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Hodge integrals, partition matrices, and the λ_g conjecture

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Abstract

We prove a closed formula for integrals of the cotangent line classes against the top Chern class of the Hodge bundle on the moduli space of stable pointed curves. These integrals are computed via relations obtained from virtual localization in Gromov-Witten theory. An analysis of several natural matrices indexed by partitions is required.

0. Introduction

0.1. *Overview.* Let $M_{g,n}$ denote the moduli space of nonsingular genus g curves with n distinct marked points (over $\mathbb{C}$). Denote the moduli point corresponding the marked curve $(C, p_1, \ldots, p_n)$ by

$$[C, p_1, \ldots, p_n] \in M_{g,n}.$$

Let ω_C be the canonical bundle of algebraic differentials on C. The rank g Hodge bundle,

$$\mathbb{E} \to M_{g,n},$$

has fiber $H^0(C, \omega_C)$ over $[C, p_1, \ldots, p_n]$. The moduli space $M_{g,n}$ is nonsingular of dimension $3g - 3 + n$ when considered as a stack (or orbifold).

There is a natural compactification $M_{g,n} \subset \overline{M}_{g,n}$ by stable curves (with nodal singularities). The moduli space $\overline{M}_{g,n}$ is also a nonsingular stack. The Hodge bundle is well-defined over $\overline{M}_{g,n}$: the fiber over a nodal curve C is defined to be the space of sections of the dualizing sheaf of C. Let λ_g be the top Chern class of $\mathbb{E}$ on $\overline{M}_{g,n}$. The main result of the paper is a formula for integrating tautological classes on $\overline{M}_{g,n}$ against λ_g.

The study of integration against λ_g has two main motivations. First, such integrals arise naturally in the degree 0 sector of the Gromov-Witten theory of one-dimensional targets. The conjectural Virasoro constraints of Gromov-Witten theory predict the λ_g integrals have a surprisingly simple form. Second, the λ_g integrals conjecturally govern the entire tautological ring of the moduli space

$$M_g^c \subset \overline{M}_g$$

of curves of compact type. A stable curve is of compact type if the dual graph of C is a tree.

0.2. *Hodge integrals.* Let $A^*(\overline{M}_{g,n})$ denote the Chow ring of the moduli space with $\mathbb{Q}$-coefficients. We will consider two types of tautological classes in $A^*(\overline{M}_{g,n})$:

- $\psi_i = c_1(\mathbb{L}_i)$ for each marking i, where

  $$\mathbb{L}_i \to \overline{M}_{g,n}$$

  denotes the cotangent line bundle with fiber T^*_{C,p_i} at the moduli point $[C, p_1, \ldots, p_n] \in \overline{M}_{g,n}$,

- $\lambda_j = c_j(\mathbb{E})$, for $j \leq g$.

Hodge integrals are defined to be the top intersection products of the ψ_i and λ_j classes in $\overline{M}_{g,n}$. Hodge integrals play a basic role in Gromov-Witten theory and the study of the moduli space $\overline{M}_{g,n}$ (see, for example, [Fa], [FaP1], [P]).

0.3. *Virasoro constraints and the λ_g conjecture.* The ψ integrals in genus 0 are determined by a well-known formula:

$$\int_{\overline{M}_{0,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} = \binom{n-3}{\alpha_1, \ldots, \alpha_n}. \tag{1}$$

The formula is a simple consequence of the string equation [W].

The ψ integrals are determined in all genera by Witten's conjecture: the generating function of the ψ integrals satisfies the KdV hierarchy (or equivalently, Virasoro constraints). Witten's conjecture has been proven by Kontsevich [K1]. A proof via Hodge integrals, Hurwitz numbers, and random trees can be found in [OP].

The Virasoro constraints for the ψ integrals over $\overline{M}_{g,n}$ were generalized to constrain tautological integrals over the moduli space of stable maps to *arbitrary* nonsingular projective varieties through the work of Eguchi, Hori, and Xiong [EHX], and Katz. This generalization of Witten's original conjecture remains open.

Tautological integrals over the moduli spaces of *constant* stable maps to nonsingular projective varieties may be expressed as Hodge integrals over $\overline{M}_{g,n}$. Hence, the Virasoro constraints of [EHX] provide (conjectural) constraints for Hodge integrals. The λ_g conjecture was found in [GeP] as a consequence of these conjectural Virasoro constraints:

$$\int_{\overline{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \lambda_g = \binom{2g+n-3}{\alpha_1, \ldots, \alpha_n} \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g, \tag{2}$$

where $g \geq 1$, $\alpha_i \geq 0$. In fact, conjecture (2) was shown to be equivalent to the Virasoro constraints for constant maps to an elliptic curve [GeP]. Equation (2) predicts the combinatorics of the integrals of the ψ classes against λ_g is parallel to the genus 0 formula (1). The integrals occurring in (2) will be called λ_g *integrals*.

0.4. *Moduli of curves of compact type.* The λ_g integrals arise naturally in the study of the moduli space of curves of compact type. Let

$$M_g^c \subset \overline{M}_g$$

denote the (open) moduli space of curves of compact type for $g \geq 2$. The class λ_g vanishes when restricted to the complement $\overline{M}_g \setminus M_g^c$ (see [FaP2]). Integration against λ_g therefore yields a canonical linear evaluation function:

$$\epsilon : A^*(M_g^c) \to \mathbb{Q},$$

$$\xi \in A^*(M_g^c), \quad \epsilon(\xi) = \int_{\overline{M}_g} \xi \cdot \lambda_g.$$

The λ_g conjecture may be viewed as governing tautological evaluations in the Chow ring $A^*(M_g^c)$.

The role of λ_g in the study of M_g^c exactly parallels the role of $\lambda_g \lambda_{g-1}$ in the study of M_g. The class $\lambda_g \lambda_{g-1}$ vanishes on the complement $\overline{M}_g \setminus M_g$. Hence, integration against $\lambda_g \lambda_{g-1}$ provides a canonical evaluation function on $A^*(M_g)$ [Fa].

There is a conjectural formula for the $\lambda_g \lambda_{g-1}$ integrals which is also related to the Virasoro constraints [Fa], [GeP]. Data for $g \leq 15$ have led to a precise conjecture for the ring of tautological classes $R^*(M_g) \subset A^*(M_g)$ [Fa]. In particular, $R^*(M_g)$ is conjectured to be Gorenstein with the $\lambda_g \lambda_{g-1}$ integrals determining the pairings into the socle. It is natural to hope the tautological ring $R^*(M_g^c) \subset A^*(M_g^c)$ will also have a Gorenstein structure with socle pairings determined by (2).

A uniform perspective on the tautological rings $R^*(\overline{M}_g)$, $R^*(M_g^c)$, and $R^*(M_g)$ may be found in [FaP2]. If the Gorenstein property holds for $R^*(M_g^c)$, the λ_g integrals determine the entire ring structure [FaP2].

0.5. *Formulas for λ_g integrals.* The main result of the paper is a proof of the λ_g conjecture for all g.

Theorem 1. *The λ_g integrals satisfy*:

$$\int_{\overline{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \lambda_g = \binom{2g+n-3}{\alpha_1, \ldots, \alpha_n} \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g \,.$$

The integrals on the right side,

$$\int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g,$$

are determined by the following formula previously proven in [FaP1]:

$$F(t,k) = 1 + \sum_{g \geq 1} \sum_{i=0}^{g} t^{2g} k^i \int_{\overline{M}_{g,1}} \psi_1^{2g-2+i} \lambda_{g-i} = \left(\frac{t/2}{\sin(t/2)} \right)^{k+1}. \tag{3}$$

In particular, we find:

$$F(t,0) = 1 + \sum_{g \geq 1} t^{2g} \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g = \left(\frac{t/2}{\sin(t/2)} \right). \tag{4}$$

Equation (4) is equivalent to the Bernoulli number formula:

$$\int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g = \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}. \tag{5}$$

Equation (5) and Theorem 1 together determine all ψ integrals against λ_g.

0.6. *An interpretation in positive characteristic.* For an effective cycle X on $\overline{M}_g$ with class equal to a multiple of λ_g, the λ_g conjecture may be viewed as the analogue of Witten's conjecture for the family of curves represented by X.

In characteristic 0, it is not known whether λ_g is effective. In characteristic $p > 0$ however, λ_g *is* effective. Over an algebraically closed field of characteristic p, define the p-rank $f(A)$ of an abelian variety by

$$p^{f(A)} = |A[p]|,$$

where $A[p]$ is the set of geometric p-torsion points. Let A_g be the moduli space of principally polarized abelian varieties of dimension g. Koblitz has shown the locus $V_0 A_g$ of p-rank 0 abelian varieties is complete and of codimension g in A_g. Van der Geer and Ekedahl [vdG] proved that the class of $V_0 A_g$ is proportional to λ_g (by a factor equal to a polynomial in p). Define the p-rank of a curve of compact type as the p-rank of its Jacobian, and define the locus $V_0 M_g^c$ of curves of p-rank 0 via pullback along the Torelli morphism. This locus is complete in $\overline{M}_g$ and of codimension g (see [FvdG]) — it may however be nonreduced. The class of $V_0 M_g^c$ is proportional to λ_g (by the same factor). Hence λ_g is effective in characteristic p. The λ_g conjecture may then be viewed as Witten's conjecture for curves of p-rank 0.

