{3.7}$$

Given $\delta > 0$ we find $\varepsilon_0 = \varepsilon_0(\delta)$ and an integer $j(\delta, \varepsilon_0)$ so that

$$\int_{S^2\setminus D_\varepsilon(\Gamma)} w^*_{\varphi(j)}\omega_{\varphi(j)} \geq \int_{S^2\setminus\Gamma} w^*_\infty\omega_{\widetilde{V}} - \delta$$

provided that $\varepsilon \in (0, \varepsilon_0)$ and $j \geq j(\delta, \varepsilon_0)$. Assume that the bubbling off point $z \in \Gamma$ is a removable puncture of w_∞ and hence that w_∞ is defined on $D_\varepsilon(z)$. We know from (3.6) that

$$\liminf_{j\to\infty}\int_{D_\varepsilon(z)} w^*_{\varphi(j)}\omega_{\varphi(j)} \geq \gamma,$$

for every sufficiently small $\varepsilon > 0$. Moreover, for a given σ and $\varepsilon > 0$ small enough

$$\int_{D_\varepsilon(z)} w^*_\infty\omega_{\widetilde{V}} \leq \sigma.$$

For j large we may slightly deform $w_{\varphi(j)}$ near the boundary (in C^∞) of $D_\varepsilon(z)$ so that we can glue it to $w_{\infty|D_\varepsilon(z)}$ and obtain a map $e_j : S^2 \to A_{\varphi(j)}$ with

$\int_{S^2} e_j^* \omega_{\varphi(j)}$ having the value exceeding $\gamma - 2\sigma$. Hence the value of $\int_{S^2} e_j^* \omega_{\varphi(j)}$ must be at least π. This shows that

$$\lim_{j\to\infty} \int_{D_\varepsilon(z)} w_{\varphi(j)}^* \omega_{\varphi(j)} \geq \pi$$

for every sufficiently small $\varepsilon > 0$. Hence

$$\begin{aligned} \pi &= \int_{S^2} w_{\varphi(j)}^* \omega_{\varphi(j)} = \int_{S^2\setminus D_\varepsilon(\Gamma)} w_{\varphi(j)}^* \omega_{\varphi(j)} + \sum_{e\in\Gamma} \int_{D_\varepsilon(e)} w_{\varphi(j)}^* \omega_{\varphi(j)} \\ &\geq \int_{S^2\setminus\Gamma} w_\infty^* \omega_{\widetilde{V}} - \delta + \int_{D_\varepsilon(z)} w_{\varphi(j)}^* \omega_{\varphi(j)}. \end{aligned}$$

Therefore, passing to the limit as $j \to \infty$, we have

$$\pi \geq \int_{S^2\setminus\Gamma} w_\infty^* \omega_{\widetilde{V}} - \delta + \pi.$$

This holds true for every $\delta > 0$. Hence

$$\int_{S^2\setminus\Gamma} w_\infty^* \omega_{\widetilde{V}} = 0,$$

contradicting (3.7). Therefore, the punctures are nonremovable. Since the map w_∞ admits negative punctures only, the second statement of the lemma follows from the asymptotics near the negative punctures. The proof of the lemma is complete. □

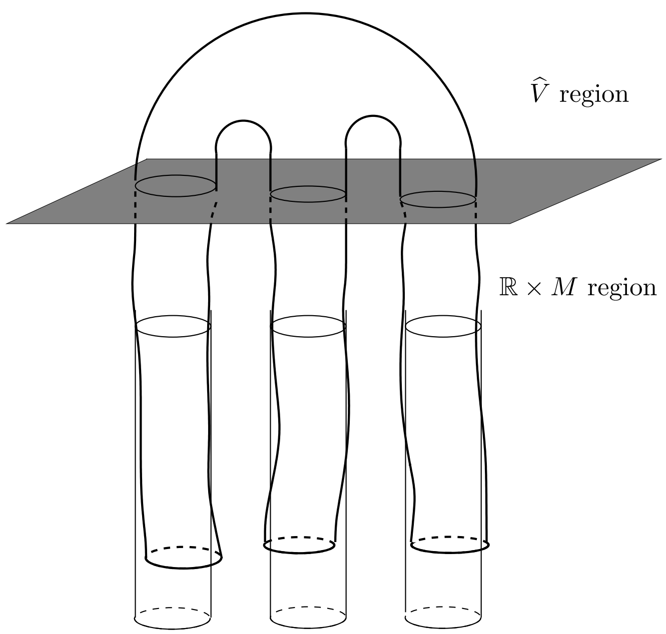

Figure 7. The finite energy sphere w_∞. In this figure there are three negative punctures.

We next introduce the mass $m(\omega_\infty, z_0)$ of a puncture $z_0 \in \Gamma$. For $\varepsilon > 0$ small enough we first set

$$m_\varepsilon(z_0) = \lim_{j\to\infty} \int_{D_\varepsilon(z_0)} w^*_{\varphi(j)}\omega_{\varphi(j)}.$$

In order to show that this limit does exist for ε small we note that the closed two-form ω_N is exact on $A_N \setminus S_\infty$. In addition, $\omega_N = d\lambda_N$ for a one-form satisfying $\lambda_N = \lambda$ on $[-N, N] \times M$ viewing λ as a form on $\mathbb{R} \times M$. Moreover, the restriction of λ_N onto $\widehat{W} \cup (\widehat{V} \setminus S_\infty)$ does not depend on N. Identifying $[-2N, 0] \times M$ with $[-N, N] \times M$ we conclude from Lemma 3.6 for sufficiently small $\varepsilon > 0$ and $j \geq j(\varepsilon)$ that

$$w_{\varphi(j)}(\partial D_\varepsilon(z_0)) \subset [-\varphi(j), \varphi(j)] \times M.$$

Therefore,

$$\int_{D_\varepsilon(z_0)} w^*_{\varphi(j)}\omega_{\varphi(j)} = \int_{\partial D_\varepsilon(z_0)} w^*_{\varphi(j)}\lambda_{\varphi(j)} = \int_{\partial D_\varepsilon(z_0)} w^*_{\varphi(j)}\lambda.$$

On $\partial D_\varepsilon(z_0)$ the maps $w_{\varphi(j)}$ converge in C^∞ to the map w_∞ so that the last integral converges as $j \to \infty$ to

$$\int_{\partial D_\varepsilon(z_0)} w^*_\infty\lambda = m_\varepsilon(z_0).$$

We also see that $m_\varepsilon(z_0)$ is decreasing as $\varepsilon \to 0^+$, which allows us to define:

Definition 3.7. *The mass of* w_∞ *at the puncture* $z_0 \in \Gamma$ is the real number

$$m(w_\infty, z_0) = \lim_{\varepsilon\to 0^+} m_\varepsilon(z_0).$$

In view of the estimate in Lemma 3.3 we have $\lim_{j\to\infty} \int_{D_\varepsilon(z_0)} w^*_{\varphi(j)}\omega_{\varphi(j)} > \gamma$, and so,

$$\gamma < m(w_\infty, z_0) < \pi.$$

Taking the limit $j \to \infty$ and then $\varepsilon \to 0^+$ in

$$\pi = \int_{S^2\setminus\bigcup_{z\in\Gamma} D_\varepsilon(z)} w^*_{\varphi(j)}\omega_{\varphi(j)} + \sum_{z\in\Gamma}\int_{D_\varepsilon(z)} w^*_{\varphi(j)}\omega_{\varphi(j)},$$

and recalling the definitions we obtain the following formula for the energy.

Lemma 3.8. *If* $w_\infty : S^2 \setminus \Gamma \to \widetilde{V}$ *is the above generalized finite energy sphere,*

$$\pi = \int_{S^2\setminus\Gamma} w^*_\infty\omega_{\widetilde{V}} + \sum_{z\in\Gamma} m(w_\infty, z).$$

The next lemma describes more precisely which part of A_N a full neighborhood of bubbling off points is mapped into under the mappings $w_{\varphi(j)} : S^2 \to A_{\varphi(j)}$.

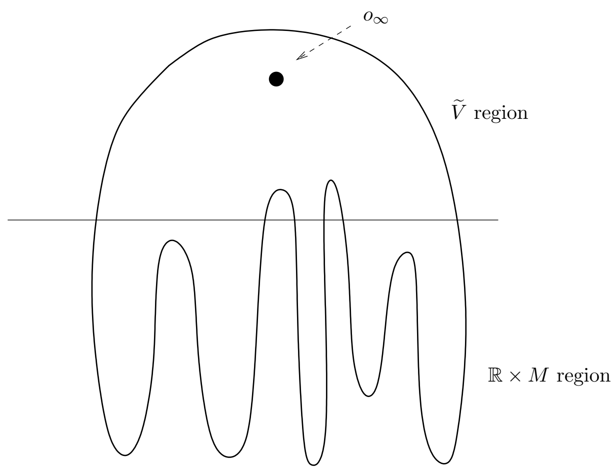

Figure 8. Near the punctures in Γ the images of $w_{\varphi(j)}$ are contained in $\widehat{W}\cup\Big([-\varphi(j),\varphi(j)]\times M\Big)$.

LEMMA 3.9. *If $z^i\in\Gamma$, then for every $\varepsilon>0$ small and $j\geq j(\varepsilon)$,*

$$w_{\varphi(j)}\Big(D_\varepsilon(z^i)\Big)\subset\widehat{W}\cup\Big([-\varphi(j),\varphi(j)]\times M\Big).$$

Proof. We fix the bubbling off point $z^i\in\Gamma\subset S^2$ of the sequence of $\widehat{J}_{\varphi(j)}$-holomorphic spheres $w_{\varphi(j)}:S^2\to A_{\varphi(j)}$. Arguing by contradiction we find (possibly going over to a subsequence denoted by the old letters) a converging sequence $\zeta_j\to z^i$ satisfying $w_{\varphi(j)}(\zeta_j)\in\widehat{V}$. For the embedded spheres $C_{\varphi(j)}$ we now choose different holomorphic parametrizations and define the maps $v_j:S^2\to A_{\varphi(j)}$ by

$$v_j(z)=w_{\varphi(j)}(\tau_j(z)),$$

where the Möbius transformations $\tau_j:S^2\to S^2$ are chosen such that the following normalization conditions are met

$$\begin{aligned} v_j(S^2) &= C_{\varphi(j)}, \quad v_j(0)=w_{\varphi(j)}(\zeta_j),\\ v_j(\infty) &= o_\infty, \quad \int_D v_j^*\omega_{\varphi(j)}=\gamma. \end{aligned}\tag{3.8}$$

In particular, $\tau_j(0)=\zeta_j$. Carrying out a bubbling off analysis for the sequence v_j we denote by $\Gamma'\subset S^2$ the finite set of its bubbling off points. Then $\Gamma'\subset S^2\setminus\operatorname{int}(D)$ in view of the last condition in (3.8). In particular, $0\notin\Gamma'$. Away from Γ' we have uniform gradient bounds. We use them in order to show that $v_j(z)\notin\widehat{W}$ if $z\notin\Gamma'$ provided that j is sufficiently large. Indeed, we can connect z with 0 by a path avoiding Γ' so that the lengths of its images under the maps $w_{\varphi(j)}$ are uniformly bounded. However, under the assumption $v_j(0)\in\widehat{V}$, this contradicts $\text{dist}_{\varphi(j)}(v_j(0), v_j(z)) \to \infty$ as $j \to \infty$ if $v_j(z) \in \widehat{W}$. Consequently, a subsequence converges,

$$v_j \to v_\infty \quad \text{in } C^\infty_{\text{loc}}(S^2 \setminus \Gamma', \widetilde{V}).$$

In particular, $v_\infty(0) \in \widehat{V}$. We next show that $\Gamma' \neq \emptyset$ by proving that $\infty \in \Gamma'$. We recall for the circles $S_\varepsilon(z^i) = \partial D_\varepsilon(z^i)$ around $z^i \in \Gamma$ that

$$w_{\varphi(j)}(S_\varepsilon(z^i)) \subset \Big[-\varphi(j), \varphi(j)\Big] \times M$$

for every $\varepsilon > 0$ small and $j \geq j(\varepsilon)$. Identifying $[-\varphi(j), \varphi(j)] \times M$ with $[-2\varphi(j), 0] \times M$ we know in view of Lemma 3.6 that given $R > 0$ we can choose $\varepsilon > 0$ small enough so that the $\mathbb{R}$-component of $w_{\varphi(j)|S_\varepsilon(z^i)}$ is smaller than $-R$ for $j \geq j(\varepsilon)$. Since $\tau_j(\infty) = \infty$, $\tau_j(0) = \zeta_j$, and $\zeta_j \to z^i$ we deduce from the fact that the mass at z^i exceeds γ that the circles $\tau_j^{-1}(S_\varepsilon(z^i))$ $\tau_j^{-1}(z) \to \infty$ converge uniformly to ∞ as $j \to \infty$ as $j \to \infty$. This follows from the fourth condition in (3.8). Since, depending on ε, the $\mathbb{R}$-component of $v_{j|\tau_j^{-1}(S_\varepsilon(z^i))}$ in $\widetilde{V}$ becomes arbitrarily negative and $v_j(\infty) = o_\infty$ we conclude the existence of a sequence $z_j \to \infty$ on S^2 satisfying $|Tv_j(z_j)| \to \infty$. Hence $\infty \in \Gamma'$ as claimed, and by the arguments of Lemma 3.4 (with $0 \in \Gamma$ replaced by $\infty \in \Gamma'$ and with the different parametrization), the generalized finite energy sphere $v_\infty : S^2 \setminus \Gamma' \to \widetilde{V}$ is not constant and has a positive $\omega_{\widetilde{V}}$-energy. Its punctures Γ' are all negative.

Assume now that v_∞ hits S_∞. This intersection cannot be isolated since otherwise $\mathcal{C}_{\varphi(j)}$ would have (together with $\infty \in S^2$) at least two intersections for j sufficiently large. Consequently, by the similarity principle (see [1]) the image of v_∞, denoted by C, is contained in S_∞. In this case all punctures are removable so that C is equal to S_∞ with finitely many points removed. In particular, C has $\omega_{\widetilde{V}}$-energy equal to π. However, the mass of every puncture is at least $\gamma > 0$ and in view of Lemma 3.8 the $\omega_{\widetilde{V}}$-energy of C is strictly smaller than π. This contradiction shows that v_∞ does not intersect S_∞. The closure of its image lies in the complement of S_∞ in $\widetilde{V}$. If all the punctures Γ' are removable we can compactify v_∞ to a sphere in $\widetilde{V} \setminus S_\infty$ of positive $\omega_{\widetilde{V}}$-energy. On the other hand, the two-form $\omega_{\widetilde{V}}$ is exact in the complement of S_∞; hence the $\omega_{\widetilde{V}}$-energy vanishes. This contradiction shows that not all the punctures are removable.

The nonremovable punctures, still denoted by Γ', are all negative. So, we obtain by Stokes' theorem using the asymptotic behavior of v_∞ near the punctures as described in Proposition 2.11 (Section 2.2),

$$0 < \int_{S^2 \setminus \Gamma'} v_\infty^* \omega_{\widetilde{V}} = -\sum_{z \in \Gamma'} T_z < 0.$$

Here $T_z > 0$ is the period of the periodic solution of X_λ on M associated with the punctures $z \in \Gamma'$. With this contradiction the proof of Lemma 3.9 is complete. □

We summarize the results of the limit procedure reached so far.

THEOREM 3.10. *Choose a point* $(r, m) \in [-N_0, N_0] \times M \subset A_{N_0}$. *Denote by* C_N, *for* $N \geq N_0$, *the unique unparametrized* $\widehat{J}_N$*-holomorphic sphere homologous to* $\mathbb{C}P^1$ *in* A_N *and containing the two points* (r, m) *and* $o_\infty \in S_\infty$. *Consider the* $\widehat{J}_N$*-holomorphic parametrization* $w_N : S^2 = \mathbb{C} \cup \{\infty\} \to C_N$ *satisfying the appropriate normalization conditions, in particular* $w_N(\infty) = o_\infty$. *Then there exists a subsequence* $w_{\varphi(j)}$ *of mappings having the following properties*:

There exists a nonempty finite set $\Gamma \subset S^2$ *of bubbling off points of the sequence* $w_{\varphi(j)}$: ($z \in \Gamma$ *if* $z = \lim z_j$ *and* $|Tw_{\varphi(j)}(z_j)| \to \infty$ *as* $j \to \infty$). *Moreover, if* $\varepsilon > 0$ *is sufficiently small, then for all* $j \geq j(\varepsilon)$ *and* $z \in \Gamma$,

$$w_{\varphi(j)}(D_\varepsilon(z)) \subset \widehat{W} \cup \Big([-\varphi(j), \varphi(j)] \times M\Big).$$

When $[-N, N] \times M$ *is identified with* $[-2N, 0] \times M$ *the sequence* $w_{\varphi(j)}$ *converges*

$$w_{\varphi(j)} \to w_\infty \quad \text{in } C^\infty_{\mathrm{loc}}(S^2 \setminus \Gamma, \widetilde{V})$$

to a nonconstant finite energy sphere $w_\infty : S^2 \setminus \Gamma \to \widetilde{V}$ *passing through* $w_\infty(\infty) = o_\infty$, *having the nonremovable negative punctures* Γ, *and the positive* $\omega_{\widetilde{V}}$*-energy determined by the formula*

$$\pi = \int_{S^2 \setminus \Gamma} w_\infty^* \omega_{\widetilde{V}} + \sum_{z \in \Gamma} m(w_\infty, z).$$

The mass of the puncture $z \in \Gamma$ *satisfies* $\gamma < m_\infty(w_\infty, z) < \pi$. *The behavior near the punctures is as follows. There exists an* $\varepsilon > 0$ *such that*

$$w_\infty\Big(D_\varepsilon(\Gamma) \setminus \Gamma\Big) \subset \mathbb{R}^- \times M \subset \widetilde{V}$$

and if $\varphi : D \subset \mathbb{C} \to U \subset S^2$ *are holomorphic coordinates near* $z \in \Gamma$ *satisfying* $\varphi(0) = z$, *then in holomorphic polar coordinates*

$$w_\infty\Big(\varphi(\mathrm{e}^{-2\pi(s+\mathrm{i}t)})\Big) = \Big(a_\infty(s,t), u_\infty(s,t)\Big)$$

for s *large, and as* $s \to \infty$

$$a_\infty(s,t) \to -\infty$$
$$u_\infty(s,t) \to x(-T_z t)$$

in $C^\infty(\mathbb{R})$. *Here* x *is a periodic solution of* X_λ *on* M *having the period* $T_z > 0$.

4. The bubbling off tree

In Section 3 we introduced the sequence $w_{\varphi(j)}$ of maps converging to the generalized finite energy sphere w_∞ in $\widetilde{V}$:

$$w_{\varphi(j)} \to w_\infty \quad \text{in } C^\infty_{\text{loc}}(S^2 \setminus \Gamma, \widetilde{V}).$$

The nonempty finite set $\Gamma \subset D$ of bubbling off points of the sequence $w_{\varphi(j)}$ consists of the negative punctures of the maps w_∞. In this section we shall look more closely at these punctures by carefully rescaling the maps $w_{\varphi(j)}$ near Γ. Proceeding inductively we shall produce generalized finite energy spheres in $\mathbb{R} \times M$ as well as in $\widetilde{W}$. As it will turn out, it will be convenient to describe the combinatorial structure of all the finite energy spheres produced by means of a graph whose vertices (dots in the figures 9 and 10) have different colors. A black dot will represent a finite energy sphere in $\widetilde{V}$, a gray dot a finite energy sphere in $\mathbb{R} \times M$ and a white dot a finite energy plane in $\widetilde{W}$. We shall draw an edge between two such dots, if the sphere represented by "dot no. 1" has a negative puncture z_0 whose negative asymptotic limit is the periodic solution of (x_0, T_0) and the "dot no. 2" has a positive puncture z_1 with the same asymptotic limit $(x_1, T_1) = (x_0, T_0)$. Figure 9 illustrates a possible configuration. Of course, our construction so far only justifies the configuration illustrated by Figure 10 below.

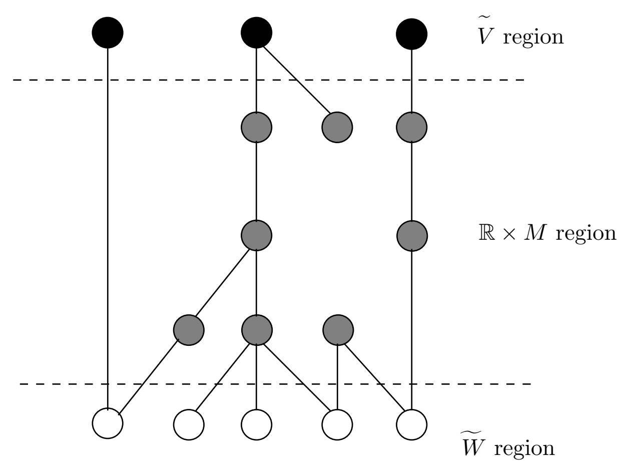

Figure 9. An *a priori* possible graph. All finite energy spheres in $\mathbb{R} \times M$ have either nonzero $d\lambda$-energy or at least two negative punctures.

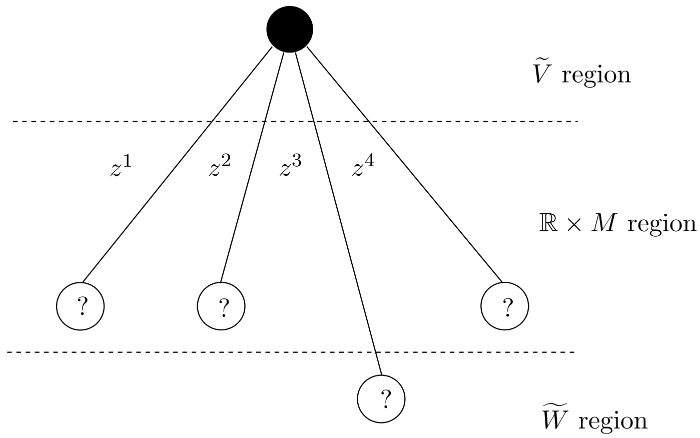

Figure 10. There are four negative punctures but the finer structure near these points is not yet known.

We start our procedure by first collecting more information about the behavior of the sequence $w_{\varphi(j)}$ of maps near the punctures Γ. We fix $z^i \in \Gamma$ and choose $\varepsilon_0 > 0$ so small that $D_{\varepsilon_0}(z^i) \cap D_{\varepsilon_0}(z^k) = \emptyset$ for $z^k \in \Gamma$ with $k \neq i$. From Lemma 3.9 we recall

$$w_{\varphi(j)}\Big(D_{\varepsilon_0}(z^i)\Big) \subset \widehat{W} \cup \Big([-\varphi(j), \varphi(j)] \times M\Big)$$

if ε_0 is sufficiently small and $j \geq j_0(\varepsilon_0)$ large. We now choose the points $\gamma_j \in \overline{D_{\varepsilon_0}(z^i)}$ in the following way: if $w_{\varphi(j)}\Big(D_{\varepsilon_0}(z^i)\Big) \cap \widehat{W} \neq \emptyset$, then $w_{\varphi(j)}(\gamma_j) \in \widehat{W}$ and if $w_{\varphi(j)}\Big(D_{\varepsilon_0}(z^i)\Big) \subset [-\varphi(j), \varphi(j)] \times M$, then $w_{\varphi(j)}(\gamma_j)$ has the smallest $\mathbb{R}$-component among the points in $w_{\varphi(j)}\Big(D_{\varepsilon_0}(z^i)\Big) \cap \Big([-\varphi(j), \varphi(j)] \times M\Big)$.

Lemma 4.1. *The sequence (γ_j) converges to z^i.*

Proof. The asymptotic behavior of $w_\infty : S^2 \setminus \Gamma \to \widetilde{V}$ near the negative puncture $z^i \in \Gamma$ is, according to Proposition 2.11 (Section 2.2), described by

$$w_\infty\Big(z^i + \mathrm{e}^{2\pi(s+\mathrm{i}t)}\Big) \to \Big(T_i s + d, x_i(T_i t)\Big) \in \mathbb{R}^- \times M$$

as $s \to -\infty$, where (x_i, T_i) is the associated periodic solution of X_λ on M. Identifying $[-N, N] \times M$ with $[-2N, 0] \times M$ we deduce that the $\mathbb{R}$-component $a_{\varphi(j)}$ of $w_{\varphi(j)}$ can be made arbitrarily negative by chosing $\varepsilon > 0$ small and j large. But away from z^i the sequence $(a_{\varphi(j)})$ is bounded, so that $\gamma_j \to z^i$ as claimed. □

Fix a number $\sigma_0 \in (0, \gamma)$. Then the following lemma is obvious.

Lemma 4.2. *There exists a sequence $\delta_j \to 0$ satisfying*

$$\int_{D_{\delta_j}(\gamma_j)} w^*_{\varphi(j)} \omega_{\varphi(j)} = m(w_\infty, z^i) - \sigma_0.$$

Next we define the subsets $\Omega_j \subset \mathbb{C}$ by

$$\Omega_j := \Big\{ z \in \mathbb{C} \mid \gamma_j + \delta_j z \in D_{\varepsilon_0}(z^i) \Big\}.$$

Then $D_R \subset \Omega_j$ for $j \geq j(R)$ since $\gamma_j \to z^i$ and $\delta_j \to 0$. With the help of the distinguished parameters γ_j and δ_j we define the rescaled maps $\widetilde{v}_j : \Omega_j \to A_{\varphi(j)}$ by

$$\widetilde{v}_j(z) = w_{\varphi(j)}\Big(\gamma_j + \delta_j z\Big). \tag{4.1}$$

These maps meet the normalization condition

$$\int_D \widetilde{v}_j^* \omega_{\varphi(j)} = \int_{D_{\delta_j}(\gamma_j)} w_{\varphi(j)}^* \omega_{\varphi(j)} = m(w_\infty, z^i) - \sigma_0. \tag{4.2}$$

Given $R > 1$ and $\sigma > 0$ we estimate, using $\delta_j \to 0$ and $\gamma_j \to z^i$,

$$\begin{aligned} \int_{D_R \setminus D} \widetilde{v}_j^* \omega_{\varphi(j)} &= \int_{D_{R\delta_j}(\gamma_j)} w_{\varphi(j)}^* \omega_{\varphi(j)} - \int_D \widetilde{v}_j^* \omega_{\varphi(j)} \\ &= \int_{D_{R\delta_j}(\gamma_j)} w_{\varphi(j)}^* \omega_{\varphi(j)} - m(w_\infty, z^i) + \sigma_0 \\ &\leq \int_{D_\sigma(z^i)} w_{\varphi(j)}^* \omega_{\varphi(j)} - m(w_\infty, z^i) + \sigma_0 \end{aligned}$$

if j is sufficiently large. Consequently,

$$\limsup_{j \to \infty} \int_{D_R \setminus D} \widetilde{v}_j^* \omega_{\varphi(j)} \leq m_\sigma(w_\infty, z^i) - m(w_\infty, z^i) + \sigma_0.$$

The estimate holds true for every $\sigma > 0$ so that

$$\limsup_{j \to \infty} \int_{D_R \setminus D} \widetilde{v}_j^* \omega_{\varphi(j)} \leq \sigma_0 \tag{4.3}$$

for every $R > 1$. Since $\sigma_0 < \gamma$ we conclude that there cannot exist a sequence $z_j \to z_0$, $|z_0| > 1$, satisfying

$$\limsup \ |T\widetilde{v}_j(z_j)| \to \infty.$$

Indeed, by rescaling, such a sequence would allow the construction of a non-trivial finite energy plane for problem (M) or (W) whose energy would be smaller than $\sigma_0 < \gamma$ in view of (4.3). However, the energy of a finite energy plane is equal to the period T of a periodic solution and $T > \gamma$ by definition of γ. This contradiction shows that we have gradient bounds for the sequence $(\widetilde{v}_j)$ on $D_R \setminus D_r$ for $R > r > 1$. From these gradient bounds we obtain the following uniform bounds for the higher order derivatives, in the metrics g_N on A_N.

LEMMA 4.3. *Every subsequence of* $(\widetilde{v}_j)$ *has uniformly bounded derivatives of order at least* 1 *on* $D_R \setminus D_r$ *for every* $R > r > 1$.

Fixing the point $2 \in \mathbb{C}$ we consider next the sequence $\widetilde{v}_j(2)$. By Lemma 3.9 we know that $\widetilde{v}_j(2) \in \widehat{W} \cup \Big([-\varphi(j), \varphi(j)] \times M\Big)$. We distinguish the two cases:

1. There exists $R > 0$ such that a subsequence of $(\widetilde{v}_j(2))$ satisfies

$$\widetilde{v}_j(2) \in \widehat{W} \cup \Big([-\varphi(j), -\varphi(j) + R] \times M\Big). \tag{4.4}$$

2. For j large we have

$$\widetilde{v}_j(2) = (r_j, m_j) \in [-\varphi(j), \varphi(j)] \times M \quad \text{and} \quad |r_j + \varphi(j)| \to \infty. \tag{4.5}$$

We begin with case 1. We denote by Γ^i the finite set of bubbling off points of the sequence $\widetilde{v}_j$. Then $\Gamma^i \subset D$ in view of Lemma 4.3. The notation Γ^i refers to the study of the original sequence $w_{\varphi(j)}$ in a neighborhood of $z^i \in \Gamma$. Identifying $[-N, N] \times M$ with $[0, 2N] \times M$ we find a subsequence $\widetilde{v}_{\psi(j)}$ of $\widetilde{v}_j$ satisfying

$$\widetilde{v}_{\psi(j)} \to \widetilde{v}_\infty \quad \text{in } C^\infty_{\text{loc}}(\mathbb{C} \setminus \Gamma^i, \widetilde{W}). \tag{4.6}$$

Note that $\widetilde{v}_{\psi(j)}$ has the form

$$\widetilde{v}_{\psi(j)} = w_{\varphi \circ \psi(j)} \circ \tau_j,$$

where τ_j is a sequence of Möbius transformations keeping ∞ fixed.

Lemma 4.4. *Considering case* 1 *in* (4.4), *we denote by* $\widetilde{v}_\infty : \mathbb{C} \setminus \Gamma^i \to \widetilde{W}$ *the generalized finite energy sphere defined by* (4.6). *Then* $\Gamma^i = \emptyset$ *so that the only puncture is the one at* ∞. *The map* $\widetilde{v}_\infty : \mathbb{C} \to \widetilde{W}$ *is a nonconstant generalized finite energy plane in* $\widetilde{W}$.

We shall prove later on that the asymptotic limit of $\widetilde{v}_\infty$ at the puncture ∞ coincides with the asymptotic limit of w_∞ at its puncture $z^i \in \Gamma$.

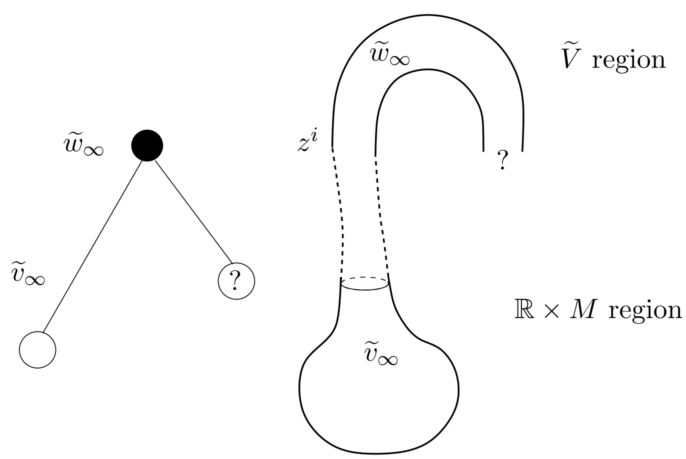

Figure 11. The figure depicts the situation in case 1.

Proof. By construction, $\widetilde{v}_{\psi(j)} \to \widetilde{v}_\infty$ in $C^\infty_{\mathrm{loc}}(\mathbb{C} \setminus \Gamma^i, \widetilde{W})$. Take a smooth function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f \equiv 1 \text{ on } [0,1], \qquad f''(s) > 0 \text{ on } (1,\infty).$$

Define $H : \widetilde{W} \to \mathbb{R}^+$ by

$$H \equiv 1 \text{ on } \widehat{W}, \qquad H(r,m) = f(r) \text{ on } \mathbb{R}^+ \times M,$$

and introduce the functions $\alpha_j : D_2 \to \mathbb{R}$,

$$\alpha_j(z) = H \circ \widetilde{v}_{\psi(j)}(z).$$

Since $\Gamma^i \subset D$, the sequence $\widetilde{v}_{\psi(j)}$ converges on the circle $S_2(0)$ of radius 2 to the map $\widetilde{v}_\infty$. Hence there exists a constant $c > 0$ such that

$$\alpha_j(z) \le c \quad \text{for all } |z| = 2 \text{ and } j \text{ large.}$$

At the point z such that $\widetilde{v}_{\psi(j)}(z) = (a_j, v_j)(z) \in [-\varphi \circ \psi(j), \varphi \circ \psi(j)] \times M$, which we identify with $[0, 2\varphi \circ \psi(j)] \times M$, we calculate

$$\begin{aligned}(\Delta\alpha_j)(z) &= \Big[f''(a_j)\cdot[(\partial_s a_j)^2 + (\partial_t a_j)^2] + f'(a_j)\Delta a_j\Big] \\ &\ge f'(a_j)|\pi\partial_s v_j|^2 \ge 0.\end{aligned}$$

If $\widetilde{v}_{\psi(j)}(z) \in \widehat{W}$, then $\Delta\alpha_j(z) = 0$. Hence the maximum principle implies

$$\sup\ \alpha_{j|D_2} \le c$$

for all j large. If there exists a point $z_0 \in D$ such that $|T\widetilde{v}_{\psi(j)}(z_j)| \to \infty$ for some sequence $z_j \to z_0$, then a bubbling off analysis produces a nontrivial finite energy plane in $\widehat{W} \cup \Big([0,c] \times M\Big)$. Its puncture at infinity is therefore removable and we obtain a nontrivial holomorphic sphere. This is not possible since $\omega_{\widetilde{W}}$ is exact on $\widehat{W} \cup \Big([0,c] \times M\Big)$. Thus $\Gamma^i = \emptyset$ and the lemma is proved. □

We next consider case 2 in (4.5). The situation is illustrated by Figure 12. Later on we shall prove that the negative asymptotic limit of w_∞ at $z^i \in \Gamma$ and the positive asymptotic limit of $\widetilde{v}_\infty$ at ∞ match up.

