Image of the Burau representation at \( d \)-th roots of unity

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Abstract

We show that the image of the braid group under the monodromy action on the homology of a cyclic covering of degree \( d \) of the projective line is an arithmetic group provided the number of branch points is sufficiently large compared to the degree \( d \). This is deduced by proving the arithmeticity of the image of the braid group on \( n+1 \) letters under the Burau representation evaluated at \( d \)-th roots of unity when \( n \geq 2d \).

Contents

1. Introduction 2
1.1. The braid group and the Burau representation 6
1.2. The Burau representations \( \rho_n(d) \) at \( d \)-th roots of unity 7
1.3. Description of the proof 9
1.4. Organisation of the paper 9
2. Algebraic groups 10
2.1. Examples of algebraic groups 10
2.2. Arithmetic groups 11
2.3. K-rank 12
2.4. Parabolic subgroups 12
2.5. The real rank 13
2.6. A criterion for arithmeticity 13
2.7. Subgroups of products of higher rank groups 15
3. Unitary groups 16
3.1. Rank of a unitary group 17
3.2. The Heisenberg group and the group \( P \) 18
3.3. An inductive step for integral unitary groups 19
4. Properties of the Burau representations \( \rho_n \) and \( \rho_n(d) \) 20
4.1. Notation 20
1. Introduction

This paper is concerned with the question of whether some natural monodromy groups are arithmetic. These questions were first considered by Griffiths and Schmid [GS75]. Following [GS75, pp. 123–124], we say that a subgroup $\Gamma \subset \text{GL}_N(\mathbb{Z})$ is an arithmetic group if $\Gamma$ has finite index in the integral points of its Zariski closure in $\text{GL}_N$. If $\Gamma$ is an arithmetic group, then by a result of Borel and Harish-Chandra, $\Gamma$ is a lattice in the group of real points of the Zariski closure. In [GS75], the groups $\Gamma$ arise as follows. Let $\pi : X \to S$ be an algebraic family of algebraic manifolds, which is a fibration. Let $\pi_0 = \pi^{-1}(\{s_0\})$ be a typical fibre and $\Gamma \subset \text{Aut}(H^m(V_{s_0}))$ for the monodromy group; i.e., the image of the action of $\pi_1(S)$ on the cohomology group (with integral coefficients) of the fibre $V_{s_0}$. Griffiths and Schmid then raise the question: is the monodromy group an arithmetic group?

The foregoing question has a negative answer in general, as was shown by Deligne and Mostow ([DM86]). (There are examples of monodromy groups...
more noncompact factor isomorphic to $U$ (ple) that the image of the projection is a lattice in $U$.

Once the projection is discrete, it follows (see [McM13, Th. (10.3)], for example) that the image of the projection of the $\Gamma$ in $U$ is 1. Assume that the g.c.d. of $\{d, k_1, \ldots, k_{n+1}\}$ is 1. Given $a \in S$, consider the set $X_{a,k}$ of solutions $(x, y)$ of the equation

$$y^d = (x - a_1)^{k_1}(x - a_2)^{k_2} \cdots (x - a_{n+1})^{k_{n+1}}.$$ 

The space $X_{a,k}$ has a natural structure of a compact Riemann surface $X_{a,k}^*$ with finitely many punctures. We then get a map $\pi$ from $S$ to $X$. We then get a map $\pi : X \to S$ described on the “affine part” by $(x, y, a) \mapsto a$, where $(x, y) \in X_{a,k}$ and $a \in S$ and where $X$ is the family of the compact Riemann surfaces $X_{a,k}^*$; we can then consider the monodromy action on $H^1(X_{a,k}^*, \mathbb{Z})$ for a typical fibre $X_{a,k}^*$.

The fundamental group of the space $S$ is well known to be the pure braid group $P_{n+1}$ on $n + 1$ strands; thus the fibration $X \to S$ yields a monodromy representation

$$\rho^*_M(k, d) : P_{n+1} \to \text{GL}(H^1(X_{a,k}^*, \mathbb{Z}))$$

of $P_{n+1}$ on the integral cohomology of the fibre $X_{a,k}^*$. If $N$ is the rank of the abelian group $H_1(X_{a,k}^*, \mathbb{Z})$, then the image $\Gamma$ of $P_{n+1}$ is a subgroup of $\text{GL}_N(\mathbb{Z})$. Form the Zariski closure $\mathcal{G}$ of $\Gamma$ in $\text{GL}_N$; this is a linear algebraic group defined over $\mathbb{Q}$.

We now give only a qualitative description of the results of [DM86]. Deligne and Mostow prove that for special choices of the integers $d, n$ and $k_1, \ldots, k_{n+1}$, the monodromy group $\Gamma$ does not have finite index in $\mathcal{G}(\mathbb{Z})$. Hence this gives a negative answer to the question of Griffiths-Schmidt mentioned before. It can be shown that the group $\mathcal{G}(\mathbb{R})$ of real points of the Zariski closure is a product $\prod_j U(p_j, q_j)$ of unitary groups $U(p_j, q_j)$. (The unitary structure on $H^1(C, \mathbb{C})$ of the curve $C = X_{a,k}$ comes from the intersection form $h(\alpha, \beta) = \int_C \alpha \wedge \beta$.) As we mentioned before, $\mathcal{G}(\mathbb{Z})$ is a discrete subgroup of $\mathcal{G}(\mathbb{R})$, and hence so is $\Gamma$. However, the projection of $\Gamma$ to one of these factors $U(p_j, q_j)$ may not be discrete. In [DM86] (see also [Mos86]) it is shown — using their INT and $\Sigma$-INT criteria — that for a finite number of special choices of $k_i, d, n$, one of these $U(p_j, q_j)$ is $U(n-1, 1)$ and that the projection of the monodromy group $\Gamma$ to this factor $U(n-1, 1)$ continues to be discrete. Once the projection is discrete, it follows (see [McM13, Th. (10.3)], for example) that the image of the projection is a lattice in $U(n-1, 1)$; if there is one more noncompact factor isomorphic to $U(p, q)$ in $\mathcal{G}(\mathbb{R})$, then it follows (see [McM13, §10, p. 48]) that the image of the projection of the $\Gamma$ in $U(n-1, 1)$ is a nonarithmetic lattice. In particular, the monodromy group $\Gamma$ does not have finite index in $\mathcal{G}(\mathbb{Z})$. (If the monodromy were to have finite index, then the
projection to $U(n - 1, 1)$ would either be nondiscrete or an arithmetic lattice; this is discussed in more detail in [McM13, Cor. 10.4].

In view of the Margulis arithmeticity theorem (that irreducible lattices in linear semi-simple Lie groups of real rank at least two are arithmetic), the above strategy to produce nonarithmetic lattices cannot work if $U(n - 1, 1)$ is replaced by $U(p, q)$ with $p, q \geq 2$; if we are to have $U(n - 1, 1)$ as a factor, it may be shown that the number $n + 1$ of branch points $a_1, \ldots, a_{n+1}$ must not exceed $2d$. (See the proof of Theorem 2, where we prove this in the special case when all the $k_i$ are 1.) However, if we take $n \geq 2d$, it is still of interest — in view of the question of Griffiths and Schmid — to know whether the monodromy $\Gamma \subset \mathcal{G}(\mathbb{Z}) \subset \text{GL}_N(\mathbb{Z})$ has finite index in $\mathcal{G}(\mathbb{Z})$. In this paper, we prove that if $n \geq 2d$ and all the $k_i$ are 1, then $\Gamma$ does have finite index in $\mathcal{G}(\mathbb{Z})$. (See Theorem 26 for the statement of a more general case.)

Consider the compactification $X_a^*$ of the affine curve $X_a$ defined by

$$y^d = (x - a_1)(x - a_2) \cdots (x - a_{n+1}),$$

with $y \neq 0$ and $x \neq a_1, \ldots, a_{n+1}$. As before, there is the monodromy action of the pure braid group $P_{n+1}$ on the cohomology of the fibre $X_a^*$. Since the equation of the curve is patently invariant under the action of the permutations of the $a_i$'s, the action extends to an action of the full braid group $B_{n+1}$ on $H_1(X_a^*, \mathbb{Z})$.

**Theorem 1.** Suppose $d \geq 3$. If $n \geq 2d$, then the image $\Gamma$ of the monodromy representation $\rho(d) : B_{n+1} \to \text{GL}(H^1(X_a^*, \mathbb{Z})) = \text{GL}_N(\mathbb{Z})$ is an arithmetic group. Moreover, the monodromy is a product of irreducible lattices, each of which is a non co-compact arithmetic group and has $\mathbb{Q}$-rank at least two.

**Remark 1.** The group $G = \mathbb{Z}/d\mathbb{Z}$, viewed as the group of $d$-th roots of unity, operates on each of the curves $X_a^*$ for varying $a$ and hence acts on the first cohomology $H^1(X_a^*, \mathbb{Z})$; therefore, one may view the first cohomology group of $X_a^*$ as a module over the group algebra $\mathbb{Z}[G] = \mathbb{Z}[q]/(q^d - 1)$. The action of $G$ commutes with the monodromy action of the braid group, and therefore, the monodromy group lies in the space of endomorphisms of $H^1(X_a^*, \mathbb{Z})$ which are $\mathbb{Z}[G]$ module maps. Moreover, the monodromy group preserves the intersection form $(\alpha, \beta) \mapsto \alpha \wedge \beta$ on $H^1(X_a^*, \mathbb{Z})$ which extends as a Hermitian form on $V = H^1(X_a^*, \mathbb{C})$ given by $(\alpha, \beta) \mapsto \alpha \wedge \overline{\beta}$. Therefore, the monodromy lies in the unitary group of this Hermitian form and preserves the eigenspaces $V_\eta$ (and hence the sum $W_\eta = V_\eta \oplus V_\overline{\eta}$) of the group $G$. Consequently, $W_\eta$ is a Hermitian space and the monodromy representation restricted to $W_\eta$ has its image in a unitary group of the form $U(r, s)$ with $r, s$ depending on $\eta$. Since the cohomology group $H^1(X_a^*, \mathbb{C})$ is a direct sum of the spaces $W_\eta$, it follows that the image of the monodromy group lies in a product of the $U(r, s)$. 

The image of \( \Gamma \) in the individual \( U(r, s) \) may not be discrete, but \( \Gamma \) as a subgroup of the product will be discrete since it preserves the integral lattice \( H^1(X_a^*, \mathbb{Z}) \).

**Remark 2.** The arithmetic group is specified (up to finite index) in Proposition 24 in Section 7. It is a little complicated to describe when \( n + 1 \) and \( d \) are not coprime, but Proposition 24 shows that the group of real points of the Zariski closure of the monodromy is a product of unitary groups \( U(r, s) \).

**Remark 3.** If \( n + 1 \leq d \), then the group of integral points of the Zariski closure of the monodromy is a product of irreducible arithmetic lattices (the integral points of the Zariski closure are as described in Proposition 24), some of which form co-compact lattices of their real Zariski closures. The Zariski closure can in fact be shown to be a product of unitary groups of suitable Hermitian forms over certain totally real number fields, and one of these Hermitian forms is definite at one of the real completions of the number field. We give a more detailed analysis of the Hermitian form later.

A result of A'Campo [A'C87] says that Theorem 1 holds even when \( d = 2 \).

In the proof of Theorem 1, we use the fact that a certain central element in \( B_{n+1} \) acts by a nontrivial scalar on the Burau representation if \( n \) is of the form \( kd - 2 \) in Lemma 17; it is here that we need that \( d \geq 3 \). The proof can be slightly modified to extend to the case \( d = 2 \) (see the remark following Lemma 17), but we will not do so here.

Theorem 1 answers a question raised in [McM13] (see Question 11.1 in [McM13]) in the affirmative when \( n \geq 2d \). When \( n \leq d - 2 \), the monodromy is, in some cases, not arithmetic, as is shown by the examples of Deligne-Mostow; cf. the example of \( d = 18 \) and \( n = 3 \) of Corollary (11.8) of [McM13].

In a different direction ([ACT02]), the arithmeticity of the image of the braid group of type \( E_6 \) into \( U(4, 1) \) is proved; in [AH10] representations of the braid group on the homology of noncyclic coverings are considered. Arithmeticity results for cyclic coverings of compact surfaces are proved in [Loo97].

The monodromy representation of the braid group \( B_{n+1} \) considered in Theorem 1 is closely related to the reduced Burau representation of the group \( B_{n+1} \). (See Section 7 for details.) Theorem 1 is deduced from the arithmeticity of the images of the Burau representation of the braid group \( B_{n+1} \) at \( d \)-th roots of unity (analogously, a more general result, namely Theorem 26, is deduced — in Section 9 — from the arithmeticity of the images of the Gassner representation of the pure braid group \( P_{n+1} \) evaluated at \( d \)-th roots of unity).

In the rest of the introduction, we concentrate only on the Burau case; the case of the Gassner representation is much more involved and we postpone dealing with it to a future occasion.
1.1. The braid group and the Burau representation.

1.1.1. Definition. The braid group \( B_{n+1} \) on \((n + 1)\)-strands is the free group on \( n \) generators \( s_1, s_2, \ldots, s_n \) modulo the relations

\[
s_j s_k = s_k s_j (|j - k| \geq 2) \quad \text{and} \quad s_j s_k s_j = s_k s_j s_k (|j - k| = 1).
\]

1.1.2. The reduced Burau representations \( \rho_n \). Let \( R = \mathbb{Z}[q, q^{-1}] \) be the Laurent polynomial ring in the variable \( q \) with integral coefficients. Let \( M = \mathbb{R}^n \) be the standard free \( R \)-module of rank \( n \) with standard generators \( e_1, e_2, \ldots, e_n \). For each \( j \), define the operator \( T_j \in \text{End}_R(M) \) by the formulae

\[
T_j(e_j) = -qe_j, \quad T_j(e_{j-1}) = e_{j-1} + qe_j, \quad T_j(e_{j+1}) = e_{j+1} + e_j,
\]

and

\[
T_j(e_k) = e_k (|k - j| \geq 2).
\]

In the above formulae, we do not attach any meaning to \( T_1(e_0) \). Similarly \( T_n(e_{n+1}) \) is left undefined. The map \( s_j \mapsto T_j \) defines a representation \( \rho_n : B_{n+1} \to \text{GL}_n(R) \). Denote by \( \Gamma_n \) the image of \( \rho_n \). The representation \( \rho_n \) is the reduced Burau representation in degree \( n \) ([Bir74]).

1.1.3. A Hermitian form on \( \mathbb{R}^n \). The ring \( R = \mathbb{Z}[q, q^{-1}] \) has an involution \( f \mapsto \overline{f} \) given by \( \overline{f}(q) = f(q^{-1}) \). The sub-ring \( S \) of invariants in \( R \) under this involution is clearly \( \mathbb{Z}[q + q^{-1}] \).

There is a unique map \( h : \mathbb{R}^n \times \mathbb{R}^n \to R \), which is a bilinear map of \( S \)-modules, so that for all \( v, w \in \mathbb{R}^n \) and all \( \lambda, \mu \in R \), we have

\[
\overline{h}(v, w) = h(w, v), \quad h(\lambda v, \mu w) = \lambda \overline{h}(v, w),
\]

and

\[
h(e_j, e_k) = 0 \quad (|j - k| \geq 2),
\]

\[
h(e_j, e_{j+1}) = -(q + 1), \quad h(e_j, e_j) = \frac{(q + 1)^2}{q}.
\]

We denote this form by \( h = h_n \). (When \( n \) is fixed, we drop the subscript, and write \( h \).) Then \( \Gamma_n \) preserves the Hermitian form \( h \) on \( \mathbb{R}^n \). We can then talk of the unitary group of the Hermitian form \( h \).

For a detailed description of a unitary group as an algebraic group defined over a field \( K \), see the beginning of Section 3. The unitary group \( U(h) \) of the Hermitian form \( h \) is an algebraic group scheme defined over \( S \) and

\[
U(h)(S) = \{ g \in \text{GL}_n(R) : h(gv, gw) = h(v, w) \ \forall v, w \in \mathbb{R}^n \}.
\]

More generally, for any commutative \( S \)-algebra \( A \), \( U(h)(A) \) is the group

\[
U(h)(A) = \{ g \in \text{GL}_n(R \otimes_S A) : h(gv, gw) = h(v, w) \ \forall v, w \in \mathbb{R}^n \otimes_S A \}.
\]
Remark 4. This Hermitian form $h$ is essentially (up to a scalar multiple and equivalence of Hermitian forms) the one constructed by Squier [Squ84] where he uses a form with coefficients in a quadratic extension $R' = \mathbb{Z}[\sqrt{q}, \sqrt{q}^{-1}]$ of $R$. We have written it in this form since we need an algebraic group defined over $S$ and not over $R$.