Perhaps this interpretation will eventually enhance our understanding of the loci V_0. For example, $V_0 A_g$ is expected to be irreducible for $g \geq 3$, but this is known only for $g = 3$ (by a result of Oort). The simple form of the Witten conjecture for $V_0 M_g^c$ suggests an analogy with genus 0 curves that may lead to new insights.

0.7. *Localization.* Our proof of the λ_g conjecture uses the Hodge integral techniques introduced in [FaP1]. Let $\mathbf{P}^1$ be equipped with an algebraic torus $\mathbf{T}$ action. The virtual localization formula established in [GrP] reduces all Gromov-Witten invariants (and their descendents) of $\mathbf{P}^1$ to explicit graph sums involving only Hodge integrals over $\overline{M}_{g,n}$. Relations among the Hodge integrals may then be found by computing invariants known to vanish. The technique may be applied more generally by replacing $\mathbf{P}^1$ with any compact algebraic homogeneous space.

The philosophical basis of this method may be viewed as follows. If M is an arbitrary smooth variety with a torus action, the fixed components of M together with their equivariant normal bundles satisfy global conditions obtained from the geometry of M. Let M be the (virtually) smooth moduli stack of stable maps $\overline{M}_{g,n}(\mathbf{P}^1)$ with the naturally induced $\mathbf{T}$-action. The $\mathbf{T}$-fixed loci are then described as products of moduli spaces of stable curves with virtual normal structures involving the Hodge bundles [K2], [GrP]. In this manner, the geometry of $\overline{M}_{g,n}(\mathbf{P}^1)$ imposes conditions on the $\mathbf{T}$-fixed loci — conditions which may be formulated as relations among Hodge integrals by [GrP].

Localization relations involving only the λ_g integrals are found in Section 1 by studying maps multiply covering an exceptional $\mathbf{P}^1$ of an algebraic surface. These relations are linear and involve a change of basis from the standard form in formula (2). However, it is not difficult to show the relations are compatible with the λ_g conjecture (see §2.4). Both the linear equations from localization and the change of basis are determined by natural matrices indexed by partitions. In Section 3, the ranks of these partition matrices are computed to prove the system of linear equations found suffices to determine all λ_g integrals (up to the scalar $\int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g$ in each genus $g \geq 1$).

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1. Localization relations

1.1. *Torus actions.* A system of linear equations satisfied by the λ_g integrals is obtained here via localization relations. These relations are found by computing vanishing integrals over moduli spaces of stable maps in terms of Hodge integrals over $\overline{M}_{g,n}$.

The first step is to define the appropriate torus actions. Let $\mathbf{P}^1 = \mathbf{P}(V)$ where $V = \mathbb{C} \oplus \mathbb{C}$. Let $\mathbb{C}^*$ act diagonally on V:

$$\xi \cdot (v_1, v_2) = (v_1, \xi \cdot v_2). \tag{6}$$

Let p_1, p_2 be the fixed points $[1,0], [0,1]$ of the corresponding action on $\mathbf{P}(V)$. An equivariant lifting of $\mathbb{C}^*$ to a line bundle L over $\mathbf{P}(V)$ is uniquely determined by the weights $[l_1, l_2]$ of the fiber representations at the fixed points

$$L_1 = L|_{p_1}, \quad L_2 = L|_{p_2}.$$

The canonical lifting of $\mathbb{C}^*$ to the tangent bundle $T_{\mathbf{P}}$ has weights $[1, -1]$. We will utilize the equivariant liftings of $\mathbb{C}^*$ to $\mathcal{O}_{\mathbf{P}(V)}(1)$ and $\mathcal{O}_{\mathbf{P}(V)}(-1)$ with weights $[1,0]$, $[0,1]$ respectively.

Let $\overline{M}_{g,n}(d) = \overline{M}_{g,n}(\mathbf{P}(V), d)$ be the moduli stack of stable genus g, degree d maps to $\mathbf{P}^1$ (see [K2], [FuP]). There are canonical maps

$$\pi : U \to \overline{M}_{g,n}(d), \quad \mu : U \to \mathbf{P}(V)$$

where U is the universal curve over the moduli stack. The representation (6) canonically induces $\mathbb{C}^*$-actions on U and $\overline{M}_{g,n}(d)$ compatible with the maps π and μ (see [GrP]).

1.2. *Equivariant cycle classes.* There are four types of Chow classes in $A^*(\overline{M}_{g,n}(d))$ which will be considered here. First, there is a natural rank $d + g - 1$ bundle on $\overline{M}_{g,n}(d)$:

$$\mathbb{R} = R^1\pi_*(\mu^*\mathcal{O}_{\mathbf{P}(V)}(-1)). \tag{7}$$

The linearization $[0,1]$ on $\mathcal{O}_{\mathbf{P}(V)}(-1)$ defines an equivariant $\mathbb{C}^*$-action on $\mathbb{R}$. Let $c_{\text{top}}(\mathbb{R})$ be the top Chern class in $A^{g+d-1}(\overline{M}_{g,n}(d))$. Second, the Hodge bundle

$$\mathbb{E} \to \overline{M}_{g,n}(d)$$

is defined by the vector space of differential forms. There is a canonical lifting of the $\mathbb{C}^*$-action on $\overline{M}_{g,n}(d)$ to $\mathbb{E}$. Let $\lambda_g \in A^g(\overline{M}_{g,n}(d))$ denote the top Chern class of $\mathbb{E}$ as before. Third, for each marking i, let ψ_i denote the first Chern class of the canonically linearized cotangent line corresponding to i. Finally, let

$$\mathrm{ev}_i : \overline{M}_{g,n}(d) \to \mathbf{P}(V)$$

denote the i^{th} evaluation morphism, and let

$$\rho_i = c_1(\mathrm{ev}_i^* \mathcal{O}_{\mathbf{P}(V)}(1)),$$

where we fix the $\mathbb{C}^*$-linearization $[1,0]$ on $\mathcal{O}_{\mathbf{P}(V)}(1)$.

1.3. *Vanishing integrals.* A series of vanishing integrals $I(g,d,\alpha)$ over the moduli space of maps to $\mathbf{P}^1$ is defined here. The parameters g and d correspond to the genus and degree of the map space. Let $g \geq 1$ (the $g = 0$ case is treated separately in §2.4). Let

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

be a (nonempty) vector of nonnegative integers satisfying two conditions:

(i) $|\alpha| = \sum_{i=1}^n \alpha_i \leq d-2$,

(ii) $\alpha_i > 0$ for $i > 1$.

By condition (i), $d \geq 2$. Condition (ii) implies α_1 is the only integer permitted to vanish. Let

$$I(g,d,\alpha) = \int_{[\overline{M}_{g,n}(d)]^{\mathrm{vir}}} \rho_1^{d-1-|\alpha|} \prod_{i=1}^n \rho_i \psi_i^{\alpha_i} \; c_{\mathrm{top}}(\mathbb{R}) \; \lambda_g. \tag{8}$$

The virtual dimension of $\overline{M}_{g,n}(d)$ equals $2g+2d-2+n$. As the codimension of the integrand equals $2g+2d-2+n$, the integrals are well-defined. Since the class ρ_1 appears in the integrand with exponent $d - |\alpha| \geq 2$ and $\rho_1^2 = 0$, the integral vanishes.

These integrals occur in the following context. Let $\mathbf{P}^1 \subset S$ be an exceptional line in a nonsingular algebraic surface. The virtual class of the moduli space of stable maps to S multiply covering $\mathbf{P}^1$ is obtained from the virtual class of $\overline{M}_{g,n}(d)$ by intersecting with $c_{\mathrm{top}}(\mathbb{R})$. Hence, the series (8) may be viewed as vanishing Hodge integrals over the moduli space of stable maps to S.

1.4. *Localization terms.* As all the integrand classes in the I series have been defined with $\mathbb{C}^*$-equivariant lifts, the virtual localization formula of [GrP] yields a computation of these integrals in terms of Hodge integrals over moduli spaces of stable curves.

The integrals (8) are expressed as a sum over connected decorated graphs Γ (see [K2], [GrP]) indexing the $\mathbb{C}^*$-fixed loci of $\overline{M}_{g,n}(d)$. The vertices of these graphs lie over the fixed points $p_1, p_2 \in \mathbf{P}(V)$ and are labelled with genera (which sum over the graph to $g - h^1(\Gamma)$). The edges of the graphs lie over $\mathbf{P}^1$ and are labelled with degrees (which sum over the graph to d). Finally, the graphs carry n markings on the vertices. The edge valence of a vertex is the number of incident edges (markings excluded).

In fact, only a very restricted subset of graphs will yield nonvanishing contributions to the I series. By our special choice of linearization on the bundle $\mathbb{R}$, a vanishing result holds: if a graph Γ contains a vertex lying over p_1 of edge valence greater than 1, then the contribution of Γ to (8) vanishes. A vertex over p_1 of edge valence at least 2 yields a trivial Chern root of $\mathbb{R}$ (with trivial weight 0) in the numerator of the localization formula to force the vanishing. This basic vanishing was first used in $g = 0$ by Manin in [Ma]. Additional applications have been pursued in [GrP], [FaP1].

By the above vanishing, only *comb* graphs Γ contribute to (8). Comb graphs contain $k \leq d$ vertices lying over p_1 each connected by a distinct edge to a unique vertex lying over p_2. These graphs carry the usual vertex genus and marking data.