Lemma 4.5. *Consider case 2 in (4.5) Then $|r_j - \varphi(j)| \to \infty$ as $j \to \infty$. Moreover, there exist a subsequence $\widetilde{v}_{\psi(j)}$ and a sequence $c_{\psi(j)} \in \mathbb{R}$ of constants such that*

$$\widetilde{v}_{\psi(j)} + c_{\psi(j)} \to \widetilde{v}_\infty \quad \textit{in} \quad C^\infty_{\mathrm{loc}}(\mathbb{C} \setminus \Gamma^i, \mathbb{R} \times M).$$

The generalized energy sphere $\widetilde{v}_\infty : \mathbb{C} \setminus \Gamma^i \to \mathbb{R} \times M$ is not constant. Moreover, $0 \in \Gamma^i$ provided $\Gamma^i \neq \emptyset$. The point at ∞ is a nonremovable puncture.

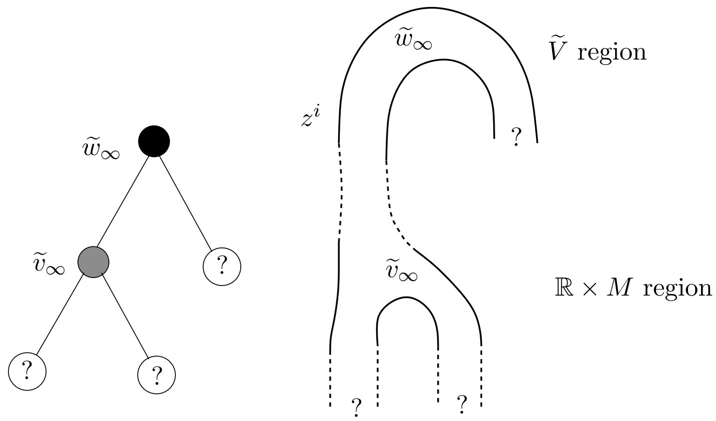

Figure 12. The figure depicts the situation in case 2.

Proof. We first note that under the assumption of this case $\widetilde{v}_j(z) \notin \widehat{W}$ if $z \notin \Gamma^i$ provided that j is sufficiently large. Here Γ^i stands for the set of bubbling off points of the sequence $\widetilde{v}_j$. Indeed, away from Γ^i we have uniform gradient bounds. We can connect z with 2 by a path avoiding Γ^i so that the lengths of its images under the maps $\widetilde{v}_j$ are uniformly bounded. However, in view of the fact that $r_j + \varphi(j) \to \infty$, this contradicts $\operatorname{dist}_{\varphi(j)}\Big(\widetilde{v}_j(2), v_j(z)\Big) \to \infty$ as $j \to \infty$ if $v_j(z) \in \widehat{W}$. Next we prove the first statement. We argue by contradiction and assume (for a subsequence) that

$$r_j - \varphi(j) \to r_\infty \in \mathbb{R}.$$

This implies, after we identify $[-N, N] \times M$ with $[-2N, 0] \times M$ and perhaps take a subsequence, that

$$\widetilde{v}_{\psi(j)} \to \widetilde{v}_\infty \quad \text{in} \ \ C^\infty_{\text{loc}}(\mathbb{C} \setminus \Gamma^i, \mathbb{R} \times M).$$

Since $\Gamma^i \subset D$ we have, in particular,

$$\widetilde{v}_{\psi(j)}(2) \to \widetilde{v}_\infty(2) = (r_\infty, m_\infty).$$

Because of Lemma 3.9 and our assumption,

$$\widetilde{v}_\infty(\mathbb{C} \setminus \Gamma^i) \subset (-\infty, 0] \times M.$$

This implies that the finite energy sphere $\widetilde{v}_\infty$ has no positive puncture. This is impossible unless $\widetilde{v}_\infty$ is constant, because it would imply that the energy is strictly negative. Hence $\widetilde{v}_\infty$ is a constant so that the mass at every puncture in Γ^i is equal to 0. The sequence $\widetilde{v}_{\psi(j)}$ does not permit any gradient blow up on D, because such a blow up would produce a puncture for $\widetilde{v}_\infty$ with a positive mass. Consequently, $\Gamma^i = \emptyset$ and $\widetilde{v}_{\psi(j)} \to \widetilde{v}_\infty$ in $C^\infty_{\text{loc}}(\mathbb{C}, \mathbb{R}^- \times M)$

where $\widetilde{v}_\infty : \mathbb{C} \to \mathbb{R}^- \times M$ is a constant map. We conclude using (4.2)

$$\begin{aligned} 0 < m(w_\infty, z^i) - \sigma_0 &= \int_D \widetilde{v}^*_{\psi(j)} \omega_{\varphi \circ \psi(j)} \\ &= \lim_{j\to\infty} \int_D \widetilde{v}^*_{\psi(j)} \omega_{\varphi \circ \psi(j)} = \int_D \widetilde{v}^*_\infty d\lambda = 0. \end{aligned}$$

Here we viewed $d\lambda$ as a 2-form on $\mathbb{R} \times M$. In fact, frequently we will write λ or $d\lambda$ for the obvious forms induced on $\mathbb{R} \times M$. The above contradiction shows that indeed

$$|r_j - \varphi(j)| \to \infty.$$

Recall now that $\widetilde{v}_{\psi(j)}(2) = (r_j, m_j)$ and assume (r_j) to be bounded, so that a subsequence converges,

$$(r_j, m_j) \to (r_\infty, m_\infty).$$

Hence

$$\widetilde{v}_{\psi(j)} \to \widetilde{v}_\infty \quad \text{in } C^\infty_{\text{loc}}(\mathbb{C} \setminus \Gamma^i, \mathbb{R} \times M).$$

By construction, $\widetilde{v}_{\psi(j)}(0)$ is either in $\widehat{W}$ or has the smallest $\mathbb{R}$-component in $[-\varphi \circ \psi(j), \varphi \circ \psi(j)] \times M$. We shall show that $0 \in \Gamma^i$ if $\Gamma^i \neq \emptyset$. Arguing indirectly, we assume that $\Gamma^i \neq \emptyset$ and $0 \notin \Gamma^i$. We can connect 0 by a path in $\mathbb{C}$ to the point 2 avoiding the points in Γ^i. Along this path the gradients are uniformly bounded so that $\widetilde{v}_{\psi(j)}(0)$ stays within a bounded distance to (r_∞, m_∞). Therefore, $\widetilde{v}_\infty$ has no negative punctures and, moreover, $\widetilde{v}_{\psi(j)}(D_2) \subset [-R, \varphi \circ \psi(j)] \times M$ for some $R > 0$. Setting $\widetilde{v}_{\psi(j)} = (a_j, u_j)$ on D_2 the functions $a_{j|\partial D_2}$ are uniformly bounded in view of Lemma 4.3. Since

$$\Delta a_j = |\pi \partial_s u_j|^2 \geq 0$$

on D_2, the maximum principle gives a uniform bound for $a_{j|D_2}$. Hence the punctures in Γ^i cannot be positive either. So, the punctures being neither positive nor negative are all removable implying as before that $\Gamma^i = \emptyset$ contrary to our assumption. We have proved that $\Gamma^i \neq \emptyset$ implies that $0 \in \Gamma^i$.

If $\Gamma^i = \emptyset$, we can take the limit $j \to \infty$ in the normalization condition (4.2) to obtain

$$\int_D \widetilde{v}^*_\infty d\lambda = m(w_\infty, z^i) - \sigma_0 > 0.$$

Hence $\widetilde{v}_\infty$ is not constant. If $\Gamma^i \neq \emptyset$, then 0 is a nonremovable puncture so that $\widetilde{v}_\infty$ is nonconstant in this case also.

Finally, if (r_j) is not bounded then still

$$\lim_{j\to\infty} \min\Big\{|r_j - \varphi(j)|,\ |r_j + \varphi(j)|\Big\} = \infty,$$

and by a slight modification of the previous argument, namely adding suitable constants to the $\mathbb{R}$-components, we find a subsequence $\widetilde{v}_{\varphi \circ \psi(j)} + c_{\psi(j)}$ converging to $\widetilde{v}_\infty$ in $C^\infty_{\mathrm{loc}}(\mathbb{C} \setminus \Gamma^i, \mathbb{R} \times M)$; moreover, $\widetilde{v}_\infty$ has the desired properties. The proof of the lemma is complete. □

Recall the definition of the mass $m(w_\infty, z^i)$ of the sphere w_∞ at its puncture $z^i \in \Gamma$. We shall define the masses $m(\widetilde{v}_\infty, z^{il})$ of the sphere $\widetilde{v}_\infty$ at its punctures $z^{il} \in \Gamma^i$ if $\Gamma^i \neq \emptyset$ similarly. We take our converging sequence $\widetilde{v}_{\psi(j)} \to \widetilde{v}_\infty$ of maps, set

$$m_\varepsilon(\widetilde{v}_\infty, z^{il}) := \lim_{j \to \infty} \int_{D_\varepsilon(z^{il})} \widetilde{v}^*_{\psi(j)} \omega_{\varphi \circ \psi(j)}$$

for $\varepsilon > 0$ small, and define

$$m(\widetilde{v}_\infty, z^{il}) := \lim_{\varepsilon \to 0} m_\varepsilon(\widetilde{v}_\infty, z^{il}).$$

Recalling $\Gamma^i \subset D$ we deduce from the normalization condition (4.2), the definition of γ and the bubbling off analysis, the estimate

$$\gamma < m(\widetilde{v}_\infty, z^{il}) < m(w_\infty, z^i)$$

for all $z^{il} \in \Gamma^i$.

PROPOSITION 4.6. *If $\widetilde{v}_\infty$ is a solution of problem* (W) (*recall that $\Gamma^i = \emptyset$ in this case*), *then*

$$m(w_\infty, z^i) = \int_{\mathbb{C}} \widetilde{v}^*_\infty \omega_{\widetilde{W}}.$$

If $\widetilde{v}_\infty$ is a solution of problem (M), *then*

$$m(w_\infty, z^i) = \int_{\mathbb{C} \setminus \Gamma^i} \widetilde{v}^*_\infty d\lambda + \sum_{z \in \Gamma^i} m(\widetilde{v}_\infty, z).$$

Proof. Considering the second case we take $z^i \in \Gamma$ and assume $\Gamma^i \neq \emptyset$. Fix $R > 0$ and $\sigma > 0$. Since $\delta_j \to 0$ and $\gamma_j \to z^i$ we find for j sufficiently large,

$$\begin{aligned} \int_{D_R \setminus D_\varepsilon(\Gamma^i)} \widetilde{v}^*_{\psi(j)} \omega_{\varphi \circ \psi(j)} + \sum_{z \in \Gamma^i} \int_{D_\varepsilon(z)} \widetilde{v}^*_{\psi(j)} \omega_{\varphi \circ \psi(j)} &= \int_{D_R} \widetilde{v}^*_{\psi(j)} \omega_{\varphi \circ \psi(j)} \\ &= \int_{D_{R\delta_j}(\gamma_j)} w^*_{\varphi \circ \psi(j)} \omega_{\varphi \circ \psi(j)} \\ &\leq \int_{D_\sigma(z^i)} w^*_{\varphi \circ \psi(j)} \omega_{\varphi \circ \psi(j)}. \end{aligned}$$

Hence, as $j \to \infty$,

$$\int_{D_R \setminus D_\varepsilon(\Gamma^i)} \widetilde{v}^*_\infty d\lambda + \sum_{z \in \Gamma^i} m_\varepsilon(\widetilde{v}_\infty, z) \leq m_\sigma(w_\infty, z^i).$$

Since this holds true for every $\sigma > 0$, $\varepsilon > 0$, and $R > 1$, we deduce

$$\int_{\mathbb{C}\setminus\Gamma^i} \widetilde{v}_\infty^* d\lambda + \sum_{z\in\Gamma^i} m(\widetilde{v}_\infty, z) \leq m(w_\infty, z^i).$$

Similarly, if $\widetilde{v}_\infty$ is a solution of problem (W), which implies $\Gamma^i = \emptyset$, then

$$\int_{\mathbb{C}} \widetilde{v}_\infty^* \omega_{\widetilde{W}} \leq m(w_\infty, z^i).$$

It remains to prove the reversed inequalities.

Denote by $A(r, R)$ the annulus $r \leq |z| \leq R$ in $\mathbb{C}$. Since $\sigma_0 < \gamma$ we deduce from (4.3) the estimates

$$\int_{A(r,R)} v_j^* \omega_{\varphi(j)} \leq \gamma$$

for all $1 \leq r \leq R$ and j sufficiently large. Recalling the definition of the constant γ, for large j the maps $\widetilde{v}_j$ meet the hypotheses of the following lemma from [31], illustrated by Figure 13.

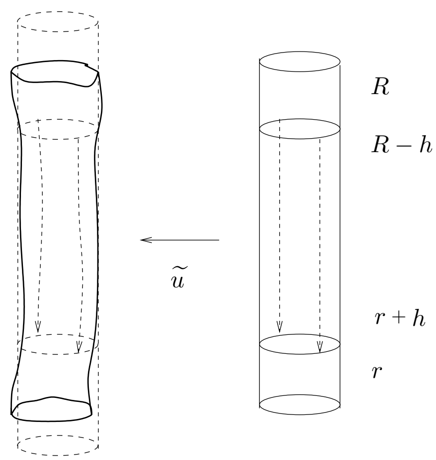

Figure 13. An arbitrary finite energy cylinder defined on some interval $[r, R] \times S^1$, with bounds on the energy and small $d\lambda$-energy, maps $[r+h, R-h] \times S^1$ almost onto some cylinder over a suitable periodic orbit.

Lemma 4.7. *Assume there exist constants $c > 0$ and $\gamma > 0$ such that all periodic orbits of X having periods $T \leq c$ are isolated (hence finite in number) and, moreover, γ is strictly smaller than the differences $|T_2 - T_1|$ of two different periods in $(0, c]$ and strictly smaller than the period of every contractible periodic orbit. Then there exists for every $\varepsilon \in (0, \gamma)$ a (large) constant $h > 0$ such that the following holds true. If a $\widetilde{J}$-holomorphic map*

$\widetilde{u} = (a, u)\colon [r, R] \times S^1 \to \mathbb{R} \times M$ *satisfies*

$$\int_{[r,R]\times S^1} u^* \, d\lambda \leq \gamma, \quad E(\widetilde{u}) \leq c, \quad r + h < R - h,$$

then

$$\int_{[r+h,R-h]\times S^1} u^* \, d\lambda \leq \varepsilon.$$

Proof. Arguing by contradiction we find a number $0 < \varepsilon < \gamma$, a sequence of pairs (r_k, R_k) with $R_k - r_k \geq 2k$, and a sequence of $\widetilde{J}$-holomorphic maps $\widetilde{u}_k = (a_k, u_k)\colon [r_k, R_k] \times S^1 \to \mathbb{R} \times M$ satisfying

$$\frac{\partial}{\partial s}\widetilde{u}_k + \widetilde{J}(\widetilde{u}_k)\frac{\partial}{\partial t}\widetilde{u}_k = 0, \quad \int_{[r_k,R_k]\times S^1} u_k^* \, d\lambda \leq \gamma, \quad E(\widetilde{u}_k) \leq c, \tag{4.7}$$

and, in addition,

$$\int_{[r_k+k,R_k-k]\times S^1} u_k^* \, d\lambda \geq \varepsilon, \tag{4.8}$$

for all k. Using the $\mathbb{R}$-invariance in s, we may assume that $s = 0$ is in the middle of the cylinder so that $r_k = -R_k$. We define the sequence $\widetilde{v}_k = (b_k, v_k)\colon [-k/2, k/2] \times S^1 \to \mathbb{R} \times M$ by

$$\widetilde{v}_k(s,t) = (a_k(s + r_k + k, t) - a_k(r_k + k, 0), u_k(s + r_k + k, t)). \tag{4.9}$$

Clearly $\widetilde{v}_k$ is $\widetilde{J}$-holomorphic and satisfies the estimates

$$E(\widetilde{v}_k) \leq c \qquad \text{and} \qquad \int_{[-k/2,k/2]\times S^1} v_k^* d\lambda \leq \gamma.$$

We claim that there are constants c_1 and c_α independent of k, such that

$$|\nabla\widetilde{v}_k(s,t)| \leq c_1 \qquad \text{on } \left[-\tfrac{k}{2} + 1, \tfrac{k}{2} - 1\right] \times S^1$$

and

$$|D^\alpha\widetilde{v}_k(s,t)| \leq c_\alpha \qquad \text{on } \left[-\tfrac{k}{2} + 2, \tfrac{k}{2} - 2\right] \times S^1.$$

Indeed, otherwise a bubbling off analysis as in [24] proves, from the assumption that the energies $E(\widetilde{v}_k) \leq c$ are bounded, the existence of a finite energy plane having an asymptotic limit of period $T \leq \gamma$, from (4.7). This, however, contradicts the definition of the number γ. Therefore, $\widetilde{v}_k$ has a C^∞_{loc} converging subsequence. Its limit $\widetilde{v} = (b, v)\colon \mathbb{R} \times S^1 \to \mathbb{R} \times M$ is a pseudoholomorphic cylinder satisfying

$$E(\widetilde{v}) \leq c \quad \text{and} \quad \int_{\mathbb{R}\times S^1} v^* \, d\lambda \leq \gamma. \tag{4.10}$$

If $\widetilde{v}$ is not constant, we conclude in view of the results in [32] and the fact that periodic orbits with period $\leq c$ are isolated, that v converges as $s \to \pm\infty$

to periodic orbits $x_\pm$ having periods $T_\pm$ bounded by c. Consequently, with $T_\pm = \int_{x_\pm} v^*\lambda$,

$$|T_+ - T_-| = \int_{\mathbb{R}\times S^1} v^*\, d\lambda \le \gamma.$$

By assumption, γ is smaller than the differences between two different periods in $(0, c]$; hence $T_+ = T_-$ and

$$\int_{\mathbb{R}\times S^1} v^*\, d\lambda = 0.$$

This and the assumption that $\widetilde{v}$ is not constant imply by Theorem 6.11 in [30] that $\widetilde{v}$ is a cylinder over a periodic orbit. Consequently, $x_- \equiv x_+ =: z_-$, and

$$\widetilde{v}(s,t) = (T_- s + c, z_-(T_- t + d))$$

for $(s,t) \in \mathbb{R}\times S^1$. Therefore, setting $s = 0$ in (4.9), we find

$$v_k(0,t) = u_k\,(r_k + k, t) \to z_-(T_- t) \quad \text{ in } C^\infty(S^1)$$

as $k \to \infty$.

In the case that $\widetilde{v}$ is constant,

$$v_k(0,t) = u_k\,(r_k + k, t) \to \text{ constant} \quad \text{ in } C^\infty(S^1).$$

The same arguments show that either

$$u_k\,(R_k - k, t) \to z_+(T_+ t) \quad \text{ in } C^\infty(S^1)$$

for a periodic solution z_+ having period T_+ or

$$u_k\,(R_k - k, t) \to \text{ constant} \quad \text{ in } C^\infty(S^1).$$

Using Stokes' theorem, (4.7) and (4.8), we can estimate

$$\gamma \ge \int_{[r_k+k, R_k-k]\times S^1} u_k^* d\lambda = \int_{S^1} v_k(r_k + k, \cdot)^*\lambda - \int_{S^1} v_k(R_k - k, \cdot)^*\lambda \ge \varepsilon.$$

Taking the limit as $k \to \infty$, we obtain a contradiction to the definition of γ. This contradiction proves Lemma 4.7. □

We continue with the proof of Proposition 4.6 applying Lemma 4.7. Fixing $\varepsilon > 0$ sufficiently small we find for every $\beta > 0$ a constant $h > 0$ such that for j sufficiently large

$$\int_{A(\mathrm{e}^h,\ \mathrm{e}^{-h}(\varepsilon/\delta_j))} \widetilde{v}_j^*\omega_{\varphi(j)} \le \beta.$$

This implies that

$$\text{(4.11)} \qquad \int_{D_{\mathrm{e}^{-h}\frac{\varepsilon}{\delta_j}}} \widetilde{v}_j^*\ \omega_{\varphi(j)} = \int_{D_{\mathrm{e}^h}} \widetilde{v}_j^*\omega_{\varphi(j)} + \int_{A(\mathrm{e}^h, \mathrm{e}^{-h}(\frac{\varepsilon}{\delta_j}))} \widetilde{v}_j^*\omega_{\varphi(j)}$$

$$\le \int_{D_{\mathrm{e}^h}\setminus D_\tau(\Gamma^i)} \widetilde{v}_j^*\omega_{\varphi(j)} + \int_{D_\tau(\Gamma^i)} \widetilde{v}_j^*\omega_{\varphi(j)} + \beta.$$

Here $\tau > 0$ is sufficiently small. Unwinding the scaling of $w_{\varphi(j)}$ and recalling $\gamma_j \to z^i$ and $\delta_j \to 0$ we obtain

$$\lim_{j\to\infty} \int_{D_{\mathrm{e}^{-h}\frac{\varepsilon}{\delta_j}}} \widetilde{v}_j^* \ \omega_{\varphi(j)} = \lim_{j\to\infty} \int_{D_{\mathrm{e}^{-h}\varepsilon}(\gamma_j)} w^*_{\varphi(j)}\omega_{\varphi(j)} = m_{\mathrm{e}^{-h}\varepsilon}(w_\infty, z^i).$$

In the notation $\widetilde{\omega} = \omega_{\widetilde{W}}$ in the W-case, and $\omega = d\lambda$ in the M-case we obtain from (4.11) taking the limit $j \to \infty$,

$$m_{\mathrm{e}^{-h}\varepsilon}(w_\infty, z^i) \le \int_{D_{\mathrm{e}^h}\setminus D_\tau(\Gamma^i)} \widetilde{v}_\infty^* \widetilde{\omega} + \sum_{z\in\Gamma^i} m_\tau(\widetilde{v}_\infty, z) + \beta.$$

Taking the limit $\tau \to 0$ we obtain in the W-case

$$m_{\mathrm{e}^{-h}\varepsilon}(w_\infty, z^i) \le \int_{D_{\mathrm{e}^h}} \widetilde{v}_\infty^* \omega_{\widetilde{W}} + \beta, \tag{4.12}$$

and in the M-case

$$\begin{aligned} m_{\mathrm{e}^{-h}\varepsilon}(w_\infty, z^i) &\le \int_{D_{\mathrm{e}^h}\setminus\Gamma^i} \widetilde{v}_\infty^* d\lambda + \sum_{z\in\Gamma^i} m(\widetilde{v}_\infty, z) + \beta \\ &\le \int_{\mathbb{C}\setminus\Gamma^i} \widetilde{v}_\infty^* d\lambda + \sum_{z\in\Gamma^i} m(\widetilde{v}_\infty, z) + \beta. \end{aligned} \tag{4.13}$$

Since $m(w_\infty, z^i) \le m_{\mathrm{e}^{-h}\varepsilon}(w_\infty, z^i)$ and since we find for every $\beta > 0$ a constant h so that the estimates (4.12) and (4.13) hold, we conclude

$$m(w_\infty, z^i) \le \int_{\mathbb{C}\setminus\Gamma^i} \widetilde{v}_\infty^* d\lambda + \sum_{z\in\Gamma^i} m(\widetilde{v}_\infty, z)$$

in the M-case, and

$$m(w_\infty, z^i) \le \int_{\mathbb{C}} \widetilde{v}_\infty^* \omega_{\widetilde{W}}$$

in the W-case. This finishes the proof of Proposition 4.6.

The punctures Γ of the generalized finite energy sphere $w_\infty : S^2 \setminus \Gamma \to \widetilde{V}$ are all negative. Associated with the punctures are periodic solutions of the Reeb vector field on M. For z near $z^i \in \Gamma$ we know that $w_\infty(z) \in \mathbb{R}^- \times M$ and when $w_\infty(z) = (a_\infty(z), u_\infty(z))$, the asymptotic behavior of the projection u_∞ of w_∞ into M is as follows:

$$u_\infty\Big(z^i + \mathrm{e}^{2\pi(s+\mathrm{i}t)}\Big) \to x_i(T_i t)$$

as $s \to -\infty$ in $C^\infty(\mathbb{R})$, for a T_i-periodic solution $x_i(t)$ of X_λ on M. At the positive puncture ∞ of the sphere $\widetilde{v}_\infty : \mathbb{C} \setminus \Gamma^i \to \mathbb{R} \times M$ (resp. $\widetilde{W}$) we have, in the notation $\widetilde{v}_\infty = (b_\infty, v_\infty)$,

$$v_\infty\Big(R\mathrm{e}^{2\pi\mathrm{i}t}\Big) \to x_\infty(T_\infty t)$$

as $R \to \infty$ in $C^\infty(\mathbb{R})$, the asymptotic limit being the T_∞-periodic solution of x_∞.

Proposition 4.8. *The asymptotic limit of w_∞ at the negative puncture $z^i \in \Gamma$ coincides with the asymptotic limit of $\widetilde{v}_\infty$ at its positive puncture ∞.*

Proof. Considering the M-case we represent $\widetilde{v}_\infty : \mathbb{C} \setminus \Gamma^i \to \mathbb{R} \times M$ by $\widetilde{v}_\infty = (a_\infty, v_\infty)$. In the following we abbreviate the mass of w_∞ at the puncture $z^i \in \Gamma$ by $m(z^i) \equiv m(w_\infty, z^i)$.

Fix $\varepsilon > 0$ and choose $\eta > 0$ so small that

$$m_\eta(z^i) \leq m(z^i) + \frac{\varepsilon}{2}.$$

This implies for j large enough

$$\int_{D_\eta(z^i)} w^*_{\varphi(j)}\omega_{\varphi(j)} \leq m(z^i) + \varepsilon. \tag{4.14}$$

Consider the sequence $\widetilde{v}_j \to \widetilde{v}_\infty$; then for large j,

$$\int_{D_R} \widetilde{v}_j^*\omega_{\varphi(j)} = \int_{D_R \setminus D_\varepsilon(\Gamma^i)} \widetilde{v}_j^* d\lambda + \int_{D_\varepsilon(\Gamma^i)} \widetilde{v}_j^* d\lambda$$

from which we obtain, using Stokes' theorem, in the limit $j \to \infty$ and then $\varepsilon \to 0$,

$$\int_{S^1} v_\infty(R\mathrm{e}^{2\pi \mathrm{i}\cdot})^*\lambda = \int_{D_R \setminus \Gamma^i} \widetilde{v}_\infty^* d\lambda + \sum_{z \in \Gamma^i} m(\widetilde{v}_\infty, z).$$

Taking the limit $R \to \infty$ we deduce using Proposition 4.6

$$m(z^i) = \lim_{R \to \infty} \int_{S^1} v_\infty(R\mathrm{e}^{2\pi \mathrm{i}\cdot})^*\lambda.$$

With $\varepsilon > 0$ as above we choose $R_0 > 1$ so large that

$$m(z^i) \geq \int_{S^1} v_\infty\Big(R_0\mathrm{e}^{2\pi \mathrm{i}\cdot}\Big)^*\lambda \geq m(z^i) - \varepsilon/2.$$

This implies

$$\int_{S^1} v_j\Big(R_0\mathrm{e}^{2\pi \mathrm{i}\cdot}\Big)^*\lambda \geq m(z^i) - \varepsilon \tag{4.15}$$

for j large enough. Setting $w_{\varphi(j)} = (a_j, u_j)$ we conclude, recalling that v_j is the M-part of $\widetilde{v}_j$ and is given by $v_j(z) = u_j(\gamma_j + \delta_j z)$, that

$$\int_{S^1} u_j\Big(\gamma_j + \delta_j R_0\mathrm{e}^{2\pi \mathrm{i}\cdot}\Big)^*\lambda \geq m(z^i) - \varepsilon.$$

Since $\gamma_j \to z^i$ and $\delta_j \to 0$, the circles $\gamma_j + \delta_j R_0 \mathrm{e}^{2\pi \mathrm{i}t}$, $t \in \mathbb{R}$, are contained in the ball $D_\eta(z^i)$ if j is large enough. Fix $\eta' \in (0, \eta)$. If j is large enough,

$$\int_{S^1} u_j\Big(\gamma_j + \eta'\mathrm{e}^{2\pi \mathrm{i}\cdot}\Big)^*\lambda \leq \int_{S^1} u_j\Big(z^i + \eta\mathrm{e}^{2\pi \mathrm{i}\cdot}\Big)^*\lambda \leq m(z^i) + \varepsilon,$$

in view of (4.14). Hence, again, if j is large enough

$$\int_{S^1} v_j\Big(\eta'\mathrm{e}^{2\pi \mathrm{i}\cdot}/\delta_j\Big)^*\lambda \leq m(z^i) + \varepsilon.$$

Since $\frac{\eta'}{\delta_j} > R_0$ for j large, we deduce, using (4.15),

$$m(z^i) - \varepsilon \leq \int_{S^1} v_j\Big(R\mathrm{e}^{2\pi \mathrm{i}\cdot}\Big)^* \lambda \leq m(z^i) + \varepsilon \quad \text{if } R \in [R_0, \eta'/\delta_j]. \tag{4.16}$$

This implies

$$\int_{A(R_0, \eta'/\delta_j)} v_j^* d\lambda \leq 2\varepsilon. \tag{4.17}$$

To summarize, for every $\varepsilon > 0$ there exist a large constant $R_0 = R_0(\varepsilon)$ and a small constant $\eta' = \eta'(\varepsilon)$ such that the estimate (4.17) holds true for every $j \geq j_0(\varepsilon)$. It follows that

$$\int_{A(R_0(\varepsilon), \infty)} v_\infty^* d\lambda \leq 2\varepsilon. \tag{4.18}$$

Observing that all the E-energies under consideration are bounded by π, in particular $E(\widetilde{v}_j) \leq \pi$ and $E(v_\infty) \leq \pi$, we see that the maps $\widetilde{v}_j$ for large j and $\widetilde{v}_\infty$ meet the assumptions of the following crucial lemma.

Lemma 4.9. *Let $\delta > 0$ and let c and γ be as in Lemma 4.7. Choose an S^1-invariant neighborhood $\mathcal{W}$ (in the loop space of M) of the distinguished loops corresponding to the contractible periodic solutions $x(Tt)$, $0 \leq t \leq 1$, having periods $T \leq c$. Choose, also, $\mathcal{W}$ so small that it separates loops of these periodic orbits. Then for any $\varepsilon \in (0, \gamma)$ there exists a constant $a > 0$ such that for every $\widetilde{J}$-holomorphic map $\widetilde{u} : A(r, R) \to \mathbb{R} \times M$ satisfying*

$$\begin{aligned} E(\widetilde{u}) &\leq c, \quad \int_{A(r,R)} u^* d\lambda \leq \varepsilon, \\ \int_{S^1} u^*\Big(\rho \mathrm{e}^{2\pi \mathrm{i}\cdot}\Big)\lambda &\geq \delta \quad \text{for } \rho \in [r, R], \quad r\mathrm{e}^a \leq R\mathrm{e}^{-a} \end{aligned}$$

the following holds:

$$u(\rho \mathrm{e}^{2\pi \mathrm{i}\cdot}) \in \mathcal{W} \quad \text{for } \rho \in [r\mathrm{e}^a, R\mathrm{e}^{-a}].$$

Hence these loops are all contained in the neighborhood component of $\mathcal{W}$ containing the loop of one of the distinguished periodic orbits.

Proof. We work in holomorphic polar coordinates. Arguing by contradiction we find an $\varepsilon \in (0, \gamma)$, a sequence (r_k, R_k) with $R_k - r_k \geq 2k$, and a sequence of pseudoholomorphic maps $\widetilde{u}_k = (a_k, u_k) : [r_k, R_k] \times S^1 \to \mathbb{R} \times M$ satisfying

$$E(\widetilde{u}_k) \leq c, \qquad \int_{[r_k, R_k] \times S^1} u_k^* d\lambda \leq \varepsilon$$

$$\int_{S^1} u_k(\rho, \cdot)^* \lambda \geq \delta \quad \text{for } \rho \in [r_k, R_k] \text{ and } u_k(s_k, \cdot) \notin \mathcal{W}$$

for some sequence $s_k \in [r_k+k, R_k-k]$. Using the $\mathbb{R}$-invariance we may assume that $s_k = 0$ so that $r_k \to -\infty$ and $R_k \to \infty$. Define the sequence of maps $\widetilde{v}_k = (b_k, u_k) : [r_k, R_k] \times S^1 \to \mathbb{R} \times M$ by setting

$$\widetilde{v}_k(s,t) = \Big(a_k(s,t) - a_k(0,0), u_k(s,t)\Big).$$

In view of the gradient bounds of $\widetilde{v}_k$, the sequence has a subsequence which converges in C^∞_{loc} to a pseudoholomorphic cylinder $\widetilde{v} = (b,v) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ which satisfies

$$E(\widetilde{v}) \leq c, \qquad \int_{\mathbb{R}\times S^1} v^* d\lambda \leq \varepsilon$$

and

$$\int_{S^1} v(\rho, \cdot)^* \lambda \geq \delta \quad \text{ for } \rho \in \mathbb{R}.$$

In particular, $\widetilde{v}$ is nonconstant. As $s \to \pm\infty$, the maps $v(s,\cdot)$ converge to periodic orbits $x_\pm$ having periods $T_\pm$ bounded by c. Applying Stokes' theorem we find

$$|T_+ - T_-| = \int_{\mathbb{R}\times S^1} v^* d\lambda \leq \varepsilon < \gamma.$$

In view of the definition of γ we deduce $T_- = T_+ =: T$ and $\int_{\mathbb{R}\times S^1} v^* d\lambda = 0$. Hence $x_+ = x_+ =: z$ and arguing as in Lemma 4.7 we conclude that $\widetilde{v}$ is the cylinder over the periodic orbit z given by $\widetilde{v}(s,t) = \Big(Ts+c, z(Tt+d)\Big)$. Setting $s = 0$ we obtain

$$u_k(0,\cdot) \to v(0,\cdot) = z(T\cdot + d) \quad \text{ in } C^\infty(S^1).$$

This leads to the contradiction that $u_k(0,\cdot) \notin \mathcal{W}$ for all k, but $z \in \mathcal{W}$. The proof of the lemma is complete. □

Continuing with the proof of Proposition 4.8 we conclude from (4.16), (4.17) and (4.18) taking $2\varepsilon = \varepsilon_0$ as in Lemma 4.9 that

$$v_j(R\mathrm{e}^{2\pi \mathrm{i}\cdot}) \in \mathcal{W}$$

if $R \in [R_0\mathrm{e}^h, \mathrm{e}^{-h}\eta'/\delta_j]$ for every $h > 0$ sufficiently large if $j \geq j_0(h)$ and moreover,

$$v_\infty\Big(R_0\mathrm{e}^h\mathrm{e}^{2\pi\mathrm{i}\cdot}\Big) \in \mathcal{W}$$

for every $h > 0$ sufficiently large. Now, as $j \to \infty$

$$v_j(R_0\mathrm{e}^h\mathrm{e}^{2\pi\mathrm{i}t}) \to v_\infty(R_0\mathrm{e}^h\mathrm{e}^{2\pi\mathrm{i}t}),$$
$$v_j\Big(\eta'\mathrm{e}^{-h}\mathrm{e}^{2\pi\mathrm{i}t}/\delta_j\Big) = u_j\Big(\gamma_j + \eta'\mathrm{e}^{-h}\mathrm{e}^{2\pi\mathrm{i}t}\Big) \to u_\infty\Big(z^i + \eta'\mathrm{e}^{-h}\mathrm{e}^{2\pi\mathrm{i}t}\Big)$$

in $C^\infty(\mathbb{R})$. We conclude for every h sufficiently large that the loops $v_\infty(R_0\mathrm{e}^h\mathrm{e}^{2\pi\mathrm{i}\cdot})$ and $u_\infty(z^i + \eta'\mathrm{e}^{-h}\mathrm{e}^{2\pi\mathrm{i}\cdot})$ belong to the closure of the S^1-neighborhood of the same periodic orbit $x(T\cdot)$. By the asymptotic behavior near the punctures,

$$u_\infty\Big(z^i + \eta' R_0 \mathrm{e}^{-h}\mathrm{e}^{2\pi \mathrm{i}t}\Big) \to x_i(T_i t),$$
$$v_\infty\Big(R_0\mathrm{e}^{h}\mathrm{e}^{2\pi \mathrm{i}t}\Big) \to x_\infty(T_\infty t)$$

as $h \to \infty$, in $C^\infty(\mathbb{R})$. Consequently, the loops $x_i(T_i\cdot)$ and $x_\infty(T_\infty\cdot)$ also belong to the closure of the S^1-neighborhood of $x(T\cdot)$. Since the neighborhood $\mathcal{W}$ separates periodic orbits we conclude

$$x_i(T_i t) = x_\infty(T_\infty t + \delta), \quad t \in \mathbb{R}$$

for some phase $\delta \in \mathbb{R}$. In particular, $T_i = T_\infty$. This completes the proof of Proposition 4.8 in the (M) case. The (W) case is proved the same way.