It is customary to use the notation $U(R, h)$ in place of $U(h)$ to denote a unitary group (see [PR94, §2.3.3]) in order to specify the quadratic extension $R/S$ with respect to which the Hermitian form $h$ is defined. Since our quadratic extension is fixed, we have used $U(h)$ to avoid complicating the notation.

1.2. The Burau representations $\rho_n(d)$ at $d$-th roots of unity. If $a \subset R$ is an ideal stable under the involution $f \mapsto \overline{f}$ and $A = R/a$ is the quotient ring, then on the $A$-module $A^n$ we get a corresponding Hermitian form $h_A$ and a corresponding representation $\rho_n(A) : B_{n+1} \to \text{GL}_n(A)$, which maps $B_{n+1}$ into $U(h_A)(B)$ where $B$ is the quotient ring $S/a \cap S$.

We can take $a$ to be the principal ideal in $R$ generated by $\Phi_d(q)$, where $\Phi_d(q)$ denotes the $d$-th cyclotomic polynomial in $q$. Then the quotient $R_d = R/a$ is an integral domain (the ring of integers in the $d$-th cyclotomic extension $E_d$ of $\mathbb{Q}$). The reduction modulo the ideal $a$ of the representation $\rho_n$ yields a representation $\rho_n(d)$ of the braid group $B_{n+1}$. This is referred to as the representation obtained by evaluating the representation $\rho_n$ at (all the) primitive $d$-th roots of unity.

Note that there is no preferred embedding of the cyclotomic field $E_d = \mathbb{Q}[q]/(\Phi_d(q))$ into the field $\mathbb{C}$ of complex numbers since there is no preferred choice of a primitive $d$-th root of unity in $\mathbb{C}$. We will therefore not consider $E_d$ as a sub-field of $\mathbb{C}$ but as being naturally embedded in the product $\prod_\mu \mathbb{C}$, where the product is over all the primitive $d$-th roots $\mu$ of unity. (The map into the product is obtained by evaluating $q$ at all the primitive $d$-th roots of unity.)

Denote by $\Gamma_n(d)$ the image of the representation $\rho_n(d)$. The image goes into the group $U(h)(O_K)$ where $K = K_d$ is the (totally real) sub-field of $E_d$ invariant under the involution $f \mapsto \overline{f}$: $K = \mathbb{Q}(2 \cos(\frac{2\pi}{d}))$ and $O_K$ is the ring of integers in $K$. We will consider arithmetic subgroups of (i.e., subgroups which have finite index in) $U(h)(O_K)$. (We refer to Section 2.2 for a discussion on why these are arithmetic groups in the sense of [GS75].)

Consider the group $\Gamma_n(d) \subset U(h)(O_K)$. The ambient Lie group in which $U(h)(O_K)$ is naturally a lattice (see the end of Section 2.2) is the product group $G_\infty = \prod_{v|\infty} U(h)(K_v)$, where the product runs through all the archimedean completions $K_v$ of $K$. Since $K$ is totally real, $K_v$ is isomorphic to $\mathbb{R}$ for each $v$. The form $h$, however, may be different for different real embeddings of $K$. When $h$ is nondegenerate, there exist nonnegative integers $r_v$ and $s_v$ with
$r_v + s_v = n$ such that the unitary group $U(h)(K_v)$ is isomorphic to $U(r_v, s_v)$ as an algebraic group over $\mathbb{R}$. (When $h$ is degenerate, the unitary group $U(h)(K_v)$ has a unipotent radical and the quotient by the unipotent radical is of the form $U(r_v, s_v)$ with $r_v + s_v \leq n - 1$.) In the course of the proof of arithmeticity of monodromy, we never need to use the ambient group $G_\infty$; we work directly with the arithmetic group $G(O_K)$ and the monodromy group $\Gamma \subset G(O_K)$. For these reasons, we do not specify the integers $r_v, s_v$ and the ambient Lie group $G_\infty$.

1.2.1. Statement of results. The main result of the paper is

**Theorem 2.** If $d \geq 3$ and $n \geq 2d$, then the image $\Gamma_n(d)$ of the Braid group $B_{n+1}$ under the reduced Burau representation $\rho_n$ evaluated at all the primitive $d$-th roots of unity — namely the image of representation $\rho_n(d) : B_{n+1} \to \text{GL}_n(\mathbb{Z}[q, q^{-1}]/(\Phi_d(q)))$ — is an arithmetic group.

More precisely, if $h$ is the Hermitian form on $A^n$ which $\Gamma_n(d)$ preserves, then $\Gamma_n(d)$ is a subgroup of finite index in $U(h)(O_K)$, where $K = \mathbb{Q}(\cos(2\pi d/5))$ is the totally real sub-field of the $d$-th cyclotomic extension of $\mathbb{Q}$.

These arithmetic groups are of $\mathbb{Q}$-rank at least two and, in particular, are not co-compact lattices.

We now take $d = 3, 4, 6$. In these cases, the $d$-th cyclotomic extension $E_d = \mathbb{Q}(e^{2\pi i/d})$ is an imaginary quadratic extension of $\mathbb{Q}$ and the totally real sub-field $K_d = \mathbb{Q}(\cos(2\pi d/5))$ is the field $\mathbb{Q}$ of rationals, $O_d = \mathbb{Z}$ and $\Gamma \subset U(h_n)(\mathbb{Z})$. Combining Theorem 2 with the results of Deligne-Mostow ([DM86], [McM13]) we will prove

**Theorem 3.** If $d = 3, 4, 6$, then for all $n$, the image of the Braid group $B_{n+1}$ under the representation $\rho_n$ evaluated at a primitive $d$-th root of unity is an arithmetic subgroup. More precisely, the image of $\rho_n(d)$ is an arithmetic subgroup of the integral unitary group $U(h)(\mathbb{Z})$.

We refer to Section 6 for a proof of Theorem 3.

Now consider the ring $A = \mathbb{Z}[q, q^{-1}]/(q^d - 1)$, where $R = \mathbb{Z}[q, q^{-1}]$ as before. The free $A$ module $A^n$ of rank $n$ may be viewed, in particular, as a free Abelian group of rank $nd$, and $\text{GL}_n(A)$ can be viewed as a subgroup of $\text{GL}_{nd}(\mathbb{Z})$. We say that a subgroup $\Gamma \subset \text{GL}_n(A)$ is arithmetic if it has finite index in its integral Zariski closure in $\text{GL}_{nd}(\mathbb{Z})$. A theorem of [A'C87] and Theorem 2 together imply the following:

**Theorem 4.** Consider the Burau representation

$$\rho : B_{n+1} \to \text{GL}_n(\mathbb{Z}[q, q^{-1}]/(q^d - 1)).$$

Then the image of $\rho$ is an arithmetic group for all $d \geq 2$ and $n \geq 2d$. 

1.3. Description of the proof. The proof of Theorem 2 is by showing that for \( n \geq 2d \), the image \( \Gamma_n(d) \) contains a large number of unipotent elements. (Precisely, \( \Gamma_n(d) \) contains an arithmetic subgroup of the unipotent radical of a parabolic \( \mathbb{Q} \)-subgroup.) By using results of Bass-Milnor-Serre and Tits (\cite{BMS67}, \cite{Tit76}) and their extensions to other groups (\cite{Rag92}, \cite{Vas73}, \cite{Ven94}) on unipotent generators for noncocompact arithmetic groups of \( \mathbb{R} \)-rank at least two, we show (in Section 5) that such groups are arithmetic if \( n \geq 2d \).

The proof that \( \Gamma_n(d) \) contains sufficiently many unipotent elements (see Section 5) is by using induction. We first prove this in the case when \( n \geq 2d \) and \( n \) is a multiple of \( d \). Then we prove an inductive step that if we can get unipotents for \( m \), then we get unipotents for \( m+1 \) (\( m \geq 2d \)). This will then cover all integers \( n \geq 2d \).

The construction of sufficiently many unipotent elements is especially easy to describe when the representation is the Burau representation of \( B_{n+1} \) at \( d \)-th roots of unity and \( n = 2d \) (or, more generally, when \( n = kd \) is a multiple of \( d \), with \( k \geq 2 \); see Section 4.5). We will exploit the fact that the representation \( \rho_{2d-1}(d) \) is not irreducible but contains a nonzero invariant vector (see Proposition 16). Let \( s_1 \) be as before. Denote by \( \Delta' \) the product element

\[
\Delta' = (s_2 s_3 \cdots s_n)(s_2 s_3 \cdots s_{n-1}) \cdots (s_2 s_3)(s_2).
\]

It can be shown that \( \Delta'^2 \) is central in \( B_n' \), the braid group generated by \( s_2, s_3, \ldots, s_n \). Form the commutator \( u = [s_1, (\Delta')^2] \). Consider the group \( U \subset B_{n+1} \) generated by conjugates of \( u \) of the form \( \{huh^{-1} : h \in B'_n\} \). We show (in Section 4.5) that the image of this group \( U \) under the Burau representation at \( d \)-th roots of unity is an arithmetic subgroup of the unipotent radical of a (maximal) parabolic subgroup of the unitary group \( U(h) \). This is enough to prove that the image of \( B_{n+1} \) under \( \rho_n(d) \) is arithmetic, by the criteria of Section 2.

In Section 7, we derive Theorem 1 from Theorem 2, by establishing the precise relationship between the monodromy representation of Theorem 1 and the Burau representation; although this connection is essentially well known (cf. \cite[Th. 5.5]{McM13}), we will give a more precise description of the connection in Section 7.

1.4. Organisation of the paper. This paper is organised as follows. In Section 2, we recall some basic notions from algebraic groups and state a criterion due to Bass-Milnor-Serre and others on unipotent generators for higher rank nonuniform lattices. In Section 3, we will apply this criterion to certain integral unitary groups.

The main section of the paper is Section 4, where we show that the image of the braid group \( B_{n+1} \) at a primitive \( d \)-th root of unity contains many unipotent
elements (more precisely, contains an arithmetic subgroup of the unipotent radical of a parabolic subgroup defined over the field $K_d$). The criteria of Section 3 then imply the arithmeticity when $n \geq 2d$ for all $d$.

In Section 8, we first consider (in Section 8.1) complex reflection groups corresponding to root systems of type $A$ and show the arithmeticity of the images of the corresponding Artin groups $A_n(q)$ (see Section 8 for references to definitions), where $q$ is a primitive $d$-th root of unity and $n \geq 2d$; the image of the group $A_n(q)$ turns out to be the same as the image of $\Gamma_n(d)$ into one of the factors of $U(h)(K \otimes \mathbb{R})$. Hence the arithmeticity of the image of $A_n(q)$ will be shown to be an immediate consequence of Theorem 2. This answers Question 5.6 in [McM13] in many cases.

In Section 8.2, we show that Theorem 2 implies the arithmeticity of the monodromy of certain one variable hypergeometric functions of type $_nF_{n-1}$, in some special cases of the parameters. The point here is that arithmeticity can be proved for an infinite family of parameters associated to the hypergeometric equations $_nF_{n-1}$.

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2. Algebraic groups

For the facts on algebraic groups stated in this section, we refer to [BT65].

2.1. Examples of algebraic groups. If $K$ is a number field and $G \subset GL_n$ is a subgroup which is the set of zeroes of a collection of polynomials in the matrix entries of $GL_n$, such that the coefficients of these polynomials lie in $K$, then $G$ is said to be an algebraic group defined over $K$. 
For example, the groups $\text{SL}_n$ and $\text{Sp}_{2g}$ are algebraic groups defined over $K$. If $E/K$ is a quadratic extension and $H : E^n \times E^n \to E$ is a $K$ bilinear form which is Hermitian with respect to the nontrivial automorphism of $E/K$, then the unitary group of the Hermitian form $h$ is an algebraic group defined over $K$.

2.2. Arithmetic groups. Let $G \subset \text{GL}_N$ be a linear algebraic group defined over $\mathbb{Q}$. A subgroup $\Gamma \subset G(\mathbb{Z})$ is said to be an arithmetic subgroup of $G(\mathbb{Q})$ if $\Gamma$ has finite index in the intersection $G(\mathbb{Z}) = G \cap \text{GL}_N(\mathbb{Z})$. (See [Rag72, Chap. X, Def. (10.12), p. 165] or [PR94, Chap. 4, (4.1), p. 171].) By a theorem of Borel and Harish-Chandra, $\Gamma$ is a lattice (a discrete subgroup with finite covolume) in the group $G(\mathbb{R})$ of real points of $G$ provided the identity component $G^0$ of the group $G$ does not have nontrivial homomorphisms into the multiplicative group $\mathbb{G}_m$ defined over $\mathbb{Q}$. (For a reference, see [PR94, Th. 4.13, p. 213].) It is also a consequence of the result of Borel and Harish-Chandra that $G(\mathbb{Z})$ is a co-compact lattice if and only if $\mathbb{Q}$-rank ($G$) = 0. (See Section 2.3 for the notion of $\mathbb{Q}$-rank.)

Let $O_K$ denote the ring of integers in $K$. The group $G(\mathbb{O}_K)$ is by definition, the intersection $\text{GL}_n(\mathbb{O}_K) \cap G$. A subgroup $\Gamma \subset G(K)$ is said to be arithmetic if the intersection $\Gamma \cap G(\mathbb{O}_K)$ has finite index in $\Gamma$ and in $G(\mathbb{O}_K)$. This definition appears to be different from the one given in the preceding paragraph. But these two definitions are equivalent. This is shown by replacing $G$ by the Weil restriction of scalars $G = R_{K/\mathbb{Q}} G$. (For a reference, see [PR94, 2.1.2, p. 49].) The theory of Weil restriction of scalars says that there exists a linear algebraic group $G$ defined over $\mathbb{Q}$ (and unique up to isomorphism) with the following property: for any commutative $\mathbb{Q}$-algebra $A$, the group $G(A)$ is naturally isomorphic to $G(K \otimes_{\mathbb{Q}} A)$, the group of $K \otimes_{\mathbb{Q}} A$-rational points of the $K$-algebraic group $G$.

Definition 1. The group $G$ is called the Weil restriction of scalars from $K$ to $\mathbb{Q}$, of the group $G$. It is denoted $G = R_{K/\mathbb{Q}}(G)$.

We then have $R_{K/\mathbb{Q}} G(\mathbb{Z}) \simeq G(\mathbb{O}_K)$, where the symbol $\simeq$ means that there is equality up to subgroups of finite index. (The two groups are commensurable.) Moreover, $G(\mathbb{R}) \simeq \prod_{v \mid \infty} G(K_v)$, where the product is over all the inequivalent archimedean completions of $K$ (see [PR94, 2.1.2, pp. 50–51]). Thus, $G(\mathbb{O}_K)$ is a lattice in the ambient Lie group $\prod G(K_v)$ where the product is over all the inequivalent nonarchimedean completions $K_v$ of $K$. We recall that any number field has $r_1$ completions $K_v$ which are isomorphic to $\mathbb{R}$ and up to complex conjugation, $r_2$ completions $K_v$ which are isomorphic to $\mathbb{C}$ such that $r_1 + 2r_2$ is the degree of the extension $K$ over $\mathbb{Q}$.
We will have occasion to deal with images of arithmetic groups under
morphisms of \( \mathbb{Q} \)-algebraic groups (see Section 7.7). In particular, we will use
the following result. Let \( f : \mathcal{G} \to \mathcal{G}' \) be a morphism of algebraic groups defined
over \( \mathbb{Q} \). This induces a homomorphism, also denoted \( f \), from \( \mathcal{G}(\mathbb{Q}) \) into \( \mathcal{G}'(\mathbb{Q}) \).
It is elementary that if \( \Gamma \subset \mathcal{G}(\mathbb{Z}) \) is a suitable subgroup (a suitable “congruence
subgroup”) of finite index, then \( f(\Gamma) \) lies in \( \mathcal{G}'(\mathbb{Z}) \). For a proof of the following
lemma, see Corollary (10.14) of [Rag72].

**Lemma 5.** The image of \( \Gamma \) under \( f \) is an arithmetic subgroup of \( \mathcal{G}'(\mathbb{Z}) \).

Let \( \theta : V \to V' \) be a linear map of \( \mathbb{Q} \)-vector spaces and \( \rho : \Delta \to \text{GL}(V) \)
be a representation of a group \( \Delta \). Then the composite \( \theta \circ \rho \) is a representation
of \( \Delta \). The following is immediate from Lemma 5.

**Proposition 6.** If the image of \( \rho \) is an arithmetic subgroup of \( \text{GL}(V) \),
then the image of the composite \( \theta \circ \rho \) is an arithmetic subgroup of \( \text{GL}(V') \).