Before deriving further restrictions on contributing graphs, a classical result due to Mumford is required [Mu].

LEMMA 1. *Let $g \geq 1$.*

$$\sum_{i=0}^{g} \lambda_i \cdot \sum_{i=0}^{g} (-1)^i \lambda_i = 1$$

in $A^*(\overline{M}_{g,n})$. In particular, $\lambda_g^2 = 0$.

The factor λ_g in the integrand of the I series forces a further vanishing: if Γ contains a vertex over p_1 of positive genus, then the contribution of Γ to the integral (8) vanishes. To see this, let v be a positive genus $g(v) > 0$ vertex lying over p_1. The integrand term $c_{\text{top}}(\mathbb{R})$ yields a factor $c_{g(v)}(\mathbb{E}^*)$ with trivial $\mathbb{C}^*$-weight on the genus $g(v)$ moduli space corresponding to the vertex v. The integrand class λ_g factors as $\lambda_{g(v)}$ on each vertex moduli space. Hence, the equation

$$\lambda_{g(v)}^2 = 0$$

yields the required vanishing by Lemma 1.

The linearizations of the classes ρ_i place restrictions on the marking distribution. As the class ρ_i is obtained from $\mathcal{O}_{\mathbf{P}(V)}(1)$ with linearization $[1, 0]$, all markings must lie on vertices over p_1 in order for the graph to contribute to (8).

Finally, we claim the markings of Γ must lie on distinct vertices over p_1 for nonvanishing contribution to the I series. Let v be a vertex over p_1 (with $g(v) = 0$). If v carries at least two markings, the fixed locus corresponding to Γ (see [K2], [GrP]) contains a product factor $\overline{M}_{0,m+1}$ where m is the number of markings incident to v. The classes $\psi_i^{\alpha_i}$ in the integrand of (8) carry trivial $\mathbb{C}^*$-weight — they are pure Chow classes. Moreover, as each $\alpha_i > 0$ for $i > 1$, we see the sum of the α_i as i ranges over the set of markings incident to v is at least $m - 1$. Since this sum exceeds the dimension of $\overline{M}_{0,m+1}$, the graph contribution to the I series vanishes.

We have now proven the main result about the localization terms of the integrals (8).

PROPOSITION 1. *The integrals in the I series are expressed via the virtual localization formula as a sum over genus g, degree d, marked comb graphs Γ satisfying*:

(i) *all vertices over p_1 are of genus* 0,

(ii) *each vertex over p_1 has at most one marking*,

(iii) *the vertex over p_2 has no markings.*

1.5. *Hodge integrals.* We introduce a new set of integrals over $\overline{M}_{g,n}$ which occur naturally in the localization terms of the I series. Let $g \geq 1$ (again the $g = 0$ case is treated separately in §2.4). Let $(d_1, \ldots, d_k)$ be a nonempty sequence of positive integers. Let

$$\langle d_1, \ldots, d_k \rangle_g = \int_{\overline{M}_{g,k}} \frac{\lambda_g}{\prod_{j=1}^{k}(1 - d_j \psi_j)}. \tag{9}$$

The value of the integral (9) clearly does not depend upon the ordering of the sequence $(d_1, \ldots, d_k)$.

Let $\mathcal{P}(d)$ denote the set of (unordered) partitions of $d > 0$ into positive integers. Elements $P \in \mathcal{P}(d)$ are unordered sets $P = \{d_1, \ldots, d_k\}$ of positive integers with possible repetition. The set $\mathcal{P}(d)$ corresponds bijectively to the set of distinct (up to reordering) *degree d integrals* by:

$$\{d_1, \ldots, d_k\} \mapsto \langle d_1, \ldots, d_k \rangle_g$$

where $\sum_{j=1}^{k} d_j = d$.

By the λ_g conjecture, we easily compute the prediction:

$$\langle d_1, \ldots, d_k \rangle_g = \left(\sum_{j=1}^{k} d_j \right)^{2g-3+k} \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g. \tag{10}$$

Equation (10) may be reduced further to the following genus independent claim: for $g \geq 1$,

$$\langle d_1, \ldots, d_k \rangle_g = d^{k-1} \langle d \rangle_g \tag{11}$$

where $\sum_{j=1}^{k} d_j = d$. In Section 2.3, we will prove prediction (11) is *equivalent* to the λ_g conjecture.

1.6. *Formulas.* The precise contributions of allowable graphs Γ to the I series are now calculated. Consider the integral $I(g, d, \alpha)$ where

$$\alpha = (\alpha_1, \ldots, \alpha_n).$$

Let Γ be a genus g, degree d, comb graph with n markings satisfying conditions (i) and (ii) of Proposition 1. By condition (ii), Γ must have $k \geq n$ edges. Γ may be described uniquely by the data

$$(d_1, \ldots, d_n) \cup \{d_{n+1}, \ldots, d_k\}, \tag{12}$$

satisfying:

$$d_j > 0, \quad \sum_{j=1}^{k} d_j = d.$$

The elements of the ordered n-tuple $(d_1, \ldots, d_n)$ correspond to the degree assignments of the edges incident to the marked vertices. The elements of the unordered partition $\{d_{n+1}, \ldots, d_k\}$ correspond to the degrees of edges incident to the unmarked vertices over p_1. Let $\mathrm{Aut}(\{d_{n+1}, \ldots, d_k\})$ be the group which permutes equal parts. The group of graph automorphisms $\mathrm{Aut}(\Gamma)$ (see [GrP]) equals $\mathrm{Aut}(\{d_{n+1}, \ldots, d_k\})$.

By a direct application of the virtual localization formula of [GrP], we find the contribution of the graph (12) to the (normalized) integral

$$(-1)^{g+1} \cdot I(g, d, \alpha)$$

equals

$$\frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{j=1}^{n} d_j^{-\alpha_j} \prod_{j=n+1}^{k} (-d_j)^{-1} \prod_{j=1}^{k} \frac{d_j^{d_j}}{d_j!} \;\; \langle d_1, \ldots, d_k \rangle_g .$$

Hence, the vanishing of $I(g, d, \alpha)$ yields the Hodge integral relation:

$$\sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{j=1}^{n} d_j^{-\alpha_j} \prod_{j=n+1}^{k} (-d_j)^{-1} \prod_{j=1}^{k} \frac{d_j^{d_j}}{d_j!} \;\; \langle d_1, \ldots, d_k \rangle_g = 0, \tag{13}$$

where the sum is over all graphs (12).

We point out two properties of the linear relations (13). First, the relations do *not* depend upon the genus $g \geq 1$ — recall that the prediction (11) is also genus independent. Second, the relations involve integrals $\langle d_1, \ldots, d_k \rangle_g$ with a fixed sum $\sum_{j=1}^{k} d_j = d$. By (5), the value $\langle d \rangle_g$ is never 0. Therefore, the integrals $\langle d_1, \ldots, d_k \rangle_g$ are given at least one scalar dimension of freedom in each degree d by the equations (13). In Section 2.6, we will show that the solution space of the relations is exactly one dimension in each degree.

1.7. *Generating functions.* Let $g \geq 1$ as above. Equation (13) may be rewritten in a generating series form. While generating series will not be used explicitly in our proof of Theorem 1, the formalism provides a concise description of the localization equations.

Let $t = \{t_1, t_2, t_3, \ldots\}$ be a set of variables indexed by the natural numbers. Let $\mathbb{Q}[t]$ denote the polynomial ring in these variables. Define a $\mathbb{Q}$-linear function

$$\langle \ \rangle : \mathbb{Q}[t] \rightarrow \mathbb{Q}$$

by the equations $\langle 1 \rangle = 1$ and

$$\langle t_{d_1} t_{d_2} \cdots t_{d_k} \rangle = \langle d_1, d_2, \ldots, d_k \rangle_g.$$

We may extend $\langle \ \rangle$ uniquely to define a q-linear function:

$$\langle \ \rangle : \mathbb{Q}[t][[q]] \rightarrow \mathbb{Q}[[q]].$$

For each nonnegative integer i, define:

$$Z_i(t, q) = \sum_{j>0} q^j t_j \frac{j^{j-i}}{j!} \in \mathbb{Q}[t][[q]].$$

The I series equations (13) are equivalent to the following constraints.

Proposition 2. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a nonempty sequence of nonnegative integers satisfying $\alpha_i > 0$ for $i > 1$. The series*

$$\langle \exp(-Z_1) \cdot Z_{\alpha_1} \cdots Z_{\alpha_n} \rangle \in \mathbb{Q}[[q]]$$

is a polynomial *of degree at most $1 + \sum_{i=1}^{n} \alpha_i$ in q.*

Proof. The coefficient terms of the expanded product

$$\langle \exp(-Z_1) \cdot Z_{\alpha_1} \cdots Z_{\alpha_n} \rangle$$

required to vanish by the proposition coincide exactly with the relations (13). □

1.8. *Example.* Consider the polynomiality constraint obtained from the sequence $\alpha = (0)$:

$$\deg_q \langle \exp(-Z_1) \cdot Z_0 \rangle \quad \leq 1.$$

After expanding the constraint, we find

$$\langle \exp(-Z_1) \cdot Z_0 \rangle = \langle t_1 \rangle_g q + (2\langle t_2 \rangle_g - \langle t_1^2 \rangle_g) q^2 + \cdots \quad .$$

The equation

$$2\langle 2\rangle_g - \langle 1,1\rangle_g = 0 \tag{14}$$

is obtained from the q^2 term. By the prediction (11), we see equation (14) is consistent with the λ_g conjecture.