Proposition 4.10. *The finite energy sphere* $\widetilde{v}_\infty : \mathbb{C} \setminus \Gamma^i \to \mathbb{R} \times M$ *produced above has the following property*:

$$\textit{Either} \quad \int_{\mathbb{C}\setminus\Gamma^i} v_\infty^* d\lambda \geq \sigma_0$$
$$\textit{or} \quad \int_{\mathbb{C}\setminus\Gamma^i} v_\infty^* d\lambda = 0 \ \ \textit{and} \ \ \sharp\Gamma^i \geq 2.$$

Proof. If $\Gamma^i \neq \emptyset$ then $0 \in \Gamma^i$ by Lemma 4.5. Assume $\Gamma^i = \emptyset$ or $\Gamma^i = \{0\}$. From the normalization condition (4.2) for $\widetilde{v}_j$ we obtain in the limit $j \to \infty$,

$$\int_{S^1} v_\infty\Big(\mathrm{e}^{2\pi \mathrm{i}\cdot}\Big)^*\lambda = m(w_\infty, z^i) - \sigma_0.$$

Since $v_\infty\Big(R\mathrm{e}^{2\pi \mathrm{i}t}\Big) \to x_i(T_i t)$ as $R \to \infty$ in $C^\infty(\mathbb{R})$, in view of Proposition 4.8, and since $T_i = m(w_\infty, z^i)$, we conclude from

$$\int_{D_R\setminus D} v_\infty^* d\lambda = \int_{S^1} v_\infty\Big(R\mathrm{e}^{2\pi \mathrm{i}\cdot}\Big)^*\lambda - \int_{S^1} v_\infty\Big(\mathrm{e}^{2\pi \mathrm{i}\cdot}\Big)^*\lambda,$$

in the limit $R \to \infty$, that

$$\int_{\mathbb{C}\setminus D} v_\infty^* d\lambda = \sigma_0 > 0.$$

We see that the $d\lambda$-energy of $\widetilde{v}_\infty$ is positive in the cases of one puncture and no punctures. Consequently, $\sharp\Gamma^i \geq 2$ if the $d\lambda$-energy of $\widetilde{v}_\infty$ vanishes, as claimed in the proposition. □

In view of Proposition 4.10 we might have $\int_{\mathbb{C}\setminus\Gamma^i} v_\infty^* d\lambda = 0$ provided $\sharp\Gamma^i \geq 2$. From the classification of finite energy surfaces in $\mathbb{R} \times M$ having vanishing $d\lambda$-energy we know that in this case the image of $\widetilde{v}_\infty$ is necessarily a multiply covered cylinder over a periodic orbit of X_λ; see the appendix of [30]. Since we have only one positive puncture, namely ∞, the associated asymptotic limit in M is l-fold covered,

$$l = \sum_{n=1}^{\sharp\Gamma^i} l_n,$$

where l_n, $1 \leq n \leq \sharp\Gamma^i$, are the covering numbers of the negative limits. Clearly, $l \geq 2$. In the following we call such a generalized finite energy surface a connector.

Definition 4.11. A *connector* is a generalized finite energy surface $\widetilde{u} = (a, u) : S \setminus \Gamma \to \mathbb{R} \times M$ having precisely one positive puncture and at least two negative punctures and satisfying

$$\int_{S\setminus\Gamma} u^* d\lambda = 0.$$

We summarize our construction so far. We used the sequence $w_{\varphi(j)}$ to produce in the limit the generalized finite energy sphere $w_\infty : S^2 \setminus \Gamma \to \widetilde{V}$. The nonempty finite set $\Gamma = \{z^1, \ldots, z^k\}$ of punctures is contained in $D \subset S^2$. These punctures are all negative. For the puncture $z^i \in \Gamma$ we constructed a rescaled subsequence $\widetilde{v}_j = w_{\varphi(j)} \circ \tau_j$ converging either to a generalized finite energy plane

$$\widetilde{v}_\infty : S^2 \setminus \{\infty\} = \mathbb{C} \to \widetilde{W}$$

or, after adding a sequence of numbers to the $\mathbb{R}$-component, to a generalized finite energy sphere

$$\widetilde{v}_\infty : \mathbb{C} \setminus \Gamma^i = S^2 \setminus \Big(\Gamma^i \cup \{\infty\}\Big) \to \mathbb{R} \times M.$$

In the first case the puncture ∞ of $\widetilde{v}_\infty$ is positive. In the second case ∞ is also a positive puncture, while all the punctures in Γ^i are negative provided $\Gamma^i \neq \emptyset$. In both cases, the asymptotic limit of the positive puncture ∞ of $\widetilde{v}_\infty$ agrees, because of Proposition 4.8, with the asymptotic limit associated with the negative puncture $z^i \in \Gamma$ of w_∞.

Now we proceed inductively. If $\widetilde{v}_\infty$ constructed in the above first step does not have punctures, i.e., $\Gamma^i = \emptyset$, so that it is either a finite energy plane in $\widetilde{W}$ or a finite energy plane in $\mathbb{R} \times M$, the induction for the puncture z^i is already finished. If $\Gamma^i \neq \emptyset$, then for the negative punctures $\Gamma^i = \{z^{i_1}, \ldots, z^{ik_i}\} \subset D$ of $\widetilde{v}_\infty$ we defined the masses $m\Big(\widetilde{v}_\infty, z^{il}\Big)$ satisfying

$$\gamma < m\Big(\widetilde{v}_\infty, z^{il}\Big) < m\Big(w_\infty, z^i\Big)$$

for all $1 \leq l \leq \sharp\Gamma^i$. For the negative puncture $z^{il} \in \Gamma^i$ we proceed as before and find by rescaling the sequence $\widetilde{v}_j$ near z^{il} precisely as we did above, a subsequence converging either to a finite energy plane $\mathbb{C} \to \widetilde{W}$ or to a finite energy sphere $\mathbb{C}\setminus\Gamma^{il} \to \mathbb{R}\times M$ having the only positive puncture ∞ and a finite, possibly empty, set $\Gamma^{il} \subset D$ of negative punctures. Moreover, the asymptotic limit associated with the positive puncture ∞ agrees with the asymptotic limit of the negative puncture z^{il}. We continue in this way. At every step either a certain amount of energy is used or one obtains a connector, where because of Proposition 4.6 the sum of the masses of the negative punctures is equal to the mass of the corresponding puncture of the previous step. At every step the new masses showing up are larger than γ but smaller than the corresponding mass of the previous step. In case of a connector, there are at least two negative punctures. The sum of their masses is the mass of the previous step. Due to the lack of further punctures the inductive procedure necessarily terminates after finitely many steps. In the last step a finite energy plane either in $\widetilde{W}$ or in $\mathbb{R} \times M$ is produced. The asymptotic limit of its positive puncture ∞ agrees with the limit of the negative puncture under consideration. All surfaces produced, except of course the initial surface w_∞, have precisely one positive puncture and a finite, possibly empty, set of negative punctures. The whole inductive procedure and its resulting configuration can be described with the help of a graph whose vertices can have three different colors, black, gray and white. A black vertex occurs only once. It is a solution of the problem (V) and represents the image of w_∞ in $\widetilde{V}$. The gray vertex represents the image of a solution of type (M) and a white vertex a solution of type (W). A white vertex is always a plane. We draw an edge between two vertices if the asymptotic limits match up. Following the inductive procedure we represent the graph as a tree, where we do not exclude the possibility that different vertices in the graph correspond to the same finite energy sphere. Figure 14 illustrates the idea with an example.

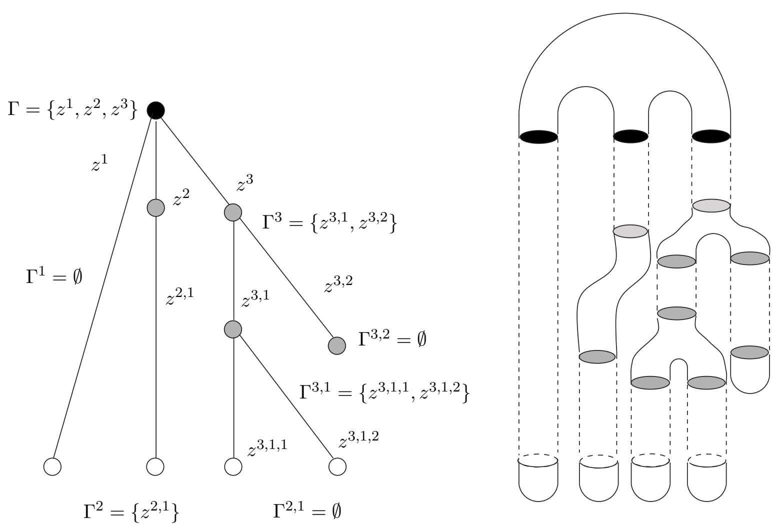

Figure 14. The evolution of a graph during the inductive construction.

5. Properties of bubbling off trees

5.1. *Fredholm indices.* In order to analyze the properties of the bubbling off tree we first recall some facts about the Fredholm index needed later on. We abbreviate the Riemann sphere by $S = S^2$ and consider the finite energy spheres

$$\widetilde{w} : \dot{S} = S^2 \setminus \Gamma \to \widetilde{A}$$

constructed in the previous section, where $\widetilde{A}$ stands for $\widetilde{V}$ or $\mathbb{R} \times M$ or $\widetilde{W}$, and where $\Gamma \neq \emptyset$. If $\widetilde{A} = \widetilde{V}$, then punctures are all negative, $\Gamma = \Gamma^-$, while $\Gamma = \Gamma^+$ for $\widetilde{A} = \widetilde{W}$. If $\widetilde{w}$ is an embedding, its Fredholm index is, according to [36], given by the formula

$$\text{Ind}(\widetilde{w}) = \mu_N(\widetilde{w}) + \chi(S^2) - \sharp\Gamma \tag{5.1}$$

for $\widetilde{A} = \widetilde{W}$ and $\widetilde{A} = \mathbb{R} \times M$. Here N denotes a normal bundle to the tangent bundle of the embedded sphere $\widetilde{w}(\dot{S}) \subset \widetilde{A}$ so that $T\widetilde{A} = T\widetilde{w}(\dot{S}) \oplus N$. Moreover, N agrees with ξ in a neighborhood of the punctures. The normal index $\mu_N(\widetilde{w}) = \mu_N^+ - \mu_N^- \in \mathbb{Z}$ is computed in a trivialization of N as explained in Appendix 8.1. If $\widetilde{A} = \widetilde{V}$ we imposed the condition $\widetilde{w}(\infty) = o_\infty$. Therefore, the Fredholm index decreased by 2 so that

$$\text{Ind}(\widetilde{w}) = \mu_N(\widetilde{w}) + \chi(S^2) - \sharp\Gamma - 2 \tag{5.2}$$

for $\widetilde{A} = \widetilde{V}$. In this case $\mu_N(\widetilde{w}) = -\mu_N^-(\widetilde{w})$ since all the punctures are negative. Recalling that M is diffeomorphic to S^3 we also have a total Conley-Zehnder index

$$\mu(\widetilde{w}) = \mu^+(\widetilde{w}) - \mu^-(\widetilde{w}),$$

which, for every periodic orbit $x(t)$ associated with the punctures Γ, is computed in trivializations of the bundles $u^*\xi$, where $u : D \to M$ are disc maps spanning the orbits so that $x(Tt) = u(e^{2\pi i t})$.

In order to establish the relation between μ_N and μ we define a new index μ_F as follows. We choose a complex line bundle E over $\dot{S}$ which is a subbundle of $\widetilde{w}^*T\widetilde{A}$ coinciding with $\mathbb{C}X$ near the punctures, where X is the Reeb vector field. In addition, we choose E so that it admits a nowhere vanishing section which near the punctures coincides with X. Let now F be a complex subbundle which near the punctures coincides with ξ and which complements E so that

$$\widetilde{w}^*T\widetilde{A} = E \oplus F.$$

Then F is also a symplectic vector bundle of real dimension 2. Therefore, we can compute the total Conley-Zehnder index $\mu_F(\widetilde{w}) \in \mathbb{Z}$ using a trivialization of the bundle F.

Concerning the computations of indices and winding numbers below it is useful to recall from [32] and [30] the special compactification of the punctured sphere $\dot{S} = S \setminus \Gamma$ to a compact surface $\bar{S}$ with boundaries. We define this by adding a circle for every puncture in Γ, taking polar coordinates centered at the puncture and distinguishing positive and negative punctures. In view of its asymptotic properties near Γ the map $\widetilde{w}$ has a smooth extension $\bar{w} : \bar{S} \to \mathbb{R} \times M$ for which the added circles parametrize the periodic orbits associated with the punctures. The vector bundles over $\dot{S}$ under consideration have unique continuous extensions to bundles over the compact surface with boundary. The following formula holds true:

Proposition 5.1.

$$\mu_N(\widetilde{w}) = \mu_F(\widetilde{w}) - 2\left[\chi(S) - \sharp\Gamma\right].$$

Proof. By construction,

$$T\widetilde{w}(\dot{S}) \oplus N = \widetilde{w}^* T\widetilde{A} = E \oplus F.$$

Choose nowhere vanishing sections t of $T\widetilde{w}(\dot{S})$ and n of N. Denote by e a nowhere vanishing section of E which near the punctures agrees with X and denote by f a nowhere vanishing section of F. Then $(t(z), n(z))$ is, in the complex basis $(e(z), f(z))$ represented by

$$\begin{aligned} t(z) &= \alpha(z)e(z) + \beta(z)f(z), \\ n(z) &= \gamma(z)e(z) + \delta(z)f(z). \end{aligned}$$

Near the punctures Γ, the matrix

$$\begin{bmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{bmatrix}, \qquad z \in \dot{S},$$

is diagonal. The sum of the winding numbers of the complex determinant at the positive punctures equals the sum at the negative punctures. The sum of the winding numbers of α at the positive punctures minus the sum at the negative punctures is equal to $\chi(S) - \sharp\Gamma$. This implies that the sum of the winding numbers of δ at the positive punctures minus the sum at the negative punctures is equal to $\sharp\Gamma - \chi(S)$. Consequently, by means of the Maslov-compatibility property of the Conley-Zehnder index in Theorem 8.1, we conclude that $\mu_N = 2[\sharp\Gamma - \chi(S)] + \mu_F$. This completes the proof of the proposition. □

We point out that the crucial property of the bundle E lies in the fact that it admits a nowhere vanishing section which near the punctures agrees with X. Given any other complex subbundle of dimension 1, also admitting a nowhere vanishing section which is X near the punctures, it is isotopic to E through complex subbundles of $\widetilde{w}^* T\widetilde{A}$, the bundles being fixed near the punctures during the deformation. All these constructions hold true under homotopies of $\widetilde{w}$ as long as the behavior near the punctures is preserved. We shall use these remarks in the proof of the next proposition.

Proposition 5.2.

- $\mu_F(\widetilde{w}) = \mu(\widetilde{w})$ *if* $\widetilde{A} = \widetilde{W}$ *or* $\widetilde{A} = \mathbb{R} \times M$.
- $\mu_F(\widetilde{w}) = \mu(\widetilde{w}) + 6$ *if* $\widetilde{A} = \widetilde{V}$, *provided that the embedded finite energy sphere* $\widetilde{w} : \dot{S} \to \widetilde{V}$ *satisfies* $\widetilde{w}(\infty) = o_\infty$ *and the intersection number with the sphere* S_∞ *at infinity is equal to* 1.

Proof. If $\widetilde{A} = \mathbb{R} \times M$ we can choose E to be the complex span of X so that F can be chosen to be ξ. Then, clearly, $\mu_F = \mu$. If $\widetilde{A} = \widetilde{W}$, then we can homotope $\widetilde{W}$ into some curve in $\mathbb{R}^+ \times M$ keeping the behavior near the punctures fixed. The same arguments apply. Assume now that $\widetilde{A} = \widetilde{V}$ and assume that it intersects the sphere at infinity precisely once transversally so that the intersection number is equal to 1. Choose the bundles E and F as described above and assume that they coincide near the punctures with $\mathbb{C}X$ and ξ respectively. Now choose a smooth map $\widetilde{v} : \dot{T} \to \widetilde{W}$ which is defined on a punctured Riemann sphere $\dot{T}$ and which near the punctures parametrizes half-cylinders over periodic orbits. The periodic orbits in question are those of $\widetilde{w}$. Gluing this Riemann sphere to $\dot{S}$ along the periodic orbits we obtain a map b from a surface Σ into $W \cup ([-R, R] \times M) \cup V \equiv \mathbb{C}P^2$ for some R sufficiently large. The map b has the intersection number 1 with the sphere at infinity. Moreover, it is homologous to $\mathbb{C}P^1$. Therefore, $c_1(b^* T\mathbb{C}P^2)[\Sigma] = 3$. If we choose complex line subbundles E' and F' over the part belonging to $\dot{T}$, so that E' admits a nonvanishing section which is X near the punctures, and so that F' is ξ near the punctures, then E' and E can be glued along the periodic orbits to a complex line bundle $\widehat{E}$ over Σ admitting a global nowhere vanishing section. Moreover, F and F' can be glued to a complex line bundle $\widehat{F}$ over Σ, so that

$$\widehat{E} \oplus \widehat{F} = b^*(T\mathbb{C}P^2).$$

Since $c_1(\widehat{E}) = 0$ we deduce

$$\begin{aligned} 3 &= c_1(b^* T\mathbb{C}P^2)([\Sigma]) \\ &= c_1(\widehat{E})([\Sigma]) + c_1(\widehat{F})([\Sigma]) \\ &= c_1(\widehat{F})([\Sigma]). \end{aligned}$$

Consequently, $\mu_F + \mu_{F'} = 2 \cdot 3 = 6$. By construction, $\mu_{F'} = \mu_\xi = \mu^-(\widetilde{w}) = -\mu(\widetilde{w})$ and the proposition follows. □

In view of Proposition 5.1 and Proposition 5.2 we find the following formulas for the Fredholm index, using the fact that $\chi(S) = 2$:

$$\text{Ind}(\widetilde{w}) = \mu(\widetilde{w}) - 2 + \sharp\Gamma, \tag{5.3}$$

if $\widetilde{A} = \widetilde{W}$ or $\widetilde{A} = \mathbb{R} \times M$, and

$$\text{Ind}(\widetilde{w}) = \mu(\widetilde{w}) + 2 + \sharp\Gamma \tag{5.4}$$

for our special surfaces in $\widetilde{A} = \widetilde{V}$.

We note that formula (5.4) can heuristically be explained by the following argument from Fredholm theory. We take smooth disc maps $\widetilde{v}_\lambda : \text{int}D_\lambda \to \widetilde{W}$ which parametrize near their boundaries half cylinders over the periodic orbits P_λ associated with the punctures $\lambda \in \Gamma$. Gluing these maps to $\widetilde{w}$ along the periodic orbits we obtain a map $\widetilde{w}\sharp\Big(\cup_\lambda \widetilde{v}_\lambda\Big) : S^2 \to A_N$ for sufficiently large N, which is homologous to $\mathbb{C}P^1$ and which intersects the sphere at infinity in the point o_∞ with intersection index 1. Hence, in view of (2.27), $\text{Ind}\Big(\widetilde{w}\sharp\Big(\cup_\lambda \widetilde{v}_\lambda\Big)\Big) = 4 - 2 = 2$. By the gluing properties of Fredholm maps,

$$\text{Ind}\Big(\widetilde{w}\sharp \cup_\lambda \widetilde{v}_\lambda\Big) = \text{Ind}(\widetilde{w}) + \sum_{\lambda\in\Gamma} \text{Ind}(\widetilde{v}_\lambda)$$

and so,

$$\text{Ind}(\widetilde{w}) = 2 - \sum_{\lambda\in\Gamma} \text{Ind}(\widetilde{v}_\lambda).$$

Now, $\text{Ind}(\widetilde{v}_\lambda) = \mu(P_\lambda) - 1$, by Theorem 2.8 in [36]. Therefore, $\text{Ind}(\widetilde{w}) = 2 + \sharp\Gamma - \sum_{\lambda\in\Gamma} \mu(P_\lambda) = 2 + \sharp\Gamma + \mu(\widetilde{w})$, which agrees with (5.4). Figure 15 illustrates schematically the construction. So far we have assumed that $\widetilde{w}$ is an embedding. This however is not needed. For a proof of this fact and also for the proof of the following genericity statement we refer to [7].

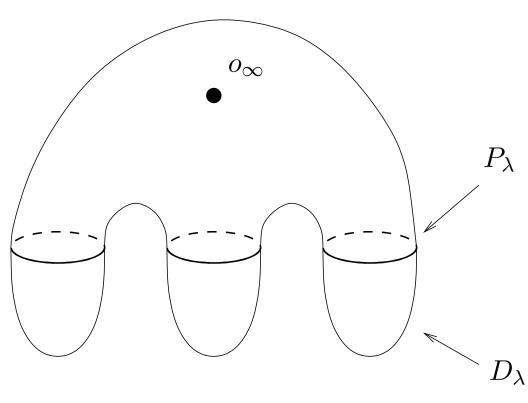

Figure 15. Construction of the map $\widetilde{w}\sharp \cup_\lambda \widetilde{u}_\lambda$.

THEOREM 5.3. *Fix a point o_∞ on the sphere $\mathbb{C}P^1$ at infinity in $\mathbb{C}P^2$. Then there exists a compatible almost complex structure $\widehat{J}$ on $\widetilde{V}$ and on $\widetilde{W}$ which on $\Phi([-\varepsilon, \varepsilon] \times M)$ coincides with $\widetilde{J}$, which in a neighborhood of $\mathbb{C}P^1$ in V coincides with the standard complex structure i, so that the following properties hold true*:

1. *For every somewhere injective finite energy sphere $\widetilde{w} : \dot{S} \to \widetilde{W}$ with respect to $\widehat{J}$,*
$$\text{Ind}(\widetilde{w}) = \mu(\widetilde{w}) - 2 + \sharp\Gamma \geq 0.$$
2. *For every finite energy sphere $\widetilde{w} : \dot{S} \to \widetilde{V}$ with respect to $\widehat{J}$, satisfying $\widetilde{w}(\infty) = o_\infty$ and intersecting the sphere at infinity once with the intersection number equal to* 1,
$$\text{Ind}(\widetilde{w}) = \mu(\widetilde{w}) + 2 + \sharp\Gamma \geq 0.$$

Here $\mu(\widetilde{w})$ is the total Conley-Zehnder index computed with respect to M. Note that in the first case $\mu(\widetilde{w}) = \mu^+(\widetilde{w})$, so that
$$\mu^+(\widetilde{w}) \geq 2 - \sharp\Gamma$$
for a finite energy sphere in $\widetilde{W}$, while $\mu(\widetilde{w}) = -\mu^-(\widetilde{w})$ in the second case, so that
$$\mu^-(\widetilde{w}) \leq 2 + \sharp\Gamma$$
for the distinguished finite energy spheres in $\widetilde{V}$.

5.2. *Analysis of bubbling off trees.* In Section 4 we constructed a tree of generalized finite energy spheres of type (W), (V) and (M). We shall show next that the asymptotic limits occurring have their Conley-Zehnder indices in the set $\{1, 2, 3\}$, in the generic case. The Conley-Zehnder or μ-index is computed with respect to a symplectic trivialization of the plane bundle ξ over a natural disc spanned by the periodic orbit under consideration. Since in our case $M = S^3$, the index neither depends on the trivialization nor on the choice of the disc. Recall that we always consider a generic structure J on ξ and a generic almost complex structure $\widehat{J}$ on $\widehat{W}$ and $\widehat{V}$ which coincides near the boundaries with the $\mathbb{R}$-invariant structure $\widetilde{J}$ determined by J.

PROPOSITION 5.4. *Consider the bubbling off tree constructed in the previous section. The bottom white dots representing generalized finite energy planes in $\widetilde{W}$ have at ∞ asymptotic limits whose Conley-Zehnder indices are at least* 1.

Proof. The generalized finite energy plane $\widetilde{u} : \mathbb{C} \to \widetilde{W}$ factors through a somewhere injective finite energy plane $\widetilde{v} : \mathbb{C} \to \widetilde{W}$ so that
$$\widetilde{u} = \widetilde{v} \circ p$$

for a polynomial map $p : \mathbb{C} \to \mathbb{C}$; see Proposition 6.2 in [30]. Since $\widehat{J}$ is generic on $\widetilde{W}$ we know from Theorem 5.3 that $\mu(\widetilde{v}) \geq 2 - \sharp\Gamma$. In the case at hand, $\sharp\Gamma = 1$ and $\mu(\widetilde{v}) = \mu^+(\widetilde{v})$ and hence $\mu^+(\widetilde{v}) \geq 1$. Since a positive μ-index cannot decrease under iteration (see Proposition 8.2 in the appendix) we conclude $\mu^+(\widetilde{u}) \geq \mu^+(\widetilde{v}) \geq 1$ as claimed. □

If the plane is contained in $\mathbb{R}\times M$ we have the following estimate from [30].

Proposition 5.5. *Let $\widetilde{u} : \mathbb{C} \to \mathbb{R} \times M$ be a $\widetilde{J}$-finite energy plane, then $\mu(\widetilde{u}) \geq 2$.*

From the Fredholm theory in Theorem 2.1, we recall:

Proposition 5.6. *If $\widetilde{u} : S^2 \setminus \Gamma \to \mathbb{R} \times M$ is a somewhere injective finite energy sphere for $\widetilde{J}$, then its Fredholm index satisfies*

$$\mathrm{Ind}(\widetilde{u}) = \mu(\widetilde{u}) - 2 + \sharp\Gamma \geq 1$$

provided $\pi \circ Tu \neq 0$.

We shall use the above information in order to prove:

Proposition 5.7. *Assume that J and $\widehat{J}$ are generic. Then the finite energy spheres occurring in the bubbling off tree have asymptotic limits whose indices belong to the set $\{1, 2, 3\}$.*

Proof. In order to show first that the indices are all ≥ 1 we proceed by induction starting at the bottom of the tree and working all the way up to w_∞. The white dots and the gray dots at the bottom represent finite energy planes in $\widetilde{W}$ and $\mathbb{R} \times M$ respectively. Here we know from Proposition 5.4 and Proposition 5.5 that the positive puncture has index ≥ 1. We now delete all these dots from the graph and attach the Conley-Zehnder indices ≥ 1 to the free edges. If there are no gray dots left with a free edge we are done. Otherwise we look at a gray dot having one or several free edges. It represents a finite energy sphere $\widetilde{u} = (a, u) : S^2 \setminus \Gamma \to \mathbb{R} \times M$ of type (M). We have to distinguish between the cases $\pi \circ Tu \neq 0$ and $\pi \circ Tu = 0$.

Consider the case $\pi \circ Tu \neq 0$ and denote by $\widetilde{v}$ the underlying somewhere injective sphere in $\mathbb{R} \times M$ satisfying $\widetilde{u} = \widetilde{v} \circ p$. The polynomial $p : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ maps the punctures Γ onto the punctures Γ' of $\widetilde{v}$. Since J is generic we conclude from Proposition 5.6 that

$$\mathrm{Ind}(\widetilde{v}) = \mu(\widetilde{v}) - 2 + \sharp\Gamma' \geq 1. \qquad (5.5)$$

Here $\mu(\widetilde{v}) = \mu^+(\widetilde{v}) - \mu^-(\widetilde{v})$. For the computation of the indices we take a trivialization of ξ over the disc spanning the positive limit in M which is defined by gluing to the image of v in M suitable discs spanning the negative limits of the previous generalized finite energy planes whose indices as we know already are ≥ 1. Since a periodic orbit with index at most 0 will have iterated indices with value at most 0, it follows that $\widetilde{v}$ has negative limits whose indices are ≥ 1. This implies that the index $\mu^+(\widetilde{v})$ of the positive puncture of $\widetilde{v}$ is at least 2. Indeed, in view of (5.5) we can estimate

$$\begin{aligned}\mu^+(\widetilde{v}) &\geq \mu^-(\widetilde{v}) + 2 + 1 - \sharp\Gamma' \\ &\geq (\sharp\Gamma' - 1) + 3 - \sharp\Gamma' = 2.\end{aligned}$$

Consequently, $\mu^+(\widetilde{u}) \geq \mu^+(\widetilde{v}) \geq 2$ in case $\deg(p) = 1$ and $\mu^+(\widetilde{u}) \geq 4$ if $\deg(p) \geq 2$.

Next consider the case $\pi \circ Tu = 0$; then $\widetilde{u}$ is a connector and hence has at least two negative punctures; see Definition 4.11. The map can be written as $\widetilde{u} = \widetilde{v} \circ p$ with a polynomial p as above and with $\widetilde{v}\left(\mathrm{e}^{2\pi(s+\mathrm{i}t)}\right) = \left(Ts + c, x(Tt)\right)$ for a simply covered T-periodic solution $x(t)$ of X_λ on M. Hence the punctures of $\widetilde{u}$ have asymptotic limits of the form (x, k_jT). Denote by

$$k_0 = \sum_{j=1}^{\sharp\Gamma^-} k_j$$

the covering number of the positive puncture of $\widetilde{u}$. Then $k_0 \geq 2$. Since the negative punctures have indices at least 1, the positive puncture also has an index at least 1. Indeed, by Proposition 8.2 if $\mu(x, k_jT) \geq 1$, then $\mu(x, T) \geq 1$ so that $\mu(x, k_0T) \geq 1$. To summarize, the positive puncture of a finite energy sphere $\widetilde{u}$ in $\mathbb{R} \times M$ has an index $\mu^+(\widetilde{u}) \geq 1$.

Since the asymptotic limits of the positive punctures agree with the asymptotic limits of the negative punctures of the next generation up the tree we conclude, iterating our procedure all the way up, that all indices are ≥ 1.

It remains to prove that the indices are < 4. Arguing by contradiction we assume that a finite energy sphere in the bubbling off tree has a negative puncture whose asymptotic limit has an index ≥ 4. We already know that all the other negative punctures of this sphere have indices ≥ 1. Hence arguing as above we conclude that the positive puncture of this sphere has index ≥ 4. Iteratively working up the tree we conclude that w_∞ has at least one negative puncture with index ≥ 4 while all the other negative punctures have indices ≥ 1. Recalling the Fredholm index formula (5.4)

$$\begin{aligned}\mathrm{Ind}(w_\infty) &= \mu(w_\infty) + \sharp\Gamma + 2 \\ &= \mu(w_\infty) - 2 + \sharp\Gamma + 4\end{aligned}$$

we obtain, in view of $\mu(w_\infty) = -\mu^-(w_\infty)$ the estimate

$$\mathrm{Ind}(w_\infty) \leq -4 - (\sharp\Gamma - 1) - 2 + \sharp\Gamma + 4 = -1.$$

This, however, contradicts the fact that $\mathrm{Ind}(w_\infty) \geq 0$ for our generic $\widehat{J}$. Having shown that all indices in the bubbling off tree are < 4 the proof of Proposition 5.7 is complete. □

From Proposition 5.7 we deduce, using Fredholm theory, the following proposition.

PROPOSITION 5.8. *Under the genericity assumptions specified above, all solutions of type* (M) *occurring in the bubbling off tree are somewhere injective, provided* $\pi \circ Tu \neq 0$.