2.3. **K-rank.** In this subsection, we define the notion of the \( K \)-rank and
\( \mathbb{Q} \)-rank of a linear algebraic group.

**Definition 2.** An algebraic group \( G \) is said to be \( K \)-isotropic if there exists
an injective morphism \( \mathbb{G}^r_m \to G \) of linear algebraic groups defined over \( K \) for
some \( r \geq 1 \). The \( r \)-fold product \( \mathbb{G}^r_m \) is called the \( K \)-split torus of dimension \( r \)
and the embedding is called a \( K \)-embedding. The \( K \)-rank of \( G \) is by definition
the maximum, call it \( r \), of the dimensions of \( K \)-split tori which are \( K \)-embedded
in \( G \). Let \( T \) be a \( K \)-split torus of dimension \( r \) which is \( K \)-embedded in \( G \). Then
\( T \) is called a maximal \( K \)-split torus. It is known that all maximal \( K \)-split tori
are conjugate under the group \( G(K) \).

For example, for any \( K \), one can compute the \( K \)-rank of the groups \( \text{SL}_n \)
and \( \text{Sp}_{2g} \): the \( K \)-rank of \( \text{SL}_n \) is \( n - 1 \); the \( K \)-rank of \( \text{Sp}_{2g} \) is \( g \). The \( K \)-rank may
depend on \( K \): if \( D \) is a quaternionic division algebra over \( \mathbb{Q} \), let \( G = \text{SL}_2(D) \)
be the algebraic group defined over \( \mathbb{Q} \). Then \( \mathbb{Q} \)-rank \( (G) = 1 \); if \( K \subset D \) is a
quadratic extension of \( \mathbb{Q} \), then as an algebraic group over \( K \), \( G \) is isomorphic
to \( \text{SL}_4 \) and \( K \)-rank \( (G) = 3 \).

It follows from definitions that \( K \)-rank \( (G) = \mathbb{Q} \)-rank \( (G) \), where \( G \) is the
Weil restriction of scalars from \( K \) to \( \mathbb{Q} \).

2.4. **Parabolic subgroups.** Suppose now that \( G \subset \text{GL}_n \) is an algebraic
group defined over a number field \( K \); embed \( K \) in \( \mathbb{C} \), and suppose that the Lie
algebra of \( G(\mathbb{C}) \) is a simple Lie algebra over \( \mathbb{C} \). Then \( G \) is said to be absolutely
almost simple.

An algebraic subgroup \( P \subset G \) is said to be a parabolic subgroup defined
over \( K \) if \( P \) is defined over \( K \) and the quotient \( G/P \) is a projective variety.
For example, a subgroup $P \subseteq \text{GL}_n = \text{GL}(V)$ is a parabolic subgroup if and only if it is the subgroup preserving a partial flag
\[ \{0\} \subset W_1 \subset W_2 \subset \cdots \subset W_{r-1} \subset V, \]
where $W_i$ form a sequence of subspaces of $V$ with $W_i \subset W_{i+1}$.

Let $G$ be an absolutely almost simple algebraic group defined over $K$. The group $G$ has positive $K$-rank if and only if $G$ contains a proper parabolic subgroup $P$ defined over $K$. Suppose that $r = \text{K-rank} (G) \geq 1$ and $T$ a maximal $K$-split torus in $G$. Then there exists a parabolic $K$-subgroup $P$ containing $T$. Furthermore, there exists a nontrivial maximal unipotent normal subgroup $U$ of $P$, called the unipotent radical of $P$. The Lie algebra of $U$ is stable under the action of the group $T$ and splits into character spaces for the adjoint action of $T$.

The structure theory of parabolic subgroups says that there exists a parabolic subgroup $P^-$ defined over $K$ containing $T$, with a unipotent radical $U^-$ such that the characters of $T$ on $\text{Lie}(U^-)$ are the inverses of the characters of $T$ on $\text{Lie}(U)$. The group $P^-$ is said to be opposite to $P$, and $U^-$ is said to be opposed to $U$. The intersection $M = P \cap P^-$ is called a Levi subgroup of $P$, and we have the Levi decomposition $P = MU$. The group $M$ normalises both $U$ and $U^-$. A $K$-parabolic subgroup $P_0$ containing $T$ is minimal if it is of the smallest dimension among the $K$-parabolic subgroups containing $T$. If $P \supset P_0$, then we have the (reverse) inclusion of the unipotent radicals $U \subset U_0$. There exists a minimal parabolic subgroup $P_0^-$ opposed to $P_0$, with unipotent radical $U_0^-$.  

2.5. The real rank. Suppose that $G$ is an absolutely almost simple algebraic group defined over a number field $K$. Write $G_\infty$ for the product group $G_\infty = \prod_v \text{G}(K_v)$ (which is a real semi-simple group), where $v$ runs over all the archimedean completions of $K$; we write
\[ \infty\text{-rank}(G) = \mathbb{R}\text{-rank}(G_\infty) = \sum_v \text{K}_v\text{-rank}(G), \]
and we call this the real rank of $G_\infty$.

2.6. A criterion for arithmeticity. Bass, Milnor and Serre proved that for any $N \geq 1$, the $N$-th powers of the upper and lower triangular unipotent matrices in $\text{SL}_n(\mathbb{Z})$ generate a subgroup of finite index in $\text{SL}_n(\mathbb{Z})$ for $n \geq 3$. A similar result holds for $\text{Sp}_{2g}(\mathbb{Z})$. In this subsection, we describe an analogous result for all higher $\mathbb{R}$-rank groups.

The following result is due to many people: for $G = \text{SL}_n$ ($n \geq 3$) or $\text{Sp}_{2g}$ ($g \geq 2$), this is due to Bass, Milnor and Serre [BMS67]; when $G$ is split over $K$ (i.e., when a maximal $K$-split torus is also a maximal $\mathbb{C}$-split torus), to Tits [Tit76] and for classical groups $G$ with $K$-rank $\geq 2$ to Vaserstein [Vas73]. The case of a general $G$ is handled in Raghunathan [Rag92] and [Ven94].
Theorem 7. Let $G$ be an absolutely almost simple algebraic group defined over a number field $K$. Assume that $K$-rank($G$) \geq 1, \mathbb{R}$-rank($G_\infty$) \geq 2 and that $P_0, P_0^-$ are minimal parabolic $K$-subgroups of $G$ with unipotent radicals $U_0$ and $U_0^-$. Then the following hold:

1. For every integer $N \geq 1$, let $\Delta_N$ be the subgroup of $G(O_K)$ generated by $N$-th powers of the elements in $U_0(O_K)$ and $U_0^-(O_K)$. Then $\Delta_N$ has finite index in $G(O_K)$.

2. If $\Delta_N' \subset \Delta_N$ is an infinite normal subgroup, then $\Delta_N'$ also has finite index in $G(O_K)$.

The above theorem says that the integral points of two maximal opposing unipotent subgroups (i.e., the unipotent radicals of two minimal parabolic $K$-subgroups) generate a finite index subgroup in $G(O_K)$ if the $\infty$-rank is at least two. We need a strengthening of this, where we assume only that the unipotent radicals which are not necessarily maximal, of two opposing parabolic subgroups are involved. This is the following:

Corollary 1. Suppose $G$ is absolutely almost simple and $K$-rank($G$) \geq 2. Let $P$ and $P^-$ be two opposite parabolic subgroups containing a maximal $K$-split torus, and let $U,U^-$ be their unipotent radicals. For any integer $N \geq 1$, the group $\Delta_N(P^\pm)$ generated by $N$-th powers of $U(O_K)$ and $U^-(O_K)$ is of finite index in $G(O_K)$.

Proof. In the notation above, let $M = P \cap P^-$. Then $M(O_K)$ normalises $U(O_K)$ and $U^-(O_K)$ and hence normalises the group $\Delta_N(P)$. Now the group generated by $M(O_K)$ and $\Delta_N(P^\pm)$ contains $(U_0 \cap M)(O_K)$ and $(U_0 \cap U)(O_K)^N = U(O_K)^N$; the decomposition $P = MU$ shows that $P(O_K) = M(O_K)U(O_K)$ and hence $U_0(O_K) = (U_0 \cap M)(O_K)U(O_K)$. (All these equalities are true only up to finite index; the decomposition $U_0 = (U_0 \cap M)U$ of algebraic groups is defined over the field $K$ but not over the integers $O_K$. This implies that there are finite index subgroups $U'_0 \subset U_0(O_K)$, $(U_0 \cap M)' \subset (U_0 \cap M)(O_K)$ and $U' \subset U(O_K)$ such that the product decomposition $U'_0 = (U_0 \cap M)'U'$ holds for the smaller groups.)

Therefore the group generated by $M(O_K)$ and $\Delta_N(P^\pm)$ contains $N$-th powers of elements of $U_0(O_K)$ and $U_0^-(O_K)$. Consequently, the group $\Delta_N(P^\pm)$ is normalised by $\Delta_N$. By the second part of the above theorem, $\Delta_N(P)$ is of finite index in $G(O_K)$. \hfill \square

Remark 5. The second part of Theorem 7 is true for any irreducible lattice in a real semi-simple group of real rank at least two. (This is the normal subgroup theorem of Margulis.)

In the next section, we will state a special case of this corollary for certain unitary groups.
2.7. Subgroups of products of higher rank groups. In Section 5, we will prove the arithmeticity of the image of the braid group in a group of the form $U(h)(\mathbb{Z}[q]/(q^d - 1))$. The latter is a product of the groups $U(h)(O_e)$ for certain rings of integers $O_e$. To deal with this case, we now prove a lemma which is a simple consequence of the super-rigidity theorem of Margulis.

**Lemma 8.** Suppose that $K_e$ are number fields for each element $e$ in a finite indexing set $X$. Suppose that $G_e$ is an absolutely almost simple semi-simple algebraic group defined over $K_e$, and suppose that $\infty$-rank($G_e$) $\geq 2$ for all $e \in X$. Suppose that $\Gamma \subset \prod G_e(O_e)$ is a subgroup such that the image of its projection to each $G_e(O_e)$ has finite index in $G_e(O_e)$. Assume, in addition, that either $K_e$ and $K_f$ are nonisomorphic or else, if $K_e$ and $K_f$ are isomorphic, the groups $G_e$ and $G_f$ (which may be thought of as groups defined over the same field $K_e$) are not isomorphic over $K_e = K_f$.

Under these assumptions, the group $\Gamma$ has finite index in the product $\prod_{e \in X} G_e(O_e)$.

**Proof.** Replacing the arithmetic groups $G_e(O_e)$ by subgroups of finite index, we may assume that these are torsion free and hence that $\Gamma$ is torsion free. We prove the lemma by induction on the number of factors. Fix an element $p \in X$. By induction, the projection of $\Gamma$ in the product $\prod_{e \in X, e \neq p} G_e(O_e)$ under the projection map $pr: \prod_{e \in X} G_e(O_e) \to \prod_{e \in X, e \neq p} G_e(O_e)$ has finite index. Suppose $N_p$ is kernel of restriction of this projection map to $\Gamma$. We will show that the kernel $N_p$ cannot be trivial.

If $N_p$ is trivial, then $\Gamma$ projects injectively into the product of the groups $G_e(O_e)$ with $e \neq p$: $\Gamma \subset \prod_{e \in X, e \neq p} G_e(O_e)$ (and by the induction assumption, its image has finite index). Therefore, $\Gamma$ is an arithmetic subgroup of the higher rank lattice $\prod_{e \neq p} G_e(O_e)$ and has a nontrivial representation (projection to the $p$-th factor) onto a finite index subgroup of the arithmetic group $G_p(O_p)$: replacing the image of $\Gamma$ by a smaller subgroup of finite index if necessary, we may assume that the image of $\Gamma$ in the “away from $p$” product is a product of finite index subgroups of $G_e(O_e)$ (with $e \neq p$). The Margulis normal subgroup theorem (applied to the image of $\Gamma$ in $G_e(O_e)$) then implies that only one of the factors in this product maps isomorphically onto its image in $G_p(O_p)$ and that the other factors map to the identity. This contradicts the Margulis super-rigidity (or Mostow rigidity): we have an isomorphism of a finite index subgroup of $G_e(O_e)$ with a finite index subgroup of $G_p(O_p)$. Such an isomorphism, by the super-rigidity theorem, is induced by first an isomorphism of $K_e$ with $K_p$ and an isomorphism of $G_e$ with $G_p$ as groups over $K_e = K_p$. This contradicts our assumptions and therefore, the kernel $N_p$ cannot be trivial.

Since the kernel $N_p$ is nontrivial, it is infinite since $\Gamma$ is assumed to be torsion free. The conjugation action of $\Gamma$ on the $p$-th factor $G_p(O_p)$ factors
through its projection to the $p$-th factor; but the $p$-th projection map has image of finite index, and hence $N_p$ is normalised by a subgroup of finite index in $G_p(O_p)$; by the Margulis normal subgroup theorem, $N_p$ has finite index in $G_p(O_p)$; therefore, $\Gamma$ maps onto a subgroup of finite index in the product of the “away from $p$” factors, and intersects the $p$-th factor in a subgroup of finite index. Therefore, $\Gamma$ has finite index. 

\[ \square \]

Remark 6. A related result is proved in [GL09] and also in [Loo97]. At the time of writing the present paper, we were unaware of these papers. Moreover, the proof here is different from the cited papers.

3. Unitary groups

Notation. Suppose that $E/K$ is a quadratic extension. Write $E = K \oplus K\sqrt{\alpha}$ for some nonsquare element $\alpha$ in $K$, and given $z \in E$, write $z = x + y\sqrt{\alpha}$ accordingly. Then $x, y$ are called the “real” and “imaginary” parts, respectively. Denote by $\overline{z}$ the element $x - \sqrt{\alpha}y$.

Given $n \geq 1$, the vector space $E^n$ may be viewed as a $2n$-dimensional vector space over $K$. Suppose $h : E^n \times E^n \to E$ is a map which is $K$-bilinear such that for all $v, w \in E^n$, we have

$$h(v, w) = \overline{h(w, v)}.$$ 

Then $h$ is said to be a Hermitian form with respect to $E/K$. By definition, the elements of the unitary group $U(h)$ satisfy the property that they commute with scalar multiplication by elements of $E$, viewed as $K$-linear endomorphisms of $E^n$; further, they preserve the real and imaginary parts of $h$. These properties characterise the elements of $U(h)$, and in this manner, the group $U(h)$ may be viewed as a $K$-algebraic subgroup of $GL_K(K^{2n}) = GL_{2n}(K)$. In particular,

$$U(h)(K) = \{ g \in GL_n(E) \subset GL_{2n}(K) : h(gv, gw) = h(v, w) \}$$

for all vectors $v, w \in E^n$. More generally, if $A$ is a commutative $K$ algebra, then

$$U(h)(A) = \{ g \in GL_n(E \otimes_K A) : h(gv, gw) = h(v, w) \}$$

for all elements $v, w$ of the $E$-module $E^n \otimes_K A$.

For example, if $K$ and $E$ are replaced by $\mathbb{R}$ and $\mathbb{C}$ and $h$ is the standard Hermitian form on $\mathbb{C}^n$, then the group of real points of the unitary group $U(h)$ are given by the compact group

$$U(h)(\mathbb{R}) = \{ g \in GL_n(\mathbb{C}) : {}^t\overline{g} g = 1 \},$$

and the group of complex points of the unitary group may easily be seen (from the above description) to be isomorphic to $GL_n(\mathbb{C})$. 
3.1. Rank of a unitary group. Let \( h_2 : E^2 \times E^2 \to E \) be a Hermitian form with respect to \( E/K \). Suppose that the Hermitian form \( h_2 \) can be written as

\[
h_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

That is, if \( v, w \) are column vectors in \( E^2 \) with entries \( v = (z_1, z_2) \) and \( w = (w_1, w_2) \), then \( h_2(v, w) = z_1 \overline{w_2} + z_2 \overline{w_1} \). Then \( h \) is called a hyperbolic form. The standard basis of \( E_2 \) may be written \( v, v^* \), and the form \( h_2 \) is such that \( h(v, v) = h(v^*, v^*) = 0 \) and \( h(v, v^*) = 1 \).

Suppose \( h \) is a nondegenerate Hermitian form on \( E^n \). Then \( (E^n, h) \) can be written as a direct sum of \( r \) copies of the hyperbolic form \( (E^2, h_2) \) and a form \( (E^{n-2r}, h_0) \) which does not represent a zero in \( E^{n-2r} \). Then \( h_0 \) is said to be anisotropic. Let \( U(h) \) be the unitary group of the Hermitian form \( h \). It can be shown that \( K \)-rank \( (U(h)) = r \).