1.9. *The λ_g conjecture.* The plan of the proof of the λ_g conjecture is as follows. We first prove (11) is equivalent to the λ_g conjecture in Section 2.3. The next step is to show the solution (11) satisfies all of our linear relations (13). This result is established in Section 2.4 via known $g = 0$ formulas. In Section 2.6, the linear relations are proven to admit at most a one-dimensional solution space in each degree. Together, these three steps prove the λ_g conjecture.

The above program relies upon the rank computations of certain natural matrices indexed by partitions. The required results for these matrices are proven in Section 3.

2. Proof of the λ_g conjecture

2.1. *String and dilaton.* The λ_g integrals satisfy the string and dilaton equations:

$$\int_{\overline{M}_{g,k+1}} \psi_1^{\alpha_1}\cdots\psi_k^{\alpha_k}\psi_{k+1}^0\lambda_g = \sum_{i=1}^k \int_{\overline{M}_{g,k}} \psi_1^{\alpha_1}\cdots\psi_i^{\alpha_i-1}\cdots\psi_k^{\alpha_k}\lambda_g,$$

$$\int_{\overline{M}_{g,k+1}} \psi_1^{\alpha_1}\cdots\psi_k^{\alpha_k}\psi_{k+1}^1\lambda_g = (2g-2+k)\cdot\int_{\overline{M}_{g,k}} \psi_1^{\alpha_1}\cdots\psi_k^{\alpha_k}\lambda_g.$$

The proofs of the string and dilaton equations given in [W] are valid in the context of λ_g integrals.

The λ_g conjecture is easily checked to be compatible with the string and dilaton equations. For genus $g = 1$, all λ_1 integrals must contain a ψ_i^0 factor in the integrand (for dimension reasons). Hence, the λ_1 conjecture is a consequence of the string equation. Alternatively, the boundary equation $12\lambda_1 = \Delta_0$ in $A^1(\overline{M}_{1,1})$ immediately reduces the λ_1 conjecture to the basic genus 0 formula (1).

The λ_g integrals for a fixed genus $g \geq 2$ may be expressed in terms of *primitive integrals* in which no factors ψ_i^0 or ψ_i^1 occur in the integrand. The distinct primitive integrals (up to ordering of the indices) are in bijective correspondence with the set $\mathcal{P}(2g-3)$ of (unordered) partitions of $2g-3$. The correspondence is given by

$$\{e_1,\ldots,e_l\} \mapsto \int_{\overline{M}_{g,l}} \psi_1^{1+e_1}\ldots\psi_l^{1+e_l}\lambda_g.$$

The value of the integral does not depend on the ordering of the markings. The λ_g integrals may thus be viewed as having $\mathcal{P}(2g-3)$ parameters in genus g.

2.2. *Matrix* **A**. Let $r \geq s > 0$. Let $\overrightarrow{\mathcal{P}}(r,s)$ be the set of *ordered* partitions of r in exactly s nonzero parts. An element $X \in \overrightarrow{\mathcal{P}}(r,s)$ is a vector $(x_1, \ldots, x_s)$. Let $\mathbf{A}$ be a matrix with row and columns indexed by $\overrightarrow{\mathcal{P}}(r,s)$. For $X, Y \in \overrightarrow{\mathcal{P}}(r,s)$, define the matrix element $\mathbf{A}(X,Y)$ by:

$$\mathbf{A}(X,Y) = \prod_{j=1}^{s} x_j^{-1+y_j}.$$

Let $\mathbb{C}^s$ be a vector space with coordinates $z_1, \ldots, z_s$. Let the partitions $X \in \overrightarrow{\mathcal{P}}(r,s)$ correspond to points in $\mathbb{C}^s$. For $Y \in \overrightarrow{\mathcal{P}}(r,s)$, let Y^- denote the vector $(-1+y_1, \ldots, -1+y_s)$. The set $\overrightarrow{\mathcal{P}}(r,s)$ corresponds bijectively to the set of degree $r-s$ monomial functions in the z variables by:

$$Y \quad \leftrightarrow \quad m_{Y^-}(z) = z_1^{-1+y_1} \cdots z_s^{-1+y_s}. \tag{15}$$

$\mathbf{A}$ is simply the matrix obtained by evaluating degree $r-s$ monomials on partition points in $\mathbb{C}^s$:

$$\mathbf{A}(X,Y) = m_{Y^-}(x_1, \ldots, x_s).$$

The following lemma needed here will be proven in Section 3.1.

LEMMA 2. *For all pairs (r,s), the matrix* $\mathbf{A}$ *is invertible.*

The symmetric group $\mathbb{S}_s$ acts naturally by permutation on the set $\overrightarrow{\mathcal{P}}(r,s)$. Let $V_{r,s}$ denote the canonically induced $\mathbb{S}_s$ permutation representation. The matrix $\mathbf{A}$ determines a natural $\mathbb{S}_s$-invariant bilinear form:

$$\phi : V_{r,s} \times V_{r,s} \to \mathbb{C}$$

by $\phi([X],[Y]) = \mathbf{A}(X,Y)$. The form ϕ is nondegenerate by Lemma 2. Let $V_{r,s}^{\mathbb{S}} \subset V_{r,s}$ denote the $\mathbb{S}_s$ invariant subspace. By an application of Schur's Lemma, the restricted form

$$\phi^{\mathbb{S}} : V_{r,s}^{\mathbb{S}} \times V_{r,s}^{\mathbb{S}} \to \mathbb{C}$$

is also nondegenerate.

Let $\mathcal{P}(r,s)$ denote the set of (unordered) partitions of r in exactly s parts. An element $P \in \mathcal{P}(r,s)$ is a set $\{p_1, \ldots, p_s\}$ of positive integers (with possible repetition). The set $\mathcal{P}(r,s)$ may be placed in bijective correspondence with a basis of $V_{r,s}^{\mathbb{S}}$ by

$$\{p_1, \ldots, p_s\} \quad \leftrightarrow \quad \sum_{\sigma \in \mathbb{S}_s} [(p_{\sigma(1)}, \ldots, p_{\sigma(s)})]. \tag{16}$$

The correspondence (15) yields an equivariant isomorphism between $V_{r,s}$ and the vector space of polynomial functions of homogeneous degree $r-s$ in the z variables. Via this isomorphism, the basis element (16) corresponds to the symmetric function:

$$\mathrm{sym}(m_{P^-}) = \sum_{\sigma \in \mathbb{S}_s} z_1^{-1+p_{\sigma(1)}} \cdots z_s^{-1+p_{\sigma(s)}}.$$

In the basis (16), the form $\phi^{\mathbb{S}}$ corresponds to the matrix $\mathbf{A}^{\mathbb{S}}$ with rows and columns indexed by $\mathcal{P}(r,s)$ and matrix element

$$\mathbf{A}^{\mathbb{S}}(P,Q) = s! \cdot \mathrm{sym}(m_{Q^-})\Big(p_1, \ldots, p_s\Big).$$

As a corollary of Lemma 2, we have proven:

LEMMA 3. *For all pairs* (r,s), *the matrix* $\mathbf{A}^{\mathbb{S}}$ *is invertible.*

2.3. *Change of basis.* The partition matrix results of Section 2.2 are required for the following proposition. This is the first step in the proof of the λ_g conjecture.

PROPOSITION 3. *Let* $g \geq 2$. *The values of the primitive* λ_g *integrals are uniquely determined by the degree* $2g-3$ *integrals*:

$$\{\langle d_1, \ldots, d_k \rangle_g\}$$

where $\sum_{j=1}^{k} d_j = 2g-3$.

Proof. Let $D = \{d_1, \ldots, d_k\} \in \mathcal{P}(2g-3, k)$. We may certainly express the integral

$$\langle D \rangle_g = \langle d_1, \ldots, d_k \rangle_g$$

in terms of the primitive λ_g integrals by:

$$\langle D \rangle_g = \sum_{l=1}^{k} \sum_{E=\{e_1,\ldots,e_l\} \in \mathcal{P}(2g-3,l)} M(D,E) \cdot \int_{\overline{M}_{g,l}} \psi_1^{1+e_1} \cdots \psi_l^{1+e_l} \lambda_g. \tag{17}$$

Note no primitive λ_g integrals corresponding to partitions of length greater than k occur in the sum. The string and dilaton equations are required to compute the values $M(D,E)$ where the length of E is strictly less than k.

Let M be the matrix with rows and columns indexed by $\mathcal{P}(2g-3)$ and matrix elements $M(D,E)$. In order to establish the proposition, it suffices to prove M is invertible.

We order the rows and columns of M by increasing length of partition (the order within a fixed length can be chosen arbitrarily). M is then block lower-triangular with diagonal blocks M_k determined by partitions of a fixed length k. Hence,

$$\det(M) = \prod_{k=1}^{2g-3} \det(M_k).$$

We will prove $\det(M_k) \neq 0$ for each k.