Proof. Assume $\widetilde{u} : S^2 \setminus \Gamma \to \mathbb{R} \times M$ is a finite energy sphere obtained through the bubbling off analysis and satisfying $\pi \circ Tu \neq 0$. Arguing by contradiction we assume that $\widetilde{u}$ is not somewhere injective. Then there exist a somewhere injective finite energy sphere $\widetilde{v} : S^2 \setminus \Gamma' \to \mathbb{R} \times M$ and a polynomial map p with $\deg(p) \geq 2$ satisfying $\widetilde{u} = \widetilde{v} \circ p$ and $p(\Gamma) = \Gamma'$. Since $\widetilde{v}$ is somewhere injective we know from Proposition 5.6 that the Fredholm index of $\widetilde{v}$ is estimated by

$$\mathrm{Ind}(\widetilde{v}) = \mu(\widetilde{v}) - 2 + \sharp\Gamma' \geq 1.$$

From Proposition 5.7 we deduce that the indices of the negative punctures of $\widetilde{v}$ are ≥ 1. Therefore, the positive puncture has an index $\mu^+(\widetilde{v}) \geq (\sharp\Gamma' - 1) + 3 - \sharp\Gamma' = 2$ and hence $\mu^+(\widetilde{u}) \geq 4$ in view of $\deg(p) \geq 2$. This contradicts $\mu^+(\widetilde{u}) < 4$ and proves Proposition 5.8. □

Another consequence of Proposition 5.7 is the following:

PROPOSITION 5.9. *Assume the solution* $\widetilde{u} = (a, u) : S^2 \setminus \Gamma \to \mathbb{R} \times M$ *of type* (M) *produced through the bubbling off analysis satisfies* $\pi \circ Tu \neq 0$. *Then* $\mathrm{wind}_\pi(\widetilde{u}) = 0$, *and every puncture* $z \in \Gamma$ *has the asymptotic winding number*

$$\mathrm{wind}_\infty(z) = 1.$$

Proof. From [30, Prop. 5.6], we recall the formula

$$\mathrm{wind}_\pi(\widetilde{u}) = \mathrm{wind}_\infty(\widetilde{u}) - 2 + \sharp\Gamma, \tag{5.6}$$

where, splitting the punctures $\Gamma = \Gamma^+ \cup \Gamma^-$ into positive and negatives ones,

$$\mathrm{wind}_\infty(\widetilde{u}) = \sum_{z \in \Gamma^+} \mathrm{wind}_\infty(z) - \sum_{z \in \Gamma^-} \mathrm{wind}_\infty(z). \tag{5.7}$$

If the asymptotic limits of the punctures have their indices in $\{1, 2, 3\}$ the inequalities given in (84) in [30] become

$$\begin{aligned} \mathrm{wind}_\infty(z) &\leq 1 \qquad \text{if } z \in \Gamma^+, \\ \mathrm{wind}_\infty(z) &\geq 1 \qquad \text{if } z \in \Gamma^-. \end{aligned} \tag{5.8}$$

This implies for the case at hand, in which $\Gamma^+ = \{\infty\}$,

$$\mathrm{wind}_\infty(\widetilde{u}) \leq 1 - (\sharp\Gamma - 1) = 2 - \sharp\Gamma.$$

From $\mathrm{wind}_\pi(\widetilde{u}) \geq 0$ we conclude

$$\mathrm{wind}_\infty(\widetilde{u}) \geq 2 - \sharp\Gamma.$$

Consequently, $\mathrm{wind}_\infty(\widetilde{u}) = 2 - \sharp\Gamma$ and hence $\mathrm{wind}_\pi(\widetilde{u}) = 0$. Therefore, in view of (5.7),

$$\mathrm{wind}_\infty(\infty) - 1 = \sum_{z\in\Gamma^-} \mathrm{wind}_\infty(z) - \sharp\Gamma^- = \sum_{z\in\Gamma^-} (\, \mathrm{wind}_\infty(z) - 1)\,.$$

Recalling (5.8), we conclude $\mathrm{wind}_\infty(z) = 1$ for every $z \in \Gamma$. □

In order to establish the crucial property of the bubbling off tree we need a result about self-linking numbers. Recalling the gluing construction in Section 2.2 we consider a compact symplectic 4-manifold A with a convex contact type boundary $\partial A = B^+$, in the following denoted by $B = B^+$. Gluing $\mathbb{R}^+ \times B$ over the boundary to A we obtain the almost complex manifold $(\widetilde{A}, \widehat{J})$. On $\mathbb{R}^+ \times B$ the almost complex structure $\widehat{J}$ agrees with the $\mathbb{R}$- invariant structure $\widetilde{J}$ determined by the contact form λ on B as defined in (2.16).

Theorem 5.10. *Consider the compact symplectic 4-manifold A with a convex contact type boundary $\partial A = B$. Let $(\widetilde{A}, \widehat{J})$ be the associated almost complex manifold. Assume that $\widetilde{u} : (D, i) \to (\widetilde{A}, \widehat{J})$ is an embedded almost complex disc-map satisfying $\widetilde{u}(\partial D) \subset \{r\} \times B$ for some $r > 0$. Moreover, assume that $u(\partial D) \subset B$ is transversal to the contact structure ξ, where $\widetilde{u}(z) = (a(z), u(z))$ if $\widetilde{u}(z) \in \mathbb{R}^+ \times B$. Assume, in addition, that $\widetilde{u} : D \to \widetilde{A}$ is homotopic (with boundary fixed) to a map $\widetilde{w} = (r, w) : D \to \{r\} \times B$ so that $\widetilde{u}(z) = (r, w(z))$ for $z \in \partial D$. Then the self-linking number of $w(\partial D)$ with respect to the disc w is $\mathrm{sl}(w, w(\partial D)) = -1$.*

Proof. Since $\widetilde{u}$ is almost complex, the symplectic vector bundle $\widetilde{u}^*(T\widetilde{A}) \to D$ admits a splitting into two complex line bundles

$$\widetilde{u}^*(T\widetilde{A}) = T\Big(\widetilde{u}(D)\Big) \oplus N,$$

where $T(\widetilde{u}(D))$ is the complex tangent bundle of $\widetilde{u}(D)$ and where N is a complex line bundle isomorphic to the normal bundle $\widetilde{u}^*(T\widetilde{A})/T(\widetilde{u}(D))$ and equal to ξ over the boundary ∂D. Abbreviate $x(t) = u(\mathrm{e}^{2\pi \mathrm{i}t})$ and consider the splitting

$$\widetilde{u}^*(T\widetilde{A}) = Y_{\mathbb{C}} \oplus Z_{\mathbb{C}}$$

defined by a nowhere vanishing section Y of $\widetilde{u}^*(T\widetilde{A})$ satisfying $Y(\mathrm{e}^{2\pi \mathrm{i}t}) = \dot{x}(t)$ at the boundary ∂D of D, and a nowhere vanishing section Z of $\widetilde{u}^*(T\widetilde{A})$ belonging to ξ over ∂D, and which together with Y spans $\widetilde{u}^*(T\widetilde{A})$ over $\mathbb{C}$. If now C is a nowhere vanishing section of N and H is a nowhere vanishing section of $T(\widetilde{u}(D)) \to D$ we have $Y(z) = a(z)H(z) + b(z)C(z)$ and $Z(z) =$

$c(z)H(z)+d(z)C(z)$ for $z \in D$. Hence we can define the map $\Phi : D \to \mathcal{GL}(\mathbb{C}^2)$ by

$$z \mapsto \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}.$$

If $z \in \partial D$, then $b(z) = c(z) = 0$. Since Φ is a disc-map, the winding number of $z \mapsto \det\Phi(z) = a(z) \cdot d(z) \in \mathbb{C} \setminus \{0\}$ over ∂D necessarily vanishes. At the boundary ∂D we have $Y(z) = a(z)H(z)$ and $Y = \dot{x}$; moreover H extends to a trivialization of the tangent bundle of the embedded disc $\widetilde{u}(D)$. Therefore, the winding number of $a : \partial D \to \mathbb{C} \setminus \{0\}$ is equal to 1 and hence the winding number of $d : \partial D \to \mathbb{C} \setminus \{0\}$ is equal to -1.

Next we push the disc $\mathcal{D} = \widetilde{u}(D)$ in the direction of Z to obtain the new disc $\mathcal{D}'$. By assumption, $\partial\mathcal{D} \subset \{r\} \times B$ and since $Z_{|\partial\mathcal{D}} \subset \xi$ we may also assume that $\partial\mathcal{D}' \subset \{r\} \times B$. Clearly, $\partial\mathcal{D} \cap \partial\mathcal{D}' = \emptyset$ and we claim that

$$\partial\mathcal{D} \cap \mathcal{D}' = \emptyset \quad \text{and} \quad \partial\mathcal{D}' \cap \mathcal{D} = \emptyset. \tag{5.9}$$

To prove this we shall show by means of the maximum principle that the interiors of $\mathcal{D}$ and $\mathcal{D}'$ lie below the level set $\{r\} \times B$, i.e., in $A \bigcup ([0, r) \times B)$. Take a smooth function $f : \widetilde{A} \to \mathbb{R}$ satisfying

$$\begin{aligned} f &= 0 \quad \text{on } A \\ f(s, b) &= \varphi(s) \quad \text{on } \mathbb{R}^+ \times B \end{aligned}$$

where $\varphi \equiv 0$ near $s = 0$, and $\varphi''(s) > 0$ otherwise. Consider the composition

$$\alpha = f \circ \widetilde{u} : D \to \mathbb{R}.$$

If $\widetilde{u}(z) = (a(z), u(z))$ at the points z where $\widetilde{u}(z) \in \mathbb{R}^+ \times B$, we conclude from our assumptions that

$$\alpha(z) = f \circ \widetilde{u}(z) = \varphi(r) \quad \text{if } z \in \partial D.$$

Moreover, for $z \in D$ near ∂D we have $\alpha(z) = f \circ \widetilde{u}(z) = \varphi(a(z))$. Since $\widetilde{u}$ is a $\widehat{J}$-holomorphic disc satisfying $\widetilde{u}(\partial D) \subset \mathbb{R}^+ \times B$ and since $\widehat{J} = \widetilde{J}$ on $\mathbb{R}^+ \times B$, the computations in Lemma 4.4 show that

$$\Delta\alpha \geq 0.$$

Since $\widetilde{u}$ is an embedding, the function $a : D \to \mathbb{R}$ is not constant. We therefore conclude by means of the strong maximum principle that

$$\begin{aligned} \alpha(z) &< \varphi(r) \quad \text{for } z \in \operatorname{int}(D), \\ \tfrac{\partial\alpha}{\partial n}(z) &> 0 \quad \text{for } z \in \partial D. \end{aligned}$$

Consequently, $\operatorname{int}\widetilde{u}(D) = \operatorname{int}(\mathcal{D}) \subset A \bigcup \Big([0, r) \times B\Big)$ as claimed. In view of (5.9) the intersection number $\operatorname{int}(\mathcal{D}, \mathcal{D}')$ is well-defined and we claim that

$$\operatorname{int}(\mathcal{D}, \mathcal{D}') = -1. \tag{5.10}$$

In order to prove this we introduce new complex coordinates in which $\mathcal{D} = D \times \{0\} \subset \mathbb{C} \times \mathbb{C}$ and $H(z) = (0,1) \in \mathbb{C} \times \mathbb{C}$ for $z \in \partial D$. Now $\mathcal{D}' = \{(z, Z(z)) \mid z \in D\}$ with $Z : D \to \mathbb{C}$. By the above discussion the winding number of Z along ∂D is equal to -1 so that Z is homotopic to $\frac{1}{z}(0,1)$ over ∂D. Consequently, $\deg(Z, D, 0) = -1$ and the claim (5.10) follows.

Now we add a piece to the disc $\mathcal{D}'$ above the level $\{r\} \times B$ to obtain the new disc $\widetilde{\mathcal{D}'}$ as illustrated in Figure 16. Clearly, $\text{int}(\mathcal{D}, \mathcal{D}') = \text{int}(\mathcal{D}, \widetilde{\mathcal{D}'})$ since the points of intersection remain the same. Recalling that $\widetilde{u}(D)$ is homotopic (with boundary fixed) to $\widetilde{w}(D)$ we obtain, moving the boundary of $\partial\widetilde{\mathcal{D}'}$ sufficiently high up, that $\text{int}(\mathcal{D}, \mathcal{D}') = \text{int}(\widetilde{w}, \widetilde{\mathcal{D}'}) = -1$. By assumptions, $\widetilde{w}(D) \subset \{r\} \times B$. Hence, by homotopy and the excision, denoting by $I \subset \mathbb{R}^+$ an open interval containing $r \in I$ we obtain:

$$
\begin{aligned}
(5.11) \quad -1 &= \text{int}(\mathcal{D}, \mathcal{D}') = \text{int}(\mathcal{D}, \widetilde{\mathcal{D}'}) = \text{int}(\widetilde{w}, \widetilde{\mathcal{D}'}) \\
&= \text{int}(\widetilde{w}, I \times \partial\mathcal{D}') = \text{int}(\widetilde{w}, \mathbb{R} \times \partial\mathcal{D}') = \text{int}(\{r\} \times w, \mathbb{R} \times \partial\mathcal{D}') \\
&= \text{int}(w, \partial\mathcal{D}').
\end{aligned}
$$

Recall now that Z is a section of ξ over ∂D. Therefore, the last integer in (5.11) is, by definition , equal to the self-linking number $\text{sl}(w, w(\partial D))$ if we show that the section $Z_{|\partial D}$ admits a nonvanishing extension over $w^*\xi$.

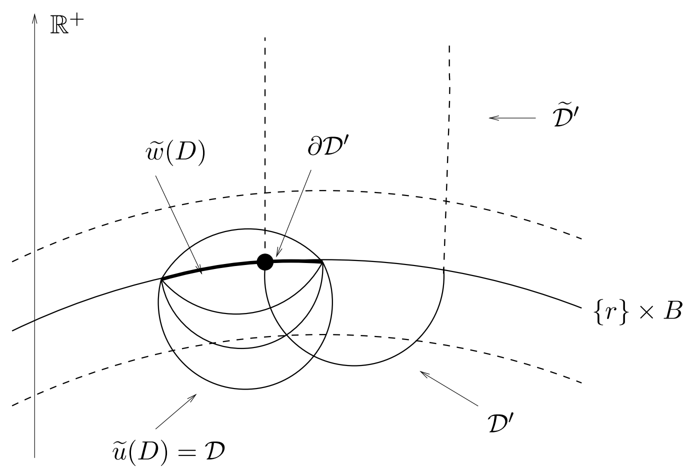

Figure 16. Schematic construction of the disc $\widetilde{D}'$ in the proof of Theorem 5.10.

In order to see this we use a sequence of homotopy arguments. First observe that along the homotopy of $\widetilde{u}$ into $\widetilde{w}$ the pair of linearly independent sections Y and Z of $\widetilde{u}^*(T\widetilde{A})$ can be homotoped to a pair $\hat{Y}$ and $\hat{Z}$ of pointwise linearly independent sections of $\widetilde{w}^*(T\widetilde{A})$ where during the homotopy the sections over the boundary ∂D remain the same. We now have two pointwise linearly independent sections $\hat{Y}$ and $\hat{Z}$ of $\widetilde{w}^*(T\widetilde{A}) = \widetilde{w}^*T(\mathbb{R}^* \times B)$. The loop x

is contained in the level set $\{r\} \times B$ where x is transversal to ξ such that $\dot{x}(t)$ and the Reeb vector field $X(w(e^{2\pi it}))$ point in the same direction. Therefore, $\hat{Y}$ can be homotoped as a nowhere vanishing section into the section $X(w(z))$ for $z \in D$. During this homotopy we can homotope $\hat{Z}$ in such a way that at any moment we have two pointwise linearly independent sections and, moreover, the section $\hat{Z}$ is fixed at the boundary. The newly obtained section $\hat{\hat{Z}}$ is now equal to Z over the boundary ∂D, and linearly independent of $X(w(z))$. Since $T(\mathbb{R} \times B) = \mathbb{C} \cdot X \oplus \xi$, the section $\hat{\hat{Z}}$ can be homotoped to a nowhere vanishing section of ξ when it is fixed over the boundary. This finishes the proof of the theorem. $\square$

We shall make use of Theorem 5.10 in the proof of the following crucial property of the bubbling off tree.

PROPOSITION 5.11. *Assume $\widetilde{u} = (a, u) : \mathbb{C} \setminus \Gamma' \to \mathbb{R} \times M$ is a solution of type* (M) *obtained through the bubbling off analysis satisfying $\pi \circ Tu \neq 0$. Denote the asymptotic limit associated with the puncture $z \in \Gamma'$ by (x_0, T_0). Then T_0 is the minimal period of x_0 and the loop $x_0(\mathbb{R})$ has self-linking number -1.*

Proof. We first consider a positive puncture $z \in \Gamma'$. By the bubbling off analysis, there exists a sequence of Möbius transformations τ_j such that

$$\widetilde{u}_j = \widetilde{w}_{\varphi(j)} \circ \tau_j \to \widetilde{u} = (a, u) \quad \text{in } C^\infty_{\text{loc}}\Big(\mathbb{C} \setminus \Gamma', \mathbb{R} \times M\Big),$$

where the $\mathbb{R}$-components of $\widetilde{u}_j$ are suitably shifted. For sufficiently large R, the loop $u(S_R) \subset M$ is embedded. Indeed, if the cylindrical end is not embedded one concludes by the asymptotic behavior of $\widetilde{u}$ near its limit that the surface is multiply covered contradicting the fact established in Proposition 5.8, that the surface is somewhere injective. Moreover, near the limit the loop $u(S_R)$ is transversal to ξ. Set $\widetilde{u}_k(z) = (a_k(z), u_k(z))$ and consider the loops $\{u_k(z) \mid a_k(z) = a\}$. Then for k and a sufficiently large, the loop is transversal to ξ and by construction bounds an embedded pseudoholomorphic disc satisfying the hypotheses of Theorem 5.10. By the invariance of the self-linking number under deformations of loops transversal to ξ, we conclude from Theorem 5.10 that the self-linking number of $u(S_R)$ is equal to -1 if R is sufficiently large.

Recall that $\text{wind}_\infty(\infty) = 1$ in view of Proposition 5.9. We now work in the local coordinates near the asymptotic limit as described in the appendix and use the asymptotic formula for $\widetilde{u}$. Then up to a small isotopy irrelevant for our arguments we may assume that

$$u(s, t) = (kt, e^{\lambda s} e(t)) \subset \mathbb{R} \times \mathbb{C}. \tag{5.12}$$

Here k is the asymptotic covering number and $\lambda < 0$. Moreover, we can find local coordinates such that the loop $e(t) \in \mathbb{C} \setminus \{0\}$ has winding number equal to 1. Note that the first component in (5.12) is to be understood $\mod \mathbb{Z}$. If s_0 is large, the self-linking number $\mathrm{sl}(y)$ of the loop $y(t) = u(s_0, t)$ as a loop in S^3 is equal to

$$\mathrm{sl}(y) = -1 \tag{5.13}$$

as we have just concluded from Theorem 5.10. Recall that the self-linking number is computed by shifting y in the direction of $Z(y(t))$ into the loop y' disjoint from y. Here Z is a nowhere vanishing section of $q^*\xi$, where $q : D \to M$ is a disc map which at the boundary ∂D satisfies

$$q(\mathrm{e}^{2\pi i t}) = y(t).$$

Then $\mathrm{sl}(y)$ is the intersection number between q and the shifted loop y'. We extend q to another disc map q' by adding the closure of $u([s_0, \infty) \times S^1)$ to the disc q. This way we obtain a disc map whose boundary parametrizes the asymptotic limit (x_0, T_0) of u. By definition of the self linking number of y',

$$\mathrm{sl}(y) = \mathrm{int}(q, y') = \mathrm{int}(q', y') - \mathrm{int}\Big(u([s_0, \infty) \times S^1), y'\Big) \tag{5.14}$$

and we claim that

$$\mathrm{int}\Big(u([s_0, \infty) \times S^1), y'\Big) = 1. \tag{5.15}$$

This will follow from the representation (5.12) for u. The loop y' may be given by $(kt, \mathrm{e}^{\lambda s_0} e(t) + \varepsilon)$ for some $\varepsilon \neq 0$ small. Hence the intersection points of $u([s_0, \infty) \times S^1)$ with y' are the solutions of the equations

$$\mathrm{e}^{\lambda s_0} e\Big(t + j/k\Big) - \mathrm{e}^{\lambda s} e(t) + \varepsilon = 0$$

for $j = 0, 1, \ldots, (k-1)$. The degree of the map is the difference between the winding numbers for s large and $s = s_0$. Since the winding number of $e(t)$ is equal to 1, we obtain that the mapping degree is equal to 1 if $j = 0$ and equal to 0 if $j = 1, \ldots, (k-1)$, hence proving the claim (5.15). Next we claim that

$$\mathrm{int}(q', y') = \mathrm{sl}(x_0, T_0) + k. \tag{5.16}$$

Indeed, choose $s_1 > s_0$. Then

$$\begin{aligned} \mathrm{int}(q', y') &= \mathrm{int}(q', y) = \mathrm{int}\Big(q', u(s_0, \cdot)\Big) = \mathrm{int}\Big(q', u(s_1, \cdot)\Big) \\ &= \mathrm{int}\Big(q, u(s_1, \cdot)\Big) + \mathrm{int}\Big(u([s_0, \infty) \times S^1), u(s_1, \cdot)\Big). \end{aligned}$$

Shifting x_0 to $x_0' = (kt, \delta)$ for some $\delta \neq 0$ small we obtain, by the definition of the self-linking number of (x_0, T_0),

$$\mathrm{sl}(x_0, T_0) = \mathrm{int}(q', x_0') = \mathrm{int}(q, x_0') + \mathrm{int}\Big(u([s_0, \infty) \times S^1), x_0'\Big).$$

Since $s_1 > s_0$, a homotopy avoiding the boundary of q shows that

$$\mathrm{int}\Big(q, u(s_1, \cdot)\Big) = \mathrm{int}(q, x_0').$$

Hence

$$\mathrm{sl}(x_0,T_0)=\mathrm{int}\Big(q,u(s_1,\cdot)\Big)+\mathrm{int}\Big(u([s_0,\infty)\times S^1),x_0'\Big).$$

Consequently,

$$\begin{aligned}\mathrm{int}(q',y')-\mathrm{sl}(x_0,T_0) &= \mathrm{int}\Big(u([s_0,\infty)\times S^1),u(s_1,\cdot)\Big)\\ &\quad-\ \mathrm{int}\Big(u([s_0,\infty)\times S^1),x_0'\Big)\\ &= -\mathrm{int}\Big(u([s_0,\infty)\times S^1),x_0'\Big).\end{aligned}$$

We have used that since the cylinder is embedded, the first term $\mathrm{int}(u([s_0,\infty)\times S^1),u(s_1,\cdot))$ vanishes. Indeed, by shifting the loop $u(s_1,\cdot)$ normal to the cylinder we obtain a loop disjoint from the cylinder. It remains to show that

$$\mathrm{int}\Big(u([s_0,\infty)\times S^1),x_0'\Big)=-k. \tag{5.17}$$

The intersection points are solutions of

$$\mathrm{e}^{\lambda s}e\Big(t+j/k\Big)-\delta=0,$$

for $j=0,1,\ldots,(k-1)$. Since the winding number of $e(t)$ is equal to 1 we obtain for every j that the mapping degree is equal to -1 hence proving (5.17). The claim (5.16) is proved. Summarizing we conclude from (5.13), (5.15) and (5.16),

$$-1=\mathrm{sl}(y)=\mathrm{sl}(x_0,T_0)-1+k$$

so that $\mathrm{sl}(x_0,T_0)=-k$. On the other hand, with $T_0=kT$, we obtain $\mathrm{sl}(x_0,T_0)=\mathrm{sl}(x_0,kT)=k^2\mathrm{sl}(x_0,T)$. Combining this with our calculation $\mathrm{sl}(x_0,T_0)=-k$ we deduce the equality

$$k^2\mathrm{sl}(x_0,T)=-k$$

which implies that $k=1$ and $\mathrm{sl}(x_0,T)=-1$. In particular, $T_0=T$ so that T_0 is the minimal period of x_0. The same arguments apply to the negative punctures and the proof of Proposition 5.11 is complete. □

Since its asymptotic limits are simply covered, the surface $\widetilde{u}$ constructed in Proposition 5.11 is an embedding near the boundary. Since it is by construction the limit of embedded curves, $\widetilde{u}$ must be an embedding by the results of McDuff in [43]. Moreover, this $\widetilde{u}$ satisfies the hypotheses of Corollary 2.3 and Theorem 2.7. Using these results we can summarize the properties of $\widetilde{u}$ in the following theorem.

THEOREM 5.12. *A solution $\widetilde{u}=(a,u):\mathbb{C}\setminus\Gamma'\to\mathbb{R}\times M$ of type* (M) *obtained through our bubbling off analysis satisfying $\pi\circ Tu\neq 0$ has the following properties*:

1. *$\widetilde{u}$ is a proper embedding.*

2. *The asymptotic limits associated with the punctures Γ' are simply covered and have self-linking numbers equal to -1.*

3. *The Conley-Zehnder index of the positive puncture is equal to 2 or to 3. The indices of the negative punctures belong to the set $\{1, 2\}$.*

4. *The Fredholm index of $\widetilde{u}$ satisfies $\mathrm{Ind}(\widetilde{u}) \in \{1, 2\}$.*

5. *The map $u : \mathbb{C} \setminus \Gamma' \to M$ is an embedding transversal to the Reeb vector field and converging at the punctures Γ' to the asymptotic limits of Γ'.*

According to Corollary 2.2 the possible configurations projected into M are illustrated in Figure 17.

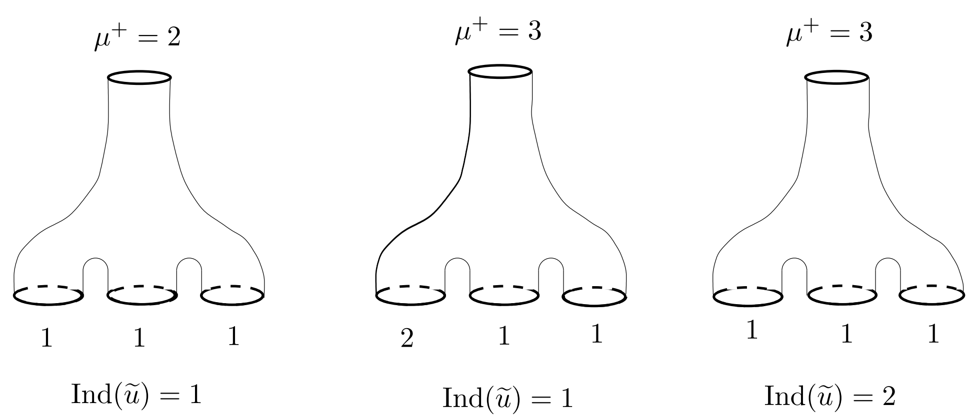

Figure 17. Possible configurations of finite energy spheres projected into M.

We point out that there are no connectors in the bubbling off tree "below" a surface of type (M) satisfying $\pi \circ Tu \neq 0$. Indeed, a connector $\widetilde{u} : \mathbb{C} \setminus \Gamma' \to \mathbb{R} \times M$ satisfies $\pi \circ Tu = 0$ and hence has at least 2 negative punctures because of Proposition 4.10. Hence the asymptotic limit of the positive puncture of a connector is always multiply covered. Therefore, it cannot be the limit of a negative puncture of an (M)-type surface satisfying $\pi \circ Tu \neq 0$ which by Theorem 5.12 is simply covered.

6. Construction of a stable finite energy foliation

We shall use the results from the previous sections in order to establish a stable finite energy foliation.

6.1. *Construction of a dense set of leaves.* We first recall our earlier bubbling off construction. Choose $m \in M$ and consider, for the given point $(0, m) \in A_N$, the sequence of finite energy spheres $C^N \in \mathcal{M}_N^0$ uniquely determined by $(0, m) \in C^N$. Choose the $\widehat{J}_N$-holomorphic parametrization $w_N : S^2 \to A_N$ used in the bubbling off analysis of Section 3 satisfying

$$w_N(S^2) = C^N, \quad w_N(\infty) = o_\infty, \quad \int_D w_N^* \omega_N = \pi - \gamma.$$

In addition, $w_N(0)$ either lies in $\widehat{W}$ or, in case $w_N(S^2) \cap \widehat{W} = \emptyset$, belongs to $[-N, N] \times M$ and has the lowest possible $\mathbb{R}$-value. We constructed an injective monotonic map $\psi : \mathbb{N} \to \mathbb{N}$ satisfying

$$w_{\psi(N)} \to w_\infty \qquad \text{in } C^\infty_{\text{loc}}(S^2 \setminus \Gamma, \widetilde{V}). \tag{6.1}$$

We can assume that ψ is already the subsequence for which all reparametrized maps occurring during the inductive bubbling off analysis do converge. Define the sequence $\zeta_N \in S^2$ by

$$w_{\psi(N)}(\zeta_N) = (0, m) \in A_N.$$

For the limit construction in (6.1) we have identified $[-N, N] \times M$ with $[-2N, 0] \times M$. Consequently, $\zeta_N \to z^{i_1}$ for a bubbling off point $z^{i_1} \in \Gamma$. Next, considering this puncture z^{i_1} we constructed special sequences $\gamma_N \to z^{i_1}$ and $\delta_N \to 0$ in order to define the rescaled maps $\widetilde{v}_N$ by

$$\widetilde{v}_N(z) = w_{\psi(N)}(\gamma_N + \delta_N z);$$

see (4.1). Define the new sequence $z_N \in \mathbb{C}$ by

$$\widetilde{v}_N(z_N) = (0, m) \in A_N.$$

Then, by Lemma 4.4 and Lemma 4.5, the sequence $\widetilde{v}_N$ converges to $\widetilde{v}_\infty$ either in $C^\infty_{\text{loc}}(\mathbb{C}, \widetilde{W})$ or, modulo adding suitable constants to the $\mathbb{R}$-components, in $C^\infty_{\text{loc}}(\mathbb{C} \setminus \Gamma^{i_1}, \mathbb{R} \times M)$. In the first case we identified $[-N, N] \times M$ with $[0, 2N] \times M$ so that $|z_N| \to \infty$. We conclude that $(0, m)$ lies on the cylinder of the distinguished periodic orbit which is the asymptotic limit of $\widetilde{v}_\infty$ associated with the puncture ∞. In the second case we have three alternatives. Either $|z_N| \to \infty$, or the sequence (z_N) remains bounded but stays away from Γ^{i_1} for large N, or has a subsequence converging to a bubbling off point in Γ^{i_1}. If the first alternative holds, then again $(0, m)$ lies on the cylinder of the periodic orbit associated with the puncture ∞. If the second alternative holds we conclude that $(0, m) \in \text{ image}(\widetilde{v}_\infty) \subset \mathbb{R} \times M$. Finally, if the third alternative holds, then we continue with the bubbling off analysis as before and again have the three alternatives.

The bubbling off analysis terminates after finitely many steps due to the lack of negative punctures. Therefore, after finitely many steps either the first or the second alternative holds. Consequently, if m does not lie on a periodic orbit, then there exists a sequence of Möbius transformations τ_N satisfying $\tau_N(\infty)=\infty$ such that the maps $g_N := w_{\psi(N)}\circ\tau_N$ have the following properties. The sequence g_N converges in $C^\infty_{\text{loc}}(\mathbb{C}\setminus\Gamma', \mathbb{R}\times M)$ to a finite energy sphere g containing the point $(0,m)$ in its image. Moreover, by Theorem 5.12 the finite energy sphere is an embedding, its asymptotic limits are simply covered, have self-linking numbers equal to -1 and indices belonging to the set $\{1,2,3\}$. Moreover, the Fredholm index $\text{Ind}(g)$ belongs to the set $\{1,2\}$.

Now, since λ is nondegenerate we can choose a dense sequence $m_k\in M$ so that no point in this sequence lies on a periodic orbit. For every k we take the uniquely determined pseudoholomorphic sphere C^k_N in $\mathcal{M}^0_N$ containing the point $(0,m_k)$. Then we find a sequence $\psi_k:\mathbb{N}\to\mathbb{N}$ consisting of injective monotonic maps, and finite energy spheres

$$C^k_\infty\subset\mathbb{R}\times M$$

such that for suitable parametrizations w^k_N of $C^k_{\psi_k(N)}$ we can pass to the limit as $N\to\infty$ in order to have a parametrization of C^k_∞. Moreover, $(0,m_k)\in C^k_\infty$. We know that each C^k_∞ is embedded, has simply covered asymptotic limits whose self-linking numbers are equal to -1. We also know that there is precisely one positive puncture but an arbitrary number of negative punctures. Further, the asymptotic limits have indices in $\{1,2,3\}$ and $\text{Ind}(C^k_\infty)\in\{1,2\}$. In addition, by the positivity of intersections of pseudoholomorphic curves, two surfaces C^i_∞ and C^j_∞ are either identical or disjoint. Indeed, if two such surfaces intersect but are not identical, then they have an isolated intersection. This implies, assuming $i<j$, that $C^j_{\psi_j(N)}$ and $C^i_{\psi_j(N)}$ intersect for large N in a point different from o_∞, which is not possible. We have proved the following result.

THEOREM 6.1. *Consider a nondegenerate contact form $\lambda=f\lambda_0$ on S^3, a generic admissible multiplication $J:\xi\to\xi$ and a dense sequence m_k on S^3 such that m_k does not lie on a periodic orbit. Then there exist a constant $c>0$ and for every point $(0,m_k)$ a finite energy sphere $C^k\subset\mathbb{R}\times M$ having the following properties*:

1. *The point $(0,m_k)$ belongs to C^k and $E(C^k)\le c$.*

2. *C^k is properly embedded and has precisely one positive puncture. The asymptotic limits are simply covered, have self-linking numbers equal to -1 and Conley-Zehnder indices in the set $\{1,2,3\}$.*

3. $\mathrm{Ind}(C^k) = \mu(C^k) - 2 + \sharp\Gamma \in \{1, 2\}$.

4. *If* $C^i \cap C^j \neq \emptyset$, *then* $C^i = C^j$.

To distinguish the types of finite energy spheres we introduce the vectors $\alpha = (\mu^+, \mu_1^-, \ldots, \mu_N^-)$, where N is the number of negative punctures, μ^+ the Conley-Zehnder index of the positive puncture, and μ_j^- the index of the j^{th} negative puncture ordered so that $\mu_j^- \geq \mu_{j+1}^-$. In view of Corollary 2.3, the following vectors are the only possibilities (see Figure 17):

$$
\begin{aligned}
\alpha &= (3, 1_1, 1_2, \ldots, 1_N), \\
\alpha &= (3, 2, 1_1, 1_2, \ldots, 1_{N-1}), \\
\alpha &= (2, 1_1, 1_2, \ldots, 1_N).
\end{aligned}
$$

In the first case $\mathrm{Ind}(C) = 2$ while $\mathrm{Ind}(C) = 1$ in the second and the third case. The numbers N, respectively $N - 1$, of negative punctures having indices equal to 1 can, of course, be zero. If this happens, the first and the third case correspond to finite energy planes, while the second case corresponds to a cylinder connecting a periodic orbit of index 3 with a periodic orbit of index 2.