With respect to this decomposition \( h = h_2 \oplus \cdots \oplus h_2 \oplus h_0 \), write the standard basis of the \( j \)-th copy of \( (E^2, h_2) \) as \( v_j, v_j^* \). We may rearrange the basis of \( E^n \) in the form

\[
v_1, v_2, \ldots, v_r, w_1, \ldots, w_s, v_r^*, \ldots, v_2^*, v_1^*,
\]

where \( w_1, \ldots, w_s \) is a basis of \( (E^{n-2r}, h_0) \). Let \( Av_1 \subset W \subset V = E^n \), where \( W \) is the \( E \)-submodule spanned by

\[
v_1, \ldots, v_r, w_1, \ldots, w_s, v_r^*, \ldots, v_2^*.
\]

In the terminology of the preceding section, if \( n \geq 2 \), then \( SU(h) \) is an absolutely almost simple algebraic group defined over \( K \); the subgroup \( P \) of \( SU(h) \) which preserves the flag \( Ev_1 \subset W \subset V \) is a parabolic subgroup defined over \( K \). Let \( U \) be the subgroup of \( SU(h) \) which preserves this flag and acts trivially on successive quotients. Then \( U \) is the unipotent radical of \( P \).

The partial flag \( Ev^* \subset Ev^* \oplus W \subset V \) defines a parabolic subgroup \( P^- \) of \( U(h) \), and \( U^- \) is the subgroup of \( P^- \) which acts trivially on the successive quotients of this flag. This parabolic subgroup \( P^- \) is opposite to \( P \), and \( U^- \) is its unipotent radical opposed to \( U \), in the sense of Section 2.4.

**Corollary 2.** Along with the preceding notation and hypotheses, suppose that \( r \geq 2 \). Then the group generated by the \( N \)-th powers of \( U(O_K) \) and \( U^-(O_K) \) is an arithmetic subgroup of \( SU(h)(O_K) \). The same conclusion holds if the \( K \)-rank of \( SU(h) \) is 1 but \( \infty \)-rank \( (SU(h)) \) is \( \geq 2 \).

**Proof.** We have already noted that \( K \)-rank \( (SU(h)) \) is the number of hyperbolic 2-planes in the decomposition of \( h \). Therefore, the \( K \)-rank of \( SU(h) \) is \( \geq 2 \). Then the corollary follows from Corollary 1 of the preceding section.

The second part follows by Theorem 7. (The group has higher real rank but \( \mathbb{Q} \)-rank one.) \( \square \)
3.2. The Heisenberg group and the group $P$. Assume now that $(V,h)$ is an $n = m + 1$-dimensional $E$ vector space with a nondegenerate Hermitian form $h$. Assume that there exists a $E$-vector subspace $X$ of $V$ of codimension two such that $h$ is nondegenerate on $X$ and that there exist isotropic vectors $v, v^*$ in $V$ which are orthogonal to $X$ such that $h(v, v^*) \neq 0$. We have a partial flag

$$0 \subset Ev \subset Ev \oplus X \subset V = Ev \oplus X \oplus Ev^*.$$ 

The subgroup of the unitary group $U(h)$ of $V$ which preserves this flag and acts trivially on successive quotients is called the Heisenberg group $H(X)$ of $X$. We write $P$ for the subgroup of $U(h)$ which preserves this flag. It is easily seen that $P$ is a maximal parabolic subgroup of the unitary group $U(h)$ defined over $K$. The Heisenberg group $H(X)$ is the unipotent radical of $P$. (We sometimes denote $H(X)$ by $U$, to be consistent with the notation of the preceding section.) Since we have an orthogonal decomposition $V = X \oplus (Ev \oplus Ev^*)$ with respect to the Hermitian form $h$, it follows that the unitary group $M = U(h|_X)$ of the restriction of $h$ to $X$ is the subgroup of $U(h)$ which fixes the vectors $v, v^*$. We have $P = H(X)M = UM$, and this gives a Levi decomposition of the parabolic subgroup $P$. Since the Hermitian form $h$ is the same on $X$ and $V$, we sometimes write $U(X)$ and $U(V)$ in place of $U(h|_X)$ and $U(h)$.

The direct sum $W = Ev \oplus X$ has the property that the Hermitian form on $W$ is degenerate with $W^\perp = Ev$. The quotient map $W \rightarrow W/Av = X$ preserves the Hermitian structure on both sides since $Ev$ is orthogonal to $W$. This induces a surjective map $U(W) \rightarrow U(X)$ of unitary groups, with kernel $U_0$, say. Consider the abelian vector group $X^* = \text{Hom}(X, Ev)$. We may view $X^*$ as a vector space over $K$ and hence as the group of $K$-rational points of a unipotent algebraic group which is isomorphic to $U_0$: $U_0(K) \simeq X^*$. Hence we refer to $U_0$ as a vector group. We have a split short exact sequence

$$0 \rightarrow U_0 \rightarrow U(W) \rightarrow U(X) \rightarrow 1,$$

and we may write $U(W) = U(X)U_0$. An element $\alpha$ of $U(W)$ may accordingly be written as a pair $\alpha = (g, x)$ with $g \in U(X)$ and $x \in U_0$. If $g \in U(X)$, then $g$ gives a transformation on the vector group $U_0$ defined by $x \mapsto xg$ (the transpose of $g$). With this notation, if $\beta = (h, y) \in U(W)$, then

$$\alpha \beta = (gh, xh + y).$$

Therefore, $\alpha^{-1} = (g^{-1}, -xg^{-1})$.

Suppose that $m \in U(X) \subset U(W)$ is of the form $(\lambda, 0)$, which is the multiplication by the scalar $\lambda$ on $X$ and acting by $0$ on $v$. Given $a, b \in U(h)$, denote by $[a, b]$ the commutator $aba^{-1}b^{-1}$. 
Lemma 9. If $\alpha = (g, x)$ and $m = (\lambda, 0)$, then the commutator $[\alpha, m]$ is of the form
\[
[\alpha, m] = (1, (1 - \lambda^{-1})xg^{-1}).
\]
In particular, if $x \neq 0$ and $\lambda \neq 1$, then the commutator $[\alpha, m]$ is a nonzero element of the vector space $U_0$.

Recall that the abelian vector group $X^* = \text{Hom}(X, Ev)$ may be viewed as a vector space over $K$ and hence as the group of $K$-rational points of a unipotent algebraic group $U_0$: $U_0(K) \cong X^*$. The nondegenerate Hermitian form $h_X$ on $X$ gives a Hermitian form on $X^*$ as well, which we again denote $h_X = h|_{X^*}$.

The group $U = H(X)$ is nonabelian with centre equal to the one-dimensional vector space $G_a$ over $K$, and we have an exact sequence of unipotent $K$-algebraic groups
\[
\{0\} \to G_a \to U \to U_0 \to \{1\}.
\]
This in turn gives a short exact sequence
\[
\{0\} \to K \to U(K) \to U_0(K) = X^* \to \{1\}
\]
at the level of $K$-rational points. (We have written $\{0\}$ and $\{1\}$ for the trivial group since one of them is written additively and the other multiplicatively.) Moreover, if $x, y \in U(K)$, and $x^*, y^*$ their images in $U_0(K)$, then the commutator $[x, y]$, as an element of $K$, is simply the imaginary part of $h(x^*, y^*)$.

Now, $P$ is a semi-direct product of $U$ with $M \simeq P/U$ and $M$ is isomorphic to $U(X)$ as in the preceding. Moreover, the centre $Z = G_a$ of $U$ is normal in $P$ and secondly the quotient $U/Z$ is isomorphic to $U_0$. We may write, using the decomposition $P = MU$, an element $\alpha$ of $P$ in the form $\alpha = (g, x)$ with $g \in M$ and $x \in U$. Let $\lambda \in U(X)$ be a scalar transformation.

Lemma 10. If $\alpha = (g, x)$ with $x \in U$, $x$ has nonzero image in $U_0$ (i.e., does not lie in the centre of $U$), and $m = (\lambda, 0) \in M$, then the commutator $[\alpha, m]$ is a noncentral element of $U$.

The proof is immediate from Lemma 9 since the image of the commutator $[\alpha, m]$ in $U_0 = U/Z$ is already nontrivial by Lemma 9.

We now state another simple observation as a lemma.

Lemma 11. If $U \to U_0$ is the quotient map, then a subgroup $N \in U(O_K)$ has finite index if and only if its image $N_0$ has finite index in $U_0(O_K)$.

3.3. An inductive step for integral unitary groups. In this subsection, we prove a result which will be used in the inductive proof of Theorem 2. This says that a subgroup of the integral unitary group which contains finite index subgroups of smaller integral unitary groups has finite index.
**Notation.** Let $V = (V, h)$ be a nondegenerate Hermitian space over $E$ such that $K\text{-rank}(U(h)) \geq 2$. Let $W, W'$ be codimension one subspaces such that the restriction of $h$ to $W, W'$ and the intersection $W \cap W'$ are all nondegenerate. We denote by $U_V$ the unitary group $U(h)$, and we similarly define $U_W, U_{W'}, U_{W \cap W'}$. If $Y$ is one of the subspaces $W, W', W \cap W'$, then by the nondegeneracy assumption, we have an orthogonal decomposition $V = Y \oplus Y^\perp$. Hence $U_Y$ may be thought of as the subgroup of $U_V$ which acts trivially on $Y^\perp$.

Suppose that $\Gamma \subset U_V(O_K)$ is a subgroup such that its intersection with $U_W(O_K)$ has finite index in $U_W(O_K)$ and such that its intersection with $U_{W'}(O_K)$ has finite index in $U_{W'}(O_K)$. Assume further that $W \cap W'$ contains a nonzero isotropic vector $v$.

**Lemma 12.** With the preceding notation (and under the assumption that $K\text{-rank}(U(h)) \geq 2$), the group $\Gamma$ has finite index in $U_V(O_K)$.

**Proof.** Since $W \cap W'$ has codimension two in $V$ and $h$ is nondegenerate on $W \cap W'$, it follows that $V$ is the direct sum of $W \cap W'$ and its orthogonal complement $(W \cap W')^\perp$ in $V$. The orthogonal complement is also a nondegenerate unitary space (of dimension two); by assumption, $W \cap W'$ contains an isotropic vector $v$, say. The nondegeneracy of $h$ on $W \cap W'$ shows that there exists a vector $v^* \in W \cap W'$ such that $h(v, v^*) \neq 0$; by replacing $v^*$ by $v^* + \lambda v$ for a suitable scalar $\lambda$ if necessary, we may assume that $v^* \in W \cap W'$ is also an isotropic vector. Write $V = (Ev + Ev^*) \oplus X$, an orthogonal decomposition. Consider the filtration

$$0 \subset Ev \subset E \oplus X \subset Ev \oplus X \oplus Ev^* = V.$$  

Denote the corresponding integral Heisenberg group (the unipotent subgroup of $U(V)$ which preserves the flag and acts trivially on successive quotients) by $H_V = H(X)(O_K)$. Similarly define the smaller Heisenberg groups $H_W = H(X \cap W)(O_K)$ and $H_{W'} = H(X \cap W')(O_K)$.

By assumption, $H_W \cap \Gamma$ has finite index in $H_W$; similarly for $H_{W'}$. The two Heisenberg groups generate $H_V$ up to finite index since two distinct vector subspaces of codimension one span the whole space. We thus find that the intersection of $\Gamma$ with the integral unipotent radical of a parabolic $K$ subgroup $H_V = H(X)(O_K)$ has finite index in $H_V$.

Similarly, we find a finite index subgroup of an opposite integral unipotent radical which lies in $\Gamma$; therefore, $\Gamma$ is arithmetic, by the Corollary 1 to Theorem 7. □

4. **Properties of the Burau representations $\rho_n$ and $\rho_n(d)$**

4.1. **Notation.** As observed in the introduction, the representation $\rho_n : B_{n+1} \to \text{GL}_n(\mathbb{Z}[q, q^{-1}])$ preserves the Hermitian form
The Burau Representation at $d$-th Roots of Unity

Let $h = h_n = \begin{pmatrix} \frac{(q+1)^2}{q} & -(1+q) & 0 & \cdots & \cdots \\ -(1+q^{-1}) & \frac{(q+1)^2}{q} & -(q+1) & \cdots & \cdots \\ 0 & -(1+q^{-1}) & \frac{(q+1)^2}{q} & -(q+1) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$

(Note that $h_{kk}$ is fixed under the involution $q \mapsto q^{-1}.$) Denote by $D_n$ the determinant of $h_n.$

**Lemma 13.** The determinant $D_n$ of the matrix $h_n$ is

$$D_n = \det(h) = \left(\frac{q+1}{q}\right)^n \left(\frac{q^{n+1}-1}{q-1}\right).$$

**Proof.** Expanding the determinant of $h_{n+1}$ using the first row, we see that

$$D_{n+1} = \frac{(q+1)^2}{q}D_n - (1+q)(1+q^{-1})D_{n-1}.$$ 

Now an easy induction implies the lemma. □

4.2. Nondegeneracy of the representation $(A^n, \rho_n(d)).$ Consider the ring $A = R/(\Phi_d(q)) \subset E = \mathbb{Q}[q]/(\Phi_d(q))$ ($q$ evaluated at all the primitive roots of unity). We then have a corresponding Hermitian form on $A^n$ which we again denote by $h_n = h_n(d).$ We will say that elements $v_1, \ldots, v_k$ are basis elements of $A^n$ if there exist vectors $v_{k+1}, \ldots, v_n$ in $A^n$ such that $A^n$ is the free module generated by $v_1, \ldots, v_n.$ We have a representation $\rho_n(d) : B_{n+1} \rightarrow U(h) \subset \text{GL}_n(A)$ as before.

**Lemma 14.** Let $A$ be as in the preceding. Assume that $d \geq 3,$ so that $q \neq \pm 1.$ Then

1. The Hermitian form $h_n$ on the module $A^n$ is nondegenerate if and only if $n$ is not congruent to $-1$ modulo $d.$
2. If $d$ divides $n+1,$ define $k$ by the formula $n = kd - 1.$ Then the vector

$$v = e_1 + \left(\frac{q^2-1}{q-1}\right) e_2 + \left(\frac{q^3-1}{q-1}\right) e_3 + \cdots + \left(\frac{q^{kd-1}-1}{q-1}\right) e_{kd-1}$$

is a basis vector and generates the null space of the degenerate Hermitian form $h$ in $V_A = A^n.$ Moreover, on the quotient module $V_A/Av,$ the form $h$ is nondegenerate.
3. The vector $v$ is fixed by the elements $s_j$ under the representation $\rho_n(d).$ We therefore get the “quotient” representation $\overline{\rho_n(d)}$ of $B_{n+1}$ on the quotient $V_A/Av$ which again preserves the (nondegenerate) Hermitian form $h$ on $V_A/Av.
Proof. Part (1) is obvious because of Lemma 13: the determinant of $h$ on $A^n$ vanishes if and only if $q^{n+1} - 1 = 0$, where now $q$ is a primitive $d$-th root of unity; therefore, this happens if and only if $n + 1$ is divisible by $d$.

In part (2), the orthogonality of $v$ with the vectors $e_j (j = 1, 2, \ldots, kd - 1)$ follows from an explicit computation; it is clear that $v, e_2, \ldots, e_{kd - 1}$ form a basis of $A^{kd - 1}$. The matrix of the Hermitian form $h_{kd - 1}$ on the quotient $A^{kd - 1}/Av$ with respect to the basis $e_2, \ldots, e_{kd - 1}$ is clearly $h_{kd - 2}$; this is nondegenerate by part (1). Thus part (2) follows.

Since $v$ is orthogonal to $e_k$, it follows that $s_k$ fixes $v$: for any $x \in A^n$, we have

$$s_k(x) = x - \frac{qh(x, e_k)}{q + 1}e_k,$$

as can be easily checked by evaluating both sides of this equality on the basis elements $e_l$. Therefore (3) follows. \qed

**Lemma 15.** Suppose $n$ is an integer not congruent to $-1$ modulo $d$, and as before let $\Gamma_n(d)$ be the image of $B_{n+1}$ under the representation $\rho_n(d)$ on $A^n$. If $0 \neq W \subset V_A = A^n$ is a subgroup which is stable under $\Gamma_n(d)$, then $W$ contains $\lambda(A^n)$ for some nonzero $\lambda \in A$; in particular, $W$ has finite index in $A^n$.

If $F \supset A$ is a field containing the integral domain $A$ and $W_F \subset F^n$ is a nonzero $F$-subspace stable under the action of $\rho_n(d)$, then $W_F = F^n$.