Let k be a fixed length. The diagonal block M_k has rows and columns indexed by $\mathcal{P}(2g-3, k)$. Let $D, E \in \mathcal{P}(2g-3, k)$. The matrix element $M_k(D, E)$ is given by:

$$M_k(D, E) = \frac{\sum_{\sigma \in \mathbb{S}_k} \prod_{j=1}^k d_j^{1+e_{\sigma(j)}}}{|\mathrm{Aut}(E)|} = \frac{\prod_{j=1}^k d_j^2}{|\mathrm{Aut}(E)|} \cdot \mathrm{sym}(m_{E^-})\Big(d_1, \ldots, d_k\Big).$$

Here $\mathrm{Aut}(E)$ is the group permuting equal parts of the partition E. This element is computed by a simple expansion of the denominator in the definition (9) of the integral $\langle D \rangle_g$. No applications of the string or dilaton equations are necessary.

Let $\mathbf{A}^{\mathbb{S}}$ be the matrix defined in Section 2.2 for $(r, s) = (2g-3, k)$. For $X, Y \in \mathcal{P}(2g-3, k)$,

$$\mathbf{A}^{\mathbb{S}}(P, Q) = k! \cdot \mathrm{sym}(m_{Q^-})\Big(p_1, \ldots, p_k\Big).$$

As M_k differs from $\mathbf{A}^{\mathbb{S}}$ only by scalar row and column operations, M_k is invertible if and only if $\mathbf{A}^{\mathbb{S}}$ is invertible. However, by Lemma 3, $\mathbf{A}^{\mathbb{S}}$ is invertible. $\square$

By Proposition 3, the λ_g conjecture is equivalent to the prediction:

$$\langle d_1, \ldots, d_k \rangle_g = d^{k-1} \langle d \rangle_g \tag{18}$$

where $\sum_{j=1}^k d_j = d$. We will prove the λ_g conjecture in form (18).

2.4. *Compatibility.* We now prove equation (18) yields a solution of the linear system of equations obtained from localization (13). Our method is to use localization equations in genus 0 together with the basic formula (1).

Define genus 0 integrals $\langle d_1, \ldots, d_k \rangle_0$ by:

$$\langle d_1, \ldots, d_k \rangle_0 = \int_{\overline{M}_{0,k+2}} \frac{\lambda_0}{\prod_{j=1}^k (1 - d_j \psi_j)} = \int_{\overline{M}_{0,k+2}} \frac{1}{\prod_{j=1}^k (1 - d_j \psi_j)}. \tag{19}$$

As $k+2 \geq 3$, these integrals are well-defined (the two extra markings of $\overline{M}_{0,k+2}$ serve to avoid the degenerate spaces $\overline{M}_{0,1}$ and $\overline{M}_{0,2}$). An easy evaluation using (1) shows:

$$\langle d_1, \ldots, d_k \rangle_0 = d^{k-1}, \quad \sum_{j=1}^k d_j = d. \tag{20}$$

In particular,

$$\langle d_1, \ldots, d_k \rangle_0 = d^{k-1} \langle d \rangle_0.$$

Relations among the integrals $\langle d_1, \ldots, d_k \rangle_0$ may be found in a manner similar to the higher genus development in Section 1. We follow the notation of the $\mathbb{C}^*$-action on $\mathbf{P}^1$ introduced in Sections 1.1–1.2. The $\mathbb{C}^*$-equivariant classes

$$c_{\text{top}}(\mathbb{R}), \psi_i, \rho_i$$

are defined on the moduli space $\overline{M}_{0,n}(d)$. Define a new class

$$\gamma_i = c_1(\text{ev}_i^*(\mathcal{O}_{\mathbf{P}(V)}(-1)))$$

with $\mathbb{C}^*$-linearization determined by the action with weights $[0, 1]$ on the line bundle $\mathcal{O}_{\mathbf{P}(V)}(-1)$.

Again, we find a series $I(0, d, \alpha)$ of vanishing integrals. We require α to satisfy conditions (i) and (ii) of Section 1.3.

$$I(0, d, \alpha) = \int_{[\overline{M}_{0,n+2}(d)]} \rho_1^{d-1-|\alpha|} \prod_{i=1}^{n} \rho_i \psi_i^{\alpha_i} \; c_{\text{top}}(\mathbb{R}) \; \gamma_{n+1}\gamma_{n+2}. \tag{21}$$

These integrals are well-defined and vanish as before.

The localization formula yields a computation of the vanishing integrals (21). The argument exactly follows the higher genus development in Sections 1. In addition to the graph restrictions found in Section 1.4, the two extra points (corresponding to the γ factors in the integrands) must lie on the unique vertex over the fixed point $p_2 \in \mathbf{P}(V)$. These extra points ensure that the unique vertex over p_2 will not degenerate in the localization formulas. The resulting graph contributions then agree exactly with the expressions found in Section 1.6.

$I(0, d, \alpha)$ yields the relation:

$$\sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{j=1}^{n} d_j^{-\alpha_j} \prod_{j=n+1}^{k} (-d_j)^{-1} \prod_{j=1}^{k} \frac{d_j^{d_j}}{d_j!} \;\; \langle d_1, \ldots, d_k \rangle_0 = 0, \tag{22}$$

where the sum is over all graphs:

$$\Gamma = (d_1, \ldots, d_n) \cup \{d_{n+1}, \ldots, d_k\}, \quad d_j > 0, \quad \sum_{j=1}^{k} d_j = d.$$

Equation (22) equals the specialization of equation (13) to genus 0. Hence, we have proven the predicted form proportional to (20) solves the linear relations obtained from localization.

2.5. *Matrix* **B**. Let $r > s > 0$. As in Section 2.2, let $\mathbb{C}^s$ be a vector space with coordinates $z_1, \ldots, z_s$. Let the set $\overrightarrow{\mathcal{P}}(r,s)$ correspond to points in $\mathbb{C}^s$ by the new association:

$$X \in \overrightarrow{\mathcal{P}}(r,s) \quad \leftrightarrow \quad \left(\frac{1}{x_1}, \ldots, \frac{1}{x_s}\right) \in \mathbb{C}^s.$$

Let $\mathcal{M}(r,s)$ be the set of monomials $m(z)$ in the coordinate variables satisfying the following two conditions:

(i) $\deg(m) \leq r-2$,

(ii) $m(z)$ omits *at most* one coordinate factor z_i.

Note the condition $\deg(m) \geq s-1$ is a consequence of condition (ii). The set $\mathcal{M}(r,s)$ is never empty.

Let **B** be a matrix with rows indexed by $\mathcal{M}(r,s)$ and columns indexed by $\overrightarrow{\mathcal{P}}(r,s)$. Let the matrix element $\mathbf{B}(m,X)$ be defined by evaluation:

$$\mathbf{B}(m,X) = m\left(\frac{1}{x_1}, \ldots, \frac{1}{x_s}\right).$$

The following lemma will be proven in Section 3.2.

Lemma 4. *For all pairs* (r,s), *the matrix* **B** *has rank equal to* $|\overrightarrow{\mathcal{P}}(r,s)|$.

There is a natural $\mathbb{S}_s$-action on the set $\mathcal{M}(r,s)$ defined by:

$$\sigma(z_1^{\alpha_1} \cdots z_s^{\alpha_s}) = z_1^{\alpha_{\sigma(1)}} \cdots z_s^{\alpha_{\sigma(s)}}. \tag{23}$$

Let $W_{r,s}$ denote the $\mathbb{S}_s$ permutation representation induced by the action (23). As before, let $V_{r,s}$ denote the $\mathbb{S}_s$ permutation representation induced by the natural group action on $\overrightarrow{\mathcal{P}}(r,s)$.

The matrix **B** determines a natural $\mathbb{S}_s$-invariant bilinear form:

$$\phi : W_{r,s} \times V_{r,s} \to \mathbb{C}$$

by $\phi([m],[X]) = \mathbf{B}(m,X)$. The form ϕ induces a canonical homomorphism of $\mathbb{S}_s$ representations:

$$W_{r,s} \to V_{r,s}^* \to 0,$$

surjective by Lemma 4. By Schur's lemma, the restricted morphism is also surjective:

$$W_{r,s}^{\mathbb{S}} \to V_{r,s}^{\mathbb{S}*} \to 0.$$

Hence the restricted form:

$$\phi^{\mathbb{S}} : W_{r,s}^{\mathbb{S}} \times V_{r,s}^{\mathbb{S}} \to \mathbb{C}$$

has rank equal to $|\mathcal{P}(r,s)|$.

Let $\mathcal{M}_{\mathrm{sym}}(r,s)$ denote the set of distinct symmetric functions obtained by symmetrizing monomials in $\mathcal{M}(r,s)$:

$$m \in \mathcal{M}(r,s) \quad \rightarrow \quad \mathrm{sym}(m) = \sum_{\sigma \in \mathbb{S}_s} \sigma(m).$$

The set $\mathcal{M}_{\mathrm{sym}}(r,s)$ corresponds to a basis of $W^{\mathbb{S}}_{r,s}$. Let the set $\mathcal{P}(r,s)$ correspond to a basis of $V^{\mathbb{S}}_{r,s}$ as before (16).

Let $\mathbf{B}^{\mathbb{S}}$ be a matrix with rows indexed by $\mathcal{M}_{\mathrm{sym}}(r,s)$, columns indexed by $\mathcal{P}(r,s)$, and matrix element:

$$\mathbf{B}^{\mathbb{S}}(\mathrm{sym}(m),P) = s! \cdot \mathrm{sym}(m)\left(\frac{1}{p_1},\ldots,\frac{1}{p_s}\right).$$

The restricted form $\phi^{\mathbb{S}}$ expressed in the bases $\mathcal{M}_{\mathrm{sym}}(r,s)$ and $\mathcal{P}(r,s)$ corresponds to the matrix $\mathbf{B}^{\mathbb{S}}$. As a corollary of Lemma 4, we have proven:

LEMMA 5. *For all pairs* (r,s), *the matrix* $\mathbf{B}^{\mathbb{S}}$ *has rank equal to* $|\mathcal{P}(r,s)|$.