We next find a dense set m_k for which only the first possibility occurs.

Proposition 6.2. *Assume that* λ *and* J *are generic as in Theorem* 6.1. *Consider*, *for a given* $c > 0$ *the set of embedded finite energy spheres* $\tilde{u} : S^2 \setminus \Gamma \to \mathbb{R} \times S^3$ *having one positive puncture and an arbitrary number of negative punctures*, *satisfying* $\pi \circ Tu \neq 0$ *and* $E(\tilde{u}) \leq c$, *having simply covered asymptotic limits whose indices belong to the set* $\{1, 2, 3\}$ *and* $\mathrm{Ind}(\tilde{u}) = 1$. *Then this set is*, *modulo reparametrizations and* $\mathbb{R}$*-action*, *finite.*

Proof. Arguing by contradiction we find a sequence $\tilde{u}_k : S^2 \setminus \Gamma_k \to \mathbb{R} \times S^3$ of embeddings all either of type $\alpha = (3, 2, 1_1, \ldots, 1_N)$ or of type $\alpha = (2, 1_1, \ldots, 1_N)$ and having the same fixed, simply covered asymptotic limits. From the complete bubbling off analysis as carried out below, one deduces for a subsequence that $\Gamma_k \to \Gamma$ with $\sharp\Gamma_k = \sharp\Gamma$, and $\tilde{u}_k \to \tilde{u}_\infty$ in $C^\infty_{\text{loc}}(S^2 \setminus \Gamma, \mathbb{R} \times S^3)$. Moreover, the limit $\tilde{u}_\infty$ is an embedded finite energy surface of type $\alpha = (3, 2, 1_1, \ldots, 1_N)$ resp. of type $\alpha = (2, 1_1, \ldots, 1_N)$ having the same fixed asymptotic limits as $\tilde{u}_k$. Hence $\mathrm{Ind}(\tilde{u}_\infty) = 1$. However, this contradicts the fact that, for generic $\tilde{J}$, such an $\tilde{u}_\infty$ is (up to the $\mathbb{R}$-action and reparametrization), isolated in view of the implicit function theorem in [36], proving the proposition. ◻

We now delete all the points m_j in Theorem 6.1 for which $(0, m_j)$ lies on a surface as described in Proposition 6.2. The remaining sequence is, of course, still dense. The new sequence we denote by m_j again. With C^j we denote the finite energy surface in $\mathbb{R}\times S^3$ guaranteed by Theorem 6.1 which contains $(0,m_j)$. The associated vectors are now all of the form

$$\alpha=(3,1_1,1_2,\ldots,1_N),$$

where $N=0$ indicates a finite energy plane. We note that the bound $E(C^j)\le c$ for the energies implies the bound $T\le c$ for the periods of the asymptotic limits.

6.2. *Bubbling off as $m_k\to m$.* First we introduce the c-spectrum $\sigma_c(\lambda)$ of the nondegenerate contact form λ. It consists of all strictly positive numbers of the form

$$T-\sum_{k=1}^{N}T_k,\qquad N\ge 0, \tag{6.2}$$

where T and T_k are periods of periodic orbits of the Reeb vector field X_λ which are bounded above by the constant c. Since λ is nondegenerate, the c-spectrum of λ is a finite set. In the following $\sigma_0>0$ denotes a number which is strictly smaller than the minimum of $\sigma_c(\lambda)$. In particular, σ_0 is smaller than the minimum of all periods bounded by c.

Now, fix $m\in S^3$ and choose an injective monotonic map $\tau:\mathbb{N}\to\mathbb{N}$ satisfying

$$m_{\tau(j)}\to m\qquad\text{on } S^3. \tag{6.3}$$

To simplify the notation we shall abbreviate $C^j\equiv C^{\tau(j)}$ for the associated embedded surfaces. In view of Proposition 6.2 we can assume that all these surfaces are of type $\alpha=(3,1_1,\ldots,1_N)$. In order to control the behavior of the surfaces as $j\to\infty$ we have to impose suitable normalizations for the $\widetilde{J}$-holomorphic embeddings

$$\widetilde{w}_j=(a_j,w_j):\mathbb{C}\setminus\Gamma_j\to\mathbb{R}\times S^3 \tag{6.4}$$

parametrizing $C^j=\widetilde{w}_j(\mathbb{C}\setminus\Gamma_j)$. Note that the set of punctures is $\Gamma_j\cup\{\infty\}$. Since the energies are uniformly bounded we have $\sharp\Gamma_j\le K<\infty$ independently of j. The normalization is as follows.

If $\Gamma_j=\emptyset$, we assume that the $\mathbb{R}$-component of $\widetilde{w}_j(0)$ has the smallest value. If $\sharp\Gamma_j=1$, we assume $\Gamma_j=\{0\}$. In both cases we require, in addition, the normalization condition

$$\int_{\mathbb{C}\setminus D}w_j^*d\lambda=\frac{\sigma_0}{2}, \tag{6.5}$$

for $\sigma_0>$ as introduced above, with D being the closed unit disc.

If $\sharp\Gamma_j \geq 2$ we assume that $0 \in \Gamma_j$ and require, in addition,

$$\int_{\mathbb{C}\setminus\Gamma_j} w_j^* d\lambda = \int_{D\setminus\Gamma_j} w_j^* d\lambda + \frac{\sigma_0}{2}, \tag{6.6}$$

so that again most of the energy lies in D. The first consequences of these normalizations are the following gradient bounds.

Lemma 6.3. *For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that for all j*

$$\sup\Big\{|\nabla\widetilde{w}_j(z)| \mid z \in \mathbb{C} \setminus \mathrm{int}D_{1+\varepsilon} \text{ and } \mathrm{dist}(z, \Gamma_j) \geq \varepsilon\Big\} \leq C.$$

Proof. Arguing by contradiction we find an $\varepsilon > 0$, a sequence of positive numbers $\varepsilon_j \to 0$ and a sequence $z_j \in \mathbb{C} \setminus \Big(D_{1+\varepsilon/2} \cup B_{\varepsilon/2}(\Gamma_j)\Big)$ satisfying

$$|\nabla\widetilde{w}_j(z_j)|\varepsilon_j \to \infty,$$
$$|\nabla\widetilde{w}_j(z)| \leq 2|\nabla\widetilde{w}_j(z_j)| \qquad \text{for } |z - z_j| \leq \varepsilon_j.$$

Here we made use of Lemma 26 in [24]. Abbreviating $R_j = |\nabla\widetilde{w}_j(z_j)|$ we introduce the rescaled maps

$$\widetilde{v}_j(z) := \Big(a_j(z_j + z/R_j) - a_j(z_j), w_j(z_j + z/R_j)\Big)$$

defined on the $\varepsilon_j R_j$-balls. In these balls we have gradient bounds from which we conclude that a subsequence converges to a nonconstant finite energy plane

$$\widetilde{v}_j \to \widetilde{v} \not\equiv \text{ constant } \quad \text{in } C^\infty_{\mathrm{loc}}(\mathbb{C}, \mathbb{R} \times S^3).$$

Given $R > 0$, the disc $D_{R/R_j}(z_j)$ lies outside of D and away from the punctures Γ_j, provided that j is sufficiently large. Hence, if j is large we can estimate, using the normalization (6.6),

$$\int_{D_R} \widetilde{v}_j^* d\lambda = \int_{D_{R/R_j}(z_j)} \widetilde{w}_j^* d\lambda \leq \int_{\mathbb{C}\setminus(D\cup\Gamma_j)} \widetilde{w}_j^* d\lambda = \frac{\sigma_0}{2}.$$

Consequently, as $j \to \infty$,

$$\int_{D_R} \widetilde{v}^* d\lambda \leq \frac{\sigma_0}{2}.$$

This holds true for every $R > 0$ and we obtain for the energy the estimate

$$\int_{\mathbb{C}} \widetilde{v}^* d\lambda \leq \frac{\sigma_0}{2}.$$

However, the energy of a nonconstant finite energy plane agrees with the period of its asymptotic limit at ∞ which (by definition of σ_0) is larger than σ_0. This contradiction proves the assertion. □

As another consequence of the normalization condition we shall see that the sets $\Gamma_j \subset \mathbb{C}$ of punctures are uniformly bounded.

LEMMA 6.4. *There exists a bounded set $B \subset \mathbb{C}$ such that*

$$\Gamma_j \subset B \quad \textit{for all } j.$$

Proof. If $\sharp\Gamma_j = 1$, the assertion is part of the required normalization. So we assume that $\sharp\Gamma_j \geq 2$ for all j. Arguing indirectly we find a subsequence, denoted again by Γ_j, containing $z_j \in \Gamma_j$ and satisfying $|z_j| \to \infty$. We may assume that

$$|z_j| = \max|\Gamma_j| \to \infty.$$

Furthermore, going over to a subsequence we may assume that the asymptotic limit (x_∞, T_∞) associated with the positive puncture ∞ stays the same for all j. Define now the rescaled maps

$$\widetilde{v}_j(z) = \Big(b_j(z), v_j(z)\Big) = \Big(a_j(z_j z) - a_j(2z_j),\ w_j(z_j z)\Big) \qquad \text{on } \mathbb{C} \setminus \widetilde{\Gamma}_j,$$

where $\widetilde{\Gamma}_j = \frac{1}{z_j}\Gamma_j$. Then

$$\widetilde{\Gamma}_j \subset D \quad \text{and} \quad 0,\ 1 \in \widetilde{\Gamma}_j.$$

Clearly,

$$\int_{D_{1/|z_j|}\setminus\widetilde{\Gamma}_j} v_j^* d\lambda = \int_{D\setminus\Gamma_j} w_j^* d\lambda.$$

Using (6.6) we deduce therefore

$$\int_{\mathbb{C}\setminus\widetilde{\Gamma}_j} v_j^* d\lambda = \int_{D_{1/|z_j|}\setminus\widetilde{\Gamma}_j} v_j^* d\lambda + \frac{\sigma_0}{2}. \tag{6.7}$$

In other words, most of the $d\lambda$-energy of v_j lies in $D_{1/|z_j|} \setminus \widetilde{\Gamma}_j$. Now by an argument as in Lemma 6.3 and the fact that $|z_j| \to \infty$ we conclude that there exists a finite set of bubbling off points $\widetilde{\Gamma} \subset D$ containing 0 and 1 so that

$$\widetilde{v}_j \to \widetilde{v} \qquad \text{in } C^\infty_{\text{loc}}(\mathbb{C} \setminus \widetilde{\Gamma}, \mathbb{R} \times S^3).$$

Denoting the period of the positive asymptotic limit by T_∞ we deduce from the normalization condition (6.7) that for $R > 0$ large enough

$$T_\infty - \sigma_0 \leq \int_{S_R} v_j^* \lambda \to \int_{S_R} v^* \lambda.$$

Hence $\widetilde{v}$ is not constant. In order to estimate the energy we use the normalization condition (6.6) and compute

$$\begin{aligned}\int_{D_R\setminus(\frac{1}{z_j}D\cup\widetilde{\Gamma}_j)} v_j^* d\lambda &= \int_{D_{|z_j|R}\setminus(D\cup\Gamma_j)} w_j^* d\lambda = \int_{D_{|z_j|R}\setminus\Gamma_j} w_j^* d\lambda - \int_{D\setminus\Gamma_j} w_j^* d\lambda \\ &\leq \int_{\mathbb{C}\setminus\Gamma_j} w_j^* d\lambda - \int_{D\setminus\Gamma_j} w_j^* d\lambda = \frac{\sigma_0}{2}.\end{aligned}$$

Consequently, as $j \to \infty$,

$$\int_{D_R \setminus \widetilde{\Gamma}} v^* d\lambda \leq \frac{\sigma_0}{2}$$

and so

$$\int_{\mathbb{C} \setminus \widetilde{\Gamma}} v^* d\lambda \leq \frac{\sigma_0}{2}. \tag{6.8}$$

The set $\widetilde{\Gamma}$ consists of negative punctures and contains 0 and 1. Moreover, in view of the energy estimate (6.6) and Lemma 4.9, the asymptotic limit of the positive puncture at ∞ is the periodic solution (x_∞, T_∞), which is simply covered. If the energy on the left-hand side vanishes, the unique positive puncture would be at least two-fold covered since we have at least two negative punctures. This follows from the classification of such surfaces in [30]. We conclude that the $d\lambda$-energy of $\widetilde{v}$ is positive. By Stokes' theorem it belongs to the c-spectrum of λ and therefore has to exceed σ_0. This contradiction proves the assertion. □

Consider now the sequence $\widetilde{w}_j = \widetilde{w}_{\tau(j)}$ of the normalized parametrizations of C^j. Recall that the set of periods is finite by the hypotheses of nondegeneracy. Passing to a subsequence we may therefore assume that the asymptotic limits associated with the punctures Γ_j of $\widetilde{w}_j$ are the same for all j. We list them as

$$(x_\infty, T_\infty), \quad (x_1, T_1), \quad \cdots, \quad (x_N, T_N), \tag{6.9}$$

where (x_∞, T_∞) is the asymptotic limit of the puncture ∞.

We may also assume that the cardinality of punctures in Γ_j having the same asymptotic limit is independent of j. In view of Lemma 6.4 we find a subsequence satisfying

$$\Gamma_j \to \Gamma \subset \mathbb{C} \tag{6.10}$$

as $j \to \infty$. In addition to the punctures Γ_j, the sequence $\widetilde{w}_j$ might posses a finite set Θ of bubbling off points disjoint from Γ. By Lemma 6.3, $\Theta \subset D$. We deduce the convergence of a subsequence

$$\widetilde{w}_j \to \widetilde{w} = (a, w) \qquad \text{in } C^\infty_{\text{loc}}(\mathbb{C} \setminus (\Gamma \cup \Theta)). \tag{6.11}$$

If $\Gamma_j \neq \emptyset$, then $0 \in \Gamma_j$; hence $0 \in \Gamma$. We claim that Γ consists of nonremovable punctures. Indeed, take $\gamma \in \Gamma$ and $\varepsilon_0 > 0$ so that $B_{\varepsilon_0}(\gamma) \cap B_{\varepsilon_0}(\gamma') = \emptyset$ for all $\gamma' \in \Gamma \setminus \{\gamma\}$. Then there exists a sequence $z_j \in \Gamma_j$ satisfying $z_j \to \gamma$, and we find, using Stokes' theorem:

$$\int_{\partial B_{\varepsilon_0}(\gamma)} w_j^* \lambda \geq \int_{\partial B_{\varepsilon'}(z_j)} w_j^* \lambda \geq \sigma_0$$

if j is large and ε' is sufficiently small. Consequently,

$$\int_{\partial B_\varepsilon(\gamma)} w^*\lambda \geq \sigma_0$$

for every $\varepsilon > 0$, proving the claim. The punctures $\Gamma \cup \Theta$ of $\widetilde{w}$ are all negative. There is a unique positive puncture, namely ∞, whose asymptotic limit is the periodic solution (x_∞, T_∞). As in Section 4 this is a consequence of the energy estimates of $\widetilde{w}_j$ outside of D and of Lemma 4.9. In particular, $\widetilde{w}$ is not constant.

Lemma 6.5. *The finite energy surface* $\widetilde{w} = (a, w) : \mathbb{C} \setminus (\Gamma \cup \Theta) \to \mathbb{R} \times S^3$ *is an embedding. The unique positive puncture* ∞ *has the asymptotic limit* (x_∞, T_∞) *which is simply covered. The asymptotic limits* (x_γ, T_γ) *of the negative punctures* $\gamma \in \Gamma \cup \Theta$ *are simply covered. The* $d\lambda$*-energy is positive*:

$$\int_{\mathbb{C}\setminus(\Gamma\cup\Theta)} w^* d\lambda > 0.$$

Proof. We first show that the energy of $\widetilde{w}$ is positive. If $\Gamma \cup \Theta = \emptyset$, then $\widetilde{w}$ is a nonconstant finite energy plane whose energy is equal to T_∞. If $\Gamma \neq \emptyset$, then $0 \in \Gamma$. Moreover, if $\Gamma = \emptyset$ and $\Theta \neq \emptyset$, then $0 \in \Theta$. Assume now that $\sharp(\Gamma \cup \Theta) = 1$, then $\widetilde{w}$ is a cylinder having the two punctures ∞ and 0, and we deduce from the normalization (6.6) that

$$\int_{\mathbb{C}\setminus D} w_j^* d\lambda = \frac{\sigma_0}{2}.$$

Consequently,

$$\int_{\mathbb{C}\setminus\{0\}} w^* d\lambda \geq \frac{\sigma_0}{2}.$$

Finally, if $\sharp(\Gamma \cup \Theta) \geq 2$, then the energy cannot vanish since the asymptotic limit of the unique positive puncture ∞ is simply covered. We have proved that the $d\lambda$-energy of $\widetilde{w}$ is positive.

It follows that $\pi \circ Tw \neq 0$ and so, $\widetilde{w}$ is not a connector. In view of the asymptotics, $\widetilde{w}$ is somewhere injective. Moreover, $\pi \circ Tw(z) \neq 0$ for every $z \in \mathbb{C} \setminus (\Gamma \cup \Theta)$. Indeed, a zero z_0 of $\pi \circ Tw(z_0) = 0$ is, in view of the similarity principle, isolated and has positive index. Consequently, for j large, the sections $\pi \circ Tw_j$ also possess zeros, which by Proposition 5.9 is not the case. Hence $\widetilde{w}$ is an immersion. Because a self-intersection has a positive intersection number, $\widetilde{w}$ cannot have any self-intersections either. Consequently, $\widetilde{w}$ must be an embedding.

Arguing as in Section 5 one sees that the asymptotic limits of $\widetilde{w}$ are simply covered. Indeed, take the orbit cylinder over an asymptotic limit (x, T). Going back to the origin of our construction we find a sequence of long stretched spheres S^2 in $A_N \equiv \mathbb{C}P^2$ approaching locally the cylinder in C^∞_{loc}. If we cut the cylinder horizontally, the preimages are simply closed curves on S^2. Take the lower parts, whose images do not contain the point o_∞ in $\mathbb{C}P^2$. This way we find a sequence of disc maps whose boundaries approach the asymptotic limit. Using the arguments of Theorem 5.10 and Proposition 5.11 we conclude that (x, T) has self-linking number -1 and minimal period T. The proof of the lemma is complete. □

The same arguments show that all the asymptotic limits showing up in the bubbling off analysis below are simply covered. In order to carry out the bubbling off analysis for the subsequence $\widetilde{w}_j$ we start introducing the masses of the punctures $\gamma \in \Gamma \cup \Theta$ of $\widetilde{w}$.

If $\vartheta \in \Theta$ we define, as before,

$$m(\vartheta) = \lim_{\varepsilon \searrow 0} m_\varepsilon(\vartheta), \quad m_\varepsilon(\vartheta) = \lim_{j\to\infty} \int_{B_\varepsilon(\vartheta)} w_j^* d\lambda. \tag{6.12}$$

If $\gamma \in \Gamma$, then its mass $m(\gamma)$ is defined as follows

$$m(\gamma) = \lim_{\varepsilon \searrow 0} m_\varepsilon(\gamma), \quad m_\varepsilon(\gamma) = \lim_{j\to\infty} \int_{B_\varepsilon(\gamma)\setminus\Gamma_j} w_j^* d\lambda. \tag{6.13}$$

The limit exists by Stokes' theorem and by the C^∞-convergence. For j large and ε small,

$$\begin{aligned}\int_{B_\varepsilon(\gamma)\setminus\Gamma_j} w_j^* d\lambda &= \int_{\partial B_\varepsilon(\gamma)} w_j^*\lambda - \lim_{\varepsilon'\to 0} \sum_{z\in\Gamma_j\cap B_\varepsilon(\gamma)} \int_{\partial B_{\varepsilon'}(z)} w_j^*\lambda \\ &= \int_{\partial B_\varepsilon(\gamma)} w_j^*\lambda - \sum_{z\in\Gamma_j\cap B_\varepsilon(\gamma)} T_z.\end{aligned}$$

Here $T_z \in \{T_1, \ldots, T_N\}$ is the period of the asymptotic limit associated with the puncture $z \in \Gamma_j$. Going over to a subsequence, we see that the last term is independent of j, and hence

$$m(\gamma) = T_\gamma - \sum_{z\in\Gamma_j\cap B_\varepsilon(\gamma)} T_z, \tag{6.14}$$

where T_γ is the period of the periodic orbit (x_γ, T_γ) associated with the puncture $\gamma \in \Gamma$.

In order to evaluate the energy of $\widetilde{w}$ we first note that by Stokes' theorem,

$$T_\infty - \sum_{z\in\Gamma_j} T_z = \int_{\mathbb{C}\setminus\Gamma_j} w_j^* d\lambda$$

for every j. Going over to a subsequence, we see that the left-hand side, by construction, does not depend on j. The right-hand side is, for j large, equal to

$$\int_{\mathbb{C}\setminus B_\varepsilon(\Gamma\cup\Theta)} w_j^* d\lambda + \int_{B_\varepsilon(\Gamma\cup\Theta)\setminus\Gamma_j} w_j^* d\lambda.$$

Taking the limit as $j \to \infty$ and $\varepsilon \to 0$ we obtain

$$T_\infty - \sum_{z \in \Gamma_j} T_z = \int_{\mathbb{C} \setminus (\Gamma \cup \Theta)} w^* d\lambda + \sum_{\gamma \in \Gamma \cup \Theta} m(\gamma). \tag{6.15}$$

Assume $\Theta \neq \emptyset$ and take the bubbling off point $\vartheta \in \Theta$. Then $m(\vartheta) \geq \sigma_0$. The bubbling off analysis for a bubbling off point is the same as in [36] and will be merely sketched. We choose $\varepsilon > 0$ so small that $D_\varepsilon(\vartheta) \cap D_\varepsilon(\gamma) = \emptyset$ for all $\gamma \neq \vartheta \in \Gamma \cup \Theta$. Then choose a sequence z_j having the property that $\widetilde{w}_j(z_j)$ has the smallest $\mathbb{R}$-component on $D_\varepsilon(\vartheta)$, and choose a sequence $\delta_j > 0$ satisfying

$$\int_{D_{\delta_j}(z_j)} \widetilde{w}_j^* d\lambda = m(\vartheta) - \frac{\sigma_0}{2}. \tag{6.16}$$

Thus $z_j \to \vartheta$ and $\delta_j \to 0$ and we define the rescaled maps

$$\widetilde{v}_j(z) := \Big(a_j(z_j + \delta_j z) - a_j(z_j), w_j(z_j + \delta_j z) \Big)$$

for $z \in D_{\varepsilon/\delta_j}$. There exists a finite, possible empty set $\Theta_1 \subset D$ of bubbling off points for the sequence $\widetilde{v}_j$ so that a subsequence converges,

$$\widetilde{v}_j \to \widetilde{v}_\vartheta \qquad \text{in } C^\infty_{\text{loc}}(\mathbb{C} \setminus \Theta_1, \mathbb{R} \times S^3)$$

to a finite energy surface $\widetilde{v}_\vartheta$ having positive energy satisfying

$$m(\vartheta) = \int_{\mathbb{C} \setminus \Theta_1} \widetilde{v}_\vartheta^* d\lambda + \sum_{\vartheta_1 \in \Theta_1} m(\vartheta_1, \widetilde{v}_\vartheta)$$

with $m(\vartheta_1) \geq \sigma_0$. The punctures Θ_1 are all negative. The asymptotic limit of the unique positive puncture ∞ agrees with the asymptotic limit of the negative puncture ϑ of $\widetilde{w}$ we started with. Moreover, the surface is embedded and its asymptotic limits are simply covered. If $\Theta_1 \neq \emptyset$ we iterate the procedure with the punctures $\vartheta_1 \in \Theta_1$ of $\widetilde{v}_\vartheta$. The iteration necessarily stops after finitely many steps due to the lack of masses and we arrive at a bubbling off tree originating from ϑ consisting of embedded finite energy surfaces all having precisely one positive puncture ∞, whose asymptotic limits are simply covered and whose energies are positive. The surfaces at the bottom of the tree are finite energy planes closing off the tree as in Figure 18. We would like to point out that our pictures are schematic representations of the actual situation. In particular, the directions of the approach of two surfaces to a common asymptotic limit are linearly independent. So the surfaces near a common asymptotic limit should be visualized as in Figure 19.

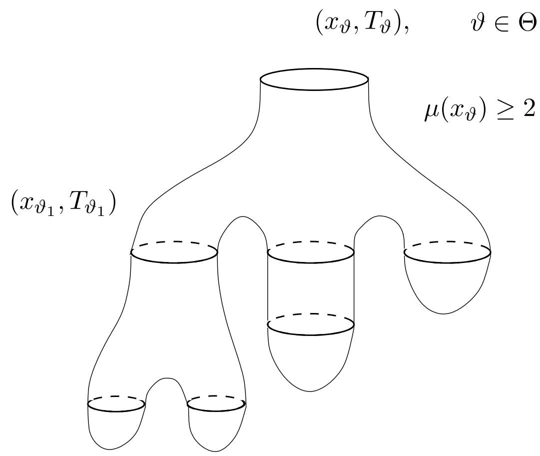

Figure 18. For a generic $\widetilde{J}$ the Conley-Zehnder index of the periodic orbit $(x_\vartheta, T_\vartheta)$ at the top of the tree is ≥ 2.

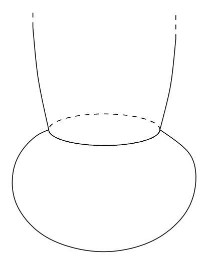

Figure 19. Two surfaces in the tree approach an asymptotic limit in linearly independent directions.

If $\widetilde{J}$ is generic, the Fredholm indices of all surfaces C in the tree are ≥ 1. Indeed, $\text{Ind}(C) = \mu^+ - \mu^- - 2 + (\sharp\Gamma^- + 1) \geq 1$, by Proposition 5.6. Recall that the Conley-Zehnder index of the asymptotic limit of a finite energy plane is ≥ 2, by Proposition 5.5. Consequently, checking successively the Fredholm indices of all the surfaces in the tree generated by ϑ from the bottom to the top we deduce for the periodic orbit $(x_\vartheta, T_\vartheta)$ at the top of the tree the estimate $\mu(x_\vartheta) \geq 2$. Moreover, $\mu(x_\vartheta) = 2$ if and only if the tree consists of a finite energy plane whose asymptotic limit is $(x_\vartheta, T_\vartheta)$ as illustrated in Figure 20.

Consider next a puncture $\gamma \in \Gamma$, and denote the associated asymptotic limit by (x_γ, T_γ). It is simply covered, by Lemma 6.5. Then either $m(\gamma) = 0$ or $m(\gamma) \geq \sigma_0$ since a positive $m(\gamma)$ belongs to the c-spectrum. Assume first $m(\gamma) > 0$. Define

$$\widehat{\Gamma}_j = \Big\{\zeta_j \in \Gamma_j \mid \zeta_j \to \gamma\Big\},$$

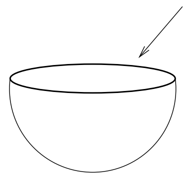

Figure 20. The tree consists of a finite energy plane whose asymptotic limit is $(x_\vartheta, T_\vartheta)$ if and only if $\mu(x_\vartheta, T_\vartheta) = 2$.

and take a sequence $z_j \in \widehat{\Gamma}_j$. Then there exists a sequence $\delta_j \to 0$ satisfying

$$\int_{D_{\delta_j}(z_j)\setminus\widehat{\Gamma}_j} w_j^* d\lambda = m(\gamma) - \frac{\sigma_0}{2}.$$

Choose $\varepsilon_0 > 0$ so small that the discs $D_{\varepsilon_0}(\gamma)$, $\gamma \in \Gamma \cup \Theta$, are disjoint, and define the rescaled maps

$$\widetilde{v}_j(z) = \Big(a_j(z_j + \delta_j z) - a_j(z_j + 2\delta_j), w_j(z_j + \delta_j z)\Big)$$

for $z \in \frac{1}{\delta_j} D_{\varepsilon_0} \setminus \widetilde{\Gamma}_j$, having the punctures

$$\widetilde{\Gamma}_j = \frac{1}{\delta_j}(\widehat{\Gamma}_j - z_j)$$

and satisfying the normalization condition

$$\int_{D\setminus\widetilde{\Gamma}_j} \widetilde{v}_j^* d\lambda = m(\gamma) - \frac{\sigma_0}{2}.$$

Using the fact that $m_\varepsilon(\gamma) \to m(\gamma)$, and choosing $D_{\varepsilon_0}(\gamma)$ sufficiently small, we deduce the estimates

$$\int_{(\frac{1}{\delta_j} D_{\varepsilon_0}\setminus D)\setminus\widetilde{\Gamma}_j} \widetilde{v}_j^* d\lambda \leq \sigma_0,$$

for j large. It follows as in Lemma 6.3 that the gradients of $\widetilde{v}_j$ are uniformly bounded away from D and the punctures $\widetilde{\Gamma}_j$. Moreover, as in Lemma 6.4, the punctures $\widetilde{\Gamma}_j$ lie in a bounded set independent of j. Here we use the fact that the asymptotic limit (x_γ, T_γ) is simply covered. Hence we find a subsequence satisfying

$$\widetilde{v}_j \to \widetilde{v}_\gamma \quad \text{in } C^\infty_{\text{loc}}\Big(\mathbb{C} \setminus (\Gamma^1 \cup \Theta^1)\Big), \qquad \widetilde{\Gamma}_j \to \Gamma^1,$$

where $\Theta^1 \subset D$ is a finite, possibly empty set of bubbling off points for the sequence $\widetilde{v}_j$. The punctures $\Gamma^1 \cup \Theta^1$ of $\widetilde{v}_\gamma$ are all negative. The unique positive puncture ∞ has the periodic orbit (x_γ, T_γ) as asymptotic limit. Moreover,

$$m(\gamma) = \int_{\mathbb{C}\setminus(\Gamma^1\cup\Theta^1)} \widetilde{v}_\gamma^* d\lambda + \sum_{\tau\in\Gamma^1\cup\Theta^1} m(\tau, \widetilde{v}_\tau).$$

The energy of $\widetilde{v}_\gamma$ is positive and hence belongs to the c-spectrum. Indeed, if it vanishes, then $\sharp(\Gamma^1 \cup \Theta^1) \leq 1$ since the limit of the unique positive puncture of $\widetilde{v}_\gamma$ is simply covered. Consequently, $\Gamma^1 = \{0\}$ and $\Theta^1 = \emptyset$ so that $\widetilde{v}_\gamma$ is a cylinder over the periodic orbit having the negative puncture 0. It follows that its mass $m(0, \widetilde{v}_\gamma)$ vanishes so that $m(\gamma) = 0$, in contradiction to $m(\gamma) > 0$.

In the case that $m(\gamma) = 0$ we argue as follows. We find $\varepsilon > 0$ small enough so that $m_\varepsilon(\gamma) \leq \sigma_0/4$. Hence for j large

$$\int_{B_\varepsilon(\gamma)\setminus\Gamma_j} w_j^* d\lambda \leq \frac{\sigma_0}{2}.$$

Denoting $\widehat{\Gamma}_j = \{\zeta_j \in \Gamma_j \mid \zeta_j \to \gamma\}$ we may assume that $\varepsilon > 0$ is so small that $B_\varepsilon \setminus \Gamma_j = B_\varepsilon \setminus \widehat{\Gamma}_j$. We first claim that $\sharp\Gamma_j \cap B_\varepsilon(\gamma) = 1$ for j large. Indeed, if $\Gamma_j \cap B_\varepsilon$ has more than one point, we pick a point $z_j \in \Gamma_j \cap B_\varepsilon(\gamma)$ and let $\hat{z}_j$ be a point in $\Gamma_j \cap B_\varepsilon(\gamma)$ with maximal distance to z_j. Then we take a sequence (τ_j) of Möbius transformations satisfying

$$\tau_j(z_j) = 0, \qquad \tau_j(\hat{z}_j) = 1, \qquad \tau_j(\infty) = \infty$$

and define

$$\widetilde{v}_j(z) = \Big(a_j(\tau_j(z)) - a_j(\tau_j(2)), w_j(\tau_j(z))\Big).$$

Perhaps, after passing to a subsequence we may assume that

$$\tau_j^{-1}(\widehat{\Gamma}_j) \to \widetilde{\Gamma} \subset D,$$

where $\widetilde{\Gamma}$ contains $\{0, 1\}$. Moreover, for evey $R > 0$ and j large we have

$$D_R \subset \tau_j^{-1}(B_\varepsilon(\gamma)).$$

Hence we may assume (modulo $\mathbb{R}$-action) that

$$\widetilde{v}_j \to \widetilde{v}_\infty \qquad \text{in } C^\infty_{\text{loc}}(\mathbb{C} \setminus \widetilde{\Gamma}, \mathbb{R} \times S^3).$$

We observe that $\widetilde{v}_\infty$ has at least two negative punctures, namely 0 and 1. The $d\lambda$-energy satisfies the estimate

$$\int_{\mathbb{C}\setminus\widetilde{\Gamma}} v_\infty^* d\lambda \leq \frac{\sigma_0}{2},$$

so that v_∞ has to be a connector. But this contradicts the fact that the positive asymptotic limit is simply covered. Hence $B_\varepsilon(\gamma) \cap \Gamma_j = \{z_j\}$, and we may assume that at the punctures z_j the surfaces w_j are asymptotic to the same limit, say (x_k, T_k). We arrange reparametrizations by a sequence τ_j of Möbius transformations keeping ∞ fixed and converging to the identity as $j \to \infty$, so that $z_j = 0$ for all j and the new puncture is $\gamma = 0$. Define now the rescaled maps

$$\widetilde{v}_j(z) = \widetilde{w}_j(\delta_j z), \qquad \text{for } z \in \mathbb{C} \setminus \tfrac{1}{\delta_j}\Gamma_j,$$

for a positive sequence $\delta_j \to 0$ specialized later on. From

$$\int_{A(\delta_j,\varepsilon/\delta_j)} v_j^* d\lambda = \int_{A(\delta_j^2,\varepsilon)} w_j^* d\lambda \le \int_{B_\varepsilon(0)} v_j^* d\lambda = m_\varepsilon(\gamma) + r_j,$$

where $r_j \to 0$ as $j \to \infty$, we deduce

$$\lim_{j\to\infty} \int_{A(\delta_j,\varepsilon/\delta_j)} v_j^* d\lambda \le m_\varepsilon(\gamma) \tag{6.17}$$

for every ε. Since $m(\gamma) = 0$, bubbling off for the sequence $\widetilde{v}_j$ is not possible in $\mathbb{C} \setminus \{0\}$. Therefore, we conclude for a subsequence,

$$\widetilde{v}_j \to \widetilde{v} = (b, v) \qquad \text{in } C^\infty_{\text{loc}}(\mathbb{C} \setminus \{0\}, \mathbb{R} \times S^3).$$

From (6.17) we deduce for the limit $\widetilde{v}$, with $m(\gamma) = 0$, that

$$\int_{\mathbb{C}\setminus\{0\}} v^* d\lambda = 0.$$

We claim that $\widetilde{v}$ is not constant. From

$$\int_{S_1(0)} v_j^* \lambda = \int_{S_{\delta_j}(0)} w_j^* \lambda = \int_{S_\varepsilon(0)} w_j^* \lambda - \int_{A(\delta_j,\varepsilon)} w_j^* d\lambda$$

we obtain

$$\lim_{j\to\infty} \int_{S_1(0)} v_j^* \lambda = \int_{S_\varepsilon(0)} w_j^* \lambda - m_\varepsilon(\gamma).$$

Consequently,

$$\int_{S_1} v^* \lambda = T_\gamma,$$

proving the claim. Hence $\widetilde{v}$ is a nonconstant cylinder over a periodic solution x having period T_γ. Choose now an S^1-invariant neighborhood $\mathcal{W}$ in the loop space separating the loops of the periodic solutions of periods $\le c$ from each other. For fixed j we know that $w_j\left(\varepsilon e^{2\pi i t}\right) \to x_k(t)$ as $\varepsilon \to 0$. We choose $\delta_j \to 0$ such that $w_j\left(\delta_j e^{2\pi i t}\right) \in \mathcal{W}$ for all j. Recalling that $w(\varepsilon e^{2\pi i t}) \to x_\gamma(t)$ as $\varepsilon \to 0$ we conclude, from estimate (6.17) and Lemma 4.9 arguing as in Section 4, that $x(t) = x_k(t + c) = x_\gamma(t + d)$, so that $\widetilde{v}$ is a cylinder over x_γ.