**Proof.** Since $n \not\equiv -1 \pmod{d}$, $h$ is nondegenerate; since $W \neq 0$, $W$ has a nonzero vector $x$, and there exists a $k$ such that $h(x, e_k) \neq 0$. The formula

$$s_k(x) = x - \frac{qh(x, e_k)}{q + 1}e_k$$

shows that a nonzero multiple $v_k$ of $e_k$ lies in $W$; applying $s_k^\pm 1$ to this multiple of $e_k$ shows that $(-q)^{\pm 1}v_k \in W$, where $q$ is a primitive root of unity; hence the $Z[q, q^{-1}]$-module generated by $v_k$ lies in $W$. Thus $Av_k \subset W$.

Applying the elements $s_k^{\pm 1}$ to $Av_k$ it follows that $W$ contains a nonzero multiple $v_l$ of $e_l$ for every $l \leq n$, and by the preceding paragraph, $Av_l \subset W$ for each $l$. Therefore the first part of the lemma follows.

The second part of the lemma is proved in the same way, by replacing $A$ by the field $F$. \qed

**Proposition 16.**

1. The representation $\rho_n(d)$ is irreducible unless $n \equiv -1 \pmod{d}$.
2. If $n \equiv -1 \pmod{d}$, then $\rho_n(d)$ contains a vector $v$ fixed under $B_{n+1}$ and the quotient $A^n/Av$ yields a representation $\rho_n(d)$ of $B_{n+1}$. Restricted to the subgroup of $B_{n+1}$ generated by $s_2, \ldots, s_n$, the representation $\rho_n(d)$ is equivalent to $\rho_{n-1}(d)$. Hence $\rho_n(d)$ is irreducible.
(3) Suppose \( n \) is not congruent to \(-1\) modulo \( d \). Consider the representation \( \rho_n(d) \) over an algebraically closed field \( \overline{E} \) containing the ring \( S = \mathbb{Z}[q + q^{-1}] \), and denote it \( \rho_n(\overline{E}) \). Then \( \rho_n(\overline{E}) \) is irreducible over \( \overline{E} \).

Proof. Part (1) of the proposition (the irreducibility) is the second part of Lemma 15. The second part is obvious. The third part again follows from the second part of Lemma 15: \( \rho_n(\overline{E}) \) is irreducible. \( \square \)

Let \( n \equiv -1 \pmod{d} \). Then the Hermitian space \( V = (A^n, h) \) contains the span \( Av \) of \( v \) as its orthogonal complement and \( \overline{V} = V/Av \) has a nondegenerate Hermitian form \( \overline{h} \) induced by \( h \). Let \( U(h) \) and \( U(\overline{h}) \) denote the unitary groups. We then have a split exact sequence of groups
\[
\{0\} \rightarrow \text{Hom}_A(\overline{V}, Av) \rightarrow U(h) \rightarrow U(\overline{V}) \rightarrow \{1\}.
\]

As before, \( \{0\} \) is the trivial group written additively and \( \{1\} \) is the trivial group written multiplicatively.

Here, the dual \( W \) of \( \overline{V} \) may be identified with \( \text{Hom}_A(\overline{V}, Av) \). Thus \( U(h) \) may be written as a semi-direct product \( U(h) = U(\overline{h})W \) with \( W \) normal in \( U(h) \) and the conjugation action of \( U(\overline{h}) \) on \( W \) is just the dual of the standard representation of \( U(\overline{h}) \).

4.3. A central element of the braid group. Consider the braid group \( B_{n+1} \) with generators \( s_1, s_2, \ldots, s_n \), consider the element
\[
\Delta = (s_1s_2 \cdots s_n)(s_1 \cdots s_{n-1}) \cdots (s_1s_2)(s_1)
\]
of \( B_{n+1} \). It is elementary to check that for every \( k \leq n \) we have: \( \Delta s_{n+1-k} = s_k \Delta \). Hence \( \Delta^2 \) is in the centre of \( B_{n+1} \).

If \( E \) is an algebraically closed field containing \( \mathbb{Z}[q + q^{-1}] \) and \( n \) is not congruent to \(-1\) mod \( d \), then by part (3) of Proposition 16, \( \rho_n(E) \) is irreducible; therefore, by Schur’s Lemma, the element \( \Delta^2 \) acts by a scalar \( \delta \), say. Since the determinant of each \( s_i \) is \(-q \) in the representation \( \rho_n(E) \), we see that the determinant of \( \Delta^2 \) is \((-q)^{n(n+1)} = \delta^n \). Therefore, \( \delta/(\delta q^{n+1}) = 1 \). On the other hand, the entries of \( \rho_n(E)(\Delta^2) \) are Laurent polynomials in \( q \) with integral coefficients. Hence the scalar \( \delta \) lies in the ring \( \mathbb{Z}[q, q^{-1}] \). Therefore, the above equation means that \( \rho_n(\Delta^2) = \delta = (\pm 1)q^{n+1} \).

Lemma 17. If \( d \geq 3 \) and \( n = kd - 2 \), then the element \( \Delta^2 \) acts by a scalar \( \lambda \neq 1 \) on the space \( A^n \) of the representation \( \rho_n(d) \).

Proof. By Proposition 16, the representation \( \rho_{kd-2} \) is irreducible. By the conclusion of the preceding paragraph, the central element \( \Delta^2 \) acts by a scalar \( \lambda = \pm q^{n+1} = \pm q^{kd-1} = \pm q^{-1} \) since \( q \) is now a \( d \)-th root of unity. This shows
that \( \lambda \neq 1 \) since, otherwise, we get
\[
q^{-2} = (\pm 1)^2 = 1.
\]
This is impossible since \( d \geq 3 \) and \( q \) is a primitive \( d \)-th root of unity. Hence \( \lambda \neq 1 \). \( \square \)

Remark 7. It can be shown, by examining the action of the Braid Group ([Bir74]) on the free group on \( n + 1 \) generators that the scalar in question is actually \( q^{n+1} \). (This is a special case of Proposition 27 of the present paper, which we do not prove.) In particular, if \( n = kd - 2 \) and \( q \) is a primitive \( d \)-th root of unity, then \( q^{n+1} = q^{-1} \neq 1 \), even when \( d = 2 \).

We have already mentioned in the introduction that the proof of Theorem 2 is by induction. We will prove Theorem 2 directly when \( n \geq 2d \) is congruent to \(-1 \) (mod \( d \)). Then we will prove that induction may be applied, which will prove Theorem 2 for all \( n \geq 2d \). To achieve this, we need to exhibit sufficiently many unipotent elements in the image of \( B_{n+1} \) under the representation \( \rho_n(d) \) for the values \( n \equiv -1 \) or \( 0 \) (mod \( d \)). This will be done in the next two subsections.

4.4. Constructing unipotent elements when \( n = kd - 1 \). Assume that \( n = kd - 1 \). Since \( \dim X = kd - 2 \), Lemma 17 and Proposition 16 imply that the square of the element
\[
\Delta' = (s_2s_3 \cdots s_n)(s_2s_3 \cdots s_{n-1})(\cdots)(s_2s_3)(s_2)
\]
lies in the subgroup generated by \( s_2, \ldots, s_n \) in \( B_{n+1} \) and acts by a nontrivial scalar \( \lambda \neq 1 \) on \( \rho_{kd-1}(d) \).

Denote the element \( [s_1, \Delta'^2] \) of \( B_{n+1} \) by \( u \). (As before, we denote by \( [g, h] \) the commutator \( ghg^{-1}h^{-1} \) of \( g \) and \( h \).) The image of this element \( u \) under the reducible representation \( \rho_{kd-1}(d) \) lies in the vector group \( A^n \), where \( U(h) = \text{Hom}_A(A^n, Av) \rtimes U(\overline{h}) \) is a semi-direct product as before.

Proposition 18. Let \( n = kd - 1 \). Under the representation \( \rho_n(d) \), the commutator element \( u = [s_1, (\Delta')^2] \) has the property that its image \( u' = \rho_{kd-1}(u) \) is a nontrivial unipotent element in the vector part \( A^{n-1} \) of \( U(h) \).

Proof. This follows from Lemma 9. Note that \( (A^n, h) \) is a degenerate Hermitian form with an isotropic vector \( v \) such that the quotient \( V/Av \) is nondegenerate; moreover, if \( X \) is the \( A \)-span of the vectors \( e_2, e_3, \ldots, e_n \), then \( A^n \) is the direct sum \( Av \oplus X \). Hence \( (V, h) \) satisfies the hypotheses of Lemma 9.

Since \( \text{Dim}(X) = kd-2 \), Lemma 17 implies that the element \( m = \rho_n((\Delta')^2) \) acts by a scalar \( \lambda \neq 1 \) on \( X \) and acts trivially on \( v \); the element \( p = \rho_n(s_1) \)
takes the element \(e_2 \in X\) into the element
\[
e_2 + e_1 = -q e_2 + v - \sum_{k=3}^{n} \left(\frac{q^k - 1}{q - 1}\right) e_k \notin X,
\]
which shows that \(p\) does not lie in \(M = U(X)\). Therefore, Lemma 9 applies, and \(u'\) is a nonzero vector in the vector part of \(U(h)\) and, in particular, is a nontrivial unipotent element.

Proposition 19. Suppose that \(n = kd - 1\). The subgroup \(U_n\) of \(B_{n+1}\) generated by the conjugates \(huh^{-1}\) where \(h\) runs through the elements of the group generated by \(s_2, s_3, \ldots, s_n\) has the property that under the representation \(\rho_n(d)\), it preserves the flag
\[
Av \subset Av + A e_2 + \cdots + A e_{n-1} \subset A^n
\]
and acts trivially on successive quotients of this flag.

Further, if \(U_0\) denotes the subgroup of \(U(h)\) which preserves the above flag and acts trivially on successive quotients, then the image of \(U_n\) is a subgroup \(N_0\) of finite index in the integral points \(U_0(\mathcal{O}_K)\).

Proof. We need only prove the last part since the image of \(u\) lies in the normal subgroup \(U_0\) and is preserved under conjugation by elements of \(U(h)\). The conjugation action of \(M\) on \(U_0\) becomes the action of \(M = U(X)\) on the dual \(X^* = \text{Hom}(X, Av)\). By Lemma 15, \(N_0\) is of finite index provided it contains a nonzero element of \(U_0\); but it was already shown (Proposition 18) that the image of \(u\) is nontrivial in \(U_0\). The proposition follows.

4.5. Constructing unipotents when \(n = kd\) with \(k \geq 1\). The space \(V = A^n\) may be written as a direct sum
\[
V = Av \oplus X \oplus Av^*, \quad v = e_1 + \sum_{k=2}^{n-1} x_k e_k, \quad v^* = e_n + \sum_{k=1}^{n-1} y_k e_{n-k}
\]
as in the preceding subsection. Then \(M = U(X)\) is the Levi part of the parabolic subgroup \(P\) of \(U(h)\) which preserves the flag
\[
0 \subset Av \subset Av \oplus X \subset V,
\]
and \(U\) is the subgroup of \(P\) which acts trivially on successive quotients of this flag.

Let \(\Delta'\) be as in the previous subsection; then \(m = \rho_n((\Delta')^2)\) lies in \(M\) since it acts trivially on \(v^*\) and \(v\) (\(v^*\) and \(v\) are orthogonal to all the vectors \(e_2, e_3, \ldots, e_{n-1}\), and hence they are fixed by \(s_2, \ldots, s_{n-1}\)). Therefore, \(v, v^*\) are fixed by \(\Delta'\), and \(\Delta'\) preserves the space \(X\). By Lemma 17, \((\Delta')^2\) acts by a scalar \(\lambda \neq 1\) on \(X\).
Proposition 20. Assume $n = kd$. Then the following hold:

1. The element $u = [s_1, (\Delta')^2]$ acts by a nontrivial unipotent element on $A^n$ under the representation $\rho_n(d)$. More precisely, the element $\rho_n(d)(u)$ lies in $U(O_K) \setminus Z$, i.e., preserves the flag $Av \subset Av \oplus X \subset V$, acts trivially on the successive quotients, and does not preserve the subspace $X$.

2. The group $N$ generated by the conjugates $huh^{-1}$ with $h \in \Gamma_{n-1}$ (the group generated by the elements $\rho_n(s_2), \ldots, \rho_n(s_{n-1})$) is a subgroup of finite index in $U(O_K)$; in particular, $\Gamma_n$ intersects the unipotent radical $U(O_K)$ in a subgroup of finite index.

Proof. The element $u$ clearly preserves the flag of the proposition, as was verified in the preceding proposition. Since $u$ acts by a unipotent element on $W$ and is a commutator, it follows that $u$ acts trivially on the one-dimensional space $V/W$, and hence $u$ acts unipotently on $V$.

By the preceding proposition, $u$ does not preserve the space $X$: it takes the basis element $e_2 \in X$ into a sum of $av$ and elements of $X$, with $a \neq 0$ a scalar. By Lemma 10, $u$ does not lie in the centre of $U$.

Let $B$ be the group generated in $U(O_K)$ by the conjugates $huh^{-1}$ with $h \in H$, where $H$ is the group generated by $s_2, \ldots, s_{n-1}$. Under the quotient map $U \to U_0$, $B$ maps onto $B_0$, and by the preceding proposition, $B_0$ has finite index in $U_0(O_K)$.

Now the proposition follows, by appealing to Lemma 11.

5. Proof of Theorem 2

5.1. Proof of Theorem 2. We prove the main theorem (Theorem 2) by induction on $n \geq 2d$; we will prove Theorem 2 directly for all $n$ which are multiples of $d$ and are at least $2d$. Then by induction, Theorem 2 follows.

5.1.1. Proof when $n = kd$, with $k \geq 2$. The representation $\rho_{n-1}$ is not irreducible. Let $Av \subset V_{n-1}$ be the subspace of invariants. The quotient $V_{n-1}/Av$ is an irreducible representation of the braid group $B(s_2, \ldots, s_{n-1})$. Hence the commutator element

$$u = [s_1, \Delta(s_2, \ldots, s_{n-1})^2]$$

is a nontrivial unipotent element under $\rho_{n-1}$ and lies in the vector group $\text{Hom}_A(V_{n-1}/Av, Av)$. Therefore, $u$ preserves the flag $0 \subset Av \subset V_{n-1} \subset V$ and acts unipotently on $V$ since $u$, being a commutator, has determinant one and is unipotent on $V_{n-1}$.

Therefore, by Proposition 20, the group generated by the elements $huh^{-1}$ with $h \in B(s_2, \ldots, s_{n-1})$ generate a subgroup commensurable with $U(O_K)$, where $U$ is the unipotent group preserving the flag $Av \subset V_{n-1} \subset V$ and acting...
trivially on successive quotients of the flag. We have therefore proved that \( \Gamma_n \supset U(O_K)^N \) for some integer \( N \).

Similarly, \( \Gamma_n \supset U^-(O_K)^N \) for some integer \( N \), where \( U^- \) is opposite to \( U \). Therefore, \( \Gamma_n \) is an arithmetic group since \( n \geq 2d \), and hence \( U(V_n) \) has \( K \)-rank at least two: the space \( V_{d-1} \) spanned by \( e_1, \ldots, e_{d-1} \) has an isotropic vector \( v \) by Lemma 14. The subspace \( V_{d-1}' \) spanned by \( e_{d+1}, \ldots, e_{2d-1} \) also has the same Hermitian form \( h_{d-1} \) and contains an isotropic vector \( v' \) by Lemma 14. Clearly, \( V_{d-1} \) and \( V_{d-1}' \) are mutually orthogonal since the indices \( j \) of the bases \( e_j \) differ by at least two. Thus we have produced two mutually orthogonal independent isotropic vectors, and hence the \( K \)-rank of the unitary group \( U(V_n) \) is at least two. Therefore, Theorem 2 follows for \( n = kd \geq 2d \) from Corollary 2.

5.1.2. Proof that Theorem 2 for \( n - 1 \) implies Theorem 2 for \( n \) when \( kd < n \leq kd+d-2 \). In this case, \( \rho_n \) and \( \rho_{n-1} \) are irreducible. Then \( V_n \) contains both the subspace \( V_{n-1} \) and the span \( W_{n-1} \) of \( e_2, e_3, \ldots, e_n \); moreover, both these subspaces \( V_{n-1} \) and \( W_{n-1} \) are nondegenerate under \( h \). The intersection \( V_{n-1} \cap W_{n-1} \) contains an isotropic vector since the intersection contains the span of \( e_2, \ldots, e_{d+1} \); recall that \( n \geq 2d \), hence \( n - 2 \geq d \).

By the induction assumption, there exists an integer \( N \) such that \( U(V_{n-1})^N \subset \Gamma_{n-1} \) and \( U(W_{n-1})^N \subset \Gamma_{n-1} \simeq (s_2, \ldots, s_n) \). It follows from Lemma 12 that \( \Gamma_n \) is arithmetic.