2.6. *Linear relations.* The rank computation of $\mathbf{B}^{\mathbb{S}}$ directly yields the final step in the proof of the λ_g conjecture.

PROPOSITION 4. *Let* $d \geq 1$. *The linear relations* (13) *admit at most a one-dimensional solution space for the integrals*

$$\langle d_1,\ldots,d_k\rangle_g, \qquad \sum_{j=1}^{k} d_j = d. \tag{24}$$

Proof. As no linear relations in (13) constrain the unique degree 1 integral $\langle 1\rangle_g$, we may assume $d \geq 2$.

Recall the distinct integrals (24) correspond to the set $\mathcal{P}(d)$. There is a unique integral of partition length d:

$$\langle 1,\ldots,1\rangle_g.$$

We will prove that the localization relations determine all degree d integrals in terms of $\langle 1,\ldots,1\rangle_g$.

We proceed by descending induction on the partition length. If $D \in \mathcal{P}(d)$ is of length $l(D) = d$, then $\langle D\rangle_g$ equals $\langle 1,\ldots,1\rangle_g$ — the base case of the induction.

Let $d > n > 0$. Assume now all integrals corresponding to partitions $D \in \mathcal{P}(d)$ of length *greater* than n are determined in terms of $\langle 1,\ldots,1\rangle_g$. Consider the integrals corresponding to the partitions $\mathcal{P}(d,n)$. For each nonempty sequence

$$\alpha = (\alpha_1,\ldots,\alpha_n)$$

satisfying

(i) $|\alpha| = \sum_{i=1}^{n} \alpha_i \leq d-2$,

(ii) $\alpha_i > 0$ for $i > 1$,

we obtain the relation:

$$\sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{j=1}^{n} d_j^{-\alpha_j} \prod_{j=n+1}^{k} (-d_j)^{-1} \prod_{j=1}^{k} \frac{d_j^{d_j}}{d_j!} \ \langle d_1, \ldots, d_k \rangle_g = 0. \tag{25}$$

Recall the sum is over all graphs:

$$\Gamma = (d_1, \ldots, d_n) \cup \{d_{n+1}, \ldots, d_k\}, \quad d_j > 0, \quad \sum_{j=1}^{k} d_j = d.$$

We note only integrals corresponding to partitions of length at least n occur in (25). By the inductive assumption, only the terms in (25) containing integrals of length *exactly* n concern us:

$$\sum_{\Gamma} \prod_{j=1}^{n} d_j^{-\alpha_j} \prod_{j=1}^{n} \frac{d_j^{d_j}}{d_j!} \ \langle d_1, \ldots, d_n \rangle_g = f_\alpha(\langle 1, \ldots, 1 \rangle_g). \tag{26}$$

The sum is over all ordered sequences:

$$(d_1, \ldots, d_n), \quad d_j > 0, \quad \sum_{j=1}^{n} d_j = d.$$

The factor $|\mathrm{Aut}(\Gamma)|$ is trivial for the terms containing integrals of length exactly n.

Let L_α denote the linear equation (26). To each α, we may associate an element of $\mathcal{M}(d,n)$ by

$$\alpha \to m_\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}.$$

Let $D \in \mathcal{P}(d,n)$. The coefficient of $\langle D \rangle_g$ in L_α is

$$\frac{\prod_{j=1}^{n} \frac{d_j^{d_j}}{d_j!}}{|\mathrm{Aut}(D)|} \ \mathrm{sym}(m_\alpha)\Big(\frac{1}{d_1}, \ldots, \frac{1}{d_n}\Big).$$

As before, $\mathrm{Aut}(D)$ is the group permuting equal parts of D. The equation L_α depends only upon the symmetric function $\mathrm{sym}(m_\alpha)$.

The set of symmetric functions $\mathrm{sym}(m_\alpha)$ obtained as α varies over all sequences satisfying conditions (i) and (ii) equals $\mathcal{M}_{\mathrm{sym}}(d,n)$. The matrix of linear equations (26) with rows indexed by $\mathcal{M}_{\mathrm{sym}}(d,n)$ and columns indexed by the variable set $\mathcal{P}(d,n)$ differs from the matrix $\mathbf{B}^{\mathbb{S}}$ defined in Section 2.5 for $(r,s) = (d,n)$ only by scalar column operations. By Lemma 5, $\mathbf{B}^{\mathbb{S}}$ has rank equal to $|\mathcal{P}(d,n)|$. Hence the linear equations (26) uniquely determine the integrals of partition length n in terms of $\langle 1, \ldots, 1 \rangle_g$. The proof of the induction step is complete. $\square$

Since we have already found a nontrivial solution (20) of the degree d localization relations (13), we may conclude all solutions are proportional to (20). By Proposition 3, the λ_g conjecture is proven.

3. Partition matrices A–E

3.1. *Proof of Lemma* 2. Let $r \geq s > 0$. Let $\mathbf{A}$ be the matrix with rows and columns indexed by $\overrightarrow{\mathcal{P}}(r,s)$ and matrix elements:

$$\mathbf{A}(X,Y) = m_{Y^-}(x_1,\ldots,x_s),$$

as defined in Section 2.2. We will prove that the matrix $\mathbf{A}$ is invertible.

The set $\overrightarrow{\mathcal{P}}(r,s)$ may be viewed as a subset of points of $\mathbb{C}^s$ (see §2.2). Matrix $\mathbf{A}$ is invertible if and only if these points impose independent conditions on the space $\mathrm{Sym}^{r-s}(\mathbb{C}^s)^*$ of homogeneous polynomials of degree $r-s$ in the variables $z_1,\ldots,z_s$.

Let $v = (v_1,\ldots,v_s)$ be s independent vectors in $\mathbb{C}^s$. Let $\overrightarrow{\mathcal{P}}(r,v)$ denote the set of points

$$\left\{ \sum_{i=1}^{s} x_i v_i \mid X = (x_1,\ldots,x_s) \in \overrightarrow{\mathcal{P}}(r,s) \right\}.$$

If v is the standard coordinate basis, the set $\overrightarrow{\mathcal{P}}(r,v)$ is the usual embedding of $\overrightarrow{\mathcal{P}}(r,s)$ in $\mathbb{C}^s$. We will prove $\overrightarrow{\mathcal{P}}(r,v)$ imposes independent conditions on $\mathrm{Sym}^{r-s}(\mathbb{C}^s)^*$ for any basis v.

If $s=1$, then the cardinality of $\overrightarrow{\mathcal{P}}(r,s)$ is 1. The point $rv_1 \neq 0$ clearly imposes a nontrivial condition on $\mathrm{Sym}^{r-1}(\mathbb{C})^*$.

Let $s>1$. By induction, we may assume $\overrightarrow{\mathcal{P}}(r', v=(v_1,\ldots,v_{s'}))$ imposes independent conditions on $\mathrm{Sym}^{r'-s'}(\mathbb{C}^{s'})^*$ for pairs (r',s') satisfying $s'<s$.

If $r=s$, then the cardinality of $\overrightarrow{\mathcal{P}}(s,s)$ is again 1. The point $\sum_{i=1}^{s} v_i$ imposes a nontrivial condition on $\mathrm{Sym}^0(\mathbb{C}^s)^*$.

Let $r>s$. By induction, we may assume $\overrightarrow{\mathcal{P}}(r', v=(v_1,\ldots,v_s))$ imposes independent conditions on $\mathrm{Sym}^{r'-s}(\mathbb{C}^s)^*$ for pairs (r',s) satisfying $r'<r$.

We must now prove the points $\overrightarrow{\mathcal{P}}(r,v)$ impose independent conditions on $\mathrm{Sym}^{r-s}(\mathbb{C}^s)^*$ for any set of independent vectors $v=(v_1,\ldots,v_s)$. Let $f(z) \in \mathrm{Sym}^{r-s}(\mathbb{C}^s)^*$ satisfy: $f(p)=0$ for all $p \in \overrightarrow{\mathcal{P}}(r,v)$. It suffices to prove $f(z)=0$.

Fix $1 \leq j \leq s$. Consider first the subset

$$\left\{ \sum_{i=1}^{s} x_i v_i \mid X = (x_1,\ldots,x_s) \in \overrightarrow{\mathcal{P}}(r,s),\ x_j = 1 \right\} \subset \overrightarrow{\mathcal{P}}(r,v). \tag{27}$$

The points (27) span a linear subspace L_j of dimension $s-1$ in $\mathbb{C}^s$. In fact, the set (27) equals the set:

$$\left\{ \sum_{i\neq j} x_i \tilde{v}_i \mid \hat{X} = (x_1, \ldots, \hat{x}_j, \ldots, x_s) \in \overrightarrow{\mathcal{P}}(r-1, s-1) \right\} \tag{28}$$

where the vectors

$$\tilde{v}_i = v_i + \frac{1}{r-1} \cdot v_j, \quad i \neq j$$

span a basis of L_j. The restriction $f|_{L_j}$ lies in $\mathrm{Sym}^{r-s}(L_j)^*$ and vanishes at the points (28). By our induction assumption on s, the restriction of f to L_j vanishes identically.