If the mass of a puncture $\gamma \in \Gamma^1$ in (6.16) is positive and if $\Theta^1 \ne \emptyset$ we repeat the process. It necessarily stops after finitely many iterations when we reach punctures with zero mass or run out of bubbling off points. We obtain a tree of embedded finite energy surfaces having a unique positive puncture whose asymptotic limit agrees with the corresponding asymptotic limit of the negative puncture belonging to the previous generation. The asymptotic limits of the surfaces are simply covered and their energies belong to the c-spectrum. At the bottom of the tree we find finite energy planes from branches of the tree originating from bubbling off points, and the periodic solutions (x_j, T_j) in the list (6.9) originating from punctures. This is illustrated in Figure 21.

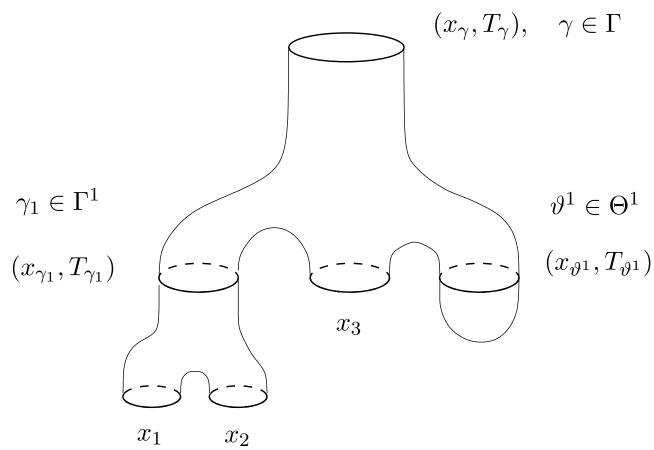

Figure 21. Possible tree originating from a puncture having a positive mass.

Since the periodic orbits at the bottom of the tree have all μ-indices equal to 1 and since the asymptotic limits of finite energy planes have μ-indices ≥ 2 we deduce from the estimates $\text{Ind}(C) \geq 1$ for the surfaces C in the generic case the inequality $\mu(x_\gamma) \geq 2$ for the asymptotic limit (x_γ, T_γ) at the top of the tree where we started.

Carrying out the bubbling off analysis for every puncture $\gamma \in \Gamma \cup \Theta$ of $\widetilde{w}$ we find that their asymptotic limits all have μ-indices ≥ 1. Recalling that $\mu(x_\infty) = 3$, the Fredholm index satisfies

$$\text{Ind}(\widetilde{w}) = 3 - \mu^-(\widetilde{w}) - 2 + (\sharp\Gamma^- + 1) \geq 1$$

for generic $\widetilde{J}$, by Theorem 2.1. If N_j denotes the number of negative punctures whose asymptotic limits have μ-indices equal to j, then $\mu^-(\widetilde{w}) = \sum_{j=1}^{l} j\ N_j$ and $\sharp\Gamma^- = \sum_{j=1}^{l}\ N_j$ so that the inequality $\text{Ind}(\widetilde{w} \geq 1$ becomes

$$\sum_{j=1}^{l} (j-1)\ N_j \leq 1,$$

implying $N_2 \leq 1$ and $N_3 = N_4 = \cdots = N_l = 0$. Therefore, $\widetilde{w}$ is necessarily either of type $\alpha = (3, 2, 1, \ldots, 1)$ or of type $\alpha = (3, 1, 1, \ldots, 1)$. In particular, $\widetilde{w}$ allows at most one negative puncture whose index is equal to 2. Hence the complete bubbling off analysis of the sequence $\widetilde{w}_j : \mathbb{C} \setminus \Gamma_j \to \mathbb{R} \times S^3$ results in a bubbling off tree which belongs to one of the following three simple types.

I. The tree consists of only one surface C of type $\alpha = (3, 1_1, \ldots, 1_N)$ having the asymptotic limits listed in (6.9). Here $\sharp\Gamma = N$ and $\Theta = \emptyset$. Moreover, $\text{Ind}(C) = 2$. We illustrate the situation for $N = 4$ in Figure 22.

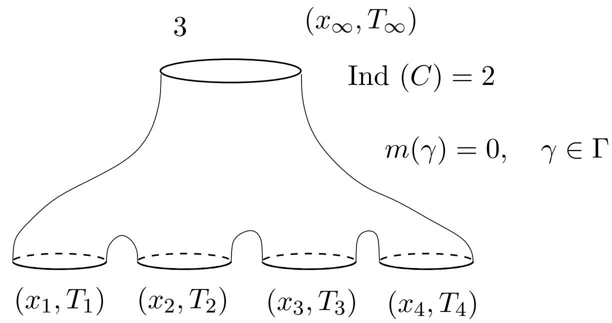

Figure 22. An example of a bubbling off tree of type I.

II. The tree consists of two surfaces, namely C_1 of type $\alpha = (3, 2, 1_1, \ldots, 1_{N-k})$ and C_2 of type $\alpha = (2, 1_{N-k+1}, \ldots, 1_N)$. Here $\sharp\Gamma < N$ and $\Theta = \emptyset$. Moreover, $\mathrm{Ind}(C_j) = 1$. This is visualized in Figure 23.

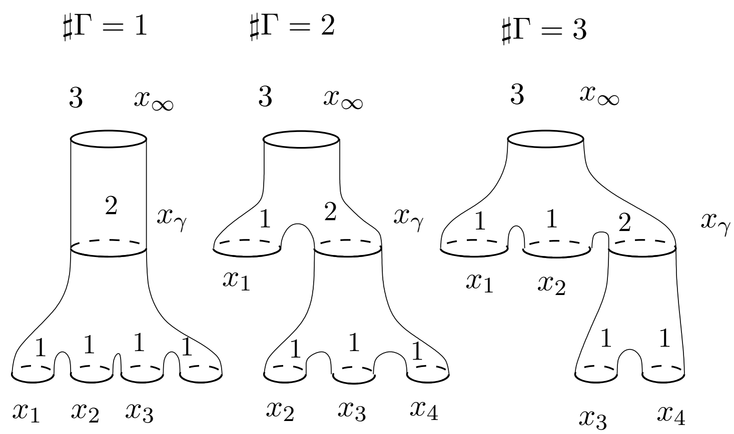

Figure 23. Possible bubbling off trees of type II.

III. The tree consists of two surfaces, namely C_1 of type $\alpha = (3, 2, 1_1, \ldots, 1_N)$ and the finite energy plane C_2 of type $\alpha = (2, \emptyset)$. Here $\sharp\Gamma = N$ and $\sharp\Theta = 1$. Moreover, $\mathrm{Ind}(C_j) = 1$. See Figure 24.

In the special case of a sequence $\widetilde{w}_j : \mathbb{C} \to \mathbb{R} \times S^3$ of finite energy planes of type $\alpha = (3, \emptyset)$ we have two types of bubbling off trees as illustrated in Figure 25.

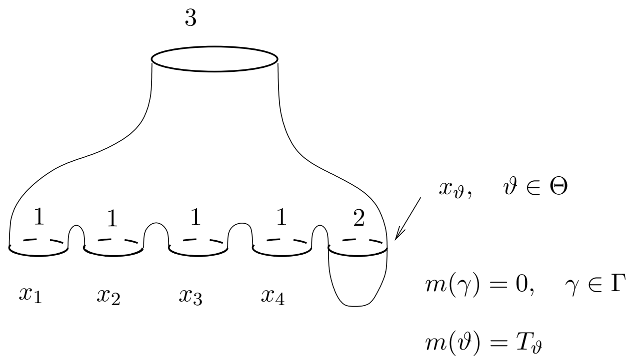

Figure 24. An example of a bubbling off tree of type III.

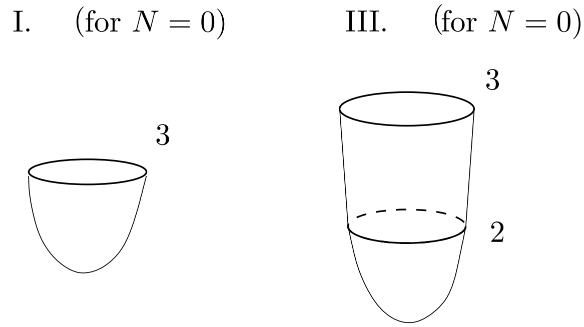

Figure 25. Bubbling off trees in the special case of finite energy planes of type $\alpha = (3, \emptyset)$.

Recall that $m \in S^3$ is given and $m_j \to m$, where $(0, m_j) \in C^{\tau(j)}$. We shall show that parametrizations of a subsequence converge in C^∞_{loc} to a parametrization of an embedded finite energy sphere C^m containing $(0, m)$. More precisely, using the above bubbling off analysis we shall prove the following result.

Theorem 6.6. *When $(0, m_j) \to (0, m)$, there exists a subsequence of C^j, $(0, m_j) \in C^j$ and parametrizations $\widetilde{u}_j$ of C^j satisfying*

$$\widetilde{u}_j(\zeta_j) = (0, m_j) \quad \text{and} \quad \zeta_j \to \zeta$$

so that the set of punctures and bubbling off points of $\widetilde{u}_j$ stay away from ζ and, moreover, near ζ the maps $\widetilde{u}_j$ converge in $C^\infty_{\text{loc}}(\mathbb{C} \setminus \widehat{\Gamma})$ to a parametrized surface $\widetilde{u} : \mathbb{C} \setminus \widehat{\Gamma} \to \mathbb{R} \times S^3$ so that

$$\widetilde{u}(\zeta) = \lim_{j \to \infty} \widetilde{u}(\zeta_j) = (0, m).$$

The map $\widetilde{u}$ is an embedding having simply covered asymptotic limits. Denote the surface obtained this way by

$$C^m = \widetilde{u}(\mathbb{C} \setminus \widehat{\Gamma}), \qquad (0, m) \in C^m.$$

Then C^m is either one of the two finite energy surfaces obtained through the bubbling off analysis or a cylinder over an asymptotic limit belonging to one of the two surfaces obtained by the bubbling off analysis.

Proof. Take the normalized parametrizations $\widetilde{w}_j : \mathbb{C} \setminus \Gamma_j \to \mathbb{R} \times S^3$ of C^j introduced in the bubbling off analysis above and define the points $\zeta_j \in \mathbb{C}$ by

$$\widetilde{w}_j(\zeta_j) = (0, m_j).$$

Either ζ_j stays away from $\Gamma \cup \Theta \cup \{\infty\}$ and hence a subsequence converges,

$$\zeta_j \to \zeta \notin \Gamma \cup \Theta \cup \{\infty\}$$

or, there is a subsequence

$$\zeta_j \to \zeta \in \Gamma \cup \Theta \cup \{\infty\}.$$

In the first case we conclude from $\widetilde{w}_j \to \widetilde{w}$ in $C^\infty_{\text{loc}}\Big(\mathbb{C} \setminus (\Gamma \cup \Theta)\Big)$ that

$$\widetilde{w}(\zeta) = \lim_{j\to\infty} \widetilde{w}_j(\zeta_j) = (0, m) \in \widetilde{w}\Big(\mathbb{C} \setminus (\Gamma \cup \Theta)\Big)$$

proving the theorem in the first case. In the second case we first study the situation

$$\zeta_j \to \zeta = \infty.$$

Reparametrizing by a sequence of Möbius transformations we define the new sequence

$$\widetilde{u}_j(z) := \widetilde{w}_j(\zeta_j z), \qquad z \in \mathbb{C} \setminus \frac{1}{\zeta_j}\Gamma_j$$

so that

$$\widetilde{u}_j(1) = \widetilde{w}_j(\zeta_j) = (0, m_j).$$

We claim for a subsequence that $\widetilde{u}_j \to \widetilde{u}$ in $C^\infty_{\text{loc}}(\mathbb{C} \setminus \{0\}, \mathbb{R} \times S^3)$, where $\widetilde{u}$ is the cylinder over the asymptotic limit (x_∞, T_∞) of the puncture ∞. It then follows that

$$\widetilde{u}(1) = \lim_{j\to\infty} \widetilde{u}_j(1) = (0, m) \in \widetilde{u}\Big(\mathbb{C} \setminus \{0\}\Big),$$

proving the theorem in this case. In order to prove the claim we first observe that due to the normalization (6.6) of $\widetilde{w}_j$,

$$\int_{\mathbb{C}\setminus\Gamma_j} w_j^* d\lambda = \int_{D\setminus\Gamma_j} w_j^* d\lambda + \frac{\sigma_0}{2},$$

we have

$$\int_{\mathbb{C}\setminus(D\cup\Gamma_j)} w_j^* d\lambda \leq \frac{\sigma_0}{2}.$$

Recalling $\Gamma_j \to \Gamma$ for a subsequence, we see that the punctures $\frac{1}{\zeta_j}\Gamma_j$ of $\widetilde{u}_j$ shrink to 0 and obtain from the above energy estimates

$$\int_{\mathbb{C}\setminus\varepsilon D} \widetilde{u}_j^* d\lambda \leq \frac{\sigma_0}{2}$$

for every $\varepsilon > 0$ provided j is sufficiently large. The energies are so small that the sequence $\widetilde{u}_j$ does not admit bubbling off points away from 0. We deduce the convergence of a subsequence

$$\widetilde{u}_j \to \widetilde{u} \qquad \text{in } C^\infty_{\text{loc}}(\mathbb{C}\setminus\{0\}, \mathbb{R}\times S^3).$$

Moreover, using the above energy estimate, the behavior of w_j at ∞, and the definition of σ_0, we conclude that the limit $\widetilde{u} = (a, u)$ satisfies

$$\int_{\mathbb{C}\setminus\{0\}} u^* d\lambda \leq \frac{\sigma_0}{2} \qquad \text{and} \qquad \lim_{R\to\infty}\int_{S_R} u^*\lambda = T_\infty.$$

Hence, arguing as in Lemma 4.9 we obtain

$$\int_{\mathbb{C}\setminus\{0\}} \widetilde{u}^* d\lambda = 0,$$

so that $\widetilde{u}$ is a nonconstant cylinder over a periodic orbit (x, T_∞). Since the asymptotic limit of the puncture ∞ of $\widetilde{w}_j$ is, independent of j, equal to (x_∞, T_∞) we find $(x, T_\infty) = (x_\infty, T_\infty)$ as claimed. We next consider the case

$$\zeta = \gamma \in \Gamma \qquad \text{and} \qquad m(\gamma) = 0.$$

In this case we find a sequence of punctures $z_j \in \Gamma_j$ satisfying $\lim_{j\to\infty} z_j = \gamma$ and $B_\varepsilon(\gamma)\cap\Gamma_j = \{z_j\}$ for some $\varepsilon > 0$. In addition, the asymptotic limits associated with the punctures z_j of $\widetilde{w}_j$ and with the puncture γ of $\widetilde{w}$ are all equal to the fixed periodic orbit (x_k, T_k). Reparametrizing by a sequence τ_j of Möbius transformations satisfying $\tau_j \to \text{Id}$ we arrange that $z_j = 0$ for all j, so that the new puncture is $\gamma = 0$. Define the parametrization

$$\widetilde{u}_j(z) = \widetilde{w}_j(\zeta_j z), \qquad \text{for } z \in \mathbb{C}\setminus \tfrac{1}{\zeta_j}\Gamma_j.$$

Then $\widetilde{u}_j(1) = \widetilde{w}_j(\zeta_j) = (0, m_j)$, and $\zeta_j \to 0$. In view of the fact that $m(\gamma) = 0$, one verifies that the small energies prevent the occurrence of bubbling off points of $\widetilde{u}_j$ in $\mathbb{C}\setminus\{0\}$. Moreover, since $\Gamma_j \to \Gamma$ and $0 \in \Gamma$, we have $\frac{1}{\zeta_j}\Gamma_j \setminus \{0\} \to \infty$. Hence a subsequence converges

$$\widetilde{u}_j \to \widetilde{u} \qquad \text{in } C^\infty_{\text{loc}}\Big(\mathbb{C}\setminus\{0\}, \mathbb{R}\times S^3\Big)$$

to the cylinder over the periodic orbit (x_k, T_k) and

$$\widetilde{u}(1) = \lim_{j\to\infty} \widetilde{u}_j(1) = \lim_{j\to\infty}(0, m_j) = (0, m),$$

proving the theorem in the case considered. In the case of positive mass,

$$\zeta = \gamma \in \Gamma \qquad \text{and} \qquad m(\gamma) > 0$$

we denote the associated asymptotic limit by (x_γ, T_γ). As a result of our bubbling off analysis there are no bubbling off points Θ for the sequence $\widetilde{w}_j$. Take the rescaled sequence $\widetilde{v}_j$ of the second round in the bubbling off analysis, defined by

$$\widetilde{v}_j(z) = \widetilde{w}_j(z_j + \delta_j z).$$

Here $z_j \in \gamma_j$ satisfies $z_j \to \gamma$ and $\delta_j \to 0$. The sequence $\widetilde{v}_j$ is normalized by (6.16). Recalling $\zeta_j \to \zeta = \gamma$ we define $\hat{z}_j$ by

$$z_j + \delta_j \hat{z}_j = \zeta_j$$

where $\widetilde{w}_j(\zeta_j) = (0, m_j)$. We know that the sequence $\widetilde{v}_j$ has no bubbling off points Θ^1. We also know that a subsequence converges,

$$\widetilde{v}_j \to \widetilde{v} \qquad \text{in } C^\infty_{\text{loc}}(\mathbb{C} \setminus \Gamma^1, \mathbb{R} \times S^3).$$

If $\hat{z}_j$ stays away from Γ^1 and $\{\infty\}$, then a subsequence converges to a point $\hat{z} \in \mathbb{C} \setminus \Gamma^1$. Consequently, $\widetilde{v}(\hat{z}) = \lim_{j\to\infty} \widetilde{v}_j(\hat{z}_j) = \lim_{j\to\infty} \widetilde{w}_j(\zeta_j) = (0, m)$. We see that $(0, m)$ belongs to the finite energy surface $\widetilde{v}$ obtained in the second round of the bubbling off analysis.

If $|\hat{z}_j| \to \infty$, we introduce the parametrization

$$\widetilde{u}_j(z) = \widetilde{v}_j(\hat{z}_j z) = \widetilde{w}_j(z_j + \delta_j \hat{z}_j z), \qquad z \in \mathbb{C} \setminus \frac{1}{\hat{z}_j}\widehat{\Gamma}_j, \;\; \widehat{\Gamma}_j = \frac{1}{\delta_j}(\Gamma_j - z_j).$$

It satisfies $\widetilde{u}_j(1) = (0, m_j)$. Proceeding as in the first case one verifies by estimating the $d\lambda$-energies of $\widetilde{u}_j$ that there is no bubbling off in $\mathbb{C} \setminus \{0\}$. Since $\widehat{\Gamma}_j \to \Gamma^1$, the punctures $\frac{1}{\hat{z}_j}\widehat{\Gamma}_j$ move to $\{0\}$ as $j \to \infty$. It follows that for a subsequence,

$$\widetilde{u}_j \to \widetilde{u} \qquad \text{in } C^\infty_{\text{loc}}\Big(\mathbb{C} \setminus \{0\}, \mathbb{R} \times S^3\Big),$$

where $\widetilde{u}$ is an orbit cylinder over the asymptotic limit (x_γ, T_γ). It contains $\widetilde{u}(1) = \lim_{j\to\infty} \widetilde{u}_j(1) = (0, m)$.

Finally, assume that $\hat{z}_j \to \gamma^1 \in \Gamma^1$. By the bubbling off analysis we know that $m(\gamma^1, \widetilde{v}_\gamma) = 0$. Hence the arguments already used in the case $m(\gamma) = 0$ for all $\gamma \in \Gamma$ show that there exist reparametrizations $\widetilde{u}_j$ of $\widetilde{v}_j$ satisfying $\widetilde{u}_j(1) = (0, m_j)$ and $\widetilde{u}_j \to \widetilde{u}$ in $C^\infty_{\text{loc}}(\mathbb{C} \setminus \{0\}, \mathbb{R} \times S^3)$, where $\widetilde{u}$ is the cylinder over the orbit (x_l, T_l) associated with the massless puncture γ^1 of $\widetilde{v}$. Again, $\widetilde{u}(1) = \lim_{j\to\infty} \widetilde{u}_j(1) = (0, m)$. The remaining case, namely $\zeta_j \to \zeta \in \Theta$ if $\Theta \neq \emptyset$, is treated similarly and we omit the details. This completes the proof of Theorem 6.6. $\square$

The possible positions of the given m in the projection of a bubbling off tree of the sequence $\widetilde{w}_j$ into S^3 is illustrated by Figure 26.

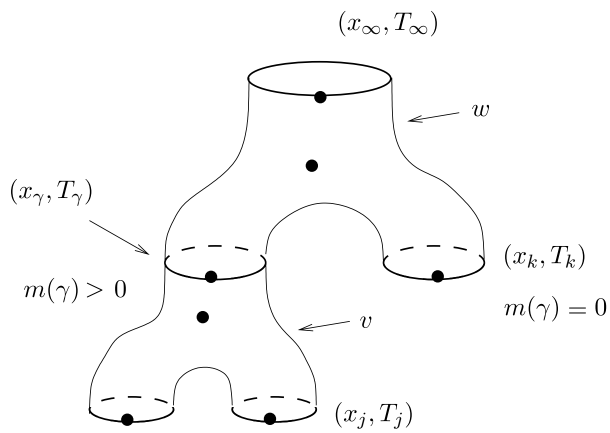

Figure 26. Possible positions of a given point m in the projection of a bubbling off tree.

The uniqueness question related to our construction will now be answered by once more using the intersection theory of pseudoholomorphic curves in dimension 4 due to McDuff [43] and Micallef and White [46]. We recall our construction. Given any injective monotonic map $\tau : \mathbb{N} \to \mathbb{N}$ satisfying $m_{\tau(j)} \to m$ and any injective map $\phi : \mathbb{N} \to \mathbb{N}$ we constructed another such map $\psi : \mathbb{N} \to \mathbb{N}$ and parametrizations of the surfaces $C^{\tau \circ \varphi \circ \psi(j)}$ which converge in C^∞_{loc} to a parametrization of a surface C containing the point $(0, m)$. Assume now that we start with a different τ' still satisfying $m(\tau'(j)) \to m$. Then for every φ' there exist ψ' and parametrizations of $C^{\tau' \circ \varphi' \circ \psi'(j)}$ converging to a parametrization of a surface C' also containing $(0, m)$. Hence $C \cap C' \neq \emptyset$, and we shall show that $C = C'$. If C and C' are not identical they have, in view of the asymptotic behavior and the similarity principle, an isolated intersection which, by the positivity of intersections has a positive local intersection index. It also follows that $C^{\tau \circ \varphi \circ \psi(j)}$ and $C^{\tau' \circ \varphi' \circ \psi'(j')}$ have an isolated intersection for j and j' large. However, this contradicts the fact, Theorem 6.1, that these two surfaces are either identical or disjoint, so that indeed $C = C'$. Assume that $m = m_k$ belongs to the dense sequence of Theorem 6.1 so that $(0, m_k) \in C^{m_k}$, and $m_{\tau(j)} \to m_k$; then there exist parametrizations of $C^{\tau \circ \varphi \circ \psi(j)}$ converging to a parametrization of a surface C containing $(0, m_k)$. The same argument as above shows that $C = C^{m_k}$. We have verified that the surface C^m through a given point $(0, m)$ is uniquely defined by our construction.

Summarizing this section, we have proved the following existence result.

THEOREM 6.7. *Given a nondegenerate contact form $\lambda = f\lambda_0$ on S^3, a generic multiplication $J : \xi \to \xi$ with associated almost complex structure $\widetilde{J}$ on $\mathbb{R} \times S^3$, then there exists for every $m \in S^3$ an embedded finite energy sphere $C^m \subset \mathbb{R} \times S^3$ containing the point $(0, m)$ and having the following properties.*

1. *There exists a constant* $c > 0$ *such that* $E(C^m) \le c$ *for every* $m \in S^3$.
2. *Two surfaces* C^m *and* $C^{m'}$ *are either disjoint or identical.*
3. *Every* C^m *has precisely one positive puncture and a finite set of negative punctures whose cardinality is bounded in terms of the c-spectrum.*
4. *The asymptotic limits of* C^m *are simply covered and have* μ*-indices in* $\{1, 2, 3\}$, *and self-linking numbers are equal to* -1.
5. *If* C^m *is not a cylinder over the asymptotic limit, then* $\mathrm{Ind}(C^m) = \mu(C^m) - 2 + \sharp\{$*punctures of* $C^m\}$ *belongs to the set* $\{1, 2\}$. *In addition, the projection of* C^m *into* S^3 *is an embedded surface, transversal to the Reeb vector field and converging at the punctures to the asymptotic limits of* C^m.

6.3. *The stable finite energy foliation.* We shall use the $\mathbb{R}$-invariance of the structure $\widetilde{J}$ on $\mathbb{R} \times S^3$ in order to extend the family $\{C^m \mid m \in S^3\}$ of surfaces by means of translation in the $\mathbb{R}$-direction. Let $\widetilde{u} = (a, u) : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times S^3$ be a finite energy surface and $r \in \mathbb{R}$; then the translated map

$$\widetilde{u}_r(z) = \Big(a(z) + r, u(z)\Big)$$

is also a finite energy surface. We can define, for every pair $(r, m) \in \mathbb{R} \times S^3$, the set

$$C^{(r,m)} = \Big\{(r + s, x) \in \mathbb{R} \times S^3 \mid (s, x) \in C^m\Big\}.$$

Then $C^{(r,m)}$ is an embedded finite energy surface parametrized by $\widetilde{u}_r$ if $\widetilde{u}$ is a parametrization of C^m. Clearly,

$$(r, m) \in C^{(r,m)} \qquad \text{and} \qquad C^{(0,m)} = C^m.$$

Hence through every point in $\mathbb{R} \times S^3$ we have an embedded finite energy surface having the properties of C^m listed in Theorem 6.7. We abbreviate this family of surfaces by

$$\mathcal{F} := \Big\{C^{(r,m)} \mid (r, m) \in \mathbb{R} \times S^3\Big\}.$$

There is a natural action $T : \mathbb{R} \times \mathcal{F} \to \mathcal{F}$ of $\mathbb{R}$ defined by the translation

$$T_r\Big(C^{(s,m)}\Big) = C^{(r+s,m)}.$$

The fixed points of this $\mathbb{R}$-action are the cylinders over periodic solutions belonging to $\mathcal{F}$, as we shall see next.

Lemma 6.8. 1. *If* $F \in \mathcal{F}$ *and* $r \neq 0$, *then either*

$$T_r(F) \cap F = \emptyset$$

or: $T_r(F) = F$ *and then* $T_s(F) = F$ *for all* $s \in \mathbb{R}$ *and* $F = \mathbb{R} \times P$, *where* P *is a periodic orbit of the Reeb vector field.*

2. *If $F \in \mathcal{F}$ is not a fixed point of the $\mathbb{R}$-action, then its projection $p(F)$ is a smooth embedded surface in S^3 transversal to the Reeb vector field. Here $p : \mathbb{R} \times S^3 \to S^3$ denotes the projection onto the second factor.*

Proof. Assume that $F \in \mathcal{F}$ is not a cylinder over a periodic orbit and let $\widetilde{u} = (a, u)$ be a parametrization of F. If $T_r(F) \cap F \neq \emptyset$, there exist two points z_1 and z_2 solving $(a(z_1) + r, u(z_1)) = (a(z_2), u(z_2))$. By Theorem 6.7, u is an embedding; hence $z_1 = z_2$ and $r = 0$. If on the other hand F is a cylinder over a periodic orbit, then clearly $T_s(F) = F$ for every s proving the first statement. The second statement follows in view of Theorem 6.7. □

LEMMA 6.9. *Any two surfaces $C^{(r,m)} \in \mathcal{F}$ are either disjoint or identical.*

Proof. Assume that F_1, $F_2 \in \mathcal{F}$ satisfy $F_1 \cap F_2 \neq \emptyset$ and $F_1 \neq F_2$. Since $F_1 = T_{r_1}(C^{m_1})$ and $F_2 = T_{r_2}(C^{m_2})$ we conclude that $C^{m_1} \neq T_r(C^{m_2})$, where $r = r_2 - r_1$. We may assume that $r > 0$. In view of the asymptotic behavior of the surfaces there is an isolated intersection having a positive intersection index. Hence C^{m_1} and $T_s(C^{m_2})$ have an isolated intersection for all $s > 0$. Consequently, $C^{m_1} \cap C^{m_2} \neq \emptyset$ and $C^{m_1} \neq C^{m_2}$. This contradicts the fact (Theorem 6.7) that C^{m_1} and C^{m_2} are either disjoint or identical, and the lemma is proved. □

Clearly, $p(F_1) = p(F_2)$ if F_1 and F_2 belong to the same orbit of the $\mathbb{R}$-action.

LEMMA 6.10. *Assume F_1 and $F_2 \in \mathcal{F}$ do not belong to the same orbit of the $\mathbb{R}$-action. Then $p(F_1) \cap p(F_2) = \emptyset$.*

Proof. Assume $T_s(F_1) \neq F_2$ for all $s \in \mathbb{R}$, and $p(F_1) \cap p(F_2) \neq \emptyset$. If $\widetilde{u}_1$ and $\widetilde{u}_2$ are parametrizations of F_1 and F_2 we find two points z_1 and z_2 solving $u(z_1) = u(z_2)$. So, $\big(a(z_1)+r, u_1(z_1)\big) = \big(a_2(z_2), u_2(z_2)\big)$ for $r = a_2(z_2) - a_1(z_1)$ and hence $T_r(F_1) \cap F_2 \neq \emptyset$. Since, by assumption, $T_r(F_1) \neq F_2$ we have a contradiction to Lemma 6.9. □

Finally, we shall show that the family $\mathcal{F}$ defines a foliation of $\mathbb{R} \times S^3$. Recall that there is a leaf $F \in \mathcal{F}$ through every point. It remains to show that locally the leaves form a 2-dimensional family of 2-dimensional surfaces.

Assume $(r_0, m_0) \in F_0$ and F_0 is of type $\alpha = (3, 1, 1, \ldots, 1)$ so that $\mathrm{Ind}(F_0) = 2$. Using the $\mathbb{R}$-action we can assume that $r_0 = 0$. In view of the implicit function theorem, Theorem 1.6 in [36], F_0 belongs to a smooth 2-parameter family $\widetilde{\mathcal{F}}_0$ of finite energy spheres having the same asymptotic limits as F_0. Moreover, one parameter accounts for the $\mathbb{R}$-action and $\widetilde{\mathcal{F}}_0$ is a foliation near F_0. Note that Theorem 1.6 in [36] is stated merely for the special case of an embedded plane. However, its proof also applies for the case at hand if we observe that for type $\alpha = (3, 1, 1, \ldots, 1)$ surfaces $\widetilde{u} = (a, u)$ we have $\pi \circ Tu(z) \neq 0$ for all $z \in S^2 \setminus \Gamma$, by Corollary 2.6. It is sufficient to show that $C^m \in \widetilde{\mathcal{F}}_0$ for m near m_0. This is a consequence of our bubbling off analysis. Indeed, take $m_k \to m_0$ and abbreviate $C^k = C^{m_k}$. Then there are parametrizations of a subsequence $C^{\psi(j)}$ converging in C^∞_{loc} to the parametrization of a surface C containing $(0, m_0)$. Consequently, $C \cap F_0 \neq \emptyset$ and hence $C = F_0 \in \mathcal{F}$ because otherwise we would find an isolated intersection and would conclude by the positivity of the intersection index, that $F_0 \cap C^{\psi(j)} \neq \emptyset$ and $F_0 \neq C^{\psi(j)}$ for j large, in contradiction to Theorem 6.7. Since $C = F_0$ we are in case I of our bubbling off analysis and conclude that there is no bifurcation so that all the surfaces $C^{\psi(j)}$ have the same asymptotic limits as F_0. Therefore, by the completeness theorem, Theorem 7.1 in [36], we deduce that indeed $C^{\psi(j)} \in \widetilde{\mathcal{F}}_0$. There are only finitely many (modulo $\mathbb{R}$-action) leaves $F \in \mathcal{F}$ satisfying $\text{Ind}(F) = 1$ or $F = \mathbb{R} \times P$. Consequently, every piece of such an F fits smoothly into the 2-parameter families nearby, when we use, as above, the C^∞_{loc} convergence and the positivity of the intersections. Having shown that $\mathcal{F}$ is a foliation, the proof of our main result, Theorem 1.6, is complete.