5.1.3. Proof that Theorem 2 for \( n - 1 \) implies Theorem 2 for \( n \) when \( n = kd - 1 \). By the previous subsection, \( \Gamma_{n-1} \) is arithmetic. Since \( n \) is congruent to \(-1 \pmod{d} \), the Hermitian form \( h_{n-1} \) is degenerate. Then, Proposition 19 implies that \( \Gamma_n \) intersects the vector part in an arithmetic group.

Since the Hermitian form is degenerate, the unitary group \( U(V_n) \) of \( V_n \) is a semi-direct product of its reductive part \( U(V_{n-1}) \) and its unipotent part \( \text{Hom}(V_{n-1}, Av) \). Then, the decomposition

\[
U(V_n)(O_K) = U(V_{n-1})(O_K) \text{Hom}_A(V_{n-1}, Av)
\]

shows that \( \Gamma_n \) is also arithmetic.

Combining the above subsections together, we obtain a proof of our main result, namely Theorem 2, for all \( n \geq 2d \).

6. Proof of Theorems 3 and 4

6.1. Proof of Theorem 3. In this subsection we prove that if \( d \) is one of the numbers 3, 4, 6, then for every integer \( n \geq 1 \), the image of the Burau representation \( \rho_n(d) \) is an arithmetic group. First note that in these cases \( d = 3, 4, 6 \), the \( d \)-th cyclotomic extension \( E_d = \mathbb{Q}[q]/(\Phi_d(q)) \simeq \mathbb{Q}(e^{2\pi i/d}) \) is an imaginary quadratic extension of \( \mathbb{Q} \), and the totally real sub-field \( K_d = \mathbb{Q} \); the ring of integers \( O_d \) is the ring \( \mathbb{Z} \) of rational integers. Therefore, we have \( \Gamma \subset U(h)(\mathbb{Z}) \) and the ambient Lie group is \( U(h)(\mathbb{R}) \) since \( K \otimes_\mathbb{Q} \mathbb{R} = \mathbb{R} \). Note
that \(U(h)(\mathbb{R})\) is of the form \(U(r, s)\) and there is only one factor involved, since \(\mathbb{Q}\) has only one archimedean completion, namely \(\mathbb{R}\).

We divide the proof into three cases.

**Case 1:** \(n \geq 2d\). In this case, this is exactly Theorem 2, and this was already proved.

**Case 2:** \(n \leq d - 2\) or \(d \leq n \leq 2d - 1\). We refer to Table 9 on page 32 of [McM13]. In these cases, the group \(SU(h)(\mathbb{R})\) is either compact or is isomorphic to \(U(n - 1, 1)\). If \(U(h)\) is compact, then \(\Gamma\) is finite and is hence “arithmetic”; this is the trivial case. If \(U(h) = U(n - 1, 1)\), then by Theorem (10.3) in [McM13] (which is actually a special case of a result of Deligne-Mostow), the image \(\Gamma = \rho_n(d)(B_{n+1})\) is a lattice, i.e., \(\Gamma\) has finite index in \(U(h)(\mathbb{Z})\).

**Case 3:** \(n = d - 1\) or \(n = 2d - 1\). Since \(n\) is congruent to \(-1\) modulo \(d\), Part (2) of Proposition 16 tells us that \(\rho_n(d)\) contains a trivial sub-representation and that the quotient is isomorphic to \(\rho_{n-1}(d)\); moreover, the unitary group \(U(h) = U(h_n)\) has a unipotent radical \(U_0\) and a smaller group \(U(h_{n-1})\) as a Levi supplement. By the preceding paragraph, the image of \(\Gamma\) in \(U(h_{n-1})(\mathbb{Z})\) is arithmetic; moreover, by Proposition 19 (note that \(n\) is of the form \(kd - 1\) with \(k = 1\) or 2), the image of a subgroup \(U_n \subset B_{n+1}\) under the representation \(\rho_n(d)\) is a subgroup of finite index in the integral unipotent radical \(U_0(\mathbb{Z})\). Hence \(\Gamma\) has finite index in the integral points of the semi-direct product group \(U(h)\).

6.2. **Proof of Theorem 4.** We will now prove Theorem 4. Let \(a\) (resp. \(a_e\)) denote the principal ideal in \(R = \mathbb{Z}[q, q^{-1}]\) generated by the polynomial \((q^d - 1)\) (resp. by the \(e\)-th cyclotomic polynomial \(\Phi_e(q)\)). Let \(A = R/a\). By the Chinese remainder theorem, the \(\mathbb{Q}\)-algebra \(A \otimes \mathbb{Q}\) is the product ring \(A \otimes \mathbb{Q} = \prod_{e|d} \mathbb{Q}[q, q^{-1}]/(a_e \otimes \mathbb{Q}) \simeq \prod E_e\) where the product runs over all the divisors of \(d\). Here \(E_e\) is the \(e\)-th cyclotomic extension. Consequently, the reduced Burau representation \(\rho_n(A)\) on the rational vector space \(A^n \otimes \mathbb{Q}\) is the direct sum \(\oplus_{e|d} \rho_n(e)\). Hence the image \(\Gamma\) of the braid group under the Burau representation \(\rho_n(A)\) lies in the product \(\prod U(h)(O_e)\), where \(O_e\) is the ring of integers in the totally real sub-field \(K_e = \mathbb{Q}(\cos \frac{2\pi}{d})\), as in the introduction.

(1) We first prove Theorem 4 in the case that \(d\) and \(n + 1\) are coprime. In that case, every divisor \(e\) of \(d\) is coprime to \(n + 1\). By Proposition 16, the Hermitian form \(h\) on \(E_e^n\) is nondegenerate and the representation \(\rho_n(e)\) is irreducible. Since \(n \geq 2d\), Theorem 2 implies that the image of \(\Gamma\) under \(\rho_n(e)\) is a subgroup of finite index in \(G_e(O_e)\), where \(G_e = U(h)\) is the unitary group of \(h\) with respect to Hermitian form \(h\) corresponding to the quadratic extension \(E_e/K_e\), provided \(e \geq 3\). Moreover, there exists a subgroup, namely \(SU(h)(O_e)\), of finite index in \(G_e(O_e)\) which is an arithmetic subgroup of a higher rank semi-simple Lie group, namely \(SU(h)(K_e \otimes \mathbb{Q})\).
If $e = 2$, then $E_e = \mathbb{Q}[q]/(\Phi_2(q)) = \mathbb{Q}$ and the image of $q$ in the field $E_e = \mathbb{Q}$ is $-1$. The form $h$ vanishes identically since $q+1$ divides all entries; we replace this zero form by first dividing $h$ by $q+1$ for a variable $q$ and then taking the resulting form evaluated at $q = -1$. This “divided” form is symplectic. Therefore, $G_e$ is the symplectic group; then by the result of A’Campo [A’C87], the image of $\Gamma$ in $G_e(O_e) = \text{Sp}(h, \mathbb{Z})$ is a higher rank arithmetic group.

If $e = 1$, then the Burau representation is evaluated at 1; i.e., the representation of the braid group $B_{n+1}$ lies in the symmetric group $S_{n+1}$ which is a finite group and may be ignored in questions on arithmeticity. Lemma 8 now implies that the image $\Gamma$ in $U(h)(A) = \prod_{e \geq 2} U(h)(O_e)$ is an arithmetic group in the product, under the assumption that $n \geq 2d$.

(2) Suppose $n + 1$ and $d$ are not coprime, and let $r \geq 2$ be the greatest common denominator of $d$ and $n + 1$. If a divisor $e$ of $d$ does not divide $r$, then it does not divide $n + 1$ either, and hence $U(h)(O_e)$ is an arithmetic group in a higher rank semi-simple group (up to finite index) and the projection to the $e$-th factor of the group $\Gamma$ has finite index in $U(h)(O_e)$.

If $e$ does divide $n + 1$, then $U(h)$ as an algebraic group over $K_e$ is not reductive; suppose $V_e$ is the unipotent radical of $U(h)$ viewed as a group over $E_e$. By Theorem 2 applied to this case, the projection of $\Gamma$ to the $e$-th factor contains a subgroup of finite index in $V_e(O_e)$. (See the proof of Theorem 2, the subsection where $n + 1$ is divisible by $e$.)

Putting the above cases together, and using Lemma 8, we get Theorem 4 in all cases.

7. Relation with a cyclic covering of $\mathbb{P}^1$ and proof of Theorem 1

In this section, we relate the monodromy representation of the braid group $B_{n+1}$ considered in Theorem 1 to the Burau representation and use this relation to prove Theorem 1.

7.1. Generalities. Suppose

$$\{1\} \to N \to F \to Q \to \{1\}$$

is an exact sequence of abstract groups. Suppose $B$ is a subgroup of the group of automorphisms of $F$ which stabilises the kernel $N$. Then $B$ acts on the quotient $Q$ also by automorphism and acts on the abelianisation $N^{ab} = N/[N,N]$; hence $B$ acts on the exact sequence

$$1 \to N/[N,N] \to F/[N,N] \to Q \to \{1\},$$

whose kernel is abelian. Moreover, the conjugation action of $F$ on the abelianisation $N/[N,N]$ is trivial on $N$ and descends to an action of the quotient $Q$ on $N^{ab}$. If $N^{ab}$ is written additively, then we have a $\mathbb{Z}[Q]$ module structure on $N^{ab}$. 
If we now assume that $B$ acts trivially on the quotient $Q$, we then have an action of the product group $B \times Q$ on the abelianisation $N^{ab}$. Therefore, the action of $B$ on $N^{ab}$ is by $\mathbb{Z}[Q]$ module endomorphisms.

7.2. Action of the braid group on the free group. Suppose $F_{n+1}$ is a free group on $n+1$ generators, $x_1, x_2, \ldots, x_{n+1}$. Recall that the braid group on $n+1$ strands was given by generators $s_i$ and relations given in the introduction. A theorem of E. Artin ([Bir74, Cor. 1.8.3]) says that the braid group $B_{n+1}$ acts on $F_{n+1}$ as follows. If $j \leq i - 1$ or if $j \geq i + 2$, then $s_i(x_j) = x_j$. If $j = i, i+1$, then the action is

$$s_i(x_i) = x_{i+1}, \quad s_i(x_{i+1}) = x_{i+1}^{-1}x_ix_{i+1}.$$  

(In [Bir74, Cor. 1.8.3], the action is on the right; to get the left action, one can make a slight modification of the formulae).

The action of $B_{n+1}$ on $F_{n+1}$ gives an action of the braid group on the abelianisation $F_{n+1}^{ab} = \mathbb{Z}^{n+1}$ of the free group; the images of $x_i$ form the standard basis of $\mathbb{Z}^{n+1}$. From the equations in the preceding paragraph, it is clear that the element $s_i$ acts by the permutation matrix interchanging $i$ and $i+1$ and fixing the rest of the basis. It is also clear that the image of $B_{n+1}$ in the automorphisms of $\mathbb{Z}^{n+1}$ is the permutation group $S_{n+1}$ on $n+1$ symbols.

The kernel, denoted $P_{n+1}$, of the map $B_{n+1} \rightarrow S_{n+1}$ is called the pure braid group.

7.3. Realisation of the Burau representation on homology. Write $G = \mathbb{Z}^{n+1}$ for the abelianisation of the free group $F_{n+1}$, where $G$ is written multiplicatively. We have a map $G \rightarrow \mathbb{Z} \simeq q^\mathbb{Z}$ given by

$$x_1^{m_1}x_2^{m_2}\cdots x_{n+1}^{m_{n+1}} \mapsto q^{-(m_1+m_2+\cdots+m_{n+1})}.$$

The group $S_{n+1}$ (and hence the braid group $B_{n+1}$) acts on $\mathbb{Z}^{n+1}$ by permutations on the standard basis and acts trivially on $q^\mathbb{Z}$. The above map is equivariant with respect to this action. Moreover, the braid group $B_{n+1}$ acts on $F_{n+1}$ and the map $F_{n+1} \rightarrow \mathbb{Z}^{n+1}$ is equivariant with respect to this action. We now have an exact sequence

$$1 \rightarrow K_{n+1} \rightarrow F_{n+1} \rightarrow q^\mathbb{Z} \rightarrow 1$$

with kernel $K_{n+1}$ stable under the action of the braid group $B_{n+1}$.

The group $F_{n+1}$ has the standard generators $x_1, x_2, \ldots, x_{n+1}$. As a normal subgroup of $F_{n+1}$, the group $K_{n+1}$ is generated by the elements

$$y_1 = x_1^{-1}x_2, \quad y_2 = x_2^{-1}x_3, \ldots, y_n = x_n^{-1}x_{n+1}.$$  

As was observed in the preceding paragraph, $B_{n+1}$ acts on $K_{n+1}$; we compute its action on the “standard basis” $y_1, y_2, \ldots, y_n$ of $K_{n+1}$. The braid group is
generated by $s_i$, and the action of $s_i$ on $F_{n+1}$ was described before. Hence, we have $s_i(y_{i-1}) = x_{i-1}^{-1}x_i y_i = y_i - y_{i-1}$,

$$s_i(y_j) = s_i(x_{j-1}^{-1}x_j) = y_j \ (j \leq i - 2 \text{ or } j \geq i + 2),$$

$$s_i(y_i) = x_{i+1}^{-1}x_i x_{i+1}^{-1} = x_{i+1}^{-1}y_i = y_i^{-1} = x_{i+1}^{-1} (y_i^{-1}),$$

and

$$s_i(y_{i+1}) = x_{i+1}^{-1}x_i x_{i+1}^{-1}x_{i+1}^{-1}x_{i+1}^{-1} = x_{i+1}^{-1} (y_i) y_{i+1}.$$ We now consider the commutator subgroup $K_{n+1}^{(1)}$ of $K_{n+1}$; it is a normal subgroup of $F_{n+1}$ and is stabilised by the action of the braid group. Therefore, we have an exact sequence of groups

$$1 \rightarrow M_n = K_{n+1}^{(1)} / K_{n+1}^{(1)} \rightarrow F_{n+1} / K_{n+1}^{(1)} \rightarrow q^Z \rightarrow 1,$$

with abelian kernel $M_n$ (namely the abelianisation of $K_{n+1}$) written additively. Since $K_{n+1}$ is generated by the $y_i$ as a normal subgroup of $F_{n+1}$ and the conjugation action of $F_{n+1}$ on the abelian group $M_n$ descends to the action of $q^Z$, it follows that $M_n$ is generated as a $q^Z$-module, by the images of the quotient map $K_{n+1} \rightarrow M_n$ of the $y_i$. Since the group law on $M_n$ is written additively, for each $x_i$, the conjugation action on $M_n$ is simply multiplication by the element $q^{-1}$. We are in the situation of Section 7.1 with $N^{ab} = M_n$, $Q = q^Z$ and $B = B_{n+1}$.

The action of the braid group on the basis $y_i$ of the group $K_{n+1}$ was computed above; this gives a description of the action of $s_i$ on the basis $y_i'$ of $M_n$ as follows. We have the formulae

$$s_i(y_j') = y_j' \ (j \leq i - 2 \text{ or } j \geq i + 2), \quad s_i(y_{i-1}') = y_{i-1}'+ y_i',$$

$$s_i(y_i') = -q y_i', \quad s_i(y_{i+1}') = y_{i+1}'+ q y_i'.$$

Now write $y_i = q^i e_i$. In the basis $\{e_i\}$, we get

$$s_i(e_j) = e_j \ (| j - i | > 2), \quad s_i(e_{i-1}) = e_{i-1} + q e_i,$$

$$s_i(e_i) = -q e_i, \quad s_i(e_{i+1}) = e_{i+1} + e_i,$$

which is exactly the reduced Burau representation defined in the introduction. We therefore have

**Theorem 21** (Burau). Let $K_{n+1}$ be the kernel of the map $F_{n+1} \rightarrow q^Z$ defined above. Then, the action of the braid group $B_{n+1}$ on the first homology of $K_{n+1}$ with integral coefficients is isomorphic to the reduced Burau representation.
7.4. Realisation of the Burau representation at d-th roots of unity. Consider now the quotient map \( q : \mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z} \). We have a surjective map \( F_{n+1} \to \mathbb{Z}/d\mathbb{Z} \). This defines an exact sequence

\[
1 \to K_{n+1}(d) \to F_{n+1} \to \mathbb{Z}/d\mathbb{Z} \to 1,
\]

of groups, with \( \mathbb{Z}/d\mathbb{Z} \) written multiplicatively. Clearly, \( K_{n+1}(d) \) is generated (as a normal subgroup of \( F_{n+1} \)) by the elements \( y_1, y_2, \ldots, y_n \) and \( x_1^d \) where, as before, \( y_i = x_i^{-1} x_{i+1} \). Therefore, \( K_{n+1}(d) \) is generated by the elements \( x_1^d \) and the collection \( \{x_1^j y_i x_1^{-j} : 0 \leq j \leq d-1, \ i \leq n\} \).