The distinct linear equations defining $L_1, \ldots, L_s$ must therefore divide f:

$$f = f' \cdot \prod_{i=1}^{s} (L_i),$$

where $f' \in \mathrm{Sym}^{r-2s}(\mathbb{C}^s)^*$. If $r < 2s$, we conclude $f = 0$.

We may assume $r \geq 2s$. The product $\prod_{i=1}^{s}(L_i)$ does not vanish at any point in the subset

$$\left\{ \sum_{i=1}^{s} x_i v_i \mid X = (x_1, \ldots, x_s) \in \overrightarrow{\mathcal{P}}(r, s),\ x_i \geq 2 \text{ forall } i \right\} \subset \overrightarrow{\mathcal{P}}(r, v). \tag{29}$$

Hence, f' must vanish at every point of (29).

Define new vectors $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_s)$ of $\mathbb{C}^s$ by

$$\tilde{v}_i = \sum_{j=1}^{s} v_j (\delta_{ij} + \frac{1}{r-s}).$$

A straightforward determinant calculation shows $\tilde{v}$ spans a basis of $\mathbb{C}^s$. The set (29) equals the set:

$$\left\{ \sum_{i=1}^{s} x_i \tilde{v}_i \mid X = (x_1,\ \ldots, x_s) \in \overrightarrow{\mathcal{P}}(r-s, s) \right\}. \tag{30}$$

By the induction assumption on r, the function f' must vanish identically. We have thus proven $f = 0$.

D. Zagier has provided us with another proof of Lemma 2 by an explicit computation of the determinant:

$$|\det(\mathbf{A})| = r^{\binom{r-1}{s}} \prod_{X \in \overrightarrow{\mathcal{P}}(r,s)} x_1^{r-s+1-x_1}.$$

We omit the derivation.

3.2. *Proof of Lemma* 4. Let $r > s > 0$. Let $\mathbf{B}$ be the matrix with rows indexed by $\mathcal{M}(r,s)$, columns indexed by $\overrightarrow{\mathcal{P}}(r,s)$, and matrix elements:

$$\mathbf{B}(m,X) = m\left(\frac{1}{x_1},\ldots,\frac{1}{x_s}\right),$$

as defined in Section 2.5. We will prove matrix $\mathbf{B}$ has rank equal to $|\overrightarrow{\mathcal{P}}(r,s)|$.

Consider first the case $s = 1$. The set $\overrightarrow{\mathcal{P}}(r,1)$ consists of a single element (r). As $r \geq 2$, the constant monomial 1 lies in $\mathcal{M}(r,1)$. Hence $\mathbf{B}$ certainly has rank equal to 1 in this case.

We now proceed by induction on s. Let $s \geq 2$. Assume Lemma 4 is true for all pairs (r', s') satisfying $s' < s$.

There is a natural inclusion of sets

$$\overrightarrow{\mathcal{P}}(r-1,s-1) \hookrightarrow \overrightarrow{\mathcal{P}}(r,s)$$

defined by:

$$(x_1,\ldots,x_{s-1}) \to (x_1,\ldots,x_{s-1},1).$$

Let $\overrightarrow{\mathcal{P}}(r-1,s-1,1)$ denote the image of this inclusion.

There is a natural inclusion of sets

$$\mathcal{M}(r-1,s-1) \hookrightarrow \mathcal{M}(r,s)$$

obtained by multiplication by z_s:

$$m(z_1,\ldots,z_{s-1}) \to m(z_1,\ldots,z_{s-1}) \cdot z_s.$$

Let $\mathcal{M}(r-1,s-1)\cdot z_s$ denote the image of this inclusion.

The submatrix of $\mathbf{B}$ corresponding to the rows $\mathcal{M}(r-1,s-1)\cdot z_s$ and columns $\overrightarrow{\mathcal{P}}(r-1,s-1,1)$ equals the matrix $\mathbf{B}_{r-1,s-1}$ for the pair $(r-1,s-1)$. By the induction assumption, we conclude the submatrix of columns of $\mathbf{B}$ corresponding to $\overrightarrow{\mathcal{P}}(r-1,s-1,1)$ has full rank equal to $|\overrightarrow{\mathcal{P}}(r-1,s-1,1)|$.

There is a natural inclusion of sets

$$\overrightarrow{\mathcal{P}}(r-1,s) \hookrightarrow \overrightarrow{\mathcal{P}}(r,s) \tag{31}$$

defined by:

$$(x_1,\ldots,x_s) \to (x_1,\ldots,x_{s-1},1+x_s).$$

Let $\overrightarrow{\mathcal{P}}(r-1,s^+)$ denote the image of this inclusion. $\overrightarrow{\mathcal{P}}(r,s)$ is the disjoint union of $\overrightarrow{\mathcal{P}}(r-1,s-1,1)$ and $\overrightarrow{\mathcal{P}}(r-1,s^+)$. We now study the columns of $\mathbf{B}$ corresponding to $\overrightarrow{\mathcal{P}}(r-1,s^+)$.

Let $T(z_1,\ldots,z_s)$ denote the polynomial function:

$$T(z) = \left(\sum_{i=1}^{s-1}\frac{1}{z_i} \; - \; \frac{r-1}{z_s}\right)\cdot\prod_{i=1}^{s} z_i.$$

Proposition 5. *The function $T(z)$ has the following properties*:

(i) *$T(z)$ is homogeneous of degree $s-1$.*

(ii) *Let $X \in \overrightarrow{\mathcal{P}}(r,s)$. Then,*

$$T\left(\frac{1}{x_1}, \ldots, \frac{1}{x_s}\right) = 0 \quad \leftrightarrow \quad X \in \overrightarrow{\mathcal{P}}(r-1, s-1, 1).$$

(iii) *Let $f(z)$ be any (possibly nonhomogeneous) polynomial function of degree at most $r-s-1$. Then,*

$$f \cdot T(z)$$

is a linear combination of monomials in $\mathcal{M}(r,s)$.

Proof. Property (i) is clear by definition. For $X \in \overrightarrow{\mathcal{P}}(r,s)$,

$$\sum_{i=1}^{s-1} \frac{1}{1/x_i} = r - x_s.$$

Hence $T(1/x_1, \ldots, 1/x_s) = 0$ if and only if

$$r - x_s = (r-1)x_s. \tag{32}$$

Equation (32) holds if and only if $x_s = 1$. Property (ii) is thus proven. Certainly the polynomial $f \cdot T(z)$ is of degree at most $r-2$. Note each monomial in $T(z)$ omits exactly 1 coordinate factor. Hence each monomial of $f \cdot T(z)$ may omit at most 1 coordinate factor. Property (iii) then holds by the definition of $\mathcal{M}(r,s)$. □

Let $\mathbb{C}_{r-s-1}[z]$ be the vector space of all polynomials of degree at most $r-s-1$ in the variables $z_1, \ldots, z_s$. Let $\mathbb{C}_{r-s-1}[z] \cdot T$ be the vector space of functions

$$\{\ f \cdot T \mid f \in \mathbb{C}_{r-s-1}[z]\ \}.$$

By property (iii) of T, after applying *row* operations to $\mathbf{B}$, we may take the first $\dim(\mathbb{C}_{r-s-1}[z])$ rows to correspond to a basis of the function space $\mathbb{C}_{r-s-1}[z] \cdot T$. Let $\mathbf{B}'$ denote the matrix $\mathbf{B}$ after these row operations. The ranks of the column spaces of a matrix do *not* change after row operations. Hence, the rank of $\mathbf{B}'$ equals the rank of $\mathbf{B}$. Moreover, the rank of the column space $\overrightarrow{\mathcal{P}}(r-1, s-1, 1)$ of $\mathbf{B}'$ remains $|\overrightarrow{\mathcal{P}}(r-1, s-1, 1)|$.

By property (ii), the block of $\mathbf{B}'$ determined by the row space $\mathbb{C}_{r-s-1}[z] \cdot T$ and columns set $\overrightarrow{\mathcal{P}}(r-1, s-1, 1)$ vanishes:

$$\mathbf{B}'[\ \mathbb{C}_{r-s-1}[z] \cdot T,\ \overrightarrow{\mathcal{P}}(r-1, s-1, 1)\ ] = 0. \tag{33}$$

Let M be the block $\mathbf{B}'[\,\mathbb{C}_{r-s-1}[z]\cdot T,\ \overrightarrow{\mathcal{P}}(r-1,s^+)\,]$. The matrix M has elements:

$$M(f\cdot T,X) = f\cdot T\ \left(\frac{1}{x_1},\ldots,\frac{1}{x_s}\right).$$

Since the column space $\overrightarrow{\mathcal{P}}(s-1,r-1,1)$ of $\mathbf{B}'$ has rank $|\overrightarrow{\mathcal{P}}(r-1,s-1,1)|$ and the vanishing (33) holds,

$$\mathrm{rk}(\mathbf{B}') \geq |\overrightarrow{\mathcal{P}}(r-1,s-1,1)| + \mathrm{rk}(M).$$

To prove the lemma, we will show that the rank of M equals $|\overrightarrow{\mathcal{P}}(r-1,s^+)|$.