The following three Figures 27–29 visualize examples of foliations of S^3.

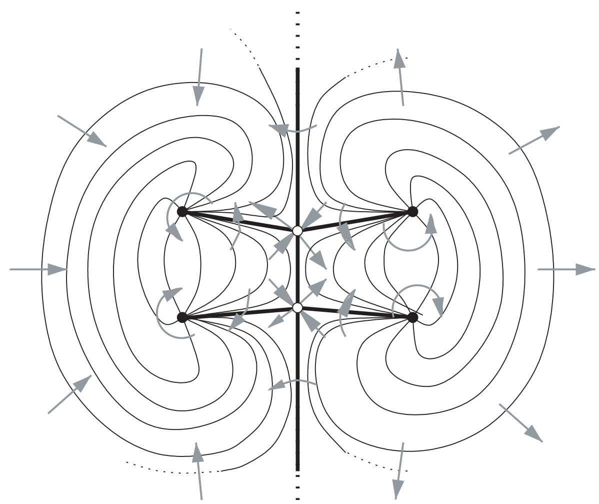

Figure 27. An example of a foliation by disk-like and annuli-like surfaces. The pair of white dots represents a hyperbolic orbit of index 2, and pairs of black dots represent periodic orbits of index 3. There are two families of disk-like surfaces asymptotic to periodic orbit of index 3. These families are represented by thin curves. The bold curves represent rigid surfaces. There are two rigid surfaces of annuli-type which connect periodic orbits of index 3 with the hyperbolic orbit, and two disk-like surfaces asymptotic to the hyperbolic orbit.

The 3-sphere is viewed as $\mathbb{R}^3 \cup \{\infty\}$. The figures show the traces of the foliations cut out by the plane $\mathbb{R}^2 \subset \mathbb{R}^3$. Periodic orbits are represented by dots. Two dots of the same color belong to the same periodic orbit, and the periodic orbits are perpendicular to the page. In these figures black dots represent periodic orbits of index 3, white dots represent hyperbolic orbits of index 2, and grey dots represent periodic orbits of index 1. The rigid surfaces are indicated by bold curves, while the families of surfaces are indicated by thin curves. The arrows indicate the Reeb flow lines.

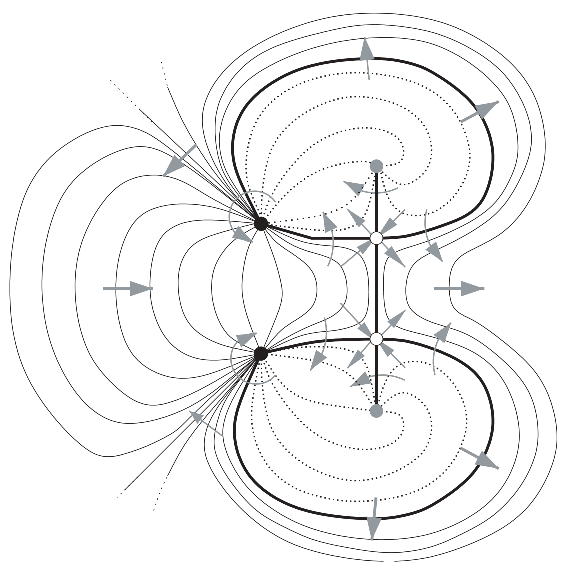

Figure 28. Here the pair of grey dots represents a periodic orbit of index 1. The periodic orbit of index 3 is indicated by the pair of black dots, and the hyperbolic orbit of index 2 by the pair of white dots. The dotted lines represent a family of annuli-like surfaces connecting the periodic orbit of index 3 with the periodic orbit of index 1. The thin curves constitute a family of disk-like surfaces asymptotic to periodic orbit of index 3. The rigid surfaces, bold curves, are of two types: the annuli-type surfaces either connecting the periodic orbit of index 3 with the periodic orbit of index 2 or connecting the periodic orbit of index 2 with the periodic orbit of index 1, and of disk-type asymptotic to the periodic orbit of index 2.

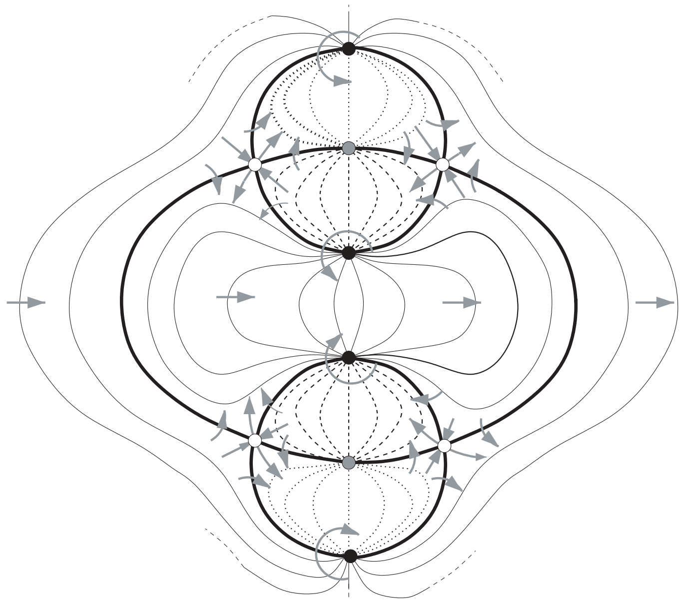

Figure 29. In this example there are two families of annuli-like surfaces connecting periodic orbits of index 2 with periodic orbits of index 1. They are represented by dotted and dashed thin curves. There are also two families of disk-like surfaces asymptotic to periodic orbits of index 3. The two rigid surfaces of disk-type are asymptotic to periodic orbits of index 2. The remaining rigid surfaces are of annuli-type. They either connect periodic orbits of index 3 with the periodic orbits of index 2 or connect the periodic orbits of index 2 with the periodic orbit of index 1.

7. Consequences for the Reeb dynamics

7.1. *Proof of Theorem* 1.9 *and its corollaries.* We first observe that the foliation $\mathcal{F}$ constructed contains a finite energy plane.

Proposition 7.1. *The foliation $\mathcal{F}$ contains a finite energy plane whose asymptotic limit P has index $\mu(P) \in \{2, 3\}$.*

Proof. For the following it is useful to recall that the foliation $\mathcal{F}$ is independent of the choice of the dense sequence $(0, m_k)$ used in the construction, as long as m_k does not belong to a periodic orbit and the finite energy surface containing $(0, m_k)$ has $\mathrm{Ind}(C_k) = 2$. This follows from the positivity of the intersection index as in the proofs of Theorems 6.1 and 6.7. In order to find the finite energy plane we take a sequence of points $(0, p_k)$, $p_k \in M$, such that the unique $\widehat{J}_{N_k}$-holomorphic sphere C^{N_k} in A_{N_k} passing through o_∞ and $(0, p_k)$ has in the part $[-N_k, N_k]$ the lowest possible $\mathbb{R}$-value equal to 0. Choose $\widehat{J}_{N_k}$-holomorphic parametrizations $\widetilde{w}_k := \widetilde{w}_{N_k} : S^2 \to A_{N_k}$ satisfying

$$\widetilde{w}_k(\infty) = o_\infty, \quad \widetilde{w}_k(0) = (0, p_k), \quad \int_D \widetilde{w}_k^* \omega_{N_k} = \gamma. \tag{7.1}$$

Then, in view of the maximum principle, we find a sequence $R_k \to \infty$ such that $D_{R_k} \subset \{z \in \mathbb{C} \mid a_k(z) \le N_k\}$. Define a sequence $\widetilde{v}_k := \widetilde{w}_{k|D_{R_k}} : D_{R_k} \to \mathbb{R} \times M$, still satisfying $\widetilde{v}_k(0) = (0, p_k)$ and converging in C^∞_{loc} to a nonconstant pseudoholomorphic map $\widetilde{v}_\infty : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times M$. In view of the normalization in (7.1) the set of punctures Γ lies outside of D so that $\widetilde{v}_\infty(0) = (0, p)$, where $p = \lim p_k \in M$. The punctures Γ are necessarily negative. Since 0 is the minimum of the $\mathbb{R}$-component of $\widetilde{v}_k$, we deduce that $\Gamma = \emptyset$ and $\widetilde{v}_\infty : \mathbb{C} \to \mathbb{R} \times M$ is the desired finite energy plane. In view of the positivity of the intersections, this plane belongs to the foliation $\mathcal{F}$. The index $\mu(P)$ of a finite energy plane is always ≥ 2; hence $\mu(P) \in \{2, 3\}$ as claimed. □

Proposition 7.2. *Assume $\mathcal{F}$ contains a finite energy plane of index $\mu(P) = 3$. Then either $\mathcal{F}$ has precisely one fixed point of the $\mathbb{R}$-action or $\mathcal{F}$ contains also a finite energy plane of index 2. In the first case the projection of $\mathcal{F}$ into M is an open book decomposition into planes having the periodic orbit of the fixed point as binding orbit.*

Proof. By the arguments in [35], the finite energy plane of index 3 is contained in a maximal 1-parameter family of such planes all having the same asymptotic limit P. By the uniqueness arguments in Theorems 6.1 and 6.7 all these planes belong to the foliation $\mathcal{F}$. The family is parametrized either by an open interval I in the noncompact case or by the circle S^1 in the compact case. As shown in [35], in the compact case the projection of the S^1-family of planes into M constitutes an open book decomposition of M into planes having P as binding orbit. Therefore, $\mathcal{F}$ possesses precisely one fixed point of the $\mathbb{R}$-action, whose projection into M is the binding orbit of the open book decomposition. If, however, the family of planes, is parametrized by an open interval, the projected surfaces degenerate towards the ends giving rise to a periodic orbit Q having index $\mu(Q) = 2$. More precisely, by our bubbling off analysis, the planes converge to a broken trajectory consisting of rigid surfaces (C^-, C^+), namely a cylinder C^- connecting the positive limit P having index $\mu(P) = 3$ with the negative limit Q having index $\mu(Q) = 2$. This periodic orbit is also the positive limit of a finite energy plane whose projection into M is the rigid surface C^+. This finishes the proof of Proposition 7.2. □

The situation is illustrated in Figure 30.

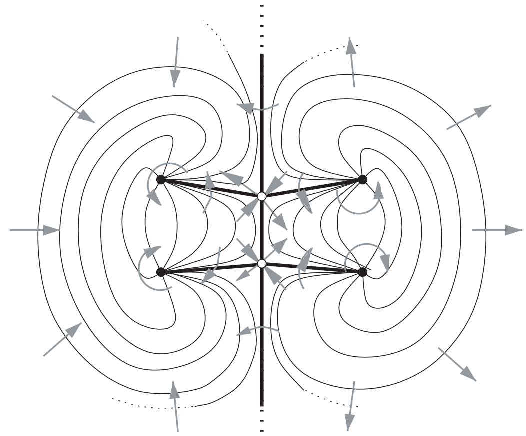

Figure 30. The figure illustrates the second case of Proposition 7.2.

The consequences of the open book decomposition of S^3 for the flow φ^t of the Reeb vector field X associated with λ are described in [35]. We recall that every plane, compactified in a natural way, is a global surface of section $\mathcal{D}$ of disc type whose boundary $\partial\mathcal{D}$ is the binding orbit P. The Poincaré section map $\psi:\ \text{int}\mathcal{D}\to\ \text{int}\mathcal{D}$ is an area-preserving diffeomorphism, the area element being $d\lambda$. The total area of $\text{int}\mathcal{D}$ is equal to the period T of the periodic orbit P and hence finite. Consequently, Brouwer's translation theorem guarantees a fixed point $p\in\ \text{int}\mathcal{D}$ of the map ψ. It is the initial condition of a periodic orbit of X different from the boundary orbit P. In addition, by a theorem of J. Franks in [20], the map ψ possesses infinitely many periodic orbits, provided ψ has at least one periodic orbit different from the fixed point p. We have proved the following proposition.

Proposition 7.3. *If the foliation $\mathcal{F}$ has precisely one fixed point of the $\mathbb{R}$-action, the Reeb vector field X on M possesses either two or infinitely many periodic orbits.*

It remains to study the dynamics of the Reeb vector field in the case where the foliation $\mathcal{F}$ contains a fixed point of the $\mathbb{R}$-action whose associated periodic orbit P has index $\mu(P)=2$. Such a periodic orbit is necessarily hyperbolic and, therefore, lies in the intersection of its oriented stable manifold $W^+(P)$ and its oriented unstable manifold $W^-(P)$. We shall first describe the positions of the local invariant manifolds $W^{\pm}_{\text{loc}}(P)$ near P, with respect to the rigid surfaces having P as positive and negative limit.

The tangent spaces of $W^{\pm}(P)$ along the periodic solution $P=(x,T)$ having minimal period T are spanned by the Reeb vector field $X(x(t))$ and vector fields $v^{\pm}(t)$ belonging to the contact planes $\xi(x(t))$. In the symplectic trivialization of the bundle ξ over P, as used in asymptotic formula for the rigid surfaces approaching P (see the appendix), the vectors $v^{\pm}(t)$ are the solutions of the following boundary value problem

$$-i\dot{v}^{\pm}(t) = A(t)v^{\pm}(t), \quad v^{+}(T) = \frac{1}{\beta}v^{+}(0), \quad v^{-}(T) = \beta v^{-}(0) \tag{7.2}$$

for some $\beta > 1$. Here $A(t) = A(t+T)$ is real and symmetric. We shall abbreviate in the following the fundamental solution by $\Phi(t) = d\varphi^t(x(0))_{|\xi(x(0))}$ so that $v^{\pm}(t) = \Phi(t)v^{\pm}(0)$. Let $e(t)$ be the asymptotic direction of a (necessarily rigid) surface having P as positive or negative limit. The vector e appears in the asymptotic formula; see the appendix. Then e is an eigenvector of the eigenvalue problem

$$-i\dot{e} = A(t)e + \lambda e, \qquad e(0) = e(T). \tag{7.3}$$

Moreover, $\lambda = \lambda^{+} < 0$ if P is the positive limit and $\lambda = \lambda^{-} > 0$ if P is the negative limit. Since the winding numbers of the asymptotic directions e are all equal, namely $\text{wind}_{\infty}(e) = 1$ according to Proposition 5.9, the Conley-Zehnder index $\mu(P) = 2$ implies that $\lambda^{+} < 0$ is the largest negative eigenvalue and $\lambda^{-} > 0$ is the smallest positive eigenvalue of (7.3). In addition, the corresponding eigenspaces are 1-dimensional. We conclude that there are precisely two directions $\pm e^{+}$ for positive surfaces and two directions $\pm e^{-}$ for negative surfaces. Since $\mathcal{F}$ is a foliation it follows from the asymptotic formula in $\mathbb{R}\times S^3$ that P is the asymptotic limit of at most four rigid leaves of $\mathcal{F}$. Figure 31 illustrates the situation. The vectors $e^{+}(t)$ and $e^{-}(t)$ constitute a basis of the contact planes $\xi(x(t)) = \mathbb{C}$ and hence define four open quadrants in $\mathbb{C}$ depending on t and denoted by I, II, III and IV. See Figure 32. The positions of the local invariant manifolds $W^{+}_{\text{loc}}(P)$ and $W^{-}_{\text{loc}}(P)$ with respect to the rigid surfaces are described by the following proposition.

Proposition 7.4. *$v^{-}(t) \in$ I or III and $v^{+}(t) \in$ II or IV.*

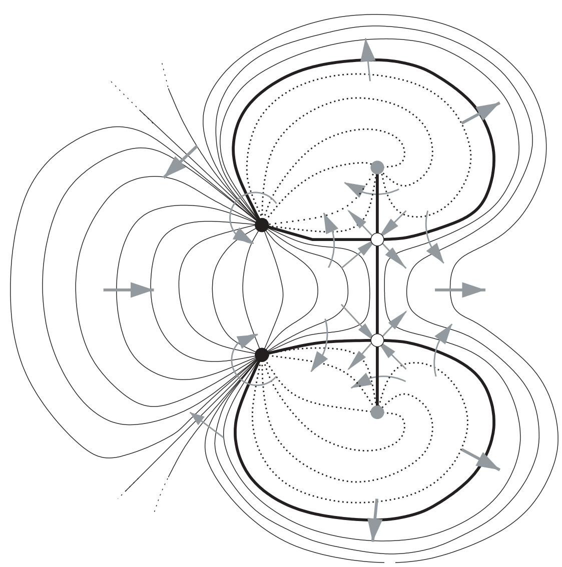

Figure 31. A hyperbolic periodic orbit may be an asymptotic limit of at most four rigid surfaces of $\mathcal{F}$.

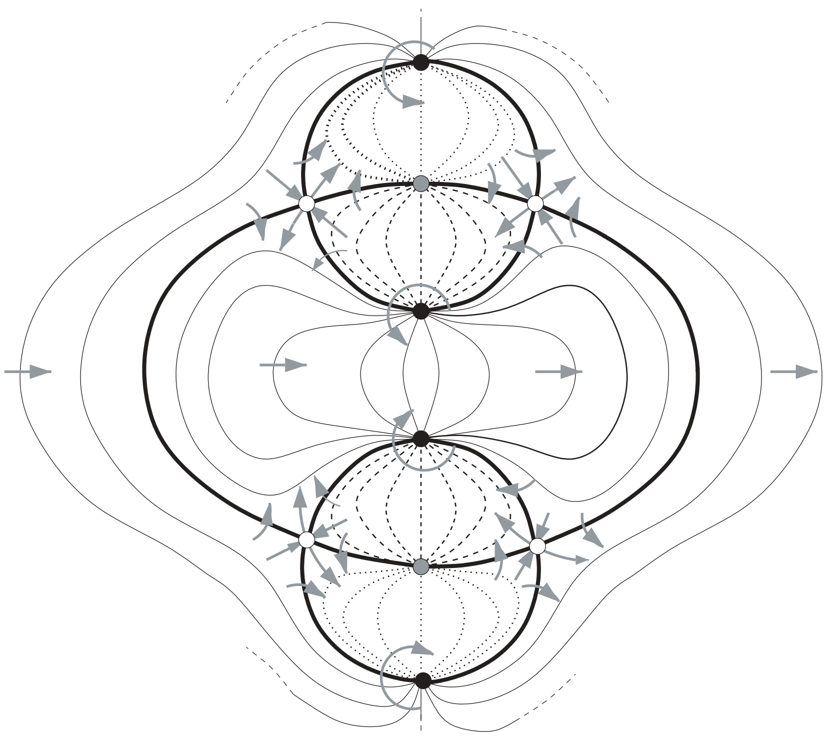

Figure 32. The vectors e^+ and e^- define four quadrants in $\mathbb{C}$.

Proof. Consider a solution $h(t)$ of $-i\dot{h} = A(t)h$. Assume $h(\tau) = e^+(\tau)$ for some τ. If $e^+(t)$ is the solution of (7.3) belonging to $\lambda^+ < 0$ we compute, abbreviating $e^+ = e$ and $\lambda^+ = \lambda$,

$$\begin{aligned}\frac{d}{dt}(h\bar{e}) &= \dot{h} \cdot \bar{e} + h\dot{\bar{e}} \\ &= i(Ah) \cdot \bar{e} - ih \cdot \overline{(Ae)} - i\lambda(h\bar{e}) = B(h, e) - i\lambda(h\bar{e}),\end{aligned}$$

where $B(h,e)=\overline{B(e,h)}$. We conclude that $h(t)$ enters I for $t>\tau$, and IV for $t<\tau$. Similarly, assuming $h(\tau)=e^-(\tau)$ we see that $h(t)$ enters I for $t>\tau$ and II for $t<\tau$. This is illustrated in Figure 33. Hence, the solution $h(t)$ starting at $h(0)=e^+(0)$ satisfies $h(t)\in$ I for all $t>0$ and $h(t)\in$ IV for all $t<0$. Representing this solution in the basis $v^+(t)$ and $v^-(t)$ and recalling $\Phi(t+nT)=\Phi(t)\Phi(T)^n$ for all $n\in\mathbb{Z}$ and $t\in\mathbb{R}$, we find

$$\begin{aligned}\lim_{n\to\infty}\frac{h(t+nT)}{|h(t+nT)|} &= \frac{v^-(t)}{|v^-(t)|}\in \text{I},\\ \lim_{n\to\infty}\frac{h(t-nT)}{|h(t-nT)|} &= \frac{v^+(t)}{|v^+(t)|}\in \text{IV}.\end{aligned}$$

A similar argument applies if the solution $h(t)$ starts at $h(0)=-e^+(0)$. This finishes the proof of the proposition. □

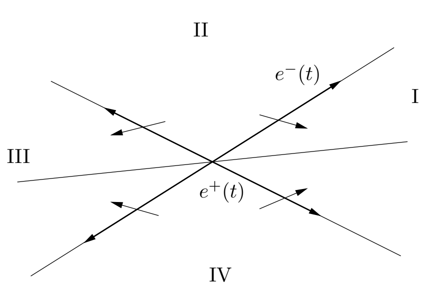

Figure 33.

The sets $W^{\pm}(P)$ are images of injective immersions of a cylinder. The transversal foliation in the complement of the binding orbit allows the study of their global behavior.

Proposition 7.5. *Assume the foliation $\mathcal{F}$ possesses a spanning orbit of index $\mu(P) = 2$. Then its unstable invariant manifold $W^-(P)$ intersects the stable invariant manifold $W^+(Q)$ for a possibly different spanning orbit Q of $\mathcal{F}$ having index $\mu(Q) = 2$.*

Proof. In view of our bubbling off analysis a binding orbit P satisfying $\mu(P) = 2$ is the positive limit of a rigid surface $C^+ \in \mathcal{F}$ and the negative limit of a rigid surface $C^- \in \mathcal{F}$. Moreover, there is a unique family of surfaces C_τ parametrized over the interval $(-1, 1)$ so that as $\tau \to -1$, the surfaces decompose into the broken trajectory (C^+, C^-). Figure 34 visualizes such a situation, when projected into S^3 and cut by a plane.

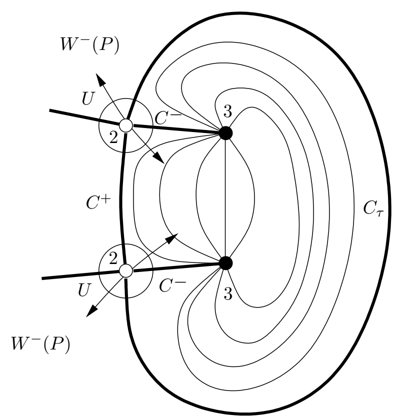

Figure 34. A family of surfaces C_τ decomposes as $\tau \to 1$ into the broken trajectory (C^+, C^-).

On S^3 we take a smooth tubular neighborhood U of the periodic orbit P equipped with a metric having the property that the distance $d(\varphi^{-t}(w), P)$ for $w \in W^-_{\text{loc}}(P) \cap U$ strictly decreases with increasing time t. Here φ^t is the flow of the Reeb vector field X. We choose U so small that $\partial U \cap W^-_{\text{loc}}(P)$ are two smoothly embedded circles on ∂U.

Lemma 7.6. *The cylinder $W^-_{\text{loc}}(P)$ cuts out a smooth embedded circle S_τ on the projected surface $p(C_\tau)$, for τ close to the left end of the interval*:

$$S_\tau = W^-_{\text{loc}}(P) \cap p(C_\tau),$$

where $p : \mathbb{R} \times S^3 \to S^3$ is the projection.

Proof. Let S^+ be a smooth circle $\partial U \cap W^-_{\mathrm{loc}}(P)$ contained in the family $p(C_\tau)$, $\tau \in (-1, 1)$, and denote by B the cylinder obtained by the backward flow starting at S^+,

$$B = \{\varphi^t(b) \mid b \in S^+, t < 0\}.$$

We shall show that $B \cap p(C_\tau) = S_\tau$ is a smoothly embedded circle for τ close to -1. Recall the bubbling off analysis leading to the broken trajectories. The surface C_τ is parametrized by a map $\widetilde{u}_\tau = (a_\tau, u_\tau)$, in appropriate coordinates given by a cylinder

$$\widetilde{u}_\tau : Z_\tau = [-R_\tau, R_\tau] \times S^1 \to \mathbb{R} \times M,$$

with $R_\tau \to \infty$ as $\tau \to -1$. The loops $u_\tau(\{-R_\tau\} \times S^1)$ and $u_\tau(\{R_\tau\} \times S^1)$ are close to $p(C^-)$ and $p(C^+)$ and near the limit P. In view of the choice of the metric and Proposition 7.4 one deduces that the algebraic intersection number is equal to

$$\mathrm{int}\Big(b \cdot \mathbb{R}^-, u_\tau(Z_\tau)\Big) = 1$$

if τ is close to -1 and $b \in S^1$. Since the local intersection indices of the pseudoholomorphic curves $(s, \varphi^t(b))$ and $\widetilde{u}_\tau$ are equal to 1, there is precisely one intersection point $b \cdot \mathbb{R}^- \cap p(C_\tau)$ for every $b \in S^1$. The flow being transversal to $p(C_\tau)$ defines a diffeomorphism of the smooth circle S^+ onto the intersection set $W^-_{\mathrm{loc}}(P) \cap p(C_\tau)$ hence proving the lemma. □

In view of the above lemma, the flow φ^t for $t > 0$ defines a family of smooth embedded circles $S_\tau \subset p(C_\tau)$ for all $-1 < \tau < 1$. We shall show that these moving loops hit the stable manifold $W^+(Q)$ of a hyperbolic spanning orbit Q of index $\mu(Q) = 2$.

Arguing indirectly we assume that this never happens. Then as $\tau \to -1$, the family C_τ decomposes along a binding orbit P_1 into two rigid surfaces (C_1^+, C_1^-) and the loop cut out by the unstable manifold $W^-(P)$ has to sit entirely on one of the two rigid surfaces $p(C_1^+)$, $p(C_1^-)$. Recall from Lemma 5.2 in [35] in case of an odd index $\mu(P_1)$, that the solutions nearby make a full turn around the binding orbit P_1 in a short interval of time. On the other side of the rigid surface hit, the foliation continues again by a 1-dimensional family and moving the loops along the transversal flow we can argue as before. There are only a finite number of rigid surfaces and 1-dimensional families. Therefore, we find a rigid surface R of the foliation hit infinitely often in forward time. By the uniqueness of the initial value problem, the loops S_j cut out on R by the unstable manifold $W^-(P)$ are mutually disjoint. Since the Reeb vector field X lies in the kernel of $d\lambda$, the λ integral over such a loop S_j is the same, namely equal to

$$T = \int_{S_j} \lambda.$$

The 2-form $d\lambda$ induces on the surface R an area form for which the total area is finite, in view of the energy estimates leading to the following contradiction. Every circle S_j on R encloses an area on R of value $\int_{S_j} \lambda$ minus the sum of the periods of the enclosed punctures of R.

Since there are only finitely many punctures the total sum of the disjoint areas enclosed by the circles S_j on R must be infinite. This contradiction shows that $W^-(P)$ necessarily intersects $W^+(Q)$ for a hyperbolic spanning orbit of index $\mu(Q) = 2$, hence proving the proposition. See Figure 35.

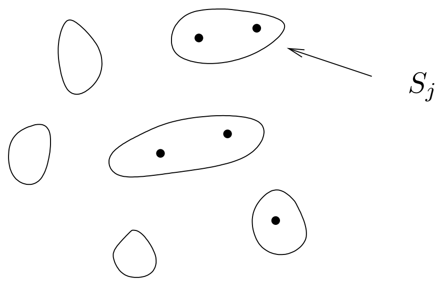

Figure 35.

Proposition 7.7. *Assume $\lambda = f\lambda_0$ is generic, i.e., $f \in \Theta_2$. If $\mathcal{F}$ has more than one fixed point of the $\mathbb{R}$-action, there exists a hyperbolic binding orbit P_0 whose stable and unstable invariant manifolds intersect each other transversally, $W^+(P_0) \cap W^-(P_0) = \{x\}$, in a homoclinic orbit $x \neq P$.*

Proof. By Proposition 7.5, there exists a hyperbolic binding orbit P whose unstable invariant manifold intersects the stable invariant manifold of a hyperbolic binding orbit P_1, also having index $\mu(P_1) = 2$. If $P_1 = P$ we are done. Otherwise we find by repeating the construction of Proposition 7.5 a heteroclinic chain between hyperbolic binding orbits P, P_1, P_2, ..., all having index equal to 2. There are only finitely many binding orbits and we deduce a hyperbolic orbit P_0 possessing a heteroclinic loop. Since the intersections are transversal, P_0 possesses a nontrivial transversal homoclinic orbit, as claimed. □

It is well known that the existence of a generic homoclinic orbit complicates the orbit structure of X considerably; see for example [47]. It allows, in particular, the construction of an embedded Bernoulli-shift for an associated local Poincaré section map. Its infinitely many periodic points are the initial conditions for infinitely many periodic solutions of the vector field X. Consequently, if in the generic case, the foliation $\mathcal{F}$ is not an open book decomposition, then the Reeb vector field X on S^3 possesses infinitely many periodic orbits. This completes the proof of Theorem 1.9 and its corollaries.

7.2. *Weakly convex contact forms.* The foliation $\mathcal{F}$ has a simple description if there are no spanning orbits of index 1. We recall from [35] the following definition.

Definition 7.8. A nondegenerate contact form λ on S^3 is called *dynamically convex* if for every periodic orbit (x, T) of the associated Reeb vector field the inequality $\mu(x, T) \geq 3$ holds.

The spanning orbits of $\mathcal{F}$ in the case of a tight dynamically convex contact form $\lambda = f\lambda_0$ have indices all equal to 3. Apart from the fixed points of the $\mathbb{R}$-action, the leaves therefore appear in 1-dimensional families of finite energy planes. Such a family is necessarily parametrized by S^1. Indeed, otherwise at the ends of the parameter interval the family necessarily splits into 2 rigid surfaces giving rise to a spanning orbit of index 2, which is excluded. Hence, arguing as above, we see that the foliation $\mathcal{F}$ possesses precisely one spanning orbit of index 3 and the projection of $\mathcal{F}$ onto S^3 is an open book decomposition into planes. Moreover, if compactified by the spanning orbit P, it is a global surface of section of disc type for the Reeb vector field. Studying the Poincaré section map we deduce by means of the area-preserving character as before the following result for the Reeb flow.

THEOREM 7.9. *The Reeb flow of a dynamically convex contact form $\lambda = f\lambda_0$ possesses either precisely two or infinitely many periodic orbits.*

An interesting example is the Hamiltonian flow on a strictly convex energy surface in $\mathbb{R}^4$. As shown in [35], such a flow is conjugated to the Reeb flow on S^3 defined by a dynamically convex tight contact form $\lambda = f\lambda_0$, and hence possesses, in the generic case either two or infinitely many periodic orbits. The result actually holds true without the assumption of genericity, as shown in [35]. We next introduce a new concept.

Definition 7.10. A nondegenerate contact form λ on S^3 is called *weakly convex* if for every periodic orbit (x, T) of the associated Reeb vector field the inequality $\mu(x, T) \geq 2$ holds.

If $\lambda = f\lambda_0$ is weakly convex the foliation $\mathcal{F}$ consists of finitely many fixed points of the $\mathbb{R}$-action, finitely many rigid cylinders having their origin in the spanning orbits of index 2, and the complement is filled with a finite number of 1-parameter families of finite energy planes, parametrized over intervals. We describe the projection of $\mathcal{F}$ into S^3 more precisely: Every spanning orbit P of index 2 is the boundary of precisely two disjoint rigid planes $C^{\pm}$. Their union $C^- \cup P \cup C^+$ is a smoothly embedded 2-sphere S^2 in S^3 whose equator is P. One of the hemispheres, C^+, is the entrance set for the Reeb flow, the other hemisphere, C^-, is the exit set of the 2-sphere. Moreover, inside the ball and in the complement of the ball there exist binding orbits of index 3 connected with the equator P by rigid cylinders. Inside of the ball there is possibly another rigid cylinder connecting the index 3 binding orbit with an index 2 binding orbit inside the sphere, which again is the equator of a smaller rigid 2-sphere inside the larger sphere already described. Proceeding this way we end up after finitely many steps with a smallest rigid sphere containing no further rigid spheres. The smallest sphere contains precisely one binding orbit of index 3 connected by a rigid cylinder with the equator of the sphere and is filled with a 1-parameter family of planes, whose boundaries agree with the index 3 binding orbit. The dynamical consequences of such a family are not yet worked out. We visualize the projection of $\mathcal{F}$ in S^3 by Figure 36. In the figure the white dots represent periodic orbits with index 2 and the black dots periodic orbits with index 3.

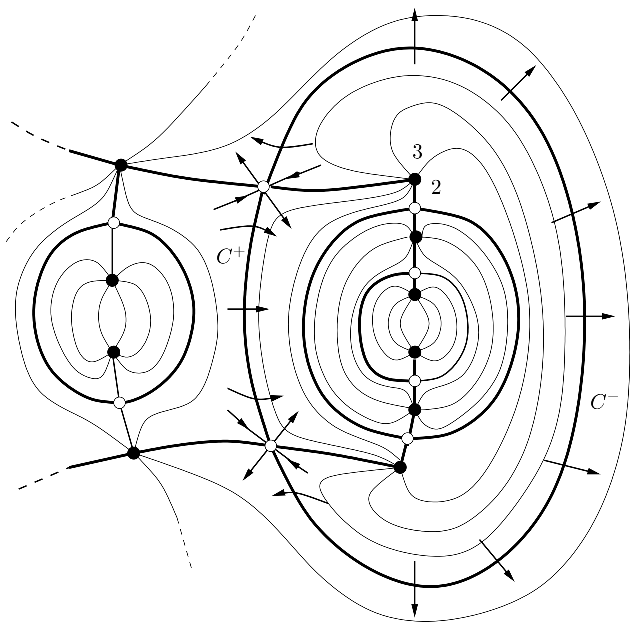

Figure 36. An example of a foliation for a weakly convex contact form. The figure shows the foliation by disk-like and annuli-like surfaces projected onto S^3 and the trace of a foliation cut by a plane. The white dots represent periodic orbits of index 2 and the black dots periodic orbits of index 3. They are perpendicular to the page and two dots belong to the same periodic orbit. The rigid surfaces are represented by bold curves. The arrows indicate the Reeb flow. The 3-sphere is viewed as $\mathbb{R}^3 \cup \{\infty\}$.