Being a subgroup of \( F_{n+1} \), \( K_{n+1}(d) \) is also free and if \( n' \) denotes the minimal number of generators of this free group, we have the formula

\[
1 - n' = d(1 - (n + 1)) = -dn, \ n' = 1 + dn.
\]

Since the cardinality of the system of generators we have exhibited is exactly \( 1 + nd \), it follows that the above generators \( x_1^d \) and \( x_1^j y_i x_1^{-j} \) freely generate \( K_{n+1}(d) \).

Therefore (Section 7.1), the abelianisation \( K_{n+1}(d)_{\text{ab}} \) is a direct sum of a free module over the ring \( \mathbb{Z}[G] = \mathbb{Z}[q]/(q^d - 1) = A \) with generators \( y_1', y_2', \ldots, y_n' \) and the trivial module \( \mathbb{Z}(x_1^d)' \). Here the prime denotes the image of the element under the quotient map \( K_{n+1}(d) \to K_{n+1}(d)_{\text{ab}} \). The map \( K_{n+1} \to K_{n+1}(d)_{\text{ab}} \) is equivariant for the action of the braid group. We have an exact sequence of \( B_{n+1} \) modules:

\[
0 \to \text{Image}(K_{n+1}^{\text{ab}}) \to K_{n+1}(d)_{\text{ab}} \to \mathbb{Z} \to 0.
\]

Therefore, it follows from Theorem 21 that the braid group acts on the abelianisation \( K_{n+1}(d)_{\text{ab}} \) and that the latter is an extension of the reduced Burau representation \( \rho_n(A) \) by the trivial representation. It follows from Theorem 4 that the image of the representation \( B_{n+1} \to \text{GL}(K_{n+1}(d)_{\text{ab}}) \) is an arithmetic group.

7.5. Some cyclic coverings of \( \mathbb{P}^1 \). Let \( a_1, a_2, \ldots, a_{n+1} \) be distinct complex numbers; write \( S_a \) for the complement in \( \mathbb{C} \) of these points: \( S_a = \mathbb{C} \setminus \{a_1, a_2, \ldots, a_{n+1}\} \). The fundamental group of \( S_a \), once a base point is chosen, may be identified with the free group on \( F_{n+1} \) generated by small circles \( x_i \) going around the point \( a_i \) counterclockwise once (and joined to the preferred base point by an arc which avoids all the other points \( a_j \) and has zero winding number around all the points \( a_j \) with \( j \neq i \)). The map \( S_a \to \mathbb{C}^* \) defined by

\[
x \mapsto (x - a_1)(x - a_2)\cdots(x - a_{n+1}) = P_a(x)
\]

induces a homomorphism \( F_{n+1} \to \mathbb{Z}/q^1 \), which sends each \( x_i \) to \( q^{-1} \). Here, \( q^{-1} \) is a small circle around zero in \( \mathbb{C}^* \) which runs counterclockwise exactly once.
For future reference, note that the loop around infinity lying in \( S_a \) represents the product element \( x_1 x_2 \cdots x_{n+1} \) and that this element is invariant under the action of the braid group \( B_{n+1} \) on the free group \( F_{n+1} \).

The affine variety \( \mathbb{C}^* = \mathbb{G}_m \) admits a cyclic covering of order \( d \) given by \( z \mapsto z^d \) from \( \mathbb{G}_m \) to \( \mathbb{G}_m \). The covering may be realised as the space \( \{(x, y) \in \mathbb{C}^* \times \mathbb{C}^* : y^d = x \} \) where the covering map is the first projection. Pulling back this covering to \( S_a \) we get a cyclic covering of \( S_a \), realised as the space \( \{ (x, y) \in \mathbb{C}^* \times S_a : y^d = (x - a_1)(x - a_2) \cdots (x - a_{n+1}) \} \), with the first projection being the covering map from \( X_a \) onto \( S_a \). Therefore, under the identification of the fundamental group of \( S_a \) with \( F_{n+1} \), the fundamental group of \( X_a \) is identified with \( K_{n+1}(d) \).

As the collection \( a \) varies, we get a collection \( \mathcal{P} \) of monic polynomials \( P_a \) of degree \( n + 1 \) which have distinct roots, and if \( Q \) denotes the variety \( \{(w, x, P) \in \mathbb{C}^* \times \mathbb{C} \times \mathcal{P} : w = P(x)\} \), then the projection on to the third coordinate gives a fibration over \( \mathcal{P} \) with fibre at \( P \) being \( S_a \). (Here \( a \) is the collection of roots of \( P \).) The fibration over \( \mathcal{P} \) has a continuous section (as can be easily seen) and hence the fundamental group of \( \mathcal{P} \) may be thought of as a subgroup of the total space \( Q \) of the fibration. We therefore get a monodromy action of the fundamental group of \( \mathcal{P} \) on the fundamental group \( F_{n+1} \) of the fibre. We have the following fundamental theorem of E. Artin ([Bir74, 1.8]):

**Theorem 22** (Artin). The fundamental group of \( \mathcal{P} \) is the braid group \( B_{n+1} \), and the monodromy action on \( F_{n+1} \) is the action of \( B_{n+1} \) on \( F_{n+1} \) defined in Section 7.2.

Consequently, the monodromy action on the fibre of the fibration \( \{(y, x, P) \in \mathbb{C}^* \times \mathbb{C} \times \mathcal{P} : y^d = P(x)\} \) over \( \mathcal{P} \) is the action of \( B_{n+1} \) defined in Section 7.2 restricted to the subgroup \( K_{n+1}(d) \simeq \pi_1(X_a) \); therefore, \( B_{n+1} \) acts on the first homology of \( X_a \): \( H_1(X_a) \simeq K_{n+1}(d)^{ab} \) by (an extension by the trivial representation of) the Burau representation \( \rho_n(A) \), and this gives the monodromy action of \( B_{n+1} \) on \( H_1(X_a) \), with image \( \Gamma' \), say.

**Theorem 23.** If \( n \geq 2d \), then the image of the representation \( B_{n+1} \rightarrow \text{GL}(H_1(X_a)) \) is an arithmetic group.

**Proof.** We have identified this representation with the extension by the trivial representation of the Burau representation \( \rho_n(A) \), where

\[
A = \mathbb{Z}[q, q^{-1}] / (q^d - 1).
\]

The theorem follows from the conclusion of the preceding Section 7.4. \( \square \)
7.6. The compactification of $X_a$. Now $X_a$ is a compact Riemann surface with finitely many punctures; denote by $X_a^*$ the smooth projective curve obtained by filling in these punctures.

The covering map $X_a \to S_a$ is such that these punctures lie over the points $a_i$ or else over the point at infinity of $S_a$. If a puncture lies over some $a_i$, then the image of a small loop around the puncture in $F_{n+1}$ is $x_i^d$; if the puncture lies above infinity, then the image of a small loop around the puncture in $F_{n+1}$ is a power of the element $x_1x_2\cdots x_{n+1}$ (represented by the loop around infinity); therefore, such an element is invariant under the action of the braid group.

The mapping of $\pi_1(X_a) \to \pi_1(X_a^*)$ is such that these loops around the punctures generate the kernel. (This is an easy consequence of the van Kampen theorem.) Note that the element $s_j$ of the braid group $B_{n+1}$ (under the action on the free group $F_{n+1}$ defined in Section 7.2) takes the loop $x_i$ to a conjugate of the loop $x_k$ for some $k$. Therefore, the braid group leaves the kernel of the map $\pi_1(X_a) \to \pi_1(X_a^*)$ stable. Hence, by Artin’s theorem (Theorem 22), the induced map $H_1(X_a) \to H_1(X_a^*)$ on homologies is equivariant under the monodromy action of the braid group.

7.7. Proof of Theorem 1. We can now prove Theorem 1. We are to prove that the image of the representation $B_{n+1} \to \text{GL}(H^1(X_a^*))$ is arithmetic where $H^1(X_a^*)$ is the cohomology with integer coefficients. The $B_{n+1}$ module $H_1(X_a^*)$ (homology of $X_a^*$ with integral coefficients) is a quotient of the module $H_1(X_a)$. By Theorem 23, the image of the braid group in $\text{GL}(H_1(X_a))$ is arithmetic. By Proposition 6, the image of the braid group in $\text{GL}(H_1(X_a^*))$ is also arithmetic. By Poincaré duality, the image of the braid group in $\text{GL}(H^1(X_a))$ is also arithmetic, proving Theorem 1.

7.8. The representation $H_1(X_a^*, \mathbb{Q})$. Denote by $A$ and $A'$ respectively the $\mathbb{Q}$-algebras $\mathbb{Q}[q]/(q^d - 1)$ and $\mathbb{Q}[q]/(1 + q + \cdots + q^{d-1})$. If $M$ is an $A$ module, denote by $(1 + q + \cdots + q^{d-1})M$ the subspace of elements of the form $(1 + q + \cdots + q^{d-1})m$ with $m \in M$, and let $M'$ be the quotient module $M/(1 + q + \cdots + q^{d-1})M$.

We have seen that $K_{n+1}(d)^{ab}$ is an extension of the image of $K_{n+1}^{ab}$ in $K_{n+1}(d)^{ab}$, by the trivial module $\mathbb{Z}$; tensoring with $\mathbb{Q}$, we have the same statement, with $\mathbb{Z}$ replaced by $\mathbb{Q}$. Clearly, $\mathbb{Q}' = 0$. Therefore, we have the equality of the “primed” modules

$$K_{n+1}^{ab} \otimes \mathbb{Q} \simeq \text{Im}(K_{n+1}^{ab} \otimes \mathbb{Q})'.$$

We have seen in the preceding subsection that the “primed” representation $H_1(X_a, \mathbb{Q})'$ of the braid group $B_{n+1}$ is isomorphic to the Burau representation $\rho_n(A')$ on $A'^n$, where $A' = \mathbb{Q}[q, q^{-1}]/(1 + q + \cdots + q^{d-1})$. Therefore, by Section 6.2, the Burau representation $\rho_n(A')$ on the $\mathbb{Q}$-vector space $(A')^n$ is the
direct sum
\[ H_1(X_a, \mathbb{Q})' \simeq \rho_n(A') = \bigoplus_{e \mid d} e \geq 2 \rho_n(e). \]
Recall from Proposition 16 that if \( n \equiv -1 \pmod{e} \), then the representation \( \rho_n(e) \) contains a one-dimensional space \( L_e \), say, of invariants, and that the quotient \( (A_e^n \otimes \mathbb{Q})/L_e \) is irreducible; in Proposition 16, this representation was denoted \( \tilde{\rho}_n(e) \). By an abuse of notation, if \( e \) does not divide \( n + 1 \), we denote \( \rho_n(e) \) also by \( \tilde{\rho}_n(e) \).

The embedding of the affine curve \( X_a \) into its compactification \( X_a^* \) induces a map \( H_1(X_a, \mathbb{Q}) \to H_1(X_a^*, \mathbb{Q}) \) on rational homology, which is equivariant for the action of the braid group. By analysing the map \( H_1(X_a) \to H_1(X_a^*) \), one can show that (compare Theorem 5.5 of [McM13, pp. 24–25], where the case \( e = d \) is treated) the monodromy representation \( H_1(X_a^*, \mathbb{Q}) \) has the decomposition
\[ H_1(X_a^*, \mathbb{Q}) \simeq \bigoplus_{e \mid d, e \geq 2} \tilde{\rho}_n(e). \]

We use the fact that loops around points in \( X_a \) which lie above \( \infty \) in the curve \( S_a \subset \mathbb{P}^1 \) lie in the kernel of the map \( H_1(X_a) \to H_1(X_a^*) \); one can show that the invariant vector in \( \rho_n(e) \) (for \( e \) dividing \( n + 1 \)) is generated by these “infinity” loops and hence lies in the kernel of the map on homology.

The following proposition is an immediate consequence of the decomposition of the representation of \( B_{n+1} \) on \( H_1(X_a^*) \) and Lemma 8.

**Proposition 24.** If \( n \geq 2d \), then the image of the monodromy representation of \( B_{n+1} \) on \( H_1(X_a^*) \) of Theorem 1 is a subgroup of finite index in the product \( \prod \mathcal{G}_e(O_e) \), where the product is over all the divisors \( e \geq 2 \) of \( d \) and \( \mathcal{G}_e \) is the unitary group of the Hermitian form \( \mathcal{H}_n \) induced by \( h = h_n \) on the quotient representation \( \tilde{\rho}_n(e) \) of the Burau representation \( \rho_n(e) \).

### 8. Applications

8.1. Some complex reflection groups. We will follow the notation of Section 5 of [McM13]. In Section 5 of [McM13], given the root system \( A_n \) (and therefore its graph), the Artin group \( A(A_n) \) is defined; given a complex number \( q = e^{2\pi i x} \) with \(-1/2 \leq x < 1/2\), there exists a representation
\[ \rho_q : A(A_n) \to \text{GL}_n(\mathbb{C}), \]
with image denoted \( A_n(q) \). The image preserves a Hermitian form and is a subgroup of a unitary group \( U(r, s) \subset \text{GL}_n(\mathbb{C}) \). The image of the braid group \( B_{n+1} \) under the Burau representation \( \rho_n : B_{n+1} \to \text{GL}_n(\mathbb{Z}[q, q^{-1}]) \) may be identified with the complex reflection group \( A_n(q) \). Question 5.6 of [McM13] asks when the image \( A_n(q) \) is a lattice in \( U(r, s) \). In the notation of Question 5.6 of [McM13], the image group \( \Gamma_n = \rho_n(d)(B_{n+1}) \) is the group \( A_n(q) \) where \( q \)
is a primitive \(d\)-th root of unity; therefore, question 5.6 asks whether \(A_n(q)\) can be a lattice in the real unitary group \(U(r, s)\); Theorem 2 answers this question in a large number of cases. We have the following corollary of Theorem 2 (and part (1) of the corollary follows from Theorem 3).

**Corollary 3.** (1) If \(q = e^{2\pi i/d}\), then \(A_n(q)\) is a lattice in \(U(r, s)\) when \(d = 3, 4, 6\) for all \(n\). If \(d = 2\), then \(A_n(q)\) is a lattice in \(Sp_n(\mathbb{Z})\).

(2) If \(d\) is not 2, 3, 4, 6 and \(n \geq 2d\), then the image \(\Gamma_n\) under the Bura representation is an irreducible lattice in the product of unitary groups \(U(h)(K_d \otimes \mathbb{Q} \mathbb{R}) \simeq \prod_{v} U(r_v, s_v)\), where the product is over all the archimedean (real) completions \(K_{d,v}\) of the totally real field \(K_d\) and the number of factors is at least two. Therefore, the projection of \(\Gamma_n\) to one of the factors is never a lattice if \(d \neq 3, 4, 6\).

In particular, the intersection \(A_n(q) \cap SU(r_v, s_v)\) is dense in \(SU(r_v, s_v)\) for each archimedean \(v\).

Thus Question 5.6 of [McM13] is open only if \(d \neq 3, 4, 6\) and if \(n \leq 2d - 1\).

8.2. Application to hypergeometric monodromy of type \(nF_{n-1}\).

**Definition 3.** Let \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n\) and \(\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n\) be complex numbers such that \(\alpha_j \neq \beta_k \pmod{1}\) for any \(j\) and \(k\). Denote by \(z\) a complex variable; write \(\theta = z\frac{d}{dz}\). Consider the differential operator \(D = D(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)\) given by

\[
D = (\theta + \beta_1 - 1)(\theta + \beta_2 - 1) \cdots (\theta + \beta_n - 1) - z(\theta + \alpha_1) \cdots (\theta + \alpha_n).
\]

The equation

\[
Du = 0
\]

is called the hypergeometric equation with parameters \(\alpha, \beta\).

The differential operator \(D\) is of the form

\[
D = a_n(z)\frac{d^n}{dz^n} + a_{n-1}(z)\frac{d^{n-1}}{dz^{n-1}} + \cdots + a_0(z),
\]

where \(a_i\) are polynomials and \(a_n(z) = z^n(1 - z)\) is the highest coefficient.

This coefficient \(a_n(z)\) vanishes at 0 and 1 (and also at infinity if we change coordinates from \(z\) to \(z^{-1}\)) but does not vanish anywhere in \(C = \mathbb{P}^1 \setminus \{0, 1, \infty\}\).