Let $\mathbf{C}$ be a matrix with rows indexed by a basis of $\mathbb{C}_{r-s-1}[z]$, columns indexed by $\overrightarrow{\mathcal{P}}(r-1,s^+)$, and matrix elements:

$$\mathbf{C}(f,X) = f\left(\frac{1}{x_1},\ldots,\frac{1}{x_s}\right).$$

As $T(1/x_1,\ldots,1/x_s)\neq 0$ for $X\in\overrightarrow{\mathcal{P}}(r-1,s^+)$, the matrix $\mathbf{C}$ differs from M only by scalar column operations. Hence,

$$\mathrm{rk}(M) = \mathrm{rk}(\mathbf{C}).$$

Matrix $\mathbf{C}$ is studied in Section 3.3 below. $\mathbf{C}$ is proven to have maximal rank $|\overrightarrow{\mathcal{P}}(r-1,s^+)|$ in Lemma 7 by extending $\mathbf{C}$ to a nonsingular square matrix $\mathbf{D}$.

The proof of Lemma 4 is complete (modulo the analysis of the matrices $\mathbf{C}$ and $\mathbf{D}$ in §3.3).

3.3. *Matrices* $\mathbf{C}$ *and* $\mathbf{D}$. Let $r>s>0$. Let $\overrightarrow{\mathcal{P}}(\leq r-1,s)$ denote the union:

$$\overrightarrow{\mathcal{P}}(\leq r-1,s) = \bigcup_{t=s}^{r-1}\overrightarrow{\mathcal{P}}(t,s).$$

The set $\overrightarrow{\mathcal{P}}(\leq r-1,s)$ may be placed in bijective correspondence with a basis of $\mathbb{C}_{r-s-1}[z]$ by:

$$X\in\overrightarrow{\mathcal{P}}(\leq r-1,s) \quad\leftrightarrow\quad m_{X^-}(z) = z_1^{-1+x_1}\cdots z_s^{-1+x_s}. \tag{34}$$

Let $\mathbf{D}$ be a matrix with rows and columns indexed by $\overrightarrow{\mathcal{P}}(\leq r-1,s)$. The matrix elements of $\mathbf{D}$ are defined by:

$$\mathbf{D}(X,Y) = m_{X^-}\left(\frac{1}{y_1},\ldots,\frac{1}{y_{s-1}},\frac{1}{1+y_s}\right).$$

Matrix $\mathbf{D}$ is invertible by the following result.

Lemma 6. *The determinant (up to sign) of* $\mathbf{D}$ *is*:

$$|\det(\mathbf{D})| = \prod_{X\in\overrightarrow{\mathcal{P}}(\leq r-1,s)} m_{X^-}\left(\frac{1}{x_1},\ldots,\frac{1}{x_{s-1}},\frac{1}{1+x_s}\right)\cdot\frac{1}{x_s}.$$

Proof. We first introduce required terminology. For

$$A=(a_1,\ldots,a_s)\in\overrightarrow{\mathcal{P}}(\leq r-1,s),$$

let $|A|=\sum_{i=1}^{s}a_i$ be the *size* of A. There is a partial ordering of $\overrightarrow{\mathcal{P}}(\leq r-1,s)$ by size. Choose a total ordering of $\overrightarrow{\mathcal{P}}(\leq r-1,s)$ which refines the size partial order (the order within each size class may be chosen arbitrarily). This total order of $\overrightarrow{\mathcal{P}}(\leq r-1,s)$ will be fixed for the entire proof.

Define another partial ordering on the set $\overrightarrow{\mathcal{P}}(\leq r-1,s)$ by:

$$A\geq B \quad\leftrightarrow\quad a_i\geq b_i \ \text{ forall } i\in\{1,\ldots,s\}.$$

If $A\geq B$, then either $|A|>|B|$ or $A=B$. Hence, B cannot appear strictly after A in the total order.

Let x_1 and x_2 be integers. Define the coefficients $e_k[x_1,x_2]$ by

$$\prod_{j=x_1}^{x_2}(t+j)=\sum_{k=0}^{x_2-x_1+1}e_k[x_1,x_2]\cdot t^{x_2-x_1+1-k}.$$

Note that $e_k[x_1,x_2]$ vanishes when $k>x_2-x_1+1$. Also, $e_k[x_1,x_2]$ vanishes when $x_1>x_2$ except for the case $e_0(x_1,x_1-1)=1$.

The key to the proof is the construction of a related matrix $\mathbf{D}'$ with rows and columns indexed by $\overrightarrow{\mathcal{P}}(\leq r-1,s)$ in the fixed total order. The matrix elements of $\mathbf{D}'$ are defined in the following manner:

(i) If $A\geq B$, then

$$\mathbf{D}'(A,B)=(-1)^{|B|}\prod_{i=1}^{s-1}\frac{e_{b_i-1}[1,a_i-1]}{(a_i-1)!}\cdot\frac{e_{b_s-1}[2,a_s]}{(a_s)!}.$$

(ii) In all other cases, $\mathbf{D}'(A,B)=0$.

$\mathbf{D}'$ is a lower-triangular matrix with diagonal elements:

$$\mathbf{D}'(A,A)=(-1)^{|A|}.$$

Hence $|\det(\mathbf{D}')|=1$.

We now study the product $\mathbf{D}'\mathbf{D}$. Consider the matrix element $\mathbf{D}'\mathbf{D}\,(A,Y)$:

$$\sum_{i=1}^{s}\sum_{b_i=1}^{a_i}(-1)^{|\sum_{i=1}^{s}b_i|}\prod_{i=1}^{s-1}\frac{e_{b_i-1}[1,a_i-1]}{(a_i-1)!}\frac{1}{y_i^{b_i-1}}\cdot\frac{e_{b_s-1}[2,a_s]}{(a_s)!}\frac{1}{(y_s+1)^{b_s-1}}.$$

The above expression may be written in a factorized form:

$$(-1)^s \prod_{i=1}^{s-1} \sum_{b_i=1}^{a_i} \frac{(-1)^{b_i-1} e_{b_i-1}[1, a_i-1]}{(a_i-1)!\ y_i^{b_i-1}} \cdot \sum_{b_s=1}^{a_s} \frac{(-1)^{b_s-1} e_{b_s-1}[2, a_s]}{(a_s)!\ (y_s+1)^{b_s-1}} .$$

These factors are easily evaluated. For $1 \leq i \leq s-1$,

$$\sum_{b_i=1}^{a_i} \frac{(-1)^{b_i-1} e_{b_i-1}[1, a_i-1]}{(a_i-1)!\ y_i^{b_i-1}} = \frac{\prod_{j=1}^{a_i-1}(y_i - j)}{(a_i-1)!\ y_i^{a_i-1}}. \tag{35}$$

For $i = s$,

$$\sum_{b_s=1}^{a_s} \frac{(-1)^{b_s-1} e_{b_s-1}[2, a_s]}{(a_s)!\ (y_s+1)^{b_s-1}} = \frac{\prod_{j=2}^{a_s}(y_s + 1 - j)}{(a_s)!\ (y_s+1)^{a_s-1}}. \tag{36}$$

We claim $\mathbf{D}'\mathbf{D}$ is upper-triangular. Suppose Y strictly precedes A in the total order. There must be a coordinate y_i which satisfies $y_i < a_i$. If $1 \leq i \leq s-1$, then the factor (35) vanishes. If $i = s$, then the factor (36) vanishes. In either case, $\mathbf{D}'\mathbf{D}\ (A, Y) = 0$.

The diagonal elements of $\mathbf{D}'\mathbf{D}$ are easily calculated by equations (35–36):

$$\mathbf{D}'\mathbf{D}\ (A, A) = (-1)^s \prod_{i=1}^{s-1} \frac{1}{a_i^{a_i-1}} \cdot \frac{1}{(a_s+1)^{a_s-1} a_s}.$$

As $\mathbf{D}'\mathbf{D}$ is upper-triangular, the determinant is the product of the diagonal entries. Since $|\det(\mathbf{D}')| = 1$, this determinant equals (up to sign) $\det(\mathbf{D})$. □

Consider the column set of $\mathbf{D}$ corresponding to the subset

$$\vec{\mathcal{P}}(r-1, s) \subset \vec{\mathcal{P}}(\leq r-1, s).$$

The submatrix of $\mathbf{D}$ obtained by restriction to this column set equals $\mathbf{C}$ via the correspondences (34) and (31). As a corollary of Lemma 6, we may conclude the required rank result for $\mathbf{C}$.

LEMMA 7. *Let $r > s > 0$. $\mathbf{C}$ has rank equal to $|\vec{\mathcal{P}}(r-1, s^+)|$.*

3.4. *Matrix* $\mathbf{E}$. Let $\mathbf{E}$ be a matrix with row and columns indexed by the set $\vec{\mathcal{P}}(\leq r-1, s)$. The matrix elements of $\mathbf{E}$ are defined by:

$$\mathbf{E}(X, Y) = m_{X^-}\left(\frac{1}{y_1}, \ldots, \frac{1}{y_s}\right).$$

While $\mathbf{E}$ is slightly more natural than $\mathbf{D}$, we do not encounter $\mathbf{E}$ in our proof of the λ_g conjecture. We note, however, the proof of Lemma 6 may be modified to prove:

LEMMA 8. *The determinant (up to sign) of* $\mathbf{E}$ *is*:

$$|\det(\mathbf{E})| = \prod_{X \in \vec{\mathcal{P}}(\leq r-1,s)} m_{X^-}\left(\frac{1}{x_1}, \ldots, \frac{1}{x_s}\right).$$

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