8. Appendix

8.1. *The Conley-Zehnder index.* For the readers' convenience we shall collect some facts about the index of a nondegenerate contractible periodic orbit of the Reeb vector field X_λ on M. Assume (x,T) is a T-periodic solution, which is nondegenerate and contractible. The linearized map $T\varphi^t(x(0)) : T_{x(0)}M \to T_{x(t)}M$ maps the contact plane $\xi_{x(0)}$ onto $\xi_{x(t)}$ and is, moreover, symplectic with respect to $d\lambda$. We choose a smooth disc map $u : D \to M$ satisfying $u\Big(\mathrm{e}^{2\pi \mathrm{i}t/T}\Big) = x(t)$, where $D = \{z \in \mathbb{C} \mid |z| \le 1\}$. Then we choose a symplectic trivialization $\beta : u^*\xi \to D \times \mathbb{R}^2$ and consider the arc $\Phi : [0,T] \to \mathrm{Sp}(1)$ of symplectic matrices $\Phi(t)$ in $\mathbb{R}^2$ defined by

$$\Phi(t) := \beta\Big(\mathrm{e}^{2\pi \mathrm{i}t/T}\Big) \circ T\varphi^t_{|\xi_{x(0)}} \circ \beta^{-1}(1).$$

The arc starts at the identity $\Phi(0) = \mathrm{Id}$ and ends at $\Phi(T)$. The integer 1 is not an eigenvalue of $\Phi(T)$ if and only if (x,T) is nondegenerate. With every such arc we shall associate an integer $\mu(\Phi)$ and then define the index of the periodic solution (x,T) by

$$\mu(x,T,[u]) = \mu(\Phi).$$

This index will only depend on the homotopy class $[u]$ of the chosen disc map having the boundary fixed. If, as in our applications, $M = S^3$, the index is independent of the disc map chosen.

In order to define $\mu(\Phi)$ we abbreviate the set of arcs under consideration by $\Sigma^*(1) = \{\Phi : [0,T] \to \mathrm{Sp}(1) \mid \ \Phi(0) = \mathrm{Id} \text{ and } \Phi(T) \in \mathrm{Sp}(1)^*\}$, where $\mathrm{Sp}(1)^* = \{A \in \mathrm{Sp}(1) \mid \det(A - \mathrm{Id}) \neq 0\}$. We recall that the eigenvalues of $A \in \mathrm{Sp}(1)^*$ occur in pairs, namely $(\lambda, \bar\lambda) \in S^1 \setminus \{1\}$ or $(\lambda, \lambda^{-1}) \in \mathbb{R}^2 \setminus \{(1,1)\}$ and $\lambda > 0$, or $(\lambda, \lambda^{-1}) \in \mathbb{R}^2$ and $\lambda < 0$. According to the spectrum of the end point $\Phi(T)$ we therefore distinguish elliptic, $(+)$-hyperbolic and $(-)$-hyperbolic arcs Φ. The characterization of the integer $\mu(\Phi)$ is as follows.

THEOREM 8.1. *The Conley-Zehnder index is the unique map*

$$\mu : \Sigma^*(1) \to \mathbb{Z}$$

characterized by the following four properties:

Homotopy invariance: *If* Φ^τ *is a homotopy of arcs in* $\Sigma^*(1)$, *then* $\mu(\Phi^\tau)$ *does not depend on* τ.

Maslov compatibility: *If* $L : [0,T] \to \mathrm{Sp}(1)$ *is a loop, i.e.,* $L(0) = L(T)$ *and* $\Phi \in \Sigma^*(1)$, *then*

$$\mu(L \cdot \Phi) = 2 \ \mathrm{maslov}(L) + \mu(\Phi).$$

Invertibility: *With* $\Phi^{-1}(t) = \Phi(t)^{-1}$,

$$\mu(\Phi) = -\mu(\Phi^{-1})$$

Normalization: *If* $\Phi_{\frac{1}{2}}(t) := \mathrm{e}^{\pi \mathrm{i} t/T}$,

$$\mu(\Phi_{\frac{1}{2}}) = 1$$

In addition, the map $\mu : \Sigma^*(1) \to \mathbb{Z}$ *is surjective.*

For a proof we refer to [26]. There are several ways to present the integer $\mu(\Phi)$ and we recall first the geometric construction from [26]. We consider a differentiable arc $\Phi(t)$. It is the resolvent of a linear Hamiltonian equation $\dot{\Phi}(t) = JA(t)\Phi(t)$ starting at $\Phi(0) = \mathrm{Id}$. Let $z \in \mathbb{C} \setminus \{0\}$ and choose for the solution $z(t) = \Phi(t)z$ a continuous argument

$$\mathrm{e}^{2\pi \mathrm{i}\varphi(t)} = \frac{z(t)}{|z(t)|}, \quad 0 \le t \le T,$$

introduce the winding number of $\Phi(t)z$ by

$$\Delta(z) = \varphi(T) - \varphi(0) \in \mathbb{R},$$

and define the winding interval of the arc Φ by

$$I(\Phi) = \{\Delta(z) \mid z \in \mathbb{C} \setminus \{0\}\}.$$

The length of this interval is strictly smaller than 1/2. Indeed, for two solutions $z(t)$ and $w(t)$, we define the curve $z_1(t) = z(t)\overline{w(t)}$ in $\mathbb{C}$ and observe that $\Delta(z_1) = \Delta(z) - \Delta(w)$. Assume that $|\Delta(z_1)| \ge 1/2$. Then we find $t_0 \in [0, T]$ satisfying $z_1(t_0) \in \mathbb{R} \setminus \{0\}$ implying $z(t_0) = \tau w(t_0)$ for some $\tau \in \mathbb{R} \setminus \{0\}$. Consequently, $z(t) = \tau w(t)$ for all $t \in [0, T]$ and hence $\Delta(z) = \Delta(w)$ in contradiction to $|\Delta(z) - \Delta(w)| \ge 1/2$. We have proved that $|I(\Phi)| < 1/2$. The winding interval either lies between two consecutive integers or contains precisely one integer and we can define

$$\mu(\Phi) = \begin{cases} 2k+1 & \text{if } I(\Phi) \subset (k, k+1) \\ 2k & \text{if } k \in I(\Phi), \end{cases}$$

for some integer $k \in \mathbb{Z}$. It is proved in [26] that $\mu(\Phi)$ satisfies all required properties in Theorem 8.1.

Clearly, the winding number $\Delta(z_0)$ is an integer if and only if $\Phi(T)z_0 = \lambda z_0$ and $\lambda > 0$. Hence (+)-hyperbolic arcs are characterized by even indices. For a (−)-hyperbolic arc we have an eigenvector $\Phi(T)z_0 = -\lambda z_0$, $\lambda > 0$, so that $\Delta(z_0) = k + 1/2$ for an integer k and hence $\mu(\Phi) = 2k + 1$ is odd. The elliptic arcs also necessarily have odd indices.

Observe now that the differentiable arc $\Phi(t)$ coming from the periodic solution (x, T) is defined for all $t \in \mathbb{R}$ and satisfies, moreover,

$$\Phi(t+T) = \Phi(t)\Phi(T), \quad t \in \mathbb{R}.$$

This allows us to define the indices $\mu(\Phi^{(n)})$ for the iterated periodic solutions assuming that $\Phi^{(n)}$ are all nondegenerate. Here $\Phi^{(n)}$ is the arc $\Phi(t)$ for $0 \leq t \leq nT$, $n \geq 1$. If $z \in \mathbb{C} \setminus \{0\}$, the winding number can be written as the sum

$$\Delta(z, \Phi^{(n)}) = \Delta(z) + \Delta(\Phi(T)z) + \cdots + \Delta(\Phi((n-1)T)z). \tag{8.1}$$

If Φ is a (+)-hyperbolic arc there is an eigenvector $\Phi(T)z_0 = \lambda z_0$ with $\lambda > 0$. Hence $k = \Delta(z_0) = \Delta(\Phi(T)z_0) = \cdots = \Delta(\Phi((n-1)T)z_0)$ and we deduce from (8.1) that

$$\mu(\Phi^{(n)}) = n\mu(\Phi), \qquad \text{for all } n \geq 1. \tag{8.2}$$

The same iteration formula holds for a ($-$)-hyperbolic arc. The iterations of elliptic arcs is more subtle. However, in the elliptic case we conclude from (8.1) the estimates

$$n[\mu(\Phi) - 1] + 1 \leq \mu(\Phi^{(n)}) \leq n[\mu(\Phi) + 1] - 1.$$

We made use in previous sections of the following monotonicity property of the index, which follows immediately from (8.1) and (8.2).

Proposition 8.2.

Either	$\mu(\Phi^{(n)}) = 0$ *for all* $n \geq 1$
or	$0 < \mu(\Phi) \leq \mu(\Phi^{(2)}) \leq \mu(\Phi^{(3)}) \leq \dots$
or	$0 > \mu(\Phi) \geq \mu(\Phi^{(2)}) \geq \mu(\Phi^{(3)}) \geq \dots$.

In previous sections we also used the fact that $\mu(\Phi) \geq 2$ implies $\mu(\Phi^{(2)}) \geq 4$. This is proved as follows. If $\mu(\Phi) = 2k$, $k \geq 1$, then by (8.2) $\mu(\Phi^{(2)}) = 4k \geq 4$. If $\mu(\Phi) = 2n+1$ is odd and $n \geq 1$, then $I(\Phi) \subset (n, n+1)$ and hence by (8.1) $I(\Phi^{(2)}) \subset (k, k+1)$ for some $k \geq 2n$. Consequently, $\mu(\Phi^{(2)}) = 2k+1 \geq 4n+1 \geq 5$.

We next recall that the index $\mu(\Phi)$ is related to the rotation of $\Phi(t)$ in $\mathrm{Sp}(1)$. There is a unique decomposition

$$\Phi(t) = O(t) \cdot P(t)$$

into an orthogonal matrix $O(t) \in \mathrm{Sp}(1)$ and a symmetric and positive definite matrix $P(t) = \mathrm{e}^{A(t)} \in \mathrm{Sp}(1)$. Since $\Phi(0) = \mathrm{Id}$, we conclude from the uniqueness of the decomposition that $O(0) = \mathrm{Id}$ and $P(0) = \mathrm{Id}$. Hence with $O(t) = \mathrm{e}^{2\pi \mathrm{i}\alpha(t)}$ we obtain for the winding number $\Delta(z)$ of $z \in \mathbb{C} \setminus \{0\}$ the representation

$$\Delta(z) = \Delta_0 + \Delta(\ \arg[P(t)z]) =: \Delta_0 + \delta(z),$$

where $\Delta_0 = \alpha(T)$ is the rotation of the arc $O(t)$, for $0 \leq t \leq T$, in $\mathrm{Sp}(1)$. From $(P(t)z, z) > 0$ we deduce the estimate $|\delta(z)| < 1/2$. This implies for an eigenvector $P(T)z_0 = \lambda z_0$ and $\lambda > 0$ that $\delta(z_0) = 0$. Hence $\Delta_0 \in I(\Phi)$.

We deform the arc Φ within $\Sigma^*(1)$ by prolonging the endpoint $\Phi(T)$, keeping the spectrum of $\Phi(T)$ fixed. Note that after conjugation with a suitable orthogonal matrix u we have,

$$u \circ \Phi(T) \circ u^{-1} = \begin{pmatrix} \cos 2\pi\Delta_0 & -\sin 2\pi\Delta_0 \\ \sin 2\pi\Delta_0 & \cos 2\pi\Delta_0 \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda > 0,$$

and the eigenvalues $\mu \in \sigma(\Phi(T))$ are

$$\mu_\pm = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)\cos 2\pi\Delta_0 \pm \sqrt{\frac{1}{4}\left(\lambda + \frac{1}{\lambda}\right)^2 \cos^2 2\pi\Delta_0 - 1}.$$

Assume at first that $\Phi(T)$ is $(+)$-hyperbolic. Then

$$\cos 2\pi\Delta_0 > 0 \qquad \text{and} \qquad \frac{1}{4}\left(\lambda + \frac{1}{\lambda}\right)^2 \cos^2 2\pi\Delta_0 - 1 > 0.$$

In particular, $\Delta_0 \in (k - 1/4, k + 1/4)$ for some integer k. Denote by $R(s)$ the rotation with angle $2\pi s$. We prolong $\Phi(T) = O(T)\mathrm{e}^{A(T)}$ keeping the spectrum fixed by the arc

$$\begin{aligned} &O(T)R(\varepsilon(s))\mathrm{e}^{A_s(T)} \\ &\quad = \begin{pmatrix} \cos 2\pi\Big(\Delta_0 + \varepsilon(s)\Big) & -\sin 2\pi\Big(\Delta_0 + \varepsilon(s)\Big) \\ \sin 2\pi\Big(\Delta_0 + \varepsilon(s)\Big) & \cos 2\pi\Big(\Delta_0 + \varepsilon(s)\Big) \end{pmatrix} \cdot \begin{pmatrix} \lambda_s & 0 \\ 0 & 1/\lambda_s \end{pmatrix}, \end{aligned}$$

for $0 \leq s \leq \varepsilon^*$. Here $\varepsilon(s) = [\ \mathrm{sign}(k - \Delta_0)]s$ and $\varepsilon^* = |k - \Delta_0|$. Adding this piece of arc we homotope $\Phi(t)$, $0 \leq t \leq T$, within $\Sigma^*(1)$ into the arc $\Psi \in \Sigma^*(1)$,

$$\Psi : \mathrm{Id} \xrightarrow{\Phi(t)} \Phi(T) \longrightarrow \mathrm{Id} \cdot \mathrm{e}^{A_0(T)}.$$

In view of the homotopy invariance in Theorem 8.1 we conclude $\mu(\Psi) = \mu(\Phi)$. The rotation of Ψ in $\mathrm{Sp}(1)$ is clearly an integer $\widetilde{\Delta} := \Delta_0 + [\ \mathrm{sign}(k - \Delta_0)]\varepsilon^* = k$ so that $\mu(\Phi) = \mu(\Psi) = 2\widetilde{\Delta}$ and $\mu(\Phi^n) = 2n\widetilde{\Delta}$. Similarly, in the $(-)$-hyperbolic case we continue the endpoint $\Phi(T)$ keeping the spectrum fixed to the matrix $-\mathrm{Id} \cdot \mathrm{e}^{A_0(T)}$. This way we homotope the arc $\Phi(t)$, $0 \leq t \leq T$, within $\Sigma^*(1)$ to the arc $\Psi \in \Sigma^*(1)$ given by

$$\Psi : \mathrm{Id} \xrightarrow{\Phi(t)} \Phi(T) \longrightarrow -\mathrm{Id} \cdot \mathrm{e}^{A_0(T)}.$$

This time the rotation of Ψ is equal to $\widetilde{\Delta} = \Delta_0 + \beta = k + 1/2$ for an integer k. Using the homotopy invariance we conclude $\mu(\Phi) = \mu(\Psi) = 2k + 1 = 2\widetilde{\Delta}$, and $\mu(\Phi^n) = 2n\widetilde{\Delta}$. Finally, if $\Phi(T)$ is elliptic, then

$$\frac{1}{4}\left(\lambda + \frac{1}{\lambda}\right)^2 \cos^2 2\pi\Delta_0 - 1 < 0$$

and there is a unique function $\varepsilon(s)$, $0 \leq s \leq 1$, satisfying $\varepsilon(0) = 0$, $\varepsilon(1) = \varepsilon$, and $|\varepsilon(s)| < 1/4$ and such that the arc

$$O(T)R(\varepsilon(s))\mathrm{e}^{A_s(T)} = \begin{pmatrix} \cos 2\pi\Big(\Delta_0 + \varepsilon(s)\Big) & -\sin 2\pi\Big(\Delta_0 + \varepsilon(s)\Big) \\ \sin 2\pi\Big(\Delta_0 + \varepsilon(s)\Big) & \cos 2\pi\Big(\Delta_0 + \varepsilon(s)\Big) \end{pmatrix} \cdot \begin{pmatrix} s + (1-s)\lambda & 0 \\ 0 & \frac{1}{s+(1-s)\lambda} \end{pmatrix}$$

has the same spectrum as $O(T) \cdot \mathrm{e}^{A(T)}$. Prolonging $\Phi(T)$ by adding this piece of arc we homotope $\Phi(t)$, $0 \leq t \leq T$, in $\Sigma^*(1)$ to the arc $\Psi \in \Sigma^*(1)$ given by

$$\Psi : \mathrm{Id} \xrightarrow{\Phi(t)} \Phi(T) \longrightarrow O(T) \cdot R(\varepsilon)$$

and hence ending at a nontrivial rotation. Since the arc is elliptic, the rotation $\widetilde{\Delta} = \Delta_0 + \varepsilon$ satisfies $k < \widetilde{\Delta} < k + 1$ for an integer k. Therefore, by homotopy invariance, $\mu(\Phi) = \mu(\Psi) = 2k + 1$ and hence

$$\mu(\Phi) = 2\widetilde{\Delta} + r_1, \quad |r_1| < 1.$$

In order to determine the index $\mu(\Phi^{(n)})$ of the iterated arc $\Phi(t)$, $0 \leq t \leq nT$, in the elliptic cases we recall that $\sigma(\Phi(T)^j) \in S^1 \setminus \{1\}$ for all iterates $j \geq 1$. Using the above prolongation of $\Phi(T)$ in $\mathrm{Sp}^*(1)$ we homotope $\Phi(t)$, $0 \leq t \leq nT$, in $\Sigma^*(1)$ to the arc Ψ of rotations,

$$\begin{aligned} \Psi : \mathrm{Id} \quad &\xrightarrow{O(t)} \quad O(T) \quad \xrightarrow{O(t)O(T)} \quad O(T)^2 \longrightarrow \cdots \\ &\xrightarrow{O(t)O(T)^{n-1}} O(T)^n \xrightarrow{(O(T)R(\varepsilon(\tau)))^n} (O(T) \cdot R(\varepsilon))^n, \end{aligned} \tag{8.3}$$

where $0 \leq t \leq T$ and $0 \leq \tau \leq 1$. The deformation Φ_s of Φ is defined as follows. We set $\Phi_s(t) = O(t)\mathrm{e}^{sA(t)}$ for $0 \leq t \leq T$ and $0 \leq s \leq 1$, and add successively the arcs $\Phi_s(t)\Phi_s(T)^j$ for $j = 1, 2, \ldots, (n-1)$, where $0 \leq t \leq T$, and finally prolong the endpoint in $\mathrm{Sp}^*(1)$ by the arc $(O(T)R(\varepsilon(t))\mathrm{e}^{sA(T)})^n$ for $s \leq t \leq 1$. From (8.3) we compute for the rotation of Ψ in $\mathrm{Sp}(1)$ the number $n\Delta_0 + n\varepsilon = n\widetilde{\Delta}$. By the homotopy invariance, $\mu(\Phi^{(n)}) = \mu(\Psi)$. Therefore, we obtain the following iteration formula for the index in the elliptic case

$$\mu(\Phi^{(n)}) = 2n\widetilde{\Delta} + r_n, \quad |r_n| < 1. \tag{8.4}$$

Note that the rotation number $\widetilde{\Delta} \in \mathbb{R}$ is uniquely determined by $\Phi(t)$, for $0 \leq t \leq T$. It splits into two parts $\widetilde{\Delta} = \Delta_0 + \delta_0$, where $\Delta_0 = \alpha(T) - \alpha(0)$ is determined by the orthogonal part of $\Phi(t) = O(t) \cdot P(t)$ over $0 \leq t \leq T$ and δ_0 is determined by the spectrum $\sigma(\Phi(T))$ of the end point. The number $\widetilde{\Delta}$ is irrational in the elliptic case, while $\widetilde{\Delta} = k \in \mathbb{Z}$ for a $(+)$-hyperbolic arc and $\widetilde{\Delta} = k + 1/2$ for a $(-)$-hyperbolic arc. To summarize the index $\mu(x, T) = \mu(\Phi)$ of a periodic solution of a Reeb vector field on S^3 has the following properties:

THEOREM 8.3. *Assume that the periodic orbit* (x,T) *and its iterates* (x,nT), $n \geq 1$, *are nondegenerate.*

1. *If* (x,T) *is* $(+)$*-hyperbolic, then*

$$\mu(X,nT) = n\mu(x,T), \quad n \geq 1$$

and $\mu(x,T) = 2k$ *is even.*

2. *If* (x,T) *is* $(-)$*-hyperbolic, then*

$$\mu(X,nT) = n\mu(x,T), \quad n \geq 1$$

and $\mu(x,T) = 2k+1$ *is odd.*

3. *If* (x,T) *is an elliptic periodic solution, then*

$$\mu(X,nT) = 2n\widetilde{\Delta} + r_n, \quad |r_n| < 1$$

with $\mu(x,T) = 2k+1$ *odd. The real number* $\widetilde{\Delta} \in (k,k+1)$ *is irrational and uniquely determined by* (x,T).

Crucial for the geometry of finite energy surfaces near the punctures is the characterization of the index $\mu(\Phi)$ in terms of spectral properties of the asymptotic linear operator. The differentiable arcs $\Phi : \mathbb{R} \to \mathrm{Sp}(1)$ satisfying $\Phi(t+T) = \Phi(t)\Phi(T)$ and $\Phi(0) = \mathrm{Id}$ are in one-to-one correspondence with the linear Hamiltonian vector fields $JA(t) = \dot{\Phi}(t)\Phi^{-1}(t)$, where $A(t+T) = A(t)$ is periodic in time and a symmetric matrix in $\mathcal{L}(\mathbb{R}^2)$. Define the linear operator L_A in $L^2(S^1,\mathbb{R}^2)$ by

$$L_A = -J\frac{d}{dt} - A(t)$$

on the domain $H^{1,2}(S^1,\mathbb{R}^2)$, where $S^1 = \mathbb{R}/T\mathbb{Z}$. The operator L_A is self-adjoint and its spectrum $\sigma(L_A)$ consists of countably many isolated eigenvalues and is unbounded from above and from below. Moreover, $\ker L_A = \{0\}$ if and only if $\Phi \in \Sigma^*(1)$.

An eigenfunction $v \neq 0$ in L^2 belonging to the eigenvalue $\lambda \in \sigma(L_A)$ solves the first order boundary value problem

$$-J\dot{v}(t) - A(t)v(t) = \lambda v(t), \quad v(0) = v(T).$$

Consequently, $v(t) \neq 0$ for all t and hence v defines a continuous map from S^1 into $\mathbb{C} \setminus \{0\}$ which has a winding number, denoted by $w(v,\lambda) \in \mathbb{Z}$. One easily verifies that two linearly independent eigenfunctions belonging to the same eigenvalue λ have the same winding number, so that with every $\lambda \in \sigma(L_A)$ we can associate the winding number $w(\lambda, A) \in \mathbb{Z}$. For every integer $k \in \mathbb{Z}$ there are precisely two eigenvalues (counted with multiplicities) λ_1 and $\lambda_2 \in \sigma(L_A)$ satisfying $k = w(\lambda_1, A) = w(\lambda_2, A)$; see [30, Lemma 3.6]. If there exists only one such eigenvalue, its multiplicity is 2. In this sense every winding number occurs twice in $\sigma(L_A)$. Moreover, the map $\lambda \mapsto w(\lambda, A)$ from $\sigma(L_A)$ onto $\mathbb{Z}$ is monotonic, see [30]. Define now two integers $\alpha(A) \in \mathbb{Z}$ and $p(A) \in \{0, 1\}$ as follows: $\alpha(A)$ is the maximum of the winding numbers $w(\lambda, A)$ belonging to the eigenvalues $\lambda < 0$ of the operator L_A, and $p(A) = (1 + (-1)^b)/2$, where b is the number of eigenvalues (counted with multiplicity) strictly less than 0 which have winding numbers equal to $\alpha(A)$.

THEOREM 8.4. *If* $\Phi \in \Sigma^*(1)$,

$$\mu(\Phi) = 2\alpha(A) + p(A).$$

The proof is given in [30, Theorem 3.10]. The arc Φ is (+)-hyperbolic if and only if $p(A) = 0$. We observe that $\mu(\Phi) = 3$ if and only if $\alpha(A) = 1$ and $p(A) = 1$. Since the asymptotic behavior of a finite energy surface near a puncture is governed by an eigenvector of the asymptotic operator L_A we can gain information about the geometry of the surface from the knowledge of the index $\mu(x, T)$ of the asymptotic limits, see [30].

Consider a finite energy surface $\widetilde{u} = (a, u) : S \setminus \Gamma \to \mathbb{R} \times M$ with the punctures $\Gamma = \Gamma^+ \cup \Gamma^-$ and assume the asymptotic limits are nondegenerate. Since the surface converges to finitely many periodic orbits we can define an index $\mu(\widetilde{u})$ as follows. We compactify the puncture surface $\dot{S} = S \setminus \Gamma$ by adding a circle for every puncture distinguishing positive and negative punctures. Using the asymptotic behavior of the surface, we see that the map $u : S \setminus \Gamma \to M$ can be extended to a smooth map $\bar{u} : \bar{S} \to M$ such that the boundary circles of $\bar{S}$ parametrize the periodic orbits associated with the punctures. Choose a symplectic trivialization Ψ of $\bar{u}^*\xi \to \bar{S}$. Then the Conley-Zehnder index μ_z for the periodic solution associated with the puncture $z \in \Gamma$ can be computed with respect to the trivialization as above and we define the index $\mu(\widetilde{u}) \in \mathbb{Z}$ by

$$\mu(\widetilde{u}) = \sum_{z \in \Gamma^+} \mu_z - \sum_{z \in \Gamma^-} \mu_z \in \mathbb{Z}.$$

This integer $\mu(\widetilde{u})$ does not depend on the choices involved, in contrast to the integers μ_z which depend on the choice of the trivialisation Ψ, [30, Prop. 5.5]. We point out that the integer $\mu(\widetilde{u})$ enters the formula for the Fredholm index $\mathrm{Ind}(\widetilde{u}) = \mu(\widetilde{u}) - \chi(S) + \sharp\Gamma$; see [36].

8.2. *Asymptotics of a finite energy surface near a nondegenerate puncture.* From [35] and [32] we recall the behavior of a nonconstant pseudoholomorphic curve near one of its punctures, assuming the energy to be bounded and the contact form to be nondegenerate. In the following, M is a three manifold equipped with the contact form λ determining the contact bundle ξ and the Reeb vector field X. Choosing a complex multiplication J on ξ we denote by

$\widetilde{J}$ the associated distinguished $\mathbb{R}$-invariant almost complex structure on $\mathbb{R}\times M$ and consider the finite energy surface

$$\widetilde{u} = (a, u) : S \setminus \Gamma \to \mathbb{R} \times M$$

with the nonempty finite set Γ of punctures. Near the puncture $z_0 \in \Gamma$ we introduce holomorphic polar coordinates. We take a holomorphic chart $h : D \subset \mathbb{C} \to U \subset S$ around z_0 satisfying $h(0) = z_0$ and define $\sigma : [0,\infty)\times S^1 \to U\setminus\{z_0\}$ by

$$\sigma(s,t) = h\Big(\mathrm{e}^{-2\pi(s+\mathrm{i}t)}\Big).$$

Then $z_0 = \lim_{s\to\infty} \sigma(s,t)$. In these coordinates, $\widetilde{u}$ becomes, near z_0, the positive half cylinder

$$\widetilde{v} = (b, v) = \widetilde{u} \circ \sigma : [0,\infty) \times S^1 \to \mathbb{R} \times M.$$

The map $\widetilde{v}$ solves the Cauchy-Riemann equation

$$\widetilde{v}_s + \widetilde{J}(\widetilde{v})\widetilde{v}_t = 0$$

and has bounded energy $E(\widetilde{v}) \leq E(\widetilde{u}) < \infty$. Because of the energy bound the following limit exists:

$$m(\widetilde{u}, z_0) = \lim_{s\to\infty} \int_{S^1} v(s,\cdot)^*\lambda.$$

Indeed, by Stokes' theorem,

$$\begin{aligned} \int_{S^1} v(s,\cdot)^*\lambda &= \int_{S^1} v(0,\cdot)^*\lambda + \int_{[0,s]\times S^1} v^* d\lambda \\ &= c_0 + \frac{1}{2}\int_{[0,s]\times S^1} \Big[|\pi v_s|_J^2 + |\pi v_t|_J^2\Big]\, dsdt \end{aligned}$$

so that the map $s \to \int_{S^1} v(s,\cdot)^*\lambda$ is monotonic and bounded. The real number $m = m(\widetilde{u}, z_0)$ is called the charge of the puncture z_0. It is positive if z_0 is a positive puncture and negative for a negative puncture. Moreover, $m = 0$ if the puncture is removable. The behavior of the surface near z_0 is determined by periodic solutions of the Reeb vector field having periods $T = |m(\widetilde{u}, z_0)|$. Namely, every sequence $s_k \to \infty$ possesses a subsequence, still denoted by s_k such that

$$u(s_k, t) \to x(mt) \qquad \text{in } C^\infty(S^1)$$

for an orbit $x(t)$ of the Reeb vector field $\dot{x}(t) = X(x(t))$. Here m is the charge of z_0. If $m \neq 0$ the solution is necessarily a periodic orbit of X having the period $T = |m|$. If the periodic orbit is nondegenerate, hence, in particular, isolated among periodic orbits having periods close to $|m|$, then

$$\lim_{s\to\infty} v(s,t) = x(Tm) \qquad \text{in } C^\infty(S^1)$$

and

$$\lim_{s\to\infty} \frac{b(s,t)}{s} = m \qquad \text{in } C^\infty(S^1).$$

In this case we call the uniquely determined periodic solution (x,T) with period $T = |m|$ the asymptotic limit of the puncture z_0.

The energy surface $\widetilde{v}$ approaches, in the nondegenerate case as $s \to \infty$, the orbit cylinder $\widetilde{v}_\infty(s,t) = (sm, x(mt))$ in $\mathbb{R} \times M$ in an exponential way. In order to describe this in detail we represent the contact structure λ in a tubular neighborhood of the asymptotic limit (x,T) in a normal form. In the following lemma we denote by

$$\lambda_0 = d\vartheta + xdy$$

the contact form on $S^1 \times \mathbb{R}^2$ with coordinates (ϑ, x, y).

Lemma 8.5. *Let M be a three-dimensional manifold equipped with a contact form λ and let (x,T) be a periodic solution of the Reeb vector field X. Denote by τ the minimal period of x so that $T = k\tau$ for an integer k. Then there exist open neighborhoods U of $S^1 \times \{0\} \subset S^1 \times \mathbb{R}^2$ and $V \subset M$ of $P = x(\mathbb{R}) \subset M$, and a diffeomorphism $\varphi : U \to V$ mapping $S^1 \times \{0\}$ onto P and satisfying*

$$\varphi^*\lambda = f\lambda_0.$$

The smooth function $f : U \to (0,\infty)$ has the properties $f(\vartheta,0,0) = \tau$ and $df(\vartheta,0,0) = 0$ for all $\vartheta \in S^1$.

Working in the covering space $\mathbb{R}$ of S^1, the curve $\widetilde{v}$ is, in the coordinates of the lemma, represented as a map

$$\begin{aligned} \widetilde{v} &= (b,v) : [0,\infty) \times \mathbb{R} \to \mathbb{R}^4, \\ \widetilde{v}(s,t) &= \Big(b(s,t), \vartheta(s,t), z(s,t)\Big), \end{aligned} \tag{8.5}$$

where the functions $b : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $z : [0,\infty) \times \mathbb{R} \to \mathbb{R}^2$ are 1-periodic in t, while $\vartheta : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfies $\vartheta(s,t+1) = \vartheta(s,t) + k$. The last factor $\mathbb{R}^2$ in (8.5) is in the contact plane along the asymptotic limit in these coordinates.

Theorem 8.6 (Asymptotics). *Let $z_0 \in \Gamma$ be a nonremovable puncture of a finite energy surface $\widetilde{u} : S\backslash\Gamma \to \mathbb{R}\times M$ whose charge is $m(\widetilde{u}, z_0) = m$ and whose nondegenerate asymptotic limit is (x,T), where $T = |m| = k\tau$ with the minimal period τ. Introduce near z_0 the cylindrical coordinates $[0,\infty) \times S^1$ and near the asymptotic limit the normal form coordinates of the lemma. In these local coordinates, the finite energy surface has the form $\widetilde{v} = \Big(b(s,t), \vartheta(s,t), z(s,t)\Big) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$, where*

$$b(s,t) = ms + c + \widehat{b}(s,t), \qquad \vartheta(s,t) = \frac{m}{\tau}t + d + \widehat{\vartheta}(s,t)$$

and either $z(s,t) \equiv 0$ *for all* $(s,t) \in [0,\infty) \times \mathbb{R}$ *with* $s \geq 0$, *or*

$$z(s,t) = \mathrm{e}^{\int_0^s \gamma(\tau) d\tau} \Big[e(t) + \widehat{r}(s,t) \Big].$$

Here, c and d are two real constants, and

$$\partial^\alpha \widehat{r}(s,t) \to 0 \qquad \text{as } s \to \infty$$

uniformly in $t \in \mathbb{R}$ *and for all derivatives* $\alpha = (\alpha_1, \alpha_2)$. *In addition, there are constants* $M_\alpha > 0$ *and* $\beta > 0$ *such that*

$$\Big| \partial^\alpha \widehat{b}(s,t) \Big|, \quad \Big| \partial^\alpha \widehat{\vartheta}(s,t) \Big| \ \leq M_\alpha \mathrm{e}^{-\beta s}$$

for $s \geq 0$ *and all derivatives* α. *Moreover, the smooth function* $\gamma : [0,\infty) \to \mathbb{R}$ *converges,* $\gamma(s) \to \mu$ *as* $s \to \infty$. *The limit* μ *is a negative eigenvalue of a self-adjoint operator* A_∞ *in* $L^2(S^1, \mathbb{R}^2)$ *if* $m > 0$, *while* $-\mu$ *is a positive eigenvalue if* $m < 0$. *The function* $e(t) = e(t+1) \neq 0$ *represents an eigenvector belonging to* μ *resp.* $-\mu$. *The operator* A_∞ *is related to the linearized flow of the Reeb vector field* X *restricted to the invariant contact bundle along the periodic orbit* (x,T).

The proofs of these statements can be found in [1], [25], [24], [32], [36], and [38]. Also, note that the above asymptotic formula is used in the Fredholm theory [36] for embedded finite energy surfaces. It also plays an important role in the geometric description of finite energy surfaces in [35], [30], [31], [33].

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(Received February 28, 2000)