Using this property of the highest coefficient, it can be shown that the space of solutions is of dimension \(n\) and that the solutions are (locally) analytic on the Riemann surface \(C\). Denote by \(\pi_1(C, \frac{1}{2})\) the fundamental group of \(C\) based at the point \(\frac{1}{2}\). Then it acts on the space \((\simeq \mathbb{C}^n)\) of solutions \(u\) of the foregoing differential equation \(Du = 0\). (Here \(u\) is analytic on \(C\).) Denote by \(\Gamma = \Gamma(\alpha, \beta)\) the image of the fundamental group \(\pi_1(C, \frac{1}{2})\) in the group \(GL_n(\mathbb{C})\) of linear automorphisms of the \(n\)-dimensional space of solutions. Denote by \(M_{\alpha, \beta}\) the resulting representation of \(\pi_1(C, \frac{1}{2})\).
If $\alpha, \beta \in \mathbb{Q}^n$, then $\Gamma$ may be conjugated into $\text{GL}_n(O)$ where $O$ is the ring of integers in a number field $F$. If $D = [F : \mathbb{Q}]$ is the degree of $F$ over $\mathbb{Q}$, then $\Gamma \subset \text{GL}_{nD}(\mathbb{Z})$. In [Sar12] the following question is considered: determine the pairs $\alpha, \beta \in \mathbb{Q}^n$ such that the associated monodromy group is arithmetic (i.e., has finite index in its integral Zariski closure in $\text{GL}_{nD}(\mathbb{Z})$).

Fuchs, Meiri and Sarnak (see [Sar12]) give an infinite family of examples of pairs $\alpha, \beta$ for which the group $\Gamma$ has a natural embedding in an integral orthogonal group $O_n(\mathbb{Z})$ with Zariski dense image and of infinite index. (In [Sar12], they are called *thin* groups.) Thus the monodromy $\Gamma$ is not always arithmetic.

By using [A'C87], an infinite family of examples can be constructed, where the monodromy is an arithmetic subgroup of $\text{Sp}_{2m}(\mathbb{Z})$. One can, using Theorem 2, give a more general formulation: let $d \geq 2$ be an integer and $1 \leq c \leq d$ be an integer coprime to $d$.

**Theorem 25.** Suppose that

$$\alpha = \left( \frac{c}{d} + \frac{1}{n+1}, \ldots, \frac{c}{d} + \frac{n-1}{n+1}, \frac{c}{d} + \frac{n}{n+1} \right),$$

$$\beta = \left( \frac{c}{d} + \frac{1}{n}, \ldots, \frac{c}{d} + \frac{n-1}{n}, 1 \right).$$

If $d = 2$ and $n = 2m$ is even, then the monodromy group $\Gamma$ (is an arithmetic group and) has finite index in the integral symplectic group $\text{Sp}_{2m}(\mathbb{Z})$.

If $d \in \{3, 4, 6\}$ and $n$ is arbitrary such that $n + 1$ is coprime to $d$, then the monodromy $\Gamma$ is an arithmetic group and is of finite index in the integral unitary group $U(h)(\mathbb{Z})$, where $h$ is a suitable Hermitian form defined over the rationals.

If $d \geq 3$ and $n \geq 2d$ is such that $n + 1$ is coprime to $d$, then $\Gamma$ is an arithmetic group. $\Gamma$ has finite index in an integral unitary group of the form $U(h)(O_d)$, where $h$ is a nondegenerate Hermitian form over the totally real number field $K_d = \mathbb{Q}(\cos(\frac{2\pi}{d}))$ and $O_d$ is the ring of integers in $K_d$.

Among these examples, only the case $d = 2$ (treated in [A'C87]) has the property that the monodromy group $\Gamma$ can be conjugated into $\text{GL}_n(\mathbb{Z})$ (in fact it already lies in $\text{Sp}_{2m}(\mathbb{Z})$ with respect to the natural representation given in [BH89]).

As we will see, Theorem 25 is an easy consequence of the fact that the image of the Burau representation of the braid group $B_{n+1}$ on $n + 1$ strands at $d$-th roots of unity is an arithmetic group in the cases stated in the theorem. In case the image of the Burau representation is not arithmetic ([DM86], [McM13]), the image of the monodromy representation defines a *thin group* in the sense of Sarnak ([Sar12]).
That the image of the Burau representation for \( d = 2 \) is a finite index subgroup of the integral symplectic group is a well known theorem of A’Campo ([A’C87]). In the other cases, this is Theorem 2.

We now sketch the relationship between Theorem 25 and the Burau representation. Suppose \( D \) is the differential operator considered at the beginning of this section. Put \( a_j = e^{2\pi i \alpha_j} \), and write \( f(X) = \prod_{j=1}^n (X - a_j) \). Consider the ring \( \mathbb{C}[X]/(f(X)) \). This is a \( \mathbb{C} \)-vector space with the basis \( 1, X, \ldots, X^{n-1} \).

The operator defined by multiplication by \( X \) on the ring \( \mathbb{C}[X]/(f(X)) \) gives a matrix \( A \) with respect to this basis and is called the companion matrix of \( f \).

Similarly, let \( B \) be the companion matrix of \( g(X) = \prod_{j=1}^n (X - b_j) \), where \( b_j = e^{2\pi i \beta_j} \). We will assume henceforth that \( a_j \neq b_k \) for any \( j, k \); i.e., \( f, g \) are coprime.

By results of Levelt ([BH89]), there exists a basis of solutions \( \{u\} = \{u_1, \ldots, u_n\} \) of the equation \( Du = 0 \) on which the monodromy action of \( \pi_1(C) = \pi_1(C, \frac{1}{2}) \) is described as follows. Let \( h_0, h_1, h_\infty \) be small loops in \( C \) going counterclockwise around 0, 1, \( \infty \) exactly once. They generate \( \pi_1(C, \frac{1}{2}) \) and satisfy the relation \( h_0 h_1 h_\infty = 1 \). Under the monodromy representation \( M_{\alpha,\beta} \), the matrix of \( h_0 \) is \( A \), that of \( h_\infty \) is \( B^{-1} \). (Then the matrix of \( h_1 \) is \( A^{-1} B \).) It follows that \( A^{-1} B \) is a complex reflection; i.e., the space of vectors fixed by \( A^{-1} B \) is a codimension one subspace.

Moreover, suppose \( X, Y \in \text{GL}_n(\mathbb{C}) \) are such that (1) the characteristic polynomial of \( X \) is \( f \), and the characteristic polynomial of \( Y \) is \( g \), (2) the matrix \( X^{-1} Y \) is a complex reflection. We then get a representation \( M' \) of \( \pi_1(C) \) by sending \( h_0 \) to \( X \) and \( h_\infty \) to \( Y^{-1} \). The result of Levelt is that the representation \( M' \) is equivalent to \( M_{\alpha,\beta} \) for some \( \alpha, \beta \) such that \( e^{2\pi i \alpha_j} \) for varying \( j \) give all the roots of \( f \); \( \beta \) is chosen similarly for \( g \).

Now consider the Burau representation \( \rho_n : B_{n+1} \rightarrow \text{GL}_n(\mathbb{Z}[q, q^{-1}]) \).

The braid group \( B_{n+1} \) is generated by the two elements \( t_0 = s_1 s_2 \cdots s_n \) and \( t_1 = s_n^{-1} \). Write \( t_\infty = (s_1 s_2 \cdots s_{n-1})^{-1} \). We then get \( t_0 t_1 t_\infty = 1 \). Therefore, we have a surjection from \( \pi_1(C, \frac{1}{2}) \) to \( B_{n+1} \) given by \( h_0 \mapsto t_0 \) and \( h_\infty \mapsto t_\infty \).

Composition of this map with the Burau representation \( \rho_n \) gives a representation \( r : \pi_1(C, \frac{1}{2}) \rightarrow \text{GL}_n(\mathbb{Z}[q, q^{-1}]) \). Put \( X = r(h_0) \) and \( Y = r(h_\infty^{-1}) \). Then \( X^{-1} Y = r(h_1) = \rho_n(t_1) = \rho_n(s_n^{-1}) \), and the formula for the Burau representation shows that the latter matrix is a complex reflection. Secondly, with respect to the standard basis \( e_i \) of the Burau representation, the element \( h_0 \) has the matrix form

\[
X = \rho_n(q)(h_0) = \begin{pmatrix}
0 & 0 & 0 & \cdots & -q \\
q & 0 & 0 & \cdots & -q \\
0 & q & 0 & \cdots & -q \\
\cdots & \cdots & \cdots & \cdots & -q \\
0 & \cdots & q & -q
\end{pmatrix}
\]
and its characteristic polynomial is of the form

\[ \text{Ch}(t, X) = \prod_{j=1}^{n} (t - q e^{2\pi ij/(n+1)}) = f(t), \]

say. Similarly, if \( Y = \rho_n(q)(h_0^{-1}) \), then the characteristic polynomial is of the form

\[ \text{Ch}(t, Y) = (t - 1) \prod_{k=1}^{n-1} (t - q e^{2\pi ik/n}) = g(t), \]

say. It is clear that the two characteristic polynomials do not have a common root. Therefore, by Levelt’s result, \( r \) is equivalent to the monodromy representation \( M_{\alpha, \beta} \) of a suitable hypergeometric equation associated to parameters \( \alpha_j, \beta_j \) where

\[ a_j = q e^{2\pi i j/(n+1)} = e^{2\pi \alpha_j} \quad \text{and} \quad b_j = q e^{2\pi i j/n} = e^{2\pi \beta_j} \]

or 1. Specialise \( q \) to any primitive \( d \)-th root of unity \( e^{2\pi i c/d} \). The resulting representation \( r \) of the group \( \pi(C, 1) \) is equivalent to the monodromy representation associated to the parameters \( \alpha, \beta \) in the theorem, and therefore, it has the same image (up to conjugacy) as the Burau representation \( \rho_n(d) \). Therefore, the arithmeticity of \( M_{\alpha, \beta} \) follows from Theorem 2.

In Theorem 25, we have proved that a very special case of the representation \( M_{\alpha, \beta} \) coincides with the monodromy of the Burau representation; therefore, its arithmeticity follows from Theorem 2 and from [A'C87].

9. Theorem 26 and its proof

Let \( d \geq 2 \) and \( k_1, k_2, \ldots, k_{n+1} \) be integers with \( 1 \leq k_i \leq d - 1 \). Let \( a_1, \ldots, a_{n+1} \) be distinct complex numbers. Consider the affine curve \( X_{a,k} = \{(x, y) \in \mathbb{C}^2\} \) given by the equation

\[ y^d = (x - a_1)^{k_1} \cdots (x - a_{n+1})^{k_{n+1}}. \]

\( X_{a,k} \) is a compact Riemann surface \( X^*_{a,k} \) with finitely many punctures. The space \( S \) of \( a = (a_1, \ldots, a_{n+1}) \in C^{n+1} \) with distinct co-ordinates has fundamental group isomorphic to the “pure braid group” denoted \( P_{n+1} \). It is the kernel to the map \( B_{n+1} \to S_{n+1} \). (See the last paragraph of Section 7.2.) As before, we have a family \( X \to S \) with the fibre over \( a \) being the compact Riemann surface \( X_{a,k}^* \) and we have the monodromy representation of \( P_{n+1} \) on the cohomology group \( H^1(X_{a,k}^*, \mathbb{Z}) \).

**Theorem 26.** If all the \( k_i \) are co-prime to \( d \) and if \( n \geq 2d \), then the image \( \Gamma \) of the monodromy representation is an arithmetic group.

We only sketch the proof. (The proof is much more involved than that of Theorem 1.) The proof of Theorem 1 used the properties of the Burau representation, which were established in Section 4. The proof of Theorem 26 is quite similar, but it uses properties of the reduced Gassner representation.
This is a representation \( g_n(X) : P_{n+1} \to \text{GL}_n(\mathbb{Z}[X_1^{\pm 1}, \ldots, X_{n+1}^{\pm 1}]) \) of the pure braid group \( P_{n+1} \) on the free module of rank \( n \) over the ring of Laurent polynomials with integral coefficients in \( n+1 \) variables \( X_1, \ldots, X_{n+1} \). If \( z_1, \ldots, z_{n+1} \) are complex numbers, then we get a specialisation \( g_n(z) \) of the reduced Gassner representation, called the reduced Gassner representation evaluated at \( z_1, \ldots, z_{n+1} \).

The properties of Gassner representation which we will use are the following. (For the second part of the proposition, see [Abd97].)

**Proposition 27.** The reduced Gassner representation is has a nondegenerate skew Hermitian form \( H \) preserved by \( P_{n+1} \). It is absolutely irreducible. The centre of \( B_{n+1} \) (which lies in \( P_{n+1} \) and is generated by \( \Delta^2 \)) acts by scalars, and \( \Delta^2 \) acts by the scalar \( X_1X_2\cdots X_{n+1} \) on the Gassner representation.

If we specialise \( X_i \mapsto z_i = q^{k_i} \), where \( k_i \) are coprime to \( d \) and \( q \) is a generator of the cyclic group \( \mathbb{Z}/d\mathbb{Z} = q\mathbb{Z}/q^d\mathbb{Z} \), then the reduced Gassner representation evaluated at these \( d \)-th roots of unity is irreducible unless \( z_1z_2\cdots z_{n+1} = 1 \).

If \( z_1\cdots z_{n+1} = 1 \), then reduction of the Hermitian form \( H \) is degenerate and its null space is one dimensional. Moreover, the quotient is irreducible.

In the Burau case, we used the fact that if \( n \equiv -1 \) (mod \( d \)), then the Hermitian form is degenerate (Proposition 16) to produce (many) unipotent elements. We similarly use the last part of Proposition 27 to obtain unipotent elements in the Gassner case.

The analogue of Theorem 2 is the following. As in the introduction, let \( A_d = \mathbb{Z}[q, q^{-1}]/(\Phi_d(q)) \); it is isomorphic to the ring of integers in the \( d \)-th cyclotomic extension \( E_d \) of \( \mathbb{Q} \). Let \( O_d \) denote the ring of integers of the totally real sub-field \( K_d = \mathbb{Q}[q + q^{-1}]/(\Phi_d(q)) \) of \( E_d \).

**Theorem 28.** Let \( g_n(d) : P_{n+1} \to \text{GL}_n(A_d) \) denote the reduced Gassner representation evaluated at \( X_i = q_0^{k_i} \), where \( k_i \) are coprime to \( d \) and \( q_0 \in A_d \) is the image of \( q \). If \( n \geq 2d \), then the image of \( g_n(d) \) is an arithmetic subgroup of a suitable unitary group.

The proof of Theorem 28 is similar to that of Theorem 2. In the case of the Burau representation, after we constructed sufficiently many unipotent elements, Theorem 2 could be proved using the fact (see the section on the proof of Theorem 2) that the \( K \)-rank of the associated unitary group was \( \geq 2 \), provided \( n \geq 2d \). The argument was as follows. Write \( n+1 = d + m + d \), where \( m \geq 1 \). Then the span of the first \( d \) basis elements \( e_1, \ldots, e_d \) of the Burau representation contains an isotropic vector \( v \), and similarly the span of the last \( d \) basis elements \( e_{n-d+1}, \ldots, e_n \) contains an isotropic vector \( v' \). If
m \geq 1$, then the two sets of basis elements are orthogonal and hence $v, v'$ are orthogonal for the hermitian form $h_n$. Hence the K-rank of the unitary group is at least 2.

We argue similarly in the case of the Gassner representation. As was remarked at the end of Proposition 27, we can get unipotent elements if there are subsets $X$ of the indexing set $1, 2, \ldots, n$ such that $\prod_{i \in X} z_i = 1$. Put $z_i = q^{k_i}$; an argument using the pigeon-hole principle implies that if $n \geq 2d$, then there are two disjoint subsets $X, Y$ of the set $\{1, 2, \ldots, n\}$ such that $\prod_{i \in X} z_i = 1$ and $\prod_{j \in Y} z_j = 1$. (In the Burau case, $X = \{1, 2, \ldots, d\}$ and $Y = \{n, n-1, \ldots, n-d+1\}$ will suffice.) By the third part of Proposition 27, it follows that the span of $e_i$ for $i \in X$ contains an isotropic vector $v_X$. Similarly, the span of $e_j$ for $j \in Y$ contains an isotropic vector $v_Y$. The Hermitian form preserved by the image of the Gassner representation is such that if $X, Y$ are disjoint and their union is a proper subset of $\{1, 2, \ldots, n\}$, then $v_X, v_Y$ are orthogonal, and hence the K-rank of the associated unitary group is at least two. Moreover, Proposition 27 applied to the set $X$ (in place of the set $1, 2, \ldots, n$) implies that we have many unipotent elements in the image of the Gassner representation. By appealing to Theorem 7, we then deduce Theorem 28.

The proof of Theorem 26 is deduced from Theorem 28, by relating the monodromy representation, to the Gassner representation. The proof of this relationship is very similar to the proof in Section 7 relating monodromy and the Burau representation (but is much more involved and we omit the details).

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